D-Modules on the Affine Flag Variety and Representations of Affine Kac-Moody Algebras

D-MODULES ON THE AFFINE FLAG VARIETY AND REPRESENTATIONS OF AFFINE KAC-MOODY ALGEBRAS

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Introduction

0.1. Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra corresponding to a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod denote the category of (continuous) $\widehat{\mathfrak{g}}$-modules at the critical level (see [FG2] for the precise definition).

It is often the case in representation theory that in order to gain a good understanding of a category of modules of some sort, one has to reinterpret it in more geometric terms, by which we mean either as the category of D-modules on an algebraic variety, or as the category of quasi-coherent sheaves on some (usually, different) algebraic variety. This is what we do in this paper for a certain subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod, thereby proving two conjectures from [FG2].

There is another angle under which this paper can be viewed: the results concerning $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod fit into the framework of the geometric local Langlands correspondence. We refer the reader to the introduction to [FG2] where this view point is explained in detail.

0.2. Localization. Let us first describe the approach via D-modules. This pattern is known as *localization*, a prime example of which is the equivalence of [BB] between the category of $\mathfrak{g}$-modules with a given central character and the category of (twisted) D-modules on the flag variety G/B.

The affine analog of G/B is the affine flag scheme $G(\!(t)\!)/I$, where $I \subset G(\!(t)\!)$ is the Iwahori subgroup. By taking sections of (critically twisted) D-modules, we obtain a functor

$$\Gamma_{\mathrm{Fl}} : \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}\,.$$

However, as in the finite-dimensional case, one immediately observes that the $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules that one obtains in this way are not arbitrary, but belong to a certain subcategory singled out by a condition on the action of the center $Z(\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{crit}})$, where $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{crit}}$ is the (completed, reduced) universal enveloping algebra at the critical level.

Namely, $\mathfrak{Z}_{\mathfrak{g}} := Z(\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{crit}})$ is a topological commutative algebra, which according to [FF], admits the following explicit description in terms of the Langlands dual group $\check{G}$: the ind-scheme $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}})$ is isomorphic to the ind-scheme $\mathrm{Op}(\mathcal{D}^\times)$ of $\check{G}$-opers on the formal punctured disc. This ind-scheme of opers was introduced in [BD], and it contains a closed subscheme denoted $\mathrm{Op}^{\mathrm{nilp}}$ which corresponds to opers with a nilpotent singularity, introduced in [FG2].

It is rather straightforward to see that the image of the functor Γ_{Fl} lands in the subcategory $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{nilp}} \subset \widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod consisting of modules, whose support over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}) \simeq \mathrm{Op}(\mathcal{D}^\times)$ is contained in $\mathrm{Op}^{\mathrm{nilp}}$. Thus, we can consider Γ_{Fl} as a functor

$$\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{nilp}}\,. \tag{0.1}$$

We should remark that it is here that the assumption that we work at the critical level becomes crucial:

For any level κ one can consider the corresponding functor $\Gamma_{\mathrm{Fl}} : \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_\kappa\text{–mod} \to \widehat{\mathfrak{g}}_\kappa$–mod, and it is again relatively easy to see that this functor cannot be essentially surjective. However, in the non-critical case the image category is much harder to describe: we cannot do this by imposing a condition on the action of the center $Z(\widetilde{U}(\widehat{\mathfrak{g}})_\kappa)$ (as we did in the finite-dimensional case, or in the affine case at the critical level) since the latter is essentially trivial. This fact prevents one from proving (or even formulating) a localization type equivalence in the non-critical case.

0.3. Non-exactness. Returning to the analysis of the functor (0.1) we observe two phenomena that distinguish the present situation from the finite-dimensional case of [BB].

First, unlike the case of the finite-dimensional flag variety, the functor Γ_{Fl} is not exact (and cannot be made exact by any additional twisting). This compels us to leave the hopes of staying within the realm of abelian categories, and pass to the corresponding derived ones. I.e., from now on we will be considering the derived functor of Γ_{Fl}, denoted by a slight abuse of notation by the same character

$$\Gamma_{\mathrm{Fl}} : \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}). \tag{0.2}$$

The necessity do work with triangulated categories as opposed to abelian ones accounts for many of the technical issues in this paper, and ultimately, for its length.

That said, we should remark that in Sect. 2 we define a new t-structure on the category $\mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ and make a conjecture that in this new t-structure the functor Γ_{Fl} is exact.

0.4. Base change. The second new phenomenon present in the case of affine Kac-Moody algebras is that the (derived) functor Γ_{Fl} is not fully faithful. The reason is very simple: the center of the category $\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ is essentially trivial, whereas that of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}$ is the algebra of functions on the scheme $\mathrm{Op}^{\mathrm{nilp}}$.

I.e., by setting the level to critical we have gained the center, which allows to potentially describe the image of Γ_{Fl}, but we have gained too much: instead of just one central character as in the finite-dimensional case, we obtain a $\mathrm{Op}^{\mathrm{nilp}}$-worth of those.

However, the non-fully faithfulness of Γ_{Fl} can be accounted for.

Let $\tilde{\check{\mathcal{N}}} := T^*(\mathrm{Fl}^{\check{G}})$ be the Springer variety corresponding to the Langlands dual group, where $\mathrm{Fl}^{\check{G}}$ denotes the flag variety of $\check{G}$. [1] Consider the stack $\tilde{\check{\mathcal{N}}}/\check{G}$. A crucial piece of structure is that the monoidal triangulated category $\mathbf{D}^b(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))$ (i.e., the $\check{G}$-equivariant derived category of coherent sheaves on $\tilde{\check{\mathcal{N}}}$) acts on the triangulated category $\mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$. This action was constructed in the paper [AB], and we denote it here by $\star$.

Another observation is that there is a natural map $\mathfrak{r}_{\mathrm{nilp}} : \mathrm{Op}^{\mathrm{nilp}} \to \tilde{\check{\mathcal{N}}}/\check{G}$, and we show that these structures are connected as follows. For $\mathcal{F} \in \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, $\mathcal{M} \in \mathbf{D}^b(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))$ we have a canonical isomorphism

$$\Gamma_{\mathrm{Fl}}(\mathcal{F} \star \mathcal{M}) \simeq \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{M}).$$

I.e., the effect of acting on $\mathcal{F}$ by $\mathcal{M}$ and then taking sections is the same as that of first taking sections and then tensoring over the algebra of functions on $\mathrm{Op}^{\mathrm{nilp}}$ by the pull-back of $\mathcal{M}$ by means of $\mathfrak{r}_{\mathrm{nilp}}$.

This should be viewed as a categorical analog of the following situation in linear algebra. Let V_1 be a vector space, acted on by an algebra A_1 by endomorphisms (i.e., V_1 is a A_1-module). Let (V_2, A_2) be another such pair; let $r_A : A_1 \to A_2$ be a homomorphism of algebras, and $r_V : V_1 \to V_2$ a map of vector spaces, compatible with the actions.

In this case, we obtain a map $A_2 \underset{A_1}{\otimes} V_1 \to V_2$.

[1] We emphasize that $\mathrm{Fl}^{\check{G}}$ denotes the finite-dimensional flag variety $\check{G}/\check{B}$ of the Langlands dual group $\check{G}$, and in this paper we will consider quasi-coherent sheaves on it. It should not be confused with the affine flag scheme $G((t))/I$ of G, denoted $\mathrm{Fl}^{\mathrm{aff}}_G$, on which we will consider D-modules.

We would like to imitate this construction, where instead of vector spaces we have categories:

$$V_1 \mapsto \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),\ V_2 \mapsto \mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}),$$

instead of algebras we have monoidal categories

$$A_1 \mapsto \mathbf{D}^b(\mathrm{Coh}(\widetilde{\tilde{\mathcal{N}}}/\check{G})),\ A_2 \mapsto \mathbf{D}^b(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}})),$$

and instead of maps we have functors:

$$r_A \mapsto \mathfrak{r}^*_{\mathrm{nilp}},\ r_V \mapsto \Gamma_{\mathrm{Fl}}.$$

Therefore, it is a natural idea to try to define a categorical tensor product

$$\mathbf{D}^b(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}})) \underset{\mathbf{D}^b(\mathrm{Coh}(\widetilde{\tilde{\mathcal{N}}}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}), \tag{0.3}$$

which can be viewed as a base change of $\mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ with respect to the morphism $\mathfrak{r}_{\mathrm{nilp}} : \mathrm{Op}^{\mathrm{nilp}} \to \widetilde{\tilde{\mathcal{N}}}/\check{G}$, and a functor from (0.3) to $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, denoted $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$, compatible with the action of $\mathbf{D}^b(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$. Unlike Γ_{Fl}, the new functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ has a chance of being an equivalence.

0.5. Localization results. The above procedure of taking the tensor product can indeed be carried out, and is the subject of Part III of this paper. I.e., we can define the category in (0.3) as well as the functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$.

We conjecture that $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ is an equivalence, which would be a complete localization result in the context of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$. Unfortunately, we cannot prove it at the moment. We do prove, however, that $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ is fully faithful; this is one of the four main results of this paper, Main Theorem 1.

In addition, we prove that $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ does induce an equivalence between certain subcategories on both sides, namely the subcategories consisting of Iwahori-monodromic objects. This is the second main result of the present paper, Main Theorem 2. The Iwahori-monodromic subcategory on the RHS, denoted $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$, can be viewed as a critical level version of the category $\mathcal{O}$. Thus, Main Theorem 2, provides a localization description at least for this subcategory.

We should remark that our inability to prove the fact that $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ is an equivalence for the ambient categories stems from our lack of any explicit information about objects of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$ other than the Iwahori-monodromic ones.

The above results that relate the categories $\mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ and $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ have analogues, when instead of $\mathrm{Fl}^{\mathrm{aff}}_G$ we consider the affine Grassmannian $\mathrm{Gr}^{\mathrm{aff}}_G$, and instead of $\mathrm{Op}^{\mathrm{nilp}} \subset \mathrm{Op}(\mathcal{D}^\times)$ we consider the sub-scheme of regular opers $\mathrm{Op}^{\mathrm{reg}} \subset \mathrm{Op}(\mathcal{D}^\times)$. These results will be recalled below in Sect. 0.7.

0.6. The quasi-coherent picture. Let us now pass to the description of a subcategory of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}$ in terms of quasi-coherent sheaves, mentioned in Sect. 0.1. The subcategory in question, or rather its triangulated version, is $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ that has appeared above.

In [FG2] we proposed (see Conjecture 6.2 of *loc. cit.*) that $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ should be equivalent to the category $\mathbf{D}^b(\mathrm{QCoh}(\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}}))$, where $\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}}$ is the scheme classifying *Miura opers with a nilpotent singularity*, introduced in [FG2], Sect. 3.14.

By definition, $\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}}$ is the fiber product

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \check{\mathrm{St}}/\check{G},$$

where $\check{\mathrm{St}}$ is the Steinberg scheme. In other words, $\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}}$ is the moduli space of pairs: an oper χ on the formal punctured disc $\mathcal{D}^\times$ with a nilpotent singularity, and its reduction to the Borel subgroup $\check{B} \subset \check{G}$ as a local system.

The motivation for the above conjecture was that for any point $\widetilde{\chi} \in \mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}}$ one can attach a specific object $\mathbb{W}_{\widetilde{\chi}} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{I^0}_{\mathrm{nilp}}$, called the Wakimoto module, and the conjecture can be viewed as saying that any object of $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ is canonically a "direct integral" of Wakimoto modules.

In this paper we prove this conjecture by combining our Main Theorem 2 and one of the main results of Bezrukavnikov's theory (Theorem 4.2 of [Bez]), which provides an equivalence

$$\mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \simeq \mathbf{D}^b(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})). \tag{0.4}$$

The proof is obtained by essentially base-changing both sides of (0.4) with respect to $\mathfrak{r}_{\mathrm{nilp}}$. This is the third main result of this paper, Main Theorem 3.

We refer the reader to the introduction to [FG2] for the explanation how the above corollary can be viewed as a particular case of the local geometric Langlands correspondence.

As a corollary we obtain the following result: let $\chi \in \mathrm{Op}^{\mathrm{nilp}}$ be an oper with a nilpotent singularity. On the one hand, let us consider $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi)^{I^0}$, which is the full subcategory of $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi)$–the derived category of $\widehat{\mathfrak{g}}$-modules with central character given by χ, consisting of I^0-integrable objects.

On the other hand, let n be an element of $\check{\mathfrak{n}}$, whose image in $\check{\mathfrak{n}}/\check{B} \simeq \tilde{\check{\mathcal{N}}}/\check{G}$ equals that of $\mathfrak{r}_{\mathrm{nilp}}(\chi)$. Let Spr_n be the derived Springer fiber over n, i.e., the Cartesian product $\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \tilde{\check{\mathcal{N}}}$, taken in the category of DG-schemes. We obtain:

Corollary. *There is an equivalence of categories*

$$\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi)^{I^0} \simeq \mathbf{D}^b(\mathrm{QCoh}(\mathrm{Spr}_n)).$$

In this paper we do not prove, however, that the functor thus obtained is compatible with the Wakimoto module construction. Some particular cases of this assertion have been established in [FG5]. We expect that the general case could be established by similar, if more technically involved, methods.

0.7. Relation to the affine Grassmannian. The main representation-theoretic ingredient in the proof of the main results of this paper is the fully faithfulness assertion, Main Theorem 1. Its proof is based on comparison between the functors Γ_{Fl} and $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ mentioned above, and the corresponding functors when the affine flag scheme $\mathrm{Fl}^{\mathrm{aff}}_G$ is replaced by the affine Grassmannian $\mathrm{Gr}^{\mathrm{aff}}_G$.

Let us recall that in [FG4] we considered the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$, corresponding to modules whose support over $\mathrm{Spec}(\mathfrak{Z}_{\mathfrak{g}}) \simeq \mathrm{Op}(\mathcal{D}^\times)$ is contained in the subscheme $\mathrm{Op}^{\mathrm{reg}}$ of regular opers.

We also considered the category $\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ and a functor $\Gamma_{\mathrm{Gr}} : \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$, which by contrast with the case of $\mathrm{Fl}^{\mathrm{aff}}_G$, was exact. In addition, the category

$\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ was naturally acted upon by $\mathrm{Rep}^{f.d.}(\check{G}) \simeq \mathrm{Coh}(\mathrm{pt}/\check{G})$, and we considered the base changed category

$$\mathrm{Coh}(\mathrm{Op}^{\mathrm{reg}}) \underset{\mathrm{Coh}(\mathrm{pt}/\check{G})}{\otimes} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod},$$

and the functor $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}$ from it to $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$.

In *loc. cit.* it was shown that on the level of derived categories, the corresponding functor

$$\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}} : \mathbf{D}^b\left(\mathrm{Coh}(\mathrm{Op}^{\mathrm{reg}}) \underset{\mathrm{Coh}(\mathrm{pt}/\check{G})}{\otimes} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}\right) \simeq$$
$$\simeq \mathbf{D}^b(\mathrm{Coh}(\mathrm{Op}^{\mathrm{reg}})) \underset{\mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$$

was fully faithful.

The relation between the $\mathrm{Fl}^{\mathrm{aff}}_G$ and the $\mathrm{Gr}^{\mathrm{aff}}_G$ pictures is provided by our fourth main result, Main Theorem 4. This theorem asserts that the base-changed category

$$\mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B})) \underset{\mathbf{D}^b(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

with respect to the map

$$\mathrm{pt}/\check{B} \simeq \mathrm{Fl}^{\check{G}}/\check{G} \hookrightarrow \tilde{\check{\mathcal{N}}}/\check{G},$$

given by the 0-section $\mathrm{Fl}^{\check{G}} \hookrightarrow \tilde{\check{\mathcal{N}}}$ is essentially equivalent to the base-changed category

$$\mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B})) \underset{\mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

This equivalence makes it possible to write down a precise relationship between the functors $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ and $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}$ (see Theorem 6.3.1), and deduce Main Theorem 1 from the fully-faithfulness of $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}$ (see Sect. 13).

0.8. Contents. This paper consists of four parts:

In Part I we perform the representation-theoretic and geometric constructions and formulate the main results.

In Sect. 1 we show how the constructions of the paper [AB] define an action of the triangulated category $\mathbf{D}^b(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))$ on $\mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$. In fact, the action at the level of triangulated categories comes naturally from a finer structure at a DG (differential graded) level. The latter fact allows to introduce the category (0.3), which is one of the main players in this paper.

In Sect. 2 we introduce a new t-structure on the category $\mathbf{D}^b(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$. It will turn out that this t-structure has a better behavior than the usual one with respect to the constructions that we perform in this paper.

In Sect. 3 we combine some results of [FG3] and [FG4] and show how the functor Γ_{Fl} extends to a functor

$$\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}} : \mathbf{D}^b(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}})) \underset{\mathbf{D}^b(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}),$$

We formulate the first main result of this paper, Main Theorem 1 that asserts that the functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ is fully faithful. As was explained above, the latter result is as close as we are currently able to get to *localization* at the critical level. We also formulate Conjecture 3.6.3 to the effect that the functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ is an equivalence.

In Sect. 4 we consider the Iwahori-monodromic subcategories in the framework of Conjecture 3.6.3. We formulate the second main result of this paper, Main Theorem 2, which asserts that the functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ induces an equivalence

$$\mathbf{D}^b(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}})) \underset{\mathbf{D}^b(\mathrm{Coh}(\widetilde{\tilde{\mathcal{N}}}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})^{I^0} \to \mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}.$$

In addition, we formulate Main Theorem 3, which sharpens the description of the category $\mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ in terms of quasi-coherent sheaves on the scheme of Miura opers, proposed in [FG2].

In Sect. 5 we formulate Main Theorem 4, which essentially expresses the category $\mathbf{D}^b(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ in terms of $\mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ and the action of $\mathbf{D}^b(\mathrm{Coh}(\widetilde{\tilde{\mathcal{N}}}/\check{G}))$ on it. More precisely, we construct a functor

$$(0.5)\quad \Upsilon : \mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B})) \underset{\mathbf{D}^b(\mathrm{Coh}(\widetilde{\tilde{\mathcal{N}}}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B})) \underset{\mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{G}))}{\otimes} \mathbf{D}^b(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}),$$

which will turn out to be "almost" an equivalence.

In Sect. 6 we formulate a theorem that connects the functors $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ and $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}$ by means of the functor Υ of Sect. 5.

In Part II we prove the theorems formulated in Part I, assuming a number of technical results, which will be the subject of Parts III and IV.

In Sect. 7 we prove a number of adjunction properties related to the functor Υ of Sect. 5, and reduce the fully faithfulness result of Main Theorem 4 to a certain isomorphism in $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$, given by Theorem 7.3.1.

In Sect. 8 we prove Theorem 7.3.1. We give two proofs, both of which use Bezrukavnikov's theory. One proof uses some still unpublished results of [Bez], while another proof uses only [AB].

In Sect. 9 we study the interaction between the new t-structure on $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ and the usual t-structure on $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$.

In Sect. 10 we use the results of Sect. 9 to complete the proof of Theorem 7.3.1.

In Sect. 11 we show how to modify the LHS of (0.5) and the functor Υ to turn it into an equivalence. We should remark that the results of this section are not needed for the proofs of the main theorems of this paper.

In Sect. 12 we prove the theorem announced in Sect. 6 on how the functor Υ intertwines between the functors $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ and $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}$.

In Sect. 13 we use the fact that Υ is fully faithful to deduce fully faithfulness of the functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$, which is our Main Theorem 1.

In Sect. 14 we prove Main Theorem 2 and Main Theorem 3.

In Part III we develop the machinery used in Parts I and II that has to do with the notion of tensor product of (triangulated) categories over a (triangulated) monoidal category. As is often the case in homotopy theory, the structure of triangulated category is not rigid enough for the constructions that we perform. For that reason we will have to deal with DG categories rather than with the triangulated ones. [2]

In Sect. 15 we recall the basics of DG categories and their relation to triangulated categories. Essentially, we summarize some of the results of [Dr].

In Sect. 16 we review the notion of homotopy monoidal structure on a DG category. Our approach amounts to considering a pseudo-monoidal structure, which yields a monoidal structure on the homotopy level; this idea was explained to us by A. Beilinson. We should note that one could consider a more flexible definition following the prescription of [Lu]; however, as was explained to us by J. Lurie, the two approaches are essentially equivalent.

Sect. 17 deals with the tensor product of categories, which is a central object for all the constructions in Part I. Given two DG categories $\mathbf{C}_1$ and $\mathbf{C}_2$ acted on by a monoidal DG category $\mathbf{A}$ on the left and on the right, respectively, we define a new DG category $\mathbf{C}_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}_2$. This construction was explained to us by J. Lurie. It essentially consists of taking the absolute tensor product $\mathbf{C}_1 \otimes \mathbf{C}_2$ and imposing the isomorphisms $(\mathbf{c}_1 \cdot \mathbf{a}) \otimes \mathbf{c}_2 \simeq \mathbf{c}_1 \otimes (\mathbf{a} \cdot \mathbf{c}_2)$, where $\mathbf{c}_i \in \mathbf{C}_i$ and $\mathbf{a} \in \mathbf{A}$.

In Sect. 18 we study the properties of the tensor product construction which can be viewed as generalizations of the projection formula in algebraic geometry.

In Sect. 19 we recollect some facts related to the notion of t-structure on a triangulated category.

In Sect. 20 we study how the tensor product construction interacts with t-structures. In particular, we study the relationship between tensor products at the triangulated and abelian levels.

In Sect. 21 we apply the constructions of Sects. 16, 17, 18, 19 and 20 in the particular case when the monoidal triangulated category is the perfect derived category of coherent sheaves on an algebraic stack. In this way we obtain the notion of triangulated category over a stack, and that of base change with respect to a morphism of stacks.

Part IV is of technical nature: we discuss the various triangulated categories arising in representation theory.

Sect. 22 contains a crucial ingredient needed to make the constructions in Part I work. It turns out that when dealing with infinite-dimensional objects such as $\mathrm{Fl}_G^{\mathrm{aff}}$ or $\widehat{\mathfrak{g}}$, the usual triangulated categories associated to them, such as the derived category $\mathbf{D}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{--mod})$ of D-modules in the former case, and the derived category $\mathbf{D}(\widehat{\mathfrak{g}}\text{--mod})$ of $\widehat{\mathfrak{g}}$-representations in the latter case, are not very convenient to work with. The reason is that these categories have too few compact objects. We show how to modify such categories "at $-\infty$" (i.e., keeping the corresponding $\mathbf{D}^+$ subcategories unchanged), so that the resulting categories are still cocomplete (i.e., contain arbitrary direct sums), but become compactly generated.

[2]Our decision to work in the framework of DG categories rather than in a better behaved world of quasi-categories stems from two reasons. One is the fact that we have not yet learnt the latter theory well enough to apply it. The other is that we are still tempted to believe that when working with linear-algebraic objects over a field of characteristic zero, on can construct a homotopy-theoretic framework based on DG categories, which will avoid some of the combinatorial machinery involved in dealing with simplicial sets when proving foundational results on quasi-categories.

In Sect. 23 we apply the discussion of Sect. 22 in the two examples mentioned above, i.e., $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod})$ and $\mathbf{D}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$, and study the resulting categories, denoted $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod})$ and $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$, respectively.

Having developed the formalism of monoidal actions, tensor products, and having defined the desired representation-theoretic categories equipped with DG models, in Sect. 24 we upgrade to the DG level the constructions from Part I, which were initially carried out at the triangulated level.

In Sect. 25 we show that imposing the condition of I-monodromicity (which in our case coincides with that of I^0-equivariance) survives the manipulations of Sects. 23 and 24.

0.9. Notation. Notation and conventions in this paper follow closely those of [FG2].

We fix G to be a semi-simple simply connected group over a ground field, which is algebraically closed and has characteristic zero. We shall denote by $\check{\Lambda}$ the lattice of coweights corresponding to G and by $\check{\Lambda}^+$ the semi-group of dominant co-weights.

We let $\check{G}$ denote the Langlands dual group of G. Let $\check{\mathfrak{g}}$ be the Lie algebra of $\check{G}$. Let $\check{B} \subset \check{N}$ be the Borel subgroup and its unipotent radical, and $\check{\mathfrak{b}} \subset \check{\mathfrak{n}}$ be their Lie algebras, respectively.

Let $\mathrm{Fl}^{\check{G}}$ be the flag variety of $\check{G}$, thought of as a scheme parameterising Borel subalgebras in $\check{\mathfrak{g}}$. For $\check{\lambda} \in \check{\Lambda}$ we denote by $\mathcal{L}^{\check{\lambda}}$ the corresponding line bundle on $\mathrm{Fl}^{\check{G}}/\check{G} \simeq \mathrm{pt}/\check{B}$. Our normalization is such that $\mathcal{L}^{\check{\lambda}}$ is ample if $\check{\lambda}$ is dominant, and $\Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{L}^{\check{\lambda}}) = V^{\check{\lambda}}$, the irreducible representation of highest weight $\check{\lambda}$.

We denote by $\tilde{\check{\mathfrak{g}}}$ Grothendieck's alteration. This is the *tautological* sub-bundle in the trivial vector bundle $\mathrm{Fl}^{\check{G}} \times \check{\mathfrak{g}}$. Let $\tilde{\check{\mathcal{N}}} \subset \tilde{\check{\mathfrak{g}}}$ be the Springer resolution, i.e., it is the variety of pairs $\{(x \in \check{\mathfrak{g}}, \check{\mathfrak{b}}' \in \mathrm{Fl}^{\check{G}}) \,|\, x \in \check{\mathfrak{n}}'\}$. We denote by π the natural projection $\tilde{\check{\mathcal{N}}} \to \mathrm{Fl}^{\check{G}}$, and by ι the zero section $\mathrm{Fl}^{\check{G}} \to \tilde{\check{\mathcal{N}}}$.

When discussing opers or Miura opers, we will mean these objects with respect to the group $\check{G}$ (and never G), so the subscript "$\check{G}$" will be omitted.

We will consider the affine Grassmannian $\mathrm{Gr}_G^{\mathrm{aff}} := G((t))/G[[t]]$ and the affine flag scheme $\mathrm{Fl}_G^{\mathrm{aff}} := G((t))/I$. We will denote by p the natural projection $\mathrm{Fl}_G^{\mathrm{aff}} \to \mathrm{Gr}_G^{\mathrm{aff}}$.

For an element $\widetilde{w}$ in the affine Weyl group, we shall denote by $j_{\widetilde{w},!}$ (resp., $j_{\widetilde{w},*}$) the corresponding standard (resp., co-standard) I-equivariant objects of $\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})$–mod. For $\check{\lambda} \in \check{\Lambda}$ we denote by $J_{\check{\lambda}} \in \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})$–mod the corresponding Mirkovic-Wakimoto D-module, which is characterized by the property that $J_{\check{\lambda}} = j_{\check{\lambda},*}$ for $\check{\lambda} \in \check{\Lambda}^+$, $J_{\check{\lambda}} = j_{\check{\lambda},!}$ for $\check{\lambda} \in -\check{\Lambda}^+$ and $J_{\check{\lambda}_1+\check{\lambda}_2} = J_{\check{\lambda}_1} \star J_{\check{\lambda}_2}$.

The geometric Satake equivalence (see [MV]) defines a functor from the category $\mathrm{Rep}(\check{G})$ to that of $G[[t]]$-equivariant objects in $\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})$–mod. We denote this functor by $V \mapsto \mathcal{F}_V$. The construction of [Ga] defines for every V as above an object $Z_V \in \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})$–mod, which is I-equivariant, and is *central*, a property that will be reviewed in the sequel.

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Part I: Constructions

1. The Arkhipov-Bezrukavnikov action

Let $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ be the bounded derived category of finitely generated critically twisted D-modules on $\mathrm{Fl}^{\mathrm{aff}}_G$. It is well-defined since $\mathrm{Fl}^{\mathrm{aff}}_G$ is a strict ind-scheme of ind-finite type.

The goal of this section is to endow $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ with a structure of triangulated category over the stack $\tilde{\check{\mathfrak{N}}}/\check{G}$ (see Sect. 21.2.1 for the precise definition of what this means). I.e., we will make the triangulated monoidal category of perfect complexes on $\tilde{\check{\mathfrak{N}}}/\check{G}$ act on $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$. The action must be understood in the sense of triangulated categories, equipped with DG models (see Sect. 16.5.4). In the present section, we will perform the construction at the triangulated level only, and refer the reader to Sect. 24.2, where it is upgraded to the DG level.

1.1. Let $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathfrak{N}}}/\check{G}))$ be the perfect derived category on the stack $\tilde{\check{\mathfrak{N}}}/\check{G}$, as introduced in Sect. 21.2, i.e.,

$$\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathfrak{N}}}/\check{G})) := \mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right) / \mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right),$$

where $\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G})$ is the DG category of bounded complexes of locally free coherent sheaves on $\tilde{\check{\mathfrak{N}}}/\check{G}$, and $\mathbf{C}^b_{acycl}(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G})$ is the subcategory of acyclic complexes. The former has a natural structure of DG monoidal category, and the latter is a monoidal ideal, making the quotient $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathfrak{N}}}/\check{G}))$ into a triangulated monoidal category.

In order to define the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathfrak{N}}}/\check{G}))$ on $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathfrak{N}}}/\check{G}))$, following [AB], we will use a different realization of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathfrak{N}}}/\check{G}))$ as a quotient of an explicit triangulated monoidal category $\mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right)$ by a monoidal ideal.

Namely, let $\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G}))$ be the monoidal DG subcategory of $\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G}))$, consisting of complexes, whose terms are direct sums of coherent sheaves of the form $\pi^*(\mathcal{L}^{\check{\lambda}})\otimes V$, where V is a finite dimensional representation of $\check{G}$, and $\mathcal{L}^{\check{\lambda}}$ is the line bundle on $\mathrm{Fl}^{\check{G}}/\check{G} \simeq \mathrm{pt}/\check{B}$ corresponding to $\check{\lambda} \in \check{\Lambda}$, and π denotes the projection $\tilde{\check{\mathfrak{N}}} \to \mathrm{Fl}^{\check{G}}$. (We remind that our normalization is that for $\check{\lambda} \in \check{\Lambda}^+$, $\Gamma(\mathrm{Fl}^{\check{G}}, \mathcal{L}^{\check{\lambda}}) \simeq V^{\check{\lambda}}$, the representation of highest weight $\check{\lambda}$.)

Let $\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G}))$ be the monoidal ideal

$$\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G})) \cap \mathbf{C}^b_{acycl}(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G})).$$

By [AB], Lemma 20, the natural functor

$$\mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right) / \mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right) \to$$
$$\to \mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right) / \mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathrm{Coh}^{loc.free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right) \mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathfrak{N}}}/\check{G}))$$

is an equivalence.

1.2. We claim, following [AB], that there exists a natural action of $\mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right)$ on $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, with $\mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\check{\mathfrak{N}}}/\check{G}))\right)$ acting trivially.

1.2.1. First, we construct a DG monoidal functor F from $\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$ to a DG monoidal subcategory of the DG category of finitely generated I-equivariant D-modules on $\mathrm{Fl}_G^{\mathrm{aff}}$. The functor F will have the property that if $\mathcal{F}_1$ and $\mathcal{F}_2$ appear as terms of complexes of some $\mathsf{F}(\mathcal{M}_1^\bullet)$ and $\mathsf{F}(\mathcal{M}_2^\bullet)$, respectively, for $\mathcal{M}_1^\bullet, \mathcal{M}_2^\bullet \in \mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$, the convolution $\mathcal{F}_1 \star \mathcal{F}_2 \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ is acyclic off cohomological degree 0.

Remark. In Sect. 2 a new t-structure on (an ind-completion of) $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ will be defined, and we will prove that the functor F is exact in this t-structure, see Corollary 4.2.3.

The functor F is characterized uniquely by the following conditions:

- For $\check{\lambda} \in \check{\Lambda}$, $\mathsf{F}(\pi^*(\mathcal{L}^{\check{\lambda}})) := J_{\check{\lambda}}$. The isomorphism $\mathcal{L}^{\check{\lambda}_1} \otimes \mathcal{L}^{\check{\lambda}_2} \simeq \mathcal{L}^{\check{\lambda}_1+\check{\lambda}_2}$ goes under F to the natural isomorphism $J_{\check{\lambda}_1} \star J_{\check{\lambda}_2} \simeq J_{\check{\lambda}_1+\check{\lambda}_2}$.
- For $V \in \mathrm{Rep}^{f.d.}(\check{G})$, $\mathsf{F}(\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V) := Z_V$, where Z_V is the corresponding central sheaf. We have commutative diagrams
$$\begin{array}{ccc} \mathsf{F}(\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V^1) \star \mathsf{F}(\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V^2) & \longrightarrow & Z_{V^1} \star Z_{V^2} \\ \downarrow & & \downarrow \\ \mathsf{F}\left((\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes (V^1 \otimes V^2)\right) & \longrightarrow & Z_{V^1 \otimes V^2} \end{array}$$
and
$$\begin{array}{ccc} \mathsf{F}(\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V) \star \mathsf{F}(\pi^*(\mathcal{L}^{\check{\lambda}})) & \longrightarrow & Z_V \star J_{\check{\lambda}} \\ \downarrow & & \downarrow \\ \mathsf{F}(\pi^*(\mathcal{L}^{\check{\lambda}})) \star \mathsf{F}(\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V) & \longrightarrow & J_{\check{\lambda}} \star Z_V, \end{array}$$
where the right vertical maps are the canonical morphisms of [Ga], Theorem 1(c) and (b), respectively.
- The tautological endomorphism N_V^{taut} of the object $\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V$ goes over under F to the monodromy endomorphism N_{Z_V} of Z_V (the latter morphism is given by Theorem 2 of [Ga] and is denoted there by M).
- For $\check{\lambda} \in \check{\Lambda}^+$ the canonical map $\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V^{\check{\lambda}} \to \pi^*(\mathcal{L}^{\check{\lambda}})$ goes over to the map $Z_{V^{\check{\lambda}}} \to J_{\check{\lambda}}$, given by Lemma 9 of [AB].

Thus, we obtain the desired functor F. It is shown in [AB], Lemma 20(a), that if $\mathcal{M}^\bullet \in \mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$, then $\mathsf{F}(\mathcal{M}^\bullet)$ is acyclic as a complex of D-modules on $\mathrm{Fl}_G^{\mathrm{aff}}$.

1.2.2. The assignment

$$\mathcal{M} \in \mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))\right), \mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \mapsto \mathcal{F} \star \mathsf{F}(\mathcal{M}) \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$$

defines a functor

$$\mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))\right) \times \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}),$$

which is the sought-for monoidal action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$ on $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$. For $\mathcal{M}$ and $\mathcal{F}$ as above, we denote the resulting object of $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ by

$$\mathcal{M} \underset{\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}}}{\otimes} \mathcal{F}. \tag{1.1}$$

As was mentioned above, in Sect. 24.2 we will upgrade this action to the DG level.

1.3. Recall the scheme classifying $\check{G}$-opers on the formal punctured disc $\mathcal{D}^\times$ with nilpotent singularities, introduced in [FG2], Sect. 2.13. We denote this scheme by $\mathrm{Op}^{\mathrm{nilp}}$. This is an affine scheme of infinite type, isomorphic to the infinite-dimensional affine space. By [FG2], Sect. 2.18, there exists a canonical smooth map

$$\mathfrak{r}_{\mathrm{nilp}} : \mathrm{Op}^{\mathrm{nilp}} \to \tilde{\check{\mathcal{N}}}/\check{G},$$

that corresponds to taking the residue of the connection corresponding to an oper.

By Sect. 21.2.2, we have a well-defined base-changed triangulated category

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}), \tag{1.2}$$

equipped with a DG model, which carries an action of the monoidal triangulated category $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$, where the latter category and the action are also equipped with DG models.

The category (1.2) is the main character of this paper.

1.3.1. As was explained in the introduction, the base change construction is a categorical version of the tensor product construction for modules over an associative algebra. In particular, it satisfies a certain universal property (see Sect. 17.5.3), which when applied to our situation reads as follows:

Let $\mathbf{D}'$ be a triangulated category *over* the scheme $\mathrm{Op}^{\mathrm{nilp}}$, i.e., $\mathbf{D}'$ is acted on by $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$ and both the category and the action are equipped with DG models. Then functors

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}',$$

compatible with the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$ (where the compatibility data is also equipped with a DG model) are in bijection with functors

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}',$$

that are compatible with the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))$, where the latter acts on $\mathbf{D}'$ via the monoidal functor $\mathfrak{r}^*_{\mathrm{nilp}} : \mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G})) \to \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$.

1.3.2. We have the tautological pull-back functor, denoted by a slight abuse of notation by the same character

$$\mathfrak{r}^*_{\mathrm{nilp}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}).$$

An additional piece of information on the category (1.2) is that we know how to calculate Hom in it between objects of the form $\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_i)$, $\mathcal{F}_i \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$, $i = 1, 2$. Namely, this is given by Corollary 18.4.2(2), and in our situation it reads as follows:

$$\mathrm{Hom}\left(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1), \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2)\right) \simeq \mathrm{Hom}(\mathcal{F}_1, (\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}_2), \tag{1.3}$$

where $(\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}})$ (resp., $(\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}_2)$) is regarded as an object of the ind-completion of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))$ (resp., the ind-completion of the category (1.2)).

The universal property of Sect. 1.3.1 and (1.3) is essentially all the information that we have about the category (1.2), but it will suffice to prove a number of results.

2. The new t-structure

As a tool for the study of the category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$, we shall now introduce a new t-structure on the ind-completion of this category. Its main property will be that the functors of convolution $? \star J_{\check\lambda}$ become exact in this new t-structure.

2.1. As the triangulated category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ is equipped with a DG model, it has a well-defined ind-completion (see Sect. 15.7.1), which we denote $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$. This is a co-complete triangulated category, which is generated by the subcategory of its compact objects, the latter being identified with $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ itself.

2.1.1. By Sect. 23.5.1, the usual t-structure on $\mathbf{D}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ induces a t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$; moreover, the latter is compactly generated (see Sect. 19.2 where this notion is introduced). We have an exact functor

$$\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}),$$

which induces an equivalence

$$\mathbf{D}^+_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}).$$

In what follows we will refer to this t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ as the "old" one.

2.1.2. Recall the general framework of defining a t-structure on a co-complete triangulated category given by Lemma 19.2.1.

We define a new t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ so that $\mathbf{D}^{\leq 0_{new}}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ is generated by objects $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ that satisfy:

$$\mathcal{F} \star J_{\check\lambda} \in \mathbf{D}^{\leq 0_{old}}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \text{ for any } \check\lambda \in \check\Lambda. \tag{2.1}$$

Since for $\check\lambda \in \check\Lambda^+$ the functor $\mathcal{F} \mapsto \mathcal{F} \star J_{\check\lambda}$ is right-exact (in the old t-structure), condition (2.1) is sufficient to check for a subset of $\check\Lambda$ of the form $\check\lambda_0 - \check\Lambda^+$ for some/any $\check\lambda_0 \in \check\Lambda$.

2.1.3. *Remarks.* At the moment we do not know how to answer some basic questions about the new t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$. For example:

Question: Is the new t-structure compatible with the subcategory $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \subset \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$, i.e., is the subcategory $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ preserved by the truncation functors? [3]

However, we propose the following conjectures:

Conjecture 2.1.4. *Let* $\mathcal{F} \in \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ *satisfy* $\mathcal{F} \star J_{\check\lambda} \in \mathbf{D}^{\leq 0_{old}}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ *for any* $\check\lambda \in \check\Lambda$. *Then* $\mathcal{F} \in \mathbf{D}^{\leq 0_{new}}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$.

The second conjecture has to do with the stability of the above t-structure with respect to base change. Let A be an associative algebra, and consider the triangulated category

$$\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}} \otimes A\text{–mod}) := \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \underset{\to}{\otimes} \mathbf{D}^{perf}(A\text{–mod}),$$

i.e., the ind-completion of $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \otimes \mathbf{D}^{perf}(A\text{–mod})$.

[3] Probably, that answer to this question is negative.

It is endowed with the usual, a.k.a. "old", t-structure, equal to the tensor product of the "old" t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ and the usual t-structure on $\mathbf{D}(A\text{–mod})$, see Sect. 20.1.2

However, there are two ways to introduce a "new" t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\otimes A\text{–mod})$. The t-structure "$\mathrm{new}_1$" is obtained by the tensor product construction of Sect. 20.1.2 from the new t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ and the usual t-structure on $\mathbf{D}(A\text{–mod})$.

The t-structure "new_2" is defined to be generated by compact objects, i.e., objects $\mathcal{F}\in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\otimes\mathbf{D}^{perf}(A\text{–mod})$, such that $\mathcal{F}\star J_{\check\lambda}$ is ≤ 0 in the old sense.

Conjecture 2.1.5. *The t-structures "new_1" and "new_2" on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\otimes A\text{–mod})$ coincide.*

We remark, that, more generally, instead of the category $\mathbf{D}(A\text{–mod})$, we could have taken any compactly generated triangulated category, equipped with a DG model, and a compactly generated t-structure.

2.2. Here are some of the basic properties of the new t-structure.

2.2.1. First, the functors $\mathcal{F}\mapsto\mathcal{F}\star J_{\check\lambda}$ are exact in the new t-structure.

Further, we have:

Proposition 2.2.2. *For* $\mathcal{M}\in\mathrm{QCoh}(\widetilde{\check{\mathcal{N}}}/\check G)$ *the functor*

$$\mathcal{F}\mapsto\mathcal{F}\star\mathsf{F}(\mathcal{M})$$

is right-exact in the new t-structure. If $\mathcal{M}$ is flat, it is exact.

Proof. Any $\mathcal{M}$ as in the proposition is quasi-isomorphic to a direct summand of a complex $\mathcal{M}^\bullet$, situated in non-positive cohomological degrees, such that each $\mathcal{M}^\bullet$ is a direct sum of line bundles $\pi^*(\mathcal{L}^{\check\lambda})$. This readily implies the first point of the proposition.

The second assertion follows from the first one, see Sect. 20.2.1.

□

2.2.3. The next lemma shows that the new t-structure induces the old one on the finite-dimensional flag variety $G/B\subset\mathrm{Fl}_G^{\mathrm{aff}}$:

Lemma 2.2.4. *Any D-module $\mathcal{F}\in\mathfrak{D}(G/B)\text{–mod}\subset\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}$ belongs to the heart of the new t-structure.*

The proof follows from the fact that the map defining the convolution $\mathcal{F}\star J_{-\check\lambda}$ with $\check\lambda\in\check\Lambda^+$ is one-to-one over the supports of the corresponding D-modules.

2.3. The identity functor on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ is tautologically left-exact when viewed as a functor from the old t-structure to the new one. We claim, however, that the deviation is by a finite amount:

Proposition 2.3.1. *Any $\mathcal{F}\in\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$, which is $\geq 0_{new}$, is $\geq -\dim(G/B)_{old}$.*

Proof. Let $\mathcal{F}$ be as in the proposition. We have to show that $\mathrm{Hom}(\mathcal{F}',\mathcal{F})=0$ for any $\mathcal{F}'\in\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$, which is $<-\dim(G/B)_{old}$.

It would be sufficient to show that for any such $\mathcal{F}'$, the objects $\mathcal{F}'\star J_{-\check\lambda}$ are $<0_{old}$ for $\check\lambda\in\check\Lambda^+$. I.e.:

Lemma 2.3.2. *For $\check\lambda\in\check\Lambda^+$, the functor $?\star J_{-\check\lambda}$ has a cohomological amplitude (in the old t-structure) bounded by $\dim(G/B)$.*

□

2.3.3. *Proof of Lemma 2.3.2.* Consider the object

$$\mathcal{O}_{\Delta_{\mathrm{pt}/\check{B}}} \in \mathrm{Coh}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{pt}/\check{B}) \simeq \mathrm{Coh}(\mathrm{Fl}^{\check{G}} \times \mathrm{Fl}^{\check{G}}/\check{G}).$$

It can be realized as a direct summand of a complex, concentrated in non-positive degrees and of length $\dim(\mathrm{Fl}^{\check{G}}) = \dim(G/B)$, whose terms are of the form $\mathcal{L}^{\check{\mu}_i} \underset{\mathcal{O}_{\mathrm{pt}/\check{G}}}{\boxtimes} \mathcal{L}^{\check{\lambda}_i}$, where with no restriction of generality we can assume that $\check{\mu}_i \in \check{\Lambda}^+$.

This implies that for any $\check{\lambda}$, the object $\mathcal{L}^{-\check{\lambda}} \in \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{B}))$ is a direct summand of an object that can be written as a successive extension of objects of the form $\mathcal{L}^{\check{\mu}} \otimes V[-k]$ with $k \leq \dim(\mathrm{Fl}^{\check{G}})$ and $\check{\mu} \in \check{\Lambda}^+$ and $V \in \mathrm{Rep}^{f.d.}(\check{G})$.

Hence, $\mathcal{F} \star J_{-\check{\lambda}}$ is a direct summand of an object which is a successive extension of objects of the form $\mathcal{F} \star J_{\check{\mu}} \star Z_V[-k]$. However, the functor $? \star J_{\check{\mu}}$ is right-exact, and $? \star Z_V$ is exact. $\square$

3. Functor to modules at the critical level

Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}$ be the abelian category of $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$, on which the center $\mathfrak{Z}_{\mathfrak{g}}$ acts through its quotient $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}}$ (see [FG2], Sect. 7.1). Let $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ be its derived category. In this section we will consider the functor of global sections $\Gamma_{\mathrm{Fl}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, and using Sect. 1.3.1 we will extend it to a functor

$$\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}} : \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\mathcal{N}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}).$$

3.1. Being the derived category of an abelian category, $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ is equipped with a natural DG model. Moreover, the abelian category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}$ has $\mathfrak{Z}_{\mathfrak{g}}^{\mathrm{nilp}} \simeq \mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$ mapping to its center. This defines on $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ a structure of triangulated category over the (affine) scheme $\mathrm{Op}^{\mathrm{nilp}}$. In particular, we have a monoidal action of the monoidal triangulated category $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$ on it.

3.1.1. In Sect. 23.2.2 we will introduce another triangulated category (also equipped with a DG model and acted on by $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}})))$, denoted $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$. This category is also co-complete and has a t-structure, but unlike $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, the new category is generated by its subcategory of compact objects denoted $\mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$.

In addition, we have a functor (equipped with a DG model, and compatible with the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}})))$

$$\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}),$$

which is exact and induces an equivalence

$$\mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \to \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}).$$

We have

$$\mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \subset \mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}),$$

so we can identify of $\mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ with its essential image in $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, denoted $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$.

By Sect. 16.7.2, all of the above triangulated categories inherit DG models and the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$, so they are triangulated categories over the scheme $\mathrm{Op}^{\mathrm{nilp}}$.

3.2. Our present goal is to construct a functor

$$\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}} : \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\mathcal{N}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}),$$

compatible with an action of $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$. Both the functor, and the compatibility isomorphisms will be equipped with DG models.

3.2.1. By Sect. 17.5.3, a functor as above would be defined once we define a functor

$$\Gamma_{\mathrm{Fl}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}), \tag{3.1}$$

which is compatible with the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\mathcal{N}}/\check{G}))$ (where the action on the RHS is via $\mathfrak{r}_{\mathrm{nilp}}$), such that again both the functor and the compatibility isomorphisms are equipped with DG models.

We define the functor

$$\Gamma_{\mathrm{Fl}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}), \tag{3.2}$$

to be $\Gamma(\mathrm{Fl}, ?)$, i.e., the derived functor of global sections of a critically twisted D-module.

In this section we will check the required compatibility on the triangulated level. In Sect. 24.3 we will upgrade this construction to the DG level. In Sect. 23.6 we will show that the image of (3.1) belongs to $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, thereby constructing the functor (3.1) with the required properties.

3.3. Let us introduce some notations:

$$\mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}} := \mathfrak{r}^*_{\mathrm{nilp}}(\pi^*(\mathcal{L}^{\check{\lambda}})), \check{\lambda} \in \check{\Lambda} \text{ and } \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}} := \mathfrak{r}^*_{\mathrm{nilp}}\left(\mathcal{O}_{\tilde{\mathcal{N}}/\check{G}} \otimes V\right), V \in \mathrm{Rep}^{f.g.}(\check{G}).$$

The data of compatibility of Γ_{Fl} with the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\mathcal{N}}/\check{G}))$ would follow from the corresponding data for $\mathrm{Ho}(\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\mathcal{N}}/\check{G})))$ (see Lemma 16.7.4).

By [AB], Proposition 4, the latter amounts to constructing the following isomorphisms for an object $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$:

- (i) For $\check{\lambda} \in \check{\Lambda}$,

$$\Gamma_{\mathrm{Fl}}(\mathcal{F} \star J_{\check{\lambda}}) \overset{\gamma_{\check{\lambda}}}{\simeq} \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}}.$$

- (ii) For $V \in \mathrm{Rep}^{f.d.}(\check{G})$, an isomorphism

$$\Gamma_{\mathrm{Fl}}(\mathcal{F} \star Z_V) \overset{\gamma_V}{\simeq} \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}},$$

and such that the following conditions hold:

- (a) For $\check{\lambda}_1, \check{\lambda}_2 \in \check{\Lambda}$, the diagram

$$\begin{CD}
\Gamma_{\mathrm{Fl}}(\mathcal{F} \star J_{\check{\lambda}_1} \star J_{\check{\lambda}_2}) @>{\gamma_{\check{\lambda}_2}}>> \Gamma_{\mathrm{Fl}}(\mathcal{F} \star J_{\check{\lambda}_1}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}_2}_{\mathrm{Op}^{\mathrm{nilp}}} \\
@VVV @V{\gamma_{\check{\lambda}_1}}VV \\
\Gamma_{\mathrm{Fl}}(\mathcal{F} \star J_{\check{\lambda}_1+\check{\lambda}_2}) @>{\gamma_{\check{\lambda}_1+\check{\lambda}_2}}>> \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}_1+\check{\lambda}_2}_{\mathrm{Op}^{\mathrm{nilp}}}
\end{CD}$$

  commutes.

- (b) The endomorphism of the object $\Gamma_{\mathrm{Fl}}(\mathcal{F}\star Z_V)$, induced by the monodromy endomorphism N_{Z_V} of Z_V goes over by means of γ_V to the endomorphism of the object $\Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}}$, induced by the tautological endomorphism of $\mathcal{O}_{\widetilde{\mathcal{N}}/\check{G}}\otimes V$.
- (c) For $V^1,V^2\in \mathrm{Rep}^{f.d.}(\check{G})$ the diagram
$$\begin{tikzcd}
\Gamma_{\mathrm{Fl}}(\mathcal{F}\star Z_{V^1}\star Z_{V^2}) \arrow[r, "\gamma_{V^2}"] \arrow[d] & \Gamma_{\mathrm{Fl}}(\mathcal{F}\star Z_{V^1}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}^2_{\mathrm{Op}^{\mathrm{nilp}}} \arrow[d, "\gamma_{V^1}"'] \\
\Gamma_{\mathrm{Fl}}(\mathcal{F}\star Z_{V^1\otimes V^2}) \arrow[r, "\gamma_{V^1\otimes V^2}"] & \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} (\mathcal{V}^1_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}^2_{\mathrm{Op}^{\mathrm{nilp}}})
\end{tikzcd}$$
commutes.
- (d) $\check{\lambda}\in\check{\Lambda}$ and $V\in \mathrm{Rep}^{f.d.}(\check{G})$, the diagram
$$\begin{tikzcd}
\Gamma_{\mathrm{Fl}}(\mathcal{F}\star J_{\check{\lambda}}\star Z_V) \arrow[r, "\gamma_V"] \arrow[d] & \Gamma_{\mathrm{Fl}}(\mathcal{F}\star J_{\check{\lambda}}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}} \arrow[d, "\gamma_{\check{\lambda}}"'] \\
\Gamma_{\mathrm{Fl}}(\mathcal{F}\star Z_V\star J_{\check{\lambda}}) \arrow[d, "\gamma_{\check{\lambda}}"'] & \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}} \arrow[d] \\
\Gamma_{\mathrm{Fl}}(\mathcal{F}\star Z_V) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}} \arrow[r, "\gamma_V"] & \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}}
\end{tikzcd}$$
commutes, where the first left vertical arrow is the isomorphism of [Ga], Theorem 1(b).
- (e) For $\check{\lambda}\in\check{\Lambda}^+$ the canonical map $Z_{V^{\check{\lambda}}}\to J_{\check{\lambda}}$ makes the following diagram commutative:
$$\begin{tikzcd}
\Gamma_{\mathrm{Fl}}(\mathcal{F}\star Z_{V^{\check{\lambda}}}) \arrow[r] \arrow[d, "\gamma_{V^{\check{\lambda}}}"'] & \Gamma_{\mathrm{Fl}}(\mathcal{F}\star J_{\check{\lambda}}) \arrow[d, "\gamma_{\check{\lambda}}"'] \\
\Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}} \arrow[r] & \Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}},
\end{tikzcd}$$
where the bottom horizontal arrow comes from the canonical map $V^{\check{\lambda}}\otimes\mathcal{O}_{\mathrm{Fl}^{\check{G}}}\to\mathcal{L}^{\check{\lambda}}$.

3.4. To construct the above isomorphisms we will repeatedly use the fact that for $\mathcal{F}'\in\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}^I$, $\mathcal{F}\in\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$

$$\Gamma_{\mathrm{Fl}}(\mathcal{F}\star\mathcal{F}')\simeq\mathcal{F}\star\Gamma_{\mathrm{Fl}}(\mathcal{F}')\in\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}). \tag{3.3}$$

By the definition of the critical twisting on $\mathrm{Fl}^{\mathrm{aff}}_G$, we have

$$\Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\simeq\mathbb{M}_{\mathrm{crit},-2\rho}. \tag{3.4}$$

Here $\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}$ is the δ-function twisted D-module on $\mathrm{Fl}^{\mathrm{aff}}_G$ at the point $1\in\mathrm{Fl}^{\mathrm{aff}}_G$, and $\mathbb{M}_{\mathrm{crit},-2\rho}$ denotes the Verma module at the critical level with highest weight -2ρ.

From (3.4), we obtain that for any $\mathcal{F}\in\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$,

$$\Gamma_{\mathrm{Fl}}(\mathcal{F})\simeq\mathcal{F}\star\mathbb{M}_{\mathrm{crit},-2\rho}.$$

3.4.1. Thus, to construct isomorphisms $\gamma_{\check{\lambda}}$ as in (i), it is enough to construct an isomorphism

$$J_{\check{\lambda}} \star \mathbb{M}_{\mathrm{crit},-2\rho} \simeq \mathbb{M}_{\mathrm{crit},-2\rho} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}}.$$

By the definition of the map $\mathfrak{r}_{\mathrm{nilp}}$ (see [FG2], Sect. 2.18),

$$\mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}} \simeq \mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}} \otimes (\mathfrak{l}')^{\check{\lambda}},$$

where $\check{\lambda} \to (\mathfrak{l}')^{\check{\lambda}}$ is a $\check{T}$-torsor. In addition, by [FG2], Corollary 13.12

$$J_{\check{\lambda}} \star \mathbb{M}_{\mathrm{crit},-2\rho} \simeq \mathbb{M}_{\mathrm{crit},-2\rho} \otimes (\mathfrak{l}'')^{\check{\lambda}},$$

where $\check{\lambda} \to (\mathfrak{l}'')^{\check{\lambda}}$ is also a $\check{T}$-torsor. Moreover, there exists a canonical isomorphism of $\check{T}$-torsors

$$(\mathfrak{l}')^{\check{\lambda}} \simeq (\mathfrak{l}'')^{\check{\lambda}},$$

given by [FG5], equations (6.3), (6.4) and Proposition 6.11(2) of *loc. cit.*

This gives rise to the isomorphism $\gamma_{\check{\lambda}}$. Condition (a) follows from the construction.

3.4.2. Isomorphism γ_V results via (3.3) from the isomorphism established in [FG3], Theorem 5.4. Namely, this theorem asserts that for every $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^I_{\mathrm{nilp}}$ and $V \in \mathrm{Rep}(\check{G})$ there exists a canonical isomorphism

$$Z_V \star \mathcal{M} \simeq \mathcal{M} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}}. \tag{3.5}$$

The fact that conditions (b) and (c) are satisfied is included in the formulation of the above result in *loc. cit.*

3.4.3. Condition (d) follows from the functoriality of the isomorphism (3.5) and the fact that the following diagram is commutative for any $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ and $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^I_{\mathrm{nilp}}$:

$$\begin{tikzcd}
(Z_V \star \mathcal{F}) \star \mathcal{M} \arrow[r, "\sim"] \arrow[d] & (\mathcal{F} \star Z_V) \star \mathcal{M} \arrow[d] \\
Z_V \star (\mathcal{F} \star \mathcal{M}) \arrow[d, "\sim"'] & \mathcal{F} \star (Z_V \star \mathcal{M}) \arrow[d, "\sim"'] \\
\mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} (\mathcal{F} \star \mathcal{M}) \arrow[r, "\sim"] & \mathcal{F} \star (\mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{M}),
\end{tikzcd} \tag{3.6}$$

The commutativity of the above diagram follows, in turn, from the construction of the isomorphism $Z_V \star \mathcal{F} \simeq \mathcal{F} \star Z_V$ and that of Theorem 5.4 of [FG3] via nearby cycles.

3.4.4. Finally, let us prove the commutativity of the diagram in (e).

As before, it suffices to consider the case of $\mathcal{F} = \delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}$. Let us choose a coordinate on a formal disc, and consider the resulting grading on the corresponding modules. We obtain that the space of degree 0 morphisms

$$\mathbb{M}_{\mathrm{crit},-2\rho} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}} \to \mathbb{M}_{\mathrm{crit},-2\rho} \otimes \mathfrak{l}^{\check{\lambda}}$$

is one-dimensional. Therefore, it is enough to show the commutativity of the following diagram instead:

$$
\begin{tikzcd}
Z_{V^{\check\lambda}} \star \mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} \arrow[r] \arrow[d, "\sim"'] & J_{\check\lambda} \star \mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} \arrow[d, "\sim"'] \\
\mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}^{\check\lambda}_{\mathrm{Op}^{\mathrm{nilp}}} \arrow[r] & \mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check\lambda}_{\mathrm{Op}^{\mathrm{nilp}}},
\end{tikzcd}
\tag{3.7}
$$

where

$$\mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} := \mathbb{M}_{\mathrm{crit},-2\rho} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}},$$

and the left vertical arrow is given by (3.5), and the right vertical arrow is given by Proposition 6.11(2) of [FG5].

Further, it is enough to show the commutativity of the following diagram, obtained from (3.7) by composing with the canonical morphism $\mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{-2\check\rho} \to \mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}}$ of equation (6.6) of [FG5], where $\mathbb{V}_{\mathrm{crit}}$ is the vacuum module at the critical level:

$$
\begin{tikzcd}
Z_{V^{\check\lambda}} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{-2\check\rho} \arrow[r] \arrow[d] & J_{\check\lambda} \star \mathbb{V}_{\mathrm{crit}} \otimes \mathfrak{l}^{-2\check\rho} \arrow[d] \\
\mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{V}^{\check\lambda}_{\mathrm{Op}^{\mathrm{nilp}}} \arrow[r] & \mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{L}^{\check\lambda}_{\mathrm{Op}^{\mathrm{nilp}}}.
\end{tikzcd}
$$

However, the latter diagram coincides with the commutative diagram (6.10) of [FG5].

3.5. We can now state the first main result of the present paper:

Main Theorem 1. *The functor*

$$\Gamma_{\mathrm{Op}^{\mathrm{nilp}}} : \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check G}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{nilp}})$$

is fully faithful.

3.6. Consider the ind-completion of $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check G}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$, which we will denote

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check G}{\underset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}). \tag{3.8}$$

The functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ extends to a functor

$$\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}} : \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check G}{\underset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{nilp}}), \tag{3.9}$$

which commutes with the formation of direct sums. Main Theorem 1 implies that the latter functor is also fully faithful.

3.6.1. By Sect. 20.1.2, the *new* t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ induces a t-structure on (3.8). Namely, the corresponding ≤ 0 category is generated by objects of the form

$$\mathcal{M} \underset{\mathcal{O}_{\tilde{\tilde{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F},$$

where $\mathcal{M} \in \mathbf{D}^{perf,\leq 0}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$ and $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \cap \mathbf{D}^{\leq 0}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$. Since $\mathrm{Op}^{\mathrm{nilp}}$ is affine, it is in fact enough to take objects just of the form $\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F})$ for $\mathcal{F}$ as above, where $\mathfrak{r}^*_{\mathrm{nilp}}$ denotes the tautological pull-back functor

$$\mathfrak{r}^*_{\mathrm{nilp}} : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}). \tag{3.10}$$

3.6.2. We propose:

Conjecture 3.6.3. *The functor* (3.9) *is an equivalence of categories, and is exact.*

There are two pieces of evidence in favor of this conjecture. One is Main Theorem 1 which says that the functor in question is fully faithful. Another is given by Main Theorem 2 (see Sect. 4), which says that the conclusion of the conjecture holds when we restrict ourselves to the corresponding I^0-equivariant categories on both sides, where I^0 is the unipotent radical of the Iwahori subgroup I.

