D-Modules on Spaces of Rational Maps and on Other Generic Data

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D-MODULES ON SPACES OF RATIONAL MAPS AND ON OTHER GENERIC DATA

Abstract

Fix an algebraic curve X. We study the problem of parametrizing geometric data over X, which is only generically defined. E.g., parametrizing generically defined maps from X to a fixed target scheme Y. There are three methods for constructing functors of points for such moduli problems (all originally due to Drinfeld), and we show that the resulting functors are equivalent in the fppf Grothendieck topology. As an application, we obtain three presentations for the category of D-modules "on" $B(K) \backslash G(\mathbb{A}) / G(\mathbb{O})$ and combine results about this category coming from the different presentations.

Contents

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1. Introduction

Let k be an algebraically closed field of characteristic 0, and let X be a smooth, projective and connected algebraic curve over k. Denote by $\mathbb{A}$, $\mathbb{O}$ and K the algebra of adeles, algebra of integer adeles, and the field of rational functions over X, respectively.

In this paper we study the problem of parametrizing geometric data over X which is only generically defined. The basic example of such a moduli problem is that of generically defined maps from X to a fixed target scheme Y. I.e., the starting point is the given set of k-points (in this case it is the set $\mathrm{Hom}\,(\mathrm{spec}\,(K)\,,Y))$ and the task at hand is that of constructing a functor of points $\mathbf{Scheme}^{\mathrm{op}} \to \mathbf{Set}$ which describes what an S-family of such generic maps is, for an arbitrary scheme S.

The main example we are interested in is motivated by the Langlands program. In the classical setting one encounters sets such as $B\,(K)\,\backslash G\,(\mathbb{A})\,/G\,(\mathbb{O})$, $N\,(K)\,\backslash G\,(\mathbb{A})\,/G\,(\mathbb{O})$ and their relatives; the premise of the geometric program is that these sets are the k-points of some space ("space" interpreted very loosely). The story goes that each point in the set $B\,(K)\,\backslash G\,(\mathbb{A})\,/G\,(\mathbb{O})$ is to be interpreted as representing a G-bundle on X, together with the data of a reduction to B at the generic point of X[1]. We wish to describe a space parametrizing such data via a functor of points, and as above our starting point is the given set of k-points, and our task is to define what an S-family of such generic reductions is.

The literature (and mathematical folklore) contains three, a-priori different, construction schemas for such moduli problems (all originally due to Drinfeld). The main result of this paper is that all three constructions give rise to functors of points which are equivalent in the fppf Grothendieck topology. Consequently, various invariants such as the categories of quasi-coherent and D-module (as well as derivative invariants such as homology groups) are equivalent.

An overview of the paper is as follows:

In section 2 we present the first construction schema, which we consider to be the conceptually fundamental one. Unfortunately, conceptual appeal notwithstanding, this approach is deficient in that invariants of the spaces so constructed are not easy to describe (directly).

In sections 3 and 4, we describe an approach which involves degenerations of regular data. This approach is starts from the notion of *quasi-maps* and the space $\overline{\mathrm{Bun}_B}$ which were first presented by Finkelberg and Mikovic in [FM99], and have received a fair amount of attention since. In particular, the construction we present is the one used by Gaitsgory in [Gai10b]. In broad terms, the upshot of this

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[1]This is admittedly equivalent to the set of global reductions to B, but this interpretation leads to a different space! See also remark 2.2.4.

construction is that the spaces in question are presented as quotients of a scheme (or Artin stack) by a proper (and schematic) equivalence relation.

In section 5 we describe an approach for parametrizing generic data using the *Ran Space*. This is the approach taken by Gaitsgory in [Gai10a, Gai10b]. This approach has the advantage that certain invariants of the spaces so constructed are amenable to computation via chiral homology methods. In particular, in [Gai11a] Gaitsgory succeeds in computing the homology groups in certain cases.

In section 6 we use Gaitsgory's initial homology computation for spaces of rational maps to obtain similar results for additional moduli problems not discussed in [Gai11a].

2. Moduli spaces of generic data

Notation 2.0.1. Recall that k is an algebraically closed field of characteristic 0, and that X is a smooth connected and projective curve over k. We denote by $\mathfrak{S}$ the category of finite type schemes over k, and by $\mathfrak{Aff}$ the category of finite type affine schemes over k.

By an ∞-*category* we mean an $(\infty, 1)$-category, see appendix 7 for more details. We denote by $\mathbf{Cat}_\infty$ the ∞-category of (small) ∞-categories. We respectively denote by $\mathbf{Set}$ and $\mathbf{Gpd}_\infty$ the full subcategories of sets and ∞-groupoids in $\mathbf{Cat}_\infty$. We shall occasionally refer to a groupoid as a *homotopy type* or a *space*.

For a category C, we let $\mathcal{P}shv(C)$ denote the ∞-category of presheaves i.e., functors $C^{\mathrm{op}} \to \mathbf{Gpd}_\infty$. In the particular case when $C = \mathfrak{Aff}$ we use the term *functor of points* to refer to a presheaf in $\mathcal{P}shv(\mathfrak{Aff})$. When C is equipped with a Grothendieck topology τ, we denote the corresponding ∞-category of sheaves by $\mathcal{S}hv(C; \tau)$ (or omit τ when it is obvious which topology is being used).

If C is a category which has been constructed to classify certain data, we shall often denote an object of C by listing the data which it classifies (and we shall say that the data *presents* the object). For example, in definition 2.1.1 below, we use the expression $(S, U_S \subseteq S \times X)$ to denote an object of the category Dom_X; it should be clear from the context what kind of data each term in the parenthesis refers to. When the data is required to satisfy certain conditions, these are implicitly assumed to hold and are not reflected in the notation.

2.1. **Families of domains.** In the interest of motivating definition 2.1.1, consider the problem of constructing a moduli space of rational functions on X (i.e., generically defined maps to $\mathbb{A}^1$), $K_X : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}$. An S-point of this functor, $f \in K_X(S)$ should be presented by a rational function on $S \times X$. Let $U \subseteq S \times X$ be the largest open subscheme on which f is defined. In order for K_X to be a functor, we must be able to pull back f along any map of schemes, $T \to S$. Requiring

that these pullbacks be defined amounts to the condition that for every $s \in S$, U intersects the fiber $s \times X$. The following definition captures this property of the domain of f:

Definition 2.1.1.

(1) A domain in X is its non-empty open subscheme[2]. Let S be a scheme, an *S-family of domains* in X is an open subscheme $U_S \subseteq S \times X$ which is *universally dense* with respect to S. I.e., for every map of schemes $T \to S$, after forming the pullback,

$$\begin{array}{ccc} U_T & \longrightarrow & U_S \\ \downarrow & & \downarrow \\ T \times X & \longrightarrow & S \times X \end{array}$$

we have that $U_T \subseteq T \times X$ is also dense. It suffices to check this condition at all closed points of S. I.e., we are simply requiring that for every closed point, the open subscheme $U_s \subseteq X$ is a domain (in particular it's non-empty).

(2) The totality of all families domains in X forms a (ordinary) category, Dom_X:

An object: consists of the data $(S, U_S \subseteq S \times X)$ where $S \in \mathfrak{Aff}$, and U_S is a family of domains in X.

A morphism: $(S, U_S \subseteq S \times X) \to (T, U_T \subseteq T \times X)$ consists of the data of a map of affine schemes $S \xrightarrow{f} T$ which induces a commutative diagram

$$\begin{array}{ccc} U_S & \longrightarrow & U_T \\ \downarrow & & \downarrow \\ S \times X & \xrightarrow{f \times id_X} & T \times X \end{array}$$

There exists an evident functor

$$\mathrm{Dom}_X \xrightarrow{\quad q \quad} \mathfrak{Aff}$$

$$(S, U_S \subseteq S \times X) \longmapsto S$$

which is a Cartesian fibration, and whose fiber, $(\mathrm{Dom}_X)_S$, is the poset of S-families of domains in X (a full subcategory of all open sub-schemes of $S \times X$).

2.2. **Abstract moduli spaces of generic data.**

---

[2]In our case such a subscheme in a-priori dense. Generalizing this notion to an arbitrary scheme, which might not be connected, one should additionally impose a density condition.

Notation 2.2.1. A functor $C \xrightarrow{f} D$ induces, via pre-composition, a pull back functor $\mathcal{P}shv\,(C) \xleftarrow{f_*} \mathcal{P}shv\,(D)$. f_* fits into an adjoint triple (LKE_f, f_*, RKE_f), where LKE_f and RKE_f are defined on the objects of $\mathcal{P}shv\,(D)$, by left and right Kan extensions (respectively) along

$$
\begin{array}{ccc}
C^{\mathrm{op}} & \xrightarrow{\ \mathcal{F}\ } & \mathbf{Gpd}_\infty \\
\llap{\scriptstyle f}\downarrow & \nearrow & \\
D^{\mathrm{op}} & &
\end{array}
$$

The following definition formalizes what we mean by a moduli problem of generic data over X :

Definition 2.2.2. The category of (abstract) *moduli problems of generic data* is $\mathcal{P}shv\,(\mathrm{Dom}_X)$. Given a presheaf $(\mathrm{Dom}_X)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Gpd}_\infty$, its *associated* functor of points is $\mathrm{LKE}_q\mathcal{F} \in \mathcal{P}shv\,(\mathfrak{Aff})$, where q denotes the fibration $\mathrm{Dom}_X \xrightarrow{q} \mathfrak{Aff}$.

Example 2.2.3. Rational functions form a moduli problem of generic data if we set $K_X^D : (\mathrm{Dom}_X)^{\mathrm{op}} \to \mathrm{Sets}$ to be the functor

$$
(S, U_S \subseteq S \times X) \mapsto \mathrm{Hom}_{\mathfrak{S}}\left(U_S, \mathbb{A}^1\right)
$$

Its associated functor of points, $K_X := \mathrm{LKE}_q\left(K_X^D\right) : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}$, sends

$$
S \mapsto \{f \in K\,(S \times X) : \text{ the domain of } f \text{ is an } S - \text{familiy of domains in } X\}
$$

where $K\,(S \times X)$ is the algebra of global section of the sheaf of total quotients. I.e., the functor K_X forgets the evidence that f is defined on a large enough domain.

Replacing $\mathbb{A}^1$ with an arbitrary target scheme Y we obtain similarly constructed presheaves classifying generically defined maps from X to Y:

$$
\mathrm{GMap}^D\,(X, Y) : \mathrm{Dom}_X^{\mathrm{op}} \to \mathbf{Set}
$$

and its associated functor of points

$$
\mathrm{GMap}\,(X, Y) : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}
$$

Remark 2.2.4. Let $\mathrm{Map}\,(X, Y)$ denote the functor of points which parametrizes families of regular maps. Since X is a curve, when Y is projective every generically defined map from X to Y admits a (unique) extension to a regular map defined across all of X. However, this is no longer the case in families, and consequently the map $\mathrm{Map}\,(X, Y) \to \mathrm{GMap}\,(X, Y)$ is not an equivalence, despite inducing an isomorphism on the set of k-points. E.g., when $Y = \mathbb{P}^1$ the functor $\mathrm{Map}\,\left(X, \mathbb{P}^1\right)$ has infinitely many components (labeled the degree of the map), but $\mathrm{GMap}\,\left(X, \mathbb{P}^1\right)$ is connected.

Example 2.2.5. Reduction spaces - Let $\mathrm{Bun}_G^{B(\mathrm{Dom}_X)} : (\mathrm{Dom}_X)^{\mathrm{op}} \to \mathbf{Gpd}$ be the functor which sends $(S, U_S \subseteq S \times X)$ to the groupoid which classifies, up to isomorphism, the data

$$\left(\mathcal{P}_G, \mathcal{P}_B^{U_S}, \mathcal{P}_B^{U_S} \times_B G \overset{\phi}{\underset{\cong}{\to}} \mathcal{P}_G\big|_{U_S} \right)$$

where $\mathcal{P}_G$ is a G-torsor over $S \times X$, $\mathcal{P}_B^{U_S}$ is a B-torsor on U_S, and ϕ is and isomorphism of G-bundles over U_S (the data of a reduction of the structure group of $\mathcal{P}_G$ to B, over U_S). Denote its associated functor of points by

$$\mathrm{Bun}_G^{B(\eta)} := \mathrm{LKE}_q \left(\mathrm{Bun}_G^{B(\mathrm{Dom}_X)} \right) \in \mathcal{P}shv\,(\mathfrak{Aff})$$

The latter is a geometrization of the set $B\,(K) \backslash G\,(\mathbb{A}) \,/ G\,(\mathbb{O})$. I.e., the isomorphism classes of the groupoid $\mathrm{Bun}_G^{B(\eta)}\,(k)$ are in bijection with this set.

More generally, if H is any subgroup of G, we define in a similar way a functor of points $\mathrm{Bun}_G^{H(\eta)}$, which classifies families of G-bundles on X with a generically defined reduction to H. In particular, $\mathrm{Bun}_G^{1(\mathrm{Dom}_X)}$ is the moduli space of G-bundles with a generic trivialization (equivalently a generic section).

Remark 2.2.6. Given a presheaf $(\mathrm{Dom}_X)^{\mathrm{op}} \overset{\mathcal{F}}{\to} \mathbf{Gpd}_\infty$, recall that its associated functor of points is the left Kan extension

$$
\begin{array}{ccc}
(\mathrm{Dom}_X)^{\mathrm{op}} & \overset{\mathcal{F}}{\longrightarrow} & \mathbf{Gpd}_\infty \\
q^{\mathrm{op}} \downarrow & \nearrow & \\
\mathfrak{Aff}^{\mathrm{op}} & \mathrm{LKE}_q \mathcal{F} &
\end{array}
$$

Ultimately, the object we wish to study is $\mathrm{LKE}_q \mathcal{F}$ and its invariants; $\mathcal{F}$ itself is no more than a presentation of the former.

Noting that q^{op} is a co-Cartesian fibration, it follows that for every $S \in \mathfrak{Aff}$, we have that

$$\mathrm{LKE}_q \mathcal{F}\,(S) \cong \mathrm{colim} \left((\mathrm{Dom}_X)_S \overset{\mathcal{F}}{\to} \mathbf{Gpd}_\infty \right)$$

I.e. the passage from $\mathcal{F}$ to $\mathrm{LKE}_q \mathcal{F}$ simply identifies generic data which agrees on a smaller domain. Noting further that $(\mathrm{Dom}_X)_S$ is weakly contractible (it is a filtered poset), we see that for any $\mathcal{G} \in \mathcal{S}hv\,(\mathfrak{Aff})$ the transformation ($\mathrm{LKE}_q \circ q_*)\,\mathcal{G} \overset{\cong}{\Rightarrow} \mathcal{G}$ is an equivalence, i.e. the functor $\mathcal{P}shv\,(\mathfrak{Aff}) \overset{q_*}{\to} \mathcal{P}shv\,(\mathrm{Dom}_X)$ is fully faithful, and that LKE_q is a localization.

We note for later use that LKE_q is left exact [3]; this follows from [Lur11b, lemma 2.4.7] after noting that Dom_X admits all finite limits, and that the functor and q preserves these.

---

[3]Preserves finite limits.

2.3. **D-modules.** For the most part the Langlands program is not as much interested in the set $B\left(K\right)\backslash G\left(\mathbb{A}\right)/G\left(\mathbb{O}\right)$, as it is in the space of functions on this set. The appropriate geometric counterpart of this space of functions should be an appropriate category of sheaves on the space chosen as the geometrization of the set. When k is of characteristic 0, this category is expected to be the category of sheaves of D-modules. We now explain how to assign to every moduli problems of generic data (presheaf on Dom_X) a category of sheaves of D-modules.

We consider the totality of D-modules to have the structure of a *stable* ∞-category, instead of a triangulated category (as is more common); the latter is the homotopy category of the former. For a short summary with references to notions, notation and ideas in this below see subsection 7.3. We denote by $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$ the ∞-category of stable ∞-categories which are co-complete, with functors which are colimit preserving (equivalently are left adjoints). The homotopy category of such a stable infinity category is a triangulated category; limits in $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$ should be thought of as homotopy limits of triangulated categories.

There exists a functor $\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\mathfrak{Dmod}} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$ which assigns to a scheme S (thought of as a presheaf via the Yoneda embedding) the a stable category $\mathfrak{Dmod}\left(S\right)$, whose homotopy category is the usual triangulated category of sheaves of D-modules on S, see 7.3.3 for more details. We think of $\mathfrak{Dmod}$ as an invariant defined on schemes (whose values are categories).

We extend $\mathfrak{Dmod}$ to arbitrary functors of points via right Kan extension along the Yoneda embedding.

$$
\begin{array}{ccc}
\mathfrak{Aff}^{\mathrm{op}} & \xrightarrow{\mathfrak{Dmod}} & \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}} \\
\downarrow{\scriptstyle j} & \nearrow & \\
& {\scriptstyle \mathfrak{Dmod}:=\mathrm{RKE}} & \\
\mathcal{P}shv\left(\mathfrak{Aff}\right)^{\mathrm{op}} & &
\end{array}
$$

So defined, $\mathfrak{Dmod}$ preserves small limits ([Lur09, lemma 5.1.5.5]).

We further extend $\mathfrak{Dmod}$ to $\mathcal{P}shv\left(\mathrm{Dom}_X\right)$ by composing

$$
\mathcal{P}shv\left(\mathrm{Dom}_X\right)^{\mathrm{op}} \xrightarrow{\mathrm{LKE}_q} \mathcal{P}shv\left(\mathfrak{Aff}\right)^{\mathrm{op}} \xrightarrow{\mathfrak{Dmod}} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}
$$

Remark 2.3.1. For $\mathcal{F} \in \mathcal{P}shv\left(\mathrm{Dom}_X\right)$, the category $\mathfrak{Dmod}\left(\mathcal{F}\right)$ can be presented as a limit over the category $\left(\left(\mathrm{Dom}_X\right)_{/\mathcal{F}}\right)^{\mathrm{op}}$

$$
\mathfrak{Dmod}\left(\mathcal{F}\right) := \mathfrak{Dmod}\left(\mathrm{LKE}_q\mathcal{F}\right) \cong \lim_{(S,U)\to\mathcal{F}}\left(\mathfrak{Dmod}\left(S\right)\right)
$$

The premise of this paper is that a presheaf on Dom_X is the conceptually natural way of classifying structures generically defined over X. However, in practice the limit presentation of $\mathfrak{Dmod}\left(\mathrm{LKE}_q\mathcal{F}\right)$ we obtain as above is unwieldy. In the

following sections we shall construct better, more economical, presentations of this invariant.

2.4. **Grothendieck topologies on** Dom_X. In [GR11, Cor 3.1.4] it is proven that D-modules may be descended along fppf covers i.e., $\mathfrak{Dmod}$ factors though fppf sheafification. For a moduli problem $\mathcal{F} \in \mathcal{P}shv\,(\mathrm{Dom}_X)$, this "continuity" property with respect to the fppf topology can be harnessed to obtain more economical presentations for the category $\mathfrak{Dmod}\,(\mathcal{F})$, than the one given in remark 2.3.1. To this end we proceed to define a few natural Grothendieck topologies on Dom_X.

Let τ be the be either the Zariski, etale or fppf Grothendieck topology on $\mathfrak{S}$. We endow Dom_X with a corresponding Grothendieck pulled back from $\mathfrak{S}$ using the functor

$$\mathrm{Dom}_X \longrightarrow \mathfrak{S}$$

$$(S, U_S \subseteq S \times X) \longmapsto U_S$$

Explicitly, a collection of morphisms in Dom_X, $\{(S_i, U_{S_i} \subseteq S_i \times X) \to (S, U_S \subseteq S \times X)\}$, is a τ-cover in Dom_X iff the collection of morphisms $\{U_{S_i} \to U_S\}$ is an τ-cover in $\mathfrak{S}$.

Observe that for all the choices of τ above, the functor $\mathrm{Dom}_X \xrightarrow{q} \mathfrak{Aff}$ is continuous in the sense that every for cover $\{(S_i, U_{S_i} \subseteq S_i \times X) \to (S, U_S \subseteq S \times X)\}$ in Dom_X, its image in $\mathfrak{Aff}$, $\{S_i \to S\}$, is a cover (this follows from the observation that $U_S \to S$ is a cover in all our topologies). Furthermore, Dom_X and $\mathfrak{Aff}$ both admit all finite limits, and q preserves these. By lemma [Lur11b, Lemma 2.4.7] q induces an adjoint pair of functors, $\mathcal{S}hv\,(\mathrm{Dom}_X) \xleftarrow{(q^*, q_*)} \mathcal{S}hv\,(\mathfrak{Aff})$ in which the functor q_* is pullback along q, and the functor q^* is the composition of LKE_q followed by sheafification and has the property of preserving finite limits[4].

The upshot is, that because $\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\mathfrak{Dmod}} \hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ satisfies fppf descent, its extension $\mathcal{P}shv\,(\mathrm{Dom}_X)^{\mathrm{op}} \xrightarrow{\mathfrak{Dmod}} \hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ factors through $\mathcal{S}hv\,(\mathrm{Dom}_X; \mathrm{fppf})$. In particular, any map of presheaves on Dom_X which induces an equivalence after sheafification, induces an equivalence D-module categories.

3. Quasi-maps

Recall example **2.2.3**, in which we constructed a moduli problem $\mathrm{GMap}\,(X, Y)$, classifying generically defined maps from X to Y. In this section we present another approach to constructing a functor of points for this moduli problem using the notion of *quasi-maps*. The latter notion is originally due to Drinfeld, was first described by Finkelberg and Mirkovic in [FM99], and has received a fair amount of attention since. This approach has the advantage of presenting the space of generic maps as a quotient of a scheme by a proper (and schematic) equivalence relation.

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[4]I.e., a geometric morphism of topoi.

The main result of this section is to prove that, up to sheafification in the Zariski topology, both approaches give equivalent functors of points. Consequently, the associated categories of D-modules are equivalent.

For the duration of this section fix $Y \hookrightarrow \mathbb{P}^n$, a scheme Y *together with* the data of a quasi-projective embedding. The space of generic maps constructed using quasi maps a-priori might depend on this embedding, however it follows from the equivalence with $\mathrm{GMap}\,(X, Y)$ which we prove that, in fact, it does not (up to sheafification).

3.1. **Definitions.** First, a minor matter of terminology. Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles, we distinguish between two properties of a map of quasi-coherent sheaves $\mathcal{V} \to \mathcal{W}$: The map is called a *sub-sheaf embedding* if it is an injective map of quasi-coherent sheaves. It is called a *sub-bundle embedding* if it is an injective map of sheaves whose co-kernel is flat (i.e., also a vector bundle). The latter corresponds to the notion of a map between geometric vector bundles which is (fiber-wise) injective.

We start by defining the notion of a *quasi map* from X to $\mathbb{P}^n$. Recall that that a regular map $X \to \mathbb{P}^n$ may be presented by the data of a line bundle $\mathcal{L}$ on X together with a sub-bundle embedding $\mathcal{L} \xrightarrow{\subseteq} \mathcal{O}_X^{n+1}$. A quasi-map from X to $\mathbb{P}^n$ is a degeneration of a regular map consisting of the data of a line bundle $\mathcal{L}$ on X, together with a sub-sheaf embedding $\mathcal{L} \hookrightarrow \mathcal{O}_X^{n+1}$ (i.e., it may not be a sub-bundle). Observe that to any quasi-map we may associate the open subscheme $U \subseteq X$ over which $\mathcal{L}\big|_U \hookrightarrow \mathcal{O}_U^{n+1}$ is a sub-bundle - where it induces a regular map $U \to \mathbb{P}^n$. In particular, to every quasi-map we may associate a generically defined map from X to $\mathbb{P}^{n}$[5].

Definition 3.1.1. Let $\mathrm{QMap}\,(X, \mathbb{P}^n) : \mathfrak{Aff}^{\mathrm{op}} \to \mathbf{Set}$ be the functor of points whose S-points are presented by the data $\left(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1}\right)$, where $\mathcal{L}$ is a line bundle over $S \times X$, and $\mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1}$ is an injection of quasi-coherent sheaves, whose co-kernel is S-flat.

If $Y \hookrightarrow \mathbb{P}^n$ is a locally closed subscheme, then a quasi-map from X to Y should be given by the data of a quasi-map from X to $\mathbb{P}^n$, with the additional property that the generic point of X maps to Y. We proceed to define this notion in a way better suited for families.

In the case when $Y \hookrightarrow \mathbb{P}^n$ is a closed subscheme, it is defined by a graded ideal $I_Y \subseteq k[x_0, \ldots, x_n]$. A regular map $X \xrightarrow{f} \mathbb{P}^n$, presented by the data of a

---

[5]Such a map of course admits an extension to a regular map (in terms of bundles, every invertible sub-bundle $\mathcal{L} \xrightarrow{\phi} \mathcal{O}_X^{n+1}$ extends to a line sub-bundle $\mathcal{L} \hookrightarrow (\mathrm{Im}\,(\phi^\vee))^\vee \subseteq \mathcal{O}_X^{n+1}$), but see remark **2.2.4.**

sub-bundle $\mathcal{L} \subseteq \mathcal{O}_X^{n+1}$, lands in Y iff the composition

$$\mathrm{Sym}_X \mathcal{L}^\vee \leftarrow \mathrm{Sym}_X \mathcal{O}_X^{n+1} \cong \mathcal{O}_X \otimes k[x_0, \dots, x_n] \leftarrow \mathcal{O}_X \otimes I_Y$$

vanishes. We degenerate the sub-bundle requirement to obtain the notion of a quasi map into Y:

Definition 3.1.2. When $Y \hookrightarrow \mathbb{P}^n$ is projective embedding, we define $\mathrm{QMap}(X, Y)$ to be the subfunctor of $\mathrm{QMap}(X, \mathbb{P}^n)$ consisting of those points presented by the data $(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}^{n+1})$ such that the composition

$$\mathrm{Sym}_{S \times X} \mathcal{L}^\vee \leftarrow \mathrm{Sym}_{S \times X} \mathcal{O}_X^{n+1} \leftarrow \mathcal{O}_{S \times X} \otimes I_Y$$

vanishes[6].

We emphasize that the definition of $\mathrm{QMap}(X, Y)$ depends on the embedding $Y \hookrightarrow \mathbb{P}^n$, and not on Y alone.

The following lemma is well known (see e.g., [FM99, lemma 3.3.1]):

Lemma 3.1.3. $\mathrm{QMap}(X, \mathbb{P}^n)$ *is representable by a scheme, which is moreover a disjoint (infinite) union of projective schemes.*

If $Y \hookrightarrow \mathbb{P}^n$ is a projective embedding, then $\mathrm{QMap}(X, Y) \to \mathrm{QMap}(X, \mathbb{P}^n)$ is a closed embedding. □

Definition 3.1.4. If $U \subseteq \mathbb{P}^n$ is an open subscheme then we define

$$\mathrm{QMap}(X, U) \subseteq \mathrm{QMap}(X, \mathbb{P}^n)$$

to be the open subscheme which is the complement of $\mathrm{QMap}(X, \mathbb{P}^n \setminus U)$ (this is independent of the closed subscheme structure given to $\mathbb{P}^n \setminus U$).

Finally, if $Y \hookrightarrow \mathbb{P}^n$ is an arbitrary locally closed subscheme we define

$$\mathrm{QMap}(X, Y) = \mathrm{QMap}(X, \overline{Y}) \cap \mathrm{QMap}(X, \mathbb{P}^n \setminus (\overline{Y} \setminus Y))$$

It is a locally closed subscheme of $\mathrm{QMap}(X, \mathbb{P}^n)$.

We point out that a map $S \to \mathrm{QMap}(X, \overline{Y})$ lands in the open subscheme $\mathrm{QMap}(X, Y)$ iff for every geometric point $s \in S(k)$, the corresponding quasi-map carries the generic point of X into Y.

3.1.5. In section 4 we shall need to replace $\mathrm{QMap}(X, Y)$ with a relative and twisted version, $\mathrm{QSect}_S(S \times X, Y)$ corresponding to a scheme Y over $S \times X$. The details are given at the end of the section in 3.3.

---

[6]The definition could have been given more economically, by replacing the entire symmetric algebras with their finite dimensional subspaces containing generators of I_Y.

3.2. **Degenerate extensions of generic maps.** Recall the presheaves $\mathrm{GMap}\,(X,Y)$ and $\mathrm{GMap}^D\,(X,Y)$ introduced in 2.2.3. There is an evident map

$$\mathrm{QMap}\,(X,Y) \to \mathrm{GMap}\,(X,Y)$$

via which we think of every quasi map as presenting a generic map. Namely, for every $S \in \mathfrak{Aff}$ it is given by the composition

$$\mathrm{QMap}\,(X,Y)\,(S) \to \coprod_{(S,U)\in\mathrm{Dom}_X} \mathrm{GMap}^D\,(X,Y)\,(S,U) \to \mathrm{GMap}\,(X,Y)\,(S)$$

where the first map is given by sending a quasi-map presented by $\mathcal{L} \hookrightarrow \mathcal{O}_{S\times X}^{n+1}$ to the the open subscheme $U \subseteq S \times X$ where it is flat, and the the regular map it defines on U. However, there is some redundancy in the presentation of because a generic map may be presented by several different quasi-maps. We introduce the equivalence relation $\mathcal{E}_Y \subseteq \mathrm{QMap}\,(X,Y) \times \mathrm{QMap}\,(X,Y)$ to be the subfunctor whose S-points are presented by those pairs

$$\left(\left(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{O}_{S\times X}^{n+1} \right), \left(\mathcal{L}', \mathcal{L}' \hookrightarrow \mathcal{O}_{S\times X}^{n+1} \right) \right) \in \left(\mathrm{QMap}\,(X,Y) \times \mathrm{QMap}\,(X,Y) \right) (S)$$

which agree over the intersection of their regularity domains. Observe that the following square is Cartesian

$$\begin{array}{ccc}
\mathcal{E}_Y & \longrightarrow & \mathrm{QMap}\,(X,Y) \\
\downarrow & & \downarrow \\
\mathrm{QMap}\,(X,Y) & \longrightarrow & \mathrm{GMap}\,(X,Y)
\end{array}$$

The following lemma is well known, we add a proof for completeness.

Lemma 3.2.1. *The equivalence relation* $\mathcal{E}_Y \to \mathrm{QMap}\,(X,Y) \times \mathrm{QMap}\,(X,Y)$ *is (representable by) a closed subscheme. Both the projections* $\mathcal{E}_Y \to \mathrm{QMap}\,(X,Y)$ *are proper.*

Proof. Let us first consider the case $Y = \mathbb{P}^n$, and show that the subfunctor

$$\mathcal{E}_{\mathbb{P}^n} \to \mathrm{QMap}\,(X,\mathbb{P}^n) \times \mathrm{QMap}\,(X,\mathbb{P}^n)$$

is a closed embedding.

We start by examining when two quasi-maps $S \xrightarrow{\phi,\psi} \mathrm{QMap}\,(X,\mathbb{P}^n)$ are generically equivalent, i.e., map to the same S-point of $\mathrm{GMap}\,(X,\mathbb{P}^n)$. Let ϕ and ψ be presented by invertible sub-sheaves

$$\mathcal{L}_\phi \xhookrightarrow{\kappa_\phi} \mathcal{O}_{S\times X}^{n+1} \quad \text{and} \quad \mathcal{L}_\psi \xhookrightarrow{\kappa_\psi} \mathcal{O}_{S\times X}^{n+1}$$

whose co-kernels are S-flat. Let $U \subseteq S \times X$ be the open subscheme where $\mathcal{L}_\phi\big|_U \hookrightarrow \mathcal{O}_U^{n+1}$ is sub-bundle, and thus a maximal invertible sub-bundle. The points ϕ and ψ are generically equivalent iff $\mathcal{L}_\psi\big|_U$ is a subsheaf of $\mathcal{L}_\phi\big|_U$ (both viewed as subsheaves of $\mathcal{O}_U^{n+1}$).

Fix a vector bundle, $\mathcal{M}$, on $S \times X$ whose dual surjects on the kernel as indicated below

$$\mathcal{L}_\phi^\vee \xleftarrow{\kappa_\phi^\vee} \left(\mathcal{O}_{S\times X}^{n+1}\right)^\vee \hookleftarrow \mathrm{Ker}\left(\kappa_\phi^\vee\right) \twoheadleftarrow \mathcal{M}^\vee$$

Dualizing and restricting to U we have

$$
\begin{array}{c}
\mathcal{L}_\psi\big|_U \\
\downarrow \\
\mathcal{L}_\phi\big|_U \xrightarrow[\kappa_\phi]{\subseteq} \mathcal{O}_{U_0}^{n+1} \twoheadrightarrow \mathrm{coker}\left(\kappa_\phi\right) \xrightarrow{\subseteq} \mathcal{M}\big|_U
\end{array}
$$

where map $\mathrm{coker}\left(\kappa\right) \to \mathcal{M}$ is injective (in fact a sub-bundle). Thus, ϕ and ψ are generically equivalent iff the composition $\mathcal{L}_\psi\big|_U \to \mathcal{M}\big|_U$ vanishes on U iff $\mathcal{L}_\psi \to \mathcal{M}$ vanishes on all of $S \times X$ (since $U \subseteq S \times X$ is dense, and both sheaves are vector bundles).

For an arbitrary quasi-projective scheme, $Y \hookrightarrow \mathbb{P}^n$, the lemma now follows from the Cartesianity of the squares below, using the fact that both right vertical maps are proper

$$
\begin{array}{ccc}
\mathcal{E}_Y & \longrightarrow & \mathcal{E}_{\mathbb{P}^n} \\
\downarrow & & \downarrow \\
\mathrm{QMap}\left(X,Y\right) \times \mathrm{QMap}\left(X,\mathbb{P}^n\right) & \longrightarrow & \mathrm{QMap}\left(X,\mathbb{P}^n\right) \times \mathrm{QMap}\left(X,\mathbb{P}^n\right) \\
\downarrow & & \downarrow{\scriptstyle \pi_1} \\
\mathrm{QMap}\left(X,Y\right) & \longrightarrow & \mathrm{QMap}\left(X,\mathbb{P}^n\right)
\end{array}
$$

$\square$

Denote the evident simplicial object in $\mathcal{P}shv\left(\mathfrak{Aff}\right)$

$$(3.1) \qquad \cdots \; \underset{\vdots}{\overset{\longrightarrow}{\longrightarrow}} \; \mathcal{E}_Y \times_{\mathrm{QMap}(X,Y)} \mathcal{E}_Y \;\underset{\longleftarrow}{\overset{\longrightarrow}{\rightrightarrows}}\; \mathcal{E}_Y \;\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}\; \mathrm{QMap}\left(X,Y\right)$$

by

$$\Delta^{\mathrm{op}} \xrightarrow{\mathcal{E}_Y^\bullet} \mathcal{P}shv\left(\mathfrak{Aff}\right)$$

$$[n] \longmapsto \mathcal{E}_Y^{(n)}$$

where

$$\mathcal{E}_Y^{(n)} := \underbrace{\mathcal{E}_Y \times_{\mathrm{QMap}(X,Y)} \cdots \times_{\mathrm{QMap}(X,Y)} \mathcal{E}_Y}_{n-\text{times}}$$

We denote by $\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y$ the functor of points which is the quotient by this equivalence relation - the colimit of this simplicial object. However, in this case it simply reduces to the naive pointwise quotient of sets

$$\left(\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y\right)(S) = \mathrm{QMap}\,(X,Y)\,(S)\,/\mathcal{E}_Y\,(S)$$

because $\mathcal{E}_Y\,(S) \subseteq \left(\mathrm{QMap}\,(X,Y)\,(S)\right)^{\times 2}$ is an equivalence relation in sets.

The functor of points $\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y$ presents another candidate for the "space of generic maps", a-priori different from $\mathrm{GMap}\,(X,Y)$. Relative to $\mathrm{GMap}\,(X,Y)$, it has the advantage of being concisely presented as the quotient of a scheme by a proper (schematic) equivalence relation. The following proposition shows that both functors are essentially equivalent, and in particular that (up to Zariski sheafification) $\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y$ is independent of the quasi-projective embedding $Y \hookrightarrow \mathbb{P}^n$.

Proposition 3.2.2. *The map* $\mathrm{QMap}\,(X,Y) \to \mathrm{GMap}\,(X,Y)$ *induces a map of presheaves*

$$\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y \to \mathrm{GMap}\,(X,Y)$$

which becomes an equivalence after Zariski sheafification.

We prove this proposition, after some preparations, in 3.2.10. First, a couple of consequences:

Corollary 3.2.3. *The Zariski sheafification of the presheaf* $\mathrm{QMap}\,(X,Y)\,/\mathcal{E}_Y$ *is independent of the quasi-projective embedding* $Y \hookrightarrow \mathbb{P}^n$. $\qquad\square$

The main invariant of $\mathrm{GMap}\,(X,Y)$ which we wish to study in this paper is homology, and by extension the category of D-modules (see 6.1). The following corollary is to be interpreted as providing a convenient presentation of this category of D-modules, and using this presentation to deduce the existence of a de-Rham cohomology functor (the left adjoint to pullback).

Corollary 3.2.4.

    (1) *Pullback induces an equivalence*

$$\lim_{[n]\in\Delta^{\mathrm{op}}} \mathfrak{Dmod}\left(\mathcal{E}_Y^{(n)}\right) \xleftarrow{\cong} \mathfrak{Dmod}\left(\mathrm{GMap}\,(X,Y)\right)$$

(2) *Consider the pullback functors*

$$\mathfrak{Dmod}\left(\mathrm{QMap}\left(X,Y\right)\right) \xleftarrow{f^!} \mathfrak{Dmod}\left(\mathrm{GMap}\left(X,Y\right)\right) \xleftarrow{t^!} \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right)$$

The functor $f^!$ always admits a left adjoint ("!-push-forward"). When $Y \hookrightarrow \mathbb{P}^n$ is a closed embedding, the functor $t^!$ also admits a left adjoint.