3.6.4. Suppose that Conjecture 3.6.3 is true. We would then obtain an equivalence of abelian categories

$$\mathrm{Heart}\left(\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right) \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}.$$

As the RHS, i.e., $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}$ is of prime interest for representation theory, let us describe the LHS more explicitly.

By Proposition 20.6.1, the category $\mathrm{Heart}\left(\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)$ is equivalent to the abelian base change

$$\mathrm{QCoh}(\mathrm{Op}^{\mathrm{nilp}}) \underset{\mathrm{QCoh}(\tilde{\tilde{\mathcal{N}}}/\check{G})}{\otimes} \mathrm{Heart}^{new}\left(\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right).$$

As in [Ga1], Sect. 22, the latter category can be described as follows.

Its objects are $\mathcal{F} \in \mathrm{Heart}^{new}\left(\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)$ endowed with an action of the algebra $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}} \simeq \mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}$ by endomorphisms together with a system of isomorphisms

$$\gamma_V : \mathcal{V}_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{F} \simeq \mathcal{F} \star Z_V, \text{ for every } V \in \mathrm{Rep}(\check{G})$$

and

$$\gamma_{\check{\lambda}} : \mathcal{L}^{\check{\lambda}}_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{F} \simeq \mathcal{F} \star J_{\check{\lambda}}, \text{ for every } \check{\lambda} \in \check{\Lambda},$$

which satisfy the conditions parallel to (a)-(e) of Sect. 3.3.

Morphisms in this category between $(\mathcal{F}^1, \{\gamma^1_V\}, \{\gamma^1_{\check\lambda}\})$ and $(\mathcal{F}^2, \{\gamma^2_V\}, \{\gamma^2_{\check\lambda}\})$ are morphisms in $\mathrm{Heart}^{new}\left(\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})\right)$ that intertwine the actions of $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$ and the data of γ_V, $\gamma_{\check\lambda}$.

3.6.5. As another corollary of Conjecture 3.6.3, we obtain:

Conjecture 3.6.6. *The functor* $\Gamma_{\mathrm{Fl}} : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{nilp}})$ *is exact for the new t-structure on the left-hand side.*

Indeed, the functor Γ_{Fl} is the composition of $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ and the functor $\mathfrak{r}^*_{\mathrm{nilp}}$ of (3.10). However, by Proposition 20.2.1, the functor $\mathfrak{r}^*_{\mathrm{nilp}}$ is exact.

4. The Iwahori-monodromic subcategory

In this section we will consider the restriction of the functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$, introduced in Sect. 3 to the corresponding I^0-equivariant subcategories.

4.1. Let $\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0}$ be the full subcategory of $\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ consisting of I^0-equivariant objects, as defined, e.g., in [FG2], Sect. 20.11.

Let $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0}$ be the full subcategory of $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ generated by $\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0}$ under the identification

$$\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \simeq \mathbf{D}^+_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}).$$

Lemma 4.1.1. *The category* $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ *is generated by the set of its compact objects, which identifies with*

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0} := \mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0} \cap \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}).$$

This lemma will be proved in Sect. 25.2.

4.1.2. The subcategory $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0} \subset \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ is preserved by the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\tilde{\mathcal{N}}}/\check G))$ on $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$, and inherits a DG model. Hence, we have a well-defined category

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check G}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0}, \tag{4.1}$$

and its ind-completion

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check G}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0}. \tag{4.2}$$

The following lemma will be proved in Sect. 25.4:

Lemma 4.1.3.

(1) The natural functor

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check G}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0} \to \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check G}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$$

is fully faithful.

(2) The image of the functor in (1) consist of all objects, whose further image under

$$(\mathfrak{r}_{\mathrm{nilp}})_* : \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check G}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$$

belongs to $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0} \subset \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$*. (Here* $(\mathfrak{r}_{\mathrm{nilp}})_*$ *is the functor right adjoint to* $(\mathfrak{r}_{\mathrm{nilp}})^*$*, see Sects 15.2.1 and 18.1.)*

4.1.4. Let $\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0} \subset \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ be the subcategory of I^0-equivariant objects, as defined, e.g., in [FG2], Sect. 20.11.

As above, we define $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ to be the full subcategory of $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, generated by $\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ under the identification

$$\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \simeq \mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}).$$

Set also:

$$\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0} := \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0} \cap \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \subset \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0},$$

which we can also regard as a subcategory

$$\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0} =: \mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0} \subset \mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0} \subset \\ \subset \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}.$$

It is clear that $\mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ consists of compact objects of $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$.

Remark. It is easy to see that the subcategory $\mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ equals the subcategory of compact objects in $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$. It will *a posteriori* true, see Corollary 4.2.5, that it actually generates $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$. However, we will be able to deduce this as a corollary of our Main Theorem 2. Unfortunately, at the moment we are unable to give an *a priori* proof of this statement, parallel to that of Lemma 4.1.1.

4.1.5. It is clear from the definitions that the functor Γ_{Fl} sends

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0} \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}.$$

Thus, it extends to a functor

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0} \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0},$$

making the diagram

$$\begin{CD}
\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0} @>>> \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \\
@V{\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}}VV @VV{\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}}V \\
\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0} @>>> \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})
\end{CD} \tag{4.3}$$

commute.

4.1.6. Recall now that the categories

$$\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}});\ \ \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod});\ \ \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

are equipped with t-structures. The t-structure that we consider on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ can be either the old or the new one, whereas the t-structure on $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ is that introduced in Sect. 3.6.1, using the *new* t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$.

We have:

Proposition 4.1.7.

(a) *The subcategory* $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})^{I^0} \subset \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})$ *is compatible with the t-structure.*

(b) *The subcategory* $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})^{I^0} \subset \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})$ *is compatible with the old t-structure.*

(c) *The subcategory* $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})^{I^0} \subset \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})$ *is compatible also with the new t-structure.*

(d) *The subcategory*

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})^{I^0} \subset \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})$$

is compatible with the t-structure.

Point (a) of the proposition follows from the fact that embedding functor

$$\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})^{I^0} \hookrightarrow \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})$$

admits a right adjoint, which fits into the commutative diagram:

$$\begin{CD}
\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})^{I^0} @<<< \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}) \\
@AAA @AAA \\
\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})^{I^0} @<{\mathrm{Av}_*^{I^0}}<< \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}),
\end{CD}$$

where $\mathrm{Av}_*^{I^0}$ is the averaging functor of [FG2], Sect. 20.10; in particular, the right adjoint in question is left-exact.

Point (b), (c) and (d) of the proposition will be proved in Sections 25.2, 25.3 and 25.4, respectively.

4.2. The second main result of this paper is the following:

Main Theorem 2. *The functor*

$$\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}} : \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})^{I^0} \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})^{I^0}$$

is an equivalence of categories, and is exact.

4.2.1. As a consequence, we obtain:

Corollary 4.2.2. *The functor* $\Gamma_{\mathrm{Fl}} : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})^{I^0} \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})^{I^0}$ *is exact (in the new t-structure on the LHS).*

As another corollary we obtain:

Corollary 4.2.3. *The functor* $\mathsf{F} : \mathbf{D}(\mathrm{QCoh}(\tilde{\check{\mathcal{N}}}/\check{G})) \to \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})$ *is exact (in the new t-structure on the RHS).*

Proof. By construction, the image of F lies in $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\operatorname{-mod})^{I^0}$. First, we claim that the composed functor

$$\Gamma_{\mathrm{Fl}} \circ \mathsf{F} \simeq \Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}} \circ \mathfrak{r}_{\mathrm{nilp}}^* \circ \mathsf{F} : \mathbf{D}(\mathrm{QCoh}(\tilde{\check{\mathcal{N}}}/\check{G})) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})^{I^0}$$

is exact. Indeed, the above functor is given by

$$\mathcal{M} \mapsto \mathcal{M} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathbb{M}_{\mathrm{crit},-2\rho},$$

and the claim follows from the fact that $\mathbb{M}_{\mathrm{crit},-2\rho}$ is $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$-flat.

Since the functor $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$ is an equivalence, the assertion of the corollary follows from the fact that the functor

$$\mathfrak{r}^*_{\mathrm{nilp}} : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is exact and conservative on the heart.

□

4.2.4. Finally, we remark that since the LHS appearing in Main Theorem 2 is compactly generated, we obtain:

Corollary 4.2.5. *The category* $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ *is generated by* $\mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$.

4.3. We will now show how Main Theorem 2 implies a conjecture from [FG2] about a description of the category $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$ is terms of quasi-coherent sheaves.

4.3.1. Let $\check{\mathrm{St}}$ denote the Steinberg scheme of $\check{G}$, i.e.,

$$\check{\mathrm{St}} := \tilde{\check{\mathcal{N}}} \underset{\check{\mathfrak{g}}}{\times} \tilde{\check{\mathfrak{g}}},$$

where $\tilde{\check{\mathfrak{g}}}$ denoted Grothendieck's alteration (see Sect. 0.9). The diagonal map $\tilde{\check{\mathcal{N}}} \to \tilde{\check{\mathcal{N}}} \times \tilde{\check{\mathfrak{g}}}$ defines a map of stacks

$$\tilde{\check{\mathcal{N}}}/\check{G} \hookrightarrow \check{\mathrm{St}}/\check{G}.$$

Let us now recall the main result of Bezrukavnikov's theory [Bez], Theorem 4.2(a):

Theorem 4.3.2. *The functor* F *extends to an equivalence*

$$\mathsf{F}_{\check{\mathrm{St}}} : \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0}$$

as categories over the stack $\tilde{\check{\mathcal{N}}}/\check{G}$.

4.3.3. Let us recall now the scheme of Miura opers with nilpotent singularities, introduced in [FG2], Sect. 3.14, and denoted $\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}}$. By definition,

$$\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}} := \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \check{\mathrm{St}}/\check{G}.$$

The following conjecture was stated in [FG2], Conjecture 6.2:

Conjecture 4.3.4. *There exists an equivalence*

$$\mathbf{D}^b(\mathrm{QCoh}(\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}})) \simeq \mathbf{D}^b(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$$

of categories over the scheme $\mathrm{Op}^{\mathrm{nilp}}$.

We will deduce this conjecture from Main Theorem 2 by base-changing the equivalence of Theorem 4.3.2 by means of the morphism $\mathfrak{r}_{\mathrm{nilp}} : \mathrm{Op}^{\mathrm{nilp}} \to \tilde{\check{\mathfrak{n}}}/\check{G}$. In fact, we will prove a slightly stronger statement:

Main Theorem 3. *There exists an equivalence*

$$\mathbf{D}^+(\mathrm{QCoh}(\mathrm{MOp}^{\mathrm{nilp}}_{\check{\mathfrak{g}}})) \simeq \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^{I^0}$$

of cohomological amplitude bounded by $\dim(G/B)$.

5. Relation to the affine Grassmannian

As was mentioned in the introduction, the main tool that will eventually allow us to prove Main Theorem 1 and, consequently, Main Theorem 2, is an explicit connection between the category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ and the category of D-modules on the affine Grassmannian, $\mathrm{Gr}^{\mathrm{aff}}_G$.

5.1. Let $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ be the derived category of finitely generated critically twisted D-modules on $\mathrm{Gr}^{\mathrm{aff}}_G$. This is a triangulated category equipped with a natural DG model.

5.1.1. As in the case of $\mathrm{Fl}^{\mathrm{aff}}_G$, we can form the ind-completion of $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, denoted $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$. This is a co-complete triangulated category, which is generated by the subcategory of its compact objects, the latter being identified with $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ itself. Moreover, $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ is equipped with a compactly generated t-structure. We have an exact functor

$$\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

which induces an equivalence

$$\mathbf{D}^+_{ren}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^+(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

5.1.2. The geometric Satake equivalence endows the abelian category $\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ with an action of the tensor category $\mathrm{Rep}^{f.d.}(G) \simeq \mathrm{Coh}(\mathrm{pt}/\check{G})$ by exact functors.

This defines on $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ an action of $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{G}))$, equipped with an (evident) DG model. I.e., $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ is a triangulated category over the stack $\mathrm{pt}/\check{G}$ (see Sect. 21 where the terminology is introduced).

5.1.3. Consider the base-changed category

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

which is a triangulated category over the stack $\mathrm{pt}/\check{B}$. (See Sect. 21.2.2 for the definition of base change.)

Consider now another triangulated category over $\mathrm{pt}/\check{B}$, namely,

$$\mathrm{pt}/\check{B} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

where

$$\mathrm{pt}/\check{B} \simeq \mathrm{Fl}^{\check{G}}/\check{G} \hookrightarrow \tilde{\check{\mathcal{N}}}/\check{G}$$

is the embedding of the zero section, denoted ι.

Our current goal is to construct a functor, denoted Υ,

$$\Upsilon : \mathrm{pt}/\check{B} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}), \tag{5.1}$$

as categories over the stack $\mathrm{pt}/\check{B}$.

5.2. By the universal property of the tensor product construction (see Sect. 1.3.1), in order to construct the functor

$$\Upsilon : \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}), \tag{5.2}$$

by Sect. 17.5.3, we have to produce a functor

$$\widetilde{\Upsilon} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}), \tag{5.3}$$

as categories over the stack $\tilde{\tilde{\mathcal{N}}}/\check{G}$.

5.2.1. Since the action of $\mathrm{Rep}(\check{G})$ on $\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}$-mod is given by right-exact (and, in fact, exact) functors, we can consider the base-changed abelian category:

$$\mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod} =: \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod},$$

see Sect. 20.5 for the definiton (the existence of this category is easily established; see, e.g., in [Ga1], Sect. 10).

By construction, $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}$-mod is an abelian category acted on by $\mathrm{Rep}(\check{B})$ by exact functors. Hence, the derived category $\mathbf{D}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ has a natural structure of category over the stack $\mathrm{pt}/\check{B}$.

5.2.2. By Sect. 17.5.3, the tautological functor

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

gives rise to a functor

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}). \tag{5.4}$$

Moreover, by Lemma 20.7.1, the functor (5.4) is fully faithful.

Hence, in order to construct the functor $\widetilde{\Upsilon}$, it suffices to construct a functor

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}), \tag{5.5}$$

as categories over $\tilde{\tilde{\mathcal{N}}}/\check{G}$, such that its image belongs to the essential image of (5.4).

By a slight abuse of notation, we will denote the functor (5.5) also by $\widetilde{\Upsilon}$. In this section we will construct $\widetilde{\Upsilon}$ as a functor between triangulated categories, compatible with the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$. In Sect. 24.4 we will upgrade the construction to the DG level.

5.2.3. Let $\mathcal{W}$ be the object of $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}^I$, introduced in [FG5], Sect. 3.15 (under the name $\mathcal{W}_{w_0}$), as well as in [FG4], Sect. 4.1 (under the name $\mathcal{F}_{w_0}$), and in [ABBGM], Sect. 3.2.13 (under the name $\mathcal{M}^1$).

We define the functor $\widetilde{\Upsilon}$ of (5.5) by

$$\widetilde{\Upsilon}(\mathcal{F}) := \mathcal{F} \star J_{2\check{\rho}} \star \mathcal{W}. \tag{5.6}$$

Lemma 5.2.4. *For a finitely generated D-module $\mathcal{F}$ on $\mathrm{Fl}_G^{\mathrm{aff}}$, the object*

$$\mathcal{F} \star J_{2\check{\rho}} \star \mathcal{W} \in \mathbf{D}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\mathrm{-mod})$$

belongs to the essential image of (5.4).

Proof. Recall (see, e.g., [ABBGM], Corollary 1.3.10 and Proposition 3.2.6) that $\mathcal{W}$, as an object of $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\mathrm{-mod}^I$, has a finite filtration with subquotients of the form

$$V \underset{\mathcal{O}_{\mathrm{pt}/\check{G}}}{\otimes} \mathcal{F}' \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\mathrm{-mod}^I$$

with $V \in \mathrm{Rep}^{f.d.}(\check{B})$ and $\mathcal{F}' \in \mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\mathrm{-mod}^I$ is finitely generated as a D-module.

Since $\mathrm{Fl}_G^{\mathrm{aff}}$ is proper, for any such $\mathcal{F}'$ and $\mathcal{F}$ as in the statement of the lemma, $\mathcal{F} \underset{I}{\star} \mathcal{F}'$ belongs to $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{crit}\mathrm{-mod})$.

□

5.3. In order to endow $\widetilde{\Upsilon}$ with the data of compatibility with respect to the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\widetilde{\check{\mathcal{N}}}/\check{G}))$ (at the triangulated level), by Lemma 16.7.4, it is enough to do so with respect to the action of $\mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{free}(\widetilde{\check{\mathcal{N}}}/\check{G}))\right)$.

This amounts to constructing isomorphisms

- (i)
$$\widetilde{\Upsilon}(\mathcal{F} \star J_{\check{\lambda}}) \simeq \mathcal{L}^{\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \widetilde{\Upsilon}(\mathcal{F}),\ \check{\lambda} \in \check{\Lambda}$$
- (ii)
$$\widetilde{\Upsilon}(\mathcal{F} \star Z_V) \simeq (\mathcal{O}_{\mathrm{pt}/\check{B}} \otimes V) \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \widetilde{\Upsilon}(\mathcal{F}),\ V \in \mathrm{Rep}^{f.d.}(\check{G})$$

so that the conditions, parallel to (a)-(e) of Sect. 3.3 hold.

5.3.1. Isomorphism (i) above follows from the basic isomorphism

$$J_{\check{\lambda}} \star \mathcal{W} \simeq \mathcal{L}^{\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{W}, \tag{5.7}$$

established in [FG4], Corollary 4.5, or [FG5], Proposition 3.19, or [ABBGM], Corollary 3.2.2.

5.3.2. To construct isomorphism (ii) we observe that for any $\mathcal{F}' \in \mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\mathrm{-mod}^I$ and $V \in \mathrm{Rep}^{f.d.}(\check{G})$ we have a canonical isomorphism

$$Z_V \underset{I}{\star} \mathcal{F}' \simeq \mathcal{F}' \underset{G[[t]]}{\star} \mathcal{F}_V, \tag{5.8}$$

where $\mathcal{F}_V$ is the spherical D-module on $\mathrm{Gr}_G^{\mathrm{aff}}$, corresponding to $V \in \mathrm{Rep}(\check{G})$. This isomorphism is a particular case of Theorem 1(b) of [Ga], combined with point (d) of the same theorem.

Thus, isomorphism (ii) is obtained from

$$\mathcal{F} \star Z_V \star J_{2\check{\rho}} \star \mathcal{W} \simeq \mathcal{F} \star J_{2\check{\rho}} \star (Z_V \star \mathcal{W}) \simeq (\mathcal{F} \star J_{2\check{\rho}} \star \mathcal{W}) \underset{G[[t]]}{\star} \mathcal{F}_V,$$

where we observe that the operation $\mathcal{F}' \mapsto \mathcal{F}' \underset{G[[t]]}{\star} \mathcal{F}_V$ corresponds by definition to

$$\mathcal{F}' \mapsto (\mathcal{O}_{\mathrm{pt}/\check{B}} \otimes V) \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{F}'$$

for $\mathcal{F}' \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$.

5.3.3. Conditions (a) and (c) follow by definition. Condition (d) follows from the functoriality of the isomorphism (5.8) and the associativity property of the isomorphism of Theorem 1(b) of [Ga], established as Property 1 in [Ga'].

Condition (b) follows from the fact that for any $\mathcal{F}' \in \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}^I$, the isomorphism, induced by N_{Z_V} on the left-hand side of (5.8), is zero.

5.3.4. To prove condition (e), we need to check the commutativity of the following diagram:

$$\begin{tikzcd}
Z_{V^{\check\lambda}} \star \mathcal{W} \arrow[rr] \arrow[d, "\sim"'] & & J_{\check\lambda} \star \mathcal{W} \arrow[d, "\sim"'] \\
\mathcal{W} \underset{G[[t]]}{\star} \mathcal{F}_{V^{\check\lambda}} \arrow[r, "\sim"] & \mathrm{Res}^{\check G}_{\check B}(V^{\check\lambda}) \underset{\mathcal{O}_{\mathrm{pt}/\check B}}{\otimes} \mathcal{W} \arrow[r] & \mathcal{L}^{\check\lambda} \underset{\mathcal{O}_{\mathrm{pt}/\check B}}{\otimes} \mathcal{W}.
\end{tikzcd}$$

Since $\mathrm{End}(\mathcal{W}) \simeq \mathbb{C}$ (see [ABBGM], Proposition 3.2.6(1)), to prove the commutativity of the latter diagram, it is enough to establish the commutativity of the following one, obtained from the map $\delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G} \to \mathcal{W}$:

$$\begin{tikzcd}
Z_{V^{\check\lambda}} \star \delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G} \arrow[rr] \arrow[d] & & J_{\check\lambda} \star \delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G} \arrow[d] \\
Z_{V^{\check\lambda}} \star \mathcal{W} \arrow[d, "\sim"'] & & J_{\check\lambda} \star \mathcal{W} \arrow[d, "\sim"'] \\
\mathcal{W} \underset{G[[t]]}{\star} \mathcal{F}_{V^{\check\lambda}} \arrow[r, "\sim"] & \mathrm{Res}^{\check G}_{\check B}(V^{\check\lambda}) \underset{\mathcal{O}_{\mathrm{pt}/\check B}}{\otimes} \mathcal{W} \arrow[r] & \mathcal{L}^{\check\lambda} \underset{\mathcal{O}_{\mathrm{pt}/\check B}}{\otimes} \mathcal{W}.
\end{tikzcd}$$

However, this last diagram is equivalent to the following one.

$$\begin{tikzcd}
\mathcal{F}_{V^{\check\lambda}} \arrow[r] \arrow[d] & J_{\check\lambda} \star \delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G} \arrow[r] & J_{\check\lambda} \star \mathcal{W} \arrow[d] \\
\mathcal{W} \underset{G[[t]]}{\star} \mathcal{F}_{V^{\check\lambda}} \arrow[r, "\sim"] & \mathrm{Res}^{\check G}_{\check B}(V^{\check\lambda}) \underset{\mathcal{O}_{\mathrm{pt}/\check B}}{\otimes} \mathcal{W} \arrow[r] & \mathcal{L}^{\check\lambda} \underset{\mathcal{O}_{\mathrm{pt}/\check B}}{\otimes} \mathcal{W}.
\end{tikzcd}$$

The latter diagram is commutative, as it is a version of the commutative diagram of Lemma 5.3 of [FG5] (or, which is the same, commutative diagram (28) of [FG4], or Corollary 3.2.3 of [ABBGM]).

5.4. Thus, the functor $\widetilde{\Upsilon}$, and hence the functor Υ, have been constructed. We are now ready to state the fourth main result of this paper:

Main Theorem 4. *The functor*

$$\Upsilon : \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is fully faithful.

Remark. As will be explained in the sequel, the functor Υ, as it is, is not an equivalence of categories: it fails to be essentially surjective. In Sect. 11 we will show how to modify the LHS by adding certain colimits so that the resulting functor becomes an equivalence. In addition, the latter functor will be exact with respect to the natural t-structures.

6. Sections over $\mathrm{Fl}^{\mathrm{aff}}_G$ vs. $\mathrm{Gr}^{\mathrm{aff}}_G$

In this section we will study how the functor Υ of Sect. 5 intertwines between the functors of taking global sections of critically twisted D-modules on $\mathrm{Fl}^{\mathrm{aff}}_G$ and $\mathrm{Gr}^{\mathrm{aff}}_G$.

6.1. Let $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$ be the abelian category of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules, on which the center $\mathfrak{Z}_{\mathfrak{g}}$ acts through its quotient $\mathfrak{Z}^{\mathrm{reg}}_{\mathfrak{g}} \simeq \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}$. Let $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$ be its derived category. It has the same properties as those discussed in Sect. 3.1.1 with "nilp" replaced by "reg".

In particular we have a diagram of categories

$$\begin{array}{ccc} \mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) & \longrightarrow & \mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \\ \sim\big\downarrow & & \sim\big\downarrow \\ \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) & \longrightarrow & \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \end{array}$$

with the rows being fully faithful functors.

6.1.1. Recall from [FG2] that we have an exact functor

$$\Gamma_{\mathrm{Gr}} : \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}},$$

compatible with the action of $\mathrm{Rep}(\check{G})$, where the action on the RHS is given via the morphism

$$\mathfrak{r}_{\mathrm{reg}} : \mathrm{Op}^{\mathrm{reg}} \to \mathrm{pt}/\check{G}.$$

This gives rise to a functor

$$\Gamma_{\mathrm{Gr}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \tag{6.1}$$

as categories over the stack $\mathrm{pt}/\check{G}$.

Repeating the argument of Sect. 23.6, one shows that the image of the functor (6.1) belongs to the subcategory $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \subset \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$. So we have a functor

$$\Gamma_{\mathrm{Gr}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \tag{6.2}$$

as categories over the stack $\mathrm{pt}/\check{G}$.

6.1.2. Consider the base-changed category

$$\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

By Sect. 17.5.3, the functor Γ_{Gr} gives rise to a functor

$$\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{nilp}}} : \mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \tag{6.3}$$

as triangulated categories over $\mathrm{Op}^{\mathrm{reg}}$.

The following has been established in [FG2], Theorem 8.17:

Theorem 6.1.3. *The functor $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{nilp}}}$ is fully faithful.*

In addition, in *loc. cit.* we formulated the conjecture to the effect that the functor $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{nilp}}}$ is an equivalence.

6.2. Recall now that the map $\mathfrak{r}_{\mathrm{reg}}$ canonically factors as

$$\mathrm{Op}^{\mathrm{reg}} \overset{\mathfrak{r}'_{\mathrm{reg}}}{\to} \mathrm{pt}/\check{B} \to \mathrm{pt}/\check{G}.$$

In fact, we have a Cartesian square with vertical arrows being smooth morphisms:

$$\begin{tikzcd} \mathrm{Op}^{\mathrm{reg}} \arrow[r, "\iota_{\mathrm{Op}}"] \arrow[d, "\mathfrak{r}'_{\mathrm{reg}}"'] & \mathrm{Op}^{\mathrm{nilp}} \arrow[d, "\mathfrak{r}_{\mathrm{nilp}}"'] \\ \mathrm{pt}/\check{B} \arrow[r, "\iota"] & \tilde{\check{\mathcal{N}}}/\check{G}, \end{tikzcd} \tag{6.4}$$

see [FG2], Lemma 2.19.

6.2.1. Hence, we can regard $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$ is a triangulated category over the stack $\mathrm{pt}/\check{B}$, and by the universal property (see Sect. 17.5.3), the functor (6.3) gives rise to a functor

$$\Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}} : \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}), \tag{6.5}$$

as triangulated categories over $\mathrm{pt}/\check{B}$.

6.2.2. Recall now that we have a functor

$$\iota^*_{\widehat{\mathfrak{g}}} : \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}),$$

obtained as a derived functor of

$$\mathcal{M} \mapsto \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{M} : \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}} \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}.$$

By construction, the above functor has a natural DG model, and as such is a functor between categories over $\mathrm{Op}^{\mathrm{nilp}}$.

In Sect. 23.3.3 we will show that the above functor sends the subcategory $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ to $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$, thereby giving rise to a functor between the above categories, as categories over $\mathrm{Op}^{\mathrm{nilp}}$. By Sect. 17.5.3, we obtain a functor

$$(\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\times} \iota^*_{\widehat{\mathfrak{g}}}) : \mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\times} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}), \tag{6.6}$$

as categories over $\mathrm{Op}^{\mathrm{reg}}$.

In Sect. 23.3.6 we will prove:

Proposition 6.2.3. *The functor* $(\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\times} \iota^*_{\widehat{\mathfrak{g}}})$ *of* (6.6) *is fully faithful.*

Remark. The functor in Proposition 6.2.3 fails to be essentially surjective for the same reasons as the functor Υ of Main Theorem 4. In Sect. 23.4.4 it will be shown how to modify the LHS to turn it into an equivalence.

6.3. Consider the base change of the functor Γ_{Fl} with respect to ι, and obtain a functor

$$\Gamma_{\mathrm{Fl},\mathrm{pt}/\check{B}} : \mathrm{pt}/\check{B} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}).$$

We will prove:

Theorem 6.3.1. *The diagram of functors between categories over* $\mathrm{pt}/\check{B}$

$$\begin{tikzcd}
\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Fl},\mathrm{pt}/\check{B}}"] \arrow[d, "\Upsilon"'] & \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \arrow[d, "(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \iota^*_{\widehat{\mathfrak{g}}})"'] \\
\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}}"] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})
\end{tikzcd}$$

is commutative.

By Proposition 17.5.3, another way to formulate the above theorem is that the diagram

$$\begin{tikzcd}
\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Fl}}"] \arrow[d] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \arrow[d, "\iota^*_{\widehat{\mathfrak{g}}}"'] \\
\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}}"] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})
\end{tikzcd} \tag{6.7}$$

commutes, as functors between categories over $\tilde{\tilde{\mathcal{N}}}/\check{G}$, where the left vertical arrow is the composition

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \xrightarrow{\iota^*} \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \xrightarrow{\Upsilon} \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

Base-changing the diagram in the theorem with respect to $\mathfrak{r}'_{\mathrm{reg}}$, we obtain:

Corollary 6.3.2. *We have the following commutative diagram of functors between categories over* $\mathrm{Op}^{\mathrm{reg}}$*:*

$$\begin{tikzcd}
\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\times} \left(\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right) \arrow[r, "\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}"] \arrow[d, "\sim"'] & \mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\times} \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \arrow[dddd, "\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\times} \iota^*_{\widehat{\mathfrak{g}}}"'] \\
\mathrm{Op}^{\mathrm{reg}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[d, "\sim"'] & \\
\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{B}}{\times} \left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right) \arrow[d, "\Upsilon"'] & \\
\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{B}}{\times} \left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right) \arrow[d, "\sim"'] & \\
\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}"] & \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}).
\end{tikzcd}$$

Part II: Proofs

The strategy of the proofs of the four main theorems is as follows. We shall first prove Main Theorem 4, which is a purely geometric assertion, i.e., involves only D-modules, but not representations of the Kac-Moody algebra.

We shall then combine Main Theorem 4 with Theorem 6.1.3 to prove Theorem 1.

Main Theorem 2 will follow from Main Theorem 1 using some additional explicit information about the category of I^0-equivariant objects in $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$. Namely, the latter category is essentially generated by Verma modules.

Finally, Main Theorem 3 will follow from Main Theorem 2 by combining the latter with the results of [Bez].

7. Fully faithfulness of Υ

In this section we will reduce the proof of Main Theorem 4 to another assertion, Theorem 7.3.1.

7.1. As will be shown in Sect. 21.5.1, we have the functors

$$\iota^* : \mathbf{D}^{perf}(\widetilde{\tilde{\mathcal{N}}}/\check{G}) \leftrightarrows \mathbf{D}^{perf}(\mathrm{pt}/\check{B}) : \iota_*$$

which give rise to a pair of mutually adjoint functors

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \leftrightarrows \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

which we denote by $(\iota_{\mathrm{Fl}})^*$ and $(\iota_{\mathrm{Fl}})_*$, respectively.

7.1.1. Recall the functor

$$\widetilde{\Upsilon} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

of (5.3). In this section, we will use an alternative notation for it, namely, $(\widetilde{\iota}_{\mathrm{Fl}})^*$. Our present goal is to construct the right adjoint of $(\widetilde{\iota}_{\mathrm{Fl}})^*$, that we will denote $(\widetilde{\iota}_{\mathrm{Fl}})_*$.

7.1.2. Consider the functor $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$

$$\mathcal{F} \mapsto p^{!*}(\mathcal{F}) := p^*(\mathcal{F})[\dim(G/B)] \simeq p^!(\mathcal{F})[-\dim(G/B)].$$

Since the action of $\mathrm{Rep}(\check{G})$ on $\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ and $\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ is given by exact functors, the above functor has an evident structure of functor between triangulated categories over $\mathrm{pt}/\check{G}$.

We regard $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ as a category over $\mathrm{pt}/\check{B}$ via the map $\pi : \widetilde{\tilde{\mathcal{N}}}/\check{G} \to \mathrm{pt}/\check{B}$.

By the universal property of base change (see Sect. 17.5.3), from $p^{!*}$ we obtain a functor

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

between triangulated categories over $\mathrm{pt}/\check{B}$. We denote it $(\widetilde{\iota}_{\mathrm{Fl}})_*$.

Proposition 7.1.3. *There exists a natural transformation $(\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \to \mathrm{Id}$, as functors between categories over $\mathrm{pt}/\check{B}$, which makes $(\widetilde{\iota}_{\mathrm{Fl}})_*$ a right adjoint of $(\widetilde{\iota}_{\mathrm{Fl}})^*$ at the level triangulated categories.*

Remark. Although both sides of

$$(\widetilde{\iota}_{\mathrm{Fl}})_* : \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$$

are categories over $\widetilde{\widetilde{\mathcal{N}}}/\check{G}$, the construction of $(\widetilde{\iota}_{\mathrm{Fl}})_*$ only makes it a functor between categories over $\mathrm{pt}/\check{B}$. The additional structure of functor between categories over $\widetilde{\widetilde{\mathcal{N}}}/\check{G}$ is not be obvious from the construction.

7.2. For the proof of Proposition 7.1.3 we will need to review two other adjunction constructions.

7.2.1. Let $\mathfrak{q}$ denote the (proper) map of stacks $\mathrm{pt}/\check{B} \to \mathrm{pt}/\check{G}$, and consider the functor

$$\mathfrak{q}^* : \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{G})) \to \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{B})).$$

We will denote this functor also by $\mathrm{Res}^{\check{G}}_{\check{B}}$ as it corresponds to the restriction functor $\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{B})$. It has an evident structure of functor between categories over $\mathrm{pt}/\check{G}$.

We claim that there exist the functors

$$\mathrm{Ind}^{\check{G}}_{\check{B}}, \text{co-}\mathrm{Ind}^{\check{G}}_{\check{B}} : \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{B})) \to \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{G})),$$

$$\mathrm{Id} \to \mathrm{Ind}^{\check{G}}_{\check{B}} \circ \mathrm{Res}^{\check{G}}_{\check{B}} \tag{7.1}$$

and

$$\text{co-}\mathrm{Ind}^{\check{G}}_{\check{B}} \circ \mathrm{Res}^{\check{G}}_{\check{B}} \to \mathrm{Id}, \tag{7.2}$$

as categories and functors over the stack $\mathrm{pt}/\check{G}$.

This follows from Sections 21.6.1 and 21.6.2, as

$$\mathrm{Ind}^{\check{G}}_{\check{B}} \simeq \mathfrak{q}_* \text{ and } \text{co-}\mathrm{Ind}^{\check{G}}_{\check{B}} \simeq \mathfrak{q}_?,$$

in the notation of *loc. cit.*

Moreover, by (21.6), we have an isomorphism of functors over $\mathrm{pt}/\check{G}$:

$$\mathrm{Ind}^{\check{G}}_{\check{B}} \simeq \text{co-}\mathrm{Ind}^{\check{G}}_{\check{B}} \circ (\mathcal{L}^{2\check{\rho}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} ?)[-\dim(\mathrm{Fl}^{\check{G}})]. \tag{7.3}$$

Hence, for any triangulated category $\mathbf{D}$ over $\mathrm{pt}/\check{G}$ we have a similar set of functors and adjunctions between $\mathbf{D}$ and $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D} \leftrightarrows \mathbf{D}$ as categories and functors over the stack $\mathrm{pt}/\check{G}$.

7.2.2. Consider the direct image functor

$$p_! = p_* : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}).$$

The construction of Sect. 24.1.2 endows it with a structure of functor between categories over $\mathrm{pt}/\check{G}$. We leave the following assertion to the reader, as it repeats the constructions carried out in Sect. 24:

Lemma 7.2.3. *The adjunction maps*

$$\mathrm{Id} \to p_* \circ p^{!*}[-\dim(G/B)] \textit{ and } p^{!*} \circ p_![-\dim(G/B)] \to \mathrm{Id}$$

and

$$p_! \circ p^{!*}[\dim(G/B)] \to \mathrm{Id} \textit{ and } \mathrm{Id} \to p^{!*} \circ p^![\dim(G/B)]$$

can be endowed with a structure of natural stransformations between functors over $\mathrm{pt}/\check{G}$.

7.2.4. *Proof of Proposition 7.1.3.* By the universal property of base change (see Sect. 17.5.3), constructing a map of functors

$$(\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \to \mathrm{Id},$$

compatible with the structure of functors over the stack $\mathrm{pt}/\check{B}$ is equivalent to constructing a map

$$(\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \circ \mathrm{Res}^{\check{G}}_{\check{B}} \to \mathrm{Res}^{\check{G}}_{\check{B}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

as functors over $\mathrm{pt}/\check{G}$.

By Sect. 7.2.1, the latter is in turn equivalent to constructing a map

$$\text{(7.4)} \quad \begin{aligned} \text{co-Ind}^{\check{G}}_{\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \circ \mathrm{Res}^{\check{G}}_{\check{B}} \simeq \text{co-Ind}^{\check{G}}_{\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^* \circ p^{!*} \to \mathrm{Id} : \\ \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}), \end{aligned}$$

as functors over $\mathrm{pt}/\check{G}$.

The construction of the arrow in (7.4) follows now from Lemma 7.2.3 and the following:

Lemma 7.2.5. *The functor*

$$\text{co-Ind}^{\check{G}}_{\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^* : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is canonically isomorphic to $p_![\dim(G/B)]$*, as functors between categories over the stack* $\mathrm{pt}/\check{G}$.

Proof. We will construct an isomorphism as a functor between triangulated categories, compatible with the action of $\mathrm{Rep}^{f.d.}(\check{G})$. The upgrading of the isomorphism to the DG level will be commented on in Sect. 24.4.1.

We have:

$$\text{co-Ind}^{\check{G}}_{\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\mathcal{F}) \simeq \text{co-Ind}^{\check{G}}_{\check{B}}(\mathcal{F} \star J_{2\rho} \star \mathcal{W}) \simeq \mathcal{F} \star \text{co-Ind}^{\check{G}}_{\check{B}}(J_{2\rho} \star \mathcal{W}).$$

The isomorphism follows now from [ABBGM], Lemma 3.2.22 (or, using (7.3), from [FG5], Proposition 3.18) where it is shown that there exists a canonical isomorphism

$$\text{(7.5)} \qquad \text{co-Ind}^{\check{G}}_{\check{B}}(J_{2\rho} \star \mathcal{W}) \simeq \delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G}[\dim(G/B)].$$

The compatibility with the action of $\mathrm{Rep}^{f.d.}(\check{G})$ follows from the construction tautologically. □

Thus, to prove the proposition, it remains to show that the map

$$\mathrm{Hom}\left(\mathcal{F}, (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}')\right) \to \mathrm{Hom}\left((\widetilde{\iota}_{\mathrm{Fl}})^*(\mathcal{F}), (\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}')\right) \to \mathrm{Hom}\left((\widetilde{\iota}_{\mathrm{Fl}})^*(\mathcal{F}), \mathcal{F}'\right).$$

is an isomorphism for any $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, $\mathcal{F}' \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$.

By the definition of the latter category, we can take $\mathcal{F}'$ of the form $\mathcal{L}^{\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}'_1)$ with $\mathcal{F}'_1 \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$. However, since the functor of tensor product with $\mathcal{L}^{\check{\lambda}}$ is an equivalence on both categories, we can replace $\mathcal{F}$ by $\mathcal{L}^{-\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{F}$, and so reduce the assertion to the case when $\mathcal{F}' = \mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}'_1)$.

In the latter case, the assertion reduces to the $(p_!, p^!)$ adjunction by Lemma 7.2.5 above. □

7.3. We claim now that Proposition 7.1.3 gives rise to a natural transformation of functors at the level of triangulated categories:

$$(\iota_{\mathrm{Fl}})_* \to (\widetilde{\iota}_{\mathrm{Fl}})_* \circ \Upsilon, \tag{7.6}$$

where both sides are functors

$$\mathrm{pt}/\check{B} \underset{\widetilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\mathrm{-mod}) \to \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\mathrm{-mod}).$$

Indeed, the natural transformation in question comes by adjunction from

$$(\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\iota_{\mathrm{Fl}})_* \simeq \Upsilon \circ (\iota_{\mathrm{Fl}})^* \circ (\iota_{\mathrm{Fl}})_* \to \Upsilon.$$

Composing the natural transformation (7.6) with the functor $(\iota_{\mathrm{Fl}})^*$, we obtain a natural transformation

$$(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^* \to (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*, \tag{7.7}$$

also at the level of triangulated categories, which makes the following diagram commute:

$$\begin{CD} \mathrm{Hom}((\iota_{\mathrm{Fl}})^*(\mathcal{F}_1), (\iota_{\mathrm{Fl}})^*(\mathcal{F}_2)) @>{\sim}>> \mathrm{Hom}(\mathcal{F}_1, (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\mathcal{F}_2)) \\ @VVV @VVV \\ \mathrm{Hom}(\Upsilon \circ (\iota_{\mathrm{Fl}})^*(\mathcal{F}_1), \Upsilon \circ (\iota_{\mathrm{Fl}})^*(\mathcal{F}_2)) @>{\sim}>> \mathrm{Hom}(\mathcal{F}_1, (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\mathcal{F}_2)) \end{CD} \tag{7.8}$$

for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\mathrm{-mod})$.

We will prove:

Theorem 7.3.1. *The natural transformation* (7.7) *is an isomorphism.*

7.3.2. Let us show how Theorem 7.7 implies fully-faithfulness of Υ.

Proof. We have to show that the map

$$\mathrm{Hom}(\mathcal{F}'_1, \mathcal{F}'_2) \to \mathrm{Hom}(\Upsilon(\mathcal{F}'_1), \Upsilon(\mathcal{F}'_2))$$

is an isomorphism for any $\mathcal{F}'_1, \mathcal{F}'_2 \in \mathrm{pt}/\check{B} \underset{\widetilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\mathrm{-mod})$.

Since the functor $\iota^* : \mathbf{D}^{perf}(\mathrm{Coh}(\widetilde{\tilde{\mathcal{N}}}/\check{G})) \to \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{B}))$ is affine (see Sect. 20.4), it is sufficient to take $\mathcal{F}'_i$, $i = 1, 2$ of the form $(\iota_{\mathrm{Fl}})^*(\mathcal{F}_i)$ with $\mathcal{F}_i \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\mathrm{-mod})$.

In the latter case the required isomorphism follows from Theorem 7.3.1 via the commutative diagram (7.8).

□

The same argument shows that Theorem 7.3.1 implies that the natural transformation (7.6) is also an isomorphism.

7.4. In Sect. 8 we will give two proofs of Theorem 7.3.1: a shorter one, which relies on some unpublished results, announced in [Bez], and a slightly longer one, which only uses [AB].

As a preparation for the latter argument we will now describe a functor, which is the right adjoint to $(\widetilde{\iota}_{\mathrm{Fl}})_*$. We will use the notation, $(\widetilde{\iota}_{\mathrm{Fl}})_! := (\widetilde{\iota}_{\mathrm{Fl}})_*$, and the right adjoint in question will be denoted $(\widetilde{\iota}_{\mathrm{Fl}})^!$.

7.4.1. Set

$$(\widetilde{\iota}_{\mathrm{Fl}})^! := \mathcal{L}^{-2\check{\rho}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} (\widetilde{\iota}_{\mathrm{Fl}})^*[-\dim(\check{G}/\check{B})].$$

In other words, the construction of $(\widetilde{\iota}_{\mathrm{Fl}})^!$ is the same as that of $(\widetilde{\iota}_{\mathrm{Fl}})^*$, where instead of the object

$$J_{2\rho} \star \mathcal{W} \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}^I$$

we use $\mathcal{W}[-\dim(\check{G}/\check{B})]$.

By [FG5], Proposition 3.18 (or [ABBGM], Proposition 3.2.16) we have

$$\mathrm{Ind}^{\check{G}}_{\check{B}}(\mathcal{W}) \simeq \delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G}. \tag{7.9}$$

Repeating the proof of Proposition 7.1.3, we obtain that there exists a natural transformation

$$\mathrm{Id} \to (\widetilde{\iota}_{\mathrm{Fl}})^! \circ (\widetilde{\iota}_{\mathrm{Fl}})_!$$

as functors between categories over $\mathrm{pt}/\check{B}$, which makes $(\widetilde{\iota}_{\mathrm{Fl}})^!$ the right adjoint of $(\widetilde{\iota}_{\mathrm{Fl}})_!$ at the level of triangulated categories.

7.4.2. According to Sect. 21.5.3, we have the functors

$$\iota_! := \iota_* : \mathbf{D}^{perf}(\mathrm{pt}/\check{B}) \leftrightarrows \mathbf{D}^{perf}(\widetilde{\widetilde{\mathcal{N}}}/\check{G}) : \iota^!$$
$$\iota^! \simeq \mathcal{L}^{-2\check{\rho}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \iota^*[-\dim(\check{G}/\check{B})],$$

over $\widetilde{\widetilde{\mathcal{N}}}/\check{G}$, and the adjunctions

$$\iota_! \circ \iota^! \to \mathrm{Id} \text{ and } \mathrm{Id} \to \iota^! \circ \iota_!,$$

defined at the triangulated level.

Hence, by *loc. cit.*, we have the corresponding functors and adjunctions between

$$(\iota_{\mathrm{Fl}})_! \simeq (\iota_{\mathrm{Fl}})_* : \mathrm{pt}/\check{B} \underset{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \leftrightarrows \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) : (\iota_{\mathrm{Fl}})^!.$$

and

$$\iota^!_{\mathrm{Fl}} \simeq \mathcal{L}^{-2\check{\rho}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \iota^*_{\mathrm{Fl}}[-\dim(\check{G}/\check{B})],$$

at the triangulated level.

7.4.3. By construction, we have:

$$(\widetilde{\iota}_{\mathrm{Fl}})^! \simeq \Upsilon \circ (\iota_{\mathrm{Fl}})^!,$$

which as in Sect. 7.3 defines a natural transformation

$$\Upsilon \circ (\widetilde{\iota}_{\mathrm{Fl}})_! \to (\iota_{\mathrm{Fl}})_!. \tag{7.10}$$

Since

$$(\widetilde{\iota}_{\mathrm{Fl}})_! \simeq (\widetilde{\iota}_{\mathrm{Fl}})_* \text{ and } (\iota_{\mathrm{Fl}})_! \simeq (\iota_{\mathrm{Fl}})_*,$$

(7.10) gives a map in the direction opposite to that of (7.6):

$$\Upsilon \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \to (\iota_{\mathrm{Fl}})_*. \tag{7.11}$$

Composing with $(\iota_{\mathrm{Fl}})^*$ we obtain a natural transformation

$$(\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^* \to (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*. \tag{7.12}$$

Theorem 7.3.1 follows from the next assertion:

Proposition 7.4.4. *Each of the two compositions*

$$(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^* \overset{(7.7)}{\to} (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^* \overset{(7.12)}{\to} (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*$$

and

$$(\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^* \overset{(7.12)}{\to} (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^* \overset{(7.7)}{\to} (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*$$

is a non-zero scalar multiple of the identity map.

8. Description of the zero section

In this section will deduce Theorem 7.3.1 from Bezrukavnikov's theory. We first give an argument, using the still unpublished results announced in [Bez].

8.1. It is clear from the construction of the map (7.7) that for $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ we have a commutative diagram of functors:

$$\begin{tikzcd}
(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\mathcal{F}) \arrow[r, "(7.7)"] \arrow[d, "\sim"'] & (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\mathcal{F}) \arrow[d, "\sim"'] \\
\mathcal{F} \star \Big((\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}_G^{\mathrm{aff}}}) \Big) \arrow[r] & \mathcal{F} \star \Big((\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}_G^{\mathrm{aff}}}) \Big) .
\end{tikzcd} \tag{8.1}$$

Hence, the assertion of Theorem 7.3.1 is equivalent to the fact that the map

$$(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}_G^{\mathrm{aff}}}) \overset{(7.7)}{\longrightarrow} (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}_G^{\mathrm{aff}}}) \tag{8.2}$$

is an isomorphism.

8.1.1. Let $\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0} \subset \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ be the categories, defined as in the case of $\mathrm{Fl}_G^{\mathrm{aff}}$, see Sect. 4.1.

The functor Υ induces a functor

$$\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0} \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}, \tag{8.3}$$

and since $\delta_{1,\mathrm{Fl}_G^{\mathrm{aff}}} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$, it suffices to show that the functor (8.3) is fully faithful.

8.1.2. Applying Theorem 4.3.2, we obtain that the LHS in (8.3) identifies with

$$\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{S}}\mathrm{t}/\check{G})).$$

Consider the Cartesian product $\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \check{\mathrm{S}}\mathrm{t}/\check{G}$, understood in the DG sense. By an argument similar to that of Sect. 14.3.3, we have a fully faithful functor

$$\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{S}}\mathrm{t}/\check{G})) \to \mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \check{\mathrm{S}}\mathrm{t}/\check{G})). \tag{8.4}$$

8.1.3. Consider now the DG scheme

$$\widetilde{\mathrm{Fl}}^{\check{G}} := \mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}}.$$

By applying the Koszul duality to the equivalence of [ABG] we obtain an equivalence

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0} \simeq \mathbf{D}^b(\mathrm{Coh}(\widetilde{\mathrm{Fl}}^{\check{G}}/\check{G})), \tag{8.5}$$

as categories over $\mathrm{pt}/\check{G}$.

By an argument similar to that of Sect. 14.3.3, there exists a fully faithful functor

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^b(\mathrm{Coh}(\widetilde{\mathrm{Fl}}^{\check{G}}/\check{G})) \to \mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \widetilde{\mathrm{Fl}}^{\check{G}}/\check{G})).$$

8.1.4. Note now that

$$\mathrm{Fl}^{\check{G}} \underset{\widetilde{\check{\mathcal{N}}}}{\times} \check{\mathrm{St}} \simeq \mathrm{Fl}^{\check{G}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\check{\mathfrak{g}}} \simeq \mathrm{Fl}^{\check{G}} \times \widetilde{\mathrm{Fl}}^{\check{G}},$$

and hence

$$\mathrm{pt}/\check{B} \underset{\widetilde{\check{\mathcal{N}}}/\check{G}}{\times} \check{\mathrm{St}}/\check{G} \simeq \left(\mathrm{Fl}^{\check{G}} \underset{\widetilde{\check{\mathcal{N}}}}{\times} \check{\mathrm{St}}\right)/\check{G} \simeq \left(\mathrm{Fl}^{\check{G}} \times \widetilde{\mathrm{Fl}}^{\check{G}}\right)/\check{G} \simeq \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \widetilde{\mathrm{Fl}}^{\check{G}}/\check{G},$$

and by the construction of the functors involved we have a commutative diagram

$$\begin{CD}
\mathrm{pt}/\check{B} \underset{\widetilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0} @>{(8.3)}>> \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0} \\
@V{\sim}VV @V{\sim}VV \\
\mathrm{pt}/\check{B} \underset{\widetilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) @. \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^b(\mathrm{Coh}(\widetilde{\mathrm{Fl}}^{\check{G}}/\check{G})) \\
@VVV @VVV \\
\mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B} \underset{\widetilde{\check{\mathcal{N}}}/\check{G}}{\times} \check{\mathrm{St}}/\check{G})) @>{\sim}>> \mathbf{D}^b(\mathrm{Coh}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \widetilde{\mathrm{Fl}}^{\check{G}}/\check{G})),
\end{CD}$$

proving that (8.3) is fully faithful.

Remark. The above interpretation in terms of coherent sheaves makes it explicit why the functor Υ is not an equivalence. The reason is that the functor (8.4) is not an equivalence.

8.2. We shall now give an alternative argument, proving Theorem 7.3.1, which avoids the reference to [Bez]. Namely, we are going to prove Proposition 7.4.4.

By the same argument as in Sect. 8.1, it is enough to show that the compositions

$$(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \tag{8.6}$$

and

$$(\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \tag{8.7}$$

are non-zero multiples of the identity map.

This, in turn, breaks into three assertions:

Proposition 8.2.1. *The compositions* (8.6) *and* (8.7) *are non-zero.*

Proposition 8.2.2. *The composition* (8.6) *is the image by means of* $(\iota_{\mathrm{Fl}})_*$ *of a map*

$$(\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})$$

in $\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$.

Proposition 8.2.3.

(1) $\mathrm{End}\Big((\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big) \simeq \mathbb{C}$.

(2) $\mathrm{End}\Big((\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big) \simeq \mathbb{C}$.

We shall now prove Proposition 8.2.1. Propositions 8.2.2 and 8.2.3 will be proved in Sect. 10.

8.3. To prove Proposition 8.2.1, we will consider a non-degenerate character $\mathfrak{n} \to \mathbb{G}_a$, and denote by ψ_0 the corresponding character of I^0:

$$\psi : I^0 \twoheadrightarrow \mathfrak{n} \to \mathbb{G}_a.$$

Consider the corresponding equivariant subcategories

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi} \subset \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

and

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi} \subset \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

where $'I^0$ is the conjugate of I^0 by means of the element $t^{\check{\rho}} \in T((t))$.

The functor Υ induces a functor

$$\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi} \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi}. \tag{8.8}$$

To prove Proposition 8.2.3 it is enough to show that the compositions (8.6) and (8.7) do not vanish, as natural transformations between functors from $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi}$ to itself.

The latter would follow, once we show that the functor (8.8) is fully faithful. We claim that the latter functor is in fact an equivalence.

8.3.1. Indeed, by [FG2], Theorem 15.8 (or [ABBGM], Corollary 2.2.3) we have a canonical equivalence

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi} \simeq \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{G})),$$

hence,

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi} \simeq \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{B})).$$

The main result of [AB] asserts that the category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi}$, viewed as a triangulated category over $\tilde{\tilde{\mathcal{N}}}/\check{G}$, is canonically equivalent to $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$. Therefore,

$$\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{'I^0,\psi} \simeq \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{B})).$$

Moreover, from the construction of the functor of [AB] and [ABBGM], Proposition 3.2.6(1), the functor (8.8) corresponds to the identity functor on $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{pt}/\check{B}))$, implying our assertion.

9. New t-structure and the affine Grassmannian

In this section we will study how the new t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ behaves with respect to the functors $(\widetilde{\iota}_{\mathrm{Fl}})^*$, $(\widetilde{\iota}_{\mathrm{Fl}})_*$ and Υ.

9.1. Let us recall that

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \tag{9.1}$$

denotes the ind-completion of the category

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$$

(see Sect. 21.2.2). By Sect. 20.1.2, the category (9.1) acquires a compactly generated t-structure. It is characterized by the property that the ≤ 0 subcategory is generated by objects of the form

$$V \underset{\mathcal{O}_{\mathrm{pt}/\check{G}}}{\otimes} \mathcal{F},$$

$V \in \mathbf{D}^{perf,\leq 0}(\mathrm{Coh}(\mathrm{pt}/\check{B}))$, $\mathcal{F} \in \mathbf{D}^{f,\leq 0}(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$.

9.1.1. Let us denote by the same characters $(\widetilde{\iota}_{\mathrm{Fl}})^*, (\widetilde{\iota}_{\mathrm{Fl}})_* = (\widetilde{\iota}_{\mathrm{Fl}})_!$, $(\widetilde{\iota}_{\mathrm{Fl}})^!$ the ind-extensions of the functors from Sect. 7

$$\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \leftrightarrows \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}).$$

They satisfy the same adjunction properties as the original functors.

Proposition 9.1.2. *With respect to the new t-structure on* $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ *we have:*
(a) $(\widetilde{\iota}_{\mathrm{Fl}})^*$ *is right-exact,*
(b) $(\widetilde{\iota}_{\mathrm{Fl}})_*$ *is exact.*
(c) $(\widetilde{\iota}_{\mathrm{Fl}})^!$ *is left-exact.*

Proof. Let us first show that $(\widetilde{\iota}_{\mathrm{Fl}})_*$ is right-exact. This amounts to the next:

Lemma 9.1.3. *The functor*

$$p^{!*} : \mathbf{D}(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$$

is right-exact (and, in fact, exact) in the new t-structure.

Remark. Note that $p^{!*}$ is exact and hence left-exact in the old t-structure, and hence is left-exact in the new t-structure. Hence, the essential image of $p^{!*}$ provides a collection of objects that belong to the hearts of both t-structures. We remind that another such collection was given by Lemma 2.2.4.

Proof. The assertion of the lemma is equivalent to the fact that

$$p^{!*}(\mathcal{F}) \star J_{-\check{\lambda}} \in \mathbf{D}_{ren}^{\leq 0_{old}}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$$

for any $\check{\lambda} \in \check{\Lambda}^+$. We have:

$$p^{!*}(\mathcal{F}) \star J_{-\check{\lambda}} \simeq \mathcal{F} \underset{G[[t]]}{\star} \left(p^{!*}(\delta_{1,\mathrm{Gr}_G^{\mathrm{aff}}}) \underset{I}{\star} J_{-\check{\lambda}} \right).$$

Since $p^{!*}(\delta_{1,\mathrm{Gr}_G^{\mathrm{aff}}}) \underset{I}{\star} J_{-\check{\lambda}} \in \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}^{G[[t]]}$, the assertion follows from the next:

Lemma 9.1.4. *For* $\mathcal{F}_1 \in \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ *and* $\mathcal{F}_2 \in \mathfrak{D}\left(G((t))\right)_{\mathrm{crit}}\text{-mod}^{G[[t]]}$, *the convolution* $\mathcal{F}_1 \underset{G[[t]]}{\star} \mathcal{F}_2 \in \mathbf{D}(\mathfrak{D}\left(G((t))\right)_{\mathrm{crit}}\text{-mod})$ *is acylcic off cohomological degree* 0.

The lemma is proved by repeating the argument of [Ga], Theorem 1(a), or [ABBGM], Sect. 2.1.3.

$\square$

The right-exactness of $(\widetilde{\iota}_{\mathrm{Fl}})_* = (\widetilde{\iota}_{\mathrm{Fl}})_!$ implies by adjunction the left-exactness of $(\widetilde{\iota}_{\mathrm{Fl}})^!$. Hence, it remains to show that $(\widetilde{\iota}_{\mathrm{Fl}})^*$ is right-exact; this would imply the left-exactness of $(\widetilde{\iota}_{\mathrm{Fl}})_*$ also by adjunction.

Lemma 9.1.5. *An object*

$$\mathcal{F}' \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is ≤ 0, *as an object of* $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ *if and only if for all* $\check{\lambda}$,

$$\text{co-Ind}^{\check{G}}_{\check{B}}(\mathcal{L}^{\check{\lambda}} \otimes \mathcal{F}) \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is ≤ 0.

Proof. This follows from the fact that for $\mathcal{F}'$ as in the lemma, which is $\leq k$ for some k, there exists $\check{\lambda}$ deep inside the dominant chamber so that

$$\mathrm{Cone}\left(\mathrm{Res}^{\check{G}}_{\check{B}} \circ \text{co-Ind}^{\check{G}}_{\check{B}}(\mathcal{L}^{-\check{\lambda}} \otimes \mathcal{F}) \to \mathcal{L}^{-\check{\lambda}} \otimes \mathcal{F}\right)$$

is $\leq k-1$.

$\square$

Hence, to prove that $(\widetilde{\iota}_{\mathrm{Fl}})^*$ is right-exact, it suffices to show that the composition

$$\text{co-Ind}^{\check{G}}_{\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^* : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is right-exact. The latter holds due to Lemma 7.2.5.

$\square$

9.2. Consider the category

$$\mathrm{pt}/\check{B} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}), \tag{9.2}$$

which is the ind-completion of the category

$$\mathrm{pt}/\check{B} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

By Sect. 20.1.2, the new t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ gives rise to a t-structure on (9.2). It is characterized by the property that the ≤ 0 category is generated by the objects of the form

$$\mathcal{M} \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}$$

for $\mathcal{M} \in \mathbf{D}^{perf,\leq 0}(\mathrm{Coh}(\mathrm{pt}/\check{B}))$ and $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \cap \mathbf{D}^{\leq 0}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$.

The ind-extension of $(\iota_{\mathrm{Fl}})^*$:

$$(\iota_{\mathrm{Fl}})^* : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathfrak{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is tautologically right-exact, and its right adjoint

$$(\iota_{\mathrm{Fl}})_* : \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathfrak{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is exact, by Proposition 20.4.1.

9.2.1. As a formal corollary of Proposition 9.1.2(a), we obtain:

Corollary 9.2.2. *The functor*

$$\Upsilon : \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathfrak{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is right-exact.

10. Calculation of endomorphisms and the functor Ω

In this section we will prove Propositions 8.2.2 and 8.2.3. This section can be considered redundant by a reader willing to accept the proof of Theorem 7.3.1, based on [Bez] given in Sect. 8.

10.1. Proof of Proposition 8.2.3(1). We have:

$$\mathrm{Hom}\left((\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\right) \simeq$$
$$\simeq \mathrm{Hom}\left((\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\right).$$

By construction, $(\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq J_{2\rho} \star \mathcal{W}$ belongs to the heart of the t-structure on the category $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$.