The second assertion above is kind of "properness" property of the (non-representable) map $\mathrm{QMap}\left(X,Y\right) \to \mathrm{GMap}\left(X,Y\right)$, and the functor of points $\mathrm{GMap}\left(X,Y\right)$ (when $Y \hookrightarrow \mathbb{P}^n$ is a closed embedding).

The following remark is not used in the rest of the article. We point it out for future use.

Remark 3.2.5. Corollary 3.2.4 implies that $\mathfrak{Dmod}\left(\mathrm{GMap}\left(X,Y\right)\right)$ is compactly generated. Namely, the pushforwards of compact generators of $\mathfrak{Dmod}\left(\mathrm{QMap}\left(X,Y\right)\right)$ are generate because $f^!$ is faithful (as is evident from (1)), and are compact because $f^!$ is colimit preserving.

Proof. (1) is an immediate consequence of proposition 3.2.2.

Regarding (2), it follows from lemma 3.2.1 that all the maps in $\mathcal{E}_Y^\bullet$ are proper, hence, on the level of D-module categories, each pull-back functor admits a left adjoint (a "!-pushforward"). Consequently, the object assignment

$$[n] \in \Delta \mapsto \mathfrak{Dmod}\left(\mathcal{E}_Y^n\right) \in \hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$$

extends to both a co-simplicial diagram (implicit in the statement of (1)) as well as a simplicial diagram. In the former, which we denote

$$\mathfrak{Dmod}^!\left(\mathcal{E}_Y^\bullet\right) : \Delta \to \hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$$

functors are given by pullback. In the latter, which we denote

$$\mathfrak{Dmod}_!\left(\mathcal{E}_Y^\bullet\right) : \Delta^{\mathrm{op}} \to \hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$$

the functors are given by the left adjoints to pullback (!-pushforward). When $Y \hookrightarrow \mathbb{P}^n$ is a closed embedding, each of the $\mathcal{E}_Y^{(n)}$'s is proper, hence the pushforward diagram is augmented over $\mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right)$.

Under the equivalence of (1), the functors whose adjoints we wish to construct are identified with

$$\mathfrak{Dmod}\left(\mathrm{QMap}\left(X,Y\right)\right) \leftarrow \lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^!\left(\mathcal{E}_Y^\bullet\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right)$$

The setup above falls into the general framework *adjoint diagrams* which we discuss in the appendix, where it is proven (8.1.1) that there exists an equivalence $\mathrm{colim}_\Delta \mathfrak{Dmod}_!\left(\mathcal{E}_Y^\bullet\right) \xrightarrow{\cong} \lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^!\left(\mathcal{E}_Y^\bullet\right)$, and that under this equivalence, the pair of

natural maps

$$\mathfrak{Dmod}\left(\mathrm{QMap}\left(X,Y\right)\right) \leftarrow \lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^{!}\left(\mathcal{E}_{Y}^{\bullet}\right)$$

$$\mathfrak{Dmod}\left(\mathrm{QMap}\left(X,Y\right)\right) \rightarrow \operatorname*{colim}_{\Delta} \mathfrak{Dmod}_{!}\left(\mathcal{E}_{Y}^{\bullet}\right)$$

are adjoint functors.

Likewise in the case when $Y \hookrightarrow \mathbb{P}^{n}$ we conclude that

$$\lim_{\Delta^{\mathrm{op}}} \mathfrak{Dmod}^{!}\left(\mathcal{E}_{Y}^{\bullet}\right) \leftarrow \mathfrak{Dmod}\left(\operatorname{spec}\left(k\right)\right); \quad \operatorname*{colim}_{\Delta} \mathfrak{Dmod}_{!}\left(\mathcal{E}_{Y}^{\bullet}\right) \rightarrow \mathfrak{Dmod}\left(\operatorname{spec}\left(k\right)\right)$$

are of adjoint functors. $\qquad\square$

We proceed with the preparations for the proof of Proposition 3.2.2.

3.2.6. *Divisor complements.* Recall that an effective Cartier divisor, on a scheme Y, is the data of a line bundle $\mathcal{L}$ together with an injection of coherent sheaves $\mathcal{L} \hookrightarrow \mathcal{O}_{Y}$. The complement of the support of $\mathcal{O}_{S \times X}/\mathcal{L}$ is an open subscheme, $U_{\mathcal{L}} \subseteq Y$. We call an open subscheme arising in this way a *divisor complement.*

Lemma 3.2.7. *Let $(S, U) \in \mathrm{Dom}_{X}$, and let $\mathcal{L}_{U}$ be a line bundle on $U \subseteq S \times X$. There exists a finite Zariski cover*

$$\left\{\left(S_{i}, U_{i}\right) \rightarrow \left(S, U\right)\right\}_{i \in I}$$

such that for every i the open subscheme $U_{i} \subseteq S_{i} \times U_{i}$ is a divisor complement. Moreover, we can choose each U_{i} so that $\mathcal{L}_{U}\big|_{U_{i}}$ is a trivial line bundle.

Proof. Since $S \times X$ is quasi-projective, the topology of its underlying topological space is generated by divisor complements Thus, we may cover U by a finite collection of open subschemes, $\{U_{i}\}_{i \in I}$, which trivialize $\mathcal{L}_{U}$, and such that each $U_{i} \subseteq S \times X$ is a divisor complement. Let $S_{i} \subseteq S$ be the open subscheme which is the image of $U_{i} \subseteq S \times X \rightarrow S$. Note that U_{i} might not be a family of domains over S, but that it is over S_{i}, and that $\left\{\left(S_{i}, U_{i}\right) \rightarrow \left(S, U\right)\right\}_{i \in I}$ is a Zariski cover in Dom_{X}. $\qquad\square$

Lemma 3.2.8. *Let $\mathcal{V}$ be a vector bundle over $S \times X$, let $\mathcal{L} \overset{\kappa}{\hookrightarrow} \mathcal{V}$ be an invertible subsheaf, and let $U \subseteq S \times X$ be the open subscheme where κ is a sub-bundle embedding. Then, the following two conditions are equivalent:*

   (1) *The coherent sheaf $\mathcal{V}/\mathcal{L}$ is S-flat.*
   (2) *The open subscheme $U \subseteq S \times X$ is universally dense relative to S. I.e., The data (S, U) defines a point of Dom_{X}.*

In particular, for an effective Cartier divisor, $\mathcal{L} \hookrightarrow \mathcal{O}_{S \times X}$, the open subscheme $U_{\mathcal{L}} \subseteq S \times X$ determines an S-point of Dom_{X} iff the coherent sheaf $\mathcal{O}_{S \times X}/\mathcal{L}$ is S-flat.

Proof. Let p and j denote the maps $U \xrightarrow{j} S \times X \xrightarrow{p} S$. Both conditions may be tested on closed points of S. I.e., it suffices to show that for every maximal sheaf of ideals $\mathcal{I}_s \subseteq \mathcal{O}_S$, corresponding to a closed point $s \in S$, we have

$$\mathrm{Tor}_S^1 \left(\mathcal{V}/\mathcal{L}, \mathcal{I}_s \right) = 0 \ \text{ iff } \ U \times_S \{s\} \neq \emptyset$$

Indeed, $\mathrm{Tor}_S^1 \left(\mathcal{V}/\mathcal{L}, \mathcal{I}_s \right)$ vanishes iff $\mathcal{L}\big|_{\{s\} \times X} \xrightarrow{\ \kappa\big|_{\{s\} \times X}\ } \mathcal{V}\big|_{\{s\} \times X}$ is injective iff $\kappa\big|_{\{s\} \times X} \neq 0$ iff $U \times_S \{s\} \neq \emptyset$. $\square$

The following lemma contains the geometric input for the proof of proposition 3.2.2:

Lemma 3.2.9. *Assume given:*

- $(S, U) \in \mathrm{Dom}_X$.
- $\mathcal{V}$ *a rank m vector bundle over $S \times X$.*
- $\mathcal{L}_U$ *a line bundle over U together with a sub-bundle embedding*

$$\mathcal{L}_U \xrightarrow{\ \kappa_U\ } \mathcal{V}\big|_U$$

Then, there exist

- *A Zariski cover $\left(\tilde{S}, \tilde{U} \right) \xrightarrow{p} \left(S, U \right)$ in Dom_X.*
- *A line bundle $\mathcal{L}$ on $\tilde{S} \times X$ together with a sub-sheaf embedding $\mathcal{L} \xhookrightarrow{\ \kappa\ } \mathcal{V}\big|_{\tilde{S} \times X}$ whose co-kernel is $\tilde{S}$-flat.*
- *An identification $\mathcal{L}\big|_{\tilde{U}} \cong \mathcal{L}_U\big|_{\tilde{U}}$ which exhibits κ as an extension of*

$$\mathcal{L}_U\big|_{\tilde{U}} \xrightarrow{\ \kappa_U\big|_{\tilde{U}}\ } \mathcal{V}\big|_{\tilde{U}}$$

Above we have used $(-)\big|_{\tilde{U}}$ to denote pullback along $\tilde{U} \to U$.

Proof. According to lemma 3.2.7, we may find a Zariski cover in Dom_X, $\left(\tilde{S}, \tilde{U} \right) \to (S, U)$, such that

- $\mathcal{L}_U\big|_{\tilde{U}}$ is a trivial line bundle.
- The open subscheme $\tilde{U} \subseteq \tilde{S} \times X$ is a divisor complement associated to a Cartier divisor $\mathcal{N} \hookrightarrow \mathcal{O}_{\tilde{S} \times X}$.

We proceed to show that the sub-bundle embedding

$$(*) \qquad \mathcal{L}_U\big|_{\tilde{U}} \xrightarrow{\ \kappa_U\big|_{\tilde{U}}\ } \mathcal{V}\big|_{\tilde{U}}$$

admits a degenerate extension across $\tilde{S} \times X$. We point out that the line bundle $\mathcal{N}$ is trivialized over $\tilde{U}$, and we fix identifications $\mathcal{N}\big|_{\tilde{U}} = \mathcal{O}_{\tilde{U}} \cong \mathcal{L}_U\big|_{\tilde{U}}$. By a standard lemma in algebraic geometry ([Har77, II.5.14]), there exists an integer l and a map of coherent sheaves $\mathcal{N}^{\otimes l} \xrightarrow{\ \kappa\ } \mathcal{V}\big|_{\tilde{S} \times X}$ whose restriction to $\tilde{U}$ may be identified with $(*)$. By lemma 3.2.8, $\mathrm{coker}\,(\kappa)$ is $\tilde{S}$-flat. $\square$

3.2.10. *Proof of proposition 3.2.2.* The following square is Cartesian

$$
\begin{array}{ccc}
\mathrm{QMap}\,(X,Y) & \longrightarrow & \mathrm{QMap}\,(X,\mathbb{P}^n) \\
\downarrow\ulcorner & & \downarrow \\
\mathrm{QMap}\,(X,Y) & \longrightarrow & \mathrm{GMap}\,(X,\mathbb{P}^n)
\end{array}
$$

hence it suffices to prove the proposition for $Y = \mathbb{P}^n$.

It suffices to fix an S-point, $S \to \mathrm{GMap}\,(X,\mathbb{P}^n)$, and show that there exists a Zariski cover $\tilde{S} \to S$, and a lift as indicated by the dotted arrow below

$$
\begin{array}{ccc}
& & \mathrm{QMap}\,(X,\mathbb{P}^n) \\
& \nearrow & \downarrow \\
\tilde{S} \dashrightarrow S & \xrightarrow{\ \phi\ } & \mathrm{GMap}\,(X,\mathbb{P}^n)
\end{array}
$$

Let ϕ be presented by the data of a point $(S, U) \in \mathrm{Dom}_X$, and a sub-bundle embedding $\mathcal{L}_U \xrightarrow[\subseteq]{\kappa_U} \mathcal{O}_U^{n+1}$ over U. Lemma 3.2.9 guarantees the existence of a cover $\tilde{S} \to S$, and an invertible sub-sheaf $\mathcal{L} \xhookrightarrow{\ \kappa\ } \mathcal{O}_{\tilde{S} \times X}^{n+1}$, which is an extension of $\kappa_U\big|_{\tilde{U}}$ to all of $\tilde{S} \times X$. The data associated with κ presents a map, $\tilde{S} \to \mathrm{QMap}\,(X,\mathbb{P}^n)$, which is the sought after lift.

3.3. **Quasi sections.** In the next section we shall need a relative and twisted generalization of the notion of quasi map, which we now define. All the results in this section, proven above, could have been stated and proven in this more general setup (at the cost of encumbering the presentation).

Fix $S \in \mathfrak{Aff}$, and let $\mathcal{V}$ be a vector bundle on $S \times X$. Denote the relative projectivization by

$$
\mathbb{P}\,(\mathcal{V}) := \mathrm{proj}_{S \times X}\left(\mathrm{sym}_{\mathcal{O}_{S \times X}} \mathcal{V}^*\right)
$$

it is a locally projective scheme over $S \times X$. We define the space of quasi-sections of $\mathbb{P}\,(\mathcal{V}) \to S \times X$, relative to S:

Definition 3.3.1.

(1) The functor

$$
\mathrm{QSect}_S\,(S \times X, \mathbb{P}\,(\mathcal{V})) : \mathfrak{Aff}_{/S}^{\mathrm{op}} \to \mathbf{Set}
$$

is defined to be the functor of points over S, whose T-points are presented by the data $\left(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{V}\big|_{T \times X}\right)$, where $\mathcal{L}$ is a line bundle over $T \times X$, and $\mathcal{L} \hookrightarrow \mathcal{V}\big|_{T \times X}$ is an injection of quasi-coherent sheaves, whose co-kernel is T-flat.

(2) For a closed embedding $Y \hookrightarrow \mathbb{P}(\mathcal{V})$, defined by a graded sheaf of ideals $\mathcal{I}_Y \subseteq \mathrm{Sym}_{T \times X} \mathcal{V}^\vee$, we define

$$\mathrm{QSect}_S (S \times X, Y) \subseteq \mathrm{QSect}_S (S \times X, \mathbb{P}(\mathcal{V}))$$

to be the subfunctor of consisting of those points presented by the data $\left(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{V}|_{T \times X} \right)$ such that the composition

$$\mathrm{Sym}_{T \times X} \mathcal{L}^\vee \leftarrow \mathrm{Sym}_{T \times X} \mathcal{V}^\vee \leftarrow \mathcal{I}_Y$$

vanishes.

When $S = \mathrm{spec}(k)$, and $\mathcal{V} = \mathcal{O}_{S \times X}^{n+1}$ this definition reduces to $\mathrm{QMap}(X, Y)$.

As for the absolute version, there exists a map

$$\mathrm{QSect}_S (S \times X, Y) \rightarrow \mathrm{GSect}_S (S \times X, Y)$$

and the counterpart of proposition 3.2.2 holds. The proof is virtually identical (after adjusting notation), and is omitted.

Proposition 3.3.2. *The map* $\mathrm{QSect}_S (S \times X, Y) \rightarrow \mathrm{GSect}_S (S \times X, Y)$ *induces a map of presheaves*

$$\mathrm{QSect}_S (S \times X, Y) / \mathcal{E}_Y \rightarrow \mathrm{GSect}_S (S \times X, Y)$$

which becomes an equivalence after Zariski sheafification. $\square$

4. D- MODULES "ON" $B(K) \backslash G(\mathbb{A}) / G(\mathbb{O})$

Recall example 2.2.5, in which we introduced a moduli problem of generic data $\mathrm{Bun}_G^{B(\mathrm{Dom}_X)} \in \mathcal{P}shv(\mathrm{Dom}_X)$, and denoted its associated functor of points by

$$\mathrm{Bun}_G^{B(\eta)} := \mathrm{LKE}_q \left(\mathrm{Bun}_G^{B(\mathrm{Dom}_X)} \right) \in \mathcal{P}shv(\mathfrak{Aff})$$

It is a geometrization[7] of $B(K) \backslash G(\mathbb{A}) / G(\mathbb{O})$. Conceptual appeal notwithstanding, this presentation of $\mathrm{Bun}_G^{B(\eta)}$ is too unwieldy to be of much value. Namely, the issue is that using it (directly) to obtain presentations of invariants such as $\mathfrak{Dmod}$ is a non-starter.

In the note [Gai10b, subsection 1.1], Gaitsgory introduces a category denoted $\mathfrak{Dmod}\left(\mathrm{Bun}_B^{\mathrm{rat}}\right)$, which is cast to play the role of the category of D-modules "on" $B(K) \backslash G(\mathbb{A}) / G(\mathbb{O})$. In this section we present the construction of Gatisgory's category, and show that it is equivalent to $\mathfrak{Dmod}\left(\mathrm{Bun}_G^{B(\eta)}\right)$. The discussion parallels that of the previous section.

Notation 4.0.1. Let G be a connected reductive affine algebraic group. Choose a Borel subgroup B, denote by N the unipotent radical of B, and by $H = B/N$

---

[7]"The geometrization" according to the premise of this paper.

the canonical Cartan. Choose a root system for G and B, and denote by Λ_G^+ the semi-group of dominant integral weights. For a dominant integral weight λ, let V^λ denote the irreducible representation of G with highest weight λ. For a H-torsor, $\mathcal{P}_H$, we denote by $\lambda(\mathcal{P}_H)$ the $\mathbb{G}_m$-torsor $\mathcal{P}_H \times_\lambda \mathbb{G}_m$ (as well as the associated line bundle - a quasi-coherent sheaf). For a G-torsor, $\mathcal{P}_G$, we denote by $\mathcal{V}^\lambda_{\mathcal{P}_G}$ the vector bundle corresponding to V^λ.

4.1. Constructions.

4.1.1. *Plucker data.* Given a scheme Y and a G-bundle $\mathcal{P}_G$ on Y, a convenient way of presenting the data of a reduction of the structure group of $\mathcal{P}_G$ to B is given by specifying an H-bundle, $\mathcal{P}_H$, together with bundle maps for every $\lambda \in \Lambda_G^+$

$$\lambda(\mathcal{P}_H) \xrightarrow[\subseteq]{\kappa_\lambda} \mathcal{V}^\lambda_{\mathcal{P}_G}$$

which satisfy the *Plucker relations.* I.e., for λ_0 the trivial character, κ^0 is the identity map

$$\mathcal{O} \cong \lambda_0(\mathcal{P}_H) \to \mathcal{V}^0_{\mathcal{P}_G} \cong \mathcal{O}$$

and for every pair of dominant integral weights the following diagram commutes

$$
\begin{array}{ccc}
(\lambda+\mu)(\mathcal{P}_H) & \xrightarrow{\kappa_{\lambda+\mu}} & \mathcal{V}^{\lambda+\mu}_{\mathcal{P}_G} \\
\downarrow & & \downarrow \\
\lambda(\mathcal{P}_H) \otimes \mu(\mathcal{P}_H) & \xrightarrow{\kappa_\lambda \otimes \kappa_\mu} & \mathcal{V}^\mu_{\mathcal{P}_G} \otimes \mathcal{V}^\lambda_{\mathcal{P}_G}
\end{array}
$$

From now on, we adopt this Plucker point of view for presenting points of $\mathrm{Bun}_G^{B(\eta)}$.

4.1.2. *Degenerate reduction spaces.* Degenerating the data of a reduction of a G-torsor to B, in a similar fashion to the degeneration of a regular map to a quasi-map, we obtain Drinfeld's (relative) compactification of $\mathrm{Bun}_B \to \mathrm{Bun}_G$:

Let $\overline{\mathrm{Bun}_G^B} \in \mathcal{P}shv(\mathfrak{Aff})$[8], be the presheaf which sends a scheme S to the groupoid which classifies the data

$$\left(\mathcal{P}_G, \mathcal{P}_H, \lambda(\mathcal{P}_H) \xrightarrow{\kappa_\lambda} \mathcal{V}^\lambda_{\mathcal{P}_G} : \lambda \in \Lambda_G^+\right)$$

where:

- $\mathcal{P}_G$ is a G-torsor on $S \times X$.
- $\mathcal{P}_H$ is an H-torsor on $S \times X$.
- For every $\lambda \in \Lambda_G^+$, κ_λ is an injection of coherent sheaves whose co-kernel is S-flat. The collection of κ_λ's is required to satisfy the Plucker relations.

---

[8]Often denoted by some variation on $\overline{\mathrm{Bun}_B}$.

Informally, this is a moduli space of G-bundles on X, with a degenerate reduction to B. There is an evident map $\mathrm{Bun}_B \to \overline{\mathrm{Bun}_G^B}$ whose image consists of those points for which the κ_λ's are sub-bundle embeddings. For more details on $\overline{\mathrm{Bun}_G^B}$ see [FM99] or [BG02].

Let Let $\{\lambda_j\}_{j \in \mathcal{J}} \subseteq \Lambda_G^+$ be a finite subset which generates Λ_G^+ over $\mathbb{Z}_{\geq 0}$. The natural map[9]

$$G/B \to \underset{j \in \mathcal{J}}{\times} \mathbb{P}\left(V^{\lambda_j}\right) \hookrightarrow \mathbb{P}\left(\otimes_{j \in \mathcal{J}} V^{\lambda_j}\right)$$

is a closed embedding. For every $j \in \mathcal{J}$ let $\mathcal{V}^{\lambda_j}$ be the vector bundle on $\mathrm{Bun}_G \times X$ corresponding to the representation V^{λ_j}, and let $\mathcal{V} := \otimes_{j \in \mathcal{J}} \mathcal{V}^{\lambda_j}$.

Lemma 4.1.3. [BG02, prop. 1.2.2] *Let* $S \to \mathrm{Bun}_G$ *classify a* G*-bundle* $\mathcal{P}_G$ *on* $S \times X$, *and denote* $\left(\overline{\mathrm{Bun}_G^B}\right)_S := S \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}_G^B}$.
    There exists a natural isomorphism

$$\left(\overline{\mathrm{Bun}_G^B}\right)_S \xrightarrow{\cong} \mathrm{QSect}_S\left(S \times X, \mathcal{P}_G/B\right)$$

where the space of quasi-sections is defined via the closed embedding

$$\mathcal{P}_G/B \hookrightarrow \mathbb{P}\left(\mathcal{V}\big|_{S \times X}\right)$$

In particular $\overline{\mathrm{Bun}_G^B}$ *is schematic and proper over* Bun_G. $\qquad\qquad\square$

Example 4.1.4. When $G = SL_2$ the presheaf $\overline{\mathrm{Bun}_{SL_2}^B}$ is equivalent to the presheaf which sends a scheme S to the groupoid $\overline{\mathrm{Bun}_{SL_2}^B}(S)$ classifying the data $(\mathcal{L}, \mathcal{V}, \mathcal{L} \hookrightarrow \mathcal{V})$, where $\mathcal{L}$ is a line bundle on $S \times X$, $\mathcal{V}$ is a rank-2 vector bundle on $S \times X$ with trivial determinant, and $\mathcal{L} \hookrightarrow \mathcal{V}$ is an injection of quasi-coherent sheaves whose co-kernel is flat over S.
    Observe that when $S = \mathrm{spec}\,(k)$, we may associate to a every degenerate reduction $(\mathcal{L}, \mathcal{V}, \mathcal{L} \hookrightarrow \mathcal{V}) \in \overline{\mathrm{Bun}_{SL_2}^B}(k)$ the genuine reduction $\left(\tilde{\mathcal{L}}, \mathcal{V}, \tilde{\mathcal{L}} \xrightarrow{\subseteq} \mathcal{V}\right) \in \mathrm{Bun}_B(k)$ where $\tilde{\mathcal{L}}$ is the maximal sub-bundle, $\mathcal{L} \hookrightarrow \tilde{\mathcal{L}} \subseteq \mathcal{V}$ extending $\mathcal{L}$. However, there may not exist such extension for an arbitrary S-family[10].

We wish to use $\overline{\mathrm{Bun}_G^B}$ to construct a geometrization for $B(K) \backslash G(\mathbb{A}) / G(\mathbb{O})$. Note that on the level of k-points there exists a surjective map

$$\pi_0\left(\overline{\mathrm{Bun}_G^B}(k)\right) \to B(K) \backslash G(\mathbb{A}) / G(\mathbb{O})$$

but that this map is not bijective.

---

[9]Which maps $1 \in G$ to the highest weight line in each component.
[10]For essentially the same reason that a continuous function $\mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ may admit a continuous extension when restricted to any path, but nonetheless fail to admit a global continuous extension.

4.1.5.   Gaitsgory's $\mathfrak{Dmod}\left(\mathrm{Bun}_B^{\mathrm{rat}}\right)$ of [Gai10b, subscetion 1.1] may be defined as follows: To every point $P \in \overline{\mathrm{Bun}_G^B}(S)$ we may associate its *regular* domain $U_P \subseteq S \times X$, this is the maximal open subscheme where the Plucker data is regular, and hence defines a genuine structure reduction of $\mathcal{P}_G\big|_{U_P}$ to B.

Define $\mathcal{H} \in \mathcal{P}shv\left(\mathfrak{Aff}\right)$ be the presheaf which sends S to the groupoid classifying the data

$$\left(P \in \overline{\mathrm{Bun}_G^B}, P' \in \overline{\mathrm{Bun}_G^B}, \phi\right)$$

where ϕ is an isomorphism of the underlying G-torsors (defined on all of $S \times X$), which commutes with the κ_λ's over $U_P \cap U_{P'}$ (hence induces an isomorphism of B-reductions there). It is evident that $\mathcal{H}$ admits a groupoid structure (in presheaves) over $\overline{\mathrm{Bun}_G^B}$. In loc. cit., $\mathfrak{Dmod}\left(\mathrm{Bun}_B^{\mathrm{rat}}\right)$ is defined to be the category of equivariant D-modules with respect to this groupoid.

On the level of points, we may define $\overline{\mathrm{Bun}_G^B}^{\mathcal{H}}$ to be the quotient of $\overline{\mathrm{Bun}_G^B}$ by this groupoid (i.e., the colimit of the associated simplicial object in $\mathcal{P}shv\left(\mathfrak{Aff}\right)$). It follows that $\mathfrak{Dmod}\left(\mathrm{Bun}_B^{\mathrm{rat}}\right) \cong \mathfrak{Dmod}\left(\overline{\mathrm{Bun}_G^B}^{\mathcal{H}}\right)$. After taking this quotient, we do have an identification of sets

$$\pi_0\left(\overline{\mathrm{Bun}_G^B}^{\mathcal{H}}(k)\right) \cong B\left(K\right)\backslash G\left(\mathbb{A}\right)/G\left(\mathbb{O}\right)$$

The main result of this section is:

Proposition 4.1.6. *There exists a map in $\mathcal{P}shv\left(\mathfrak{Aff}\right)$*

$$\left(\overline{\mathrm{Bun}_G^B}\right)^{\mathcal{H}} \to \mathrm{Bun}_G^{B(\eta)}$$

which becomes an equivalence after sheafification in the Zariski topology.

The following corollary is of particular interest in the geometric Langlands program:

Corollary 4.1.7.

(1) *Pullback along the map constructed in 4.1.6 gives rise to an equivalence*

$$\lim_{[n]\in\Delta^{\mathrm{op}}}\left(\mathfrak{Dmod}\left(\mathcal{H}^{(n)}\right)\right) \cong \mathfrak{Dmod}\left(\left(\overline{\mathrm{Bun}_G^B}\right)^{\mathcal{H}}\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{Bun}_G^{B(\eta)}\right)$$

where

$$\mathcal{H}^{(n)} := \underbrace{\mathcal{H} \times_{\overline{\mathrm{Bun}_G^B}} \cdots \times_{\overline{\mathrm{Bun}_G^B}} \mathcal{H}}_{n-\text{times}}$$

(2) *The pullback functors*

$$\mathfrak{Dmod}\left(\overline{\mathrm{Bun}_G^B}\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{Bun}_G^{B(\eta)}\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{Bun}_G\right)$$

admit left adjoints ("!-push-forward").

In Theorem 6.2.4 we shall prove that the pullback functor is moreover fully-faithful. The proof of this corollary is completely analogous to that of corollary 3.2.4.

4.1.8. *Proof of proposition 4.1.6.* We proceed to reduce the statement to proposition 3.3.2. For every $S \to \mathrm{Bun}_G$ denote

$$\left(\overline{\mathrm{Bun}_G^B} \right)_S := S \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}_G^B}$$

and denote similarly for $\mathrm{Bun}_G^{B(\eta)}$ and $\mathcal{H}$.

It follows from lemma 4.1.3 that

$$\left(\overline{\mathrm{Bun}_G^B} \right)_S \cong \mathrm{QSect}_S \left(S \times X, \mathcal{P}_G^S / B \right)$$

and it is evident that

$$\left(\mathrm{Bun}_G^{B(\eta)} \right)_S \cong \mathrm{GSect}_S \left(S \times X, \mathcal{P}_G^S / B \right)$$

and that $\mathcal{H}_S$ is equivalent to the fiber product

$$
\begin{array}{ccc}
\mathcal{H}_S & \longrightarrow & \mathrm{QSect}_S \left(S \times X, \mathcal{P}_G^S / B \right) \\
\downarrow & & \downarrow \\
\mathrm{QSect}_S \left(S \times X, \mathcal{P}_G^S / B \right) & \longrightarrow & \mathrm{GSect}_S \left(S \times X, \mathcal{P}_G^S / B \right)
\end{array}
$$

Thus we obtain maps, for every $S \to \mathrm{Bun}_G$,

$$\left(\overline{\mathrm{Bun}_G^{B(\eta)}} \right)_S \Big/ \mathcal{H}_S \xrightarrow{\cong} \left(\mathrm{Bun}_G^{B(\eta)} \right)_S$$

which become equivalences after sheafification in the Zariski topology by proposition 3.3.2. These maps are all natural in $S \to \mathrm{Bun}_G$, and we conclude the existence of a map of presheaves

$$\overline{\mathrm{Bun}_G^B}^{\mathcal{H}} \cong \operatorname*{colim}_{S \to \mathrm{Bun}_G} \left(\overline{\mathrm{Bun}_G^B} \right)_S \Big/ \mathcal{H}_S \to \operatorname*{colim}_{S \to \mathrm{Bun}_G} \left(\mathrm{Bun}_G^{B(\eta)} \right)_S \cong \mathrm{Bun}_G^{B(\eta)}$$

which becomes an equivalence after sheafification in the Zariski topology.   □

Remark 4.1.9. *Drinfeld's Parabolic structures.* In [BG02, 1.3], Braverman and Gaitsgory consider two different notions (attributed to Drinfeld in loc. cit.) of a degenerate reduction of a G-torsor (on X) to P. These two notions agree in the case when $P = B$, but differ in general. Correspondingly, they construct two different relative compactification of the map $\mathrm{Bun}_P \to \mathrm{Bun}_G$, denoted $\overline{\mathrm{Bun}_P}$ and $\widetilde{\mathrm{Bun}_P}$, both schematic and proper over Bun_G. The categories of D-modules $\mathfrak{Dmod} \left(\overline{\mathrm{Bun}_P} \right)$ and $\mathfrak{Dmod} \left(\widetilde{\mathrm{Bun}_P} \right)$ have received a fair amount of attention (e.g., in [BG02, BFGM02]) due to their part in the construction of a geometric "Eisenstein series" functor

$$\mathfrak{Dmod} \left(\mathrm{Bun}_G \right) \xleftarrow{\mathrm{Eis}_M^G} \mathfrak{Dmod} \left(\mathrm{Bun}_M \right)$$

where M is the Levi factor of P.

It can be shown that $\overline{\mathrm{Bun}_P}$ and $\widetilde{\mathrm{Bun}_P}$ give rise to two different presentations of $\mathrm{Bun}_G^{P(\eta)}$ (up to fppf sheafification) as a quotient of a scheme (relative to Bun_G) by a schematic and proper equivalence relation

$$\left(\overline{\mathrm{Bun}_P}\right)/\overline{\mathcal{H}_P} \to \mathrm{Bun}_G^{P(\eta)} \quad \text{and} \quad \left(\widetilde{\mathrm{Bun}_P}\right)/\widetilde{\mathcal{H}_P} \to \mathrm{Bun}_G^{P(\eta)}$$

Consequently, we obtain two different presentations for the category of D-modules on $\mathrm{Bun}_G^{P(\eta)}$ as a category of equivariant objects

$$\mathfrak{Dmod}\left(\left(\overline{\mathrm{Bun}_P}\right)\right)^{\overline{\mathcal{H}_P}} \overset{\cong}{\Longleftarrow} \mathfrak{Dmod}\left(\mathrm{Bun}_G^{P(\eta)}\right)$$

and

$$\mathfrak{Dmod}\left(\left(\widetilde{\mathrm{Bun}_P}\right)\right)^{\widetilde{\mathcal{H}_P}} \overset{\cong}{\Longleftarrow} \mathfrak{Dmod}\left(\mathrm{Bun}_G^{P(\eta)}\right)$$

5. The Ran Space Approach to Parametrizing Domains

In this section we describe an approach to presenting moduli problems of generic data using presheaves over the *Ran space*. This approach has the advantage that D-modules presented this way are amenable to Chiral Homology techniques. After presenting the framework we prove that it is equivalent, in an appropriate sense, to the Dom_X approach.

5.1. **The Ran Space.**

Let $\mathfrak{Fin}_{\mathrm{sur}}$ denote the category of finite sets with surjections as morphisms. The *Ran space,* denoted $\mathcal{Ran}_X$, is the colimit of the diagram

$$\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}} \xrightarrow{I \mapsto X^I} \mathcal{P}shv\left(\mathfrak{Aff}\right)$$

in which a surjection of finite sets $J \twoheadleftarrow I$ maps to the corresponding diagonal embedding $X^J \hookrightarrow X^I$. In the appendix (8.3.1) it is proven that a point $S \to \mathcal{Ran}_X$ is equivalent to the data of a finite subset $F \subset \mathrm{Hom}\,(S, X)$, i.e., $\mathcal{Ran}_X\,(S) \in \mathbf{Gpd}_\infty$ is the set of finite subsets of $\mathrm{Hom}\,(S, X)$. Note that $\mathcal{Ran}_X$ is not a sheaf even in the Zariski topology (it is not separated), and in any case its sheafifications are not representable. by a scheme or ind-scheme.

Morally, the Ran space should be thought of as the moduli space for finite subsets[11] of X, as is reflected in the fact that a closed point $\mathrm{spec}\,(k) \to \mathcal{Ran}_X$ corresponds to a finite subset $F \subset X\,(k)$. More generally, to a point $S \to \mathcal{Ran}_X$ classified by $F = \{f_1, \ldots, f_n\} \subseteq \mathrm{Hom}\,(S, X)$, we associate the closed subspace $\Gamma_F := \cup \Gamma_{f_i} \subseteq S \times X$, where $\Gamma_{f_i} \subseteq S \times X$ is the graph of $S \xrightarrow{f_i} X$. However, since we are concerned with generic data, we take the opposite perspective and interpret

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[11]We emphasize the distinction between finite subsets and finite subschemes.

such a point as parametrizing the complement open subscheme

$$U_F := (S \times X) \setminus \Gamma_F$$

which is family of domains in the sense of 2.1.1. We point out that because X is a curve, every open subscheme is the complement of a finite collection of points, whence we are justified in thinking of $\mathcal{R}\mathrm{an}_X$ as a moduli of open subschemes of X. It would seem that for a higher dimensional scheme in place of X, this approach would not be reasonable.

There are two differences between $\mathcal{R}\mathrm{an}_X$ and Dom_X. The first concerning objects, is that not every family of domains $(S, U) \in \mathrm{Dom}_X$ may be presented using a map $S \to \mathcal{R}\mathrm{an}_X$, thus $\mathcal{R}\mathrm{an}_X$ classifies a restrictive collection of domain families - *graph complements*. The second difference, concerning morphisms, is that while the fibers Dom_X over $\mathfrak{Aff}$ are posets, $\mathcal{R}\mathrm{an}_X$ takes values in sets, and thus does not account for the inclusion of one finite subset in another.

5.1.1. *Preview.* Consider the moduli problem of classifying generically defined maps from X to Y. Construct a presheaf

$$\mathrm{GMap}\,(X, Y)_{\mathcal{R}\mathrm{an}_X} \in \mathcal{P}shv\,(\mathfrak{Aff})$$

by defining that a map $S \to \mathrm{GMap}\,(X, Y)_{\mathcal{R}\mathrm{an}_X}$ is presented by the data

$$\left(S \xrightarrow{F} \mathcal{R}\mathrm{an}_X, U_F \xrightarrow{f} Y \right)$$

where f is regular map. I.e., there is a map of presheaves

$$\mathrm{GMap}\,(X, Y)_{\mathcal{R}\mathrm{an}_X} \to \mathcal{R}\mathrm{an}_X$$

and the points of $\mathrm{GMap}\,(X, Y)_{\mathcal{R}\mathrm{an}_X}$, lying over a point $S \xrightarrow{F} \mathcal{R}\mathrm{an}_X$, classify those generic maps from $S \times X$ to Y which are regular on U_F. We would like to endow the functor of points $\mathrm{GMap}\,(X, Y)_{\mathcal{R}\mathrm{an}_X}$ with additional structure which reflects the fact that a pair of its points may be parametrizing the same generic data, but with different domain data.

In the next subsection we shall define the Ran version of the category of *moduli problems of generic data* over X, and compare it to the Dom_X version defined in 2.2.2 - they will be almost equivalent. Namely, the respective functors of points in $\mathcal{P}shv\,(\mathfrak{Aff})$ associated to each formulation will be proven to be equivalent after fppf sheafification.