We have the following assertion

Proposition 10.1.1. *For $\mathcal{F}' \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, which is ≤ 0, the adjunction map $(\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}') \to \mathcal{F}'$ induces an isomorphism*

$$H^0\left((\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}')\right) \to \mathcal{F}'.$$

From the proposition we obtain that the map

$$\mathrm{Hom}\left((\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\right) \to \mathrm{Hom}\left((\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\right)$$

is surjective (in fact, an isomorphism). Hence, it suffices to show that

$$\mathrm{Hom}\left((\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\right) := \mathrm{End}(J_{2\rho} \star \mathcal{W}) \simeq \mathbb{C}.$$

This, however, follows from the fact that $\mathrm{End}(\mathcal{W}) \simeq \mathbb{C}$, which is easy to deduce from the definitions (or, alternatively, from [ABBGM], Proposition 3.2.6(1)).

10.1.2. *Proof of Proposition 10.1.1.* By Sect. 20.6.1, the heart of the t-structure on the category $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ is the abelian category

$$\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}.$$

Since the functor $(\widetilde{\iota}_{\mathrm{Fl}})_*$ is exact and $(\widetilde{\iota}_{\mathrm{Fl}})^*$ is right-exact, it sufficient to take $\mathcal{F}'$ to be one of the generators of the category, i.e., of the form $\mathcal{L}^{\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}'')$ with $\mathcal{F}'' \in \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}$. However, since the functors $(\widetilde{\iota}_{\mathrm{Fl}})_*$ and $(\widetilde{\iota}_{\mathrm{Fl}})^*$ are compatible with the action of $\mathrm{Coh}(\mathrm{pt}/\check{B})$, we can assume $\check{\lambda} = 0$, i.e., $\mathcal{F}' \simeq \mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}'')$.

Thus, we are reduced to showing that for $\mathcal{F}'' \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$, the map

$$H^0(\widetilde{\Upsilon} \circ p^{!*}(\mathcal{F}'')) \to \mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}'')$$

is an isomorphism.

We have:

$$\widetilde{\Upsilon} \circ p^{!*}(\mathcal{F}'') \simeq p^{!*}(\mathcal{F}'') \star J_{2\rho} \star \mathcal{W} \simeq \mathcal{F}'' \underset{G[[t]]}{\star} \left(p^{!*}(\delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G}) \star J_{2\rho} \star \mathcal{W}\right).$$

By [ABBGM], Proposition 3.2.6 (or which can be otherwise easily proved directly),

$$H^0\left(p^{!*}(\delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G}) \star J_{2\rho} \star \mathcal{W}\right) \simeq \mathrm{Res}^{\check{G}}_{\check{B}}(\delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G}),$$

and $H^{-k}\left(\delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G}) \star J_{2\rho} \star \mathcal{W}\right)$ is a successive extension of objects of the form

$$\mathcal{L}^{\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathrm{Res}^{\check{G}}_{\check{B}}(\delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G})$$

(in fact, each $\check{\lambda}$ appears the number of times equal to $\dim(\Lambda^k(\check{\mathfrak{n}}^*)_{\check{\lambda}})$).

Hence, we obtain that

$$H^0(\widetilde{\Upsilon} \circ p^{!*}(\mathcal{F}'')) \simeq \mathcal{F}'' \underset{G[[t]]}{\star} \mathrm{Res}^{\check{G}}_{\check{B}}(\delta_{1,\mathrm{Gr}^{\mathrm{aff}}_G}) \simeq \mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}'').$$

□

10.2. Proof of Proposition 8.2.3(2). By adjunction,

$$\mathrm{Hom}\left((\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\right) \simeq \mathrm{Hom}\left(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}, (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\right).$$

We have

$$(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq \mathsf{F}(\mathcal{O}_{\mathrm{pt}/\check{B}}),$$

and let us represent $\mathcal{O}_{\mathrm{pt}/\check{B}} \in \mathbf{D}^{perf}(\mathrm{Coh}(\widetilde{\widetilde{\mathcal{N}}}/\check{G}))$ by the Koszul complex $\mathrm{Kosz}^{\bullet}_{\mathrm{pt}/\check{B},\widetilde{\widetilde{\mathcal{N}}}/\check{G}}$, corresponding to the vector bundle $\widetilde{\widetilde{\mathcal{N}}}/\check{G} \xrightarrow{\pi} \mathrm{pt}/\check{B}$ and its zero section $\mathrm{pt}/\check{B} \xrightarrow{\iota} \widetilde{\widetilde{\mathcal{N}}}/\check{G}$.

The terms $\mathrm{Kosz}^k_{\mathrm{pt}/\check{B},\widetilde{\widetilde{\mathcal{N}}}/\check{G}}$ are of the form $\pi^*(\Lambda^k(\check{\mathfrak{n}}^*))$, where $\Lambda^k(\check{\mathfrak{n}}^*)$ is naturally an object of $\mathrm{Rep}(\check{B})$.

To prove the proposition, it suffices to show that

$$\mathrm{Hom}\left(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}, \mathrm{Kosz}^k_{\mathrm{pt}/\check{B},\widetilde{\widetilde{\mathcal{N}}}/\check{G}}[k] \underset{\mathcal{O}_{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}}{\otimes} \delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}\right) = 0$$

for $k \neq 0$.

Note that $\Lambda^k(\check{\mathfrak{n}}^*) \in \operatorname{Rep}(\check{B})$ has a filtration with 1-dimensional subquotients $\mathcal{L}^{\check{\lambda}}$, with $\check{\lambda} \in \check{\Lambda}^{pos}$, and $\check{\lambda} = 0$ is excluded if $k \neq 0$. Hence, the proposition follows from the next lemma:

$\square$

Lemma 10.2.1. *$R\operatorname{Hom}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}, J_{\check{\lambda}}) = 0$ if $\check{\lambda} \in \check{\Lambda}^{pos} - 0$.*

Proof. For any $\check{\mu}$ we have:

$$R\operatorname{Hom}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}, J_{\check{\lambda}}) \simeq R\operatorname{Hom}(J_{\check{\mu}}, J_{\check{\lambda}+\check{\mu}}).$$

Let $\check{\mu} \in \check{\Lambda}^+$ be such that $\check{\lambda} + \check{\mu} \in \check{\Lambda}^+$. In this case

$$J_{\check{\mu}} = j_{\check{\mu},*} \text{ and } J_{\check{\lambda}+\check{\mu}} = j_{\check{\lambda}+\check{\mu},*}.$$

The required vanishing follows now from the fact that $R\operatorname{Hom}(j_{\widetilde{w},*}, j_{\widetilde{w}',*}) \neq 0$ only when $\widetilde{w}' \leq \widetilde{w}$ as elements in the affine Weyl group, in particular, $|\widetilde{w}'| \leq |\widetilde{w}|$. However, the assumptions on $\check{\mu}$ and $\check{\lambda}$ imply that $|\check{\lambda} + \check{\mu}| > |\check{\mu}|$.

$\square$

10.3. In order to prove Proposition 8.2.2, we will need the following construction, which will be useful also in the sequel.

Namely, we will construct a functor

(10.1)

$$\Omega : \operatorname{Heart}\left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right) \to \operatorname{Heart}\left(\mathrm{pt}/\check{B} \underset{\widetilde{\mathcal{N}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right).$$

Note, however, that by Sect. 20.6.1, the abelian category on the LHS of (10.1) identifies with

$$\operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}.$$

10.3.1. For

$$\mathcal{F} \in \operatorname{Heart}\left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)$$

we set

$$\Omega(\mathcal{F}) := H^0\left((\iota_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F})\right).$$

We claim:

Proposition 10.3.2. *$(\iota_{\mathrm{Fl}})_*(\Omega(\mathcal{F})) \simeq (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F})$.*

Since the functor $(\iota_{\mathrm{Fl}})_*$ is exact, the above proposition immediately follows from the next one:

Lemma 10.3.3. *For $\mathcal{F} \in \operatorname{Rep}(\check{B}) \underset{\operatorname{Rep}(\check{G})}{\otimes} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$, the object*

$$(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*\left((\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F})\right) \in \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is canonically a direct sum

$$\bigoplus_k \Lambda^k(\check{\mathfrak{n}}^*) \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F})[k].$$

10.3.4. *Proof of Lemma 10.3.3.* We calculate $(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^* \left((\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}')\right)$ by means of

$$\mathrm{Kosz}^\bullet_{\mathrm{pt}/\check{B},\widetilde{\widetilde{\mathcal{N}}}/\check{G}} \underset{\mathcal{O}_{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}}{\otimes} (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}'),$$

where $\mathrm{Kosz}^\bullet_{\mathrm{pt}/\check{B},\widetilde{\widetilde{\mathcal{N}}}/\check{G}}$ is as in Sect. 10.2.

Note that the differential of the Koszul complex has the property the maps

$$\mathrm{Kosz}^k_{\mathrm{pt}/\check{B},\widetilde{\widetilde{\mathcal{N}}}/\check{G}} \to \mathrm{Kosz}^{k-1}_{\mathrm{pt}/\check{B},\widetilde{\widetilde{\mathcal{N}}}/\check{G}}$$

take value zero on the zero section $\mathrm{pt}/\check{B} \overset{\iota}{\hookrightarrow} \widetilde{\widetilde{\mathcal{N}}}/\check{G}$. Hence, the assertion of the next lemma follows from the next one:

Lemma 10.3.5. *Let $\mathcal{N}_1 \to \mathcal{N}_2$ be a map of vector bundles on $\widetilde{\widetilde{\mathcal{N}}}/\check{G}$ whose value on $\mathrm{pt}/\check{B}$ is zero. Then for any $\mathcal{F}' \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ of the form $(\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F})$ for*

$$\mathcal{F} \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

the map

$$\mathcal{N}_1 \underset{\mathcal{O}_{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}' \to \mathcal{N}_2 \underset{\mathcal{O}_{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}'$$

is zero.

Proof. Since the functor $(\widetilde{\iota}_{\mathrm{Fl}})^*$ is compatible with the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\widetilde{\widetilde{\mathcal{N}}}/\check{G}))$, by adjunction, we obtain that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{N}_1 \underset{\mathcal{O}_{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}}{\otimes} (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}) \arrow[r] \arrow[d, "\sim"'] & \mathcal{N}_2 \underset{\mathcal{O}_{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}}{\otimes} (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}) \arrow[d, "\sim"'] \\
(\widetilde{\iota}_{\mathrm{Fl}})_*\left(\iota^*(\mathcal{N}_1) \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{F}'\right) \arrow[r] & (\widetilde{\iota}_{\mathrm{Fl}})_*\left(\iota^*(\mathcal{N}_2) \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{F}\right).
\end{tikzcd}$$

□

10.4. Proof of Proposition 8.2.2. Consider the object

$$\Omega \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \in \mathrm{Heart}\left(\mathrm{pt}/\check{B} \underset{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right).$$

We claim that there exist canonical maps

$$(\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to \Omega \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \text{ and } \Omega \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}). \tag{10.2}$$

The former map follows by adjunction from

$$\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G} \to (\iota_{\mathrm{Fl}})_* \circ \Omega \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}),$$

whereas the latter map is obtained similarly by adjunction from

$$(\iota_{\mathrm{Fl}})_! \circ \Omega \circ (\widetilde{\iota}_{\mathrm{Fl}})^!(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq (\widetilde{\iota}_{\mathrm{Fl}})_! \circ (\widetilde{\iota}_{\mathrm{Fl}})^!(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to \delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}.$$

By adjunction, the maps in (10.2) give rise to non-zero maps

$$(\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\iota_{\mathrm{Fl}})_* \circ \Omega \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})$$

and

$$(\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq (\iota_{\mathrm{Fl}})_* \circ \Omega \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\iota_{\mathrm{Fl}})^* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}).$$

To prove the proposition, it is enough to show that the above maps coincide, up to non-zero scalars, with the maps (7.7) and (7.12), respectively.

This, in turn, follows from the next lemma:

Lemma 10.4.1. *The* Hom *spaces*

$$\mathrm{Hom}\Big((\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big)$$

and

$$\mathrm{Hom}\Big((\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big)$$

are 1-dimensional.

Proof. We have:

$$\mathrm{Hom}\Big((\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big) \simeq$$
$$\mathrm{Hom}\Big((\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big) \simeq$$
$$\mathrm{Hom}\Big(\Upsilon \circ (\iota_{\mathrm{Fl}})^* \circ (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), \Upsilon \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big)$$

We have a canonical map

$$(\iota_{\mathrm{Fl}})^* \circ (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \to (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}),$$

whose cone is filtered by objects of the form $\mathcal{L}^{\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}\,/\check{B}}}{\otimes} (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})[k]$, $k > 0$. Hence, when we apply the functor Υ to this cone, we obtain an object which is < 0.

Hence, the above Hom is isomorphic to $\mathrm{End}\Big(\Upsilon \circ (\iota_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})\Big)$, and the latter identifies with $\mathrm{End}(\mathcal{W}) \simeq \mathbb{C}$.

The second assertion of the lemma follows by a similar manipulation involving the pair $(\iota_{\mathrm{Fl}})^!, (\iota_{\mathrm{Fl}})_!$.

□

11. Turning Υ into an equivalence

In this section we will show how to modify the functor Υ to turn it into an equivalence. The results of this section will not be used elsewhere in the paper.

11.1. As a first step, we will modify the functor Υ so that the new functor defines an equivalence between the corresponding $\mathbf{D}^+$ categories. [4]

11.1.1. We begin with the following observation:

Proposition 11.1.2. *Suppose that* $\mathcal{F} \in \mathrm{pt}\,/\check{B} \underset{\widetilde{\check{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ *belongs to* $\mathbf{D}^+$*. Then for* $i \ll 0$*, the truncation* $\tau^{<i}(\Upsilon(\mathcal{F}))$ *is an acyclic object of* $\mathrm{pt}\,/\check{B} \underset{\mathrm{pt}\,/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$*, i.e., is cohomologically* $\leq -n$ *for any* $n \in \mathbb{N}$*.*

Proposition 11.1.2 follows from the next more precise estimate:

[4] Here and elsewhere, for a triangulated category $\mathbf{D}$ equipped with a t-structure, we denote by $\mathbf{D}^b$, $\mathbf{D}^+$ and $\mathbf{D}^-$, respectively, the corresponding bounded (resp., bounded from below, bounded from above) subcategories, see Sect. 19.1.

Proposition 11.1.3. *If* $\mathcal{F} \in \mathrm{pt}/\check{B} \underset{\widetilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ *is cohomologically* ≥ 0*, then* $\tau^{<0}(\Upsilon(\mathcal{F})) \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ *is acyclic.*

11.2. Proof of Proposition 11.1.3. We will deduce the proposition from the next assertion:

Lemma 11.2.1. *The functor*

$$(\widetilde{\iota}_{\mathrm{Fl}})_* : \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$$

is conservative when restricted to $\mathbf{D}^+$*.*

Proof. It is sufficient to show that for $\mathcal{F}'$, belonging to the heart of the t-structure on the category $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$, the object $(\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}')$ is non-zero. For that, it is sufficient to show that $(\widetilde{\iota}_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F}') \neq 0$. However, the latter follows from Proposition 10.1.1. $\square$

Returning to the proof of the proposition, it is sufficient to show that the composed functor $(\widetilde{\iota}_{\mathrm{Fl}})_* \circ \Upsilon$ sends an object $\mathcal{F}$ as in the proposition to an object of $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$, which is acyclic in degrees < 0.

However, by Theorem 7.3.1, the map $(\iota_{\mathrm{Fl}})_* \to (\widetilde{\iota}_{\mathrm{Fl}})_* \circ \Upsilon$ of (7.6) is an isomorphism. So, our assertion follows from the fact that the functor $(\iota_{\mathrm{Fl}})_*$ is exact, see Proposition 20.4.1. $\square$

11.3. Using Proposition 11.1.2, we define a new functor

$$\Upsilon^+ : \left(\mathrm{pt}/\check{B} \underset{\widetilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})\right)^+ \to \left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})\right)^+$$

by

$$\Upsilon^+(\mathcal{F}) := \tau^{\geq i}(\Upsilon(\mathcal{F})) \text{ for some/any } i \ll 0.$$

We claim:

Theorem 11.3.1. *The functor* Υ^+ *is an exact equivalence of categories.*

Before beginning the proof of this theorem, let us make several remarks:

11.3.2. First, the fact that Υ^+ is right-exact follows from the right-exactness of Υ (Corollary 9.2.2). The fact that Υ^+ is left-exact follows from Proposition 11.1.3.

11.3.3. Secondly, we claim that the fully-faithfulness of Υ^+ follows from that of Υ (Main Theorem 4).

Indeed, we need to show that for $\mathcal{F}_1, \mathcal{F}_2 \in \left(\mathrm{pt}/\check{B} \underset{\widetilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})\right)^+$, the map

$$\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}(\Upsilon^+(\mathcal{F}_1), \Upsilon^+(\mathcal{F}_2))$$

is an isomorphism.

By construction, $\Upsilon^+(\mathcal{F}_2)$ belongs to $\mathbf{D}^+$, and $\mathrm{Cone}(\Upsilon(\mathcal{F}_1) \to \Upsilon^+(\mathcal{F}_1))$ belongs to $\mathbf{D}^{<i}$ for any i. Hence,

$$\mathrm{Hom}(\Upsilon^+(\mathcal{F}_1), \Upsilon^+(\mathcal{F}_2)) \to \mathrm{Hom}(\Upsilon(\mathcal{F}_1), \Upsilon^+(\mathcal{F}_2))$$

is an isomorphism.

We claim now that

$$\mathrm{Hom}(\Upsilon(\mathcal{F}_1), \Upsilon(\mathcal{F}_2)) \to \mathrm{Hom}(\Upsilon(\mathcal{F}_1), \Upsilon^+(\mathcal{F}_2))$$

is an isomorphism for any $\mathcal{F}_1 \in \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathfrak{D}^f(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}$. Indeed, with no restriction of generality we can take $\mathcal{F}_1$ from $\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}^f(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}$, and further of the form $\iota^*_{\mathrm{Fl}}(\mathcal{F}'_1)$ for some $\mathcal{F}'_1 \in \mathfrak{D}^f(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}$.

Thus, by adjunction, it suffices to show that the map

$$(\widetilde{\iota}_{\mathrm{Fl}})_* \circ \Upsilon \to (\widetilde{\iota}_{\mathrm{Fl}})_* \circ \Upsilon^+$$

is an isomorphism, which follows from the construction.

11.3.4. Thirdly, we claim that the functor Υ^+ commutes with colimits taken within $\mathbf{D}^{\geq -i}$ for any i.

Thus, to prove Theorem 11.3.1 it suffices to show the functor Υ^+ is essentially surjective onto

$$\mathrm{Heart}\left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})\right). \tag{11.1}$$

We will do so by exhibiting a right inverse functor on this subcategory. In fact, we claim that the functor Ω constructed in Sect. 10.3 provides such an inverse. Namely, we claim:

Proposition 11.3.5. *There exists a canonical isomorphism*

$$\Upsilon^+(\Omega(\mathcal{F})) \simeq \mathcal{F}.$$

By the above discussion, Proposition 11.3.5 implies Theorem 11.3.1.

11.3.6. *Proof of Proposition 11.3.5.* Since the functor Υ^+ is exact, it suffices to show that

$$H^0\left(\Upsilon^+\left((\iota_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F})\right)\right) \simeq \mathcal{F}.$$

By the definition of Υ^+, the LHS of the latter expression is isomorphic to

$$H^0\left(\Upsilon\left((\iota_{\mathrm{Fl}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})_*(\mathcal{F})\right)\right) \simeq \mathcal{F}.$$

Hence, our assertion follows from Proposition 10.1.1.

$\square$

11.4. Let

$$\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B}\underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times}\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}\right) \tag{11.2}$$

be the full subcategory of $\mathrm{pt}/\check{B}\underset{\overset{\to}{\tilde{\tilde{\mathcal{N}}}/\check{G}}}{\times}\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$, consisting of objects $\mathcal{F}$, for which $(\iota_{\mathrm{Fl}})_*(\mathcal{F})$ belongs to $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})\subset\mathbf{D}^f_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$.

Let

$$\mathbf{D}_{ren}\left(\mathrm{pt}/\check{B}\underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times}\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}\right) \tag{11.3}$$

be the corresponding renormalized triangulated category, given by the procedure of Sect. 22.1, i.e., the ind-completion of (11.2). By Sect. 22.2, the category (11.3) acquires a t-structure.

Let us denote by

$$((\iota_{\mathrm{Fl}})_*)_{ren}:\mathbf{D}_{ren}\left(\mathrm{pt}/\check{B}\underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times}\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}\right)\to\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$$

the ind-extension of the functor

$$(\iota_{\mathrm{Fl}})_*:\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B}\underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times}\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}\right)\to\mathbf{D}^f_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}).$$

By the construction of the t-structure on (11.3), the exactness of the functor $(\iota_{\mathrm{Fl}})_*$ (see Proposition 20.4.1) implies the exactness of $((\iota_{\mathrm{Fl}})_*)_{ren}$.

11.4.1. We shall now construct a functor

$$\Upsilon_{ren}:\mathbf{D}_{ren}\left(\mathrm{pt}/\check{B}\underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times}\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}\right)\to\mathrm{pt}/\check{B}\underset{\overset{\to}{\mathrm{pt}/\check{G}}}{\times}\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}).$$

It will be defined as the ind-extension of a functor

$$\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B}\underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times}\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}\right)\to\mathrm{pt}/\check{B}\underset{\overset{\to}{\mathrm{pt}/\check{G}}}{\times}\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$$

that we denote by the same character Υ_{ren}. The latter functor is defined as the restriction of Υ^+ to the subcategory

$$\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B}\underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times}\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}\right)\subset\left(\mathrm{pt}/\check{B}\underset{\overset{\to}{\tilde{\tilde{\mathcal{N}}}/\check{G}}}{\times}\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})\right)^+.$$

We are now ready to formulate the main result of this section:

Theorem 11.4.2. *The functor Υ_{ren} is an equivalence of categories. It is exact with respect to the t-structures defined on both sides.*

The rest of this section is devoted to the proof of this theorem.

Remark 1. Let us note that it is *a priori* not clear, although ultimately true, that the restriction of the functor Υ_{ren} to

$$\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})\right)^+ \simeq \mathbf{D}_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}\right)^+$$

is isomorphic to Υ^+. This is isomorphism is one of the finiteness issues we will have to come to grips with in the proof that follows.

Remark 2. As another manifestation of the fact that the original functor

$$\Upsilon : \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$$

is not an equivalence of categories (even after passing to Karoubian envelopes) is that the object $\Omega(\mathrm{Res}^{\check{G}}_{\check{B}}(\delta_{1,\mathrm{Gr}_G^{\mathrm{aff}}})) \in \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$ is not compact.

11.4.3. The main step in the proof of the theorem is the following:

Proposition 11.4.4. *The functor Υ_{ren} sends $\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}\right)$ to the category of compact objects in $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$.*

11.4.5. Let us show how Proposition 11.4.4 implies Theorem 11.4.2.

First, we claim that the functor Υ_{ren} is fully faithful. Indeed, by Proposition 11.4.4, it is enough to check that the restriction of Υ_{ren} to $\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}\right)$ is fully faithful, and the latter assertion follows from the fully-faithfulness of Υ^+.

Thus, to prove that Υ_{ren} is an equivalence, it is sufficient to check that it is essentially surjective onto the generators of $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$. Up to tensoring with $\mathcal{L}^{\check{\lambda}}$, it is sufficient to check that an object of the form $\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F})$ for $\mathcal{F}$ a finitely generated D-module on $\mathrm{Gr}_G^{\mathrm{aff}}$, is in the image of Υ_{ren}.

Consider the object

$$\Omega(\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F})) \in \mathrm{Heart}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathfrak{D}^f(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}\right) \subset \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathfrak{D}^f(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}.$$

By Proposition 10.3.2, $\Omega(\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}))$ belongs to $\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}\right)$, since $p^{!*}(\mathcal{F})$ is finitely generated. Therefore,

$$\Upsilon_{ren}(\Omega(\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}))) \simeq \Upsilon^+(\Omega(\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}))),$$

and the latter identifies with $\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F})$, by Proposition 11.3.5.

It remains to show that Υ_{ren} is exact. The right-exactness follows by construction. Hence, it is enough to show that the inverse functor $(\Upsilon_{ren})^{-1}$ is also right-exact. For that it is enough to show that $(\Upsilon_{ren})^{-1}$ sends generators of

$$\mathrm{Heart}\left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\rightarrow}{\times}} \mathfrak{D}^f(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}\right) \simeq \mathrm{Rep}(\check{B}) \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$$

to objects that are ≤ 0. Hence, it is sufficient to show that $(\Upsilon_{ren})^{-1}(\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}))$ is ≤ 0 for $\mathcal{F}$ a finitely generated D-module on $\mathrm{Gr}^{\mathrm{aff}}_G$. However, we have seen above that $(\Upsilon_{ren})^{-1}(\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F})) \simeq \Omega(\mathrm{Res}^{\check{G}}_{\check{B}}(\mathcal{F}))$, which is in the heart of the t-structure.

□

11.4.6. As a remark, let us now show that the functors Υ_{ren} and Υ^+ are canonically isomorphic when restricted to

$$\mathbf{D}^+_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}\right) \simeq \left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)^+,$$

(where the latter identification of the categories is given by Proposition 22.2.1).

Proof. By construction, there is a natural transformation $\Upsilon_{ren} \to \Upsilon^+$. Since both functors send $\mathbf{D}^+$ to $\mathbf{D}^+$ is suffices to check that the above map induces an isomorphism an individual cohomologies.

Let Ψ denote the canonical functor

$$\mathbf{D}_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}\right) \to \mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

see Sect. 22.1. By construction, we have a natural transformation $\Upsilon \circ \Psi \to \Upsilon_{ren}$.

The composed map $\Upsilon \circ \Psi \to \Upsilon_{ren} \to \Upsilon^+$, applied to objects from $\mathbf{D}^+$, is an isomorphism, as follows from the definition of Υ^+. Hence, it is enough to check that $H^i(\Upsilon \circ \Psi) \to H^i(\Upsilon_{ren})$ is an isomorphism.

Both functors are defined on the whole of $\mathbf{D}_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}\right)$ and commute with direct limits. Hence, it is enough to prove the assertion for their restrictions to $\mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}\right)$. The isomorphism in the latter case again follows from the definition of Υ^+.

□

11.5. Proof of Proposition 11.4.4. The proof is based on the following:

Lemma 11.5.1. *There exists a finite collection of elements $\check{\lambda}_i$, such that an object*

$$\mathcal{F} \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is compact if and only if

$$\mathrm{Ind}^{\check{G}}_{\check{B}}\left(\mathcal{L}^{\check{\lambda}_i} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{F}\right) \in \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is compact for every i.

11.5.2. Let us show how to deduce Proposition 11.4.4 from the lemma. By construction, for $\mathcal{F} \in \mathbf{D}^f_{ren}\left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}\right)$, the object $\Upsilon_{ren}(\mathcal{F})$ belongs to $\mathbf{D}^b$. In addition, we claim that $\Upsilon_{ren}(\mathcal{F})$ is *almost compact*, see Sect. 22.3, where the latter notion is introduced. Indeed, this follows from Theorem 11.3.1 and the fact that the functor Ω commutes with direct limits on the abelian category.

Now, we claim that any almost compact object of $\left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)^b$ is compact. Indeed, by Lemma 11.5.1, it is sufficient to check that $\mathcal{F}' := \mathrm{Ind}^{\check{G}}_{\check{B}}\left(\mathcal{L}^{\check{\lambda}} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{F}\right)$ is compact for any (or, in fact, a finite collection of) $\check{\lambda}$. However, $\mathcal{F}'$ is also almost compact by adjunction. But since the category $\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$ is Noetherian, it is clear that every almost compact object in $\mathbf{D}^b(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ is compact.

□

11.5.3. *Proof of Lemma 11.5.1.* The argument given below belongs to Jacob Lurie:

We will prove the lemma in the general context of a triangulated category $\mathbf{D}$ over the stack $\mathrm{pt}/\check{G}$, where in our case we take $\mathbf{D} := \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$.

Let us represent $\mathcal{O}_{\Delta_{\mathrm{pt}/\check{B}}} \in \mathrm{Coh}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{pt}/\check{B}) \simeq \mathrm{Coh}(\mathrm{Fl}^{\check{G}} \times \mathrm{Fl}^{\check{G}}/\check{G})$ as a direct summand of a complex as in Sect. 2.3.3.

This implies that any object $\mathcal{F} \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}$ (or, $\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\overset{\to}{\times}} \mathbf{D}$) is a direct summand of an object, which is a successive extensions of objects of the form

$$\mathcal{L}^{\check{\mu}_i} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathrm{Ind}^{\check{G}}_{\check{B}}\left(\mathcal{L}^{\check{\lambda}_i} \underset{\mathcal{O}_{\mathrm{pt}/\check{B}}}{\otimes} \mathcal{F}\right).$$

If all of the latter are compact, then so is $\mathcal{F}$.

□

12. Compatibility of Γ_{Fl} and Υ

In this section we will prove Theorem 6.3.1.

12.1. As was mentioned earlier, it suffices to show that the following diagram commutes

$$\begin{tikzcd}
\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Fl}}"] \arrow[d, "\widetilde{\iota}^*_{\mathrm{Fl}}"'] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \arrow[d, "\iota^*_{\widehat{\mathfrak{g}}}"'] \\
\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}}"] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}),
\end{tikzcd} \tag{12.1}$$

as functors between categories over $\widetilde{\check{\mathcal{N}}}/\check{G}$.

12.1.1. Let $(\iota_{\widehat{\mathfrak{g}}})_*$ be the evident functor

$$\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}).$$

This is a functor between categories over $\mathrm{Op}^{\mathrm{nilp}}$, and in particular over $\widetilde{\check{\mathcal{N}}}/\check{G}$. By Sect. 23.3.3, it sends

$$\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}),$$

making the latter also into a functor between categories over $\mathrm{Op}^{\mathrm{nilp}}$ and $\widetilde{\check{\mathcal{N}}}/\check{G}$; it is in fact the right adjoint of $(\iota_{\widehat{\mathfrak{g}}})^*$.

Proposition 12.1.2. *We have a commutative diagram of functors between categories over* $\mathrm{pt}/\check{B}$*:*

$$\begin{tikzcd}
\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Fl}}"] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \\
\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}}"] \arrow[u, "(\widetilde{\iota}_{\mathrm{Fl}})_*"] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}). \arrow[u, "(\iota_{\widehat{\mathfrak{g}}})_*"']
\end{tikzcd}$$

Proof. By Sect. 17.5.3, it suffices to prove the commutativity of the following diagram of functors between categories over $\mathrm{pt}/\check{G}$,

$$\begin{tikzcd}
\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Fl}}"] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \\
\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \arrow[r, "\Gamma_{\mathrm{Gr}}"] \arrow[u, "p^{!*}"] & \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}), \arrow[u, "(\iota_{\widehat{\mathfrak{g}}})_*"']
\end{tikzcd}$$

which is manifest.

□

12.2. As the next step in establishing the commutativity of the diagram (12.1), we will prove the following:

Proposition 12.2.1. *The composition of the functors*

$$(\iota_{\widehat{\mathfrak{g}}})^* \circ \Gamma_{\mathrm{Fl}} \text{ and } \Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^* : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \rightrightarrows \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$$

with $(\iota_{\widehat{\mathfrak{g}}})_*$*, yield isomorphic functors*

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \rightrightarrows \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$$

at the triangulated level.

12.2.2. *Proof of Proposition 12.2.1.* On the one hand, the composition

$$(\iota_{\widehat{\mathfrak{g}}})_* \circ (\iota_{\widehat{\mathfrak{g}}})^* : \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$$

identifies with the functor

$$\mathcal{M} \mapsto \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{M} \simeq \mathcal{O}_{\mathrm{pt}/\check{B}} \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{M}.$$

Hence, since the functor Γ_{Fl} respects the action of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))$, we obtain that the composition $(\iota_{\widehat{\mathfrak{g}}})_* \circ (\iota_{\widehat{\mathfrak{g}}})^* \circ \Gamma_{\mathrm{Fl}}$ is isomorphic to

$$\Gamma_{\mathrm{Fl}} \circ (\mathcal{O}_{\mathrm{pt}/\check{B}} \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} -) \simeq \Gamma_{\mathrm{Fl}} \circ (\iota_{\mathrm{Fl}})_* \circ (\iota_{\mathrm{Fl}})^*.$$

On the other hand, by Proposition 12.1.2, the composition $(\iota_{\widehat{\mathfrak{g}}})_* \circ \Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^*$ identifies with

$$\Gamma_{\mathrm{Fl}} \circ (\widetilde{\iota}_{\mathrm{Fl}})_* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*,$$

and the assertion follows from Theorem 7.3.1.

□

12.3. We are now ready to prove the commutativity of the diagram of functors given by (12.1).

12.3.1. Since the tautological functor $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$ is fully faithful, it is sufficient to construct an isomorphism between the corresponding functors

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \rightrightarrows \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$$

as functors between categories over $\tilde{\check{\mathcal{N}}}/\check{G}$.

Consider the two objects

$$(\iota_{\widehat{\mathfrak{g}}})^* \circ \Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \text{ and } \Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \in \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})^I.$$

Proposition 12.3.2. *The above objects belong to* $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^I_{\mathrm{nilp}}$ *and are isomorphic.*

Proof. Since the functor

$$(\iota_{\widehat{\mathfrak{g}}})_* : \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$$

is exact and conservative, and in the diagram of functors between *abelian* categories

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^I_{\mathrm{reg}} & \xrightarrow{(\iota_{\widehat{\mathfrak{g}}})_*} & \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^I_{\mathrm{nilp}} \\ \downarrow & & \downarrow \\ \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}} & \xrightarrow{(\iota_{\widehat{\mathfrak{g}}})_*} & \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}} \end{array}$$

all the arrows are fully faithful, it is sufficient to show that

$$(\iota_{\widehat{\mathfrak{g}}})_* \circ (\iota_{\widehat{\mathfrak{g}}})^* \circ \Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \text{ and } (\iota_{\widehat{\mathfrak{g}}})_* \circ \Gamma_{\mathrm{Gr},\mathrm{pt}/\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G})$$

are isomorphic as objects of $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, and that the LHS belongs to the abelian category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}$.

The first assertion follows readily from Proposition 12.2.1. To prove the second assertion we note that

$$\Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq \mathbb{M}_{\mathrm{crit},-2\rho},$$

and since $\mathbb{M}_{\mathrm{crit},-2\rho}$ is $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$-flat (see [FG2], Corollary 13.9),

$$(\iota_{\widehat{\mathfrak{g}}})_* \circ (\iota_{\widehat{\mathfrak{g}}})^*(\mathbb{M}_{\mathrm{crit},-2\rho}) \simeq \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathbb{M}_{\mathrm{crit},-2\rho} =: \mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}}$$

belongs to $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$.

□

Remark. Note that the above assertion, which amounts to the isomorphism

$$\mathbb{M}_{\mathrm{crit},-2\rho,\mathrm{reg}} \simeq \Gamma_{\mathrm{Gr},\mathrm{pt}\,/\check{B}}(J_{2\rho} \star \mathcal{W})),$$

coincides with that of Theorem 15.6 of [FG2], which was proven by a rather explicit and tedious calculation. Thus, Proposition 12.3.2, can be regarded as an alternative proof of this fact, in a way more conceptual. Note, however, that the main ingredient in the proof of Proposition 12.3.2 was Theorem 7.3.1, which was far from tautological.

12.3.3. As in Sections 24.3 and 24.4 the functors

$$\mathcal{F} \mapsto \mathcal{F} \star \Big((\iota_{\widehat{\mathfrak{g}}})^* \circ \Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}_G^{\mathrm{aff}}})\Big) \text{ and } \mathcal{F} \mapsto \mathcal{F} \star \Big(\Gamma_{\mathrm{Gr},\mathrm{pt}\,/\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\delta_{1,\mathrm{Fl}_G^{\mathrm{aff}}})\Big) : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})$$

can be upgraded to a DG level and endowed with the structure of functors between categories over $\widetilde{\widetilde{\mathcal{N}}}/\check{G}$. Moreover, the above functors identify with the functors

$$(\iota_{\widehat{\mathfrak{g}}})^* \circ \Gamma_{\mathrm{Fl}} \text{ and } \Gamma_{\mathrm{Gr},\mathrm{pt}\,/\check{B}} \circ (\widetilde{\iota}_{\mathrm{Fl}})^*,$$

respectively.

Thus, the commutativity of the diagram (12.1) follows from Proposition 12.3.2.

13. Fully-faithfulness of $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}$

In this section we will prove Main Theorem 1.

13.1. By definition, the theorem says that the map

$$\mathrm{Hom}_{\mathrm{Op}^{\mathrm{nilp}} \underset{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})}(\mathcal{F}'_1, \mathcal{F}'_2) \to \mathrm{Hom}_{\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}\Big(\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\mathcal{F}'_1), \Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\mathcal{F}'_2)\Big) \tag{13.1}$$

is an isomorphism for any $\mathcal{F}'_i \in \mathrm{Op}^{\mathrm{nilp}} \underset{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$, $i = 1, 2$.

By a slight abuse of notation we will denote by $\mathfrak{r}^*_{\mathrm{nilp}}$ the pull-back functor

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod} \to \mathrm{Op}^{\mathrm{nilp}} \underset{\widetilde{\widetilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod}).$$

13.1.1. We shall now perform a reduction step, showing that it is enough to establish isomorphism (13.1) for $\mathcal{F}'_i = \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_i)$, where each $\mathcal{F}'_i$ is a single finitely generated I-equivariant D-module on $\mathrm{Fl}_G^{\mathrm{aff}}$.

13.1.2. *Step 1.* By the definition of $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$, we can take $\mathcal{F}'_i$ in (13.1) to be of the form $\mathcal{M}_i \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_i)$ with $\mathcal{M}_i \in \mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$.

Since $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$ is a polynomial algebra, every $\mathcal{M}_i$ as above is a direct summand of a finite complex consisting of free $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$-modules. So, we can assume that $\mathcal{F}'_i = \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_i)$, with $\mathcal{F}'_i \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$. I.e, we are reduced to showing that the map

$$\mathrm{Hom}_{\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1), \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2)) \to \\ \to \mathrm{Hom}_{\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}\left(\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1)), \Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2))\right) \tag{13.2}$$

is an isomorphism.

13.1.3. *Step 2.* Recall that $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_i)) \simeq \Gamma_{\mathrm{Fl}}(\mathcal{F}_i)$ for $i = 1, 2$. So, the RHS of (13.2) is isomorphic to

$$\mathrm{Hom}_{\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}\left(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), \Gamma_{\mathrm{Fl}}(\mathcal{F}_2)\right). \tag{13.3}$$

By Sect. 22.1, the above expression is isomorphic to Hom taken in the usual category $\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$.

We rewrite the LHS of (13.2) using Corollary 18.4.2(2), and we obtain that it is isomorphic to

$$\mathrm{Hom}_{\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})}\left(\mathcal{F}_1, (\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}_2\right), \tag{13.4}$$

where $(\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \in \mathbf{D}(\mathrm{QCoh}(\tilde{\check{\mathcal{N}}}/\check{G})) \simeq \underset{\to}{\mathbf{D}}^{perf}(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))$.

By Proposition 2.2.2, the object

$$(\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}_2 \simeq \mathcal{F} \star \mathsf{F}((\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}))$$

belongs to $\mathbf{D}^+$, *in the new t-structure*. By Proposition 2.3.1, we obtain that it belongs to $\mathbf{D}^+$ also in the *old t-structure*. Therefore, by Sect. 22.2, we can regard it as an object of the usual category $\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$, and the expression in (13.4) can also be rewritten as Hom in $\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})$.

The map from (13.4) to (13.3) can thus be identified with the composition

$$\mathrm{Hom}_{\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{-mod})}(\mathcal{F}_1, (\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}_2) \to \\ \to \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}\left(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), \Gamma_{\mathrm{Fl}}(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}} \underset{\mathcal{O}_{\tilde{\check{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}_2)\right) \simeq \\ \simeq \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}\left(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), \mathfrak{r}^*_{\mathrm{nilp}}((\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}})) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \Gamma_{\mathrm{Fl}}(\mathcal{F}_2)\right) \to \\ \to \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}\left(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), \Gamma_{\mathrm{Fl}}(\mathcal{F}_2)\right), \tag{13.5}$$

where the last arrow comes from the canonical map

$$\mathfrak{r}^*_{\mathrm{nilp}}((\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}})) \to \mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}.$$

13.1.4. *Step 3.* By the adjunction [FG2], Proposition 22.22, we have the isomorphisms:

$$\mathrm{Hom}_{\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})}(\mathcal{F}_1, (\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\tilde{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}_2) \simeq$$

$$\simeq \mathrm{Hom}_{\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^I}\left(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}, (\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \otimes \mathcal{F}\right),$$

and

$$\mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), \Gamma_{\mathrm{Fl}}(\mathcal{F}_2)) \simeq \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^I}\left(\Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), \Gamma_{\mathrm{Fl}}(\mathcal{F})\right),$$

with $\mathcal{F} \simeq \mathcal{F}_1^* \star \mathcal{F}_2 \in \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^I$, where $\mathcal{F}_1^*$ denotes the dual D-module on $I\backslash G((t))$, see [FG2], Sect. 22.21.

Hence, it is sufficient to show that the map

$$\mathrm{Hom}_{\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^I}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}, (\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\tilde{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}) \to$$

$$\to \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})^I}(\Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), \Gamma_{\mathrm{Fl}}(\mathcal{F})),$$

given by a I-equivariant version of (13.5), is an isomorphism for $\mathcal{F} \in \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^I$.

13.1.5. *Step 4.* Finally, using the spectral sequence that expresses Hom in the I-equivariant category in terms of the usual Hom, we obtain that it is enough to show that the map

$$\mathrm{Hom}_{\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}, (\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\tilde{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F}) \to \tag{13.6}$$

$$\to \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}(\Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}), \Gamma_{\mathrm{Fl}}(\mathcal{F})),$$

of (13.5) is an isomorphism for any $\mathcal{F} \in \mathbf{D}^b(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^I$.

Using the fact that $\Gamma_{\mathrm{Fl}}(\delta_{1,\mathrm{Fl}^{\mathrm{aff}}_G}) \simeq \mathbb{M}_{-2\rho}$ is almost compact as an object of $\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$ (see Sect. 22.3), we can use devissage on $\mathcal{F}$ and assume that it consists of a single finitely generated I-equivariant D-module.

Thus, the reduction announced in Sect. 13.1.1 has been performed.

13.2. Note that the categories $\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ and $\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, appearing on the two sides of (13.6) carry a weak $\mathbb{G}_m$-action by loop rotations; moreover, the action on the former category is strong (a.k.a. of Harish-Chandra type), see [FG2], Sect. 20, where these concepts are introduced.

The category $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$ carries a weak $\mathbb{G}_m$-action (via the action of $\mathbb{G}_m$ on $\check{\mathfrak{g}}$ by dilations), and the map $\mathfrak{r}_{\mathrm{nilp}} : \mathrm{Op}^{\mathrm{nilp}} \to \tilde{\tilde{\mathcal{N}}}/\check{G}$ is easily seen to be $\mathbb{G}_m$-equivariant. In addition, the construction of the functor

$$\mathsf{F} : \mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\tilde{\mathcal{N}}}/\check{G})) \to \mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

implies that it has a natural $\mathbb{G}_m$-equvariant structure.

With no restriction of generality we can assume that $\mathcal{F}$ that appears in (13.6) consists of a single twisted D-module, which is strongly equivariant with respect to $\mathbb{G}_m$. By the $\mathbb{G}_m$-equivariance of $\mathfrak{r}_{\mathrm{nilp}}$, the object $(\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \in \mathbf{D}^+(\mathrm{QCoh}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$ is $\mathbb{G}_m$-equivariant; hence

$$(\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}) \underset{\mathcal{O}_{\tilde{\tilde{\mathcal{N}}}/\check{G}}}{\otimes} \mathcal{F} \simeq \mathcal{F} \star \mathsf{F}((\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}})) \in \mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is weakly $\mathbb{G}_m$-equivariant.

13.2.1. Thus, both sides of (13.6) compute Hom between weakly $\mathbb{G}_m$-equivariant objects in categories endowed with weak $\mathbb{G}_m$-actions. Hence, the resulting Hom groups are acted on by $\mathbb{G}_m$, i.e., are graded complexes. Furthermore, it is easy to see that the map in (13.6) preserves the gradings. We claim now that the grading on both cases is bounded from below.

Right-hand side. It was shown in [FG2] that $\mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod}_{\mathrm{nilp}})}(\mathbb{M}_{-2\rho}, \mathcal{M})$ is related by a spectral sequence to $\mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod})}(\mathbb{M}_{-2\rho}, \mathcal{M})$ for $\mathcal{M} \in \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod}_{\mathrm{nilp}})$. Moreover, if $\mathcal{M}$ is graded, then the grading on the former Hom is bounded from below on the former if and only if it is so on the latter. We apply this to $\mathcal{M} := \Gamma_{\mathrm{Fl}}(\mathcal{F})$.

Left-hand side. Since the action of $\mathbb{G}_m$ on $\mathbf{D}^+(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})$ is strong our assertion follows from the next:

Lemma 13.2.2. *The defect of strong equivariance on the object* $\mathsf{F}((\mathfrak{r}_{\mathrm{nilp}})_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}))$ *is bounded from below.*

Proof. Consider the following Cartesian square:

$$\begin{array}{ccc}
\mathrm{Op}^{\mathrm{nilp}} & \longrightarrow & \widetilde{\check{\mathcal{N}}}/\check{G} \times \mathrm{Op}^{\mathrm{nilp}} \\
\big\downarrow & & \big\downarrow \\
\widetilde{\check{\mathcal{N}}}/\check{G} & \longrightarrow & \widetilde{\check{\mathcal{N}}}/\check{G} \times \widetilde{\check{\mathcal{N}}}/\check{G}.
\end{array}$$

We can realize $\mathcal{O}_{\widetilde{\check{\mathcal{N}}}/\check{G}}$ as a $\mathbb{G}_m$-equivariant coherent sheaf on $\widetilde{\check{\mathcal{N}}}/\check{G} \times \widetilde{\check{\mathcal{N}}}/\check{G}$, up to quasi-isomorphism, as a direct summand of a finite complex $\mathcal{M}^\bullet$, such that each $\mathcal{M}^i$ is a direct sum of sheaves of the form $\pi^*(\mathcal{L}^{\check{\lambda}}) \boxtimes \pi^*(\mathcal{L}^{\check{\mu}})(k)$, where (k) indicates a twist of the grading.

We obtain that $(\mathfrak{r}_{\mathrm{nilp}})_*(\mathrm{Op}^{\mathrm{nilp}})$, as a $\mathbb{G}_m$-equivariant quasi-coherent sheaf on $\widetilde{\check{\mathcal{N}}}/\check{G}$, up to quasi-isomorphism, is a direct summand of a finite complex $\mathcal{N}^\bullet$, such that each $\mathcal{N}^i$ is a direct sum of sheaves of the form

$$\pi^*(\mathcal{L}^{\check{\lambda}}) \otimes \Gamma(\mathrm{Op}^{\mathrm{nilp}}, \mathcal{L}^{\check{\mu}}_{\mathrm{Op}^{\mathrm{nilp}}})(k).$$

Both assertions of the lemma follow, since $\mathsf{F}(\mathcal{L}^{\check{\lambda}})$ is a (single) D-module, and the grading on each $\Gamma(\mathrm{Op}^{\mathrm{nilp}}, \mathcal{L}^{\check{\mu}}_{\mathrm{Op}^{\mathrm{nilp}}})$ is bounded from below.

□

13.2.3. Let us recall now the following observation, which is a Nakayama-type lemma (see [FG2], Lemma 16.5):

Lemma 13.2.4. *Let* $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ *be a graded map of two complexes of* $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$*-modules, such that the grading on both sides is bounded from below. Assume that the induced map*

$$\mathcal{M}_1 \overset{L}{\underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \to \mathcal{M}_2 \overset{L}{\underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}$$

is a quasi-isomorphism. Then ϕ *is a quasi-isomorphism.*

Thus, in order to establish that (13.6) is an isomorphism (for $\mathcal{F}$ assumed strongly equivariant with respect to $\mathbb{G}_m$), we need to know that the map in question induces an isomorphism after tensoring with $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}$ over $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$.

We will prove more generally that the map (13.2) becomes an isomorphism after tensoring with $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}$ over $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$ for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})$.

13.2.5. Let $\mathcal{F}_1, \mathcal{F}_2$ as above, we have:

$$\mathrm{Hom}_{\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1), \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2)) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\overset{L}{\otimes}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \simeq$$
$$\simeq \mathrm{Hom}_{\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1), \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2)) \simeq$$
$$\simeq \mathrm{Hom}_{\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})} \left(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1), (\iota_{\mathrm{Op}})_* \circ \iota^*_{\mathrm{Op}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2))\right) \simeq$$
$$\simeq \mathrm{Hom}_{\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\times} \left(\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)} \left(\iota^*_{\mathrm{Op}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1)), \iota^*_{\mathrm{Op}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2))\right) \simeq$$
$$\simeq \mathrm{Hom}_{\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{B}}{\times} \left(\mathrm{pt}/\check{B} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)} \left((\mathfrak{r}'_{\mathrm{reg}})^*(\iota^*_{\mathrm{Fl}}(\mathcal{F}_1)), (\mathfrak{r}'_{\mathrm{reg}})^*(\iota^*_{\mathrm{Fl}}(\mathcal{F}_2))\right)$$

and

$$\mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})} \left(\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_1)), \Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\mathfrak{r}^*_{\mathrm{nilp}}(\mathcal{F}_2))\right) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\overset{L}{\otimes}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \simeq$$
$$\mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})} \left(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), \Gamma_{\mathrm{Fl}}(\mathcal{F}_2)\right) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\overset{L}{\otimes}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \simeq$$
$$\simeq \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})} \left(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \Gamma_{\mathrm{Fl}}(\mathcal{F}_2)\right) \simeq$$
$$\simeq \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})} \left(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1), (\iota_{\widehat{\mathfrak{g}}})_* \circ (\iota_{\widehat{\mathfrak{g}}})^*(\Gamma_{\mathrm{Fl}}(\mathcal{F}_2))\right) \simeq$$
$$\simeq \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})} \left((\iota_{\widehat{\mathfrak{g}}})^*(\Gamma_{\mathrm{Fl}}(\mathcal{F}_1)), (\iota_{\widehat{\mathfrak{g}}})^*(\Gamma_{\mathrm{Fl}}(\mathcal{F}_2))\right),$$

where the last isomorphism follows from Proposition 6.2.3.

By Corollary 6.3.2, we have:

$$(\iota_{\widehat{\mathfrak{g}}})^*(\Gamma_{\mathrm{Fl}}(\mathcal{F}_i)) \simeq \Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}} \left((\mathfrak{r}'_{\mathrm{reg}})^*(\Upsilon \circ \iota^*_{\mathrm{Fl}}(\mathcal{F}_1))\right),$$

and if we denote

$$\mathcal{F}'_i = (\iota_{\mathrm{Fl}})^*(\mathcal{F}_i) \in \mathrm{pt}/\check{B} \underset{\tilde{\mathcal{N}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

the map (13.2) after tensoring with $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}$ identifies with

$$\mathrm{Hom}_{\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{B}}{\times} \left(\mathrm{pt}/\check{B} \underset{\tilde{\mathcal{N}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)} \left((\mathfrak{r}'_{\mathrm{reg}})^*(\mathcal{F}'_1), (\mathfrak{r}'_{\mathrm{reg}})^*(\mathcal{F}'_2)\right) \to$$
$$\to \mathrm{Hom}_{\mathrm{Op}^{\mathrm{reg}} \underset{\mathrm{pt}/\check{B}}{\times} \left(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)} \left((\mathfrak{r}'_{\mathrm{reg}})^*(\Upsilon(\mathcal{F}'_1)), (\mathfrak{r}'_{\mathrm{reg}})^*(\Upsilon(\mathcal{F}'_2))\right) \to$$
$$\to \mathrm{Hom}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}})} \left(\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}\left((\mathfrak{r}'_{\mathrm{reg}})^*(\Upsilon(\mathcal{F}'_1))\right), \Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}\left((\mathfrak{r}'_{\mathrm{reg}})^*(\Upsilon(\mathcal{F}'_2))\right)\right).$$

The first arrow in the above composition is an isomorphism by Main Theorem 4 (combined with Corollary 18.4.2(2)). The second arrow is an isomorphism by Theorem 6.1.3. This establishes the required isomorphism property of (13.2).

14. Equivalence for the I^0-categories

In this section we will prove Main Theorems 2 and 3.

14.1. The proof of Main Theorem 2 will be carried out in the following general framework.

Let $\mathbf{D}_i$, $i = 1, 2$ be two triangulated categories, equipped with DG models. We assume that the following conditions hold:

- Cat(i) $\mathbf{D}_1$ and $\mathbf{D}_2$ are co-complete.
- Cat(ii) $\mathbf{D}_1$ is generated by the subcategory $\mathbf{D}_1^c$ of compact objects.

Let $\mathbf{D}_i$ for $i = 1, 2$ be equipped with a t-structure. As usual, we denote

$$\mathbf{D}_i^+ = \underset{k}{\cup} \mathbf{D}_i^{\geq k}.$$

We assume that the following conditions hold:

- Cat(a) The t-structures on both $\mathbf{D}_1$ and $\mathbf{D}_2$ are compatible with colimits (see Sect. 19.1.2).
- Cat(b) $\mathbf{D}_2^+$ generates $\mathbf{D}_2$.

Let now $\mathsf{T} : \mathbf{D}_1 \to \mathbf{D}_2$ be a functor, equipped with a DG model. Assume:

- Funct(i) T commutes with direct sums.
- Funct(ii) T sends $\mathbf{D}_1^c$ to $\mathbf{D}_2^c$.

Finally, assume:

- Funct(a) $\mathsf{T} : \mathbf{D}_1^c \to \mathbf{D}_2^c$ is fully faithful.
- Funct(b) T is right-exact.
- Funct(c) For $X \in \mathbf{D}_1^c$, we have $\mathsf{T}\left(\tau^{\geq 0}(X)\right) \in \mathbf{D}_2^+$.
- Funct(d) For any $Y \in \text{Heart}(\mathbf{D}_2)$ there exists $X \in \mathbf{D}_1^{\leq 0}$ with a non-zero map $\mathsf{T}(X) \to Y$.

Proposition 14.1.1. *Under the above circumstances, the functor* T *is an exact equivalence of categories.*

The proof will be given in Sect. 19.4.

14.2. Proof of Main Theorem 2. We apply Proposition 14.1.1 to

$$\mathbf{D}_1 := \operatorname{Op}^{\text{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\operatorname{Fl}_G^{\text{aff}})_{\text{crit}}\text{-mod})^{I^0},$$

$$\mathbf{D}_2 := \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{nilp}})^{I^0},$$

and $\mathsf{T} := \Gamma_{\text{Fl},\operatorname{Op}^{\text{nilp}}}$. Thus, we need to verify that the conditions of Proposition 14.1.1 hold.

14.2.1. Conditions Cat(i) and Cat(ii) hold by definition. Conditions Cat(a) and Cat(b) follow by Proposition 4.1.7(a,d) from the corresponding properties of

$$\operatorname{Op}^{\text{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\operatorname{Fl}_G^{\text{aff}})_{\text{crit}}\text{-mod}) \text{ and } \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{nilp}}),$$

respectively.

Conditions Funct(i) and Funct(ii) also follow from the constructions.

14.2.2. Condition Funct(a) follows using the commutative diagram (4.3) from Lemma 4.1.3 and Main Theorem 1.

14.2.3. *Condition Funct(b).* By the definition of the t-structure on the LHS, our assertion is equivalent to Γ_{Fl} being right-exact, when restricted to $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0}$ in the new t-structure. In other words, we need to show that for $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})^{I^0}$, which is ≤ 0 in the new t-structure, the higher cohomologies $H^i(\Gamma_{\mathrm{Fl}}(\mathcal{F}))$ vanish. Suppose not, and let i be the maximal such i.

Any $\mathcal{F}$ as above in monodromic with respect to the action of $\mathbb{G}_m$ by loop rotations. Hence, all $H^i(\Gamma_{\mathrm{Fl}}(\mathcal{F}))$ acquire a natural grading. Moreover, as in [BD], Sect. 9.1, the grading on each $H^i(\Gamma_{\mathrm{Fl}}(\mathcal{F}))$ is bounded from below.

By Lemma 13.2.4, this implies that

$$H^i(\Gamma_{\mathrm{Fl}}(\mathcal{F})) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\otimes} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \neq 0.$$

The maximality assumption on i implies that

$$H^i\left(\Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\overset{L}{\otimes}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}\right) \neq 0.$$

By Theorem 6.3.1,

$$\Gamma_{\mathrm{Fl}}(\mathcal{F}) \underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}}{\overset{L}{\otimes}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}} \simeq \Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}\left((\mathfrak{r}'_{\mathrm{reg}})^* \circ (\widetilde{\iota}_{\mathrm{Fl}})^*(\mathcal{F})\right).$$

However, by Proposition 9.1.2(a), $(\widetilde{\iota}_{\mathrm{Fl}})^*(\mathcal{F})$ is ≤ 0, and the functor $\Gamma_{\mathrm{Gr},\mathrm{Op}^{\mathrm{reg}}}$ is right-exact by the main theorem of [FG1]; in fact, by [FG4], Theorem 1.7, the latter functor is exact when restricted to the I^0-equivariant category. This is a contradiction.

14.2.4. *Condition Funct(c).* We will show more generally that for $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, the object $\Gamma_{\mathrm{Fl},\mathrm{Op}^{\mathrm{nilp}}}(\tau^{\geq 0}(\mathcal{F}))$ belongs to $\mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$.

Since $\mathcal{O}_{\mathrm{Op}^{\mathrm{nilp}}}$ is a polynomial algebra, we can represent $\mathrm{Op}^{\mathrm{nilp}}$ as an inverse limit of affine schemes of finite type S

$$\mathrm{Op}^{\mathrm{nilp}} \xrightarrow{\psi} S \xrightarrow{\phi} \tilde{\tilde{\mathcal{N}}}/\check{G}$$

with ψ flat and ϕ smooth. Hence, any $\mathcal{F}$ as above is in the image of the functor

$$\psi^* : S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),$$

for some such scheme S.

The ind-completion of $S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, denoted $S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\underset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$, acquires a t-structure, and by Proposition 20.2.1 the functor

$$S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\underset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \xrightarrow{\psi^*} \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\underset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is exact. Hence, it is enough to show that the composed functor

$$S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\underset{\to}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$$

sends $\mathbf{D}^+$ to $\mathbf{D}^+$.

The functor in question is isomorphic to the composition

$$
\begin{aligned}
(14.1)\quad & S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\operatorname{-mod}) \overset{\mathrm{Id}_S \times \Gamma_{\mathrm{Fl}}}{\to} S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}) \simeq \\
& \simeq (S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathrm{Op}^{\mathrm{nilp}}) \underset{\mathrm{Op}^{\mathrm{nilp}}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}) \overset{(\psi\times \mathrm{id}_{\mathrm{Op}^{\mathrm{nilp}}})^*}{\to} \mathrm{Op}^{\mathrm{nilp}} \underset{\mathrm{Op}^{\mathrm{nilp}}}{\overset{\rightarrow}{\times}} \mathbf{D}^f_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}) \simeq \\
& \simeq \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}).
\end{aligned}
$$

We claim that both the first and the third arrows in (14.1) send $\mathbf{D}^+$ to $\mathbf{D}^+$.

The first arrow in (14.1) fits into a commutative diagram

$$
\begin{tikzcd}
S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\operatorname{-mod}) \arrow[r, "\mathrm{Id}_S \times \Gamma_{\mathrm{Fl}}"] \arrow[d, "\phi_*"'] & S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}) \arrow[d, "\phi_*"'] \\
\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\operatorname{-mod}) \arrow[r, "\Gamma_{\mathrm{Fl}}"] & \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}}),
\end{tikzcd}
$$

where the vertical arrows are exact and conservative by Proposition 20.4.1. Hence, it is enough to show that the functor Γ_{Fl} sends $\mathbf{D}^{+_{new}}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\operatorname{-mod})$ to $\mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})$. Clearly, Γ_{Fl} sends $\mathbf{D}^{+_{old}}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\operatorname{-mod})$ to $\mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}_{\mathrm{nilp}})$. The required assertion results now from Proposition 2.3.1.

The assertion concerning the third arrow in (14.1) follows from Sect. 21.5.5, since the map

$$
\mathrm{Op}^{\mathrm{nilp}} \overset{\psi\times \mathrm{id}_{\mathrm{Op}^{\mathrm{nilp}}}}{\to} S \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathrm{Op}^{\mathrm{nilp}}
$$

is a regular immersion.

14.2.5. *Condition Funct(d).* We will show that for every object $\mathcal{M} \in \widehat{\mathfrak{g}}_{\mathrm{crit}}\operatorname{-mod}^{I^0}_{\mathrm{nilp}}$ there exists an object $\mathcal{F} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\operatorname{-mod})^{I^0}$, that belongs to the heart of the new t-structure, such that

$$
\mathrm{Hom}(\Gamma_{\mathrm{Fl}}(\mathcal{F}), \mathcal{M}) \neq 0.
$$

Indeed, for $\mathcal{M}$ as above, some Verma module $\mathbb{M}_{w,\mathrm{crit}}$ maps to it non-trivially. However,

$$
\mathbb{M}_{w,\mathrm{crit}} \simeq \Gamma_{\mathrm{Fl}}(j_{w,!}).
$$

(Indeed, $\mathbb{M}_{w,\mathrm{crit}} := \mathrm{Ind}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}(M_w)$, where M_w is the Verma module over $\mathfrak{g}$ isomorphic to $\Gamma(G/B, j_{w,!})$, and for any D-module $\mathcal{F}$ on G/B we have: $\Gamma(\mathrm{Fl}_G, \mathcal{F}) \simeq \mathrm{Ind}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}(\Gamma(G/B, \mathcal{F}))$.)

The object $j_{w,!}$ evidently belongs to the heart of the new t-structure since for $\check{\lambda} \in \check{\Lambda}^+$,

$$
j_{w,!} \star J_{-\check{\lambda}} = j_{w,!} \star j_{-\check{\lambda},!} \simeq j_{w\cdot(-\check{\lambda}),!},
$$

since $\ell(w) + \ell(-\check{\lambda}) = \ell(w\cdot(-\check{\lambda}))$. (For a more general assertion see Lemma 2.2.4.)

This concludes the proof of Main Theorem 2.

14.3. We are now going to prove Main Theorem 3.

14.3.1. First, from Theorem 4.3.2, we obtain that there exists an equivalence:

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \simeq \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0},$$

and hence an equivalence of the corresponding ind-completions

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \simeq \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}. \tag{14.2}$$

Moreover, we claim that the above equivalence has a cohomological amplitude (with respect the t-structures defined on both sides) bounded by $\dim(G/B)$. Indeed, by the definition of the t-structure on a base-changed category, it suffices to establish the corresponding property of the equivalence

$$\mathbf{D}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})) \simeq \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0},$$

where $\mathbf{D}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G}))$ is the ind-completion of $\mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G}))$.

I.e., we have to show that $\mathcal{M} \in \mathbf{D}^{\leq 0}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})) \cap \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G}))$ goes over to an object of $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$, which is $\leq \dim(G/B)_{new}$. And vice versa, that an object $\mathcal{F} \in \mathbf{D}^{\leq 0}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0} \cap \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$ goes over to an object of $\mathbf{D}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G}))$, which is $\leq \dim(G/B)$.

Both assertions follow from Proposition 2.3.1, combined with the fact that the functor of Theorem 4.3.2 is right-exact, when viewed with respect to the *old* t-structure on the category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$, and has a cohomological amplitude bounded by $\dim(G/B)$.

Taking into account Main Theorem 2, we obtain that in order to prove Main Theorem 3, it remains to prove the following:

Proposition 14.3.2. *There exists an exact equivalence between the* $\mathbf{D}^+$ *part of*

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G}))$$

and $\mathbf{D}^+(\mathrm{QCoh}(\mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{nilp}}))$.

14.3.3. *Proof of Proposition 14.3.2.* By Proposition 21.4.3, since the morphism $\mathfrak{r}_{\mathrm{nilp}} : \mathrm{Op}^{\mathrm{nilp}} \to \tilde{\tilde{\mathcal{N}}}/\check{G}$ is flat, we have an exact equivalence

$$\mathbf{D}(\mathrm{QCoh}(\mathrm{MOp}_{\check{\mathfrak{g}}}^{\mathrm{nilp}})) \simeq \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^{perf}(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})).$$

Recall the functor $\Psi : \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \to \mathbf{D}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G}))$, see Sect. 22.1.3. To prove the proposition, it suffices, therefore, to show that the functor

$$\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\times} \Psi : \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \to \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\tilde{\mathcal{N}}}/\check{G}}{\overset{\rightarrow}{\times}} \mathbf{D}^{perf}(\mathrm{Coh}(\check{\mathrm{St}}/\check{G}))$$

induces an exact equivalence of the corresponding $\mathbf{D}^+$ categories.

We have a commutative diagram of functors

$$
\begin{tikzcd}
\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \arrow[r, "\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Psi"] \arrow[d, "(\mathfrak{r}_{\mathrm{nilp}})_*"'] & \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^{perf}(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \arrow[d, "(\mathfrak{r}_{\mathrm{nilp}})_*"'] \\
\mathbf{D}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})) \arrow[r, "\Psi"] & \mathbf{D}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})),
\end{tikzcd}
\tag{14.3}
$$

with the vertical arrows being exact and conservative. Since Ψ is exact (see Sect. 22.2), we obtain that so is $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Psi$. In particular, it sends $\mathbf{D}^+$ to $\mathbf{D}^+$.

Let us now construct the inverse functor. Let us denote by Ξ the tautological functor

$$\mathbf{D}^{perf}(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \to \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})),$$

and its ind-extension

$$\Xi : \mathbf{D}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})) \to \mathbf{D}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})).$$

By Sect. 22.1.3, Ξ is a left adjoint and right inverse of Ψ. Consider now the functor $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Xi$, which fits into the commutative diagram

$$
\begin{tikzcd}
\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \arrow[d, "(\mathfrak{r}_{\mathrm{nilp}})_*"'] & \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^{perf}(\mathrm{Coh}(\check{\mathrm{St}}/\check{G})) \arrow[l, "\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Xi"'] \arrow[d, "(\mathfrak{r}_{\mathrm{nilp}})_*"'] \\
\mathbf{D}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})) & \mathbf{D}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})). \arrow[l, "\Xi"']
\end{tikzcd}
$$

We define a functor $(\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Xi)_{ren}$ from the $\mathbf{D}^+$ part of $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^{perf}(\mathrm{Coh}(\check{\mathrm{St}}/\check{G}))$ to the $\mathbf{D}^+$ part of $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^b(\mathrm{Coh}(\check{\mathrm{St}}/\check{G}))$ as follows. For $\mathcal{M} \in \mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\overset{\to}{\times}} \mathbf{D}^{perf}(\mathrm{Coh}(\check{\mathrm{St}}/\check{G}))$, which is $\geq i$, we set

$$(\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Xi)_{ren}(\mathcal{M}) := \tau^{\geq j}\left((\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Xi)(\mathcal{M})\right),$$

for some/any $j < i$.

The fact that $(\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Xi)_{ren}$ and $\mathrm{Op}^{\mathrm{nilp}} \underset{\tilde{\check{\mathcal{N}}}/\check{G}}{\times} \Psi$ are mutually inverse follows from the corresponding assertion for $\mathbf{D}_{ren}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G})) \leftrightarrows \mathbf{D}(\mathrm{QCoh}(\check{\mathrm{St}}/\check{G}))$, since the functor $(\mathfrak{r}_{\mathrm{nilp}})_*$ is conservative.

Part III: Tensor products of categories

15. DG categories and triangulated categories: a reminder

In this section, whenever we will discuss the category of functors between two categories, the source category will be assumed essentially small.

15.1. DG categories and modules. Recall that a DG category is a k-linear category $\mathbf{C}$, enriched over $\mathbf{Comp}_k$. I.e., for every $X, Y \in \mathbf{C}$, the vector space $\mathrm{Hom}(X,Y)$ is endowed with a structure of complex, denoted $\mathrm{Hom}^\bullet(X,Y)$, in a way compatible with compositions. Unless specified otherwise, our DG categories are pre-triangulated. In this case, the homotopy category $\mathrm{Ho}(\mathbf{C})$ is a triangulated category.

It is clear what a DG functor between DG categories is. A DG functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ induces a triangulated functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}_2)$. We say that F is a quasi-equivalence if $\mathrm{Ho}(F)$ is an equivalence.

It is also clear what a DG natural transformation between DG functors is.

15.1.1. For a DG category $\mathbf{C}$ we let $\mathbf{C}^{op}$–mod (resp., $\mathbf{C}$–mod) denote the DG category of contravariant (resp., covariant) functors $\mathbf{C} \to \mathbf{Comp}_k$.

We will also consider the derived category $\mathbf{D}(\mathbf{C}^{op}\text{–mod})$, which is the triangulated quotient of $\mathrm{Ho}(\mathbf{C}^{op}\text{–mod})$ by the subcategory of modules $M^\bullet$ such that $M^\bullet(X) \in \mathbf{Comp}_k$ is acyclic for every $X \in \mathbf{C}$.

15.1.2. Inside $\mathbf{C}^{op}$–mod one singles out a full DG subcategory $\underset{\to}{\mathbf{C}}$ of *semi-free* modules. Namely, its objects are $\mathbf{C}^{op}$-modules of the following form:

Let $X_k, k \geq 0$ be objects of $\mathbf{C}^{op}$–mod that are free (i.e., representable by a formal direct sum of objects of $\mathbf{C}$), and consider a strictly upper triangular [5] matrix $\Phi : \underset{k\geq 0}{\oplus} X_k \to \underset{k\geq 0}{\oplus} X_k$ of morphisms of degree 1, such that $2d(\Phi) + \Phi \circ \Phi = 0$. The data $\{\underset{k\geq 0}{\oplus} X_k, \Phi\}$ defines a $\mathbf{C}^{op}$-module by

$$X \mapsto \underset{k}{\oplus} \mathrm{Hom}^\bullet(X, X_k),$$

with the differential given by the sum of the original differential and Φ.

15.1.3. According to [Dr], Sect. 4.1, the functor

$$\mathrm{Ho}(\underset{\to}{\mathbf{C}}) \to \mathbf{D}(\mathbf{C}^{op}\text{–mod}) \tag{15.1}$$

is an equivalence.

Note also that the definition of $\underset{\to}{\mathbf{C}}$ makes sense whether or not $\mathbf{C}$ is small, so when it is not, $\mathrm{Ho}(\underset{\to}{\mathbf{C}})$ serves as a replacement for $\mathbf{D}(\mathbf{C}^{op}\text{–mod})$ via (15.1).

[5] i.e., we have morphisms $\phi_{i,j} : X_i \to X_j$ for $i > j$

15.1.4. We have fully faithful (Yoneda) embeddings $\mathbf{C} \to \underset{\to}{\mathbf{C}}$ and $\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\underset{\to}{\mathbf{C}}) \simeq \mathbf{D}(\mathbf{C}^{op}\text{–mod})$. In addition, we have:

- $\mathrm{Ho}(\underset{\to}{\mathbf{C}})$ is co-complete (i.e., admits arbitrary direct sums).
- Every object of $\mathrm{Ho}(\mathbf{C})$, considered as an object of $\mathrm{Ho}(\underset{\to}{\mathbf{C}})$, is compact (we remind that an object X of a triangulated category is compact if the functor of $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(X, ?)$ commutes with direct sums),
- $\mathrm{Ho}(\mathbf{C})$ generates $\mathrm{Ho}(\underset{\to}{\mathbf{C}})$ (i.e., the former is not contained in a proper full co-complete triangulated subcategory of the latter).

Such pairs of categories are a convenient setting to work in.

15.2. Pseudo and homotopy functors. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two DG categories. A (left) DG pseudo functor is by definition an object $M^\bullet_F \in (\mathbf{C}_1^{op} \times \mathbf{C}_2\text{–mod})^{op}$, i.e., a bi-additive functor $\mathbf{C}_1^{op} \times \mathbf{C}_2 \to \mathbf{Comp}_k$. We shall denote this category of pseudo-functors $\mathbf{C}_1 \to \mathbf{C}_2$ by $\mathrm{PFunct}(\mathbf{C}_1, \mathbf{C}_2)$.

A homotopy functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ is by definition a DG pseudo functor that satisfies the following property:

For every $X \in \mathbf{C}_1$, the object of $\mathbf{C}_2\text{–mod}$, given by $Y \mapsto M^\bullet_F(X, Y)$ is such that its image in $\mathbf{D}(\mathbf{C}_2\text{–mod})$ lies in the essential image of $\mathrm{Ho}(\mathbf{C}_2^{op})$.

I.e., we require that for every object $X \in \mathrm{Ho}(\mathbf{C}_1)$, the functor on $\mathrm{Ho}(\mathbf{C}_2)$ given by $Y \mapsto H^0\left(M^\bullet_F(X, Y)\right)$ be co-representable.

For example, a DG functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ gives rise to a homotopy functor by setting $M^\bullet_F(X, Y) = \mathrm{Hom}^\bullet_{\mathbf{C}_2}(F(X), Y)$. For a homotopy functor F we will sometimes use the notation

$$\mathrm{Hom}^\bullet_{\mathbf{C}_2}(\text{"}F(X)\text{"}, Y) := M^\bullet_F(X, Y).$$

By definition, a homotopy functor as above defines a triangulated functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}_2)$.

15.2.1. For a pseudo functor F as above and $Y \in \mathbf{C}_2^{op}\text{–mod}$ we can form

$$Y \overset{L}{\underset{\mathbf{C}_2}{\otimes}} M^\bullet_F \in \mathbf{D}(\mathbf{C}_1^{op}\text{–mod}).$$

This gives rise to a triangulated functor $F_* : \mathbf{D}(\mathbf{C}_2^{op}\text{–mod}) \to \mathbf{D}(\mathbf{C}_1^{op}\text{–mod})$.

In addition, F naturally extends to a pseudo functor $\underset{\to}{\mathbf{C}}_1 \to \underset{\to}{\mathbf{C}}_2$. Indeed, for $X = \{\underset{i}{\oplus} X_i, \Phi\} \in \underset{\to}{\mathbf{C}}_1$, $Y = \{\underset{j}{\oplus} Y_j, \Psi\} \in \underset{\to}{\mathbf{C}}_2$, we set

$$M^\bullet_F(X, Y) = \prod_i \bigoplus_j M^\bullet_F(X_i, Y_j).$$

If F was a homotopy functor, then so is the above extension; thus we obtain a functor

$$\mathbf{D}(\mathbf{C}_1^{op}\text{–mod}) \simeq \mathrm{Ho}(\underset{\to}{\mathbf{C}}_1) \to \mathrm{Ho}(\underset{\to}{\mathbf{C}}_2) \to \mathbf{D}(\mathbf{C}_2^{op}\text{–mod}),$$

which will be denoted F^*; it is the left adjoint of F_*.

15.2.2. Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a homotopy functor, and $\mathbf{C}'_2 \subset \mathbf{C}_2$ a full DG subcategory. We can tautologically define a pseudo functor $F' : \mathbf{C}_1 \to \mathbf{C}'_2$. It is easy to see that F' is a homotopy functor if and only if the essential image of $\mathrm{Ho}(F)$ belongs to $\mathrm{Ho}(\mathbf{C}'_2)$.

15.2.3. Let us note that homotopy functors can be represented by "huts" of DG functors, and vice versa.

Namely, given a diagram

$$\mathbf{C}_1 \overset{\Psi}{\leftarrow} \widetilde{\mathbf{C}}_1 \overset{\widetilde{F}}{\to} \mathbf{C}_2, \tag{15.2}$$

where Ψ and $\widetilde{F}$ are DG functors with Ψ being a quasi-equivalence, we define a homotopy functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ up to a derived natural isomorphism as $(\Psi \otimes \mathrm{Id})^*(M^\bullet_{\widetilde{F}})$.

15.2.4. Vice versa, given a homotopy functor F, we define a diagram such as (15.2) as follows. We set $\widetilde{\mathbf{C}}_1$ to have as objects triples $\{X \in \mathbf{C}_1, Y \in \mathbf{C}_2, f \in M^0_F(X, Y)\}$, where f is closed and such that it induces an isomorphism $Y \to F(X) \in \mathbf{D}(\mathbf{C}_2\text{–mod})$.

Morphisms are defined by

$$\begin{aligned}&\mathrm{Hom}^\bullet_{\widetilde{\mathbf{C}}_1}(\{X', Y', f'\}, \{X'', Y'', f''\}) := \\ &\qquad \{\alpha \in \mathrm{Hom}^\bullet_{\mathbf{C}_1}(X', X''), \beta \in \mathrm{Hom}^\bullet_{\mathbf{C}_2}(Y', Y''), \gamma \in M^\bullet_F(X', Y'')[-1]\}.\end{aligned}$$

The differential arises from the differentials on $\mathrm{Hom}^\bullet$, $M^\bullet$ and f.

The functors $\widetilde{F}$ and Ψ send $\{X, Y, f\}$ as above to X and Y, respectively.

15.2.5. Finally, let us note that if we have a diagram

$$\mathbf{C}_1 \overset{\widetilde{G}}{\to} \widetilde{\mathbf{C}}_2 \overset{\Phi}{\leftarrow} \mathbf{C}_2, \tag{15.3}$$

with Φ a quasi-equivalence, it gives rise to a diagram as in (15.2) by first defining a homotopy functor $G : \mathbf{C}_1 \to \mathbf{C}_2$, namely,

$$M^\bullet_G(X, Y) = \mathrm{Hom}^\bullet_{\widetilde{\mathbf{C}}_2}(\widetilde{G}(X), \Phi(Y)),$$

and the applying the procedure of Sect. 15.2.4.

15.3. Natural transformations. Let F' and F'' be two pseudo functors, corresponding to bimodules $M^\bullet_{F'}$ and $M^\bullet_{F''}$, respectively. A DG natural transformation $g : F' \Rightarrow F''$ is by definition a closed morphism of degree zero $g : M^\bullet_{F''} \to M^\bullet_{F'}$ in $(\mathbf{C}_1^{op} \times \mathbf{C}_2\text{–mod})^{op} \simeq \mathrm{PFunct}(\mathbf{C}_1, \mathbf{C}_2)$.

A derived natural transformation between F' and F'' is a morphism between $M^\bullet_{F'}$ and $M^\bullet_{F''}$ in the triangulated category $\mathrm{PFunct}^{\mathrm{Ho}}(\mathbf{C}_1, \mathbf{C}_2) := \mathbf{D}(\mathbf{C}_1^{op} \times \mathbf{C}_2\text{–mod})^{op}$. We shall denote the full subcategory of $\mathrm{PFunct}^{\mathrm{Ho}}(\mathbf{C}_1, \mathbf{C}_2)$ consisting of homotopy functors by $\mathrm{HFunct}(\mathbf{C}_1, \mathbf{C}_2)$.

15.3.1. It is clear that a derived natural transformation between homotopy functors gives rise to a natural transformation

$$\mathrm{Ho}(F') \Rightarrow \mathrm{Ho}(F'') : \mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}_2),$$

and also

$$F''_* \Rightarrow F'_* \text{ and } F'^* \Rightarrow F''^*.$$

15.3.2. Let now $\mathbf{C}'_2 \subset \mathbf{C}_2$ be a full DG subcategory. Let $\mathrm{HFunct}(\mathbf{C}_1, \mathbf{C}_2)'$ be the full subcategory of $\mathrm{HFunct}(\mathbf{C}_1, \mathbf{C}_2)$, consisting of those homotopy functors F, such that $\mathrm{Ho}(F) : \mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}'_2)$.

Restriction (see Sect. 15.2.2) defines a functor

$$\mathrm{HFunct}(\mathbf{C}_1, \mathbf{C}_2)' \to \mathrm{HFunct}(\mathbf{C}_1, \mathbf{C}'_2).$$

Lemma 15.3.3. *The above functor is an equivalence. Its inverse is given by composing with the tautological object in* $\mathrm{HFunct}(\mathbf{C}'_2, \mathbf{C}_2)$ *(see Sect. 15.4).*

15.4. Compositions. Let $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ be three DG categories, and $F' : \mathbf{C}_1 \to \mathbf{C}_2$, $F'' : \mathbf{C}_2 \to \mathbf{C}_3$, $G : \mathbf{C}_1 \to \mathbf{C}_3$ be pseudo functors. We define the complex $\mathrm{Hom}^\bullet_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_3)}(G, ``F'' \circ F'")$ to consist of maps

$$\mathrm{Hom}^\bullet_{\mathbf{C}_2}(``F'(X)", Y) \otimes \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``F''(Y)", Z) \to \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``G(X)", Z), \tag{15.4}$$

functorial in $X \in \mathbf{C}_1, Z \in \mathbf{C}_3$, and functorial in $Y \in \mathbf{C}_2$ in the sense that for any $Y', Y'' \in \mathbf{C}_2$, the composition

$$\begin{aligned}&\mathrm{Hom}^\bullet_{\mathbf{C}_2}(``F'(X)", Y') \otimes \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``F''(Y'')", Z) \otimes \mathrm{Hom}^\bullet_{\mathbf{C}_2}(Y', Y'') \to \\ &\qquad \to \mathrm{Hom}^\bullet_{\mathbf{C}_2}(``F'(X)", Y'') \otimes \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``F''(Y'')", Z) \to \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``G(X)", Z)\end{aligned}$$

equals

$$\begin{aligned}&\mathrm{Hom}^\bullet_{\mathbf{C}_2}(``F'(X)", Y') \otimes \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``F''(Y'')", Z) \otimes \mathrm{Hom}^\bullet_{\mathbf{C}_2}(Y', Y'') \to \\ &\qquad \to \mathrm{Hom}^\bullet_{\mathbf{C}_2}(``F'(X)", Y') \otimes \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``F''(Y')", Z) \to \mathrm{Hom}^\bullet_{\mathbf{C}_3}(``G(X)", Z).\end{aligned}$$

A DG natural transformation

$$G \Rightarrow ``F'' \circ F'"$$

is by definition a 0-cycle in $\mathrm{Hom}^\bullet_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_3)}(G, ``F'' \circ F'")$. We denote the set of DG natural transformations as above by $\mathrm{Hom}_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_3)}(G, ``F'' \circ F'")$.

If F', F'', G are homotopy functors, then a DG natural transformation as above defines a natural transformation

$$\mathrm{Ho}(G) \Rightarrow \mathrm{Ho}(F'') \circ \mathrm{Ho}(F').$$

We say that G is a homotopy composition of F' and F'' if the latter map is an isomorphism of functors $\mathrm{Ho}(\mathbf{C}_1) \rightrightarrows \mathrm{Ho}(\mathbf{C}_3)$.

15.4.1. In a similar way one defines the complex

$$\mathrm{Hom}^\bullet_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_{n+1})}(G, ``F_n \circ ... \circ F_1"),$$

where F_i, $j = 1, ..., n$ are pseudo functors $\mathbf{C}_i \to \mathbf{C}_{i+1}$ and G is a pseudo functor $\mathbf{C}_1 \to \mathbf{C}_{n+1}$. As above, this allows to introduce the notion of homotopy composition of $F_1, ..., F_n$.

A DG natural transformation

$$G \Rightarrow ``F_n \circ ... \circ F_1"$$

is by definition a 0-cycle in $\mathrm{Hom}^\bullet_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_{n+1})}(G, ``F_n \circ ... \circ F_1")$. We denote the set of DG natural transformations by $\mathrm{Hom}_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_{n+1})}(G, ``F_n \circ ... \circ F_1")$.

15.4.2. Let $\mathbf{C}_1, ..., \mathbf{C}_{n+1}$ and $F_1, ..., F_n$ be above. For a pseudo functor $G : \mathbf{C}_1 \to \mathbf{C}_{n+1}$, let us denote by $\mathrm{Hom}_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_{n+1})}(G, ``F_n \circ ... \circ F_1")$ the set of DG natural transformations as above.

We define a functor

$$\mathrm{Hom}_{\mathrm{PFunct}^{\mathrm{Ho}}(\mathbf{C}_1,\mathbf{C}_{n+1})}(G, ``F_n \circ ... \circ F_1") : \mathrm{PFunct}^{\mathrm{Ho}}(\mathbf{C}_1, \mathbf{C}_{n+1})^{op} \to \mathrm{Sets}$$

by

$$G \mapsto \underset{F_i \to F_i', i=1,...,n}{colim} H^0\left(\mathrm{Hom}^\bullet_{\mathrm{PFunct}(\mathbf{C}_1,\mathbf{C}_{n+1})}(G, ``F_n' \circ ... \circ F_1'")\right),$$

where the colimit is taken over the index category of DG natural transformations $F_i \to F_i'$ that are quasi-isomorphisms.

Lemma 15.4.3. *The functor* $G \mapsto \mathrm{Hom}_{\mathrm{PFunct}^{\mathrm{Ho}}(\mathbf{C}_1,\mathbf{C}_{n+1})}(G, ``F_n \circ ... \circ F_1")$ *is representable.*

Proof. Let $M^\bullet_{F_i} \in \mathbf{C}_i^{op} \times \mathbf{C}_{i+1}$–mod be the bi-module representing F_i. Then it is easy to see that the object of $\mathbf{D}(\mathbf{C}_1^{op} \times \mathbf{C}_{n+1}$–mod), given by

$$M^\bullet_{F_n} \underset{\mathbf{C}_n}{\overset{L}{\otimes}} M^\bullet_{F_{n-1}} \otimes ... \otimes M^\bullet_{F_2} \underset{\mathbf{C}_2}{\overset{L}{\otimes}} M^\bullet_{F_1}$$

satisfies the requirements of the lemma. □

Let us denote by $F_n \circ .... \circ F_1$ the universal object in $\mathrm{PFunct}^{\mathrm{Ho}}(\mathbf{C}_1, \mathbf{C}_{n+1})$; we shall call it the pseudo composition of $F_1, ..., F_n$.

Lemma 15.4.4. *If $F_1, ..., F_n$ are homotopy functors, their pseudo-composition is their homotopy composition.*

15.4.5. We shall view (essentially small) DG categories as a 2-category with objects being DG categories and 1-morphisms being the categories $\mathrm{HFunct}(\mathbf{C}_1, \mathbf{C}_2)$. We shall denote this 2-category by **DGCat**.

15.5. Karoubization. Let $\mathbf{C}$ be a DG category. A homotopy Karoubian envelope of $\mathbf{C}$ is a pair $(\mathbf{C}', F)$, where $\mathbf{C}'$ is another DG category equipped with a homotopy functor $F : \mathbf{C} \to \mathbf{C}'$, such that $\mathrm{Ho}(F)$ is fully faithful and makes $\mathrm{Ho}(\mathbf{C}')$ into the Karoubian envelope of $\mathrm{Ho}(\mathbf{C})$, i.e., $\mathrm{Ho}(\mathbf{C}')$ contains images of all projectors, and every object of $\mathrm{Ho}(\mathbf{C}')$ is isomorphic to a direct summand of an object of the form $\mathrm{Ho}(F)(X)$, $X \in \mathrm{Ho}(\mathbf{C})$.

By [BV], 1.6.2, a homotopy Karoubian envelope of $\mathbf{C}$ is well-defined as an object of **DGCat**. We shall denote it by $\mathbf{C}^{Kar}$. Here is an explicit construction:

Lemma 15.5.1. *([BV], 1.4.2) For a DG category $\mathbf{C}$, any compact object in $\mathrm{Ho}(\underset{\to}{\mathbf{C}})$ is isomorphic to a direct summand of an object of $\mathrm{Ho}(\mathbf{C})$.*

Thus, $\mathbf{C}^{Kar}$ can be defined as the preimage in $\underset{\to}{\mathbf{C}}$ of the subcategory $\mathrm{Ho}(\underset{\to}{\mathbf{C}})^c \subset \mathrm{Ho}(\mathbf{C})$ consisting of compact objects.

15.6. Quotients. Let $\mathbf{C}$ be a DG category, and $\mathbf{C}'$ a full DG subcategory. Following [Dr], Sect. 4.9, one defines an object of **DGCat**, denoted $\mathbf{C}/\mathbf{C}'$, equipped with a 1-morphism $p_{can} : \mathbf{C} \to \mathbf{C}/\mathbf{C}'$, such that the induced functor $\mathrm{Ho}(p_{can}) : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C}/\mathbf{C}')$ identifies $\mathrm{Ho}(\mathbf{C}/\mathbf{C}')$ with the quotient of $\mathrm{Ho}(\mathbf{C})$ by the triangulated subcategory $\mathrm{Ho}(\mathbf{C}')$. Moreover, the pair $(\mathbf{C}/\mathbf{C}', p_{can})$ satisfies a natural universal property of [Dr], Theorem 1.6.2, see also Sect. 15.7.3 below.

15.6.1. Here is a concrete construction of $\mathbf{C}/\mathbf{C}'$. Consider the triangulated category $\mathrm{Ho}(\underset{\to}{\mathbf{C}'})$, which is a full subcategory of $\mathrm{Ho}(\underset{\to}{\mathbf{C}})$. Let $\mathrm{Ho}(\underset{\to}{\mathbf{C}'})^\perp \subset \mathrm{Ho}(\underset{\to}{\mathbf{C}})$ be its right orthogonal. By [Dr], Proposition 4.7, the subcategory $\mathrm{Ho}(\underset{\to}{\mathbf{C}'}) \subset \mathrm{Ho}(\underset{\to}{\mathbf{C}})$ is right-admissible, i.e., the tautological functor

$$\mathrm{Ho}(\underset{\to}{\mathbf{C}'})^\perp \to \mathrm{Ho}(\underset{\to}{\mathbf{C}})/\,\mathrm{Ho}(\underset{\to}{\mathbf{C}'})$$

is an equivalence.

Let $\underset{\to}{\mathbf{C}'}^\perp \subset \underset{\to}{\mathbf{C}}$ be the DG subcategory equal to the preimage of $\mathrm{Ho}(\underset{\to}{\mathbf{C}'})^\perp$. We let $\mathbf{C}/\mathbf{C}'$ be the full subcategory of $\underset{\to}{\mathbf{C}'}^\perp$ consisting of objects Y, such that their image in the homotopy category

$$\mathrm{Ho}(\underset{\to}{\mathbf{C}'}^\perp) \simeq \mathrm{Ho}(\underset{\to}{\mathbf{C}'})^\perp \simeq \mathrm{Ho}(\underset{\to}{\mathbf{C}})/\,\mathrm{Ho}(\underset{\to}{\mathbf{C}'})$$

has the property that it is isomorphic to the image of an object $Y' \in \mathrm{Ho}(\mathbf{C})$ under

$$\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\underset{\to}{\mathbf{C}}) \to \mathrm{Ho}(\underset{\to}{\mathbf{C}}) / \mathrm{Ho}(\underset{\to}{\mathbf{C}'}).$$

The homotopy functor p_{can} is defined tautologically: for $X \in \mathbf{C}$ and $Y \in \mathbf{C}/\mathbf{C}' \subset \underset{\to}{\mathbf{C}'}^{\perp} \subset \underset{\to}{\mathbf{C}}$ we let

$$M^{\bullet}_{p_{can}}(X, Y) = \mathrm{Hom}^{\bullet}_{\underset{\to}{\mathbf{C}}}(X, Y).$$

15.6.2. A part of the universal property of $\mathbf{C}/\mathbf{C}'$ is the following assertion (see [Dr], Proposition 4.7):

Proposition 15.6.3. *The functor*

$$(p_{can})^* : \mathbf{D}(\underset{\to}{\mathbf{C}}^{op}\text{–mod}) \simeq \mathrm{Ho}(\underset{\to}{\mathbf{C}}) \to \mathrm{Ho}(\underset{\to}{\mathbf{C}/\mathbf{C}'}) \simeq \mathbf{D}((\mathbf{C}/\mathbf{C}')^{op}\text{–mod})$$

induces an equivalence

$$\mathrm{Ho}(\underset{\to}{\mathbf{C}}) / \mathrm{Ho}(\underset{\to}{\mathbf{C}'}) \simeq \mathrm{Ho}(\underset{\to}{\mathbf{C}/\mathbf{C}'})$$

and the functor $(p_{can})_* : \mathrm{Ho}(\underset{\to}{\mathbf{C}/\mathbf{C}'}) \simeq \mathbf{D}((\mathbf{C}/\mathbf{C}')^{op}\text{–mod}) \to \mathbf{D}(\underset{\to}{\mathbf{C}}^{op}\text{–mod}) \simeq \mathrm{Ho}(\underset{\to}{\mathbf{C}})$ *induces an equivalence*

$$\mathrm{Ho}(\underset{\to}{\mathbf{C}/\mathbf{C}'}) \simeq \mathrm{Ho}(\underset{\to}{\mathbf{C}'})^{\perp}.$$

15.6.4. The following construction will be useful in the sequel. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be DG categories, and $F : \mathbf{C}_1 \to \mathbf{C}_2$ a DG pseudo functor. Let $\mathbf{C}'_1 \subset \mathbf{C}_1$ and $\mathbf{C}'_2 \subset \mathbf{C}_2$ be DG subcategories. Assume that the following holds:

For any $X \in \mathbf{C}_1$ the functor on $\mathrm{Ho}(\mathbf{C}_2)/\mathrm{Ho}(\mathbf{C}'_2)$ given by

$$Y \mapsto \underset{f:Y \to Y'}{colim} \, H^0 \left(M_F(X, Y') \right),$$

is co-representable, where the colimit is taken over the set of morphisms f with $\mathrm{Cone}(f) \in \mathbf{C}'_2$. Assume also that the above functor is zero for $X \in \mathbf{C}'_1$.

We claim that the above data gives rise to a 1-morphism $F' : \mathbf{C}_1/\mathbf{C}'_1 \to \mathbf{C}_2/\mathbf{C}'_2$. Namely, let us realize the above categories as in Sect. 15.6.1 as full subcategories of $\underset{\to}{\mathbf{C}'_i}^{\perp}$, $i = 1, 2$, respectively. We define the sought-for quasi-functor by setting

$$M_{F'}(X, Y) := M_F(X, Y)$$

for $X \in \mathbf{C}_1/\mathbf{C}'_1 \subset \underset{\to}{\mathbf{C}'_1}^{\perp}, Y \in \mathbf{C}_2/\mathbf{C}'_2 \subset \underset{\to}{\mathbf{C}'_2}^{\perp}$.

The required co-representability on the homotopy level follows from the assumptions.

15.7. DG models of triangulated categories. Let **TrCat** be the 2-category of triangulated categories. We have an evident 2-functor $\mathrm{Ho} : \mathbf{DGCat} \to \mathbf{TrCat}$ that sends a DG category $\mathbf{C}$ to $\mathrm{Ho}(\mathbf{C})$.

Given a triangulated category $\mathbf{D}$, its DG model is an object of the 2-category **DGCat** equal to the fiber of Ho over $\mathbf{D}$. Similarly, given an arrow in **TrCat** (i.e., a triangulated functor $F_{tr} : \mathbf{D}_1 \to \mathbf{D}_2$) by a model of F_{tr} we shall mean a lift of this functor to **DGCat** (i.e., if $\mathbf{D}_i = \mathrm{Ho}(\mathbf{C}_i)$, then a model for F_{tr} is a homotopy functor $F_h : \mathbf{C}_1 \to \mathbf{C}_2$).

15.7.1. Let $\mathbf{D}$ be a triangulated category, equipped with a DG model. In this case, we can form a new triangulated category $\underset{\to}{\mathbf{D}}$ and a fully faithful triangulated functor $\mathbf{D} \to \underset{\to}{\mathbf{D}}$, both equipped with models, such that the pair $(\mathbf{D}, \underset{\to}{\mathbf{D}})$ satisfies the three properties of Sect. 15.1.4. Namely, if $\mathbf{D} = \mathrm{Ho}(\mathbf{C})$, we set $\underset{\to}{\mathbf{D}} := \mathrm{Ho}(\underset{\to}{\mathbf{C}})$.

We will informally call $\underset{\to}{\mathbf{D}}$ the "ind-completion" of $\mathbf{D}$.

If $F : \mathbf{D}_1 \to \mathbf{D}_2$ are triangulated categories and a functor between them, all equipped with DG models, we have the corresponding functors

$$F^*, F_* : \underset{\to}{\mathbf{D}}_1 \leftrightarrows \underset{\to}{\mathbf{D}}_2,$$

also equipped with models.

Similarly, the Karoubian envelope $\mathbf{D}^{Kar}$ of $\mathbf{D}$ and the functor $\mathbf{D} \to \mathbf{D}^{Kar}$ are both equipped with models.

15.7.2. Let $\mathbf{D}$ be a triangulated category equipped with a model, and let $\mathbf{D}' \subset \mathbf{D}$ be a full triangulated subcategory. Note that $\mathbf{D}'$ is also naturally equipped with a model.

Indeed, if $\mathbf{D} = \mathrm{Ho}(\mathbf{C})$, we define $\mathbf{C}' \subset \mathbf{C}$ to be the full subcategory consisting of objects, whose image in $\mathbf{D}$ is isomorphic to an object from $\mathbf{D}'$.

In this case, by Sect. 15.6, the triangulated category $\mathbf{D}/\mathbf{D}'$ and the projection functor $\mathbf{D} \to \mathbf{D}/\mathbf{D}'$ also come equipped with models.

15.7.3. Let $\mathbf{D}$ and $\mathbf{D}'$ be as above, and let $\mathbf{D}_1$ be yet another triangulated category equipped with a model. The following is a version of [Dr], Theorem 1.6.2:

Lemma 15.7.4. *For $\mathbf{D}' \subset \mathbf{D}$ and $\mathbf{D}_1$ in $\mathbf{DGCat}$ the following two categories are equivalent:*

(a) 1-morphisms $F : \mathbf{D} \to \mathbf{D}_1$ in $\mathbf{DGCat}$, such that $F|_{\mathbf{D}'} = 0$.

(b) 1-morphisms $\mathbf{D}/\mathbf{D}' \to \mathbf{D}_1$ in $\mathbf{DGCat}$.

15.8. Homotopy colimits. Let $\mathbf{D}$ be a co-complete triangulated category equipped with a model. In this subsection we will review the notion of homotopy colimit, following a recipe, explained to us by J. Lurie.

15.8.1. Let us first consider the simplest case of a sequence of objects $X_i \overset{f_{i,i+1}}{\to} X_{i+1}, i = 1, 2, ...$ In this case we define *hocolim*$(\{X_i\})$ as the cone of the map $\underset{i \geq 1}{\oplus} X_i \to \underset{i \geq 1}{\oplus} X_i$, where the map is

$$X_i \overset{\mathrm{id}_{X_i} \oplus -f_{i,i+1}}{\longrightarrow} X_i \oplus X_{i+1}.$$

In this definition, *hocolim*$(\{X_i\})$ is defined up to a non-canonical isomorphism, and one does not even need a DG model.

If the DG model $\mathbf{C}$ of $\mathbf{D}$ was itself co-complete (which we can assume, up to replacing a given model by a quasi-equivalent one), and if we lift the morphisms $f_{i,i+1}$ to closed morphisms of degree 0 in the DG model, the above construction becomes canonical.

15.8.2. The above homotopy colimit construction implies also the following. Let $\{\underset{k\geq 0}{\oplus} X_k, \Phi\}$ be an object of $\underset{\rightarrow}{\mathbf{C}}$. Consider the $\mathbf{C}$-module given by

$$X \mapsto \prod_k \mathrm{Hom}^\bullet(X_k, X),$$

with the differential given by Φ.

Lemma 15.8.3. *Assume that* $\mathrm{Ho}(\mathbf{C})$ *is co-complete. Then the image of the above* $\mathbf{C}$*-module in* $\mathbf{D}(\mathbf{C}\text{–mod})$ *is in the Yoneda image of* $\mathrm{Ho}(\mathbf{C}^{op})$*.*

Thus, for $\mathbf{C}$ with $\mathrm{Ho}(\mathbf{C})$ co-complete, we obtain that the identity functor on $\mathrm{Ho}(\mathbf{C})$ naturally extends to a functor

$$\mathbf{D}(\mathbf{C}^{op}\text{–mod}) \simeq \mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}) \to \mathrm{Ho}(\mathbf{C})$$

that commutes with direct sums, and which is the left adjoint to the tautological embedding.

15.8.4. In the general case we proceed as follows. Let I be a small category. Let I^{DG} be the free (non-pretriangulated) DG category, spanned by I. Consider the DG category

$$\mathrm{POb}(I, \mathbf{C}) := \mathrm{PFunct}(I^{op\,DG}, \mathbf{C}^{op})^{op} := \mathbf{D}((I \times \mathbf{C}^{op})\text{–mod}),$$

and the corresponding triangulated categories

$$\mathrm{HOb}(I, \mathbf{C}) := \mathrm{HFunct}(I^{op\,DG}, \mathbf{C}^{op})^{op} \subset \mathrm{PFunct}^{\mathrm{Ho}}(I^{op\,DG}, \mathbf{C}^{op})^{op}.$$

By definition, a homotopy I-object of a DG category $\mathbf{C}$ is an object $X_I \in \mathrm{HOb}(I, \mathbf{C})$. For X_I as above and $i \in I$ we will denote by X_i be the corresponding object of $\mathrm{Ho}(\mathbf{C})$.

Being a full triangulated subcategory of $\mathbf{D}((I \times \mathbf{C}^{op})\text{–mod})$, the category $\mathrm{HOb}(I, \mathbf{C})$ acquires a natural DG model, by Sect. 15.7.2.

15.8.5. For a functor $F : I_1 \to I_2$ we have a pair of adjoint functors

$$(F \times \mathrm{Id})^* : \mathrm{POb}(I_1, \mathbf{C}) \leftrightarrows \mathrm{POb}(I_2, \mathbf{C}) : (F \times \mathrm{Id})_*,$$

and it is easy to see that $(F \times \mathrm{Id})_*$ sends $\mathrm{HOb}(I_2, \mathbf{C})$ to $\mathrm{HOb}(I_1, \mathbf{C})$.

Applying this to $I_1 = I$ and $I_2 = \mathrm{pt}$ we recover the tautological functor $\mathbf{C} \to \mathrm{HOb}(I, \mathbf{C})$, and its left adjoint with values in $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}) \simeq \mathrm{POb}(\mathrm{pt}, \mathbf{C})$. We denote the latter functor

$$\mathrm{HOb}(I, \mathbf{C}) \to \mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}})$$

by "$\underset{I}{hocolim}$".

Assume now that $\mathrm{Ho}(\mathbf{C})$ is co-complete. We define the functor

$$\underset{I}{hocolim} : \mathrm{HOb}(I, \mathbf{C}) \to \mathrm{Ho}(\mathbf{C})$$

as the composition of "$\underset{I}{hocolim}$" and the functor $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}) \to \mathrm{Ho}(\mathbf{C})$ of Sect. 15.8.2.

By construction, *hocolim* is the left adjoint to the above functor $\mathbf{C} \to \mathrm{HOb}(I, \mathbf{C})$.

15.8.6. Here are some basic properties of the homotopy colimit construction.

Let $\Psi : \mathbf{C}_1 \to \mathbf{C}_2$ be a 1-morphism in $\mathbf{DGCat}$, such that $\mathrm{Ho}(\mathbf{C}_1)$ and $\mathrm{Ho}(\mathbf{C}_2)$ are both co-complete.

Lemma 15.8.7. *Assume that* $\mathrm{Ho}(\Psi) : \mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}_2)$ *commutes with direct sums. Then for any* I *the diagram of functors*

$$\begin{array}{ccc} \mathrm{HOb}(I, \mathbf{C}_1) & \xrightarrow{\Psi\circ ?} & \mathrm{HOb}(I, \mathbf{C}_2) \\ {\scriptstyle hocolim}\Big\downarrow & & {\scriptstyle hocolim}\Big\downarrow \\ \mathrm{Ho}(\mathbf{C}_1) & \xrightarrow{\Psi} & \mathrm{Ho}(\mathbf{C}_2) \end{array}$$

commutes.

In what follows, when talking about homotopy colimits, we will always assume that I is filtered.

Lemma 15.8.8. *Let* Y *be an object of* $\mathbf{C}$*, such that the corresponding object of* $\mathbf{D}$ *is compact. Then*

$$\mathrm{Hom}_{\mathbf{D}}(Y, hocolim(X_I)) \simeq \underset{i\in I}{colim}\ \mathrm{Hom}_{\mathbf{D}}(Y, X_i).$$

15.8.9. The following assertion will be useful in the sequel.

Lemma 15.8.10. *Let* $\mathbf{C}$ *be a DG category. Assume that* $\mathbf{D} := \mathrm{Ho}(\mathbf{C})$ *is co-complete, and let* $\mathbf{D}' \subset \mathbf{D}$ *be a triangulated subcategory that generates it. Then every object* $X \in \mathbf{D}$ *can be represented as a homotopy colimit of* $X_I \in \mathrm{HOb}(I, \mathbf{C})$ *for some* I*, where* $X_i \in \mathbf{D}'$ *for every* $i \in I$*.*

16. Homotopy monoidal categories and actions

16.1. Let $\mathbf{A}$ be a DG category. A DG pseudo monoidal structure on $\mathbf{A}$ is a collection of DG functors $(\mathbf{A}^{\times I})^{op} \times \mathbf{A} \to \mathbf{Comp}_k$

$$X_I, Y \mapsto \mathrm{Hom}^\bullet(\text{"}X_I^{\otimes}\text{"}, Y),$$

for an ordered finite set I. Here X_I stands for an I-object of $\mathbf{A}$, and the symbol "$X_I^{\otimes}$" stands for the a priori non-existing tensor product $\underset{i\in I}{\otimes} X_i$. The above functors must be endowed with the appropriate natural transformations, see [CHA], Sect. 1.1.1. For $I = \{1\}$ we must be given an identification $\mathrm{Hom}^\bullet(X_I^{\otimes}, Y) \simeq \mathrm{Hom}^\bullet(X, Y)$. We require $\mathbf{A}$ to be homotopy unital. I.e., there should exist an object $\mathbf{1_A} \in \mathbf{A}$ and functorial quasi-morphisms

$$\mathrm{Hom}^\bullet(\text{"}X_I^{\otimes}\text{"}, Y) \to \mathrm{Hom}^\bullet(\text{"}X_{I\cup\mathrm{pt}}^{\otimes}\text{"}, Y).$$

where $X_{I\cup\mathrm{pt}}$ corresponds to the insertion of the unit in $\mathbf{A}$ in any place in I with respect to its order.

We say that a DG pseudo monoidal structure is a homotopy monoidal structure if the induced pseudo monoidal structure on $\mathrm{Ho}(\mathbf{A})$ given by

$$H^0\left(\mathrm{Hom}^\bullet(\text{"}X_I^{\otimes}\text{"}, Y)\right)$$

is a monoidal.

Evidently, a usual DG monoidal structure on $\mathbf{A}$ gives rise to a homotopy one. A homotopy monoidal structure on $\mathbf{A}$ defines a structure of monoidal triangulated category on $\mathrm{Ho}(\mathbf{A})$.

16.2. Functors. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two DG pseudo monoidal categories. A DG pseudo monoidal functor F between them is the following data:

For a finite ordered set I we must be given a complex

$$\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F(X_I^{\otimes})\text{"}, Y) \tag{16.1}$$

that depends functorially on both arguments.

The above functors must be equipped with the following system of natural transformations. Let $I \twoheadrightarrow J \twoheadrightarrow K$ be surjections of finite ordered sets. For $k \in K$ (resp., $j \in J$) let $J^k \subset J$ (resp., $I^j \subset I$) denote its pre-image. Fix objects $X_I \in \mathbf{A}_1^I, X_J \in \mathbf{A}_1^J, Y_K \in \mathbf{A}_2^K, Y \in \mathbf{A}_2$.

We must be given a map

$$\begin{split}\left(\underset{j\in J}{\otimes}\mathrm{Hom}^\bullet_{\mathbf{A}_1}(\text{"}X_{I^j}^{\otimes}\text{"}, X_j)\right) \bigotimes \left(\underset{k\in K}{\otimes}\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F(X_{J^k}^{\otimes})\text{"}, Y_k)\right) \bigotimes \\ \bigotimes \mathrm{Hom}^\bullet_{\mathbf{A}_2}(\text{"}Y_K^{\otimes}\text{"}, Y) \to \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F(X_I^{\otimes})\text{"}, Y).\end{split} \tag{16.2}$$

In addition, we must be given natural quasi-isomorphisms that correspond to insertions of the unit object. These natural transformations must satisfy the natural axioms that we will not spell out here.

16.2.1. Assume now that on both $\mathbf{A}_1$ and $\mathbf{A}_2$ the DG pseudo monoidal structure is homotopy monoidal. We say that F is a homotopy monoidal functor if the functors

$$\{X_I \in \mathbf{A}_1^I, Y \in \mathbf{A}_2\} \mapsto H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F(X_I^{\otimes})\text{"}, Y)\right) : \mathrm{Ho}(\mathbf{A}_1^{op})^I \times \mathrm{Ho}(\mathbf{A}_2) \to \mathrm{Vect}_k$$

and the maps

$$\begin{split}H^0\left(\underset{j\in J}{\otimes}\mathrm{Hom}^\bullet_{\mathbf{A}_1}(\text{"}X_{I^j}^{\otimes}\text{"}, X_j)\right) \bigotimes H^0\left(\underset{k\in K}{\otimes}\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F(X_{J^k}^{\otimes})\text{"}, Y_k)\right) \bigotimes \\ \bigotimes H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A}_2}(\text{"}Y_K^{\otimes}\text{"}, Y)\right) \to H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F(X_I^{\otimes})\text{"}, Y)\right).\end{split}$$

come from a (automatically uniquely determined) monoidal structure on the functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathbf{A}_1) \to \mathrm{Ho}(\mathbf{A}_2)$.

In the homotopy monoidal case we say that F is a monoidal quasi-equivalence if the functor $\mathrm{Ho}(F)$ is an equivalence of categories.

16.2.2. Let F' and F' be two DG pseudo monoidal functors $\mathbf{A}_1 \to \mathbf{A}_2$. A DG natural transformation $F' \Rightarrow F''$ is a data of a functorial map of complexes

$$\phi : \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F''(X_I^{\otimes})\text{"}, Y) \to \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F'(X_I^{\otimes})\text{"}, Y), \tag{16.3}$$

defined for all finite ordered sets I, (preserving the degree and commuting with the differential), which makes the diagrams coming from (16.2) commute.

We say that ϕ is a quasi-isomorphism if maps (16.3) are quasi-isomorphisms for all I. In the case when $\mathbf{A}_1, \mathbf{A}_2$ and both functors are homotopy monoidal, the quasi-isomorphism condition is enough to check for $I = \{1\}$.

16.2.3. For two DG pseudo monoidal categories $\mathbf{A}_1, \mathbf{A}_2$ we denote the category whose objects are DG pseudo monoidal functors $\mathbf{A}_1 \to \mathbf{A}_2$ and arrows DG natural transformations by $\mathrm{PMon}(\mathbf{A}_1, \mathbf{A}_2)$.

We claim that $\mathrm{PMon}(\mathbf{A}_1, \mathbf{A}_2)$ has a structure of closed model category, with weak equivalences being quasi-isomorphisms, and cofibrations being those natural transformations, for which the maps ϕ that are surjective for all I (in particular, all objects are cofibrant). Let us denote the corresponding homotopy category by $\mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1, \mathbf{A}_2)$. We shall refer to maps between objects of $\mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1, \mathbf{A}_2)$ as homotopy natural transformations.

Remark. The category $\mathrm{PMon}(\mathbf{A}_1, \mathbf{A}_2)^{op}$ is akin to that of DG associative algebras. Indeed, an object $F \in \mathrm{PMon}(\mathbf{A}_1, \mathbf{A}_2)^{op}$ is given by specifying a collection of vector spaces (16.1) and multiplication maps (16.2).

16.2.4. Let us construct a supply of fibrant objects in $\mathrm{PMon}(\mathbf{A}_1, \mathbf{A}_2)^{op}$ (by the above analogy, these play the role of DG associative algebras that are free as plain associative algebras).

By a *graded* (vs. DG) pseudo monoidal functor $F : \mathbf{A}_1 \to \mathbf{A}_2$ we will understand the same data as in (16.1) and (16.2), with the difference that the $\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}F(X_I^{\otimes})\text{"}, Y)$'s are just graded vector spaces, with no differential. Morphisms in the category are defined as in (16.3). Let us denote the corresponding category by $\mathrm{PMon}_{\mathrm{gr}}(\mathbf{A}_1, \mathbf{A}_2)$.

For $n > 0$ consider also the category $\mathrm{PFunct}_{\mathrm{gr}}(\mathbf{A}_1^{\times n}, \mathbf{A}_2)$ being the opposite of that of multi-additive functors $\mathbf{A}_1^{op} \times ... \times \mathbf{A}_1^{op} \times \mathbf{A}_2 \to \mathrm{Vect}_k^{\mathbb{Z}}$ I.e., for $M^\bullet \in \mathrm{PFunct}_{\mathrm{gr}}(\mathbf{A}_1^{\times n}, \mathbf{A}_2)$, $X_1, ..., X_n \in \mathbf{A}_1, Y \in \mathbf{A}_2$, each $M^\bullet(X_1, ...X_n, Y)$ is just a graded vector space, without a differential.

The evident forgetful functor $\mathrm{PMon}_{\mathrm{gr}}(\mathbf{A}_1, \mathbf{A}_2) \to \prod\limits_{n>0} \mathrm{PFunct}_{\mathrm{gr}}(\mathbf{A}_1^{\times n}, \mathbf{A}_2)$ has a right adjoint; we denote it

$$M^\bullet \mapsto \mathrm{Free}(M^\bullet).$$

Suppose an object $F \in \mathrm{PMon}(\mathbf{A}_1, \mathbf{A}_2)$ has the following properties:

- The image of F in $\mathrm{PMon}_{\mathrm{gr}}(\mathbf{A}_1, \mathbf{A}_2)$ is isomorphic to $\mathrm{Free}(M^\bullet)$ for some
$$M^\bullet \in \prod_{n>0} \mathrm{PFunct}_{\mathrm{gr}}(\mathbf{A}_1^{\times n}, \mathbf{A}_2).$$
- The above object $M^\bullet$ can be represented as a direct sum $M^\bullet = \bigoplus\limits_{i \geq 0} M_i^\bullet$, where each $M_i^\bullet$ is in turn a direct sum of representable functors
$$\bigoplus_a \mathrm{Hom}_{\mathbf{A}_1^{\times n_a} \times \mathbf{A}_2^{op}}(?, \left(X_1^a, ..., X_{n_a}^a, Y^a\right))$$
- The discrepancy between the natural differential on every $M_i^\bullet$ and one coming by restriction from $M_i^\bullet \subset M^\bullet \subset F$, is as a map
$$M_i^\bullet \to \mathrm{Free}\left(\bigoplus_{i>j\geq 0} M_j^\bullet\right) \in \prod_{n>0} \mathrm{PFunct}_{\mathrm{gr}}(\mathbf{A}_1^{\times n}, \mathbf{A}_2)^{op}.$$

Then such F is fibrant.

16.2.5. When both $\mathbf{A}_1$ and $\mathbf{A}_2$ are homotopy monoidal we can consider the full subcategory category of $\mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1, \mathbf{A}_2)$ whose objects are homotopy monoidal functors. We shall denote this category by $\mathrm{HMon}(\mathbf{A}_1, \mathbf{A}_2)$.

16.3. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be DG pseudo-monoidal categories. There is a natural notion of lax DG monoidal functor between them. By definition, this is a DG functor $F_{DG} : \mathbf{A}_1 \to \mathbf{A}_2$ endowed with a system of morphisms

$$\mathrm{Hom}^{\bullet}_{\mathbf{A}_1}("X_I^{\otimes}", Y) \to \mathrm{Hom}^{\bullet}_{\mathbf{A}_2}("F_{DG}(X_I^{\otimes})", F_{DG}(Y)), \tag{16.4}$$

compatible with the associativity constraints.

In other words, this is a pseudo-monoidal functor F, for which there exists a DG functor $F_{DG} : \mathbf{A}_1 \to \mathbf{A}_2$ and isomorphisms

$$\mathrm{Hom}^{\bullet}_{\mathbf{A}_2}("F_{DG}(X_I^{\otimes})", Y) \simeq \mathrm{Hom}^{\bullet}_{\mathbf{A}_1,\mathbf{A}_2}("F(X_I^{\otimes})", Y).$$

We say that a lax DG monoidal functor is a lax DG monoidal quasi-equivalence, if F_{DG} is a quasi-equivalence at the level of plain categories and the maps (16.4) are quasi-isomorphisms.

16.3.1. Let $G : \mathbf{A}_2 \to \mathbf{A}_3$ be a pseudo monoidal functor. For $F_{DG} : \mathbf{C}_1 \to \mathbf{C}_2$, which is a lax DG monoidal functor, one can define the composition $G \circ F_{DG}$ as the pseudo monoidal functor $\mathbf{C}_1 \to \mathbf{C}_3$ given by

$$\mathrm{Hom}^{\bullet}_{\mathbf{A}_1,\mathbf{A}_3}("G \circ F_{DG}(X_I^{\otimes})", Z) := \mathrm{Hom}^{\bullet}_{\mathbf{A}_2,\mathbf{A}_3}("G(F_{DG}(X_I^{\otimes}))", Z).$$

This operation defines the functor $? \circ F$

$$\mathrm{PMon}(\mathbf{A}_2, \mathbf{A}_3) \to \mathrm{PMon}(\mathbf{A}_1, \mathbf{A}_3),$$

and if the categories $\mathbf{A}_i$ and the functor F_{DG} are homotopy monoidal, we also obtain a functor

$$\mathrm{HMon}(\mathbf{A}_2, \mathbf{A}_3) \to \mathrm{HMon}(\mathbf{A}_1, \mathbf{A}_3).$$

Lemma 16.3.2. *Assume that F_{DG} is a lax DG monoidal quasi-equivalence. Then the induced functor*

$$G \mapsto G \circ F_{DG} : \mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_2, \mathbf{A}_3) \to \mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1, \mathbf{A}_3)$$

is an equivalence.

16.3.3. We claim that as in Sect. 15.2.3, for any homotopy monoidal functor $F : \mathbf{A}_1 \to \mathbf{A}_2$ one can find a "hut"

$$\mathbf{A}_1 \overset{\Psi}{\leftarrow} \widetilde{\mathbf{A}}_1 \overset{\widetilde{F}}{\to} \mathbf{A}_2,$$

where Ψ is lax DG monoidal and is a quasi-equivalence, $\widetilde{F}$ is also lax DG monoidal, and a DG natural transformation $F \circ \Psi \to \widetilde{F}$, which is a quasi-isomorphism.

Namely, we take $\widetilde{\mathbf{A}}_1$ to be the DG category from Sect. 15.2.4, i.e., its objects are triples

$$\widetilde{X} = \{X \in \mathbf{A}_1, Y \in \mathbf{A}_2, f \in \mathrm{Hom}^0_{\mathbf{A}_1,\mathbf{A}_2}("F(X)", Y)\},$$

where f is a closed morphism that induces an isomorphism in $\mathrm{Ho}(\mathbf{A}_2)$. The pseudo monoidal structure is given by

$$\mathrm{Hom}^{\bullet}_{\widetilde{\mathbf{A}}_1}("\widetilde{X}_I^{\otimes}", \widetilde{X}) :=$$
$$\{\alpha \in \mathrm{Hom}^{\bullet}_{\mathbf{A}_1}("X_I^{\otimes}", X),\ \beta \in \mathrm{Hom}^{\bullet}_{\mathbf{A}_2}("Y_I^{\otimes}", Y),\ \gamma \in \mathrm{Hom}^{\bullet}_{\mathbf{A}_1,\mathbf{A}_2}("F(X_I^{\otimes})", Y)[-1].\}$$

The DG functors Ψ and $\widetilde{F}$ are defined in an evident way. The DG natural transformation $F \circ \Psi \to \widetilde{F}$ is

$$\mathrm{Hom}^{\bullet}_{\widetilde{\mathbf{A}}_1,\mathbf{A}_2}("\widetilde{F}(\widetilde{X}_I^{\otimes})", Y) =: \mathrm{Hom}^{\bullet}_{\mathbf{A}_2}("Y_I^{\otimes}", Y) \to$$
$$\to \mathrm{Hom}^{\bullet}_{\mathbf{A}_1,\mathbf{A}_2}("F(X_I^{\otimes})", Y) =: \mathrm{Hom}^{\bullet}_{\widetilde{\mathbf{A}}_1,\mathbf{A}_2}("F \circ \Psi(\widetilde{X}_I^{\otimes})", Y),$$

where the second arrow is given via (16.2) by the data of $\underset{i\in I}{\otimes} f_i$.

16.4. Compositions. Let us be given three pseudo monoidal categories $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ and pseudo monoidal functors $F' : \mathbf{A}_1 \to \mathbf{A}_2$, $F'' : \mathbf{A}_2 \to \mathbf{A}_3$ and $G : \mathbf{A}_1 \to \mathbf{A}_3$.

A DG natural transformation $G \Rightarrow {}''F'' \circ F'{}''$ is a collection of morphisms defined for $I \twoheadrightarrow J \twoheadrightarrow K$, $X_I \in \mathbf{C}_1^I$, $Y_J \in \mathbf{C}_2^J$, $Z \in \mathbf{C}_3$

$$\left(\underset{j\in J}{\otimes} \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}({}''F'(X_{I^j}^{\otimes}){}'', Y_j)\right) \bigotimes \left(\underset{j\in J}{\otimes} \mathrm{Hom}^\bullet_{\mathbf{A}_2,\mathbf{A}_3}({}''F''(Y_J^{\otimes}){}'', Z)\right) \to \\ \to \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_3}({}''G(X_I^{\otimes}){}'', Z), \tag{16.5}$$

preserving the degree and commuting with the differential. These morphisms are required to make the corresponding diagrams commute.