This Ran version will be defined as the category of modules for monad acting on $\mathcal{P}shv\,(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$. This formulation, using the Ran space and the monad, is more economical than the one via presheaves on Dom_X, and gives rise to presentations of various invariants of the moduli problem which are more approachable.
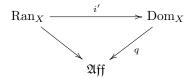
A first approximation to the construction of the Ran version is as follows: Let

$$\mathrm{Ran}_X \to \mathfrak{Aff}$$

be the Cartesian fibration which is the Grothendieck un-straightening of the functor

$$\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\mathcal{R}\mathrm{an}_X} \mathbf{Set}$$

I.e., Ran_X is the category of points of the functor of points $\mathcal{R}\mathrm{an}_X$. There is an evident functor i' making the following diagram commute

The monad which we will use is a more economical version of the monad on $\mathcal{P}shv\,(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X} \cong \mathcal{P}shv\,(\mathrm{Ran}_X)$ induced by the adjunction

$$\mathcal{P}shv\,(\mathrm{Ran}_X) \underset{i'_*}{\overset{\mathrm{LKE}_{i'}}{\rightleftarrows}} \mathcal{P}shv\,(\mathrm{Dom}_X)$$

The monad we shall use will be constructed using an intermediate domain category presented in 5.2.

It turns out that for $\mathcal{F} \in \mathcal{P}shv\,(\mathrm{Dom}_X)$, invariants such as its homology are equivalent to those of $i'_*\mathcal{F} \in \mathcal{P}shv\,(\mathrm{Ran}_X) \cong \mathcal{P}shv\,(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$, which are often computable. In particular, in the example of generic maps, Gaitsgory computes the homology of $\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}$ (for certain choices of Y), from which we deduce the homology of $\mathrm{GMap}\,(X,Y)$ (this is the topic of section 6).

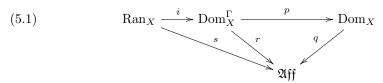### 5.2. Dom_X^{Γ} - A more economical category of domains.
Recall the Cartesian fibration $\mathrm{Ran}_X \to \mathfrak{Aff}$ defined in 5.1.1. As mentioned above, an object of Ran_X lying over a scheme S may be interpreted as presenting a family of domains in $S \times X$ (the *graph complement*), however Ran_X doesn't include morphisms which account for the inclusion of one domain in another. We construct the category Dom_X^{Γ} by adding the appropriate morphisms:

5.2.1. *Construction.*
Let Dom_X^{Γ} be the following category:

An object: consists of the data $(S, F \subseteq \mathrm{Hom}\,(S, X))$, where S is an affine scheme, and F is a non-empty finite subset.

A morphism: $(S, F \subseteq \mathrm{Hom}\,(S, X)) \to (T, G \subseteq \mathrm{Hom}\,(T, X))$ is a map of schemes $S \xrightarrow{f} T$ such that pre-composition with f carries G into F.

It is evident that Dom_X^Γ is sandwiched in a commuting diagram

(5.1)
$$\begin{array}{ccccc} \mathrm{Ran}_X & \xrightarrow{\ i\ } & \mathrm{Dom}_X^\Gamma & \xrightarrow{\ \ p\ \ } & \mathrm{Dom}_X \\ & {}_{s}\searrow & \downarrow{\scriptstyle r} & {}_{q}\swarrow & \\ & & \mathfrak{Aff} & & \end{array}$$

in which p associates to each $(S, F \subseteq \mathrm{Hom}\,(S, X)) \in \mathrm{Dom}_X^\Gamma$ the family of domains $U_F := S \times X \setminus \Gamma_F$, where Γ_F is the union of the graphs of the maps in $F \subseteq \mathrm{Hom}\,(S, X)$. Note that all three diagonal maps are Cartesian fibrations, and that i and p preserve morphisms which are Cartesian over $\mathfrak{Aff}$. We remark that p is not full (but it is faithful).

We endow Dom_X^Γ with the fppf Grothendieck topology pulled back from Dom_X along p. I.e., a collection of morphisms

$$\{(S_i, F_i \subseteq \mathrm{Hom}\,(S_i, X)) \to (S, F \subseteq \mathrm{Hom}\,(T, X))\}$$

is a cover iff the collection of scheme morphisms $\{U_{F_i} \to U_F\}$ is an fppf cover in $\mathfrak{Aff}$.

Proposition 5.2.2. *The adjunction*

$$\mathcal{P}shv\left(\mathrm{Dom}_X^\Gamma\right) \underset{p_*}{\overset{\mathrm{LKE}_p}{\rightleftarrows}} \mathcal{P}shv\left(\mathrm{Dom}_X\right)$$

induces mutually inverse equivalences after sheafification in the fppf Grothendieck topology.

The proof is given in 5.5. The upshot of this subsection is the following corollary:

Corollary 5.2.3. *There exists a naturally commuting triangle*

$$\begin{array}{ccc} \mathcal{P}shv\left(\mathrm{Dom}_X^\Gamma\right) & \xleftarrow{\qquad p_* \qquad} & \mathcal{P}shv\left(\mathrm{Dom}_X\right) \\ {}_{\mathrm{LKE}_r}\searrow & & \swarrow{\scriptstyle \mathrm{LKE}_q} \\ & \mathcal{S}hv\left(\mathfrak{Aff}; \mathrm{fppf}\right) & \end{array}$$

Consequently, for every $\mathcal{F} \in \mathcal{P}shv\left(\mathrm{Dom}_X\right)$ D-module pullback gives rise to an equivalence

$$\mathfrak{Dmod}\left(p_*\mathcal{F}\right) \overset{\cong}{\leftarrow} \mathfrak{Dmod}\left(\mathcal{F}\right)$$

The point being, that when formulating moduli problems of generic data as functors of points, it suffices to describe their points over Dom_X^Γ, rather than over the much larger category Dom_X.

5.3. The Ran formulation of moduli problems of generic data.

5.3.1. *The monad.* Recall the functor $\mathrm{Ran}_X \xrightarrow{i} \mathrm{Dom}_X^\Gamma$ introduced in 5.1. Let $\mathcal{M}$ denote the monad on $\mathcal{P}shv\left(\mathfrak{Aff}\right)_{/\mathcal{R}an_X} \cong \mathcal{P}shv\left(\mathrm{Ran}_X\right)$ induced by the adjunction

$$\mathcal{P}shv\left(\mathrm{Ran}_X\right) \underset{i_*}{\overset{\mathrm{LKE}_i}{\rightleftarrows}} \mathcal{P}shv\left(\mathrm{Dom}_X^\Gamma\right) \ . \text{ I.e., its underlying endofunctor is } i_* \circ \mathrm{LKE}_i,$$

and its unit and action transformations are induced by the adjunction unit and co-unit.

Remark 5.3.2. The action of $\mathcal{M}$ is pretty simple. Let $\mathcal{G} \in \mathcal{P}shv\left(\mathrm{Ran}_X\right)$, let $(S, F) \in \mathrm{Ran}_X$, and let us compute the value of $\mathcal{M}\left(\mathcal{G}\right)$ at (S, F) (we implicitly identify the objects of Ran_X and Dom_X^Γ):

$$\mathcal{M}\left(\mathcal{G}\right)(S, F) = \mathrm{LKE}_i \mathcal{G}\left(S, F\right) = \mathrm{colim}\left(\left(\left(\mathrm{Ran}_X\right)_{(S,F)/}\right)^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathbf{Gpd}_\infty\right) \cong$$

Since both Ran_X and Dom_X^Γ are fibered over $\mathfrak{Aff}$ we can fix S so that

$$\cong \mathrm{colim}\left(\left(\left(\mathrm{Ran}_X\left(S\right)\right)_{(S,F)/}\right)^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathbf{Gpd}_\infty\right) \cong$$

The category $\left(\left(\mathrm{Ran}_X\left(S\right)\right)_{(S,F)/}\right)^{\mathrm{op}}$ is discrete so

$$\cong \coprod_{G \subseteq F} \mathcal{G}\left(S, G\right)$$

The following definition is the counterpart of (the first part of) definition 2.2.2:

Definition 5.3.3. The *Ran formulation* for *moduli problems of generic data* is the category of modules for the monad $\mathcal{M}$ which we denote by

$$\mathrm{Mod}_\mathcal{M} = \mathrm{Mod}_\mathcal{M}\left(\mathcal{P}shv\left(\mathfrak{Aff}\right)_{/\mathcal{R}an_X}\right)$$

Remark 5.3.4. Recall the functors in diagram 5.1. Starting from a presheaf $\mathcal{F} \in \mathcal{P}shv\left(\mathrm{Dom}_X\right)$, we may construct two different functors of points:

$$\mathrm{LKE}_q \mathcal{F} \text{ and } \mathrm{LKE}_s\left(i_* p_* \mathcal{F}\right)$$

The one on the left is the one we have been referring to as the associated functor of points (cf. 2.2.2). It should be interpreted as the functor of points obtained by quotienting out domain data. The one on the right should be interpreted as the functor of points obtained by retaining the domain data, but "forgetting" how to restrict data to a smaller domain.

The simplicial resolution in the following theorem is the main result of the section. We recall that the functors denoted p, q, r, s and i were introduced in diagram 5.1. Also recall, that the functor $\mathcal{P}shv\left(\mathrm{Ran}_X\right) \xleftarrow{i_*} \mathcal{P}shv\left(\mathrm{Dom}_X^\Gamma\right)$ canonically factors through $\mathrm{Mod}_\mathcal{M}$.

Theorem 5.3.5. *The canonical functor*

$$\mathrm{Mod}_{\mathcal{M}} \longleftarrow \mathcal{P}shv\left(\mathrm{Dom}_X^{\Gamma}\right)$$

is an equivalence.

For every $\mathcal{F} \in \mathcal{P}shv\left(\mathrm{Dom}_X\right)$ *there exists an augmented simplicial object in* $\mathcal{P}shv\left(\mathfrak{Aff}\right)$

(5.2)

$$\cdots \; \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}} \; \mathrm{LKE}_s\left(\mathcal{M}^2\left(i_*p_*\mathcal{F}\right)\right) \underset{\longleftarrow}{\overset{\longrightarrow}{\Longrightarrow}} \mathrm{LKE}_s\left(\mathcal{M}\left(i_*p_*\mathcal{F}\right)\right) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \mathrm{LKE}_s\left(\left(i_*p_*\mathcal{F}\right)\right)$$

$$\downarrow$$

$$\mathrm{LKE}_q\mathcal{F}$$

which becomes a colimit diagram, after sheafification in the fppf Grothendieck topology.

In the simplicial complex above, $\mathcal{M}$ refers to the endofunctor $i_* \circ \mathrm{LKE}_i$ underlying the eponymous monad acting on $\mathcal{P}shv\left(\mathfrak{Aff}\right)_{/\mathcal{R}\mathrm{an}_X}$. We remark that despite the vagueness in the existence statement of the simplicial complex, it is actually quite explicit, as will be explained in 5.3.6.

Proof. The first assertion is a consequence of the Bar-Beck-Lurie theorem [Lur11a, Thm 6.2.0.6], since i_* is conservative and colimit preserving (it admits a right adjoint given by right Kan extension).

For the second assertion, the Bar construction for $p_*\mathcal{F} \in \mathcal{P}shv\left(\mathrm{Dom}_X^{\Gamma}\right) \cong \mathrm{Mod}_{\mathcal{M}}$ yields an augmented simplicial complex in $\mathcal{P}shv\left(\mathrm{Dom}_X^{\Gamma}\right)$

$$\cdots \; \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}} \; \mathrm{LKE}_i\mathcal{M}^2\left(i_*p_*\mathcal{F}\right) \overset{\longrightarrow}{\underset{\longrightarrow}{\Longrightarrow}} \mathrm{LKE}_i\mathcal{M}\left(i_*p_*\mathcal{F}\right) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \mathrm{LKE}_i\left(i_*p_*\mathcal{F}\right) \longrightarrow p_*\mathcal{F}$$

which is a colimit diagram [Lur11a, Thm 4.3.5.8 or Prop 6.2.2.12].

The sought after complex in $\mathcal{P}shv\left(\mathfrak{Aff}\right)$ is obtained by applying the functor LKE_r (note that $r \circ i = s$), and composing the augmentation with $\mathrm{LKE}_r p_*\mathcal{F} \to \mathrm{LKE}_q\mathcal{F}$, to obtain

$$\cdots \mathrm{LKE}_s\mathcal{M}^2\left(i_*p_*\mathcal{F}\right) \underset{\longleftarrow}{\overset{\longrightarrow}{\Longrightarrow}} \mathrm{LKE}_s\mathcal{M}\left(i_*p_*\mathcal{F}\right) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \mathrm{LKE}_s\left(i_*p_*\mathcal{F}\right)$$

$$\downarrow$$

$$\mathrm{LKE}_q\mathcal{F}$$

This augmented complex becomes a colimit diagram after sheafification in the fppf topology, since LKE_r is colimit preserving and by Proposition 5.2.2.   □

5.3.6.   We make the resolution constructed in the theorem explicit. For the sake of concreteness, let us consider the case $\mathcal{F} = \mathrm{GMap}^D\left(X,Y\right) \in \mathcal{P}shv\left(\mathrm{Dom}_X\right)$. Recall

the presheaf $\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}$ introduced in 5.1.1, and observe that

$$i_* p_* \mathrm{GMap}^D (X,Y) = \mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X} \in \mathcal{P}shv\,(\mathfrak{Aff})_{/\mathcal{R}\mathrm{an}_X}$$

We denote an S-point of $\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X} \times (\mathcal{R}\mathrm{an}_X)^n$ by $(f; F_0, \cdots, F_n)$, where it is understood that each F_i is a finite subset of $\mathrm{Hom}\,(S,X)$, and that f is a generic map from $S \times X$ to Y, defined on the open subscheme determined by F_0.

Using remark 5.3.2, we see that the n'th term of the simplicial complex 5.2 is the subsheaf

$$\mathrm{LKE}_s\left(\mathcal{M}^n\left(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}\right)\right) \subseteq \mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X} \times (\mathcal{R}\mathrm{an}_X)^n$$

whose S-points are the tuples $(f,; F \subseteq F_1 \cdots \subseteq F_n)$ (i.e., in which the finite subsets are increasing) . The maps are given as follows:

(1) For a degeneracy $[n+1] \xrightarrow{\ d_i\ } [n]$ $(i, i+1 \mapsto i)$ we have

$$\mathrm{LKE}_s\left(\mathcal{M}^{n+1}\left(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}\right)\right) \longleftarrow \mathrm{LKE}_s\left(\mathcal{M}^n\left(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}\right)\right)$$

$$(f; F_0 \subseteq \ldots \subseteq F_i \subseteq F_i \subseteq \ldots \subseteq F_n) \longleftarrow\!\!\!| \ (f; F_0 \subseteq \ldots \subseteq F_n)$$

(2) For a face map $[n+1] \xleftarrow{\ s_i\ } [n]$ (skip $i \in [n+1]$) we have

$$\left(\mathcal{M}^{n+1}\left(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}\right)\right)_0 \longrightarrow \left(\mathcal{M}^n\left(\mathrm{GMap}\,(X,Y)_{\mathcal{R}\mathrm{an}_X}\right)\right)_0$$

$$(f; F_0 \subseteq \ldots \subseteq F_{n+1}) \longmapsto (f; F_0 \subseteq \ldots \hat{F}_i, \ldots \subseteq F_{n+1})$$

where the hat over $\hat{F}_i$ denotes that the i'th term has been omitted. We point out that, since $F_i \subseteq F_{i+1}$ the i'th term in $(f; F_0, \ldots, \hat{F}_i, \ldots, F_{n+1})$ is the equal to $F_i \cup F_{i+1}$, which how it should be morally interpreted.

Remark 5.3.7. There is another closely related way of describing the category $\mathrm{Mod}_{\mathcal{M}}$. The presheaf $\mathcal{R}\mathrm{an}_X$ has the structure of a semi-group in presheaves of sets, and the category $\mathrm{Mod}_{\mathcal{M}}$ is equivalent to a certain category of its modules (in $\mathbf{Gpd}_\infty$). The approach will be taken up a future note.

5.4. **D-module fully-faithfulness.** In the proposition below we compare two categories of D-modules which may be constructed from functors of points associated to a given moduli problem. This result will be used in section 6.

Proposition 5.4.1. *Let* $\mathcal{F} \in \mathcal{P}shv\left(\mathrm{Dom}_X^\Gamma\right)$. *The map on D-module categories induced by pullback along the adjunction co-unit*

$$\mathfrak{Dmod}\,(\mathrm{LKE}_s\,(\mathcal{F})) \cong \mathfrak{Dmod}\,(\mathrm{LKE}_i \circ i_* \mathcal{F}) \leftarrow \mathfrak{Dmod}\,(\mathcal{F})$$
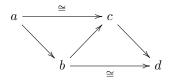
is fully faithful.

Likewise, for $\mathcal{F}' \in \mathcal{P}shv \left(\mathrm{Dom}_X \right)$ *the functor*

$$\mathfrak{Dmod} \left(\mathrm{LKE}_s \left(p_* \mathcal{F}' \right) \right) \cong \mathfrak{Dmod} \left(\mathrm{LKE}_i \circ i_* \circ p_* \mathcal{F}' \right) \leftarrow \mathfrak{Dmod} \left(\mathcal{F}' \right)$$

is fully faithful.

In the proof we use the following general fact: if C is an ∞-category, then equivalences in C satisfy "2-out-of-6". I.e., given a commutative diagram in C

in which the horizontal morphisms are equivalences, we may conclude that that all the morphisms are equivalences (the 6th being the composition $a \to d$).

Proof. We start by reducing to the case when $\mathcal{F}$ is in the essential image if the Yoneda functor $\mathrm{Dom}_X^{\Gamma} \to \mathcal{P}shv \left(\mathrm{Dom}_X^{\Gamma} \right)$. Denote the Yoneda image of point $(S, F) \in \mathrm{Dom}_X^{\Gamma}$ by $\mathcal{Y}_{(S,F)} \in \mathcal{P}shv \left(\mathrm{Dom}_X^{\Gamma} \right)$. Present the presheaf $\mathcal{F}$ as the "colimit of its points" i.e.,

$$\operatorname*{colim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}} \left(\mathcal{Y}_{(S,F)} \right) \xrightarrow{\cong} \mathcal{F}$$

Noting that both LKE_i and i_* preserve colimits (since both admit right adjoints), we also have

$$\operatorname*{colim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}} \left(\mathrm{LKE}_i \circ i_* \mathcal{Y}_{(S,F)} \right) \xrightarrow{\cong} \mathrm{LKE}_i \circ i_* \mathcal{F}$$

Consequently, it suffices to show that the functor

$$\operatorname*{lim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}} \mathfrak{Dmod} \left(\mathrm{LKE}_i \circ i_* \mathcal{Y}_{(S,F)} \right) \leftarrow \operatorname*{lim}_{\mathcal{Y}_{(S,F)} \to \mathcal{F}} \mathfrak{Dmod} \left(\mathcal{Y}_{(S,F)} \right)$$

is fully faithful. The latter will follow if we show that for every $(S, F) \in \mathrm{Dom}_X^{\Gamma}$ the functor

$$\mathfrak{Dmod} \left(\mathrm{LKE}_i \circ i_* \mathcal{Y}_{(S,F)} \right) \leftarrow \mathfrak{Dmod} \left(\mathcal{Y}_{(S,F)} \right)$$

is fully faithful, or equivalently that

$$\mathfrak{Dmod} \left(\mathrm{LKE}_s \left(\mathcal{Y}_{(S,F)} \right) \right) = \mathfrak{Dmod} \left(\mathrm{LKE}_{r \circ i} \circ i_* \mathcal{Y}_{(S,F)} \right) \leftarrow \mathfrak{Dmod} \left(\mathrm{LKE}_r \mathcal{Y}_{(S,F)} \right) = \mathfrak{Dmod} \left(S \right)$$

is fully faithful.

The latter functor is induced by the map in $\mathcal{P}shv \left(\mathfrak{Aff} \right)$

$$\mathrm{LKE}_s \left(\mathcal{Y}_{(S,F)} \right) \to S$$

The functor of points $\mathrm{LKE}_s \left(\mathcal{Y}_{(S,F)} \right)$ sends a scheme T, to the set

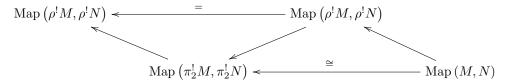$$\left\{ \left((T, G), T \xrightarrow{f} S \right) : G \subset \mathrm{Hom} \left(T, X \right) \text{ finite}, G \supseteq f^* F \right\}$$

"Union with F" gives rise to a map

$$\mathcal{R}an_X \times S \xrightarrow{\ \cup F\ } \mathrm{LKE}_s\left(\mathcal{Y}_{(S,F)}\right)$$

$$\left((T,G),T \xrightarrow{f} S\right) \longmapsto \left(T \xrightarrow{f} S, G \cup f^*F\right)$$

which fits into the commutative diagram

$$\left(\mathcal{Y}_{(S,F)}\right)_0 \xrightarrow{\ \ id\ \ } \left(\mathcal{Y}_{(S,F)}\right)_0$$

(diagram) with $\subseteq$, $\cup F$, ρ, $\mathcal{R}an_X \times S \xrightarrow{\pi_2} S$

Passing to D-modules, pullback along the bottom map is fully-faithful by [Gai11a, Thm 1.6.5] (or [BD04, Prop 4.3.3]). We conclude by a "2-out-of-6" argument: for every pair $M, N \in \mathfrak{Dmod}\,(S)$, the maps above give rise to a diagram of ∞-groupoids

$$\mathrm{Map}\left(\rho^! M, \rho^! N\right) \xleftarrow{\ \ =\ \ } \mathrm{Map}\left(\rho^! M, \rho^! N\right)$$

$$\mathrm{Map}\left(\pi_2^! M, \pi_2^! N\right) \xleftarrow{\ \ \cong\ \ } \mathrm{Map}\left(M, N\right)$$

By "2-out-of-6", for equivalences in $\mathbf{Gpd}_\infty$, it follows that $\mathrm{Map}\left(\rho^! M, \rho^! N\right) \leftarrow \mathrm{Map}\left(M, N\right)$ is an equivalence of ∞-groupoids, so that $\mathfrak{Dmod}\left(i_* \mathcal{Y}_{(S,F)}\right) \xleftarrow{\rho^!} \mathfrak{Dmod}\,(S)$ is fully-faithful.

For $\mathcal{F}' \in \mathcal{P}shv\,(\mathrm{Dom}_X)$, the second assertion now follows from the fact that the functor

$$\mathfrak{Dmod}\,(p_*\,(\mathcal{F}')) \leftarrow \mathfrak{Dmod}\,(\mathcal{F}')$$

is an equivalence (proposition 5.2.2). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.5. **The proof of proposition 5.2.2.** The following lemma contains the geometric input for the proof of proposition 5.2.2:

Lemma 5.5.1. *The functor $\mathrm{Dom}_X^\Gamma \to \mathrm{Dom}_X$ has dense image with respect to the fppf topology. I.e., every point of Dom_X has a cover by points in the essential image of Dom_X^Γ.*

Proof. Let $(S,U) \in \mathrm{Dom}_X$; we must show that it admits a cover by points in the essential image of Dom_X^Γ.

We may assume that S is connected, and by lemma 3.2.7 we may also assume that $U \subseteq S \times X$ is a divisor complement. Let $\mathcal{L} \to \mathcal{O}_{S \times X}$ be an effective Cartier divisor whose complement is U.

Since S is connected, the data of the divisor $\mathcal{L} \to \mathcal{O}_{S \times X}$ is equivalent to a map $S \to \mathcal{H}ilb_X^n$ for some n, where $\mathcal{H}ilb_X^n{}^{[12]}$ is the degree n component of the Hilbert scheme of X. The standard map $X^n \to \mathcal{H}ilb_X^n$ is an fppf cover (it is faithfully flat and finite of index $n!$). Form the pullback

$$
\begin{array}{ccc}
\tilde{S} & \longrightarrow & X^n \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{H}ilb_X^n
\end{array}
$$

The components of the top map gives rise to a subset $F \subseteq \mathrm{Hom}\left(\tilde{S}, X\right)$, which in turn determines a point $\left(\tilde{S}, F\right) \in \mathrm{Dom}_X^\Gamma$. Observing that $U_F = \tilde{S} \times_S U$ we get a map $\left(\tilde{S}, U_F\right) \to \left(S, U\right)$ which is an fppf cover in Dom_X, and whose domain is in the essential image of Dom_X^Γ. $\qquad\square$

Factor p as
$$
\mathrm{Dom}_X^\Gamma \xrightarrow{p'} \mathrm{Dom}_X^{00} \xrightarrow{j} \mathrm{Dom}_X
$$
where Dom_X^{00} is the essential image of p - the full subcategory of Dom_X consisting of "graph complements". We endow Dom_X^{00} with the Grothendieck topology pulled back from the fppf topology on Dom_X. We will prove that j and p' both induce equivalences on sheaf categories, whence proposition 5.2.2 will follow.

Regarding p', informally, the idea is that every fiber of p' is weakly contractible, and that every map in such a fiber is a cover. Thus, it is reasonable to suspect that p' might be a site equivalence. The necessary accounting is a little involved, and the relevant site-theoretic properties of p', which allow the argument to go through, are embodied in the hypothesis of lemma **5.5.3**. Before stating the lemma, we introduce some notation:

Notation 5.5.2. For a category D and an object $d \in D$, we use $D_{/d}$ to denote the overcategory, and we use $D_{d/}$ to denote the undercategory. We shall denote an object of $D_{/d}$ by $(d', d' \to d)$ where d' is an object of D, and $d' \to d$ is a morphism in D (similarly for undercategories).

If C is another category and $C \xrightarrow{F} D$ is a functor, C_d denotes the *fiber* of f over d *i.e.*, the fibered product $C \times_D \{d\}$ in $\mathbf{Cat}_\infty$. We denote $C_{/d} := C \times_D D_{/d}$, it is a relative overcategory. We denote an object of this category by the data $(c, F(c) \to d)$ where it is implicitly understood that c is an object in C, and that $F(c) \to d$ is a morphism in D. Dually, we denote by $C_{d/} = C \times_D D_{d/}$, it is a relative undercategory. This notation is slightly abusive since obviously these categories are dependent on the functor F, and not only on C and d.

---

[12]Since X is a curve, $\mathcal{H}ilb_X^n \cong X^{(n)}$, the n'th symmetric power.

Lemma 5.5.3. *Let C and D be small sites whose underlying categories admit all finite non-empty limits, and whose Grothendieck topologies are generated by finite covers. Let $C \xrightarrow{p} D$ be a functor such that:*

(1) *The Grothendieck topology on C is the pullback of the topology on D.*

(2) *The functor p is essentially surjective.*

(3) *For every $c \in C$, and for every morphism in D, $d \xrightarrow{\tilde{f}} p(c)$, there exists a morphism in C, $c' \xrightarrow{\tilde{f}} c$, which lifts[13] f.*

(4) *p preserves finite limits.*

(5) *For every $d \in D$, the functor $(C_d)^{\mathrm{op}} \to \left(C_{d/}\right)^{\mathrm{op}}$ is cofinal.*

(6) *For every $d \in D$, the category C_d is a co-filtered poset.*

Then the functor

$$\mathcal{S}hv\,(C) \xleftarrow{p_*} \mathcal{S}hv\,(D)$$

is an equivalence, and left Kan extension along p is its inverse (no sheafification necessary).

In 5.5.4 we will show that $\mathrm{Dom}_X^{\Gamma} \xrightarrow{p'} \mathrm{Dom}_X^{00}$ satisfies the hypothesis of this lemma.

Proof. We will show that the left Kan extension

$$\mathcal{S}hv\,(C) \xrightarrow{\mathrm{LKE}_p} \mathcal{P}shv\,(D)$$

lands in sheaves, and prove that the resulting adjoint functors (LKE_p, p_*)

$$\mathcal{S}hv\,(C) \underset{p_*}{\overset{\mathrm{LKE}_p}{\rightleftarrows}} \mathcal{S}hv\,(D)$$

are mutually inverse equivalences.

The following is the key observation: Let $\mathcal{G} \in \mathcal{S}hv\,(C)$, and let $d \in D$. Then $\mathcal{G}$ is constant on the fiber C_d. First we point out that (4) implies that C_d admits all finite non-empty limits, which may be computed in C. Let $c' \xrightarrow{f} c$ be a morphism in C_d; it is a cover by (1). The value of $\mathcal{G}$ at c may be computed using the Čech complex of f. However, (6) implies that this Cech complex is the constant simplicial object with value c', since $c' \times_c c' = c'$ because C_d is a poset. It follows that $\mathcal{G}(c') \leftarrow \mathcal{G}(c)$ is an equivalence. Since C_d it is weakly contractible (being a co-filtered poset), the observation follows.

Let $\mathcal{G} \in \mathcal{S}hv\,(C)$, and let us show that the co-unit transformation (a-priori in $\mathcal{P}shv\,(C)$)

$$p_* \circ \mathrm{LKE}_p \mathcal{G} \to \mathcal{G}$$

---

[13]But we do not assume that a Cartesian lift exists.

is an equivalence. Fix $c \in C$, and let us prove that the map

$$p_* \circ \mathrm{LKE}_p \mathcal{G}(c) \to \mathcal{G}(c)$$

is an equivalence of groupoids. We compute

$$p_* \mathrm{LKE}_p \mathcal{G}(c) = \mathrm{LKE}_p \mathcal{G}(p(c)) = \mathrm{colim}\left(\left(C_{p(c)/}\right)^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathbf{Gpd}_\infty \right) \cong$$

Since $\left(C_{p(c)}\right)^{\mathrm{op}} \to \left(C_{p(c)/}\right)^{\mathrm{op}}$ is co-final by (5),

$$\cong \mathrm{colim}\left(\left(C_{p(c)}\right)^{\mathrm{op}} \xrightarrow{\mathcal{G}} \mathbf{Gpd}_\infty \right) \cong$$

Because $\mathcal{G}$ is constant on the fibers, and these fibers are weakly contractible we conclude

$$\cong \mathcal{G}(c)$$

Next we show that for every $\mathcal{G} \in \mathcal{S}hv(C)$, the presheaf $\mathrm{LKE}_p \mathcal{G}$ is in fact a sheaf. Let $d \in D$, and let $\{d_i \to d\}_{i=1}^k$ be a cover in D. Let $c \in C$ be such that $p(c) = d$, and let $\left\{ c_i \xrightarrow{\tilde{f}_i} c \right\}_{i=1}^k$ be a lift of the f_i's; it is a cover of c by (1). For every n-tuple of indexes in $\{1, \ldots, k\}$, $\bar{i}$, we let $c_{\bar{i}}$ and $d_{\bar{i}}$ denote the corresponding n-fold fibered products over c and d, and we note that $p(c_{\bar{i}}) \cong d_{\bar{i}}$ by (4). Consequently, forming the Čech covers associated with the covers, we obtain a commutative square

$$
\begin{array}{ccc}
\displaystyle\lim_{[n]\in\Delta^{\mathrm{op}}} \left(\coprod_{|\bar{i}|=n} \mathcal{G}(c_{\bar{i}}) \right) & \xleftarrow{\ \cong\ } & \mathcal{G}(c) \\
{\scriptstyle \cong} \Big\downarrow & & \Big\downarrow {\scriptstyle \cong} \\
\displaystyle\lim_{[n]\in\Delta^{\mathrm{op}}} \left(\coprod_{|\bar{i}|=n} \mathrm{LKE}_p \mathcal{G}(d_{\bar{i}}) \right) & \longleftarrow & \mathrm{LKE}_p \mathcal{G}(d)
\end{array}
$$

in which the vertical maps are equivalences by computation above, and the top map is an equivalence because $\mathcal{G}$ is a sheaf. We conclude that the bottom map is an equivalence for every cover of d, thus $\mathrm{LKE}_p \mathcal{G}$ is a sheaf.

We complete the proof of the lemma by observing that we have exhibited adjoint functors

$$\mathcal{S}hv(C) \underset{p_*}{\overset{\mathrm{LKE}_p}{\rightleftarrows}} \mathcal{S}hv(D)$$

for which the co-unit transformation is an equivalence. In addition, Since p is essentially surjective, p_* is conservative, whence we conclude that the unit transformation is also an natural equivalence. The equivalence of sheaf categories follows.   □

5.5.4. *Proof of proposition 5.2.2.* Below, all sites are endowed with their (respective) fppf Grothendieck topologies, and we suppress the topology in the notation. E.g., $\mathcal{S}hv\,(\mathrm{Dom}_X) := \mathcal{S}hv\,(\mathrm{Dom}_X;\mathrm{fppf})$ etc..

Recall the factorization

$$\mathrm{Dom}_X^{\Gamma} \xrightarrow{p'} \mathrm{Dom}_X^{00} \xrightarrow{j} \mathrm{Dom}_X$$

We endow Dom_X^{00} with the Grothendieck topology pulled back from the fppf topology on Dom_X. We treat p' and j separately.

(1) We prove that $\mathcal{S}hv\left(\mathrm{Dom}_X^{00}\right) \xleftarrow{j_*} \mathcal{S}hv\,(\mathrm{Dom}_X)$ is an equivalence by showing that it satisfies the hypothesis of a general criterion for the inclusion of a sub-site to induce an equivalence on sheaf categories (often referred to as the "comparison lemma"). A statement and proof of this criterion is included in the appendix (lemma 8.2.1).

The functor j has dense image by lemma 5.5.1. The category Dom_X admits all finite limits. In particular, fibered products in Dom_X are given by squares of the form

$$
\begin{array}{ccc}
(R \times_T S, U \times_T W) & \longrightarrow & (R, W) \\
\downarrow & & \downarrow{\scriptstyle g} \\
(S, U) & \xrightarrow{f} & (T, V)
\end{array}
$$

Whence it is evident whenever $U \subseteq S \times X$ and $W \subseteq T \times X$ are graph complements (i.e., present points in Dom_X^{00}), then so is

$$U \times_T W \subseteq R \times_S T \times X$$

whence it follows that $(R \times_T S, U \times_T W) \in \mathrm{Dom}_X^{00}$. These are precisely the hypothesis of the comparison lemma (8.2.1), and we conclude that j_* is an equivalence of sheaf categories.

(2) We prove that $\mathcal{S}hv\left(\mathrm{Dom}_X^{\Gamma}\right) \xleftarrow{p'_*} \mathcal{S}hv\left(\mathrm{Dom}_X^{00}\right)$ is an equivalence by showing that the functor p' satisfies the hypothesis of lemma 5.5.3. Aside from (5), which we will show, the rest of the hypothesis are immediate.

Fix $(S, U) \in \mathrm{Dom}_X^{00}$. In order to prove that

$$\left(\mathrm{Dom}_X^{\Gamma}\right)_{(S,U)}^{\mathrm{op}} \to \left(\left(\mathrm{Dom}_X^{\Gamma}\right)_{(S,U)/}\right)^{\mathrm{op}}$$

is cofinal, it suffices to show that for every point $Q \in \left(\left(\mathrm{Dom}_X^{\Gamma}\right)_{(S,U)/}\right)^{\mathrm{op}}$ we have that the category

$$\left(\left(\mathrm{Dom}_X^{\Gamma}\right)_{(S,U)}^{\mathrm{op}}\right)_{Q/}$$

is weakly contractible. Or equivalently, that its opposite category

(5.3) $$\left(\left(\mathrm{Dom}_X^\Gamma \right)_{(S,U)} \right)_{/Q}$$

is weakly contractible. The object Q is presented by the data of

$$(T,G) \in \mathrm{Dom}_X^\Gamma$$

$$\uparrow {}_{p'}$$

$$(S,U) \xrightarrow{f:S \to T} (T,U_G) \in \mathrm{Dom}_X^{00}$$

and the category 5.3 classifies all the ways of lifting f to a "commutative" square

$$(S,F) \longrightarrow (T,G) \in \mathrm{Dom}_X^\Gamma$$

$$\uparrow {}_{p'} \qquad\qquad \uparrow {}_{p'}$$

$$(S,U) \xrightarrow{f:S \to T} (T,U_G) \in \mathrm{Dom}_X^{00}$$

It is equivalent to the category of all finite subsets $F \subseteq \mathrm{Hom}\,(S,X)$ whose associated open subscheme is U, and which contain $\{g \circ f : T \to X \ : \ g \in G\}$, with morphisms being the opposite of inclusion. This category is non-empty, because the assumption that $(S,U) \in \mathrm{Dom}_X^{00}$ implies that it is the image of some (S,F'), and then $(S,F' \cup f^*G)$ completes the square. It also admits finite products (given by the union of F's), thus is weakly contractible by [Lur11b, lemma 2.4.6]. $\qquad\square$

6. Some "Homological Contractibility" Results

In this section we present a few results which relate the D-module categories associated to different moduli spaces of the kind we have been considering. Namely, we prove that certain maps between the spaces induce, via pullback, fully-faithful functors on D-module categories. These results are of interest to the geometric Langlands program, because the D-module categories involved are the counterparts, in the geometric setting, of function spaces that appear on the automorphic side of the correspondence in the classical setting.

Fully faithfulness of D-module pullback has implications for classical[14], invariants such as homology groups, and we start by pointing these out in subsection 6.1.

We emphasize the difference between the results we will discuss below, and those discussed in subsection 5.4. Previously we compared the D-module categories associated with different functor-of-points formulations of the same moduli problem. Below we will compare D-module categories associated to different moduli problems.

---

[14]In contrast with "higher" invariants, such as D-module categories.

6.1. **The homology of a functor of points.** In this subsection we define the homology groups of an arbitrary functor of points, and relate this classical invariant to the higher invariant $\mathfrak{Dmod}$.

6.1.1. *Motivation.* To every scheme S, of finite type over $\mathbb{C}$, we may associate its analytic topological space, S^{an}. By the *homology* of the scheme S, we mean the topological (singular) homology of S^{an} with coefficients in $\mathbb{C}$.