In a similar way one defines the notion of DG natural transformation $G \Rightarrow {}''F^n \circ ... \circ F^1{}''$, where F^i are pseudo monoidal functors $\mathbf{A}_i \to \mathbf{A}_{i+1}$, and G is a pseudo monoidal functor $\mathbf{A}_1 \to \mathbf{A}_{n+1}$.

Assume that the categories $\mathbf{A}_1, ..., \mathbf{A}_{n+1}$ and the functors $F_1, ..., F_n, G$ are homotopy monoidal. By construction, a DG natural transformation as above gives rise to a natural transformation between the monoidal functors

$$\mathrm{Ho}(G) \Rightarrow \mathrm{Ho}(F_n) \circ ... \circ \mathrm{Ho}(F_1).$$

We say that G is a homotopy composition of $F_1, ..., F_n$ if the latter natural transformation is an isomorphism.

16.4.1. For $F_1, ..., F_n, G$ as above let us denote by $\mathrm{Hom}_{\mathrm{PMon}(\mathbf{A}_1,\mathbf{A}_{n+1})}(G, {}''F_n \circ ... \circ F_1{}'')$ the set of natural transformations as above. Keeping $F_1, ..., F_n$ fixed, we define the set $\mathrm{Hom}_{\overline{\mathrm{PMon}}(\mathbf{A}_1,\mathbf{A}_{n+1})}(G, {}''F_n \circ ... \circ F_1{}'')$ as a quotient of $\mathrm{Hom}_{\mathrm{PMon}(\mathbf{A}_1,\mathbf{A}_{n+1})}(G, {}''F_n \circ ... \circ F_1{}'')$ by the equivalence relation defined by homotopy.

We define the functor

$$\mathrm{Hom}_{\mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1,\mathbf{A}_{n+1})}(G, {}''F_n \circ ... \circ F_1{}'') : \mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1, \mathbf{A}_{n+1})^{op} \to \mathrm{Sets}$$

by

$$G \mapsto \underset{F_i \to F'_i, i=1,...,n}{colim} \mathrm{Hom}_{\overline{\mathrm{PMon}}(\mathbf{A}_1,\mathbf{A}_{n+1})}(G, {}''F'_n \circ ... \circ F'_1{}''),$$

where the colimit is taken over the index category of homotopy natural transformations $F_i \to F'_i, i = 1, ..., n$ that are quasi-isomorphisms.

Lemma 16.4.2. *The functor* $G \mapsto \mathrm{Hom}_{\mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1,\mathbf{A}_{n+1})}(G, {}''F_n \circ ... \circ F_1{}'')$ *is representable.*

We shall denote the resulting universal object of $\mathrm{PMon}^{\mathrm{Ho}}(\mathbf{A}_1, \mathbf{A}_{n+1})$ by $F_n \circ ... \circ F_1$, and call it the pseudo composition of $F_1, ..., F_n$.

Lemma 16.4.3. *The pseudo-composition of* $F_1, ..., F_n$ *induces their pseudo-composition as functors between plain DG categories (see Sect. 15.4). If* $F_1, ..., F_n$ *are homotopy monoidal functors, then the pseudo composition is their homotopy composition.*

16.4.4. Thus, we can introduce the 2-category, whose 0-objects are homotopy monoidal categories, and 1-morphisms are the categories $\mathrm{HMon}(\mathbf{A}_1, \mathbf{A}_2)$. We shall denote this 2-category by **DGMonCat**.

We can also consider the 2-category **TrMonCat** of triangulated monoidal categories. We have an evident forgetful 2-functor **DGMonCat** $\to$ **TrMonCat**.

For a triangulated monoidal category, by its DG model we will understand the fiber of the above functor.

16.5. Actions. Let $\mathbf{A}$ be a DG pseudo monoidal category and $\mathbf{C}$ another DG category. A (left) pseudo action of $\mathbf{A}$ on $\mathbf{C}$ is the following data:

For a finite ordered set I and $X_I : I \to \mathbf{A}$, $Y', Y'' \in \mathbf{C}$ we must be given a complex

$$X_I, Y \mapsto \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_I^{\otimes} \otimes Y'\text{"}, Y''),$$

which functorially depends on all arguments. The symbol "$X_I^{\otimes} \otimes Y'$" stands for the non-existing object $\underset{i\in I}{\otimes} X_i \otimes Y' \in \mathbf{C}$. These functors must be equipped with the following system of natural transforations:

For $I \twoheadrightarrow \{1, ..., n\}$, $X_I \in \mathbf{A}^I$, $Y_1, ..., Y_n \in \mathbf{C}$ we must be given a map

$$\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_{I^n}^{\otimes} \otimes Y_{n-1}\text{"}, Y_n) \bigotimes \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_{I^{n-1}}^{\otimes} \otimes Y_{n-2}\text{"}, Y_{n-1}) \bigotimes ...$$
$$... \bigotimes \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_{I^1}^{\otimes} \bigotimes Y_1\text{"}, Y_2) \to \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_I^{\otimes} \otimes Y_1\text{"}, Y_n),$$

and for a surjection of finite ordered sets $I \twoheadrightarrow J$, $X_I \in \mathbf{A}^I$, $X_J \in \mathbf{A}^J$, $Y', Y'' \in \mathbf{C}$ we must be given a map

$$\underset{j\in J}{\otimes} \mathrm{Hom}^\bullet_{\mathbf{A}}(\text{"}X_{I^j}^{\otimes}\text{"}, X_j) \bigotimes \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_J^{\otimes} \otimes Y'\text{"}, Y'') \to \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_I^{\otimes} \otimes Y'\text{"}, Y'').$$

In addition, we need to be given a natural quasi-isomorphism corresponding to $I \hookrightarrow I \cup \mathrm{pt}$. These natural transformations must satisfy the usual associativity axioms.

Assume that the pseudo monoidal structure on $\mathbf{A}$ is a homotopy monoidal structure. We say that a pseudo action of $\mathbf{A}$ on $\mathbf{C}$ is a homotopy action if the data of functors

$$\{X_I, Y', Y''\} \mapsto H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}(\text{"}X_I^{\otimes} \otimes Y'\text{"}, Y'')\right) : \mathrm{Ho}(\mathbf{A})^{I,op} \times \mathrm{Ho}(\mathbf{C})^{op} \times \mathrm{Ho}(\mathbf{C}) \to \mathrm{Vect}_k$$

and natural transformations come from a (automatically uniquely defined) monoidal action of $\mathrm{Ho}(\mathbf{A})$ on $\mathrm{Ho}(\mathbf{C})$.

In a similar way one defines the notion of pseudo action and homotopy action on the right. Any pseudo (resp., homotopy) monoidal category carries a pseudo (resp., homotopy) action on itself on both right and left; moreover, these two structures commute in a natural sense.

16.5.1. If $\mathbf{A}$ is a pseudo monoidal category with pseudo actions on $\mathbf{C}_1$ and $\mathbf{C}_2$, a DG pseudo functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ compatible with the action is a functorial assignment for any $X_I \in \mathbf{A}^I$, $Y_1 \in \mathbf{C}_1, Y_2 \in \mathbf{C}_2$ of a complex

$$\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}(\text{"}X_I^{\otimes} \otimes F(Y_1)\text{"}, Y_2), \tag{16.6}$$

where we should think of "$X_I^{\otimes} \otimes F(Y_1)$" as the corresponding non-existing object of $\mathbf{C}_2$. (In the above formula I might be empty.)

We must be given the following system of natural transformations. Given three ordered finite sets I_1, I_2 and I_3, $X_{I_j} \in \mathbf{A}^{I_j}$, $j = 1, 2, 3$, $Y_1', Y_1'' \in \mathbf{C}_1$, $Y_2', Y_2'' \in \mathbf{C}_2$, consider the concatenation $I = I_3 \cup I_2 \cup I_1$. We should have a map

$$\begin{aligned}(16.7) \quad \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1}({}''X^{\otimes}_{I_1} \otimes Y_1'{}'', Y_1'') \bigotimes \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}({}''X^{\otimes}_{I_2} \otimes F(Y_1''){}'', Y_2') \bigotimes \\ \bigotimes \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_2}({}''X^{\otimes}_{I_3} \otimes Y_2'{}'', Y_2'') \to \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}({}''X^{\otimes}_{I} \otimes F(Y_1'){}'', Y_2'').\end{aligned}$$

We should also be given natural transformations corresponding insertions of unit objects. There natural transformations must satisfy the natural axioms that we will not spell out here.

We say that F is a homotopy functor compatible with the action of $\mathbf{A}$ if the data of $H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}({}''X^{\otimes}_{I} \otimes F(Y_1){}'', Y_2)\right)$ comes from a functor $\mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}_2)$, compatible with the action of the monoidal category $\mathrm{Ho}(\mathbf{A})$.

We say that F is a quasi-equivalence, if the underlying functor $\mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}_2)$ is an equivalence.

16.5.2. Pseudo functors $\mathbf{C}_1 \to \mathbf{C}_2$ compatible with the action of $\mathbf{A}$ naturally form a DG category. Namely, for two DG pseudo functors F' and F'' we set $\mathrm{Hom}^\bullet_{\mathbf{A}}(F', F'')$ to be the be the sub-complex in

$$\prod_{I; X_I, Y_1, Y_2} \mathrm{Hom}^\bullet_{\mathbf{Comp}_k}\left(\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}({}''X^{\otimes}_{I} \otimes F''(Y_1){}'', Y_2), \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}({}''X^{\otimes}_{I} \otimes F'(Y_1){}'', Y_2)\right),$$

that makes all diagrams corresponding to (16.7) commute.

Let us denote this DG category by $\mathrm{PFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2)$. We call an object of $\mathrm{PFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2)$ acyclic of all the complexes $\mathrm{Hom}^\bullet({}''X^{\otimes}_{I} \otimes Y_1{}'', Y_2)$ are acyclic. The resulting quotient traingulated category will be denoted

$$\mathbf{D}(\mathrm{PFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2)) :=: \mathrm{PFunct}^{\mathrm{Ho}}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2).$$

When the pseudo action of $\mathbf{A}$ on both $\mathbf{C}_1$ and $\mathbf{C}_2$ is a homotopy action, we will denote the full subcategory of $\mathbf{D}(\mathrm{PFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2))$ formed by homotopy functors by $\mathrm{HFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2)$.

16.5.3. The structure of DG category on $\mathrm{PFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2)$ is part of a closed model category structure, where cofibrations are those maps $F' \Rightarrow F''$, for which all maps

$$\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}({}''X^{\otimes}_{I} \otimes F''(Y_1){}'', Y_2) \to \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}({}''X^{\otimes}_{I} \otimes F'(Y_1){}'', Y_2)$$

are surjective.

The model category structure on $\mathrm{PFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2)^{op}$ is akin to that on the category of DG modules over a DG associative algebra.

A supply of fibrant objects in $\mathrm{PFunct}_{\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2)$ is provided as in Sect. 16.2.4 by the pair of adjoint functors

$$\mathrm{PFunct}_{\mathrm{gr}\,\mathbf{A}}(\mathbf{C}_1, \mathbf{C}_2) \leftrightarrows \prod_{n>0} \mathrm{PFunct}_{\mathrm{gr}}(\mathbf{A}^n \times \mathbf{C}_1, \mathbf{C}_2),$$

where the subscript "gr" stands for the graded-without-differential versions of the corresponding categories.

16.5.4. One defines compositions of homotopy functors compatible with the action of $\mathbf{A}$ following the pattern of Sect. 15.4 and Sect. 16.4.

Thus, given a homotopy monoidal category $\mathbf{A}$, we can speak about the 2-category $\mathbf{DGmod}(\mathbf{A})$. Its 0-objects are (essentially small) DG categories, endowed with a homotopy action of $\mathbf{A}$ and 1-morphisms are the categories $\mathrm{HFunct}_{\mathbf{A}}(?,?)$.

We can also have the 2-category $\mathbf{Trmod}(\mathrm{Ho}(\mathbf{A}))$ of triangulated categories equipped with an action of $\mathrm{Ho}(\mathbf{A})$. There exists an evident forgetful 2-functor $\mathbf{DGmod}(\mathbf{A}) \to \mathbf{Trmod}(\mathrm{Ho}(\mathbf{A}))$.

In what follows, for a 0-object $\mathbf{D} \in \mathbf{Trmod}(\mathrm{Ho}(\mathbf{A}))$, by its DG model we will mean the fiber of the above map.

16.6. Changing the acting category. Let $\mathbf{A}_1, \mathbf{A}_2$ be two pseudo monoidal categories, and let $F_{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$ be a pseudo monoidal functor. Let $\mathbf{C}$ be a DG category equipped with a homotopy action of $\mathbf{A}_2$. In this subsection we shall construct the restriction 2-functor $\mathbf{DGmod}(\mathbf{A}_2) \to \mathbf{DGmod}(\mathbf{A}_1)$.

16.6.1. Let $\mathbf{A}_1$, $\mathbf{A}_2$ be two DG pseudo monoidal categories, and $\mathbf{C}_1, \mathbf{C}_2$ be two DG categories, endowed with pseudo actions of $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Let $F_{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$ a DG pseudo-monoidal functor.

A DG pseudo functor $F_{\mathbf{C}} : \mathbf{C}_1 \to \mathbf{C}_2$ compatible with $F_{\mathbf{A}}$ is a functorial assignment to $X_{1I} \in \mathbf{A}_1^I$, $Y_1 \in \mathbf{C}_1$, $Y_2 \in \mathbf{C}_2$ of a complex

$$\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{C}_1,\mathbf{C}_2}(\text{"}{X_{1I}}^{\otimes} \otimes F_{\mathbf{C}}(Y_1)\text{"}, Y_2),$$

endowed with the following system of natural transformations:

For finite ordered sets I, J, K, L, a surjection $J \twoheadrightarrow K$, $X_{1I} \in \mathbf{A}_1^I$, $X_{1J} \in \mathbf{A}_1^J$, $X_{1L} \in \mathbf{A}_1^L$ $X_{2K} \in \mathbf{A}_2^K$ $Y_1', Y_1'' \in \mathbf{C}_1$, $Y_2', Y_2'' \in \mathbf{C}_2$, we need to be given a map

$$\begin{aligned}&\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{C}_1}(\text{"}{X_{1L}}^{\otimes} \otimes Y_1'\text{"}, Y_1'') \bigotimes \\ &\bigotimes \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{C}_1,\mathbf{C}_2}(\text{"}{X_{1I}}^{\otimes} \otimes F_{\mathbf{C}}(Y_1'')\text{"}, Y_2') \bigotimes \left(\underset{k\in K}{\otimes} \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{A}_2}(\text{"}{X_{1J^k}}^{\otimes}\text{"}, X_{2k})\right) \bigotimes \\ &\bigotimes \mathrm{Hom}^\bullet_{\mathbf{A}_2,\mathbf{C}_2}(\text{"}{X_{2K}}^{\otimes} \otimes Y_2'\text{"}, Y_2'') \to \mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{C}_1,\mathbf{C}_2}(\text{"}X{1}^{\otimes}_{J\cup I\cup L} \otimes F_{\mathbf{C}}(Y_1)\text{"}, Y_2'').\end{aligned}$$

These natural transformations are required to satisfy the natural axioms.

If $\mathbf{A}_1$, $\mathbf{A}_2$ are homotopy monoidal categories, $F_{\mathbf{A}}$ is a homotopy monoidal functor, and the pseudo actions of $\mathbf{A}_i$ on $\mathbf{A}_i$ for $i = 1, 2$ are homotopy actions, one says that $F_{\mathbf{C}}$ is a homotopy functor, if the data of $H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A}_1,\mathbf{C}_1,\mathbf{C}_2}(\text{"}{X_{1I}}^{\otimes} \otimes F_{\mathbf{C}}(Y_1)\text{"}, Y_2)\right)$ and natural transformations comes from a functor $\mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C}_2)$, compatible with the action of $\mathrm{Ho}(\mathbf{A}_1)$ via $\mathrm{Ho}(F_{\mathbf{A}})$.

16.6.2. Proceeding as in Sect. 16.5.2, for $\mathbf{A}_1, \mathbf{A}_2, F_{\mathbf{A}}, \mathbf{C}_1, \mathbf{C}_2$ we introduce the DG category $\mathrm{PFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2)$, which has a closed model structure, and its homotopy category

$$\mathbf{D}(\mathrm{PFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2)) :=: \mathrm{PFunct}^{\mathrm{Ho}}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2),$$

which is a triangulated category. In the case of homotopy monoidal structures and actions, the latter contains a triangulated subcategory that consists of homotopy functors, denoted $\mathrm{HFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2)$.

In addition, one has the functors:

$$\circ : \mathrm{PFunct}^{\mathrm{Ho}}_{\mathbf{A}_1}(\mathbf{C}_1', \mathbf{C}_1) \times \mathrm{PFunct}^{\mathrm{Ho}}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2) \to \mathrm{PFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1', \mathbf{C}_2)$$

and

$$\circ : \mathrm{PFunct}^{\mathrm{Ho}}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2) \times \mathrm{PFunct}^{\mathrm{Ho}}_{\mathbf{A}_2}(\mathbf{C}_2, \mathbf{C}'_2) \to \mathrm{PFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}'_2),$$

and in the case of homotopy monoidal structures, the functors

$$\circ : \mathrm{HFunct}_{\mathbf{A}_1}(\mathbf{C}'_1, \mathbf{C}_1) \times \mathrm{HFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2) \to \mathrm{HFunct}_{F_{\mathbf{A}}}(\mathbf{C}'_1, \mathbf{C}_2)$$

and

$$\circ : \mathrm{HFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2) \times \mathrm{HFunct}_{\mathbf{A}_2}(\mathbf{C}_2, \mathbf{C}'_2) \to \mathrm{HFunct}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}'_2),$$

defined by the procedure analogous to that of Sect. 15.4 and Sect. 16.4.

16.6.3. Given pseudo monoidal categories $\mathbf{A}_1, \mathbf{A}_2$, a pseudo monoidal functor $F_{\mathbf{A}}$ and a category $\mathbf{C}_2$ with a pseudo action of $\mathbf{A}_2$, a DG category $\mathbf{C}_{1,can}$ with a pseudo action of $\mathbf{A}_1$ is called a restriction of $\mathbf{C}_2$ with respect to $F_{\mathbf{A}}$ if we are given an object $F_{\mathbf{C},can} \in \mathrm{PFunct}^{\mathrm{Ho}}_{F_{\mathbf{A}}}(\mathbf{C}_{1,can}, \mathbf{C}_2)$, such that for any $\mathbf{C}_1$ with a pseudo action of $\mathbf{A}_1$, the functor

$$G \mapsto F_{\mathbf{C},can} \circ G : \mathrm{PFunct}^{\mathrm{Ho}}_{\mathbf{A}_1}(\mathbf{C}_1, \mathbf{C}_{1,can}) \to \mathrm{PFunct}^{\mathrm{Ho}}_{F_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_2)$$

is an equivalence. By Yoneda's lemma, if a restriction exists, it is canonically defined as a 0-obect in the appropriate 2-category of DG categories with a pseudo action of $\mathbf{A}_1$, up to quasi-equivalence; we shall denote by $\mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_2)$.

Lemma 16.6.4. *For any $\mathbf{C}_2$ with a pseudo action of $\mathbf{A}_2$, the restriction $\mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_2)$ exists. The pseudo functor*

$$F_{\mathbf{C}_2,can} : \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_2) \to \mathbf{C}_2,$$

when regarded as a pseudo functor between plain DG categories, is a quasi-equivalence.

16.6.5. Let $F'_{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$, $F''_{\mathbf{A}} : \mathbf{A}_2 \to \mathbf{A}_3$, $G_{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_3$ be pseudo monoidal functors between pseudo monoidal categories, and let us be given a DG natural transformation $\phi_{\mathbf{A}} : G_{\mathbf{A}} \Rightarrow$ "$F''_{\mathbf{A}} \circ F'_{\mathbf{A}}$".

Let $\mathbf{C}_i$ be a DG category with a pseudo-action of $\mathbf{A}_i$, $i = 1, 2, 3$. Let us be given pseudo functors $F'_{\mathbf{C}} : \mathbf{C}_1 \to \mathbf{C}_2$ and $F''_{\mathbf{C}} : \mathbf{C}_2 \to \mathbf{C}_3$, compatible with $F'_{\mathbf{A}}$ and $F''_{\mathbf{A}}$, respectively.

For a pseudo functor $G_{\mathbf{C}} : \mathbf{C}_1 \to \mathbf{C}_3$, compatible with $G_{\mathbf{A}}$, we define the set

$$\mathrm{Hom}_{\mathrm{PFunct}_{\phi_{\mathbf{A}}}(\mathbf{C}_1,\mathbf{C}_3)}(G_{\mathbf{C}}, \text{"}F''_{\mathbf{C}} \circ F'_{\mathbf{C}}\text{"}).$$

Proceeding as in Sects. 15.4.2 and 16.4.1, we define also the set

$$\mathrm{Hom}_{\mathrm{PFunct}^{\mathrm{Ho}}_{\phi_{\mathbf{A}}}(\mathbf{C}_1,\mathbf{C}_3)}(G_{\mathbf{C}}, \text{"}F''_{\mathbf{C}} \circ F'_{\mathbf{C}}\text{"}).$$

Lemma 16.6.6. *The functor*

$$G_{\mathbf{C}} \mapsto \mathrm{Hom}_{\mathrm{PFunct}^{\mathrm{Ho}}_{\phi_{\mathbf{A}}}(\mathbf{C}_1,\mathbf{C}_3)}(G_{\mathbf{C}}, \text{"}F''_{\mathbf{C}} \circ F'_{\mathbf{C}}\text{"})$$

on $\mathrm{PFunct}^{\mathrm{Ho}}_{G_{\mathbf{A}}}(\mathbf{C}_1, \mathbf{C}_3)$ is representable. The universal object, denoted $F'_{\mathbf{C}} \circ F''_{\mathbf{C}}$ and called the pseudo composition of $F'_{\mathbf{C}}$ and $F''_{\mathbf{C}}$, induces a pseudo composition at the level of plain categories. If the categories, actions and functors in question are homotopy monoidal, then the map

$$\mathrm{Ho}(\phi_{\mathbf{C}}) : \mathrm{Ho}(F''_{\mathbf{C}} \circ F'_{\mathbf{C}}) \Rightarrow \mathrm{Ho}(F''_{\mathbf{C}}) \circ \mathrm{Ho}(F'_{\mathbf{C}})$$

is an isomorphism.

16.6.7. In the above setting, let us take $\mathbf{C}_2 := \mathrm{Res}^{\mathbf{A}_3}_{\mathbf{A}_2}(\mathbf{C}_3)$ with $F''_{\mathbf{C}}$ being $F''_{\mathbf{C}_3,can}$, and $\mathbf{C}_1 := \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_2)$ with $F'_{\mathbf{C}}$ being $F'_{\mathbf{C}_2,can}$. Assume that the arrow $\phi_{\mathbf{A}}$ defines an isomorphism $G_{\mathbf{A}} \simeq F''_{\mathbf{A}} \circ F'_{\mathbf{A}}$.

Lemma 16.6.8. *Under the above circumstances the 1-morphism*

$$\mathbf{C}_1 \to \mathrm{Res}^{\mathbf{A}_3}_{\mathbf{A}_1}(\mathbf{C}_3)$$

is a 1-isomorphism in the 2-category of DG categories with a pseudo action of $\mathbf{A}_1$, up to quasi-equivalence.

In particular, if the categories and actions are homotopy monoidal, then the above 1-morphism is an isomorphism in $\mathbf{DGmod}(\mathbf{A}_1)$.

The upshot of the lemma is that we have a canonical equivalence in $\mathbf{DGmod}(\mathbf{A}_1)$:

$$\mathrm{Res}^{\mathbf{A}_3}_{\mathbf{A}_1}(\mathbf{C}_3) \simeq \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathrm{Res}^{\mathbf{A}_3}_{\mathbf{A}_2}(\mathbf{C}_3)).$$

16.7. Let $\mathbf{A}$ be a DG pseudo monoidal category. We claim that $\underset{\to}{\mathbf{A}}$ naturally acquires a DG pseudo monoidal structure. Namely, for a finite order set I and $X_i \in \underset{\to}{\mathbf{A}}$, $i \in I$, $Y \in \underset{\to}{\mathbf{A}}$, given by

$$\{\underset{k_i \geq 0}{\oplus} X_i^{k_i}, \Phi_i\} \text{ and } Y = \{\underset{m \geq 0}{\oplus} Y^m, \Psi\}, \text{ respectively}$$

set

$$\mathrm{Hom}^\bullet(\text{"}X_I^{\otimes}\text{"}, Y) := \underset{\alpha}{\Pi} \underset{m}{\oplus} \mathrm{Hom}^\bullet(\text{"}X_{I_\alpha}^{\otimes}\text{"}, Y^m),$$

where α runs over the set of maps $I \to \mathbb{Z}^{\geq 0}$, each defining a map $I \overset{X_{I_\alpha}}{\to} \mathbf{A} : i \mapsto X_i^{\alpha(i)}$. The differential in the above complex is given using the maps Φ and Ψ.

If the DG pseudo monoidal structure on $\mathbf{A}$ is a homotopy monoidal structure, then so will be the case for $\underset{\to}{\mathbf{A}}$.

Similarly, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are two DG pseudo monoidal categories and $F : \mathbf{A}_1 \to \mathbf{A}_2$ is a pseudo monoidal functor, it extends to a pseudo monoidal functor $\underset{\to}{\mathbf{A}}_1 \to \underset{\to}{\mathbf{A}}_2$. If F is a homotopy monoidal functor between homotopy monoidal categories, so will be its extension.

16.7.1. Let $\mathbf{C}$ be a DG category equipped with a DG pseudo action of a pseudo monoidal category $\mathbf{A}$. Then it extends to a DG pseudo action of $\underset{\to}{\mathbf{A}}$ on $\underset{\to}{\mathbf{C}}$, preserving the property of being a homotopy action.

By a similar procedure, given a 1-morphism (resp., homotopy functor) $F : \mathbf{C}_1 \to \mathbf{C}_2$ in $\mathbf{DGmod}(\mathbf{A})$ categories we extend it to a 1-morphism $\underset{\to}{\mathbf{C}}_1 \to \underset{\to}{\mathbf{C}}_2$ in $\mathbf{DGmod}(\underset{\to}{\mathbf{A}})$.

16.7.2. If $\mathbf{A}$ is a DG pseudo monodal category, and $\mathbf{A}' \subset \mathbf{A}$ is a full DG subcategory, it automatically inherits a DG pseudo monodal structure. If $\mathbf{A}$ is a homotopy monoidal category, then $\mathbf{A}'$ will be such if and only if $\mathrm{Ho}(\mathbf{A}')$ is a monoidal subcategory of $\mathrm{Ho}(\mathbf{A})$.

If $F : \mathbf{A}_1 \to \mathbf{A}_2$ is a pseudo monoidal functor and $\mathbf{A}'_2 \subset \mathbf{A}_2$ a DG subcategory, we obtain a pseudo monoidal functor $F' : \mathbf{A}_1 \to \mathbf{A}'_2$. If F is a homotopy monoidal functor between homotopy monoidal categories, and $\mathbf{A}'_2$ is also a homotopy monoidal category, then F' is a homotopy monoidal functor if and only if $\mathrm{Ho}(F)$ sends $\mathrm{Ho}(\mathbf{A}_1)$ to $\mathrm{Ho}(\mathbf{A}'_2)$. Analogously to Sect. 15.3.2, this establishes an equivalence between the category of 1-morphisms $\mathbf{A}_1 \to \mathbf{A}'_2$ and the full subcategory of the category of those 1-morphisms $\mathbf{A}_1 \to \mathbf{A}_2$ whose essential image on the homotopy level belongs to $\mathrm{Ho}(\mathbf{A}'_2)$.

By the same token if we have a pseudo action of $\mathbf{A}$ on $\mathbf{C}$ and $\mathbf{C}' \subset \mathbf{C}$ is a full DG subcategory, we have a pseudo action of $\mathbf{A}$ on $\mathbf{C}'$. If initially we had a homotopy action of $\mathbf{A}$ on $\mathbf{C}$, then it will be the case for $\mathbf{C}'$ if and only if the $\mathrm{Ho}(\mathbf{A})$-action on $\mathrm{Ho}(\mathbf{C})$ preserves $\mathrm{Ho}(\mathbf{C}')$.

A similar discussion applies to DG pseudo functors and homotopy functors $\mathbf{C}_1 \to \mathbf{C}_2$ in $\mathbf{DGmod}(\mathbf{A})$.

16.7.3. Let $\mathbf{A}$ be a DG pseudo monoidal category, and $\mathbf{A}' \subset \mathbf{A}$ a full subcategory. By the construction of quotients in Sect. 15.6 and the above discussion, the DG category $\mathbf{A}/\mathbf{A}'$ acquires a natural DG pseudo monoidal structure. If $\mathbf{A}$ was a homotopy monoidal category, then $\mathbf{A}/\mathbf{A}'$ will be such if and only if $\mathrm{Ho}(\mathbf{A}') \subset \mathrm{Ho}(\mathbf{A})$ is a monoidal ideal.

In the situation of a DG pseudo monoidal functor $F : \mathbf{A}_1 \to \mathbf{A}_2$ we obtain a pseudo monoidal functor $\mathbf{A}_1/\mathbf{A}'_1 \to \mathbf{A}_2/\mathbf{A}'_2$. If the initial situaition was homotopy monoidal, then the latter functor will be homotopy monoidal if and only if $\mathrm{Ho}(F)(\mathbf{A}'_1) \subset \mathrm{Ho}(\mathbf{A}'_2)$.

In particular, the canonical homotopy functor $\mathbf{A} \to \mathbf{A}/\mathbf{A}'$ naturally extends to a pseudo monoidal functor, which is homotopy monoidal if $\mathrm{Ho}(\mathbf{A}')$ is an ideal.

A similar discussion applies to the situation when we have an action of $\mathbf{A}$ on $\mathbf{C}$ and a DG subcategory $\mathbf{C}'$. In particular, we obtain:

Lemma 16.7.4.

(1) Let $\mathbf{A}$ be a homotopy monoidal category with a homotopy action on a DG category $\mathbf{C}$. Let $\mathbf{A}' \subset \mathbf{A}$, $\mathbf{C}' \subset \mathbf{C}$ be DG subcategories. Assume that

$$\mathrm{Ho}(\mathbf{A}') \times \mathrm{Ho}(\mathbf{C}) \mapsto \mathrm{Ho}(\mathbf{C}') \text{ and } \mathrm{Ho}(\mathbf{A}) \times \mathrm{Ho}(\mathbf{C}') \mapsto \mathrm{Ho}(\mathbf{C}').$$

Then we have a well-defined homotopy action of $\mathbf{A}/\mathbf{A}'$ on $\mathbf{C}/\mathbf{C}'$.

(2) For a DG category $\mathbf{C}_1$ with a homotopy action of $\mathbf{A}/\mathbf{A}'$ the following two categories are equivalent:

(a) $\mathrm{HFunct}_{\mathbf{A}/\mathbf{A}'}(\mathbf{C}/\mathbf{C}', \mathbf{C}_1)$.

(b) The full subcategory of $\mathrm{HFunct}_{\mathbf{A}}(\mathbf{C}, \mathrm{Res}^{\mathbf{A}/\mathbf{A}'}_{\mathbf{A}}(\mathbf{C}_1))$, *consisting of homotopy functors compatible with the action, for which the underlying plain homotopy functor $\mathbf{C} \to \mathbf{C}_1$ factors through $\mathbf{C}/\mathbf{C}'$.*

16.7.5. In the sequel we will need a generalization of the above discussion along the lines of Sect. 15.6.4. Let $\mathbf{A}$ be a DG pseudo-monoidal category (resp., $\mathbf{C}$ a DG category with a pseudo-action of $\mathbf{A}$; $F : \mathbf{C}_1 \to \mathbf{C}_2$ a pseudo functor between such categories, compatible with the pseudo-actions of $\mathbf{A}$.)

Let $\mathbf{A}' \subset \mathbf{A}$ (resp., $\mathbf{C}' \subset \mathbf{C}$; $\mathbf{C}'_i \subset \mathbf{C}_i$) be DG subcategories. Suppose that the following conditions hold:

- For $X_1, ..., X_n \in \mathbf{A}$, the functor on $\mathrm{Ho}(\mathbf{A})/\mathrm{Ho}(\mathbf{A}')$ given by
$$X \mapsto \underset{f:X\to X'}{colim} H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A}}("X_1 \otimes ... \otimes X_n", X')\right)$$
is co-representable, and if one of the X_i's belongs to $\mathbf{A}'$, then it equals zero. (The colimit is taken over the set of arrows f with $\mathrm{Cone}(f) \in \mathbf{A}'$).
- For $X_1, ..., X_n \in \mathbf{A}$, $'Y \in \mathbf{C}$, the functor on $\mathrm{Ho}(\mathbf{C})/\mathrm{Ho}(\mathbf{C}')$ given by
$$''Y \mapsto \underset{f:''Y\to ''Y'}{colim} H^0\left(\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}}("X_1 \otimes ... \otimes X_n \otimes {}'Y", ''Y')\right)$$
is co-representable, and if one of the X_i's belongs to $\mathbf{A}'$ or $'Y$ belongs to $\mathbf{C}'$, then it equals zero.

- For $X_1, ..., X_n \in \mathbf{A}$, $Y_1 \in \mathbf{C}_1$, the functor on $\mathrm{Ho}(\mathbf{C}_2)/\mathrm{Ho}(\mathbf{C}'_2)$ given by
$$Y_2 \mapsto \underset{f:Y_2 \to Y'_2}{colim} H^0 \left(\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}_1,\mathbf{C}_2}(" X_1 \otimes ... \otimes X_n \otimes F(Y_1), Y'_2)\right)$$
is co-representable, and if one of the X_i's belongs to $\mathbf{A}'$ or Y_1 belongs to $\mathbf{C}'_1$, then it equals zero.

Then the construction of Sect. 15.6.4 endows $\mathbf{A}/\mathbf{A}'$ with a structure of homotopy monoidal category, the category $\mathbf{C}/\mathbf{C}'$ with a homotopy action of $\mathbf{A}/\mathbf{A}'$ and the homotopy functor $\mathbf{C}_1/\mathbf{C}'_1 \to \mathbf{C}_2/\mathbf{C}'_2$ with the structure of compatibility with the homotopy action of $\mathbf{A}/\mathbf{A}'$.

16.7.6. Finally, from Sect. 15.5, we obtain that if $\mathbf{A}$ is a DG pseudo monoidal (resp., homotopy monoidal) category, then so is $\mathbf{A}^{Kar}$. Any DG pseudo monoidal (resp., homotopy monoidal) functor $\mathbf{A}_1 \to \mathbf{A}_2$ gives rise to a DG pseudo monoidal (resp., homotopy monoidal) functor $\mathbf{A}_1^{Kar} \to \mathbf{A}_2^{Kar}$, and similarly for actions.

17. Tensor products of categories

17.1. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two DG categories. We form a non-pretriangulated DG category $(\mathbf{C}_1 \otimes \mathbf{C}_2)^{non-pretr}$ to have as objects pairs X_1, X_2 with $X_i \in \mathbf{C}_i$ and
$$\mathrm{Hom}((X_1, X_2), (Y_1, Y_2)) = \mathrm{Hom}_{\mathbf{C}_1}(X_1, Y_1) \otimes \mathrm{Hom}_{\mathbf{C}_2}(X_2, Y_2).$$
We define $\mathbf{C}_1 \otimes \mathbf{C}_2$ as the strongly pre-triangulated envelope of $(\mathbf{C}_1 \otimes \mathbf{C}_2)^{non-pretr}$.

If $F_1 : \mathbf{C}_1 \to \mathbf{C}'_1$ and $F_2 : \mathbf{C}_2 \to \mathbf{C}'_2$ are 1-morphisms in $\mathbf{DGCat}$, we have a well-defined 1-morphism
$$F_1 \otimes F_2 : \mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{C}'_1 \otimes \mathbf{C}'_2.$$

Note that for any DG category $\mathbf{C}$, we have:
$$\mathbf{C} \otimes \mathbf{Comp}^f_k \simeq \mathbf{C},$$
where $\mathbf{Comp}^f_k$ is the DG category of finite-dimensional complexes.

17.1.1. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two DG categories equipped with a DG pseudo monoidal (resp., homotopy monoidal) structure. Then their tensor product $\mathbf{A}_1 \otimes \mathbf{A}_2$ is naturally a DG pseudo monoidal (resp., homotopy monoidal) category.

Indeed, for $(X_1^1, ..., X_1^n) \in \mathbf{A}_1$, $(X_2^1, ..., X_2^n) \in \mathbf{A}_2$, $Y_1 \in \mathbf{A}_1$, $Y_2 \in \mathbf{A}_2$, we set
$$\mathrm{Hom}^\bullet_{\mathbf{A}_1 \otimes \mathbf{A}_2}\left(" (X_1^1, X_2^1) \otimes ... \otimes (X_1^n, X_2^n)", (Y_1, Y_2)\right) :=$$
$$= \mathrm{Hom}^\bullet_{\mathbf{A}_1}(" X_1^1 \otimes ... \otimes X_1^n", Y_1) \otimes \mathrm{Hom}^\bullet_{\mathbf{A}_2}(" X_2^1 \otimes ... \otimes X_2^n", Y_2),$$
and this uniquely extends onto arbitrary objects of $\mathbf{A}_1 \otimes \mathbf{A}_2$.

Similarly, if in the above situation $\mathbf{A}_1$ is endowed with a DG pseudo action (resp., homotopy action) on $\mathbf{C}_1^l$ and similarly for the pair $(\mathbf{A}_2, \mathbf{C}_2^l)$ we have a DG pseudo action (resp., homotopy action) of $\mathbf{A}_1 \otimes \mathbf{A}_2$ on $\mathbf{C}_1^l \otimes \mathbf{C}_2^l$.

As a particular case, we obtain that given a DG pseudo action (resp., homotopy action) of $\mathbf{A}$ on $\mathbf{C}^l$, for an arbitrary DG category $\mathbf{C}'$, the tensor product $\mathbf{C}^l \otimes \mathbf{C}'$ carries a DG pseudo action (resp., homotopy action) of $\mathbf{A}$.

17.2. Suppose now that $\mathbf{A}$ is a DG category with a homotopy monoidal structure. Let $\mathbf{C}^r$ and $\mathbf{C}^l$ be two DG categories equipped with homotopy right and left actions of $\mathbf{A}$, respectively. Our present goal is to define a new DG category, which would be the tensor product of $\mathbf{C}^r$ and $\mathbf{C}^l$ *over* $\mathbf{A}$, denoted

$$\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l. \tag{17.1}$$

The construction given below was explained to us by J. Lurie.

17.2.1. First, we define a DG category $\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)^{non-pretr}$. Its objects are

$$(Y^r, \overline{X}^n, Y^l) := (Y^r, \underbrace{X^1, ..., X^n}_{n}, Y^l), \tag{17.2}$$

where $n \in \{0, 1, 2...\}$, $Y^r \in \mathbf{C}^r$, $Y^l \in \mathbf{C}^l$ and $X^i \in \mathbf{A}$. In what follows, for a fixed n, we shall denote the corresponding subcategory by $\mathrm{Bar}^n(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$.

For an object $(Y^r_1, \overline{X}^n_1, Y^l_1)$ and an object $(Y^r_2, \overline{X}^m_2, Y^l_2)$ and a non-decreasing map $\phi : \{0, 1, ..., m\} \to \{0, 1, ..., n\}$ the set of ϕ-morphisms

$$\mathrm{Hom}^\bullet_\phi((Y^r_1, \overline{X}^n_1, Y^l_1), (Y^r_2, \overline{X}^m_2, Y^l_2))$$

to be by definition

$$\begin{gathered}\mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}^r}(\text{"}Y^r_1 \otimes X^1_1 \otimes ... \otimes X^{\phi(0)}_1\text{"}, Y^r_2) \otimes \mathrm{Hom}_{\mathbf{A}}(\text{"}X^{\phi(0)+1}_1 \otimes ... \otimes X^{\phi(1)}_1\text{"}, X^1_2) \otimes ... \\ \otimes \mathrm{Hom}^\bullet_{\mathbf{A}}(\text{"}X^{\phi(i-1)+1}_1 \otimes ... \otimes X^{\phi(i)}_1\text{"}, X^i_2) \otimes ... \\ ... \otimes \mathrm{Hom}^\bullet_{\mathbf{A}}(\text{"}X^{\phi(m-1)+1}_1 \otimes ... \otimes X^{\phi(m)}_1\text{"}, X^m_2) \otimes \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}^l}(\text{"}X^{\phi(m)+1}_1 \otimes ... \otimes X^n_1 \otimes Y^l_1\text{"}, Y^l_2),\end{gathered}$$

where if $\phi(i-1) = \phi(i)$ in the corresponding term we make an insertion of $\mathbf{1_A}$.

We define

$$\mathrm{Hom}^\bullet_{\mathrm{Bar}(\mathbf{C}^r,\mathbf{A},\mathbf{C}^l)^{non-pretr}}((Y^r_1, \overline{X}^n_1, Y^l_1), (Y^r_2, \overline{X}^m_2, Y^l_2)) := \underset{\phi}{\oplus} \mathrm{Hom}^\bullet_\phi((Y^r_1, \overline{X}^n_1, Y^l_1), (Y^r_2, \overline{X}^m_2, Y^l_2)),$$

as ϕ runs over the set of all non-decreasing maps $\phi : \{0, 1, ..., m\} \to \{0, 1, ..., n\}$.

For $\psi : \{0, 1, ..., k\} \to \{0, 1, ..., m\}$ and $(Y^r_3, \overline{X}^k_3, Y^l_3)$ we have a natural composition map,

$$\begin{aligned}\mathrm{Hom}^\bullet_\phi((Y^r_1, \overline{X}^n_1, Y^l_1), (Y^r_2, \overline{X}^m_2, Y^l_2)) \otimes \mathrm{Hom}^\bullet_\psi((Y^r_2, \overline{X}^m_2, Y^l_2), (Y^r_3, \overline{X}^k_3, Y^l_3)) \to \\ \to \mathrm{Hom}^\bullet_{\phi\circ\psi}((Y^r_1, \overline{X}^n_1, Y^l_1), (Y^r_3, \overline{X}^k_3, Y^l_3)).\end{aligned}$$

This defines a structure of (non-pretriangulated) DG category on $\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)^{non-pretr}$. We denote by $\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l)$ its strongly pre-triangulated envelope.

17.2.2. Let $(Y^r_1, \overline{X}^n_1, Y^l_1)$ and $(Y^r_2, \overline{X}^m_2, Y^l_2)$ be two objects as above. Let f be an element of $\mathrm{Hom}^\bullet_\phi((Y^r_1, \overline{X}^n_1, Y^l_1), (Y^r_2, \overline{X}^m_2, Y^l_2))$ equal to the tensor product of

$$\begin{aligned}&f^r \in \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}^r}(\text{"}Y^r_1 \otimes X^1_1 \otimes ... \otimes X^{\phi(0)}_1\text{"}, Y^r_2),\\ &f^i \in \mathrm{Hom}^\bullet_{\mathbf{A}}(\text{"}X^{\phi(i-1)+1}_1 \otimes ... \otimes X^{\phi(i)}_1\text{"}, X^i_2), i = 1, ..., m\\ &f^l \in \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}^l}(\text{"}X^{\phi(m)+1}_1 \otimes ... \otimes X^n_1 \otimes Y^l_1\text{"}, Y^l_2).\end{aligned}$$

We say that f is a *quasi-isomorphism* if all f^i are cocycles of degree 0 that induce isomorphisms between the corresponding objects on the homotopy level:

$$Y_1^r \otimes X_1^1 \otimes ... \otimes X_1^{\phi(0)} \to Y_2^r \in \mathrm{Ho}(\mathbf{C}^r),$$
$$X_1^{\phi(i-1)+1} \otimes ... \otimes X_1^{\phi(i)} \to X_2^i \in \mathrm{Ho}(\mathbf{A}),$$
$$X_1^{\phi(m)+1} \otimes ... \otimes X_1^n \otimes Y_1^l \to Y_2^l \in \mathrm{Ho}(\mathbf{C}^l).$$

Let $\mathrm{Ho}(\mathbf{I}_{\mathbf{C}^r,\mathbf{A},\mathbf{C}^l})) \subset \mathrm{Ho}(\mathrm{Bar}(\mathbf{C}^r,\mathbf{A},\mathbf{C}^l)$ be the triangulated subcategory generated by the cones of all quasi-isomorphisms. Let $\mathbf{I}_{\mathbf{C}^r,\mathbf{A},\mathbf{C}^l}$ be its preimage in $\mathrm{Bar}(\mathbf{C}^r,\mathbf{A},\mathbf{C}^l)$.

We consider the quotient

$$\mathrm{Bar}(\mathbf{C}^r,\mathbf{A},\mathbf{C}^l)/\mathbf{I}_{\mathbf{C}^r,\mathbf{A},\mathbf{C}^l},$$

as a 0-object of **DGCat**. By definition,

$$\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l := \mathrm{Bar}(\mathbf{C}^r,\mathbf{A},\mathbf{C}^l)/\mathbf{I}_{\mathbf{C}^r,\mathbf{A},\mathbf{C}^l}.$$

I.e., $\mathrm{Ho}\left(\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l\right) \simeq \mathrm{Ho}(\mathrm{Bar}(\mathbf{C}^r,\mathbf{A},\mathbf{C}^l)/\,\mathrm{Ho}(\mathbf{I}_{\mathbf{C}^r,\mathbf{A},\mathbf{C}^l}).$

17.2.3. Let $\mathbf{C}_1^r,\mathbf{C}_1^l,\ \mathbf{C}_2^r,\mathbf{C}_2^l$ be two pairs of DG categories with homotopy actions of $\mathbf{A}$. It is clear from Sect. 16.5.1 that we obtain well-defined functors

$$\mathrm{HFunct}_{\mathbf{A}}(\mathbf{C}_1^l,\mathbf{C}_2^l) \times \mathrm{HFunct}_{\mathbf{A}}(\mathbf{C}_1^r,\mathbf{C}_2^r) \to \mathrm{HFunct}\left(\mathrm{Bar}(\mathbf{C}_1^r,\mathbf{A},\mathbf{C}_1^l),\mathrm{Bar}(\mathbf{C}_2^r,\mathbf{A},\mathbf{C}_2^l)\right)$$

and

$$\mathrm{HFunct}_{\mathbf{A}}(\mathbf{C}_1^l,\mathbf{C}_2^l) \times \mathrm{HFunct}_{\mathbf{A}}(\mathbf{C}_1^r,\mathbf{C}_2^r) \to \mathrm{HFunct}\left(\mathbf{C}_1^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l, \mathbf{C}_2^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_2^l\right).$$

In particular, if $F^r : \mathbf{C}_1^r \to \mathbf{C}_2^r$ and $F^l : \mathbf{C}_1^l \to \mathbf{C}_2^l$ are quasi-equivalences, then so are the resulting functors

$$\mathrm{Bar}(\mathbf{C}_1^r,\mathbf{A},\mathbf{C}_1^l) \to \mathrm{Bar}(\mathbf{C}_2^r,\mathbf{A},\mathbf{C}_2^l) \text{ and } \mathbf{C}_1^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l \to \mathbf{C}_2^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_2^l.$$

17.2.4. Let $(\mathbf{C}_1^l,\mathbf{C}_1^r,\mathbf{A}_1)$ and $(\mathbf{C}_2^l,\mathbf{C}_2^r,\mathbf{A}_2)$ be two triples as above, $F_{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$ be a homotopy monoidal functor, and $F^l : \mathbf{C}_1^l \to \mathbf{C}_2^l$, $F^r : \mathbf{C}_1^r \to \mathbf{C}_2^r$ be homotopy functors compatible with $F_{\mathbf{A}}$. In this case, we have well-defined 1-morphisms

$$\mathrm{Bar}(\mathbf{C}_1^r,\mathbf{A}_1,\mathbf{C}_1^l) \to \mathrm{Bar}(\mathbf{C}_2^r,\mathbf{A}_2,\mathbf{C}_2^l) \text{ and } \mathbf{C}_1^r \underset{\mathbf{A}_1}{\otimes} \mathbf{C}_1^l \to \mathbf{C}_2^r \underset{\mathbf{A}_2}{\otimes} \mathbf{C}_1^l. \tag{17.3}$$

In particular, this applies to the case when

$$\mathbf{C}_1^l = \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_2^l) \text{ and } \mathbf{C}_1^r = \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_2^r).$$

If the functors $F_{\mathbf{A}}, F^l, F^r$ are quasi-equivalences, then so are the 1-morphisms in (17.3).

17.3. Let us take $\mathbf{C}^r := \mathbf{A}$ with the standard right action on itself. We claim:

Proposition 17.3.1. *There exists a canonical quasi-equivalence*

$$\mathbf{A} \underset{\mathbf{A}}{\otimes} \mathbf{C}^l \simeq \mathbf{C}^l.$$

17.3.2. We construct a DG functor $\iota : \mathbf{C}^l \to \mathbf{A} \underset{\mathbf{A}}{\otimes} \mathbf{C}^l$ as the composition

$$\mathbf{C}^l \to \mathrm{Bar}^0(\mathbf{A},\mathbf{A},\mathbf{C}^l) \hookrightarrow \mathrm{Bar}(\mathbf{A},\mathbf{A},\mathbf{C}^l) \to \mathbf{C}^l \underset{\mathbf{A}}{\otimes} \mathbf{C}^l,$$

where the first functor is

$$Y^l \mapsto (\mathbf{1}_{\mathbf{A}}, Y^l), \text{ where } \mathbf{1}_{\mathbf{A}} \in \mathbf{A} \simeq \mathbf{C}^r.$$

17.3.3. We define a DG pseudo functor $F : \mathrm{Bar}(\mathbf{A}, \mathbf{A}, \mathbf{C}^l) \to \mathbf{C}^l$ as follows. For an object $(X^0, \overline{X}^n, Y^l) := (X^0, \underbrace{X^1, ..., X^n}_{n}, Y^l)$ of $\mathrm{Bar}^n(\mathbf{A}, \mathbf{A}, \mathbf{C}^l)$ and $Y^l_1 \in \mathbf{C}^l$ we set

$$\mathrm{Hom}^\bullet\left(\text{"}F(X^0, \overline{X}^n, Y^l)\text{"}, Y^l_1\right) := \mathrm{Hom}^\bullet_{\mathbf{A},\mathbf{C}^l}(\text{"}X^0 \otimes X^1 \otimes ... \otimes X^n \otimes Y^l\text{"}, Y^l_1).$$

This assignment is clearly a pseudo functor $\mathrm{Bar}(\mathbf{A}, \mathbf{A}, \mathbf{C}^l)^{non-pretr} \to \mathbf{C}^l$, which uniquely extends to a pseudo functor $\mathrm{Bar}(\mathbf{A}, \mathbf{A}, \mathbf{C}^l) \to \mathbf{C}^l$.

Lemma 17.3.4. *The above pseudo functor F is a homotopy functor.*

Proof. By construction, it suffices to see that for $(X^0, \overline{X}^n, Y^l) \in \mathrm{Bar}^n(\mathbf{A}, \mathbf{A}, \mathbf{C}^l)$, the corresponding $\mathbf{C}^l$-module is corepresentable up to homotopy. But this is ensured by the fact that the DG pseudo action of $\mathbf{A}$ on $\mathbf{C}^l$ was a homotopy action. □

We claim that the resulting 1-morphism $\mathrm{Bar}(\mathbf{A}, \mathbf{A}, \mathbf{C}^l) \to \mathbf{C}^l$ in **DGCat** gives rise to a 1-morphism $F' : \mathbf{A} \underset{\mathbf{A}}{\otimes} \mathbf{C}^l \to \mathbf{C}^l$. Indeed, by Sect. 15.7.3, it suffices to check that the functor

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathbf{I}_{\mathbf{A},\mathbf{A},\mathbf{C}^l}) \to \mathrm{Ho}(\mathbf{C}^l)$$

is 0, which follows from the definition of $\mathrm{Ho}(\mathbf{I}_{\mathbf{A},\mathbf{A},\mathbf{C}^l})$.

17.3.5. We claim now that ι and F' are mutually quasi-inverse 1-morphisms in **DGCat**.

The fact that $F' \circ \iota \simeq \mathrm{Id}_{\mathbf{C}^l}$ is evident from the construction. We will now show that $\mathrm{Ho}(F')$ is the left adjoint of $\mathrm{Ho}(\iota)$. For $Y^l_1 \in \mathrm{Ho}(\mathbf{C}^l)$ and $Z \in \mathrm{Ho}(\mathbf{A} \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)$ consider the map

$$\begin{aligned}(17.4) \quad \mathrm{Hom}_{\mathrm{Ho}(\mathbf{A}\underset{\mathbf{A}}{\otimes}\mathbf{C}^l)}(Z, \mathrm{Ho}(\iota)(Y^l_1)) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C}^l)}(\mathrm{Ho}(F')(Z), \mathrm{Ho}(F') \circ \mathrm{Ho}(\iota)(Y^l_1)) \simeq \\ \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C}^l)}(\mathrm{Ho}(F')(Z), Y^l_1).\end{aligned}$$

We claim that this map is an isomorphism. To check this we can assume that Z comes from an object $(X^0, \underbrace{X^1, ..., X^n}_{n}, Y^l)$ of $\mathrm{Bar}^n(\mathbf{A}, \mathbf{A}, \mathbf{C}^l)$. In this case we will construct a map inverse to (17.4). Namely, consider

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C}^l)}(\mathrm{Ho}(F')(Z), Y^l_1) := H^0\left(\mathrm{Hom}^\bullet_{\mathbf{C}^l}(\text{"}X^0 \otimes X^1 \otimes ... \otimes X^n \otimes Y^l\text{"}, Y^l_1)\right) \to \\ \to H^0\left(\mathrm{Hom}^\bullet_\phi\left((\mathbf{1}_\mathbf{A}, X^0, X^1, ..., X^n, Y^l), (\mathbf{1}_\mathbf{A}, Y^l_1)\right)\right) \hookrightarrow \\ \hookrightarrow H^0\left(\mathrm{Hom}^\bullet_{\mathrm{Bar}(\mathbf{A},\mathbf{A},\mathbf{C}^l)}\left((\mathbf{1}_\mathbf{A}, X^0, X^1, ..., X^n, Y^l), (\mathbf{1}_\mathbf{A}, Y^l_1)\right)\right) \simeq \\ \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Bar}(\mathbf{A},\mathbf{A},\mathbf{C}^l))}\left((\mathbf{1}_\mathbf{A}, X^0, X^1, ..., X^n, Y^l), (\mathbf{1}_\mathbf{A}, Y^l_1)\right),\end{aligned}$$

where ϕ is the map $0 \in \{0\} \mapsto 0 \in \{0, 1, ..., n+1\}$.

We compose the above map with

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Bar}(\mathbf{A},\mathbf{A},\mathbf{C}^l))}\left((\mathbf{1}_\mathbf{A}, X^0, X^1, ..., X^n, Y^l), (\mathbf{1}_\mathbf{A}, Y^l_1)\right) \to \\ \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{A}\underset{\mathbf{A}}{\otimes}\mathbf{C}^l)}\left((\mathbf{1}_\mathbf{A}, X^0, X^1, ..., X^n, Y^l), (\mathbf{1}_\mathbf{A}, Y^l_1)\right) \simeq \\ \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathbf{A}\underset{\mathbf{A}}{\otimes}\mathbf{C}^l)}\left((X^0, X^1, ..., X^n, Y^l), (\mathbf{1}_\mathbf{A}, Y^l_1)\right),\end{aligned}$$

the latter isomorphism is due to the canonical isomorphism between $(\mathbf{1}_\mathbf{A}, X^0, X^1, ..., X^n, Y^l) \in \mathrm{Bar}^{n+1}(\mathbf{A}, \mathbf{A}, \mathbf{C}^l)$ and $(X^0, X^1, ..., X^n, Y^l) \in \mathrm{Bar}^n(\mathbf{A}, \mathbf{A}, \mathbf{C}^l)$ as objects of $\mathrm{Ho}(\mathbf{A} \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)$.

17.3.6. We are now ready to finish the proof of Proposition 17.3.1. Consider the adjunction map

$$\mathrm{Id}_{\mathrm{Ho}(\mathbf{A}\underset{\mathbf{A}}{\otimes}\mathbf{C}^l)} \to \mathrm{Ho}(\iota) \circ \mathrm{Ho}(F'), \tag{17.5}$$

and it suffices to show that this map is an isomorphism. For that it is sufficient to evaluate it on objects Z of the form

$$(X^0, \underbrace{X^1, ..., X^n}_{n}, Y^l) \in \mathrm{Bar}^n(\mathbf{A}, \mathbf{A}, \mathbf{C}^l).$$

By construction, for such an object the map in (17.5) is represented by the following "hut" in $\mathrm{Bar}(\mathbf{A}, \mathbf{A}, \mathbf{C}^l)$:

$$(X^0, X^1, ..., X^n, Y^l) \leftarrow (\mathbf{1_A}, X^0, X^1, ..., X^n, Y^l) \to (\mathbf{1_A}, Y^l_1),$$

where $Y^l_1 \in \mathbf{C}^l$ is an object, whose image in $\mathrm{Ho}(\mathbf{C}^l)$ is isomorphic to $X^0 \otimes X^1 \otimes ... \otimes X^n \otimes Y^l$. Since the arrow $\to$ in the above formula is a quasi-isomorphism, the assertion follows.

17.3.7. We note that the same argument proves the following generalization of Proposition 17.3.1. Namely, let $\mathbf{C}_1$ be a DG category, and let us consider $\mathbf{C}_1 \otimes \mathbf{A}$, which has natural right homotopy module structure. We have:

$$(\mathbf{C}_1 \otimes \mathbf{A}) \underset{\mathbf{A}}{\otimes} \mathbf{C}^l \simeq \mathbf{C}_1 \otimes \mathbf{C}^l. \tag{17.6}$$

As a particular case we obtain that for any two DG categories $\mathbf{C}_1$ and $\mathbf{C}_2$, which can be seen as acted on by the DG monoidal category $\mathbf{Comp}^f_k$, we have:

$$\mathbf{C}_1 \underset{\mathbf{Comp}^f_k}{\otimes} \mathbf{C}_2 \simeq \mathbf{C}_1 \otimes \mathbf{C}_2.$$

17.4. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two homotopy monoidal categories. We have the tautological DG monoidal functors

$$\mathbf{A}_1 \to \mathbf{A}_1 \otimes \mathbf{A}_2 \leftarrow \mathbf{A}_2.$$

Assume that in the situation of Sect. 17.2 the right action of $\mathbf{A} =: \mathbf{A}_2$ on $\mathbf{C}^r$ has been extended to a right action of $\mathbf{A}^o_1 \otimes \mathbf{A}_2$, where the superscript "o" stands for the opposite homotopy monoidal structure.

Then, by Sect. 17.1.1, the DG category $\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}_2, \mathbf{C}^l)$ carries an action of $\mathbf{A}_1$. The subcategory $\mathbf{I}_{\mathbf{C}^r,\mathbf{A}_2,\mathbf{C}^l} \subset \mathrm{Bar}(\mathbf{C}^r, \mathbf{A}_2, \mathbf{C}^r)$ has the property that its image in $\mathrm{Ho}(\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}_2, \mathbf{C}^l))$ is preserved by the action of $\mathrm{Ho}(\mathbf{A}_1)$.

Hence, by Lemma 16.7.4, we obtain that $\mathbf{C}^r \underset{\mathbf{A}_2}{\otimes} \mathbf{C}^l$ is a well-defined object of $\mathbf{DGmod}(\mathbf{A}_1)$

17.4.1. Suppose in addition that $\mathbf{C}^r_1$ is a DG category, equipped with a right homotopy action of $\mathbf{A}_1$. From the constriction we have the following natural 1-isomorphisms in $\mathbf{DGCat}$:

$$\mathrm{Bar}(\mathbf{C}^r_1, \mathbf{A}_1, \mathrm{Bar}(\mathbf{C}^r, \mathbf{A}_2, \mathbf{C}^l)) \simeq \mathrm{Bar}((\mathbf{C}^r_1, \mathbf{A}_1, \mathbf{C}^r), \mathbf{A}_2, \mathbf{C}^l)$$

and

$$\mathbf{C}^r_1 \underset{\mathbf{A}_1}{\otimes} (\mathbf{C}^r \underset{\mathbf{A}_2}{\otimes} \mathbf{C}^l) \simeq (\mathbf{A}^r_1 \underset{\mathbf{A}_1}{\otimes} \mathbf{C}^r) \underset{\mathbf{A}_2}{\otimes} \mathbf{C}^l. \tag{17.7}$$

In other words, we have a well-defined objects of $\mathbf{DGCat}$: $\mathrm{Bar}(\mathbf{C}^r_1, \mathbf{A}_1, \mathbf{C}^r, \mathbf{A}_2, \mathbf{C}^l)$ and

$$\mathbf{C}^r_1 \underset{\mathbf{A}_1}{\otimes} \mathbf{C}^r \underset{\mathbf{A}_2}{\otimes} \mathbf{C}^l.$$

17.4.2. In the above situation let us change the notations slightly and denote $\mathbf{C}_2^l := \mathbf{C}^l$ and $\mathbf{C} := \mathbf{C}^r$. We have:

Lemma 17.4.3. *Under the above circumstances, we have a canonical 1-equivalence in* $\mathbf{DGCat}$*:*

$$\mathbf{C}_1^r \underset{\mathbf{A}_1}{\otimes} \mathbf{C} \underset{\mathbf{A}_2}{\otimes} \mathbf{C}_2^l \simeq (\mathbf{C}_1^r \otimes \mathbf{C}_2^l) \underset{\mathbf{A}_1 \otimes \mathbf{A}_2^o}{\otimes} \mathbf{C}.$$

Proof. Consider the categories

$$\mathrm{Bar}(\mathbf{C}_1^r, \mathbf{A}_1, \mathbf{C}, \mathbf{A}_2, \mathbf{C}_2^l) \text{ and } \mathrm{Bar}\left((\mathbf{C}_1^r \otimes \mathbf{C}_2^l), (\mathbf{A}_1 \otimes \mathbf{A}_2^o), \mathbf{C}\right).$$

We construct DG functors in both directions as follows:

For $\to$ we send

$$(Y_1^r, X_1^1, ..., X_1^n, Y, X_2^1, ..., X_2^m, Y_2^l) \mapsto \\ \left((Y_1^r, Y_2^l), (X_1^1, \mathbf{1}_{\mathbf{A}_2}), ..., (X_1^n, \mathbf{1}_{\mathbf{A}_2}), (\mathbf{1}_{\mathbf{A}_1}, X_2^m), ..., (\mathbf{1}_{\mathbf{A}_1}, X_2^1), Y\right).$$

For $\leftarrow$ we send

$$\left((Y_1^r, Y_2^l), (X_1^1, X_2^1), ..., (X_1^k, X_2^k), Y\right) \mapsto (Y_1^r, X_1^1, ..., X_1^k, Y, X_2^k, ..., X_2^1, Y_2^l).$$

These functors are easily seen to descend to 1-morphisms

$$\mathbf{C}_1^r \underset{\mathbf{A}_1}{\otimes} \mathbf{C} \underset{\mathbf{A}_2}{\otimes} \mathbf{C}_2^l \leftrightarrows (\mathbf{C}_1^r \otimes \mathbf{C}_2^l) \underset{\mathbf{A}_1 \otimes \mathbf{A}_2^o}{\otimes} \mathbf{C},$$

which are mutually quasi-inverse. $\square$

17.5. Induction. Let $F_{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$ be a homotopy monoidal functor between homotopy monoidal categories. Consider the 0-object

$$\mathrm{Res}^{\mathbf{A}_2 \otimes \mathbf{A}_2^o}_{\mathbf{A}_2 \otimes \mathbf{A}_1^o}(\mathbf{A}_2) \in \mathbf{DGmod}(\mathbf{A}_2 \otimes \mathbf{A}_1^o).$$

17.5.1. For a DG category $\mathbf{C}_1^l$ equipped with a homotopy action of $\mathbf{A}_1$ we define

$$\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_1^l) := \mathrm{Res}^{\mathbf{A}_2 \otimes \mathbf{A}_2^o}_{\mathbf{A}_2 \otimes \mathbf{A}_1^o}(\mathbf{A}_2) \underset{\mathbf{A}_1}{\otimes} \mathbf{C}_1^l,$$

as an object of $\mathbf{DGmod}(\mathbf{A}_2)$.

Moreover, for a homotopy functor $\mathbf{C}_1^l \to \widetilde{\mathbf{C}}_1^l$ compatible with an action of $\mathbf{A}_1$, from Sect. 17.2.3 we obtain a homotopy functor

$$\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}_1^l) \to \mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\widetilde{\mathbf{C}}_1^l).$$

In other words, the above construction defines a 2-functor $\mathbf{DGmod}(\mathbf{A}_1) \to \mathbf{DGmod}(\mathbf{A}_2)$.

17.5.2. Let us consider a particular case when $\mathbf{A}_2 = \mathbf{A}_1$. We claim that the above 2-functor of induction is 1-isomorphic to the identity functor. Namely, we claim that the map

$$\iota : \mathbf{C}_1^l \to \mathbf{A}_1 \underset{\mathbf{A}_1}{\otimes} \mathbf{C}_1^l$$

of Sect. 17.3.2 has a natural structure of homotopy functor compatible with the action of $\mathbf{A}_1$. The above map is a 1-isomorphism by Proposition 17.3.1.

17.5.3. Let $\mathbf{C}^l_2$ be a 0-object of $\mathbf{DGmod}(\mathbf{A}_2)$, and consider the corresponding 0-object $\mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_2) \in \mathbf{DGmod}(\mathbf{A}_1)$.

Proposition 17.5.4. *There exists an equivalence of categories*

$$\mathrm{HFunct}_{\mathbf{A}_1}(\mathbf{C}^l_1, \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_2)) \simeq \mathrm{HFunct}_{\mathbf{A}_2}(\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_1), \mathbf{C}^l_2).$$

Proof. We need to construct the adjunction 1-morphisms

$$\mathbf{C}^l_1 \mapsto \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}\left(\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_1)\right) \tag{17.8}$$

and

$$\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}\left(\mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_2)\right) \to \mathbf{C}^l_2. \tag{17.9}$$

The former follows by the functoriality of the tensor product construction from the 1-morphism

$$\mathbf{A}_1 \to \mathrm{Res}^{\mathbf{A}_2\otimes\mathbf{A}^o_1}_{\mathbf{A}_1\otimes\mathbf{A}^o_1}\left(\mathrm{Res}^{\mathbf{A}_2\otimes\mathbf{A}^o_2}_{\mathbf{A}_2\otimes\mathbf{A}^o_1}(\mathbf{A}_2)\right) \simeq \mathrm{Res}^{\mathbf{A}_2\otimes\mathbf{A}^o_2}_{\mathbf{A}_1\otimes\mathbf{A}^o_1}(\mathbf{A}_2) \in \mathbf{DGmod}(\mathbf{A}_1 \otimes \mathbf{A}^o_1),$$

which in turn results from the functor $F_{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$, viewed as compatible with $F_{\mathbf{A}} \otimes F_{\mathbf{A}}$.

The 1-morphism (17.9) results from (17.3):

$$\mathrm{Res}^{\mathbf{A}_2\otimes\mathbf{A}^o_2}_{\mathbf{A}_2\otimes\mathbf{A}^o_1}(\mathbf{A}_2) \underset{\mathbf{A}_1}{\otimes} \mathrm{Res}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_2) \to \mathbf{A}_2 \underset{\mathbf{A}_2}{\otimes} \mathbf{C}^l_2 \simeq \mathbf{C}^l_2.$$

The fact that these 1-morphisms satisfy the adjunction property follows from (17.7).

□

17.5.5. Suppose that $\mathbf{A}_3$ is a third homotopy monoidal category, equipped with a homotopy monoidal functor $\mathbf{A}_2 \to \mathbf{A}_3$.

From Proposition 17.5.4 and Lemma 16.6.8, we obtain a 1-isomorphism of two 2-functors

$$\mathbf{DGmod}(\mathbf{A}_1) \rightrightarrows \mathbf{DGmod}(\mathbf{A}_3) : \mathrm{Ind}^{\mathbf{A}_3}_{\mathbf{A}_2} \circ \mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1} \simeq \mathrm{Ind}^{\mathbf{A}_3}_{\mathbf{A}_1}.$$

18. Adjunctions and tightness

18.1. In this subsection we will fix some notation. Let $\mathbf{A}$, $\mathbf{C}^l$ and $\mathbf{C}^r$ be as in Sect. 17.2. By construction, we have a canonical 1-morphism

$$m_{\mathbf{C}^r,\mathbf{C}^l} : \mathbf{C}^r \otimes \mathbf{C}^l \to \mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l.$$

Consider the corresponding functors

$$m^*_{\mathbf{C}^r,\mathbf{C}^l} : \mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod}) \to \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)^{op}\text{–mod})$$

and its right adjoint

$$(m_{\mathbf{C}^r,\mathbf{C}^l})_* : \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)^{op}\text{–mod}) \to \mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod}).$$

We will also use the notation

$$(Y^r, Y^l) \mapsto Y^r \underset{\mathbf{A}}{\otimes} Y^l := m^*_{\mathbf{C}^r,\mathbf{C}^l}(Y^r, Y^l)$$

for $Y^r \in \mathbf{D}(\mathbf{C}^{r,op}\text{–mod})$ and $Y^l \in \mathbf{D}(\mathbf{C}^{l,op}\text{–mod})$.

A particular case of the above situation is when we have a homotopy monoidal category $\mathbf{A}$ equipped with a homotopy action on a DG category $\mathbf{C}^l$. In particular, we have a homotopy functor

$$act : \mathbf{A} \otimes \mathbf{C}^l \to \mathbf{C}^l.$$

We will consider the corresponding functor

$$act^* : \mathbf{D}((\mathbf{A} \times \mathbf{C}^l)^{op}\text{–mod}) \to \mathbf{D}(\mathbf{C}^{l,op}\text{–mod})$$

and its right adjoint

$$act_* : \mathbf{D}(\mathbf{C}^{l,op}\text{–mod}) \to \mathbf{D}((\mathbf{A} \times \mathbf{C}^l)^{op}\text{–mod}).$$

We will also use the notation

$$(X, Y) \mapsto X \underset{\mathbf{A}}{\otimes} Y := act^*(X, Y)$$

for $X \in \mathbf{D}(\mathbf{A}^{op}\text{–mod})$, $Y \in \mathbf{D}(\mathbf{C}^{l,op}\text{–mod})$.

18.2. Let now $F : \mathbf{C}^l_1 \to \mathbf{C}^l_2$ be a homotopy functor compatible with a homotopy action of $\mathbf{A}$. The following diagram of functors evidently commutes:

$$\begin{array}{ccc}
\mathbf{D}((\mathbf{A} \times \mathbf{C}^l_1)^{op}\text{–mod}) & \xrightarrow{act^*_{\mathbf{C}^l_1}} & \mathbf{D}(\mathbf{C}^{l,op}_1\text{–mod}) \\
{\scriptstyle (\mathrm{Id}_{\mathbf{A}} \times F)^*}\big\downarrow & & {\scriptstyle F^*}\big\downarrow \\
\mathbf{D}((\mathbf{A} \times \mathbf{C}^l_2)^{op}\text{–mod}) & \xrightarrow{act^*_{\mathbf{C}^l_2}} & \mathbf{D}(\mathbf{C}^{l,op}_2\text{–mod}).
\end{array}$$

The next diagram, however, does *not* necessarily commute:

$$(18.1) \qquad \begin{array}{ccc}
\mathbf{D}((\mathbf{A} \times \mathbf{C}^l_1)^{op}\text{–mod}) & \xrightarrow{act^*_{\mathbf{C}^l_1}} & \mathbf{D}(\mathbf{C}^{l,op}_1\text{–mod}) \\
{\scriptstyle (\mathrm{Id}_{\mathbf{A}} \times F)_*}\big\uparrow & & {\scriptstyle F_*}\big\uparrow \\
\mathbf{D}((\mathbf{A} \times \mathbf{C}^l_2)^{op}\text{–mod}) & \xrightarrow{act^*_{\mathbf{C}^l_2}} & \mathbf{D}(\mathbf{C}^{l,op}_2\text{–mod}).
\end{array}$$

However, we have the natural transformation:

$$(18.2) \qquad act^*_{\mathbf{C}^l_1} \circ (\mathrm{Id}_{\mathbf{A}} \times F)_* \to F_* \circ act^*_{\mathbf{C}^l_2} : \mathbf{D}((\mathbf{A} \times \mathbf{C}^l_2)^{op}\text{–mod}) \rightrightarrows \mathbf{D}(\mathbf{C}^{l,op}_1\text{–mod}).$$

We shall say that the functor F is *tight* if (18.2) is an isomorphism.

18.3. Let $\mathbf{A}, \mathbf{C}^r, \mathbf{C}^l_1, \mathbf{C}^l_2, F : \mathbf{C}^l_1 \to \mathbf{C}^l_2$ be as above. The following diagram of functors tautologically commutes:

$$\begin{array}{ccc}
\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l_1)^{op}\text{–mod}) & \xrightarrow{m^*_{\mathbf{C}^r,\mathbf{C}^l_1}} & \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_1)^{op}\text{–mod}) \\
{\scriptstyle (\mathrm{Id}_{\mathbf{C}^r} \times F)^*}\big\downarrow & & {\scriptstyle (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F)^*}\big\downarrow \\
\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l_2)^{op}\text{–mod}) & \xrightarrow{m^*_{\mathbf{C}^r,\mathbf{C}^l_2}} & \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_2)^{op}\text{–mod}).
\end{array}$$

However, the diagram

$$(18.3) \qquad \begin{array}{ccc}
\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l_1)^{op}\text{–mod}) & \xrightarrow{m^*_{\mathbf{C}^r,\mathbf{C}^l_1}} & \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_1)^{op}\text{–mod}) \\
{\scriptstyle (\mathrm{Id}_{\mathbf{C}^r} \times F)_*}\big\uparrow & & {\scriptstyle (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F)_*}\big\uparrow \\
\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l_2)^{op}\text{–mod}) & \xrightarrow{m^*_{\mathbf{C}^r,\mathbf{C}^l_2}} & \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_2)^{op}\text{–mod})
\end{array}$$

does not a priori commute. However, we have a natural transformation:

$$m^*_{\mathbf{C}^r,\mathbf{C}^l_1} \circ (\mathrm{Id}_{\mathbf{C}^r} \times F)_* \to (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F)_* \circ m^*_{\mathbf{C}^r,\mathbf{C}^l_2}. \tag{18.4}$$

Proposition 18.3.1. *The following conditions are equivalent:*

(a) F is tight.

(b) For any $\mathbf{C}^r$ the natural transformation (18.4) *is an isomorphism.*

Proof. For (b)⇒(a) let us take $\mathbf{C}^r = \mathbf{A}$. Then the diagram (18.3) coincides with (18.1).

For (b)⇒(a) consider first the diagram

$$\begin{array}{ccc} \mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l_1)^{op}\text{–mod}) & \xrightarrow{\iota^*_{\mathbf{C}^r,\mathbf{C}^l_1}} & \mathbf{D}(\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l_1)^{op}\text{–mod}) \\ {\scriptstyle (\mathrm{Id}_{\mathbf{C}^r} \times F)_*}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{Bar}(\mathrm{Id}_{\mathbf{C}^r}, \mathrm{Id}_{\mathbf{A}}, F)_*} \\ \mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l_2)^{op}\text{–mod}) & \xrightarrow{{\iota^*_{\mathbf{C}^r,\mathbf{C}^l_2}}^*} & \mathbf{D}(\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l_2)^{op}\text{–mod}), \end{array}$$

where $\iota_{\mathbf{C}^r,\mathbf{C}^l}$ denotes the canonical functor $\mathbf{C}^r \otimes \mathbf{C}^l \to \mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l_1)$. It is easy to see that the above diagram commutes.

Hence, it is enough to show that the following diagram

$$\begin{array}{ccc} \mathbf{D}(\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^l_1)^{op}\text{–mod}) & \longrightarrow & \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_1)^{op}\text{–mod}) \\ {\scriptstyle \mathrm{Bar}(\mathrm{Id}_{\mathbf{C}^r}, \mathbf{A}, F)_*}\Big\uparrow & & \Big\uparrow{\scriptstyle (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\times} F)_*} \\ \mathbf{D}(\mathrm{Bar}(\mathbf{C}^r, \mathbf{A}, \mathbf{C}^2_1)^{op}\text{–mod}) & \longrightarrow & \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_2)^{op}\text{–mod}) \end{array}$$

is commutative, where the horizontal arrows are the ind-limits of the tautological projections of the "Bar" categories to the tensor products.

The latter is equivalent to the fact that the functor $(\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\times} F)_*$ sends $\underset{\to}{\mathbf{I}}_{\mathbf{C}^r,\mathbf{A},\mathbf{C}^l_1}$ to $\underset{\to}{\mathbf{I}}_{\mathbf{C}^r,\mathbf{A},\mathbf{C}^l_2}$, which in turn follows from the tightness condition.

□

Corollary 18.3.2. *Let $\mathbf{C}^r_i$, $\mathbf{C}^l_i$, $i = 1, 2$ be two pairs of categories as above, and $F^r : \mathbf{C}^r_1 \to \mathbf{C}^r_2$, $F^l : \mathbf{C}^l_1 \to \mathbf{C}^l_2$ be homotopy functors. Assume that the functor F^r is tight. Then the following diagram of functors commutes:*

$$\begin{array}{ccc} \mathbf{D}\left((\mathbf{C}^r_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_1)^{op}\text{–mod}\right) & \xrightarrow{(\mathrm{Id}_{\mathbf{C}^r_1} \underset{\mathbf{A}}{\otimes} F^l)^*} & \mathbf{D}\left((\mathbf{C}^r_1 \underset{\mathbf{A}}{\otimes} \mathbf{C}^2_1)^{op}\text{–mod}\right) \\ {\scriptstyle (F^r \underset{\mathbf{A}}{\otimes} \mathrm{Id}_{\mathbf{C}^l_1})_*}\Big\uparrow & & \Big\uparrow{\scriptstyle (F^r \underset{\mathbf{A}}{\otimes} \mathrm{Id}_{\mathbf{C}^l_2})_*} \\ \mathbf{D}\left((\mathbf{C}^r_2 \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_1)^{op}\text{–mod}\right) & \xrightarrow{(\mathrm{Id}_{\mathbf{C}^r_2} \underset{\mathbf{A}}{\otimes} F^l)^*} & \mathbf{D}\left((\mathbf{C}^r_2 \underset{\mathbf{A}}{\otimes} \mathbf{C}^l_2)^{op}\text{–mod}\right). \end{array}$$

18.4. Let us consider some examples of the above situation.

18.4.1. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two homotopy monoidal categories and $F_{\mathbf{A}}:\mathbf{A}_1\to\mathbf{A}_2$ a homotopy monoidal functor between them. We say that $F_{\mathbf{A}}$ is tight if the canonical 1-morphism $\mathbf{A}_1\to\mathrm{Res}^{\mathbf{A}_2^o}_{\mathbf{A}_1^o}(\mathbf{A}_2^o)$ is tight as a functor between categories, acted on the *right* by $\mathbf{A}_1$. Recall also that as plain DG categories $\mathrm{Res}^{\mathbf{A}_2^o}_{\mathbf{A}_1^o}(\mathbf{A}_2^o)\simeq\mathbf{A}_2^o$.

Consider the functors

$$F_{\mathbf{A}}^*:\mathbf{D}(\mathbf{A}_1^{op}\text{–mod})\to\mathbf{D}(\mathbf{A}_2^{op}\text{–mod}) \text{ and } F_{\mathbf{A}*}:\mathbf{D}(\mathbf{A}_2^{op}\text{–mod})\to\mathbf{D}(\mathbf{A}_1^{op}\text{–mod}).$$

From Proposition 18.3.1 we obtain the following:

Corollary 18.4.2. *Let $\mathbf{C}^l_1$ be a category equipped with a homotopy action of $\mathbf{A}_1$ on the left. Assume that $F_{\mathbf{A}}$ is tight.*

(1) The following diagram of functors is commutative:

$$\begin{array}{ccc}
\mathbf{D}((\mathbf{A}_1\otimes\mathbf{C}^l_1)^{op}\text{–mod}) & \xrightarrow{act^*_{\mathbf{C}^l}} & \mathbf{D}(\mathbf{C}^{l,op}_1\text{–mod}) \\
{\scriptstyle (F_{\mathbf{A}}\times\mathrm{Id}_{\mathbf{C}^l_1})_*}\uparrow & & \uparrow{\scriptstyle (F_{\mathbf{C}^l_1})_*} \\
\mathbf{D}((\mathbf{A}_2\otimes\mathbf{C}^l_1)^{op}\text{–mod}) & \xrightarrow{act^*_{\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_1)}\circ(\mathrm{Id}_{\mathbf{A}_2}\otimes F_{\mathbf{C}^l_1})} & \mathbf{D}((\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_1))^{op}\text{–mod}),
\end{array}$$

where $F_{\mathbf{C}^l_1}$ denotes the canonical 1-morphism $\mathbf{C}^l_1\to\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_1)$.

(2) For $Y_1,Y_2\in\mathbf{C}^l_1$, $X\in\mathbf{A}_2$,

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(\mathbf{C}^l_1))}(F^*_{\mathbf{C}^l_1}(Y_1),X\underset{\mathbf{A}_2}{\otimes}F^*_{\mathbf{C}^l_1}(Y_2))\simeq\mathrm{Hom}_{\mathbf{D}(\mathbf{C}^{l,op}_1\text{–mod})}(Y_1,F_{\mathbf{A}*}(X)\underset{\mathbf{A}_1}{\otimes}Y_2).$$

Corollary 18.4.3. *Let $F_{\mathbf{A}}:\mathbf{A}_1\to\mathbf{A}_2$, be as in Corollary 18.4.2. Let $'\mathbf{C}^l_1,''\mathbf{C}^l_1$ be DG categories equipped with a homotopy action of $\mathbf{A}_1$. Let $F:{'\mathbf{C}^l_1}\to{''\mathbf{C}^l_1}$ be a homotopy functor. Then the diagram of functors*

$$\begin{array}{ccc}
\mathbf{D}((\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}({'\mathbf{C}^l_1}))^{op}\text{–mod}) & \xrightarrow{(\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}(F))^*} & \mathbf{D}((\mathrm{Ind}^{\mathbf{A}_2}_{\mathbf{A}_1}({''\mathbf{C}^l_1}))^{op}\text{–mod}) \\
{\scriptstyle (F_{'\mathbf{C}^l_1})_*}\downarrow & & \downarrow{\scriptstyle (F_{''\mathbf{C}^l_2})_*} \\
\mathbf{D}(({'\mathbf{C}^l_1})^{op}\text{–mod}) & \xrightarrow{F^*} & \mathbf{D}(({''\mathbf{C}^l_1})^{op}\text{–mod})
\end{array}$$

commutes.

18.4.4. We will say that $\mathbf{A}$ *has a tight diagonal* if the tensor product homotopy functor $m_{\mathbf{A}}:\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}$, considered as a homotopy functor between categories endowed with a homotopy action of $\mathbf{A}\otimes\mathbf{A}^o$ is tight.

Assume that $\mathbf{A}$ has a tight diagonal. We obtain that for any $\mathbf{C}^l$ and $\mathbf{C}^r$ as above the following diagram of functors commutes:

$$\begin{array}{ccc}
\mathbf{D}\left((\mathbf{C}^r\otimes\mathbf{C}^l\otimes\mathbf{A}\otimes\mathbf{A})^{op}\text{–mod}\right) & \xrightarrow{(act_{\mathbf{C}^r}\otimes act_{\mathbf{C}^l})^*} & \mathbf{D}((\mathbf{C}^r\otimes\mathbf{C}^l)^{op}\text{–mod}) \\
{\scriptstyle (\mathrm{Id}_{\mathbf{C}^r\times\mathbf{C}^l}\otimes m_{\mathbf{A}})_*}\uparrow & & \uparrow{\scriptstyle (m_{\mathbf{C}^r,\mathbf{C}^l})_*} \\
\mathbf{D}((\mathbf{C}^r\otimes\mathbf{C}^l\otimes\mathbf{A})^{op}\text{–mod}) & \longrightarrow & \mathbf{D}((\mathbf{C}^r\underset{\mathbf{A}}{\otimes}\mathbf{C}^l)^{op}\text{–mod}),
\end{array}$$

where the lower horizontal arrow is induced by either of the two compositions

$$\begin{tikzcd}
\mathbf{C}^r \otimes \mathbf{A} \otimes \mathbf{C}^l \arrow[r, "\mathrm{Id}_{\mathbf{C}^r} \otimes act_{\mathbf{C}^l}"] \arrow[d, "act_{\mathbf{C}^r} \otimes \mathrm{Id}_{\mathbf{C}^l}"'] & \mathbf{C}^r \otimes \mathbf{C}^l \arrow[d, "m_{\mathbf{C}^r,\mathbf{C}^l}"] \\
\mathbf{C}^r \otimes \mathbf{C}^l \arrow[r, "m_{\mathbf{C}^r,\mathbf{C}^l}"] & \mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l.
\end{tikzcd}$$

Let $\mathrm{Diag}_{\mathbf{A}}$ denote the object of $\mathbf{D}((\mathbf{A} \otimes \mathbf{A})^{op}\text{–mod})$ equal to $(m_{\mathbf{A}})_*(\mathbf{1}_{\mathbf{A}})$. We obtain:

Corollary 18.4.5. *If* $\mathbf{A}$ *has a tight diagonal, for* $Y_1^l, Y_2^l \in \mathbf{C}^l$, $Y_1^r, Y_2^r \in \mathbf{C}^r$ *we have:*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)}\left(Y_1^r \underset{\mathbf{A}}{\otimes} Y_1^l, Y_2^r \underset{\mathbf{A}}{\otimes} Y_2^l\right) \simeq$$

$$\simeq \mathrm{Hom}_{\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod})}\left((Y_1^r, Y_1^l), (\mathrm{Diag}_{\mathbf{A}}) \underset{\mathbf{A} \otimes \mathbf{A}}{\otimes} (Y_2^r, Y_2^l)\right).$$

18.4.6. Let us assume now that in the situation of Proposition 18.3.1, the functor F_* sends $\mathrm{Ho}(\mathbf{C}_2^l)$, regarded a full subcategory of $\mathbf{D}(\mathbf{C}_2^{l,op}\text{–mod})$ to $\mathrm{Ho}(\mathbf{C}_1^l)$, regarded as a subcategory of $\mathbf{D}(\mathbf{C}_1^{l,op}\text{–mod})$. By Lemma 16.7.4, the resulting functor, denoted $G : \mathrm{Ho}(\mathbf{C}_2^l) \to \mathrm{Ho}(\mathbf{C}_1^l)$ naturally lifts to a 1-morphism in $\mathbf{DGmod}(\mathbf{A})$.

We can then consider the functor

$$(\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} G) : \mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_2^l \to \mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l,$$

and its ind-extension

$$(\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} G)^* : \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_2^l)^{op}\text{–mod}) \to \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l)^{op}\text{–mod}).$$

Proposition 18.4.7. *Suppose* F *is tight. Then at the triangulated level, the functor* $(\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} G)^*$ *is the right adjoint of* $(\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F)^*$, *i.e., we have an isomorphism of functors at the triangulated level:*

$$(\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} G)^* \simeq (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F)_*.$$

.

Proof. We have an evidently defined 2-morphism $\mathrm{Id}_{\underset{\to}{\mathbf{C}_1^l}} \to F_* \circ F^*$ in $\mathbf{DGmod}(\mathbf{A})$, and by Sect. 16.7.2, also a 2-morphism $\mathrm{Id}_{\mathbf{C}_1^l} \to G \circ F$. The latter gives rise to a 2-morphism

$$\mathrm{Id}_{\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l} \to (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} G) \circ (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F),$$

and to a morphism

$$\mathrm{Id}_{\mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l)^{op}\text{–mod})} \to (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} G)^* \circ (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F)^*,$$

Thus, for $\widetilde{X_1} \in \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l)^{op}\text{–mod})$ and $\widetilde{X_2} \in \mathbf{D}((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l)^{op}\text{–mod})$ we obtain a map

$$\mathrm{Hom}((\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} F)^*(\widetilde{X_1}), \widetilde{X_2}) \to \mathrm{Hom}(\widetilde{X_1}, (\mathrm{Id}_{\mathbf{C}^r} \underset{\mathbf{A}}{\otimes} G)^*(\widetilde{X_2}),$$

and we have to show that the latter is an isomorphism.

With no restriction of generality we can assume that

$$\widetilde{X_1} \in \mathrm{Ho}(\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_1^l) \text{ and } \widetilde{X_2} \in \mathrm{Ho}(\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}_2^l).$$

Further, we can assume that $\widetilde{X_2}$ is of the form $X^r \underset{\mathbf{A}}{\otimes} X_2$ with $X^r \in \mathrm{Ho}(\mathbf{C}^r)$ and $X_2 \in \mathrm{Ho}(\mathbf{C}^l_2)$.

In the latter case the desired isomorphism follows from Proposition 18.3.1.

□

18.5. Rigidity. Here is a way to insure that any functor F is tight. We shall say that $\mathbf{A}$ is rigid of the triangulated monoidal category $\mathrm{Ho}(\mathbf{A})$ has this property. I.e., if there exists a self anti-equivalence of $\mathrm{Ho}(\mathbf{A}) : X \mapsto X^\vee$, and maps $\mathbf{1_A} \to X \otimes X^\vee$ and $X^\vee \otimes X \to \mathbf{1_A}$ such that the two compositions

$$X \to X \underset{\mathbf{A}}{\otimes} X^\vee \underset{\mathbf{A}}{\otimes} X \to X \text{ and } X^\vee \to X^\vee \underset{\mathbf{A}}{\otimes} X \underset{\mathbf{A}}{\otimes} X^\vee \to X^\vee$$

are the identity maps in $\mathrm{Ho}(\mathbf{A})$.

Lemma 18.5.1. *For $\mathbf{C}$ endowed with a homotopy action of $\mathbf{A}$ and $X \in \mathbf{A}$ as above the functor $act^*_{\mathbf{C}}(X, ?)$ is the right adjoint of $act^*_{\mathbf{C}}(X^\vee, ?)$.*

We have:

Proposition 18.5.2. *If $\mathbf{A}$ is rigid, then any functor $F : \mathbf{C}^l_1 \to \mathbf{C}^l_2$ is tight.*

Proof. It is sufficient to show that the map

$$X \underset{\mathbf{A}}{\otimes} F_*(Y) \to F_*(X \underset{\mathbf{A}}{\otimes} Y)$$

is an isomorphism for any $X \in \mathrm{Ho}(\mathbf{A})$, $Y \in \mathbf{D}(\mathbf{C}^{l,op}_2\text{–mod})$.

In this case, we we will construct the inverse map to the one above. By Lemma 18.5.1, constructing a map

$$F_*(X \underset{\mathbf{A}}{\otimes} Y) \to X \underset{\mathbf{A}}{\otimes} F_*(Y)$$

is equivalent to constructing a map

$$X^\vee \underset{\mathbf{A}}{\otimes} F_*(X \underset{\mathbf{A}}{\otimes} Y) \to F_*(Y).$$

The latter equals the composition

$$X^\vee \underset{\mathbf{A}}{\otimes} F_*(X \underset{\mathbf{A}}{\otimes} Y) \to F_*\left(X^\vee \underset{\mathbf{A}}{\otimes} (X \underset{\mathbf{A}}{\otimes} Y)\right) \simeq F_*\left((X^\vee \underset{\mathbf{A}}{\otimes} X) \underset{\mathbf{A}}{\otimes} Y\right) \to F_*(Y).$$

□

19. t-STRUCTURES: A REMINDER

19.1. Recall the notion of t-structure on a triangulated category. Given a t-structure on $\mathbf{D}$ we will use the standard notations:

$$\mathbf{D}^+ := \underset{k}{\cup} \mathbf{D}^{\geq k},\ \mathbf{D}^- := \underset{k}{\cup} \mathbf{D}^{\leq k},\ \mathbf{D}^b := \mathbf{D}^+ \cap \mathbf{D}^-.$$

19.1.1. If $\mathbf{D}$ is a triangulated category equipped with a t-structure, and $\mathbf{D}' \subset \mathbf{D}$ is a full triangulated subcategory, we will say that $\mathbf{D}'$ is compatible with the t-structure, if it is preserved by the truncation functors. In this case, $\mathbf{D}'$ inherits a t-structure: it is the unique t-structure for which the inclusion functor is exact.

19.1.2. Let $\mathbf{D}$ be a co-complete triangulated category equipped with a DG model and a t-structure. We say that the t-structure is compatible with colimits if for for every homotopy I-object X_I with $X_i \in \mathbf{D}^{\leq 0}$ (resp., $X_i \in \mathbf{D}^{\geq 0}$) for all $i \in I$, we have $hocolim(X_I) \in \mathbf{D}^{\leq 0}$ (resp., $hocolim(X_I) \in \mathbf{D}^{\geq 0}$).

19.1.3. The following assertion generalizes Lemma 15.8.10. Suppose that $\mathbf{D}$ is co-complete, is equipped with a DG model and a t-structure. Assume that the t-structure is compatible with colimits.

Let $\mathbf{D}' \subset \mathbf{D}$ be triangulated subcategory that generates $\mathbf{D}$. Note that we are not assuming that $\mathbf{D}'$ is compatible with the t-structure.

Lemma 19.1.4. *Under the above circumstances every object of $\mathbf{D}^{\leq 0}$ (resp., $\mathbf{D}^{\geq 0}$) can be represented as a homotopy colimit of a homotopy I-object X_I such that the image of every X_i in $\mathbf{D}$ is of the form $\tau^{\leq 0}(X_i')$ (resp., $\tau^{\geq 0}(X_i')$) for $X_i' \in \mathbf{D}'$.*

19.2. Let us recall a general construction of t-structures on a co-complete triangulated category. This construction was explained to us by J. Lurie.

Let $\mathbf{D}$ be a co-complete triangulated category. Let $X_a \in \mathbf{D}$ be a collection of compact objects, indexed by some set A. [6]

Lemma 19.2.1. *Under the above circumstances, there exists a unique t-structure on $\mathbf{D}$, such that $\mathbf{D}^{>0}$ consists of all objects Y such that $\mathrm{Hom}(X_a[k], Y) = 0$ for all $k \geq 0$. In this case $\mathbf{D}^{\leq 0}$ is the minimal subcategory of $\mathbf{D}$, stable under extensions and direct sums that contains the objects $X_a[k]$, $k \geq 0$.*

We will call t-structures that arise by the procedure of the above lemma *compactly generated*. Tautologically, we have:

Lemma 19.2.2. *Let $\mathbf{D}$ be a co-complete triangulated category equipped with a t-structure, and with $\mathbf{D}^c$ (the subcategory of compact objects) essentially small. Then the t-structure is compactly generated if and only if*

$$X \in \mathbf{D}^{>0} \Leftrightarrow \mathrm{Hom}(X', X) = 0,\ \forall X \in \mathbf{D}^c \cap \mathbf{D}^{\leq 0}.$$

The following results immediately from Lemma 15.8.8:

Lemma 19.2.3. *Let $\mathbf{D}$ be a co-complete triangulated category equipped with a DG model and a compactly generated t-structure. Then this t-structure is compatible with colimits.*

19.3. Let us recall that whenever we have a triangulated category $\mathbf{D}$ equipped with a t-structure and a DG model, we have an exact functor:

$$\mathbf{D}^b(\overset{\circ}{\mathbf{C}}) \to \mathbf{D}^b, \tag{19.1}$$

where $\overset{\circ}{\mathbf{C}} := \mathrm{Heart}(\mathbf{D})$, equipped with a DG model.

19.3.1. Let us recall the construction. Let $\mathbf{C}$ be a DG category such that $\mathbf{D} = \mathrm{Ho}(\mathbf{C})$. The DG model for $\mathbf{D}^b(\mathrm{Heart}(\mathbf{D}))$ is the standard one, resulting from the identification $\mathrm{Ho}(\mathbf{C}^b(\mathrm{Heart}(\mathbf{D})))/\,\mathrm{Ho}(\mathbf{C}^b_{acylc}(\mathrm{Heart}(\mathbf{D})))$ (see Sect. 15.7.2).

Consider the following DG category, denoted $\mathbf{C}^{double}$. Its objects are finite diagrams

$$\{X^{-\infty} = X^{\geq -n} \supset X^{\geq -n+1} \supset ... \supset X^{\geq m} \supset X^{\geq m+1} = 0\}$$

for some $m, n \in \mathbb{Z}^{\geq 0}$, $X^{\geq i} \in \mathbf{C}$, such that for each i we are given a splitting

$$X^{\geq i} \simeq X^{\geq i+1} \oplus X^i,\ X^i \in \mathbf{C}$$

[6] As was explained to us by J. Lurie, for what follows one does not in fact need to require that X_a be compact, if some general set-theoretic assumption on $\mathbf{D}$ is satisfied.

as functors $\mathbf{C}^{op} \to \mathrm{Vect}_k^{\mathbb{Z}}$ (where $\mathrm{Vect}^{\mathbb{Z}}$ denotes the category of $\mathbb{Z}$-graded vector spaces). We require that the image of each X^i in $\mathrm{Ho}(\mathbf{C})$ belong to $\mathrm{Heart}(\mathbf{D})[-i]$. Morphisms between $X^\bullet$ and $Y^\bullet$ are compatible families of maps $X^{\geq i} \to Y^{\geq i}$.

We have the evident forgetful functor $\mathbf{C}^{double} \to \mathbf{C}$. In addition, the boundary map for the t-structure defines a DG functor $\mathbf{C}^{double} \to \mathbf{C}^b(\mathrm{Heart}(\mathbf{D}))$.

Let $\mathbf{C}^{double}_{acycl}$ be the preimage of $\mathbf{C}_{acycl}(\mathrm{Heart}(\mathbf{D}))$ under $\mathbf{C}^{double} \to \mathbf{C}^b(\mathrm{Heart}(\mathbf{D}))$.

Lemma 19.3.2.
(a) *The functor*

$$\mathrm{Ho}\left(\mathbf{C}^{double}\right) / \mathrm{Ho}\left(\mathbf{C}^{double}_{acycl}\right) \to \mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Heart}(\mathbf{D}))\right) / \mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathrm{Heart}(\mathbf{D}))\right) \simeq \mathbf{D}^b(\mathrm{Heart}(\mathbf{D}))$$

is an equivalence.
(b) *The functor* $\mathrm{Ho}\left(\mathbf{C}^{double}\right) \to \mathrm{Ho}(\mathbf{C})$ *factors as*

$$\mathrm{Ho}\left(\mathbf{C}^{double}\right) \to \mathrm{Ho}\left(\mathbf{C}^{double}\right) / \mathrm{Ho}\left(\mathbf{C}^{double}_{acycl}\right) \to \mathrm{Ho}(\mathbf{C}).$$

The desired 1-morphism is obtained from the diagram

$$\mathbf{C}^b(\mathrm{Heart}(\mathbf{D}))/\mathbf{C}^b_{acycl}(\mathrm{Heart}(\mathbf{D})) \overset{\sim}{\leftarrow} \mathbf{C}^{double}/\mathbf{C}^{double}_{acycl} \to \mathbf{C}.$$

19.3.3. Let us recall the necessary and sufficient conditions for the functor of (19.1) to be fully faithful:

Lemma 19.3.4. *The following conditions are equivalent:*
(1) The functor (19.1) *is fully faithful.*
(2) The functor (19.1) *is equivalence.*
(3) The functor (19.1) *induces an isomorphism* $\mathrm{Ext}^i_{\overset{\circ}{\mathbf{C}}}(X', X) \to \mathrm{Hom}_{\mathbf{D}}(X', X[i])$ *for any* $X, X' \in \overset{\circ}{\mathbf{C}}$.
(4) For any X, X' *as above,* $i > 0$ *and an element* $\alpha \in \mathrm{Hom}_{\mathbf{D}}(X', X[i])$ *there exists a surjection* $X'_1 \twoheadrightarrow X'$ *in* $\overset{\circ}{\mathbf{C}}$, *such that the image of* α *in* $\mathrm{Hom}_{\mathbf{D}}(X'_1, X[i])$ *vanishes.*
(5) For any X, X' *as above,* $i > 0$ *and an element* $\alpha \in \mathrm{Hom}_{\mathbf{D}}(X', X[i])$ *there exists an injection* $X \hookrightarrow X_1$ *in* $\overset{\circ}{\mathbf{C}}$, *such that the image of* α *in* $\mathrm{Hom}_{\mathbf{D}}(X, X_1[i])$ *vanishes.*

19.4. In this subsection we will prove Proposition 14.1.1. First, we note that conditions Cat(i,ii) and Funct(i,ii) imply that T is fully faithful.

19.4.1. T *is left-exact.* Let X be an object of $\mathbf{D}_1^{\geq 0}$. Using conditions Cat(i,ii) for $\mathbf{D}_1$, Cat(a) for $\mathbf{D}_1$ and $\mathbf{D}_2$, Lemma 19.1.4 for $\mathbf{D}_1$, Funct(i) and Lemma 15.8.7 we conclude that it is sufficient to show that $\mathsf{T}(\tau^{\geq 0}(X))$ is in $\mathbf{D}_2^{\geq 0}$ for $X \in \mathbf{D}_1^c$.

By condition Funct(c), we know that for X as above, $\mathsf{T}(\tau^{\geq 0}(X)) \in \mathbf{D}_2^+$. Let k be the minimal integer such that $H^k\left(\mathsf{T}(\tau^{\geq 0}(X))\right) \neq 0$. Assume by contradiction that $k < 0$.

By Funct(d), we can find $X' \in \mathbf{D}_1^{\leq 0}$ with a non-zero map $\mathsf{T}(X') \to H^k\left(\mathsf{T}(\tau^{\geq 0}(X))\right)$. I.e., we obtain a non-zero map

$$\mathsf{T}(X'[-k]) \to \mathsf{T}(\tau^{\geq 0}(X)).$$

However, this contradicts the fact that T is fully faithful.

19.4.2. T *induces an equivalence* $\mathbf{D}_1^+ \to \mathbf{D}_2^+$. It is easy to see that T admits a right adjoint. We shall denote it by S. Fully faithfulness of T means that the composition

$$\mathrm{Id}_{\mathbf{D}_2} \to \mathsf{S} \circ \mathsf{T}$$

is an isomorphism. Hence, it is sufficient to show that for $Y \in \mathbf{D}_2^+$, the adjunction map $\mathsf{T}(\mathsf{S}(Y)) \to Y$ is an isomorphism.

Being a right adjoint of a right-exact functor (condition Funct(b)), S is left-exact. In particular, it maps $\mathbf{D}_2^+$ to $\mathbf{D}_1^+$. We have: $\mathsf{T}(\mathsf{S}(Y)) \in \mathbf{D}_2^+$, by the already established left-exactness of T. Hence, $Y' = \mathrm{Cone}(\mathsf{T}(\mathsf{S}(Y)) \to Y) \in \mathbf{D}_2^+$. By the fully faithfulness of T, we have $\mathsf{S}(Y') = 0$. However, condition Funct(d) implies that S is conservative on $\mathbf{D}_2^+$. Hence, $Y' = 0$.

19.4.3. *End of the proof.* It remains to show that T is essentially surjective. This is equivalent to the fact that the image of T generates $\mathbf{D}_2$. Using Cat(b), it is enough to show that $\mathbf{D}_2^+$ is in the image of Ψ. However, this has been established above.

20. Tensor products and t-structures

20.1. t-structure on the tensor product. Let $\mathbf{A}$ be a DG category equipped with a homotopy monoidal structure. Set $\mathbf{D}_\mathbf{A} := \mathrm{Ho}(\underset{\to}{\mathbf{A}})$. We assume that $\mathbf{D}_\mathbf{A}$ is equipped with a t-structure. We assume that this t-structure is compactly generated (see Sect. 19.2), and the tensor product functor

$$\mathbf{D}_\mathbf{A} \times \mathbf{D}_\mathbf{A} \to \mathbf{D}_\mathbf{A}$$

is right-exact, and $\mathbf{1}_\mathbf{A} \in \mathrm{Heart}(\mathbf{D}_\mathbf{A})$.

20.1.1. Let $\mathbf{C}^l$ and $\mathbf{C}^r$ be DG categories, equipped with homotopy actions of $\mathbf{A}$ on the left and on the right, respectively.

Let $\mathbf{D}^l := \mathrm{Ho}(\underset{\to}{\mathbf{C}^l})$ and $\mathbf{D}^r := \mathrm{Ho}(\underset{\to}{\mathbf{C}^r})$ be equipped with t-structures. We are assume that these t-structures are compactly generated.

We assume also that the action functors

$$act^*_{\mathbf{C}^l} : \mathbf{D}_\mathbf{A} \times \mathbf{D}^l \to \mathbf{D}^l \text{ and } act^*_{\mathbf{C}^r} : \mathbf{D}^r \times \mathbf{D}_\mathbf{A} \to \mathbf{D}^r$$

are right-exact.

20.1.2. Consider the DG category $\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l$, and the triangulated category

$$\mathbf{D}^t := \mathbf{D}\left((\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)^{op}\text{–mod} \right) =: \mathrm{Ho}\left(\underset{\longrightarrow}{\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l} \right).$$

We define a t-structure on $\mathbf{D}^t$ by the procedure of Lemma 19.2.1 with $\mathbf{D}^{t,\leq 0}$ being generated by objects of the form $Y^r \underset{\mathbf{A}}{\otimes} Y^l \in \mathrm{Ho}(\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)$ with $Y^r \in \mathrm{Ho}(\mathbf{C}^r) \cap \mathbf{D}^{r,\leq 0}$ and $Y^l \in \mathrm{Ho}(\mathbf{C}^l) \cap \mathbf{D}^{l,\leq 0}$.

20.1.3. Consider the particular case when $\mathbf{C}^r = \mathbf{A}$. By Proposition 17.3.1, we have an equivalence $\mathbf{D}^t \simeq \mathbf{D}^l$, and by construction the t-structure on the LHS equals the given one on the RHS.

20.1.4. The following question appears to be natural, but we do not know how to answer it:

Assume that the t-structures on $\mathbf{D}_{\mathbf{A}}$, $\mathbf{D}^l$ and $\mathbf{D}^r$ are compatible with the subcategories $\mathbf{D}^{l,c} \simeq (\mathrm{Ho}(\mathbf{C}^l))^{Kar}$ and $\mathbf{D}^{r,c} \simeq (\mathrm{Ho}(\mathbf{C}^r))^{Kar}$. Under what conditions is the above t-structure on $\mathbf{D}^t$ compatible with $\mathbf{D}^{t,c} \simeq \mathrm{Ho}(\mathbf{C}^r \underset{\mathbf{A}}{\otimes} \mathbf{C}^l)^{Kar}$?

20.2. Flatness. Let $\mathbf{A}$, $\mathbf{C}^l$, $\mathbf{C}^r$ be as above. We will say that an object $Y^l \in \mathrm{Heart}(\mathbf{D}^l)$ is $\mathbf{A}$-flat if the functor

$$\mathbf{D}_{\mathbf{A}} \to \mathbf{D}^l : X \mapsto X \underset{\mathbf{A}}{\otimes} Y \tag{20.1}$$

is exact. (A priori, this functor is right-exact.)

Proposition 20.2.1. *Assume that $\mathbf{A}$ has a tight diagonal. Let $Y^l \in \mathrm{Heart}(\mathbf{D}^l)$ be flat. Then the functor*

$$\mathbf{D}^r \to \mathbf{D}^t : Y^r \mapsto Y^r \underset{\mathbf{A}}{\otimes} Y^l$$

is exact.

Proof. A priori, the functor in question is right-exact. To prove the left-exactness, we have to show that for $Y^r \in \mathbf{D}^{r,\geq 0}$, $Y_1^r \in \mathbf{D}^{r,<0}$ and $Y_1^l \in \mathbf{D}^{l,<0}$, we have

$$\mathrm{Hom}_{\mathbf{D}^t}(Y_1^r \underset{\mathbf{A}}{\otimes} Y_1^l, Y^r \underset{\mathbf{A}}{\otimes} Y^l) = 0.$$

By Corollary 18.4.5, the LHS of the above expression can be rewritten as

$$\mathrm{Hom}_{\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod})}\left((Y_1^r, Y_1^l), \mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y^r, Y^l)\right).$$

Thus, it is sufficient to show that the object

$$\mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y^r, Y^l) \in \mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod})$$

is ≥ 0.

By the flatness assumption on Y^l, the functor

$$(\mathrm{Id} \times act(?, Y^l))^* : \mathbf{D}((\mathbf{C}^r \times \mathbf{A})^{op}\text{–mod}) \to \mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod}),$$

given by $(Y^r, X) \mapsto (Y^r, X \underset{\mathbf{A}}{\otimes} Y^l)$, is exact. We have

$$\mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y^r, Y^l) \simeq (\mathrm{Id} \times act(?, Y^l))^* \left(\mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y^r, \mathbf{1}_{\mathbf{A}})\right).$$

Thus, it remains to see that

$$\mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y^r, \mathbf{1}_{\mathbf{A}}) \in \mathbf{D}((\mathbf{C}^r \times \mathbf{A})^{op}\text{–mod})$$

is ≥ 0.

However, since $\mathbf{A}$ has a tight diagonal, the latter object is isomorphic to $(act_{\mathbf{C}^r})_*(Y^r)$, and the assertion follows.

□

20.3. Base change. We will mostly consider a particular case of the above situation, where $\mathbf{C}^r = \mathrm{Res}_{\mathbf{A}}^{\mathbf{A}_1}(\mathbf{A}_1)$, where $\mathbf{A}_1$ is another homotopy monoidal category, equipped with a homotopy monoidal functor $F : \mathbf{A} \to \mathbf{A}_1$. We will assume that the t-structure on $\mathbf{D}_{\mathbf{A}_1}$ satisfies the assumptions of Sect. 20.1. In particular, the functor F is right-exact.

20.3.1. Denote $\mathbf{C}^l_1 := \mathrm{Ind}_{\mathbf{A}}^{\mathbf{A}_1}(\mathbf{C}^l)$. We will call this category the base change of $\mathbf{C}^l$ with respect to F. We will call the above t-structure on $\mathbf{D}^l_1 := \mathbf{D}(\mathbf{C}^{l,op}_1\text{–mod})$ the base-changed t-structure.

The following results from the definitions:

Lemma 20.3.2. *The functor*

$$(F_{\mathbf{C}^l})^* : \mathbf{D}^l \to \mathbf{D}^l_1$$

is right-exact.

Hence, by adjunction, the functor

$$(F_{\mathbf{C}^l})_* : \mathbf{D}^l_1 \to \mathbf{D}^l$$

is left-exact.

We shall say that F is flat if it is exact. From Proposition 20.2.1 we obtain:

Corollary 20.3.3. *Assume that F is flat. Assume also that $\mathbf{A}$ has a tight diagonal. Then the functor $(F_{\mathbf{C}^l})^*$ is exact.*

20.4. Affiness. Let F be as above. We will say that F is affine if the functor

$$F_* : \mathbf{D}_{\mathbf{A}_1} \to \mathbf{D}_{\mathbf{A}}$$

is exact and conservative.

Proposition 20.4.1. *Assume that F is affine and tight. Then for $\mathbf{C}^l$ as above, the functor $(F_{\mathbf{C}^l})_*$ is also exact and conservative.*

Proof. The fact that F_* is conservative is equivalent to the fact that $F(\mathrm{Ho}(\mathbf{A}))$ generates $\mathbf{D}_{\mathbf{A}_1}$. By construction, this implies that the image of $(F_{\mathbf{C}^l})^*$ generates $\mathbf{D}^l_1$. The latter is equivalent to $(F_{\mathbf{C}^l})_*$ being conservative.

It remains to show that $(F_{\mathbf{C}^l})_*$ is right-exact. By the definition of the t-structure, we have to show that

$$(F_{\mathbf{C}^l})_*(X_1 \underset{\mathbf{A}}{\otimes} Y^l) \in \mathbf{D}^{l,\leq 0}$$

for $X_1 \in \mathbf{D}^{\leq 0}_{\mathbf{A}_1}$ and $Y^l \in \mathbf{D}^{l,\leq 0}$.

By Corollary 18.4.2, we have:

$$(F_{\mathbf{C}^l})_*(X_1 \underset{\mathbf{A}_1}{\otimes} Y^l) \simeq F_*(X_1) \underset{\mathbf{A}}{\otimes} Y^l.$$

However, by assumption, $F_*(X_1) \in \mathbf{D}^{\leq 0}_{\mathbf{A}}$, implying our assertion. $\square$

20.5. Tensor product of abelian categories. In this subsection we will be concerned with the following situation:

Let $\overset{\circ}{\mathbf{A}}$ be a abelian category equipped with a monoidal structure. We will assume that the following conditions hold:

(A*) $\overset{\circ}{\mathbf{A}}$ is a Grothendieck category, and that the action functor $\overset{\circ}{\mathbf{A}} \times \overset{\circ}{\mathbf{A}} \to \overset{\circ}{\mathbf{A}}$ is right-exact, and commutes with direct sums.

Let $\overset{\circ}{\mathbf{C}}{}^l$ be another abelian category, equipped with a monoidal action $\overset{\circ}{\mathbf{A}}$. We will assume that the following conditions hold:

(C*) $\overset{\circ}{\mathbf{C}}{}^l$ is also a Grothendieck category, and that the action functor $\overset{\circ}{\mathbf{A}} \times \overset{\circ}{\mathbf{C}}{}^l \to \overset{\circ}{\mathbf{C}}{}^l$ is right-exact, and commutes with direct sums.

Let $\overset{\circ}{\mathbf{C}}{}^r$ be another abelian category with a right action of $\overset{\circ}{\mathbf{A}}$, satisfying (C*).

20.5.1. We shall say that a Grothendieck abelian category $\overset{\circ}{\mathbf{C}}{}^t$ is the tensor product $\overset{\circ}{\mathbf{C}}{}^r \underset{\overset{\circ}{\mathbf{A}}}{\otimes} \overset{\circ}{\mathbf{C}}{}^l$ if it has the following universal property:

For any Grothendieck abelian category $\overset{\circ}{\mathbf{C}}{}'$, the category of right-exact functors $Q^t : \overset{\circ}{\mathbf{C}}{}^t \to \overset{\circ}{\mathbf{C}}{}'$ that commute with direct sums is equivalent to the category of bi-additive functors

$$Q^{r,l} : \overset{\circ}{\mathbf{C}}{}^r \times \overset{\circ}{\mathbf{C}}{}^l \to \overset{\circ}{\mathbf{C}}{}'$$

that are right-exact and commute with direct sums in both arguments, and equipped with functorial isomorphisms

$$Q^{r,l}(Y^r \underset{\overset{\circ}{\mathbf{A}}}{\otimes} X, Y^l) \simeq Q^{r,l}(Y^r, X \underset{\overset{\circ}{\mathbf{A}}}{\otimes} Y^l),\ X \in \overset{\circ}{\mathbf{A}}, Y^r \in \overset{\circ}{\mathbf{C}}{}^r, Y^l \in \overset{\circ}{\mathbf{C}}{}^l.$$

20.5.2. Let us consider a special case of the above situation, when $\overset{\circ}{\mathbf{C}}{}^r$ is itself an abelian monoidal category $\overset{\circ}{\mathbf{A}}_1$ satisfying (A*), and the action of $\overset{\circ}{\mathbf{A}}$ on $\overset{\circ}{\mathbf{A}}_1$ comes from a right-exact monoidal functor $\overset{\circ}{\mathbf{A}} \to \overset{\circ}{\mathbf{A}}_1$.

Assume that for $\overset{\circ}{\mathbf{C}}{}^l$ as above, the category $\operatorname{Ind}^{\overset{\circ}{\mathbf{A}}_1}_{\overset{\circ}{\mathbf{A}}}(\overset{\circ}{\mathbf{C}}{}^l) := \overset{\circ}{\mathbf{A}}_1 \underset{\overset{\circ}{\mathbf{A}}}{\otimes} \overset{\circ}{\mathbf{C}}{}^l$ exists. By the universal property, $\operatorname{Ind}^{\overset{\circ}{\mathbf{A}}_1}_{\overset{\circ}{\mathbf{A}}}(\overset{\circ}{\mathbf{C}}{}^l)$ carries an action of $\overset{\circ}{\mathbf{A}}_1$, satisfying (C*).

Let now $\overset{\circ}{\mathbf{C}}{}^l_1$ be another abelian category endowed with an action of $\overset{\circ}{\mathbf{A}}_1$, satisfying (C*). We have:

Lemma 20.5.3. *The category of functors $\overset{\circ}{\mathbf{C}}{}^l \to \overset{\circ}{\mathbf{C}}{}^l_1$ that are compatible with the action of $\overset{\circ}{\mathbf{A}}$, and are right-exact and commute with direct sums is equivalent to the category of functors $\operatorname{Ind}^{\overset{\circ}{\mathbf{A}}_1}_{\overset{\circ}{\mathbf{A}}}(\overset{\circ}{\mathbf{C}}{}^l) \to \overset{\circ}{\mathbf{C}}{}^l_1$ that are compatible with the action of $\overset{\circ}{\mathbf{A}}_1$, and are right exact and commute with direct sums.*

20.5.4. Consider a specific example of the above situation. Let $\overset{\circ}{\mathbf{C}}$ be a Grothenieck abelian category, and A a commutative algebra that maps to the center of $\overset{\circ}{\mathbf{C}}$. Then the abelian monoidal category $\overset{\circ}{\mathbf{A}} := A$–mod acts on $\overset{\circ}{\mathbf{C}}$.

Let $A \to A'$ be a homomorphism of commutative algebras, and set $\overset{\circ}{\mathbf{A}}_1 := A_1$–mod.

Under the above circumstances, the category $\operatorname{Ind}^{\overset{\circ}{\mathbf{A}}_1}_{\overset{\circ}{\mathbf{A}}}(\overset{\circ}{\mathbf{C}})$ exists and can be described as follows. Its objects are objects $X \in \overset{\circ}{\mathbf{C}}$, endowed with an additional action of A_1, such that the two actions of A on X coincide. Morphisms are $\overset{\circ}{\mathbf{C}}$-morphisms $X_1 \to X_2$ that intertwine the A_1-actions.

20.6. We shall now study the compatibility of the notions of tensor products in the DG and abelian settings.

Let now $\mathbf{A}$, $\mathbf{C}^l$, $\mathbf{C}^r$ be as in Sect. 20.1. Then the abelian categories $\overset{\circ}{\mathbf{A}} = \mathrm{Heart}(\underset{\to}{\mathbf{A}})$, $\overset{\circ}{\mathbf{C}}{}^l = \mathrm{Heart}(\underset{\to}{\mathbf{C}}{}^l)$ and $\overset{\circ}{\mathbf{C}}{}^r = \mathrm{Heart}(\underset{\to}{\mathbf{C}}{}^r)$ satisfy the conditions of Sect. 20.5.

We will assume that $\mathbf{A}$ has a tight diagonal. In addition, we will assume that has $\mathbf{A}$ *has an affine diagonal*, by which we mean that the functor

$$(m_{\mathbf{A}})_* : \mathrm{Ho}(\underset{\to}{\mathbf{A}}) \times \mathrm{Ho}(\underset{\to}{\mathbf{A}}) \to \mathrm{Ho}(\underset{\to}{\mathbf{A}})$$

is exact (a priori, it is only left-exact).

Proposition 20.6.1. *Under the above circumstances, we have:*

$$\mathrm{Heart}(\mathbf{D}^t) \simeq \overset{\circ}{\mathbf{C}}{}^r \underset{\overset{\circ}{\mathbf{A}}}{\otimes} \overset{\circ}{\mathbf{C}}{}^r.$$

Proof. The universal right-exact functor

$$\overset{\circ}{\mathbf{C}}{}^r \times \overset{\circ}{\mathbf{C}}{}^l \to \overset{\circ}{\mathbf{C}}{}^t := \mathrm{Heart}(\mathbf{D}^t)$$

comes from the functor

$$(m_{\mathbf{C}^r,\mathbf{C}^l})_* : \mathbf{D}^r \times \mathbf{D}^l \to \mathbf{D}^t.$$

Conversely, given a right-exact functor $Q^{r,l} : \overset{\circ}{\mathbf{C}}{}^r \times \overset{\circ}{\mathbf{C}}{}^l \to \overset{\circ}{\mathbf{C}}{}'$ we produce the functor $Q^t : \overset{\circ}{\mathbf{C}}{}^t \to \overset{\circ}{\mathbf{C}}{}'$ as follows. First, we tautologically extend $Q^{r,l}$ to a functor

$$'Q^{r,l} : \mathrm{Ho}^{\leq 0}(\mathbf{C}^r \underset{\longrightarrow}{\otimes} \mathbf{C}^l) \simeq \mathbf{D}^{\leq 0}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod}) \to \overset{\circ}{\mathbf{C}}{}'$$

that vanishes on $\mathbf{D}^{\leq -1}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod})$.