Let $\mathcal{F} \in \mathcal{P}shv\,(\mathfrak{Aff})$ be any functor of points over $\mathbb{C}$. We define the homotopy type of $\mathcal{F}$ to be the homotopy colimit, over all the points of $\mathcal{F}$

$$\mathrm{type}\,(\mathcal{F}) := \operatorname*{hocolim}_{S \to \mathcal{F}} (S^{\mathrm{an}})$$

It is the homology groups of this homotopy type which we are after (when over $\mathbb{C}$). The point of the circuitous definition for the homology of $\mathcal{F}$ given below, is to have it presented in terms of D-module categories. In proposition 6.1.7 we prove that (over $\mathbb{C}$) both notions of homology agree.

Notation 6.1.2. For a functor of points, $\mathcal{F} \in \mathcal{P}shv\,(\mathfrak{Aff})$, and a pair of D-modules $M, N \in \mathfrak{Dmod}\,(\mathcal{F})$ we denote the mapping space (an ∞-groupoid) by

$$\mathrm{Map}_{\mathcal{F}}\,(M, N) := \mathrm{Map}_{\mathfrak{Dmod}(\mathcal{F})}\,(M, N)$$

6.1.3. Let $\mathcal{F} \in \mathcal{P}shv\,(\mathfrak{Aff})$ be an arbitrary functor of points, and let

$$\mathcal{F} \xrightarrow{t} \mathrm{spec}\,(k) =: \mathrm{pt}$$

denote the map to the terminal object. We denote by **Vect** the stable ∞-category of chain complexes of vector spaces over k, mod quasi-isomorphism (whose homotopy category is equivalent to the derived category of the the ordinary category of k-vector spaces). We shall identify $\mathfrak{Dmod}\,(\mathrm{spec}\,(k)) = \mathfrak{Dmod}\,(\mathrm{pt}) \cong \mathbf{Vect}$.

A left adjoint, $t_!$, to the pullback functor $\mathfrak{Dmod}\,(\mathcal{F}) \xleftarrow{t^!} \mathbf{Vect}$, may not be globally defined, but nonetheless makes sense as a partial functor, defined on the full subcategory of those $\mathcal{G} \in \mathfrak{Dmod}\,(\mathcal{F})$ for which the functor

(6.1) $\mathbf{Vect} \longrightarrow \mathbf{Gpd}_{\infty}$

$$V \longmapsto \mathrm{Map}_{\mathcal{F}}\,\big(\mathcal{G}, t^! V\big)$$

is co-representable. For such $\mathcal{G}$, the object $t_! \mathcal{G}$ is such a co-representing object in **Vect**.

Definition 6.1.4. The *canonical sheaf* of a functor of points, $\mathcal{F} \in \mathcal{P}shv\,(\mathfrak{Aff})$, is

$$\omega_{\mathcal{F}} := t^! k$$

Lemma 6.1.5. *Let $\mathcal{F} \in \mathcal{P}shv\,(\mathfrak{Aff})$. The partial functor $t_!$ is defined on $\omega_{\mathcal{F}}$.*

Proof. Define an object of **Vect**

$$H := \operatorname*{colim}_{S \xrightarrow{s} \mathcal{F}} (t_! \omega_S)$$

where the index diagram is the category of points of $\mathcal{F}$ (so each S is an affine scheme). We remark that $t_! \omega_S \in$ **Vect** is well-defined because ω_S is bounded holonomic.

We show that H co-represents the functor 6.1. Indeed

$$\operatorname{Map}_{\mathcal{F}} \left(\omega_{\mathcal{F}}, t^! V \right) \cong \lim_{S \xrightarrow{s} \mathcal{F}} \operatorname{Map}_S \left(\omega_S, t^! V \right) \cong \lim_{S \xrightarrow{s} \mathcal{F}} \operatorname{Map}_{\mathrm{pt}} \left(t_! \omega_S, V \right) \cong$$

$$\operatorname{Map}_{\mathrm{pt}} \left(\operatorname*{colim}_{S \xrightarrow{s} \mathcal{F}} (t_! \omega_S), V \right) = \operatorname{Map}_{\mathrm{pt}} (H, V)$$

we conclude that $t_! \omega_{\mathcal{F}}$ is defined. $\qquad\qquad\square$

Definition 6.1.6. We define the *homology* of $\mathcal{F}$ to be

$$\mathrm{H}_\bullet (\mathcal{F}; k) := t_! \omega_{\mathcal{F}} \in \mathbf{Vect}$$

It follows from the proof of lemma 6.1.5, that

$$\mathrm{H}_\bullet (\mathcal{F}; k) \cong \operatorname*{colim}_{S \xrightarrow{s} \mathcal{F}} (t_! \omega_S) = \operatorname*{colim}_{S \xrightarrow{s} \mathcal{F}} \mathrm{H}_\bullet (S; k)$$

The following well known proposition justifies our use of the word 'homology' (we include a proof for completeness).

Proposition 6.1.7. *Assume $k = \mathbb{C}$, and let $\mathcal{F} \in \mathcal{P}shv(\mathfrak{Aff})$. Then*

$$\mathrm{H}_\bullet (\mathcal{F}; \mathbb{C}) \cong \mathrm{H}_\bullet^{\mathrm{top}} (\mathrm{type}(\mathcal{F}); \mathbb{C})$$

where $\mathrm{H}_\bullet^{\mathrm{top}}$ denotes topological homology.

Proof. Since both homology theories are the left Kan extensions from affine schemes (equivalently, they are colimit preserving), it suffices to consider the case when $\mathcal{F}$ is representable by an affine scheme S.

For an affine scheme S, ω_S is a bounded holonomic complex, and using the Riemann-Hilbert correspondence we obtain an equivalence

$$\mathrm{H}_\bullet (S; \mathbb{C}) = t_! t^! \mathbb{C}_{\mathrm{pt}} \cong t_!^c t_c^! \mathbb{C}_{\mathrm{pt}} \cong$$

where $t_c^!$ and $t_!^c$ denote the !-functors on the (derived) category of constructible sheaves of vector spaces on S^{an}. Denote the duality functor on constructible sheaves by $\mathbb{D}$, and topological co-homology by $\mathrm{H}_{\mathrm{top}}^\bullet$. By Verdier duality we have an equivalence

$$\cong t_!^c t_c^! \mathbb{D} \mathbb{C}_{\mathrm{pt}} \cong \mathbb{D} t_*^c t_c^* \mathbb{C}_{\mathrm{pt}} = \mathbb{D} \left(\mathrm{H}_{\mathrm{top}}^\bullet (S^{\mathrm{an}}; \mathbb{C}) \right) \cong$$

Using the universal coefficient theorem (and that S^{an} has finite dimensional cohomologies) we conclude

$$\cong \mathrm{H}_\bullet^{\mathrm{top}}\left(S^{\mathrm{an}}; \mathbb{C}\right)$$

as claimed. $\qquad\square$

Remark 6.1.8. If a map between functors of points $\mathcal{F} \xrightarrow{f} \mathcal{G}$ induces a fully faithful pullback functor on homology

$$\mathfrak{Dmod}\left(\mathcal{F}\right) \underset{\supseteq}{\xleftarrow{f^!}} \mathfrak{Dmod}\left(\mathcal{G}\right)$$

then

$$\mathrm{H}_\bullet\left(\mathcal{F}; k\right) \cong \mathrm{H}_\bullet\left(\mathcal{G}; k\right)$$

since

$$\mathrm{Map}_{\mathrm{pt}}\left(k, t_!\omega_\mathcal{F}\right) \cong \mathrm{Map}_\mathcal{F}\left(\omega_\mathcal{F}, \omega_\mathcal{F}\right) \underset{\cong}{\xleftarrow{f^!}} \mathrm{Map}_\mathcal{G}\left(\omega_\mathcal{G}, \omega_\mathcal{G}\right) \cong \mathrm{Map}_{\mathrm{pt}}\left(k, t_!\omega_\mathcal{G}\right)$$

In the particular case of $\mathcal{F} \xrightarrow{t} \mathrm{spec}\left(k\right)$, the fully faithfulness of $t^!$ is equivalent to $\mathrm{H}_\bullet\left(\mathcal{F}; k\right) \cong t_!t^!\left(k_{\mathrm{pt}}\right) \to k_{\mathrm{pt}}$ being an equivalence.

In a certain sense the following remark summarizes the essential thesis of this paper:

Remark 6.1.9. For each concrete moduli problem of generic data we have introduced numerous candidates for a functor of points presenting a moduli space - with or without domain data, allowing general domains or only graph-complements, presheaves over $\mathcal{R}\mathrm{an}_X$ and modules for $\mathcal{M}$ therein, and the compactification constructions of sections 3 and 4. The underlying theme of this paper is, that in many ways, these different choices are all equivalent. E.g., a main application of propositions 5.2.3, 5.4.1, 3.2.2, and 4.1.6, about fully-faithfulness of D-module pullback, is that the homology groups of all these functors of points are isomorphic.

6.2. **Back to D-modules.** The following theorem of Gaitsgory is the prototype for the main result of this section, as well as its foundation:

Theorem 6.2.1. [Gai11a, Thm 1.8.2] *Let Y be a connected affine scheme which can be covered by open subschemes U_α, each of which is isomorphic to an open subscheme of the affine space $\mathbb{A}^n$ (for some integer n). Then, the pullback functor*

$$\mathfrak{Dmod}\left(\mathrm{GMap}\left(X, Y\right)_{\mathcal{R}\mathrm{an}_X}\right) \xleftarrow{t^!} \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right) = \mathbf{Vect}$$

is fully faithful. $\qquad\square$

In particular, we conclude that under the assumptions of the theorem

$$\mathrm{H}_\bullet\left(\mathrm{GMap}\left(X, Y\right)_{\mathcal{R}\mathrm{an}_X}; k\right) \cong k$$

(see 6.1.8).

In this section we use theorem 6.2.1 to obtain more results of a similar nature.

Recall that the a-priori premise of this paper is that "the correct" (from a conceptual point of view) space of generic maps is presented by the functor of points GMap (X, Y), introduced in 2.2.3.

Corollary 6.2.2. *Let Y be as in theorem 6.2.1. The pullback functor*

$$\mathfrak{Dmod}\left(\mathrm{GMap}\left(X, Y\right)\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right) = \mathbf{Vect}$$

is fully faithful.

Proof. Consider the pull back functors

$$
\begin{array}{c}
\mathbf{Vect} \\
\downarrow \cong \\
\mathfrak{Dmod}\left(\mathrm{LKE}_r p_* \mathrm{GMap}^D\left(X, Y\right)\right) \xleftarrow{\alpha^!} \mathfrak{Dmod}\left(\mathrm{GMap}\left(X, Y\right)\right) \xleftarrow{t^!} \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right) \\
\beta^! \downarrow \\
\mathfrak{Dmod}\left(\mathrm{GMap}\left(X, Y\right)_{\mathcal{R}\mathrm{an}_X}\right)
\end{array}
$$

The composition is fully faithful by theorem 6.2.1. $\alpha^!$ is an equivalence by proposition 5.2.2, and $\beta^!$ is fully faithful by proposition 5.4.1. We conclude that $t^!$ is fully faithful. $\qquad\square$

The next result is a minor extension of theorem 6.2.1, in which we remove the requirement that the target be affine:

Theorem 6.2.3. *Let Y be a connected and separated scheme which can be covered by open subschemes U_α, each of which is isomorphic to an open subscheme of the affine space $\mathbb{A}^n$ (for some integer n). Then, the pullback functor*

$$\mathfrak{Dmod}\left(\mathrm{GMap}\left(X, Y\right)\right) \xleftarrow{t^!} \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right) = \mathbf{Vect}$$

is fully faithful. When Y is projective, $t^!$ admits a (globally defined) left adjoint.

The main examples to consider for Y (aside from $\mathbb{A}^n$), are $\mathbb{P}^n$, a connected affine algebraic group G, and its flag variety G/B. We prove this theorem in 6.2.7.

Recall the functors of points $\mathrm{Bun}_G^{H(\eta)}$ and $\mathrm{Bun}_G^{1(\eta)} \in \mathcal{P}shv\left(\mathfrak{Aff}\right)$ which were introduced in 2.2.5. The following theorem is the main result of this section:

Theorem 6.2.4. *Let G be a connected reductive algebraic group. Let H be a subgroup of G such that G/H is rational (e.g., $H = 1$, N, or any parabolic subgroup). Then, the pull back functor*

$$\mathfrak{Dmod}\left(\mathrm{Bun}_G^{H(\eta)}\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{Bun}_G\right)$$

is fully faithful. When $H = B$, this pullback functor admits a (globally defined) left adjoint.

Theorem 6.2.4 is proven in 6.2.11, after some preparations.

The existence of the left adjoint can be extended to include any parabolic subgroup, if the statement (and proof) of proposition 4.1.6 is extended accordingly.

We remark that the existence of the left adjoint above (and in theorem 6.2.7) is a kind of "properness" of the map $\mathrm{Bun}_G^{B(\eta)} \to \mathrm{Bun}_G$ $(\mathrm{GMap}\,(X, Y) \to \mathrm{spec}\,(k))$, though this map between functor of points is not schematic. We also emphasize, as a concrete application, that pullback fully faithfulness implies that the homology of the spaces in question is equivalent (see remark 6.1.8).

The rest of this section is contains the proofs (and supporting lemmas) of theorems 6.2.3 and 6.2.4.

By a *Zariski cover* of presheaves we mean a morphism of presheaves, which becomes an effective epimorphism after sheafification in the Zariski Grothendieck topology.

Lemma 6.2.5. *The functor*

$$\mathrm{GMap}\,(X, -) : \mathfrak{S} \to \mathcal{P}shv\,(\mathfrak{Aff})$$

carries Zariski covers to Zariski covers.

Proof. Let Y be a scheme, and $\{Y_i \to Y\}_{i \in I}$ its finite cover by open subschemes. We must show that for every point $S \xrightarrow{s} \mathrm{GMap}\,(X, Y)$, there exists a Zariski cover $\tilde{S} \to S$, and a lift

$$
\begin{array}{ccc}
 & & \coprod_{i \in I} \mathrm{GMap}\,(X, Y_i) \\
 & \nearrow & \downarrow \\
\tilde{S} \longrightarrow S & \longrightarrow & \mathrm{GMap}\,(X, Y)
\end{array}
$$

The point s is presented by a point point $(S, U) \in \mathrm{Dom}_X$, together with a regular map $U \to Y$. For every $i \in I$, let $U_i := U \times_{Y_i} Y \subseteq U$ (it is an open subscheme of U), and let $S_i \subseteq S$ be the open subscheme which is the image of $U_i \to S \times X \to S$. The composition $U_i \to U \to Y$ lands in Y_i, and thus determines a lift

$$
\begin{array}{ccc}
 & & \mathrm{GMap}\,(X, Y_i) \\
 & \nearrow & \downarrow \\
S_i \longrightarrow S & \longrightarrow & \mathrm{GMap}\,(X, Y)
\end{array}
$$

Taking $\tilde{S} = \coprod S_i$, the map $\coprod S_i \to \coprod \mathrm{GMap}\,(X, Y_i)$ is the sought after lift of s. $\square$

Lemma 6.2.6. *The functor*

$$\text{GMap}\,(X,-) : \mathfrak{Aff} \to \mathcal{P}shv\,(\mathfrak{Aff})$$

preserves finite limits.

Proof. $\text{GMap}\,(X,-)$ is the composition

$$\mathfrak{Aff} \xrightarrow{\text{GMap}^D(X,-)} \mathcal{P}shv\,(\text{Dom}_X) \xrightarrow{\text{LKE}_q} \mathcal{P}shv\,(\mathfrak{Aff})$$

$\text{GMap}^D\,(X,-)$ preserves (all) limits, and LKE_q preserves finite limits. $\qquad\square$

6.2.7. *Proof of Theorem 6.2.3.* The theorem is now an almost immediate result of lemmas 6.2.6 and 6.2.5.

Let $\{U_i \to Y\}_{i \in I}$ be a cover of Y by its affine open subschemes, which are each isomorphic to an open subscheme of $\mathbb{A}^n$. We note that since Y is separated, every intersection of the U_i's has the same property.

Construct the Čech complex corresponding to the cover

$$\Delta^{\text{op}} \xrightarrow{\ \ U_\bullet\ \ } \mathfrak{Aff}$$

$$[n] \longmapsto \coprod_{|\bar{i}|=n} U_{\bar{i}}$$

where $\bar{i} = (i_1, \ldots, i_n)$ is a multi-index of elements in I, and $U_{\bar{i}} = \cap_{k=1}^n U_{i_k}$. We have that $Y = \underset{[n]\in\Delta^{\text{op}}}{\text{colim}} \left(\coprod_{|\bar{i}|=n} U_{\bar{i}} \right)$. By lemma 6.2.6, the simplicial object

$$\Delta^{\text{op}} \xrightarrow{\text{GMap}(X,U_\bullet)} \mathcal{P}shv\,(\mathfrak{Aff})$$

$$[n] \longmapsto \coprod_{|\bar{i}|=n} \text{GMap}\,(X, U_{\bar{i}})$$

is the Čech nerve of $\{\text{GMap}\,(X,U_i) \to \text{GMap}\,(X,Y)\}_{i\in I}$, which is a Zariski cover by lemma 6.2.5. We conclude that the homology of $\text{GMap}\,(X,Y)$ is isomorphic to that of a point, being the colimit

$$\text{H}_\bullet\,(\text{GMap}\,(X,Y)\,;k) \cong \underset{[n]\in\Delta^{\text{op}}}{\text{colim}} \text{H}_\bullet\left(\coprod_{|\bar{i}|=n} \text{GMap}\,(X,U_{\bar{i}})\,;k \right) \cong$$

$$\cong \underset{[n]\in\Delta^{\text{op}}}{\text{colim}} \text{H}_\bullet\left(\coprod_{|\bar{i}|=n} \text{spec}\,(k)\,;k \right) \cong \text{H}_\bullet\,(\text{spec}\,(k)\,;k)$$

Finally, the equivalence $H_\bullet(\mathcal{F}) \cong H_\bullet(\mathrm{spec}\,(k); k)$ implies that the fully faithfulness of $t^!$ (see remark 6.1.8).

Regarding the existence of the left adjoint, when Y is projective, this is a restatement of corollary **3.2.4** (2). $\qquad\qquad\square$

We continue with the preparations for the proof of the theorem 6.2.4. The following is a corollary of lemma 6.2.6:

Corollary 6.2.8. *Let be G an algebraic group.*

  (1) $\mathrm{GMap}\,(X, G)$ is a group object in $\mathcal{P}shv\,(\mathfrak{Aff})$.
  (2) If Y is a scheme acted on by G, then $\mathrm{GMap}\,(X, Y)$ is acted on by $\mathrm{GMap}\,(X, G)$.