Note that, by construction, for every object $Z \in \mathbf{D}^{t,\leq 0}$ there exist $Y^r \in \overset{\circ}{\mathbf{C}}{}^r$, $Y^l \in \overset{\circ}{\mathbf{C}}{}^l$ and maps $Y^r \underset{\mathbf{A}}{\otimes} Y^l \to Z$ that induces a surjection

$$H^0\left(Y^r \underset{\mathbf{A}}{\otimes} Y^l\right) \to H^0(Z).$$

Hence, in order to construct Q^t it suffices to define a functorial map

$$\mathrm{Hom}_{\mathbf{D}^{t,\leq 0}}(Y_1^r \underset{\mathbf{A}}{\otimes} Y_1^l, Y_2^r \underset{\mathbf{A}}{\otimes} Y_2^l) \to \mathrm{Hom}_{\overset{\circ}{\mathbf{C}}{}'}({}'Q^{r,l}(Y_1^r, Y_1^l), {}'Q^{r,l}(Y_2^r, Y_2^l)). \tag{20.2}$$

We rewrite the LHS of (20.2) as

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C}^r \underset{\longrightarrow}{\otimes} \mathbf{C}^l)}((Y_1^r, Y_1^l), \mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y_2^r, Y_2^l)).$$

By assumption, the object $\mathrm{Diag}_{\mathbf{A}}$ belongs to $\mathbf{D}^{\leq 0}((\mathbf{A} \otimes \mathbf{A})^{op}\text{–mod})$. Hence, it suffices to construct a functorial isomorphism

$$'Q^{r,l}\left(\mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y^r, Y^l)\right) \to {}'Q^{r,l}(Y^r, Y^l).$$

However, the latter follows from the fact that for any $X^{r,l} \in \mathrm{Ho}^{\leq 0}(\mathbf{A} \times \mathbf{A})$ we have:

$$'Q^{r,l}\left(X^{r,l} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (Y^r, Y^l)\right) \simeq {}'Q^{r,l}(Y^r, m_{\mathbf{A}}^*(X^{r,l}) \underset{\mathbf{A}}{\otimes} Y^l) \simeq {}'Q^{r,l}(m_{\mathbf{A}}^*(X^{r,l}) \underset{\mathbf{A}}{\otimes} Y^r, Y^l).$$

$\square$

20.7. Let $\mathbf{A}$, $\mathbf{C}^r$, $\mathbf{C}^l$ be as in Sect. 20.6. We will assume that the triangulated categories $\mathbf{D}^l$, $\mathbf{D}^r$ and $\mathbf{D}_{\mathbf{A}}$ satisfy the equivalent conditions of Lemma 19.3.4. We are interested in the following question: under what circumstances will $\mathbf{D}^t$ also satisfy these conditions?

Lemma 20.7.1. *Assume that the following condition holds: for every $Y^l \in \overset{\circ}{\mathbf{C}}{}^l, Y^r \in \overset{\circ}{\mathbf{C}}{}^r$ and $i > 0$ there exist surjections $Y_1^l \twoheadrightarrow Y^l$, $Y_1^r \twoheadrightarrow Y^r$ in $\overset{\circ}{\mathbf{C}}{}^l$ and $\overset{\circ}{\mathbf{C}}{}^r$, respectively, such that the map*

$$H^{-i}\left(Y_1^r \underset{\mathbf{A}}{\otimes} Y_1^l\right) \to H^{-i}\left(Y^r \underset{\mathbf{A}}{\otimes} Y^r\right)$$

is zero. Then $\mathbf{D}^t$ satisfies the equivalent conditions of Lemma 19.3.4.

Proof. We will show that $\mathbf{D}^t$ satisfies condition (4) of Lemma 19.3.4. Since every object of $\overset{\circ}{\mathbf{C}}{}^t$ receives a surjection from some $H^0\left(Y^r \underset{\mathbf{A}}{\otimes} Y^l\right)$ for Y^l, Y^r as above, it is enough to show that for any $Z \in \overset{\circ}{\mathbf{C}}{}^t$, $i > 0$ and $\alpha \in \mathrm{Hom}_{\mathbf{D}^t}\left(H^0\left(Y^r \underset{\mathbf{A}}{\otimes} Y^l\right), Z[i]\right)$ there exist Y'^l, Y'^r such that the image α' of α in

$$\mathrm{Hom}_{\mathbf{D}^t}\left(H^0\left(Y'^r \underset{\mathbf{A}}{\otimes} Y'^l\right), Z[i]\right)$$

vanishes.

Consider the map $Y^r \underset{\mathbf{A}}{\otimes} Y^l \to H^0(Y^r \underset{\mathbf{A}}{\otimes} Y^l)$, and let β be the image of α in

$$\mathrm{Hom}_{\mathbf{D}^t}(Y^r \underset{\mathbf{A}}{\otimes} Y^l, Z[i]) \simeq \mathrm{Hom}_{\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod})}\left((Y^r, Y^l), \mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} Z[i]\right).$$

Since $\mathbf{C}^l$ and $\mathbf{C}^r$ satisfy condition (4) of Lemma 19.3.4, and

$$\mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} Z \simeq (m_{\mathbf{C}^r,\mathbf{C}^l})_*(Z) \in \mathbf{D}^{\leq 0}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod}),$$

there exist surjections $Y''^r \to Y^r$ and $Y''^l \to Y^l$, such that the image β'' of β in

$$\mathrm{Hom}_{\mathbf{D}^t}(Y''^r \underset{\mathbf{A}}{\otimes} Y''^l, Z[i]) \simeq \mathrm{Hom}_{\mathbf{D}((\mathbf{C}^r \times \mathbf{C}^l)^{op}\text{–mod})}\left((Y''^r, Y''^l), \mathrm{Diag}_{\mathbf{A}} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} Z[i]\right)$$

vanishes.

Let α'' be the image of α in $\mathrm{Hom}_{\mathbf{D}^t}\left(H^0(Y''^r \underset{\mathbf{A}}{\otimes} Y''^l), Z[i]\right)$. By construction its image in

$$\mathrm{Hom}_{\mathbf{D}^t}(Y''^r \underset{\mathbf{A}}{\otimes} Y''^l, Z[i])$$

vanishes; so α'' is the image of some class

$$\gamma \in \mathrm{Hom}_{\mathbf{D}^t}\left(\tau^{\leq -1}(Y''^r \underset{\mathbf{A}}{\otimes} Y''^l), Z[i-1]\right) \simeq \mathrm{Hom}_{\mathbf{D}^t}\left(\tau^{\geq -i+1, \leq -1}(Y''^r \underset{\mathbf{A}}{\otimes} Y''^l), Z[i-1]\right).$$

By the assumption of the lemma, there exist surjections $Y_1^r \to Y''^r$ and $Y_1^r \to Y''^l$, so that the restriction of γ to $\mathrm{Hom}_{\mathbf{D}^t}\left(\tau^{\geq -i+1, \leq -1}(Y_1^r \underset{\mathbf{A}}{\otimes} Y_1^l), Z[i-1]\right)$ is such that its further restriction to $\mathrm{Hom}_{\mathbf{D}^t}\left(H^{-i+1}(Y_1^r \underset{\mathbf{A}}{\otimes} Y_1^l), Z[i-1]\right)$ vanishes. So, the above restriction comes from a class

$$\gamma_1 \in \mathrm{Hom}_{\mathbf{D}^t}\left(\tau^{\geq -i+2, \leq -1}(Y_1^r \underset{\mathbf{A}}{\otimes} Y_1^l), Z[i-1]\right).$$

By induction, we find sequences of surjections

$$Y^r_{i-1} \to ... \to Y^r_1 \text{ and } Y^l_{i-1} \to ... \to Y^l_1$$

so that the restriction of γ to $\mathrm{Hom}_{\mathbf{D}^t}\left(\tau^{\geq -i+j, \leq -1}(Y^r_j \underset{\mathbf{A}}{\otimes} Y^l_j), Z[i-1]\right)$ vanishes. This implies that the restriction of α'' to $\mathrm{Hom}_{\mathbf{D}^t}\left(H^0(Y^r_{i-1} \underset{\mathbf{A}}{\otimes} Y^l_{i-1}), Z[i]\right)$ vanishes. I.e., the objects $Y'^r = Y^r_{i-1}$, $Y'^l = Y^l_{i-1}$ have the desired properties.

□

21. Categories over stacks

21.1. Let $\mathcal{Y}$ be an Artin stack. We assume that $\mathcal{Y}$ is of finite type, i.e., can be covered with a smooth map by an affine scheme of finite type. In addition, we shall assume that the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine.

An example of such a stack is when $\mathcal{Y} = Z/G$, where Z is a scheme of finite type and G an affine algebraic group, acting on it. In fact, all algebraic stacks in this paper will be of this form.

21.1.1. The abelian categories $\mathrm{Coh}(\mathcal{Y})$ and $\mathrm{QCoh}(\mathcal{Y})$ have an evident meaning. We consider the DG categories of complexes $\mathbf{C}^b(\mathrm{Coh}(\mathcal{Y}))$, $\mathbf{C}(\mathrm{QCoh}(\mathcal{Y}))$, and the corresponding subcategories of acyclic complexes

$$\mathbf{C}^b_{acycl}(\mathrm{Coh}(\mathcal{Y})) \subset \mathbf{C}^b(\mathrm{Coh}(\mathcal{Y})),\ \mathbf{C}_{acycl}(\mathrm{QCoh}(\mathcal{Y})) \subset \mathbf{C}(\mathrm{QCoh}(\mathcal{Y})).$$

Denote the corresponding quotients, regarded as 0-objects in $\mathbf{DGCat}$ by $\mathbf{C}^b(\mathcal{Y})$ and $\mathbf{C}(\mathcal{Y})$, respectively (see Sect. 15.7.2). Their homotopy categories identify with

$$\mathbf{D}^b(\mathrm{Coh}(\mathcal{Y})) := \mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}(\mathcal{Y}))\right) / \mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathrm{Coh}(\mathcal{Y}))\right),$$
$$\mathbf{D}(\mathrm{QCoh}(\mathcal{Y})) := \mathrm{Ho}\left(\mathbf{C}(\mathrm{QCoh}(\mathcal{Y}))\right) / \mathrm{Ho}\left(\mathbf{C}_{acycl}(\mathrm{QCoh}(\mathcal{Y}))\right),$$

respectively.

As in the case of schemes, one shows that the natural functor

$$\mathbf{D}^b(\mathrm{Coh}(\mathcal{Y})) \to \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}))$$

is fully faithful, and its essential image consists of objects with coherent cohomologies.

21.1.2. We shall now make the following additional assumptions on $\mathcal{Y}$:

(*a) *The functor of global sections on* $\mathbf{D}(\mathrm{QCoh}(\mathcal{Y}))$ *is of finite cohomological dimension.*

(*b) *Every coherent sheaf on* $\mathcal{Y}$ *admits a non-zero map from a locally free coherent sheaf.*

In the example $\mathcal{Y} = Z/G$, assumption (*a) is satisfied e.g. if the ground field is of characteristic 0. Assumption (*b) is satisfied e.g. if Z admits a G-equivariant locally closed embedding into the projective space.

Remark. Assumption (*a) implies that the structure sheaf $\mathcal{O}_{\mathcal{Y}}$ is a compact object in $\mathbf{D}(\mathrm{QCoh}(\mathcal{Y}))$.

Assumption (*b) expresses the fact that $\mathcal{Y}$ carries enough vector bundles. It implies that every object $\mathcal{M} \in \mathbf{D}^-(\mathrm{Coh}(\mathcal{Y}))$ can be represented by a (possible infinite to the left) complex whose terms are locally free. This complex can be chosen to be finite if $\mathcal{M} \in \mathbf{D}^b(\mathrm{Coh}(\mathcal{Y}))$, and $H^i(\mathcal{M})$ are of finite Tor-dimension.

21.1.3. Let $\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\mathcal{Y}))$ be the DG category of bounded complexes of locally free sheaves, and let $\mathbf{C}^b_{acycl}(\mathrm{Coh}^{loc.free}(\mathcal{Y})) \subset \mathbf{C}^b(\mathrm{Coh}^{loc.free}(\mathcal{Y}))$ be the subcategory of acyclic complexes. Let $\mathbf{C}^{perf}(\mathcal{Y})$ be the corresponding quotient, regarded as a 0-object of **DGCat**; denote $\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y})) := \mathrm{Ho}(\mathbf{C}^{perf}(\mathcal{Y}))$.

We will use the following general assertion:

Lemma 21.1.4. *Let $\mathbf{D}_1 \subset \mathbf{D} \supset \mathbf{D}'$ be triangulated categories, and set $\mathbf{D}'_1 := \mathbf{D}_1 \cap \mathbf{D}'$. Consider the functor*

$$\mathbf{D}_1/\mathbf{D}'_1 \to \mathbf{D}/\mathbf{D}'. \tag{21.1}$$

Assume that in the above situation one of the following two conditions holds:

(1) *For every object $X \in \mathbf{D}$ there exists an object $X_1 \in \mathbf{D}_1$ and an arrow $X_1 \to X$ with* $\mathrm{Cone}(X_1 \to X) \in \mathbf{D}'$.

(2) *For every object $X \in \mathbf{D}$ there exists an object $X_1 \in \mathbf{D}_1$ and an arrow $X \to X_1$ with* $\mathrm{Cone}(X \to X_1) \in \mathbf{D}'$.

Then the functor (21.1) *is an equivalence.*

Assumption (*b) and Lemma 21.1.4 imply that the natural functor

$$\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y})) \to \mathbf{D}^b(\mathrm{Coh}(\mathcal{Y}))$$

is fully faithful, and its essential image consists of objects, whose cohomologies have a finite Tor-dimension. In particular, the above functor is an equivalence if $\mathcal{Y}$ is smooth.

As in the case of schemes which have enough locally free sheaves, one shows:

Lemma 21.1.5. *The functor*

$$\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y})) \to \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}))$$

extends to an equivalence between the ind-completion of $\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}))$ *and* $\mathbf{D}(\mathrm{QCoh}(\mathcal{Y}))$.

21.2. The category $\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\mathcal{Y}))$ is naturally a DG monoidal category, and

$$\mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathrm{Coh}^{loc.free}(\mathcal{Y}))\right) \subset \mathrm{Ho}\left(\mathbf{C}^b(\mathrm{Coh}^{loc.free}(\mathcal{Y}))\right)$$

is an ideal. Therefore, by Sect. 16.7.2, $\mathbf{C}^{perf}(\mathcal{Y})$ is a 0-object of **DGMonCat**, which is a DG model for the triangulated monoidal category $\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}))$.

The category $\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}))$ is rigid (see Sect. 18.5). The category $\mathbf{D}(\mathrm{QCoh}(\mathcal{Y}))$ has a natural t-structure, and the functor of tensor product is right-exact. I.e., the above DG model of $\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}))$ satisfies the assumptions of Sect. 20.1.

In addition, the assumption that $\mathcal{Y}$ has an affine diagonal implies that $\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}))$ has an affine diagonal, see Sect. 20.6.

21.2.1. By a triangulated category *over* the stack $\mathcal{Y}$ we will understand a triangulated category, equipped with a DG model, which is a 0-object of

$$\mathbf{DGmod}\left(\mathbf{C}^{perf}(\mathcal{Y})\right). \tag{21.2}$$

For two triangulated categories over $\mathcal{Y}$, by a functor (resp., natural transformation) *over* $\mathcal{Y}$ we will mean a 1-morphism (resp., 2-morphism) between the corresponding objects in the 2-category (21.2).

21.2.2. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between stacks. It defines an evident DG monoidal functor $f^* : \mathbf{C}(\mathrm{Coh}^{loc.free}(\mathcal{Y}_2)) \to \mathbf{C}(\mathrm{Coh}^{loc.free}(\mathcal{Y}_1))$, and by Lemma 16.7.4, it gives rise to a 1-morphism in **DGMonCat**,

$$f^* : \mathbf{C}^{perf}(\mathcal{Y}_2) \to \mathbf{C}^{perf}(\mathcal{Y}_1),$$

providing a model for the pull-back functor $f^* : \mathbf{D}^{perf}(\mathrm{QCoh}(\mathcal{Y}_2)) \to \mathbf{D}^{perf}(\mathrm{QCoh}(\mathcal{Y}_1))$.

If $\mathbf{D}_2$ is a triangulated category over $\mathcal{Y}_2$, we will denote by $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}_2$ the induction

$$\mathrm{Ind}^{\mathbf{C}^{perf}(\mathcal{Y}_1)}_{\mathbf{C}^{perf}(\mathcal{Y}_2)}(\mathbf{C}_2),$$

(see Sect. 17.5.1), where $\mathbf{C}_2$ is a DG model of $\mathbf{D}_2$. We will refer to $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}_2$ as the base change of $\mathbf{D}_2$ with respect to f. We have a tautological functor

$$(f^* \underset{\mathcal{Y}}{\times} \mathrm{Id}_{\mathbf{D}_2}) : \mathbf{D}_2 \to \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}_2.$$

Since $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}_2$ is defined as an object of **DGCat**, its ind-completion (see Sect. 15.7.1) is well-defined; we will denote it by $\mathcal{Y}_1 \underset{\underset{\mathcal{Y}_2}{\rightarrow}}{\times} \mathbf{D}_2$. By a slight abuse of notation, we will denote by the same character $(f^* \underset{\mathcal{Y}}{\times} \mathrm{Id}_{\mathbf{D}_2})$ the functor

$$\underset{\rightarrow}{\mathbf{D}}_2 \to \mathcal{Y}_1 \underset{\underset{\mathcal{Y}_2}{\rightarrow}}{\times} \mathbf{D}_2.$$

21.2.3. Assume that $\underset{\rightarrow}{\mathbf{D}}_2$ is equipped with a t-structure, satisfying the conditions of Sect. 20.1. Then the category $\mathcal{Y}_1 \underset{\underset{\mathcal{Y}_2}{\rightarrow}}{\times} \mathbf{D}_2$ acquires a t-structure, such that the functor $(f^* \underset{\mathcal{Y}}{\times} \mathrm{Id}_{\mathbf{D}_2})$ is right-exact.

21.3. Consider now the following situation. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map of stacks as before, and let $g : \mathcal{Y}'_2 \to \mathcal{Y}_2$ be another map. We assume that all three are Artin stacks of finite type that satisfy assumptions (*a) and (*b) of Sect. 21.1.2.

Assume also that the following holds:

(**) *The tensor product* $\mathcal{O}_{\mathcal{Y}_1} \overset{L}{\underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes}} \mathcal{O}_{\mathcal{Y}'_2}$ *is acyclic off cohomologically degree* 0.

This ensures that the naive fiber product $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}'_2$ coincides with the derived one.

21.3.1. Set $\mathcal{Y}'_1 := \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}'_2$. Let f' denote the map $\mathcal{Y}'_1 \to \mathcal{Y}'_2$. The pull-back functor $f'^* : \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2)) \to \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_1))$, which coincides with the derived pull-back by virtue of assumption (**), is naturally a 1-morphism between categories over $\mathcal{Y}_2$. Hence, by Sect. 17.5.3 we can consider the induced 1-morphism

$$\mathrm{Ind}^{\mathcal{Y}_1}_{\mathcal{Y}_2}(f'^*) : \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2)) \to \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_1))$$

of categories over $\mathcal{Y}_1$. By ind-extension we obtain a functor

$$\mathrm{Ind}^{\mathcal{Y}_1}_{\mathcal{Y}_2}(f'^*) : \mathcal{Y}_1 \underset{\underset{\mathcal{Y}_2}{\rightarrow}}{\times} \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2)) \to \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}'_1)). \tag{21.3}$$

We shall now make the following additional assumption:

(***) *Every quasi-coherent sheaf on $\mathcal{Y}'_1$ admits a non-zero map from a sheaf of the form*

$$\mathcal{M}_1 \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \mathcal{M}'_2, \tag{21.4}$$

where $\mathcal{M}_1$ (resp., $\mathcal{M}'_2$) is a locally free coherent sheaf on $\mathcal{Y}_1$ (resp., $\mathcal{Y}'_2$).

Proposition 21.3.2. *Under the above circumstances the functor $\mathrm{Ind}^{\mathcal{Y}_1}_{\mathcal{Y}_2}(f'^*)$ of (21.3) is an exact equivalence of categories.*

To prove the proposition, we are going to check that the conditions of Proposition 14.1.1 hold. Conditions Cat(i,ii), Cat(a,b) and Funct(i,ii) evidently hold. The functor in question is by construction right-exact, so Funct(b) holds as well.

21.3.3. *Condition Funct(a).* Let is verify Funct(a), i.e., the fact that the functor in question is fully faithful. Since objects of the form

$$\mathcal{M}_1 \underset{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_2))}{\otimes} \mathcal{M}'_2,\ \mathcal{M}_1 \in \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_1)),\ \mathcal{M}'_2 \in \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2))$$

generate $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2))$, it suffices to show that the functor $\mathrm{Ind}^{\mathcal{Y}_1}_{\mathcal{Y}_2}(f'^*)$ induces an isomorphism on Hom's between such objects.

Note that assumption (**) is equivalent to the fact that

$$\mathrm{Ind}^{\mathcal{Y}_1}_{\mathcal{Y}_2}(f'^*)\left(\mathcal{M}_1 \underset{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_2))}{\otimes} \mathcal{M}'_2\right) \simeq \mathcal{M}_1 \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \mathcal{M}'_2.$$

Thus, we have to verify that

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2))}\left(\mathcal{M}_1 \underset{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_2))}{\otimes} \mathcal{M}'_2, \widetilde{\mathcal{M}}_1 \underset{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_2))}{\otimes} \widetilde{\mathcal{M}}'_2\right) \to \\ &\qquad \to \mathrm{Hom}_{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_1))}\left(\mathcal{M}_1 \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \mathcal{M}'_2, \widetilde{\mathcal{M}}_1 \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \widetilde{\mathcal{M}}'_2\right) \end{aligned} \tag{21.5}$$

is an isomorphism. Using Lemma 18.5.1, we can assume that $\mathcal{M}_1 \simeq \mathcal{O}_{\mathcal{Y}_1}$.

By Corollary 18.4.2, the LHS of (21.5) identifies with

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{QCoh}(\mathcal{Y}'_2))}\left(\mathcal{M}'_2, f_*(\widetilde{\mathcal{M}}_1) \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \widetilde{\mathcal{M}}'_2\right).$$

By adjunction and the projection formula,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_1))}\left(f'^*(\mathcal{M}'_2), \widetilde{\mathcal{M}}_1 \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \widetilde{\mathcal{M}}'_2\right) &\simeq \mathrm{Hom}_{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2))}\left(\mathcal{M}'_2, f'_*(\widetilde{\mathcal{M}}_1 \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \widetilde{\mathcal{M}}'_2)\right) \simeq \\ &\simeq \mathrm{Hom}_{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}'_2))}\left(\mathcal{M}'_2, f_*(\widetilde{\mathcal{M}}_1) \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \widetilde{\mathcal{M}}'_2\right), \end{aligned}$$

implying the desired isomorphism.

21.3.4. *End of proof of Proposition 21.3.2.* Note that assumption (***) implies condition Funct(d). Moreover, it implies that every object in $\mathbf{C}^-(\mathrm{QCoh}(\mathcal{Y}_1'))$ is quasi-isomorphic to a complex whose terms are of the form (21.4). This observation, combined with the fully-faithfulness of $\mathsf{T} := \mathrm{Ind}_{\mathcal{Y}_2}^{\mathcal{Y}_1}(f'^*)$, imply that the above functor induces an equivalence $\mathbf{D}_1^- \to \mathbf{D}_2^-$.

We claim that this implies the left-exactness of T, and in particular, condition Funct(c). Indeed, for $X \in \mathbf{D}_1^{\geq 0}$ consider $\tau^{\leq 0}(\mathsf{T}(X))$. If this object is non-zero, it is of the form $\mathsf{T}(X')$ for some $X' \in \mathbf{D}_1^{<0}$, and by fully faithfulness $\mathrm{Hom}_{\mathbf{D}_1}(X', X) \neq 0$, which is a contradiction.

□

21.4. Let A be an associative algebra over the ground field. Consider the DG category of finite complexes of free A-modules of finite rank, denoted $\mathbf{C}^b(A - mod^{free, fin.rk.})$. Note that $\mathrm{Ho}\left(\underset{\rightarrow}{\mathbf{C}}^b(A\text{–mod}^{free,fin.rk.})\right)$ identifies with $\mathbf{D}(A\text{–mod})$–the usual derived category of A-modules.

The Karoubian envelope of $\mathrm{Ho}(\mathbf{C}^b(A\text{–mod}^{free,fin.rk.}))$ is the full subcategory of $\mathbf{D}(A\text{–mod})$ consisting of complexes quasi-isomorphic to finite complexes of projective finitely generated A-modules; we will denote this category by $\mathbf{D}^{perf}(A\text{–mod})$; it comes equipped with a natural DG model.

21.4.1. Assume now that A is commutative. Let $Z = \mathrm{Spec}(A)$; this is an affine scheme, possibly of infinite type.

In this case $\mathbf{C}^b(A - mod^{free,fin.rk.})$ has a naturally a DG monoidal category structure. Then $\mathbf{D}^{perf}(\mathrm{Coh}(Z)) := \mathbf{D}^{perf}(A\text{–mod})$ admits a natural lifting to a 0-object of **DGMonCat**, denoted $\mathbf{C}^{perf}(Z)$.

By a triangulated category $\mathbf{D}$ over Z we will mean an object of $\mathbf{DGmod}\left(\mathbf{C}^{perf}(Z)\right)$.

For example, if $\mathbf{C}$ is a DG category with a map $Z \to \mathrm{End}(\mathrm{Id}_{\mathbf{C}})$ and such that $\mathrm{Ho}(\mathbf{C})$ is Karoubian, it gives rise to a well-define object of $\mathbf{DGmod}\left(\mathbf{C}^{perf}(Z)\right)$.

21.4.2. Let $\mathcal{Y}$ be an Artin stack as in Sect. 21.1.2, and let us be given a map $g : Z \to \mathcal{Y}$. We will assume that this map factors $Z \to Z' \to \mathcal{Y}$, where Z' is a scheme of finite type. The the functor g^* defines a 1-morphism $\mathbf{C}^{perf}(\mathcal{Y}) \to \mathbf{C}^{perf}(Z)$ in **DGMonCat**.

For a triangulated category $\mathbf{D}_{\mathcal{Y}}$ over $\mathcal{Y}$, we shall denote by

$$Z \underset{\mathcal{Y}}{\times} \mathbf{D}_{\mathcal{Y}}$$

the base change of $\mathcal{Y}$ with respect to the 1-morphism g^*.

21.4.3. We conclude this subsection by the following observation used in the main body of the paper.

Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be a representable map of stacks, and let Z be an affine scheme of infinite type, endowed with a flat morphism g to $\mathcal{Y}_2$. We will assume that Z can be represented as $\underset{\longleftarrow}{lim}\, Z_i$, with the transition maps $Z_j \to Z_i$ flat, and such that g factors through a flat morphism $g_i : Z_I \to \mathcal{Y}_2$ for some i. Consider the scheme $\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} Z$.

Proposition 21.4.4. *Under the above circumstances there exists an exact equivalence*

$$Z \underset{\underset{\mathcal{Y}_2}{\rightarrow}}{\times} \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_1)) \to \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} Z)).$$

The proof is obtained by applying Proposition 14.1.1, repeating the argument proving Proposition 21.3.2.

21.5. In this subsection we will assume that $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a regular closed immersion.

21.5.1. By Lemma 16.7.4, we obtain that the direct image functor f_* gives rise to a well-defined 1-morphism in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$:

$$f_* : \mathbf{C}^{perf}(\mathcal{Y}_1) \to \mathbf{C}^{perf}(\mathcal{Y}_2).$$

Let now $\mathbf{D}_2$ be a category over $\mathcal{Y}_2$. We obtain a 1-morphism over $\mathcal{Y}_2$

$$(f_* \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2}) : \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}_2 \to \mathbf{D}_2.$$

From Sect. 18.4.6 we obtain:

Lemma 21.5.2. *The functor* $(f_* \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2})$ *is the right adjoint of*

$$(f^* \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2}) : \mathbf{D}_2 \to \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D}_2$$

at the triangulated level.

21.5.3. Note that $f_! := f_*$ admits a right adjoint at the triangulated level, which by Lemma 16.7.4 give rise to a 1-morphism in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$. Moreover, by Sect. 18.4.6, for any category $\mathbf{D}_2$ over $\mathcal{Y}_2$, we obtain a pair of functors

$$(f_* \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2}) : \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathbf{D} \leftrightarrows \mathbf{D} : (f^! \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2}),$$

which are mutually adjoint at the triangulated level.

Note that at the triangulated level we have an isomorphism

$$f^!(\mathcal{O}_{\mathcal{Y}_2}) \simeq f^*(\mathcal{O}_{\mathcal{Y}_2}) \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} \Lambda^n(\mathrm{Norm}_{\mathcal{Y}_1/\mathcal{Y}_2})[-n] \in \mathbf{C}^{perf}(\mathcal{Y}_1).$$

However, since $\mathbf{C}^{perf}(\mathcal{Y}_2)$ is a free 0-object of $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$, generated by $\mathcal{O}_{\mathcal{Y}_2}$, by Proposition 17.5.4, we obtain that this isomorphism takes place also at the level of 1-morphisms in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$.

Hence, we obtain:

Corollary 21.5.4. *For a category* $\mathbf{D}_2$ *over* $\mathcal{Y}_2$*, we have an isomorphism:*

$$(f^! \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2}) \simeq (f^* \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2}) \circ (\Lambda^n(\mathrm{Norm}_{\mathcal{Y}_1/\mathcal{Y}_2}) \underset{\mathcal{O}_{\mathcal{Y}_2}}{\otimes} ?)[-n].$$

21.5.5. Let $\mathbf{D}_2$ be as above, and let us assume that it is equipped with a t-structure, satisfying the conditions of Sect. 20.1, and consider the corresponding t-structure on $\mathcal{Y}_1 \underset{\overrightarrow{\mathcal{Y}_2}}{\times} \mathbf{D}_2$.

Let us assume now that f is a regular closed immersion of codimension k. The following assertion is used in the main body of the paper, and follows immediately from Corollary 21.5.4.

Lemma 21.5.6. *Under the above circumstances the functor*

$$(f^* \underset{\mathcal{Y}_2}{\times} \mathrm{Id}_{\mathbf{D}_2}) : \underset{\rightarrow}{\mathbf{D}_2} \to \mathcal{Y}_1 \underset{\overrightarrow{\mathcal{Y}_2}}{\times} \mathbf{D}_2$$

has a cohomological amplitude bounded by k*.*

21.6. Returning to the general context of Artin stacks, assume that a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is representable.

21.6.1. Let us call a quasi-coherent sheaf on $\mathcal{Y}_1$ *adapted* if the higher cohomologies of its direct image on $\mathcal{Y}_2$ vanish. Let $\mathbf{C}_{adapt}(\mathrm{QCoh}(\mathcal{Y}_1))$ denote the full DG subcategory of $\mathbf{C}(\mathrm{QCoh}(\mathcal{Y}_1))$ consisting of complexes of adapted quasi-coherent sheaves.

We will make the following assumption on the morphism f: any coherent sheaf $\mathcal{Y}_2$ can be embedded into an adapted quasi-coherent sheaf.

We claim that under the above circumstances, we can lift the direct image functor $f_* : \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}_1)) \to \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}_2))$ to a 1-morphism in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$.

Indeed f is of finite type, the above assumption implies that any object of $\mathbf{C}(\mathrm{QCoh}(\mathcal{Y}_1))$ admits a quasi-isomorphism into an object of $\mathbf{C}_{adapt}(\mathrm{QCoh}(\mathcal{Y}_1))$. Hence, by Lemma 21.1.4, the functor

$$\begin{aligned}&\mathrm{Ho}\left(\mathbf{C}_{adapt}(\mathrm{QCoh}(\mathcal{Y}_1))\right) / \mathrm{Ho}\left(\mathbf{C}_{adapt,acycl}(\mathrm{QCoh}(\mathcal{Y}_1))\right) \to \\ &\qquad\qquad \mathrm{Ho}\left(\mathbf{C}(\mathrm{QCoh}(\mathcal{Y}_1))\right) / \mathrm{Ho}\left(\mathbf{C}(\mathrm{QCoh}_{acycl}(\mathcal{Y}_1))\right) =: \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}_1))\end{aligned}$$

is an equivalence, where

$$\mathbf{C}_{adapt,acycl}(\mathrm{QCoh}(\mathcal{Y}_1)) := \mathbf{C}_{adapt}(\mathrm{QCoh}(\mathcal{Y}_1)) \cap \mathbf{C}(\mathrm{QCoh}_{acycl}(\mathcal{Y}_1)).$$

Since,

$$f_*(\mathbf{C}_{adapt,acycl}(\mathrm{QCoh}(\mathcal{Y}_1))) \subset \mathbf{C}_{acycl}(\mathrm{QCoh}(\mathcal{Y}_2)),$$

the lifting of f_* to the DG level follows from Lemma 16.7.4.

Moreover, we claim that the adjunction morphism $\mathrm{Id}_{\mathbf{D}(\mathrm{QCoh}(\mathcal{Y}_2))} \to f_* \circ f^*$ at the level of triangulated categories can also be lifted to a 2-morphism in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$. This follows again from the fact that $\mathbf{C}^{perf}(\mathcal{Y}_2)$ is free as a 1-object of $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$, so by Lemma 17.5.4, this 2-morphism is determined by the map

$$\mathcal{O}_{\mathcal{Y}_2} \to f_* \circ f^*(\mathcal{O}_{\mathcal{Y}_2}) \in \mathbf{D}(\mathrm{QCoh}(\mathcal{Y}_2)).$$

21.6.2. Finally, let us consider the following situation. Assume that the map f is proper and smooth. We define the functor $f_? : \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_1)) \to \mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_2))$ by

$$f_?(\mathcal{N}) := f_*(\mathcal{N} \otimes \Omega^n_{\mathcal{Y}_1/\mathcal{Y}_2})[n],$$

where n is the relative dimension.

By Sect. 21.6.1, $f_?$ lifts to a 1-morphism in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$.

Note that the assumptions on f imply that $f_?$ admits a right adjoint at the triangulated level. As in Sect. 21.5.3, we obtain that there exists a 1-morphism $f^? : \mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2)) \to \mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_1))$ in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$, such that $(f_?, f^?)$ form a pair of mutually adjoint functors at the triangulated level. As in Sect. 21.6.1, the adjunction morphism

$$f_? \circ f^? \to \mathrm{Id}_{\mathbf{D}^{perf}(\mathrm{Coh}(\mathcal{Y}_2))}$$

lifts to a 2-morphism in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$.

Finally, we note that, as in *loc. cit.*, from the usual Serre duality, we obtain a 2-isomorphism in $\mathbf{DGmod}(\mathbf{C}^{perf}(\mathcal{Y}_2))$:

$$f^? \simeq f^*. \tag{21.6}$$

Part IV: Triangulated categories arising in representation theory

22. A renormalization procedure

22.1. Let $\mathbf{D}$ be a co-complete triangulated category, equipped with a DG model. Let $X_a, a \in A$ be a collection of objects in $\mathbf{D}$. Starting with this data we will construct a pair of new triangulated categories, both equipped with models.

22.1.1. Let $\mathbf{D}^f$ be the Karoubian envelope of the triangulated subcategory of $\mathbf{D}$ *strongly generated* by the objects X_a. I.e., this is the smallest full triangulated subcategory of $\mathbf{D}$, containing all these objects and closed under direct summands. By Sect. 15.7.2, $\mathbf{D}^f$, and its embedding into $\mathbf{D}$, come equipped with models.

Set $\mathbf{D}_{ren}$ be the ind-completion of $\mathbf{D}^f$, i.e., $\underset{\rightarrow}{\mathbf{D}^f}$, see Sect. 15.7.1. This is a co-complete triangulated category, which is also equipped with a model. When thinking of $\mathbf{D}^f$ as a subcategory of $\mathbf{D}_{ren}$, we will sometimes denote it also by $\mathbf{D}^f_{ren}$.

22.1.2. We claim that there exists a pair of functors

$$\Psi : \mathbf{D}_{ren} \to \mathbf{D} \text{ and } \Phi : \mathbf{D} \to \mathbf{D}_{ren},$$

such that Φ is the right adjoint of Ψ.

Indeed, let $\mathbf{D} = \mathrm{Ho}(\mathbf{C})$, we set $\mathbf{C}^f \subset \mathbf{C}$ be the preimage of $\mathbf{D}^f \subset \mathbf{D}$. Then $\mathbf{D}_{ren} \simeq \mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}^f})$.

The functor

$$\Phi : \mathbf{D} \simeq \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}((\mathbf{C}^f)^{op}\text{–mod}) \simeq \mathbf{D}_{ren}$$

is defined tautologically. Namely, an object $Y \in \mathrm{Ho}(\mathbf{C})$ maps to the $(\mathbf{C}^f)^{op}$-module

$$X \mapsto \mathrm{Hom}^\bullet_{\mathbf{C}}(X, Y).$$

The restriction of Ψ to $\mathbf{D}^f_{ren} \subset \mathbf{D}_{ren}$ is by definition the tautological embedding $\mathbf{D}^f_{ren} = \mathbf{D}^f \hookrightarrow \mathbf{D}$. For $X \in \mathbf{D}^f$, the adjunction

$$\mathrm{Hom}_{\mathbf{D}}(\Psi(X), Y) \simeq \mathrm{Hom}_{\mathbf{D}_{ren}}(X, \Phi(Y))$$

is evident.

Since $\mathbf{D}^f_{ren}$ generates $\mathbf{D}_{ren}$, the functor Ψ extends canonically onto the entire $\mathbf{D}_{ren}$ by the adjunction property.

22.1.3. Suppose for a moment that the category $\mathbf{D}$ was itself generated by its subcategory $\mathbf{D}^c$ of compact objects. [7]

Assume that $\mathbf{D}^c \subset \mathbf{D}^f$. The ind-extension of the above embedding defines a functor, that we shall denote $\Xi : \mathbf{D} \to \mathbf{D}_{ren}$. It is easy to see that Ξ is the left adjoint of Ψ. In addition, Ξ is fully faithful, which implies that the adjunction map $\mathrm{Id}_{\mathbf{D}} \to \Psi \circ \Xi$ is an isomorphism.

It is easy to see that Ψ is an equivalence if and only if the objects X_a, $a \in A$ generate $\mathbf{D}$ and are compact.

[7] This assumption is *not* satisfied in the examples for which the notion of $\mathbf{D}_{ren}$ is developed here, i.e., D-modules on infinite-dimensional schemes, or Kac-Moody representations. However, it is satisfied in the example of the derived category of quasi-coherent sheaves on a singular algebraic variety.

22.1.4. Assume that in the set-up of Sect. 22.1, $\mathbf{D}$ is equipped with a t-structure. As usual, let us denote $\mathbf{D}^+ := \underset{n}{\cup} \mathbf{D}^{\geq -n}$, $\mathbf{D}^- := \underset{n}{\cup} \mathbf{D}^{\leq n}$, and $\mathbf{D}^b := \mathbf{D}^+ \cap \mathbf{D}^-$. Assume that $\mathbf{D}^f \subset \mathbf{D}^+$.

Assume that the following condition holds:

(*) *There exists an exact and conservative functor* $\mathsf{F} : \mathbf{D} \to \mathrm{Vect}$ *(conservative means* $\mathsf{F}(Y[n]) = 0\ \forall n \in \mathbb{Z} \Rightarrow Y = 0$*), which commutes with colimits, and a filtered inverse system* $\{\mathsf{Z_k}\}, \mathsf{k} \in \mathsf{K}$ *of objects from* $\mathbf{D}^f$ *such that for* $Y \in \mathbf{D}^+$ *we have a functorial isomorphism*

$$\mathsf{F}(Y) \simeq \underset{k}{colim}\ \mathrm{Hom}_{\mathbf{D}}(\mathsf{Z_k}, Y) = 0.$$

Proposition 22.1.5. *Under the above circumstances, the adjunction map* $\Psi \circ \Phi|_{\mathbf{D}^+} \to \mathrm{Id}_{\mathbf{D}^+}$ *is an isomorphism.*

Proof. For $Y \in \mathbf{D}^{\geq 0}$, let X denote $\Phi(Y)$. By Lemma 15.8.10, $X \simeq \underset{I}{hocolim}(X_I)$ for some set I and a homotopy I-object X_I of a DG model of $\mathbf{D}_{ren}$, with all X_i being in $\mathbf{D}^f_{ren}$. Denote $Y_I = \Psi(X_I)$, so $\Psi \circ \Phi(Y) \simeq \underset{I}{hocolim}(Y_I)$.

It suffices to show that the arrow

$$\mathsf{F}(\underset{I}{hocolim}(Y_I)) \to \mathsf{F}(Y)$$

is an isomorphism.

We have

$$\mathsf{F}(\underset{I}{hocolim}(Y_I)) \simeq \underset{i}{colim}\, \mathsf{F}(Y_i) \simeq \underset{i,k}{colim}\ \mathrm{Hom}_{\mathbf{D}}(\mathsf{Z_k}, Y_i),$$

since F commutes with colimits and $Y_i \in \mathbf{D}^+$, and

$$\mathsf{F}(Y) \simeq \underset{k}{colim}\ \mathrm{Hom}_{\mathbf{D}}(\mathsf{Z_k}, Y) \simeq \underset{k}{colim}\ \mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z_k}, X) \simeq \underset{k,i}{colim}\ \mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z_k}, X_i),$$

and the isomorphism is manifest.

□

22.2. Assume again that in the set-up of Sect. 22.1, $\mathbf{D}$ is equipped with a t-structure. [8]

Proposition 22.2.1. *Assume that the adjunction map* $\Psi \circ \Phi(Y) \to Y$ *is an isomorphism for* $Y \in \mathbf{D}^+$ *(in particular, the functor* Φ *restricted to* $\mathbf{D}^+$ *is fully faithful).*

(a) *There exists a unique t-structure on* $\mathbf{D}_{ren}$ *such that the functor* Φ *induces an exact equivalence* $\mathbf{D}^+ \to \mathbf{D}^+_{ren}$.

(b) *The functor* Ψ *is exact with respect the t-structure of point (a).*

Proof. The requirement on the t-structure implies that $\mathbf{D}^{>0}_{ren}$ equals the essential image of $\Phi|_{\mathbf{D}^{>0}}$. Hence, $\mathbf{D}^{\leq 0}_{ren}$, being its left orthogonal, consists of

$$\{X \in \mathbf{D}_{ren},\, |\, \Psi(X) \in \mathbf{D}^{\leq 0}\}.$$

To prove (a) we need to show that every $X \in \mathbf{D}_{ren}$ admits a truncation triangle. Consider the map $X \to \Phi(\tau^{>0}(\Psi(X)))$. We have $\Phi(\tau^{>0}(\Psi(X))) \in \mathbf{D}^{>0}_{ren}$, and it remains to see that

$$\mathrm{Cone}\left(X \to \Phi(\tau^{>0}(\Psi(X)))\right)[-1] \in \mathbf{D}^{\leq 0}_{ren}.$$

We have

$$\Psi\left(\mathrm{Cone}\left(X \to \Phi(\tau^{>0}(\Psi(X)))[-1]\right)\right) \simeq \mathrm{Cone}\left(\Psi(X) \to \tau^{>0}(\Psi(X))\right)[-1] \simeq \tau^{\leq 0}(\Psi(X)),$$

[8] We are grateful to Jacob Lurie and Amnon Neeman for help with the material in this subsection.

as required.

Point (b) of the proposition follows from the construction.

$\square$

Remark. Assume for a moment that the t-structure on $\mathbf{D}$ has the property that $\underset{k\geq 0}{\cap} \mathbf{D}^{\leq -k} = 0$, and Φ is fully faithful. Then we obtain $\mathbf{D}$ is the triangulated quotient of $\mathbf{D}_{ren}$ by the subcategory of acyclic objects, i.e., $\underset{k\geq 0}{\cap} \mathbf{D}_{ren}^{\leq -k}$.

22.3. Let $\mathbf{D}$ be a co-complete triangulated category equipped with a DG model and a t-structure. Assume that the t-structure is compatible with colimits.

We say that an object $Y \in \mathbf{D}$ is almost compact if for any k and a homotopy I-object X_I, the map

$$\underset{i\in I}{colim} \operatorname{Hom}_{\mathbf{D}}(Y[k], X_i) \to \operatorname{Hom}_{\mathbf{D}}(Y[k], hocolim(X_i))$$

is an isomorphism, *provided that* $X_i \in \mathbf{D}^{\geq 0}$ for all $i \in I$.

22.3.1. Let $\mathbf{D}$ be a triangulated category as in Sect. 22.2, satisfying the assumption of Proposition 22.2.1.

Proposition 22.3.2. *Assume that the t-structure on* $\mathbf{D}$ *is compatible with colimits, and that the objects* $X_a \in \mathbf{D}$ *are almost compact. Then the t-structure on* $\mathbf{D}_{ren}$ *is compatible with colimits.*

Proof. It follows from Lemma 15.8.7 that if X_I is a homotopy I-object of (a DG model of) $\mathbf{D}_{ren}$ with $X_i \in \mathbf{D}_{ren}^{\leq 0}$, then $hocolim(X_I) \in \mathbf{D}_{ren}^{\leq 0}$. This does not require the objects X_a to be almost compact.

Assume now that $X_i \in \mathbf{D}_{ren}^{\geq 0}$. We have $\Psi(X_i) \in \mathbf{D}^{\geq 0}$ and $hocolim(\Psi(X_I)) \in \mathbf{D}^{\geq 0}$. Thus, it suffices to show that the map

$$hocolim(X_I) \to \Phi(hocolim(\Psi(X_I))) \tag{22.1}$$

is an isomorphism.

The next assertion results from the definitions.

Lemma 22.3.3. *Assume in the circumstances of Proposition 22.2.1 that the objects* X_a *are almost compact. Then, if* Y_I *is a homotopy* I*-object of (a DG model of)* $\mathbf{D}$ *with* $Y_i \in \mathbf{D}^{\geq 0}$*, the natural map*

$$hocolim(\Phi(Y_I)) \to \Phi(hocolim(Y_I))$$

is an isomorphism.

Applying the lemma to $Y_I = \Psi(X_I)$, we obtain

$$\Phi(hocolim(\Psi(X_I))) \simeq hocolim(\Phi(\Psi(X_I))) \simeq hocolim(X_I).$$

$\square$

Remark. It is easy to see that under the above circumstances, the t-structure on $\mathbf{D}_{ren}$ is compatible with colimits if *and only if* each X_a is almost compact in $\mathbf{D}$.

22.4. Let $\mathbf{D}$ be again a category as in Sect. 22.2. We would like now to give a criterion for when the t-structure on it is compactly generated.

Assume that there exists an inverse family $\{\mathsf{Z}_k\}$ as in Sect. 22.1.4, with the following additional properties:

- (i) $\mathsf{Z}_\mathsf{k} \in \mathbf{D}^{\leq 0}$ and for $\mathsf{k}_2 \geq \mathsf{k}_1$, $\mathrm{Cone}(Z_{\mathsf{k}_2} \to Z_{\mathsf{k}_1}) \in \mathbf{D}^{<0}$.
- (ii) Any $Y \in \mathbf{D}^f$ belongs to $\mathbf{D}^b$ and if $Y \in \mathbf{D}^{\leq 0} \cap \mathbf{D}^f$, there exists an object Z, equal to a finite direct sum of Z_k's, and a map $\mathsf{Z} \to Y$ with $\mathrm{Cone}(\mathsf{Z} \to Y) \in \mathbf{D}^{<0}$.

Proposition 22.4.1. *Under the above circumstances, the t-structure on $\mathbf{D}_{ren}$ is compactly generated.*

Proof. By Lemma 19.2.2, we have to show that if $X \in \mathbf{D}_{ren}$ satisfies $\mathrm{Hom}_{\mathbf{D}_{ren}}(Y, X)$ for all $Y \in \mathbf{D}^f_{ren} \cap \mathbf{D}^{\leq 0}_{ren}$, then $X \in \mathbf{D}^{>0}_{ren}$.

The proof of Proposition 22.1.5 shows that $\mathsf{F} \circ \Psi(X[n]) = 0$ for $n > 0$, and hence $\Psi(X) \in \mathbf{D}^{>0}$. Consider the object $X' := \mathrm{Cone}(X \to \Phi \circ \Psi(X))[-1]$. It satisfies the same assumption as X, and also $\Psi(X') = 0$. We claim that any such object equals 0.

Indeed, suppose, by contradiction, that $X' \neq 0$. Let n be the minimal integer such that $\mathrm{Hom}_{\mathbf{D}_{ren}}(Y[-n], X') \neq 0$ for $Y \in \mathbf{D}^f_{ren} \cap \mathbf{D}^{\leq 0}_{ren}$. By assumption, $n \geq 0$, and it exists since $\mathbf{D}^f_{ren}$ generates $\mathbf{D}_{ren}$. Consider a map $\mathsf{Z} \to Y$ as in (ii). By the minimality assumption on n, the map $\mathrm{Hom}_{\mathbf{D}_{ren}}(Y[-n], X') \to \mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z}[-n], X')$ is injective. So, there exists an index k' and a non-zero element in $\mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z}_{k'}[-n], X')$. By (i), for $k_2 \geq k_1$, he map

$$\mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z}_{k_1}[-n], X') \to \mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z}_{k_2}[-n], X')$$

is injective. Hence, we obtain that

$$\underset{\mathsf{k} \in \mathsf{K}}{colim}\ \mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z}_\mathsf{k}[-n], X') \neq 0.$$

However, by assumption (*),

$$\underset{\mathsf{k} \in \mathsf{K}}{colim}\ \mathrm{Hom}_{\mathbf{D}_{ren}}(\mathsf{Z}_\mathsf{k}[-n], X') \simeq \underset{\mathsf{k} \in \mathsf{K}}{colim}\ \mathrm{Hom}_{\mathbf{D}}(\mathsf{Z}_\mathsf{k}[-n], \Psi(X')) \simeq \mathsf{F} \circ \Psi(X[n]),$$

which is a contradiction.

□

22.5. Let us consider an example of the situation described in Sect. 22.2. (Another example relevant to representations of Kac-Moody algebras will be considered in Sect. 23).

22.5.1. Let $\overset{\circ}{\mathbf{C}}$ be a Grothendieck abelian category, i.e., it is closed under inductive limits, and the functor of the inductive limit over a filtered index category is exact. We take $\mathbf{D} := \mathbf{D}(\overset{\circ}{\mathbf{C}})$ be the usual derived category of $\overset{\circ}{\mathbf{C}}$.

Let $\overset{\circ}{\mathbf{C}}{}^f \subset \overset{\circ}{\mathbf{C}}$ be a small abelian subcategory, satisfying the following two conditions:

- (a) Every object of $\overset{\circ}{\mathbf{C}}$ can be presented as an inductive limit of objects the $\overset{\circ}{\mathbf{C}}{}^f$.
- (b) Every object $X \in \overset{\circ}{\mathbf{C}}{}^f$ is almost compact as an object of $\mathbf{D}$, i.e., the functors $\mathrm{Ext}^i(X_a, ?)$ commute with filtered inductive limits for $i = 0, 1, 2, ...$

Note that condition (a), combined with (b) for just $i = 0, 1$ imply that $\overset{\circ}{\mathbf{C}}$ identifies with the category $\mathrm{Ind}(\overset{\circ}{\mathbf{C}}{}^f)$.

22.5.2. We take the objects $X_a \in \mathbf{D}$ to be the objects from $\overset{\circ}{\mathbf{C}}{}^f$, and let us form the corresponding categories $\mathbf{D}^f = \mathbf{D}^f_{ren} \subset \mathbf{D}_{ren}$. Note that we have a natural equivalence:

$$\mathbf{D}^f_{ren} \simeq \mathbf{D}^b(\overset{\circ}{\mathbf{C}}{}^f).$$

We claim that the conditions of Proposition 22.2.1 hold. Indeed, we only have to check that the adjunction

$$\Psi \circ \Phi(Y) \to Y \tag{22.2}$$

is an isomorphism for $Y \in \mathbf{D}^{\geq 0}$.

Lemma 22.5.3. *Every object $Y \in \mathbf{D}^+$ can be represented as hocolim(Y_I), where $Y_i \in \mathbf{D}^b(\overset{\circ}{\mathbf{C}}{}^f)$.*

This lemma, combined with Lemma 22.3.3 reduces (22.2) to the case of $Y \in \mathbf{D}^f$, for which it follows from the definitions.

In addition, we claim that the t-structure on $\mathbf{D}_{ren}$ is compactly generated. Indeed, by Lemma 19.1.4, it is enough to show that for any $X \in \mathbf{D}^f_{ren}$, the object $\tau^{\leq 0}(X)$ belongs to the subcategory generated by extensions, direct sums and non-positive shifts by $\overset{\circ}{\mathbf{C}}{}^f$. However, $\tau^{\leq 0}(X) \in \mathbf{D}^{b,\leq 0}(\overset{\circ}{\mathbf{C}}{}^f)$.

Remark. As was explained to us by A. Neeman, the above construction reproduces one of [Kr]. Namely, one can show that $\mathbf{D}_{ren}$ is equivalent to the homotopy category of complexes of injective objects in $\overset{\circ}{\mathbf{C}}$.

22.5.4. Let us consider two specific examples of the situation described in Sect. 22.5.1. Let $\mathfrak{Y}$ be a strict ind-scheme of ind-finite type. I.e., $\mathfrak{Y}$ is a union $\underset{i\geq 0}{\cup} \mathfrak{Y}_i$, where $\mathfrak{Y}_i$ are schemes of finite type, and the maps $\mathfrak{Y}_i \to \mathfrak{Y}_{i+1}$ are closed embeddings. We let $\overset{\circ}{\mathbf{C}}$ be either

$$\mathrm{QCoh}(\mathfrak{Y}) \text{ or } \mathfrak{D}(\mathfrak{Y})\text{–mod}.$$

The corresponding categories $\overset{\circ}{\mathbf{C}}{}^f$ identify with

$$2\text{-}colim\ \mathrm{Coh}(\mathfrak{Y}_i) \text{ and } 2\text{-}colim\, \mathfrak{D}^f(\mathfrak{Y}_i)\text{–mod},$$

respectively, where $\mathfrak{D}^f(\cdot)$–mod denotes the category of finitely generated (i.e., coherent) D-modules over a scheme of finite type. We shall denote the corresponding categories by $\mathbf{D}_{ren}(\mathrm{QCoh}(\mathfrak{Y})$ and $\mathbf{D}_{ren}(\mathfrak{D}(\mathfrak{Y})\text{–mod})$, respectively.

In both cases, the derived functor of global sections $\Gamma : \mathbf{D}^f \to \mathbf{D}(\mathrm{Vect}_k)$ admits a DG model and gives rise to a functor

$$\Gamma : \mathbf{D}_{ren} \to \mathbf{D}(\mathrm{Vect}_k).$$

Note that the latter functor does not, in general, factor through $\mathbf{D}_{ren} \twoheadrightarrow \mathbf{D}$.

For example, let assume that each is $\mathfrak{Y}_i$ smooth and projective, and $\dim(\mathfrak{Y}_{i+1}) > \dim(\mathfrak{Y}_i)$. Let $\overset{\circ}{\mathbf{C}} = \mathrm{QCoh}(\mathfrak{Y})$. We take $X \in \mathbf{D}_{ren}$ to be the dualizing complex $K_{\mathfrak{Y}}$, which identifies with $\underset{i}{colim}\, K_{\mathfrak{Y}_i}$. The image of $K_{\mathfrak{Y}}$ in $\mathbf{D}(\mathrm{QCoh}(\mathfrak{Y}))$ is zero, however, $\Gamma(\mathfrak{Y}, K_{\mathfrak{Y}}) \simeq k$.

23. The derived category of Kac-Moody modules

23.1. Let $\widehat{\mathfrak{g}}_\kappa$–mod be the abelian category of modules over the Kac-Moody algebra $\widehat{\mathfrak{g}}$ at level κ. Let $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ be the usual derived category of $\widehat{\mathfrak{g}}_\kappa$–mod, i.e., the triangulated quotient of the homotopy category of complexes of objects $\widehat{\mathfrak{g}}_\kappa$–mod by the subcategory of acyclic ones. By Sect. 15.7.2, $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ naturally comes equipped with a DG model.

23.1.1. By construction, $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is co-complete, and is equipped with a t-structure compatible with colimits. The difficulty in working with $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is that it is not generated by compact objects.

For $i \geq 0$ let us denote by $\mathbb{V}_{\kappa,i} \in \widehat{\mathfrak{g}}_\kappa\text{–mod}$ the induced module $\mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{g}\otimes(t^i\cdot\mathbb{C}[[t]])}(\mathbb{C})$. By [FG2], Proposition 23.12, these objects are almost compact (see Sect. 22.2) in $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$, but they are not compact.

23.1.2. We are now going to apply a renormalization procedure described in Sect. 22.1 and obtain a better behaved triangulated category:

We take $\mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ *to be the Karoubian envelope of the subcategory of* $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ *strongly generated by the objects* $\mathbb{V}_{\kappa,i}$, $i = 0, 1, \ldots$.

By Sect. 22.1, we obtain a triangulated category that we shall denote $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$, equipped with a DG model, which is co-complete, and endowed with a pair of mutually adjoint functors

$$\Psi : \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \leftrightarrows \mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) : \Phi.$$

We claim that the conditions of Sections 22.1.4 and 22.4 are satisfied. Indeed, we take F to be the usual forgetful functor, and $\mathsf{Z}_k := \mathbb{V}_{\kappa,k}$. Thus we obtain that

$$\Psi \circ \Phi|_{\mathbf{D}^+(\widehat{\mathfrak{g}}_\kappa\text{–mod})} \simeq \mathrm{Id}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_\kappa\text{–mod})}.$$

Moreover, $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ has a compactly generated t-structure, in which Ψ exact and $\Phi|_{\mathbf{D}^+(\widehat{\mathfrak{g}}_\kappa\text{–mod})}$ exact, and we have an equivalence

$$\mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \leftrightarrows \mathbf{D}^+(\widehat{\mathfrak{g}}_\kappa\text{–mod}).$$

The kernel of Ψ is the subcategory of $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$, consisting of acyclic objects with respect to this t-structure.

Remark. We do not know whether the t-structure on $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ (or, equivalently, on $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$) induces a t-structure on the subcategory $\mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{–mod})$. [9] I.e., the formalism of Sect. 22.5.1 is a priori not applicable in this case.

23.2. The critical level case. Let us specialize to the case $\kappa = \kappa_{\mathrm{crit}}$. Let $\mathfrak{Z}_\mathfrak{g}$ denote the center of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$–mod. This is a topological commutative algebra isomorphic to the inverse limit of $\mathfrak{Z}^i_\mathfrak{g}$ (see [FG2], Sect. 7.1), where each $\mathfrak{Z}^i_\mathfrak{g}$ is isomorphic to a polynomial algebra on infinitely many variables, and the ideals $\ker(\mathfrak{Z}^{i_2}_\mathfrak{g} \to \mathfrak{Z}^{i_1}_\mathfrak{g})$ are finitely presented and regular. By [BD], Theorem 3.7.9, the action of $\mathfrak{Z}_\mathfrak{g}$ on $\mathbb{V}_{\mathrm{crit},i}$ factors through $\mathfrak{Z}^i_\mathfrak{g}$, and $\mathbb{V}_{\mathrm{crit},i}$ is flat as a $\mathfrak{Z}^i_\mathfrak{g}$-module.

[9]Most probably, it does not.

23.2.1. Let Z be a discrete quotient algebra of $\mathfrak{Z}_{\mathfrak{g}}$, such that for some (=any) i the map $\mathfrak{Z}_{\mathfrak{g}} \to \mathsf{Z}$ factors through $\mathfrak{Z}^i_{\mathfrak{g}} \to \mathsf{Z}$, with the ideal $\ker(\mathfrak{Z}^i_{\mathfrak{g}} \to \mathsf{Z})$ being finitely presented and regular.

Let $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}}$ be the full abelian subcategory of $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}$, consisting of modules on which the action of the center $\mathfrak{Z}_{\mathfrak{g}}$ factors through $\mathfrak{Z}$. Let $\mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$ be the usual derived category of this abelian category; it is co-complete and has a t-structure compatible with colimits, and naturally comes equipped with a DG model.

We have a tautological functor

$$(\iota^{\mathsf{Z}}_{\mathfrak{g}})_* : \mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}).$$

23.2.2. We shall now define a renormalized version of the category $\mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$, denoted $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$:

We define $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$ *to consist of those objects of* $\mathcal{M} \in \mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$*, for which* $(\iota^{\mathsf{Z}}_{\mathfrak{g}})_*(\mathcal{M}) \in \mathbf{D}^f(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod})$.

By Sect. 22.1, we obtain a category which we shall denote $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$, and a pair of mutually adjoint functors

$$\Psi^{\mathsf{Z}} : \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}}) \leftrightarrows \mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}}) : \Phi^{\mathsf{Z}}.$$

We claim that conditions of Sect. 22.1.4 and Sect. 22.4 are satisfied. Indeed, we take F to be again the forgetful functor, and we take Z_k to be the objects

$$\mathbb{V}^{\mathsf{Z}}_{\text{crit},i} := \mathbb{V}_{\text{crit},i} \underset{\mathfrak{Z}^i_{\mathfrak{g}}}{\otimes} \mathsf{Z}$$

for i such that Z is a quotient of $\mathfrak{Z}^i_{\mathfrak{g}}$. (Note that $\mathbb{V}^{\mathsf{Z}}_{\text{crit},i}$ is an object of the abelian category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}}$.)

Thus, we obtain that $\Psi^{\mathsf{Z}} \circ \Phi^{\mathsf{Z}}|_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})} \to \text{Id}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})}$, and that $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$ acquires a compactly generated t-structure, such that the functors Ψ^{Z} and $\Phi^{\mathsf{Z}}_{\mathbf{D}^+(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})}$ are exact, and induce mutually quasi-inverse equivalences

$$\Psi^{\mathsf{Z}} : \mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}) \leftrightarrows \mathbf{D}^+(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}) : \Phi^{\mathsf{Z}}.$$

23.2.3. By construction, the category $\mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$ is realized as a quotient

$$\text{Ho}\left(\mathbf{C}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})\right) / \text{Ho}\left(\mathbf{C}_{acycl}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})\right),$$

so by Sect. 16.7.4, it lifts to an object of $\mathbf{DGmod}\left(\mathbf{C}^b(\mathsf{Z}\text{-mod}^{free,fin.rk.})\right)$, see Sect. 21.4. Since $\mathbf{D}(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$ is Karoubian, this structure extends to that of triangulated category over $\text{Spec}(\mathsf{Z})$.

Hence, the category $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\mathsf{Z}})$, which is also Karoubian, inherits this structure.

23.3. Changing the central character. Let Z' be another discrete and regular quotient of $\mathfrak{Z}_{\mathfrak{g}}$, such that the projection $\mathfrak{Z}_{\mathfrak{g}} \twoheadrightarrow \mathsf{Z}'$ factors through Z. We have a regular closed immersion

$$\text{Spec}(\mathsf{Z}') \hookrightarrow \text{Spec}(\mathsf{Z})$$

that we shall denote by $\iota^{\mathsf{Z}',\mathsf{Z}}$.

23.3.1. The functor

$$(\iota^{\mathsf{Z}',\mathsf{Z}})^* : \mathbf{D}^{perf}(\mathsf{Z}\text{–mod}) \to \mathbf{D}^{perf}(\mathsf{Z}'\text{–mod})$$

lifts naturally to a 1-morphism in **DGMonCat**, which we denote by the same character. Since $\iota^{\mathsf{Z}',\mathsf{Z}}$ is a regular embedding, the adjoint functor

$$\iota^{\mathsf{Z}',\mathsf{Z}}_* : \mathbf{D}(Z'\text{–mod}) \to \mathbf{D}(Z\text{–mod})$$

sends $\mathbf{D}^{perf}(\mathsf{Z}'\text{–mod})$ to $\mathbf{D}^{perf}(\mathsf{Z}\text{–mod})$.

Consider the base-changed category

$$\mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}}). \tag{23.1}$$

This is a triangulated category over $\mathrm{Spec}(\mathsf{Z}')$. Consider also its ind-completion

$$\mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\underset{\to}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}}).$$

By Sect. 18.4.6, the functors $(\iota^{\mathsf{Z}',\mathsf{Z}})^*$ and $(\iota^{\mathsf{Z}',\mathsf{Z}})_*$ induce a pair of mutually adjoint functors, denoted

$$(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})^*,\ (\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})_*,$$

respectively:

$$\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}}) \leftrightarrows \mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}})$$

and

$$\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}}) \leftrightarrows \mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\underset{\to}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}})$$

The functor $(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})^*$ is right-exact. By Proposition 20.4.1, the functor $(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})_*$ is exact and conservative.

23.3.2. Consider now the tautological functor

$$(\iota^{\mathsf{Z}',\mathsf{Z}}_{\widehat{\mathfrak{g}}})_* : \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}'}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}}). \tag{23.2}$$

It is naturally equipped with a DG model, and as such is compatible with the action of $\mathbf{C}^b(\mathsf{Z}\text{–mod}^{free,fin.rk.})$, and hence is a functor between categories over $\mathrm{Spec}(\mathsf{Z})$.

By definition, the above functor sends $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}'})$ to $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}})$. Hence, by Sect. 16.7.2, we obtain a functor

$$(\iota^{\mathsf{Z}',\mathsf{Z}}_{\widehat{\mathfrak{g}}})_* : \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}'}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}})$$

between triangulated categories over $\mathrm{Spec}(\mathsf{Z})$. We shall denote by

$$(\iota^{\mathsf{Z}',\mathsf{Z}}_{\widehat{\mathfrak{g}},ren})_* : \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}'}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}})$$

its ind-extension.

23.3.3. The functor (23.2) admits a right adjoint, denoted $(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})^*$. The following assertion is established in [FG2], Lemma 7.5:

Lemma 23.3.4. *For* Z', Z *as above we have a commutative diagram of functors*

$$\begin{array}{ccc} \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}}) & \xrightarrow{(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})^*} & \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}'}) \\ \downarrow & & \downarrow \\ \mathbf{D}(\mathsf{Z}\text{--mod}) & \xrightarrow{(\iota^{\mathsf{Z}',\mathsf{Z}})^*} & \mathbf{D}(\mathsf{Z}'\text{--mod}). \end{array}$$

The functor $(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z},\mathsf{Z}'})^*$ is naturally equipped with a DG model, and as such is compatible with the action of $\mathbf{C}^b(\mathsf{Z}\text{--mod}^{free,fin.rk.})$, and hence is a functor between categories over $\mathrm{Spec}(\mathsf{Z})$.

Lemma 23.3.5. *The functor*

$$(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})^* : \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}}) \to \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}'})$$

sends $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}})$ *to* $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod})$.

Proof. For $\mathcal{M} \in \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}})$,

$$(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}'})_* \circ (\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})^*(\mathcal{M}) \simeq (\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}})_* \left((\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})_* \circ (\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})^*(\mathcal{M})\right) \simeq (\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}})_* \left(\mathsf{Z}' \underset{\mathsf{Z}}{\otimes} \mathcal{M}\right),$$

and the assertion follows from the fact that Z' admits a finite resolution by locally free Z-modules.

□

Thus, we obtain a 1-morphism

$$(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})^* : \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}'})$$

over $\mathrm{Spec}(\mathsf{Z})$.

23.3.6. By Sect. 17.5.3, the functor $(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z}',\mathsf{Z}})^*$ gives rise to a functor

$$(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z},\mathsf{Z}'})^* : \mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}'}) \tag{23.3}$$

and by ind-extension a functor

$$(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z},\mathsf{Z}'})^* : \mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\overset{\to}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}'}) \tag{23.4}$$

Proposition 23.3.7. *The functors* (23.3) *and* (23.4) *are fully faithful.*

Proof. It suffices to prove the assertion concerning the functor (23.3). Moreover, it is easy to see that it suffices to show that for $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}})$, the functor $(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z},\mathsf{Z}'})^*$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}})} \left((\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})^*(\mathcal{M}_1), (\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})^*(\mathcal{M}_2)\right) \to$$
$$\to \mathrm{Hom}_{\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{--mod}_{\mathsf{Z}'})} \left(\iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z},\mathsf{Z}'})^*(\mathcal{M}_1), \iota_{\widehat{\mathfrak{g}}}^{\mathsf{Z},\mathsf{Z}'})^*(\mathcal{M}_2)\right).$$

We rewrite the LHS using Corollary 18.4.2(2) as

$$\mathrm{Hom}_{\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}\left(\mathcal{M}_1, (\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})_* \circ (\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})^*(\mathcal{M}_2)\right) \simeq$$
$$\simeq \mathrm{Hom}_{\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}(\mathcal{M}_1, \mathsf{Z}' \underset{\mathsf{Z}}{\otimes} \mathcal{M}_2),$$

and the RHS using Lemma 23.3.4 as

$$\mathrm{Hom}_{\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}(\mathcal{M}_1, (\iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})_* \circ (\iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^*(\mathcal{M}_2)) \simeq \mathrm{Hom}_{\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}(\mathcal{M}_1, \mathsf{Z}' \underset{\mathsf{Z}}{\otimes} \mathcal{M}_2),$$

implying the desired isomorphism.

□

23.4. The functor (23.4) that appears in Proposition 23.3.7 is not an equivalence of categories. We shall now repeat the manipulation of Sect. 11 and turn it into an equivalence by modifying the LHS.

23.4.1. Consider the functor

$$(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_* : \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})$$

and its ind-extension

$$(\iota^{\mathsf{Z}',\mathsf{Z}}_{\widehat{\mathfrak{g}},ren})_* : \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}).$$

Proposition 23.4.2. *The functor $(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}},ren})_*$ is exact, and is conservative when restricted to $\mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$.*

The proof will be based on the following lemma, established in [FG2], Proposition 23.11:

Lemma 23.4.3. *For $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$ with $\mathcal{M}_1$ almost compact, the natural map*

$$\underset{i}{colim}\ \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{Z}^i_{\mathfrak{g}}})}\left((\iota^{\mathsf{Z},\mathfrak{Z}^i_{\mathfrak{g}}}_{\widehat{\mathfrak{g}}})_*(\mathcal{M}_1), (\iota^{\mathsf{Z},\mathfrak{Z}^i_{\mathfrak{g}}}_{\widehat{\mathfrak{g}}})_*(\mathcal{M}_2)\right) \to$$
$$\to \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})}\left((\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_*(\mathcal{M}_1), (\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_*(\mathcal{M}_2)\right)$$

is an isomorphism.

Proof. The right-exatness of $(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}},ren})_*$ follows by the definition of the t-structures on both sides from the following diagram

$$\begin{array}{ccc}
\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}}) & \xrightarrow{(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}},ren})_*} & \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}) \\
{\scriptstyle\Psi^{\mathsf{Z}}}\downarrow & & {\scriptstyle\Psi}\downarrow \\
\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}}) & \xrightarrow{(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_*} & \mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}),
\end{array}$$

which is commutative by construction.

To prove that $(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}},ren})_*$ is left-exact and conservative on $\mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$, it suffices to show that the following diagram also commutes:

$$\begin{array}{ccc}
\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}}) & \xrightarrow{(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}},ren})_*} & \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}) \\
{\scriptstyle\Phi^{\mathsf{Z}}}\uparrow & & {\scriptstyle\Phi}\uparrow \\
\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}}) & \xrightarrow{(\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_*} & \mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}).
\end{array}$$

In other words, we have to show the following:

Let $\mathcal{N}$ be an object of $\mathbf{D}^+(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$, represented as $\underset{A}{hocolim}\, \mathcal{M}_A$ for some set A, such that for $a \in A$, $\mathcal{M}_a \in \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$, and so that the arrow

$$\underset{a}{colim}\ \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}(\mathcal{M}', \mathcal{M}_a) \to \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}(\mathcal{M}', \mathcal{N})$$

is an isomorphism for any $\mathcal{M}' \in \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$. We have to show that in this case the arrow

$$\underset{a}{colim}\ \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})}\left(\mathcal{M}'', (\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_*(\mathcal{M}_a)\right) \to \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})}\left(\mathcal{M}'', (\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_*(\mathcal{N})\right)$$

is also an isomorphism for any $\mathcal{M}'' \in \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod})$.

By Lemma 23.4.3, the latter would follow once we show that

$$\underset{a}{colim}\ \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{z}^i_{\mathfrak{g}}})}\left(\mathcal{M}'', (\iota^{\mathsf{Z},\mathfrak{z}^i_{\mathfrak{g}}}_{\widehat{\mathfrak{g}}})_*(\mathcal{M}_a)\right) \to$$
$$\to \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{z}^i_{\mathfrak{g}}})}\left(\mathcal{M}'', (\iota^{\mathsf{Z},\mathfrak{z}^i_{\mathfrak{g}}}_{\widehat{\mathfrak{g}}})_*(\mathcal{N})\right)$$

is an isomorphism for any $i \gg 0$; in particular i is such that we can consider $\mathcal{M}''$ as an object of $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{z}^i_{\mathfrak{g}}})$.

We rewrite both sides of the above expression as

$$\underset{a}{colim}\ \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}\left((\iota^{\mathsf{Z},\mathfrak{z}^i_{\mathfrak{g}}}_{\widehat{\mathfrak{g}}})^*(\mathcal{M}''), \mathcal{M}_a\right) \to$$
$$\to \mathrm{Hom}_{\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})}\left((\iota^{\mathsf{Z},\mathfrak{z}^i_{\mathfrak{g}}}_{\widehat{\mathfrak{g}}})^*(\mathcal{M}''), \mathcal{N}\right).$$

The assertion follows now from the fact that $(\iota^{\mathsf{Z},\mathfrak{z}^i_{\mathfrak{g}}}_{\mathfrak{g}})^*(\mathcal{M}'') \in \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$, established above.

$\square$

23.4.4. We define a new functor

$$(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^{*,+} : \left(\mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\overset{\to}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})\right)^+ \to \mathbf{D}^+_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}'}) \tag{23.5}$$

as follows.

For $\mathcal{M} \in \mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\overset{\to}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathsf{Z}})$, which is $\geq i$, we set

$$(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^{*,+}(\mathcal{M}) := \tau^{\geq j}\left((\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^*(\mathcal{M})\right)$$

for some/any $j < i$. The independence of the choice of j is assured by Proposition 23.4.2, since we have an isomorphism of functors:

$$(\iota^{\mathsf{Z}'}_{\widehat{\mathfrak{g}}})_* \circ (\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^* \simeq (\iota^{\mathsf{Z}}_{\widehat{\mathfrak{g}}})_* \circ (\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})_* : \tag{23.6}$$
$$\mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}),$$

and the latter functor is exact.

As in Theorem 11.3.1 one shows that the functor $(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^{*,+}$ is an equivalence of categories.

Let $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}',\mathsf{Z}})$ be the full subcategory of $\mathrm{Spec}(\mathsf{Z}') \underset{\mathrm{Spec}(\mathsf{Z})}{\overset{\rightarrow}{\times}} \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}})$ consisting of objects $\mathcal{M}$, such that

$$(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\otimes} \mathrm{Id})_*(\mathcal{M}) \in \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}}).$$

Let $\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}',\mathsf{Z}})$ denote its ind-completion.

Consider the restriction of the functor $(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^{*,+}$ to $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}',\mathsf{Z}})$. The isomorphism (23.6) implies that the image of this functor belongs to $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}'})$. Let $(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^*_{ren}$ denote the ind-extension

$$\mathbf{D}_{ren}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}',\mathsf{Z}}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathsf{Z}'}).$$

Proposition 23.4.5. *The functor $(\iota^{\mathsf{Z}',\mathsf{Z}} \underset{\mathrm{Spec}(\mathsf{Z})}{\times} \iota^{\mathsf{Z},\mathsf{Z}'}_{\widehat{\mathfrak{g}}})^*_{ren}$ is an exact equivalence of categories.*

We omit the proof as it essentially repeats the proof of Theorem 11.4.2.

23.5. From D-modules to $\widehat{\mathfrak{g}}$-modules.

23.5.1. For any level κ we consider the abelian category $\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa$–mod, and its derived category $\mathbf{D}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$.

Let $\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod} \subset \mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa$–mod be the abelian subcategory of finitely generated D-modules. This pair of categories satisfies the conditions of Sect. 22.5.1. We obtain the renormalized category $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$,

$$\mathbf{D}^f_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) := \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \simeq \mathbf{D}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$$

and

$$\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \simeq \mathrm{Ind}\left(\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right).$$

Our present goal is to construct a functor

$$\Gamma_{\mathrm{Fl}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{–mod}),$$

and its ind-extension

$$\Gamma_{\mathrm{Fl}} : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \mathbf{D}_{ren}(\widehat{\mathfrak{g}}_\kappa\text{–mod}).$$

In order to do this we will use a particular DG model for the category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$.

23.5.2. Let $\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ be the category consisting of finite complexes of objects from $\mathfrak{D}^f(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}$. Let $\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \subset \mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ be the DG subcategory of acyclic complexes. By definition, $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ is the triangulated quotient

$$\mathrm{Ho}\left(\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right) / \mathrm{Ho}\left(\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right),$$

which by Sect. 15.7.2 endows $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ with a DG model. This is the standard DG model for $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$.

Let now $'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ be a full DG subcategory of $\mathbf{C}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ that consists of finite complexes $\mathcal{F}^\bullet$, with cohomologies belonging to $\mathbf{D}^{fg}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$, and such that each $\mathcal{F}^k$ is supported on a finite-dimensional subscheme of $\mathrm{Fl}^{\mathrm{aff}}_G$. Let $'\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \subset {}'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ be the DG subcategory of acyclic complexes.

We have a canonical 1-morphism in **DGCat**:

$$\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})/\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \\ \to {}'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})/{}'\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}).$$

It is easy to see that it induces an isomorphism on the level of homotopy categories. We obtain an equivalence

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \simeq \mathrm{Ho}\left({}'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right) / \mathrm{Ho}\left({}'\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right),$$

which equips $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ with a DG model within the same equivalence class.

23.5.3. Let now

$${}'\mathbf{C}^b_{adapt}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \subset {}'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$$

be a DG subcategory, whose objects are complexes $\mathcal{F}^\bullet$ such that for each k

$$R^i\Gamma_{\mathrm{Fl}}(\mathcal{F}^k) = 0, \ \forall i > 0.$$

Set

$${}'\mathbf{C}^b_{adapt,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) := {}'\mathbf{C}^b_{adapt}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \cap {}'\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}).$$

We have a canonical 1-morphism:

$${}'\mathbf{C}^b_{adapt}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})/{}'\mathbf{C}^b_{adapt,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \\ \to {}'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})/{}'\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}).$$

Lemma 23.5.4. *The functor*

$$\mathrm{Ho}\left({}'\mathbf{C}^b_{adapt}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right) / \mathrm{Ho}\left({}'\mathbf{C}^b_{adapt,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right) \to \\ \to \mathrm{Ho}\left({}'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right) / \mathrm{Ho}\left({}'\mathbf{C}^b_{acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right)$$

is an equivalence.

Proof. The assertion of the lemma follows from Lemma 21.1.4(2), since every D-module $\mathcal{F}$ over a finite-dimensional subscheme $\mathcal{Y}$ of $\mathrm{Fl}^{\mathrm{aff}}_G$ admits a Cech resolution with respect to an affine cover of $\mathcal{Y}$.

□

Thus, we have an equivalence

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \simeq \\ \mathrm{Ho}\left({}'\mathbf{C}^b_{adapt}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right) / \mathrm{Ho}\left({}'\mathbf{C}^b_{adapt,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right),$$

which equips $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})$ with yet another DG model within the same equivalence class. It is that latter DG model that we will use to construct the functor of sections.

23.5.5. We have a DG functor

$$\Gamma_{\mathrm{Fl}} : {}'\mathbf{C}^b_{adapt}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \mathbf{C}(\widehat{\mathfrak{g}}_\kappa\text{–mod}),$$

obtained by restriction from the evident functor

$$\Gamma_{\mathrm{Fl}} : \mathbf{C}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \mathbf{C}(\widehat{\mathfrak{g}}_\kappa\text{–mod}).$$

By construction,

$$\Gamma\left({}'\mathbf{C}^b_{adapt,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod})\right) \subset \mathbf{C}_{acycl}(\widehat{\mathfrak{g}}_\kappa\text{–mod}),$$

which endows the (usual) derived functor

$$\Gamma_{\mathrm{Fl}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod}) \tag{23.7}$$

with a DG model.

Proposition 23.5.6. *The image of the functor* (23.7) *belongs to* $\mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{–mod})$.

Since the DG model on $\mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ is inherited from that on $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$, we obtain that the desired functor

$$\Gamma^f : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa\text{–mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{–mod}),$$

equipped with a DG model.

23.5.7. *Proof of Proposition 23.5.6.* It is enough to show that for a finitely generated D-module $\mathcal{F}$ on Fl, the object $\Gamma_{\mathrm{Fl}}(\mathcal{F}) \in \mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{–mod})$ belongs to $\mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{–mod})$.

Let $\mathcal{F}$ be supported over a closed finite-dimensional subscheme $\mathcal{Y} \subset \mathrm{Fl}^{\mathrm{aff}}_G$. Then $\mathcal{F}$ admits a finite resolution by D-modules of the form $\mathrm{Ind}^{\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa}_{\mathcal{O}(\mathrm{Fl}^{\mathrm{aff}}_G)}(\mathcal{M})$, where $\mathcal{M}$ is a coherent sheaf on $\mathcal{Y}$. Thus, we can assume that $\mathcal{F}$ has this form.

Let $n > 0$ be sufficiently large so that

$$\mathfrak{g} \otimes (t^n \cdot \mathbb{C}[[t]]) \subset \mathrm{Lie}(I_y)$$

for all $y \in \mathcal{Y}$, where I_y denotes the conjugate of the Iwahori subgroup I corresponding to a point $y \in \mathrm{Fl}^{\mathrm{aff}}_G \simeq G(\!(t)\!)/I$. Consider the vector bundle $I_{\mathcal{Y}}/\mathfrak{g} \otimes (t^n \cdot \mathbb{C}[[t]])$ over $\mathcal{Y}$, whose fiber at $y \in \mathcal{Y}$ is $\mathrm{Lie}(I_y)/\mathfrak{g} \otimes (t^n \cdot \mathbb{C}[[t]])$. This vector bundle carries an action of the group $G(t^n \cdot \mathbb{C}[[t]])$. For an integer i consider its i-th exterior power $\Lambda^i(I_{\mathcal{Y}}/\mathfrak{g} \otimes (t^n \cdot \mathbb{C}[[t]]))$, and the $G(t^n \cdot \mathbb{C}[[t]])$-module

$$\Gamma\left(\mathcal{Y}, \Lambda^i(I_{\mathcal{Y}}/\mathfrak{g} \otimes (t^n \cdot \mathbb{C}[[t]])) \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} \mathcal{M}\right). \tag{23.8}$$

Consider the $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathbb{V}_{\kappa,n,i}(\mathcal{Y}, \mathcal{M}) := \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}\otimes(t^n\cdot\mathbb{C}[[t]])}\left(\Gamma\left(\mathcal{Y}, \Lambda^i(I_{\mathcal{Y}}/\mathfrak{g} \otimes (t^n \cdot \mathbb{C}[[t]])) \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} \mathcal{M}\right)\right).$$

The relative version of the Chevalley complex construction gives rise to a map of $\widehat{\mathfrak{g}}_\kappa$-modules

$$\mathbb{V}^i_{\kappa,n}(\mathcal{Y}, \mathcal{M}) \to \mathbb{V}^{i-1}_{\kappa,n}(\mathcal{Y}, \mathcal{M}),$$

such that the composition

$$\mathbb{V}^i_{\kappa,n}(\mathcal{Y},\mathcal{M}) \to \mathbb{V}^{i-1}_{\kappa,n}(\mathcal{Y},\mathcal{M}) \to \mathbb{V}^{i-2}_{\kappa,n}(\mathcal{Y},\mathcal{M})$$

vanishes. Denote the resulting complex $\mathbb{V}^\bullet_{\kappa,n}(\mathcal{Y},\mathcal{M})$.

Lemma 23.5.8. *Assume that for all i and $j>0$*

$$R^j\Gamma\left(\mathcal{Y},\Lambda^i(I_{\mathcal{Y}}/\mathfrak{g}\otimes(t^n\cdot\mathbb{C}[[t]]))\underset{\mathcal{O}_{\mathcal{Y}}}{\otimes}\mathcal{M}\right)=0.$$

Then $\mathbb{V}^\bullet_{\kappa,n}(\mathcal{Y},\mathcal{M})$ is quasi-isomorphic to $\Gamma_{\mathrm{Fl}}\left(\mathrm{Ind}^{\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_\kappa}_{\mathcal{O}(\mathrm{Fl}^{\mathrm{aff}}_G)}(\mathcal{M})\right)$.

This implies the assertion of the proposition:

Indeed, resolving $\mathcal{M}$ by coherent sheaves, we can assume that the vanishing condition of the above lemma holds. Hence, it suffices to see that the $\widehat{\mathfrak{g}}_\kappa$-modules $\mathbb{V}^i_{\kappa,n}(\mathcal{Y},\mathcal{M})$ belong to $\mathbf{D}^f(\widehat{\mathfrak{g}}_\kappa\text{-mod})$.

However, since $\mathcal{Y}$ is proper, the $G(t^n\cdot\mathbb{C}[[t]])$-module (23.8) is finite-dimensional, and hence admits a finite filtration with trivial quotients. Hence, $\mathbb{V}^i_{\kappa,n}(\mathcal{Y},\mathcal{M})$ admits a finite filtration with quotients isomorphic to $\mathbb{V}_{\kappa,n}$.

□

23.6. Sections at the critical level. Let us take now $\kappa=\kappa_{\mathrm{crit}}$. Our goal is to show that the functor Γ_{Fl} of Sect. 23.5.1 factors through a functor

$$\Gamma_{\mathrm{Fl}}:\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\to\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}}).$$

First, we recall that by [FG2], Sect. 7.19, for any $\mathcal{F}\in\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}$, the individual cohomologies $R^i\Gamma_{\mathrm{Fl}}(\mathcal{F})$ belong to the full subcategory

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}}\subset\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}.$$

Hence, the procedure of Sect. 23.5.5 defines a functor

$$\begin{aligned}&\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\simeq\\ &\simeq\mathrm{Ho}\left({}'\mathbf{C}^b_{adapt}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)/\,\mathrm{Ho}\left({}'\mathbf{C}^b_{adapt,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right)\to\\ &\to\mathrm{Ho}\left(\mathbf{C}(\widehat{\mathfrak{g}}_\kappa\text{-mod}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}})\right)/\,\mathrm{Ho}\left(\mathbf{C}_{acycl}(\widehat{\mathfrak{g}}_\kappa\text{-mod}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}})\right)\simeq\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}}),\end{aligned}$$

equipped with a DG model.

Thus, it remains to see that on the triangulated level, its essential image is contained in $\mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathfrak{Z}^{\mathrm{nilp}}_{\mathfrak{g}}})$. The latter results from Proposition 23.5.6 and the definition of the latter category (see Sect. 23.2.2).

24. DG model for the [AB] action

24.1. In this subsection we will discuss several different, but equivalent, DG models for the category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ that are needed to upgrade to the DG level various triangulated functors from the main body of the paper.

24.1.1. Recall the category $'\mathbf{C}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ and its subcategory of acyclic objects, denoted $'\mathbf{C}_{acycl}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$.

Let

$$\mathbf{C}^b_{aff}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \subset {}'\mathbf{C}^b(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$$

be the DG subcategory consisting of complexes $\mathcal{F}^\bullet$ with the additional condition that each $\mathcal{F}^i$ is a direct sum of D-modules, each being the direct image of a D-module on a finite-dimensional locally closed *affine* sub-scheme of $\mathrm{Fl}_G^{\mathrm{aff}}$. (Note that $\mathbf{C}^b_{aff}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$ is contained is the subcategory $\mathbf{C}_{adapt}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$, introduced earlier.)

Set

$$\mathbf{C}^b_{aff,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) := \mathbf{C}^b_{aff}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \cap {}'\mathbf{C}_{acycl}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}).$$

The proof of Lemma 23.5.4 shows that the natural functor

$$\begin{aligned}\mathrm{Ho}\left(\mathbf{C}^b_{aff}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right) / \mathrm{Ho}\left(\mathbf{C}^b_{aff,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right) \to \\ \to \mathrm{Ho}\left({}'\mathbf{C}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right) / \mathrm{Ho}\left({}'\mathbf{C}_{acycl}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right)\end{aligned}$$

is an equivalence.

Hence,

$$\mathbf{C}^b_{aff}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})/\mathbf{C}^b_{aff,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$$

defines a different, but equivalent, DG model for $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$.

The above DG model gives a different, but again equivalent, DG model for the functor $\Gamma_{\mathrm{Fl}} : \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \to \mathbf{D}^f(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{–mod}_{\mathrm{nilp}})$.

24.1.2. Let

$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \subset \mathbf{C}^b_{aff}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$$

be the DG subcategory, consisting of complexes $\mathcal{F}^\bullet$, where we impose the additional condition that each $\mathcal{F}^i$ is of the form $\mathrm{Ind}^{\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}}_{\mathcal{O}_{\mathrm{Fl}_G^{\mathrm{aff}}}}(\mathcal{M})$ for a quasi-coherent sheaf $\mathcal{M}$ on $\mathrm{Fl}_G^{\mathrm{aff}}$.

Set

$$\begin{aligned}\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) :=& \\ \mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) &\cap \mathbf{C}^b_{aff,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}).\end{aligned}$$

Lemma 24.1.3. *The natural functor*

$$\begin{aligned}\mathrm{Ho}\left(\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right) / \mathrm{Ho}\left(\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right) \to \\ \to \mathrm{Ho}\left(\mathbf{C}^b_{aff}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right) / \mathrm{Ho}\left(\mathbf{C}^b_{aff,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})\right)\end{aligned}$$

is an equivalence.

Proof. It suffices to show that any D-module equal to the direct image from a (sufficiently small) finite-dimensional affine subscheme $\mathcal{Y}$ of $\mathrm{Fl}_G^{\mathrm{aff}}$ admits a finite left resolution consisting of induced D-modules. Since the operation of induction commutes with that of direct image under a locally closed map, it suffices to construct such a resolution in the category of D-modules on $\mathcal{Y}$ itself.

We can assume that $\mathcal{Y}$ is small enough so that it is contained in an open ind–subscheme of $\mathrm{Fl}^{\mathrm{aff}}_G$, isomorphic to the ind-affine space $\mathbb{A}^\infty$; then $\mathcal{Y} \subset \mathbb{A}^n \subset \mathbb{A}^\infty$. The required resolution is given by the De Rham complex on $\mathbb{A}^n$.

□

Thus, we can use
$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})/\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$$
as yet another DG model for the category $\mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ and the functor Γ_{Fl}. In what follows, it will be this model that we will use to perform our constructions.

24.2. We shall now define a homotopy action of
$$\mathbf{C}^{free}(\mathrm{Coh}(\tilde{\check{\mathcal{N}}}/\check{G}))/\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\check{\mathcal{N}}}/\check{G}))$$
on
$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})/\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}).$$

We will use the paradigm of Sect. 16.7.5. First, we note:

Lemma 24.2.1. *Assume that $\mathcal{F}$ is a D-module on $\mathrm{Fl}^{\mathrm{aff}}_G$, which is induced from a quasi-coherent sheaf, and isomorphic to the direct image of a D-module from a finite-dimensional affine subscheme. Then*
$$H^j(\mathcal{F} \star J_{\check{\lambda}}) = 0,\ \forall j \neq 0 \text{ and } \check{\lambda} \in \check{\Lambda}.$$

Proof. Write $\check{\lambda} = \check{\lambda}_1 - \check{\lambda}_2$ with $\check{\lambda}_i \in \check{\Lambda}^*$.

Recall that the functor $? \star J_{\check{\mu}}$ is right-exact for $\check{\mu} \in \check{\Lambda}^+$, since in this case $J_{\check{\mu}} = j_{\check{\mu},*}$. The functor $? \star J_{-\check{\mu}}$, being the right (but in fact also left) adjoint of $J_{\check{\mu}}$, is left-exact.

Let us first show that $H^j(\mathcal{F} \star J_{\check{\lambda}}) = 0$ for $j > 0$. For that it suffices to see that $\mathcal{F} \star J_{-\check{\lambda}_2} \in \mathbf{D}^{\leq 0}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$. However, this is true for any convolution $\mathcal{F} \star \mathcal{F}'$ for $\mathcal{F}' \in \mathfrak{D}(\mathcal{Y})\text{–mod}^I$, where $\mathcal{Y}$ is an ind-scheme with an action of $G((t))$, provided that $\mathcal{F}$ is somorphic to the direct image of a D-module from a finite-dimensional affine subscheme.

Let us now show that $H^j(\mathcal{F} \star J_{\check{\lambda}}) = 0$ for $j < 0$. For that it suffices to see that $\mathcal{F} \star J_{\check{\lambda}_1} \in \mathbf{D}^{\geq 0}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$. But this is again true for any convolution $\mathcal{F} \star \mathcal{F}'$ as above, provided that $\mathcal{F}$ is induced.

□

24.2.2. The sought-for pseudo-action of $\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathcal{N}}}/\check{G}))$ on $\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ is defined as follows: for a collection of objects $\mathcal{M}^\bullet_1, ..., \mathcal{M}^\bullet_n \in \mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathcal{N}}}/\check{G}))$, $\mathcal{F}^\bullet_1, \mathcal{F}^\bullet_2 \in \mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$ we set

$$\mathrm{Hom}^\bullet_{\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\check{\mathcal{N}}}/\check{G})),\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})}(\text{"}\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n \otimes \mathcal{F}^\bullet_1\text{"}, \mathcal{F}^\bullet_2) := \mathrm{Hom}_{\mathbf{C}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})}(\mathcal{F}^\bullet_1 \star \mathsf{F}(\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n), \mathcal{F}^\bullet_2), \tag{24.1}$$

where

$$\mathcal{F}^\bullet_1 \star \mathsf{F}(\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n) \tag{24.2}$$

is regarded as a complex of D-modules on $\mathrm{Fl}^{\mathrm{aff}}_G$ (which does not in general belong to $\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$), obtained by *term-wise* application of the convolution functor.

We claim that the the required co-representability and vanishing conditions of Sect. 16.7.5 hold with respect to the subcategories

$$\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G})) \subset \mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$$

and

$$\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \subset \mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

Indeed, by Lemma 24.2.1, for fixed $\mathcal{M}^\bullet_1, ..., \mathcal{M}^\bullet_n$ and $\mathcal{F}^\bullet_1$ the co-representing object in

$$\mathrm{Ho}\left(\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right) / \,\mathrm{Ho}\left(\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})\right) \simeq \\ \simeq \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is represented by the complex (24.2).

Thus, we obtain a required homotopy action of

$$\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))/\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$$

on

$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})/\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}).$$

24.3. We shall now upgrade the functor Γ_{Fl} to a DG functor, compatible with the action of

$$\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))/\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G})).$$

To do so, it is sufficient to construct the corresponding structure on Γ_{Fl} as a homotopy functor between the DG categories

$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})/\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

and

$$\mathbf{C}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})/\mathbf{C}_{acycl}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}}).$$

24.3.1. We first consider the pseudo-functor

$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{C}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$$

as categories with a pseudo-action of $\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$. For $\mathcal{M}^\bullet_1, ..., \mathcal{M}^\bullet_n \in \mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$, $\mathcal{F}^\bullet \in \mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ and $\mathcal{V}^\bullet \in \mathbf{C}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})$, we set

$$(24.3) \quad \mathrm{Hom}_{\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G})),\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}),\mathbf{C}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})} \\ (\text{"}\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n \otimes \Gamma_{\mathrm{Fl}}(\mathcal{F}^\bullet)\text{"}, \mathcal{V}^\bullet) := \mathrm{Hom}_{\mathbf{C}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{nilp}})}\left(\Gamma_{\mathrm{Fl}}(\mathcal{F}^\bullet \star \mathsf{F}(\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n)), \mathcal{V}^\bullet\right),$$

where again

$$\Gamma_{\mathrm{Fl}}(\mathcal{F}^\bullet \star \mathsf{F}(\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n)) \tag{24.4}$$

is by definition obtained by applying the functor Γ_{Fl} term-wise to (24.2) (with $\mathcal{F}^\bullet_1$ replaced by $\mathcal{F}^\bullet_2$) as a complex of D-modules on $\mathrm{Fl}^{\mathrm{aff}}_G$.

The construction of the required natural transformations has been carried out in Sect. 3.3.

24.3.2. Now we claim that the co-representability conditions of Sect. 16.7.5 hold with respect to the subcategories:

$$\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G})) \subset \mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G})),$$
$$\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}) \subset \mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}),$$

and

$$\mathbf{C}_{acycl}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod}_{\mathrm{nilp}}) \subset \mathbf{C}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod}_{\mathrm{nilp}}).$$

Indeed, for $\mathcal{M}^\bullet_1, ..., \mathcal{M}^\bullet_n$, $\mathcal{F}^\bullet$ as above, the required object of $\mathbf{D}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod}_{\mathrm{nilp}})$ is the image of the complex (24.4).

Thus, we have constructed the required 1-morphism

$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})/\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}) \to \\ \to \mathbf{C}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod}_{\mathrm{nilp}})/\mathbf{C}_{acycl}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\,\mathrm{-mod}_{\mathrm{nilp}}),$$

in $\mathbf{DGmod}\left(\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))/\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))\right)$.

24.4. Recall the functor

$$\widetilde{\Upsilon}: \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}) \to \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}).$$

We will now upgrade it to the DG level, in a way compatible with the homotopy action of $\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))/\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$ on both sides. This amounts to constructing a 1-morphism

$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})/\mathbf{C}^b_{aff,ind,acycl}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}) \to \\ \to \mathbf{C}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})/\mathbf{C}_{acycl}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}). \tag{24.5}$$

in $\mathbf{DGmod}\left(\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))/\mathbf{C}^b_{acycl}(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))\right)$.

Proceeding as above, we first define a pseudo-functor

$$\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}) \to \mathbf{C}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})$$

with a compatibility data with respect to the action of $\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$.

For $\mathcal{M}^\bullet_1, ..., \mathcal{M}^\bullet_n \in \mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G}))$, $\mathcal{F}^\bullet \in \mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})$ and

$${}'\mathcal{F}^\bullet \in \mathbf{C}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}),$$

we set

$$\mathrm{Hom}^\bullet_{\mathbf{C}^b(\mathrm{Coh}^{free}(\tilde{\tilde{\mathcal{N}}}/\check{G})),\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}),\mathbf{C}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})} \\ \left(\text{"}\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n \otimes \widetilde{\Upsilon}(\mathcal{F}^\bullet)\text{"}, {}'\mathcal{F}^\bullet\right) := \\ \mathrm{Hom}_{\mathbf{C}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod})}\left(\mathcal{F}^\bullet \star \mathsf{F}(\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n) \star J_{2\rho} \star \mathcal{W}, {}'\mathcal{F}^\bullet\right), \tag{24.6}$$

where

$$\mathcal{F}^\bullet \star \mathsf{F}(\mathcal{M}^\bullet_1 \otimes ... \otimes \mathcal{M}^\bullet_n) \star J_{2\rho} \star \mathcal{W} \in \mathbf{C}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\,\mathrm{-mod}) \tag{24.7}$$

is given by term-wise convolution.

The co-representability condition of Sect. 16.7.5 is satisfied because for any D-module $\mathcal{F}$ appearing as a term of an object of $\mathbf{C}^b_{aff,ind}(\mathfrak{D}^{fg}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ and $\mathcal{M} \in \mathrm{Coh}^{free}(\tilde{\check{\mathcal{N}}}/\check{G})$, the convolution

$$\mathcal{F} \star \mathsf{F}(\mathcal{M}) \star \mathcal{F}' \in \mathbf{D}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

is acyclic off cohomological degree 0 for any $\mathcal{F}' \in \mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}^I$.

This defines the desired 1-morphism (24.5).

24.4.1. Finally, we remark that the above construction defines also the lifting of the natural transformation of Lemma 7.2.5 to a 2-morphism over $\mathrm{pt}/\check{G}$, once we lift the functor

$$\text{co-Ind} : \mathbf{D}(\mathrm{pt}/\check{B} \underset{\mathrm{pt}/\check{G}}{\times} \mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod}) \to \mathbf{D}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$$

to a 1-morphism over $\mathrm{pt}/\check{G}$, by the procedure of Sect. 21.6.2.

25. The I^0-equivariant situation

25.1. Let $\mathbf{D}$ be either of the categories $\mathbf{D}(\widehat{\mathfrak{g}}_\kappa\text{-mod})$, or $\mathbf{D}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{-mod})$ (the latter being considered with the *old* t-structure).

We define $\mathbf{D}^{I^0,+} \subset \mathbf{D}^+$ to be the full triangulated subcategory, consisting of complexes whose cohomologies are strongly I^0-equivariant objects of the corresponding abelian category, see [FG2], Sect. 20.11.

The goal of this subsection is to define the I^0-equivariant version of the corresponding category $\mathbf{D}_{ren}$. This will be done in the following abstract set-up.

25.1.1. Let $\mathbf{D}^f \subset \mathbf{D}$ be as in Sect. 22.2, so that the conditions of Proposition 22.2.1 hold.

Let $\mathbf{D}^+_1 \subset \mathbf{D}^+$ be a full triangulated subcategory. We assume that the following conditions hold:

- (1) The tautological functor $\mathrm{emb} : \mathbf{D}^+_1 \to \mathbf{D}^+$ admits a right adjoint, denoted $\mathrm{Av} : \mathbf{D}^+ \to \mathbf{D}^+_1$, such that the composition $\mathrm{emb} \circ \mathrm{Av} : \mathbf{D}^+ \to \mathbf{D}^+$ is left-exact.
- (2) For every $X \in \mathbf{D}$ belonging to a strongly generating set of objects of $\mathbf{D}^f$ there exists an inverse family $X \to \{... \to X^2 \to X^1\}$ with $X^k \in \mathbf{D}^+_1 \cap \mathbf{D}^f$ such that for any $Z \in \mathbf{D}^+_1$, the arrow

$$colim\ \mathrm{Hom}_{\mathbf{D}^+_1}(X^k, Z) \to \mathrm{Hom}_{\mathbf{D}^+}(X, Z)$$

  is an isomorphism.

Denote $\mathbf{D}^f_1 := \mathbf{D}^f_{1,ren} := \mathbf{D}^+_1 \cap \mathbf{D}^f$; being a triangulated subcategory of $\mathbf{D}$, it acquires a DG model. Hence, its ind-completion, denoted $\mathbf{D}_{1,ren}$, is well-defined. It comes equipped with a functor (which is also equipped with a DG model)

$$\mathrm{emb}_{ren} : \mathbf{D}_{1,ren} \to \mathbf{D}_{ren},$$

which is fully faithful, sends compact objects to compact ones, and commutes with direct sums. Hence, emb_{ren} admits a right adjoint, denoted Av_{ren}, which also commutes with direct sums.

Proposition 25.1.2. *Under the above circumstances we have:*

(a) The functor $\mathrm{emb}_{ren} \circ \mathrm{Av}_{ren} : \mathbf{D}_{ren} \to \mathbf{D}_{ren}$ *is left-exact.*

(b) The category $\mathbf{D}_{1,ren}$ *acquires a unique t-structure, for which the functor* emb_{ren} *is exact. (The functor* Av_{ren} *is then automatically left-exact.)*

(c) The (mutually adjoint) functors $\Psi : \mathbf{D}^+_{ren} \leftrightarrows \mathbf{D}^+ : \Phi$ *send the categories* $\mathbf{D}^+_{1,ren} \subset \mathbf{D}^+_{ren}$ *and* $\mathbf{D}^+_1 \subset \mathbf{D}^+$ *to one another. (We shall denote the resulting pair of mutually adjoint functors by* Ψ_1, Φ_1*, respectively.)*

(d) We have the isomorphisms of functors

$$\mathrm{Av}_{ren} \circ \Phi \simeq \Phi_1 \circ \mathrm{Av} : \mathbf{D}^+ \to \mathbf{D}^+_{1,ren} \text{ and } \mathrm{Av} \circ \Psi \simeq \Psi_1 \circ \mathrm{Av}_{ren} : \mathbf{D}^+_{ren} \to \mathbf{D}^+_1.$$

25.1.3. *Proof of Proposition 25.1.2.* First, we claim that there exists a natural transformation:

$$\mathrm{emb}_{ren} \circ \mathrm{Av}_{ren} \circ \Phi \to \Phi \circ \mathrm{emb} \circ \mathrm{Av} \tag{25.1}$$

between functors $\mathbf{D}^+ \to \mathbf{D}_{ren}$. To construct it, it is sufficient to construct a natural transformation

$$\Psi \circ \mathrm{emb}_{ren} \circ \mathrm{Av}_{ren} \circ \Phi \to \mathrm{emb} \circ \mathrm{Av} : \mathbf{D}^+ \rightrightarrows \mathbf{D}. \tag{25.2}$$

For $Y \in \mathbf{D}^+$ consider the maps from the LHS of (25.2) to the distinguished triangle

$$\mathrm{emb} \circ \mathrm{Av}(Y) \to Y \to \mathrm{Cone}(\mathrm{emb} \circ \mathrm{Av}(Y) \to Y).$$

We have a canonical map

$$\Psi \circ \mathrm{emb}_{ren} \circ \mathrm{Av}_{ren} \circ \Phi(Y) \to Y.$$

Hence, in order to construct the morphism in (25.2), it suffices to show that for any $Y' \in \ker(\mathrm{Av})$ and any $X \in \mathbf{D}_{1,ren}$, we have

$$\mathrm{Hom}_{\mathbf{D}}(\Psi \circ \mathrm{emb}_{ren}(X), Y') = 0,$$

which follows from the definitions.

Now, we claim that the natural transformation (25.1) is an isomorphism. To check it, it suffices to show that for any $Y \in \mathbf{D}^+$ and $X \in \mathbf{D}^f$ as in Condition (2) of Sect. 25.1.1, the map

$$\mathrm{Hom}_{\mathbf{D}_{ren}}(X, \mathrm{emb}_{ren} \circ \mathrm{Av}_{ren} \circ \Phi(Y)) \to \mathrm{Hom}_{\mathbf{D}_{ren}}(X, \Phi \circ \mathrm{emb} \circ \mathrm{Av}(Y)) \tag{25.3}$$

is an isomorphism. Let us write $\mathrm{Av}_{ren} \circ \Phi(Y)$ as $colim\, Z_i$, $Z_i \in \mathbf{D}^f_1$, i.e., for every $X_1 \in \mathbf{D}^f_1$ the arrow

$$\underset{i}{colim}\ \mathrm{Hom}_{\mathbf{D}^f_1}(X_1, Z_i) \to \mathrm{Hom}_{\mathbf{D}}(X_1, Y)$$

is an isomorphism. Let X^k be the corresponding inverse system for X. Then the LHS of (25.3) identifies with

$$\underset{i}{colim}\ \mathrm{Hom}_{\mathbf{D}^f}(X, Z_i) \simeq \underset{i}{colim}\, \underset{k}{colim}\ \mathrm{Hom}_{\mathbf{D}^f_1}(X^k, Z_i) \simeq$$
$$\simeq \underset{k}{colim}\, \underset{i}{colim}\ \mathrm{Hom}_{\mathbf{D}^f_1}(X^k, Z_i) \simeq \underset{k}{colim}\ \mathrm{Hom}_{\mathbf{D}}(X^k, Y).$$

The RHS of (25.3) identifies with

$$\mathrm{Hom}_{\mathbf{D}}(X, \mathrm{emb} \circ \mathrm{Av}(Y)) \simeq \underset{k}{colim}\ \mathrm{Hom}_{\mathbf{D}_1}(X^k, \mathrm{emb} \circ \mathrm{Av}(Y)) \simeq \underset{k}{colim}\ \mathrm{Hom}_{\mathbf{D}}(X^k, Y),$$

implying our assertion.

The isomorphism (25.1) readily implies point (a) of the proposition. Indeed, for $X \in \mathbf{D}^{\geq 0}_{ren}$, write $X = \Phi(Y)$ for $Y \in \mathbf{D}^{\geq 0}$. We have:

$$\mathrm{emb}_{ren} \circ \mathrm{Av}_{ren}(X) \simeq \Phi\left(\mathrm{emb} \circ \mathrm{Av}(Y)\right),$$

and as $\mathrm{emb} \circ \mathrm{Av}(Y) \in \mathbf{D}^{\geq 0}$ (by assumption), the assertion follows.

Point (b) is a formal corollary of point (a). Indeed, we claim that for $Z \in \mathbf{D}_{1,ren}$, the terms of the distinguished triangle

$$\tau^{\leq 0}(\mathrm{emb}_{ren}(Z)) \to \mathrm{emb}_{ren}(Z) \to \tau^{>0}(\mathrm{emb}_{ren}(Z)) \tag{25.4}$$

belong to the essential image of emb_{ren}, which implies our assertion. To prove the claim we compare the distinguished triangle (25.4) with

$$\mathrm{emb}_{ren} \circ \mathrm{Av}_{ren}\left(\tau^{\leq 0}(\mathrm{emb}_{ren}(Z))\right) \to \mathrm{emb}_{ren}(Z) \to \mathrm{emb}_{ren} \circ \mathrm{Av}_{ren}\left(\tau^{>0}(\mathrm{emb}_{ren}(Z))\right). \tag{25.5}$$

The adjunction map $\mathrm{emb}_{ren} \circ \mathrm{Av}_{ren} \to \mathrm{Id}$ gives rise to a map of triangles (25.5) $\to$ (25.4). On the other hand, since $\mathrm{emb}_{ren} \circ \mathrm{Av}_{ren}\left(\tau^{>0}(\mathrm{emb}_{ren}(Z))\right) \in \mathbf{D}^{>0}_{ren}$ and $\tau^{\leq 0}(\mathrm{emb}_{ren}(Z)) \in \mathbf{D}^{\leq 0}_{ren}$, we have a unique map (25.4) $\to$ (25.5). Moreover, the composition (25.4) $\to$ (25.5) $\to$ (25.4) equals the identity map. This implies that the terms of (25.4) are direct summands of the terms of (25.5). However, a direct summand of an object in the essential image of emb_{ren} itself belongs to the essential image of emb_{ren}.

Points (c) and (d) of the proposition follow formally from (a) and (b) and (25.1). □

25.2. We are going to apply Proposition 25.1.2 to $\mathbf{D}$ being one of the categories of Sect. 25.1 and $\mathbf{D}_1 = \mathbf{D}^{I^0}$.

Note that in the case of $\mathbf{D} = \mathbf{D}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$, this would establish Lemma 4.1.1 and Proposition 4.1.7(b).

25.2.1. Let us show that conditions (1) and (2) of Sect. 25.1.1 hold. This will be done in the following context:

Let $\mathbf{D}$ be the derived category of an abelian category $\mathcal{C}$ acted on by $G(\!(t)\!)$ of [FG2], Sect. 22.1. Note that condition (1) of Sect. 25.1.1 is given by [FG2], Sect. 20.10. Let H be a group sub-scheme of $G[[t]]$.

Let $\mathbf{D}^+(\mathcal{C})^{w,H}$ and $\mathbf{D}^+(\mathcal{C})^{s,H} := \mathbf{D}^+(\mathcal{C})^H$ be the corresponding weak and strong equivariant categories, respectively. Let

$$\mathbf{D}^+(\mathcal{C})^{s,H} \xrightarrow{\mathrm{emb}^{s,w}} \mathbf{D}^+(\mathcal{C})^{w,H} \xrightarrow{\mathrm{emb}^{w}} \mathbf{D}^+(\mathcal{C})$$

be the corresponding functors, and let $\mathrm{emb} := \mathrm{emb}^s := \mathrm{emb}^w \circ \mathrm{emb}^{s,w}$.

Let $\mathbf{D}^f(\mathcal{C}) \subset \mathbf{D}(\mathcal{C})$ be a full subcategory, contained in $\mathbf{D}^b(\mathcal{C})$, such that every object of $\mathbf{D}^f(\mathcal{C})$ is strongly equivariant with respect to some congruence subgroup $G(t^n \cdot \mathbb{C}[[t]])$.

We will make the following additional assumption, satisfied for the categories appearing in Sect. 25.1:

($\star$) *The category $\mathbf{D}^f(\mathcal{C})$ is strongly generated by objects that belong to the essential image of the functor $\mathbf{D}^+(\mathcal{C})^{w,H} \to \mathbf{D}^+(\mathcal{C})$.*

This assumption is satisfied for both examples under consideration.

For any X as above we will construct a family of objects X_k that satisfy condition (2) of Sect. 25.1.1.

25.2.2. Indeed, let $X = \mathrm{emb}^w(X_1)$ with $X_1 \in \mathbf{D}^{\leq 0}(\mathcal{C})^{w,H}$. Let X_1 be strongly equivariant with respect to $G(t^n \cdot \mathbb{C}[[t]])$.

Let H_n be the group $H/H \cap G(t^n \cdot \mathbb{C}[[t]])$, $\mathfrak{h}_n := \mathrm{Lie}(H_n)$. Consider the H_n-module $\mathrm{Fun}(H_n)$ and let us represent it as a union of finite-dimensional modules $\mathrm{Fun}(H_n)^k$. Let $(\mathrm{Fun}(H_n)^k)^*$ be the dual representations.

Then the desired objects X^k are given by

$$X^k := \Lambda^\bullet(\mathfrak{h}_n) \otimes \left(X_1 \otimes (\mathrm{Fun}(H_n)^k)^*\right),$$

where $X_1 \otimes (\mathrm{Fun}(H_n)^k)^*$ is regarded as an object of $\mathbf{D}(\mathcal{C})^{w,H}$, and for $Y \in \mathbf{D}(\mathcal{C})^{w,H}$

$$Y \mapsto \Lambda^\bullet(\mathfrak{h}_n) \otimes Y$$

is the (homological) Chevalley complex of $\mathfrak{h}_n$ with coefficients in Y, which by [FG2], Sect. 20.10 is the left adjoint functor to $\mathrm{emb}^{s,w}$, restricted to the strongly $G(t^n \cdot \mathbb{C}[[t]])$-equivariant subcategory.

25.3. In this subsection we will prove point (c) of Proposition 4.1.7, i.e., that the subcategory $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$ is compatible with the new t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$.

25.3.1. Let us consider a t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$, denoted new' by letting $\mathbf{D}_{ren}^{\leq 0_{new'}}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$ be generated by

$$\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0} \cap \mathbf{D}_{ren}^{\leq 0_{new}}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \cap \mathbf{D}^f_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}). \tag{25.6}$$

It is sufficient to show that the functor

$$\mathrm{emb}_{ren} : \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0} \to \mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$$

is exact in the new t-structures. The above functor is evidently right-exact. Hence, it remains to show the following:

(*) If $\mathcal{F}$ is an object of $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})^{I^0}$ such that $\mathrm{Hom}(\mathcal{F}', \mathcal{F}) = 0$ for any $\mathcal{F}'$ belonging to (25.6), then $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}) = 0$ for any

$$\mathcal{F}_1 \in \mathbf{D}_{ren}^{\leq 0_{new}}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}) \cap \mathbf{D}^f_{ren}(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod}).$$

25.3.2. For an element w of the affine Weyl group, let j_w denote the embedding of the corresponding I-orbit $\mathrm{Fl}_{w,G}^{\mathrm{aff}} \subset \mathrm{Fl}_G^{\mathrm{aff}}$. We shall say that w is right-maximal if w is the element of maximal length in its right coset with respect to the finite Weyl group. We have the following assertion ([AB], Lemma 15):

Lemma 25.3.3. *For any fixed* $\mathcal{F}_1 \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}_G^{\mathrm{aff}})_{\mathrm{crit}}\text{–mod})$*, for all* $\check{\lambda}$ *sufficiently deep inside the dominant cone,* $j_w^!\left(\mathcal{F}_1 \star j_{-\check{\lambda},*}\right) = 0$ *unless* w *is right-maximal.*

Let $\mathcal{F}$ and $\mathcal{F}_1$ be as in (*). Suppose by contradiction that we have a non-zero morphism $\mathcal{F}_1 \to \mathcal{F}$. Let $\check{\lambda}$ be as in the above lemma. Since $- \star j_{-\check{\lambda},!}$ is fully faithful, then the morphism

$$\mathcal{F}_1 \star j_{-\check{\lambda},!} \to \mathcal{F} \star j_{-\check{\lambda},!} \tag{25.7}$$

is non-zero either. Then there exists an element w in the affine Weyl group, such that the morphism

$$\mathcal{F}_1 \star j_{-\check{\lambda},!} \to j_{w*}\left(j_w^!(\mathcal{F} \star j_{-\check{\lambda},!})\right) \tag{25.8}$$

is non-zero. By the choice of $\check{\lambda}$, we obtain that w must be right-maximal. By the assumption on $\mathcal{F}$, the LHS in (25.8) is ≤ 0 in the old t-structure. Hence, to obtain a contradiction, it suffices to show that the RHS in(25.8) is > 0 in the old t-structure, or, which is the same, that $j^!_w(\mathcal{F} \star j_{-\check{\lambda},!})$ is > 0 as an object of the derived category of twisted D-modules on $\mathrm{Fl}^{\mathrm{aff}}_{w,G}$.

Since $\mathcal{F}$ is I^0-equivariant, the object $j^!_w(\mathcal{F} \star j_{-\check{\lambda},!})$ is an extension (in fact, a direct sum) of copies of the shifted constant D-module on $\mathrm{Fl}^{\mathrm{aff}}_{w,G}$; let $j_{w,!}$ denote the !-extension of the latter onto $\mathrm{Fl}^{\mathrm{aff}}_G$. Thus, it is enough to show that $\mathrm{Hom}(j_{w,!}[k], \mathcal{F} \star j_{-\check{\lambda},!}) = 0$ for $k \geq 0$. However, this follows from the assumption on $\mathcal{F}$ and the next assertion:

Lemma 25.3.4. *If w is right-maximal, then the object $j_{w,!} \in \mathbf{D}^f(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0}$ belongs to $\mathbf{D}^{\leq 0_{new}}_{ren}(\mathfrak{D}(\mathrm{Fl}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$.*

Proof. Indeed, for w, which is right-maximal, we have $j_{w,!} \star j_{-\check{\lambda},!} \simeq j_{w\cdot(-\check{\lambda}),!}$.

□

25.4. In order to prove Proposition 4.1.3 we will consider the following general context.

Let $F : \mathbf{A} \to \mathbf{A}'$ be a 1-morphism in **DGMonCat**. Let $G : \mathbf{C}_1 \to \mathbf{C}$ be a 1-morphism in $\mathbf{DGmod}(\mathbf{A})$. Assume that the functor $\mathrm{Ho}(G) : \mathrm{Ho}(\mathbf{C}_1) \to \mathrm{Ho}(\mathbf{C})$ is fully faithful.

Set $\mathbf{C}'_1 = \mathrm{Ind}^{\mathbf{A}'}_{\mathbf{A}}(\mathbf{C}_1)$ and $\mathbf{C}' = \mathrm{Ind}^{\mathbf{A}'}_{\mathbf{A}}(\mathbf{C})$. Let G' denote the resulting 1-morphism $\mathbf{C}'_1 \to \mathbf{C}'$ in $\mathbf{DGmod}(\mathbf{A}')$, and

$$F_{\mathbf{C}_1} : \mathbf{C}_1 \to \mathbf{C}'_1 \text{ and } F_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}'$$

be the corresponding morphisms in **DGCat**.

Proposition 25.4.1. *Assume that* $\mathrm{Ho}(\mathbf{A})$ *is rigid. We have:*

(1) *The functor* $\mathrm{Ho}(G') : \mathrm{Ho}(\mathbf{C}'_1) \to \mathrm{Ho}(\mathbf{C}')$ *is fully faithful.*

(2) *The natural transformation* $G^* \circ F_{\mathbf{C}_1 *} \to F_{\mathbf{C}*} \circ G'^*$ *is an isomorphism.*

(3) *If* $F_* : \mathrm{Ho}(\mathbf{A}') \to \mathrm{Ho}(\mathbf{A})$ *is conservative, then the essential image of* $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}'_1)$ *in* $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}')$ *under* G'^* *equals the pre-image of* $G^*(\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}_1)) \subset \mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}})$ *under* $F_{\mathbf{C}*}$.

Proof. Point (1) follows from Corollary 18.4.2(2). Point (2) follows from Corollary 18.4.3.

To prove point (3), we note that the functor $F_{\mathbf{C}}$ is also conservative (see Proposition 20.4.1), hence, it suffices to show that the natural transformation

$$G^* \circ G_* \circ F_{\mathbf{C}*} \to F_{\mathbf{C}*} \circ G'^* \circ G'_*$$

is an isomorphism. However, this follows from point (2).

□

25.4.2. To prove Lemma 4.1.3 we apply Proposition 25.4.1 to $\mathbf{A}$ and $\mathbf{A}'$ being DG models of $\mathbf{D}^{perf}(\mathrm{Coh}(\tilde{\mathcal{N}}/\check{G}))$ and $\mathbf{D}^{perf}(\mathrm{Coh}(\mathrm{Op}^{\mathrm{nilp}}))$, respectively, and $\mathbf{C}_1 \subset \mathbf{C}$ being DG models of

$$\mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})^{I^0} \subset \mathbf{D}^f(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod}).$$

25.4.3. *Proof of Proposition 4.1.7(d).* Assume now that in the context of Sect. 25.4, the categories $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}), \mathrm{Ho}(\underset{\rightarrow}{\mathbf{A}}), \mathrm{Ho}(\underset{\rightarrow}{\mathbf{A}}')$ are equipped with t-structures, satisfying the assumptions of Sect. 20.1. Consider the resulting t-structures on $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}')$.

The next assertion follows immediately from Proposition 25.4.1(2) and Proposition 20.4.1:

Corollary 25.4.4. *Assume that the subcategory* $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}_1) \subset \mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}})$ *is compatible with the t-structure. Assume also that the functor F is affine. Then the subcategory* $\mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}'}_1) \subset \mathrm{Ho}(\underset{\rightarrow}{\mathbf{C}}')$ *is also compatible with the t-structure.*

We apply the above corollary to the same choice of the categories as in Sect. 25.4.2, and the new t-structure on $\mathbf{D}_{ren}(\mathfrak{D}(\mathrm{Gr}^{\mathrm{aff}}_G)_{\mathrm{crit}}\text{–mod})$. The required compatibility with the t-structure is insured by Proposition 4.1.7(c).

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