Definition 6.2.9. A map of presheaves $\mathcal{E} \to \mathcal{B}$ is an *fppf-locally trivial fibration with fiber $\mathcal{F}$,* if there exists an fppf cover $\mathcal{B}' \to \mathcal{B}$ (i.e., a morphism of presheaves which becomes an effective epimorphism after fppf sheafification), and a map

$$\mathcal{B}' \times_\mathcal{B} \mathcal{E} \to \mathcal{F}$$

which exhibits the former as a product $\mathcal{B}' \times_\mathcal{B} \mathcal{E} \cong \mathcal{F} \times \mathcal{B}'$.

Lemma 6.2.10. *Let $\mathcal{E} \xrightarrow{p} \mathcal{B}$ be an fppf-locally trivial fibration with fiber $\mathcal{F}$ of presheaves in $\mathcal{P}shv\,(\mathfrak{Aff})$.*

If $\mathfrak{Dmod}\,(\mathcal{F}) \xleftarrow{t^!} \mathfrak{Dmod}\,(\mathrm{spec}\,(k))$ is fully faithful, then $\mathfrak{Dmod}\,(\mathcal{E}) \xleftarrow{p^!} \mathfrak{Dmod}\,(\mathcal{B})$ is fully faithful.

Proof. Let $M, N \in \mathfrak{Dmod}\,(\mathcal{B})$. We must show that

$$(*) \qquad \mathrm{Map}_\mathcal{E}\left(p^! M, p^! N\right) \leftarrow \mathrm{Map}_\mathcal{B}\,(M, N)$$

is an equivalence of ∞-groupoids.

Fix a Cartesian square

$$
\begin{array}{ccc}
\mathcal{F} \times \mathcal{B}_0 & \longrightarrow & \mathcal{E} \\
\downarrow & \ulcorner & \downarrow \\
\mathcal{B}_0 & \longrightarrow & \mathcal{B}
\end{array}
$$

in which $\mathcal{B}_0 \to \mathcal{B}$ is an fppf cover. Denote the Čech simplicial complex associated with the cover $\mathcal{B}_0 \to \mathcal{B}$ by

$$\Delta^{\mathrm{op}} \xrightarrow{\;\mathcal{B}_\bullet\;} \mathcal{P}shv\,(\mathfrak{Aff})$$

$$[n] \longmapsto \mathcal{B}_n := \underbrace{\mathcal{B}_0 \times_\mathcal{B} \cdots \times_\mathcal{B} \mathcal{B}_0}_{n-\text{times}}$$

and the one associated with the cover $\mathcal{B}_0 \times \mathcal{F} \to \mathcal{E}$ by

$$\Delta^{\mathrm{op}} \xrightarrow{(\mathcal{B}_0 \times \mathcal{F})_\bullet} \mathcal{P}shv\,(\mathfrak{Aff})$$

$$[n] \longmapsto (\mathcal{B}_0 \times \mathcal{F})_n := \underbrace{(\mathcal{B}_0 \times \mathcal{F}) \times_\mathcal{E} \cdots \times_\mathcal{E} (\mathcal{B}_0 \times \mathcal{F})}_{n-\text{times}}$$

There exist an equivalences of stable ∞-categories

$$\mathfrak{Dmod}\,(\mathcal{B}) \cong \varprojlim_{[n] \in \Delta} \mathfrak{Dmod}\,(\mathcal{B}_n) \quad \text{and} \quad \mathfrak{Dmod}\,(\mathcal{E}) \cong \varprojlim_{[n] \in \Delta} \mathfrak{Dmod}\,((\mathcal{B}_0 \times \mathcal{F})_n)$$

and $p^!$ is induced by a transformation of the co-simplicial diagrams.

Let M_n and N_n denote the images of M and N in $\mathfrak{Dmod}\,(\mathcal{B}_n)$. Let $\left(p^! M\right)_n$ and $\left(p^! N\right)_n$ denote the images of M and N in $\mathfrak{Dmod}\,((\mathcal{B}_0 \times \mathcal{F})_n)$. We have equivalences of ∞-groupoids

$$\mathrm{Map}_\mathcal{B}\,(M, N) \cong \varprojlim_{[n] \in \Delta} \mathrm{Map}_{\mathcal{B}_n}\,(M_n, N_n)$$

and

$$\mathrm{Map}_\mathcal{E}\,\left(p^! M, p^! N\right) \cong \varprojlim_{[n] \in \Delta} \mathrm{Map}_{(\mathcal{B}_0 \times_\mathcal{B} \mathcal{E})_n}\,\left(\left(p^! M\right)_n, \left(p^! N\right)_n\right)$$

We have that $\left(p^! M\right)_n \cong p_n^! N_n$, where p_n is the map $(\mathcal{B}_0 \times_\mathcal{B} \mathcal{E})_n \xrightarrow{p_n} \mathcal{B}_n$. Furthermore, the map $(*)$ is the limit of the maps

$$(**) \quad \mathrm{Map}_{(\mathcal{B}_0 \times_\mathcal{B} \mathcal{E})_n}\,\left(p_n^! M_n, p_n^! N_n\right) \leftarrow \mathrm{Map}_{\mathcal{B}_n}\,(M_n, N_n')$$

Finally, observing that for each n we have a commuting diagram

$$\begin{array}{ccc} (\mathcal{B}_0 \times_\mathcal{B} \mathcal{E})_n & \xrightarrow{\;\cong\;} & \mathcal{F} \times \mathcal{B}_n \\ {\scriptstyle p_n}\big\downarrow & & \big\downarrow \\ \mathcal{B}_n & \xrightarrow{\;=\;} & \mathcal{B}_n \end{array}$$

which implies that, for every $[n] \in \Delta$, the functor $p_n^!$ is fully faithful. Thus, the map $(**)$ is an equivalence, whence we conclude that the map $(*)$ is an equivalence. $\square$

The proof below uses the symmetric properties of the map $\mathrm{Bun}_G^{H(\eta)} \to \mathrm{Bun}_G$. After the proof we indicate a strategy for another proof, similar to that of theorem 6.2.3..

6.2.11. *Proof of Theorem 6.2.4.* Observe that there exists a Cartesian square

$$\begin{array}{ccc} \mathrm{GMap}\,(X, G/H) \times \mathrm{Bun}_G^{1(\eta)} & \xrightarrow{\;a\;} & \mathrm{Bun}_G^{H(\eta)} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow \\ \mathrm{Bun}_G^{1(\eta)} & \longrightarrow & \mathrm{Bun}_G \end{array}$$

The functor $\mathrm{Bun}_G^{1(\eta)} \to \mathrm{Bun}_G$ becomes an effective epimorphism after Etale sheafification. Indeed, if $\mathcal{P}_G$ is G-torsor on $S \times X$ then, by the Drinfeld-Simpson theorem [DS95, Thm 2], there exists an Etale base change $S' \to S$, such that $\mathcal{P}_G \times_S S'$ is Zariski locally trivial, hence admits a generic trivialization.

Our assumptions on H imply that it may be covered by open subschemes which are isomorphic to open subschemes of affine space. Thus by theorem 6.2.1,

$$\mathfrak{Dmod}\left(\mathrm{GMap}\left(X, H\right)\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right)$$

is fully faithful. The fully faithfulness of the pullback functor

$$\mathfrak{Dmod}\left(\mathrm{Bun}_G^{H(\eta)}\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{Bun}_G\right)$$

now follows from 6.2.10.

In the case when $H = B$, the existence of a left adjoint is a restatement of corollary 4.1.7 (2). $\qquad\qquad\square$

Remark 6.2.12. A different proof of the theorem may be deduced from the following statement:

Let $Y \to S \times X$ be an fppf fiber bundle with fiber F, which becomes Zariski-locally trivial, after a suitable fppf base change $\tilde{S} \to S$. And such that $\mathfrak{Dmod}\left(F\right) \leftarrow \mathfrak{Dmod}\left(\mathrm{spec}\left(k\right)\right)$ is fully faithful.

Then the pullback functor

$$\mathfrak{Dmod}\left(\mathrm{GSect}_S\left(S \times X, Y\right)\right) \leftarrow \mathfrak{Dmod}\left(S\right)$$

is fully faithful.

7. Appendix - The quasi functor $\mathfrak{Dmod}$ and other abstract nonsense

The main purpose of this section is to sketch out the construction of the functorial assignment

$$\mathcal{F} \mapsto \mathfrak{Dmod}\left(\mathcal{F}\right)$$

There are two catches, the first is that we wish to allow $\mathcal{F} \in \mathcal{P}shv\left(\mathfrak{Aff}\right)$ to be an arbitrary functor of points without assuming any representability properties (such as being a scheme or Artin stack). The second is that by $\mathfrak{Dmod}\left(\mathcal{F}\right)$ we mean a *stable* ∞-category whose homotopy category is the eponymous triangulated category which is usually considered.

What follows is intended as a broad overview only; we shall point to references where details may be found.

7.1. Motivation. We think of the quasi-functor

$$\mathfrak{S}^{\mathrm{op}} \xrightarrow{\mathfrak{Dmod}^{\triangle}} \{\text{triangulated cateogries}\}$$

which assigns to a scheme S its category of D-module sheaves, as an invariant defined on schemes whose values are triangulated categories. $\mathfrak{Qco}^{\triangle}$, which assigns the category of quasi-coherent sheaves, and $\mathfrak{Ico}^{\triangle}$ which assigns the category of ind-coherent sheaves are other triangulated category valued invariants commonly considered in the study of schemes. We wish to study spaces presented by arbitrary functors of points (such as the ones considered in this paper) by the same kind of invariants, thus we wish to extend their domains of definition to include all functors of points - representable or not.

Consider first a diagram of schemes $I \xrightarrow{F} \mathfrak{S}$, where I is a small index category. Informally, the value of $\mathfrak{Dmod}^{\triangle}$ on such a diagram is $\lim_I \mathfrak{Dmod}^{\triangle}(F(i))$. As a first approximation, this limit is the category whose objects consist of an assignment of D-modules

$$i \mapsto \mathcal{F}_i \in \mathfrak{Dmod}^{\triangle}(F(i)),$$

which is required to be pullback compatible in the sense that for every morphism $i \xrightarrow{f} j$ in I, the D-modules $\mathcal{F}_i$ and $F(f)^*(\mathcal{F}_j)$ on the scheme $F(i)$ are identified (compatibly).

However, the totality of triangulated categories is not suited for taking such limits. In practice, we take a kind of homotopy limit and it is the formalism of this process which is the topic of this section. This homotopy limit will manifest itself as a (coherent) limit within the ∞-category of *stable* ∞-categories (see below).

7.2. **∞-categories.** We use the term ∞-*category* to refer to the abstract (model independent) notion of an $(\infty, 1)$-category - a collection of objects with an infinite hierarchy of morphisms, in which all n n-morphisms for $n \geq 2$ are invertible. For the most part, we use the notation of [Lur09] and [Lur11a].

To an ∞-category, C, one can associate an ordinary category hC - its homotopy category. It is the left adjoint to the inclusion of ordinary categories in ∞-categories ([Lur09, 1.2.3]).

Being a *stable ∞-category* is a property an ∞-category (rather than structure, see [Lur11a, 1.1.1.14]). In setting of ∞-categories, these stable categories play a role similar to that of abelian categories in the ordinary setting. The homotopy category of a stable ∞-category is naturally a triangulated category ([Lur11a, 1.1.2.13]). As such, we think of a stable ∞-category as remembering higher morphisms which its triangulated homotopy category has forgotten. *Exactness* is a property of functors between stable ∞-categories ([Lur11a, 1.1.4]). The functor it induces between the triangulated homotopy categories is then a triangulated functor.

Remark 7.2.1. Set theory must be dealt with in some way. We consider three sizes of sets: small, large and abominable[15]. One can make sense of these by considering

---

[15]cf. french alpine grades.

a chain of universes of sets $\mathcal{U} \subseteq \hat{\mathcal{U}} \subseteq \hat{\hat{\mathcal{U}}}$, each a set in its super universes. Set size corresponds to universe size (small sets are the objects of $\mathcal{U}$, which is itself a large set. Large sets are the objects of $\hat{\mathcal{U}}$, which is itself an abominable set etc).

It is the theory of $\mathcal{U}$ we are interested in, the others are auxiliary universes introduced to make sense of "large constructions". In particular k is a field in $\mathcal{U}$, and $\mathfrak{S}$ refers to the category of finite type schemes over k which belong to $\mathcal{U}$. Note that due to the "finite type" restriction, $\mathfrak{S}$ is an essentially small category.

A large ∞-category may have the property of being *presentable*. For the precise definition see [Lur09, section 5.5]. Let us only say that this property allows us to deal with a large category from within the small universe and that most large categories we shall deal with will shall have this property.

7.2.2. *Categories of ∞-cat's.* The totality of all small ∞-categories is itself given the structure of an (large) ∞-category, denoted $\mathbf{Cat}_\infty$. We denote by $\mathbf{Gpd}_\infty \subseteq \mathbf{Cat}_\infty$ the full subcategory (small) ∞-groupoids, also known as spaces or homotopy-types.

Likewise, the totality of all large ∞-categories is denoted $\hat{\mathbf{Cat}}_\infty$. Let $\hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}} \subseteq \hat{\mathbf{Cat}}_\infty$ be the full subcategory of presentable stable categories and functors which preserve small colimits (equivalently, are left adjoints, whence the L in the notation).
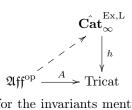
Proposition 7.2.3. [Lur11a, 1.1.4.4][Lur09, 5.5.3.13] *The (abominably large) infinity category $\hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$ admits all small limits. its inclusion into $\hat{\mathbf{Cat}}_\infty$ is limit preserving.* □

7.3. **Stable invariants.** Classically one considers triangulated invariants (i.e. Tricat valued). However, the $(2,1)$-category Tricat is not well suited for taking limits as we wish to do. In order to remedy this shortcoming we compute our limits in $\hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$. The stable and triangulated invariants are related as follows:

Definition 7.3.1. A *stable invariant* of schemes is a functor $\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\mathcal{A}} \hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}}$. Its associated Triangulated invariant is defined to be the composition

$$\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\mathcal{A}} \hat{\mathbf{Cat}}_\infty^{\mathrm{Ex,L}} \xrightarrow{h} \mathrm{Tricat}$$

The particular example we consider in this paper is when $\mathcal{A}$ is $\mathfrak{Dmod}$, but along the way we will also mention $\mathfrak{Ico}$. In these cases the associated triangulated invariant is a familiar notion, and we would like to have a stable model for it. I.e. a lift

$$\mathbf{\hat{Cat}}_{\infty}^{\mathrm{Ex,L}}$$

$$\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\quad A \quad} \mathrm{Tricat}$$

with h on the right vertical arrow.

It seems to be the case that for the invariants mentioned above, such a lift exists and is in fact quite natural.

7.3.2. *IndCoh.* Our initial input for the construction of $\mathfrak{Dmod}$, is the functor which assigns to a scheme its category of ind-coherent sheaves. Ind-coherent sheaves are a convenient substitute for quasi-coherent sheaves, for an in-depth discussion see [Gai10c].

In [Gai10c, Cor 3.3.6] Gaitsgory constructs a functor

$$\mathfrak{Ico} : \mathfrak{Aff} \to \mathbf{\hat{Cat}}_{\infty}^{\mathrm{Ex,L}}$$

assigning to a scheme a higher category model of its triangulated category of ind-coherent sheaves, and to every morphism of schemes the $!-$pullback. There, pre-triangulated DG-categories are taken as models for stable ∞-categories.

We extend $\mathfrak{Ico}$ to arbitrary functors of points via a right Kan extension

$$\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\quad \mathfrak{Ico} \quad} \mathbf{\hat{Cat}}_{\infty}^{\mathrm{Ex}}$$

with j down the left, $\mathfrak{Ico}:=\mathrm{RKE}$, and $\mathcal{P}shv\,(\mathfrak{Aff})^{\mathrm{op}}$ below.

where j is the Yoneda embedding. We abusively continue to denote the extension by $\mathfrak{Ico}$. So defined, $\mathfrak{Ico}$ preserves small limits ([Lur09, lemma 5.1.5.5]).

7.3.3. $\mathfrak{Dmod}$. We follow [GR11, 2.3.1]. First we define the de-Rham functor

$$\mathcal{P}shv\,(\mathfrak{Aff}) \xrightarrow{\quad dR \quad} \mathcal{P}shv\,(\mathfrak{Aff})$$
$$\mathcal{F} \mapsto \mathcal{F}^{dR}$$

where the presheaf $\mathcal{F}^{dR}$ is defined by $\mathcal{F}^{dR}\,(S) = \mathcal{F}\,(S_{\mathrm{red}})$.

We define $\mathfrak{Dmod}$ on $\mathcal{P}shv\,(\mathfrak{Aff})$ to be the composition

$$\overbrace{(\mathcal{P}shv\,(\mathfrak{Aff}))^{\mathrm{op}} \xrightarrow{\quad dR \quad} (\mathcal{P}shv\,(\mathfrak{Aff}))^{\mathrm{op}} \xrightarrow{\quad \mathfrak{Ico} \quad} \mathbf{\hat{Cat}}_{\infty}^{\mathrm{Ex,L}}}^{\mathfrak{Dmod}:=}$$

Remark 7.3.4. The functor $\mathcal{P}shv\,(\mathfrak{Aff}) \xrightarrow{\mathrm{dR}} \mathcal{P}shv\,(\mathfrak{Aff})$ is simply the pullback along the composition $\mathfrak{Aff} \xrightarrow{\mathrm{red}} \mathfrak{Aff}$, thus is colimit preserving. It follows that $\mathfrak{Dmod}$ is

limit preserving, consequently it is equivalent to the right Kan extension

$$
\begin{array}{ccc}
\mathfrak{Aff}^{\mathrm{op}} & \xrightarrow{\ \mathrm{dR}\ } \mathcal{P}shv\,(\mathfrak{Aff})^{\mathrm{op}} \xrightarrow{\ \mathcal{I}\mathfrak{co}\ } \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}} \\
\Big\downarrow{\scriptstyle j} & \nearrow \\
\mathcal{P}shv\,(\mathfrak{Aff})^{\mathrm{op}} & \mathfrak{Dmod}
\end{array}
$$

so that for every $\mathcal{F} \in \mathcal{P}shv\,(\mathfrak{Aff})$ we have $\mathfrak{Dmod}\,(\mathcal{F}) \cong \varprojlim\limits_{S \to \mathcal{F}} \mathfrak{Dmod}\,(S)$.

Remark 7.3.5. What we have defined above should rightfully be called the category of *crystalline* sheaves. It is well known that for a smooth scheme S there exist equivalences between the homotopy category of $\mathfrak{Dmod}\,(S)$ (as we have defined it), and the usual derived category of sheaves of right modules for the algebra of differential operators on S (see e.g., [GR11, 4.7]). Moreover, this equivalence is compatible with !-pullback, and with Kashiwara's theorem.

8. Appendix - odds and ends

8.1. **Limits and and colimits of adjoint diagrams.**
Let

$$
\begin{array}{c}
I \xrightarrow{\ \ G\ \ } \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}} \\[2mm]
i \longmapsto C_i
\end{array}
$$

be a small diagram. If for every morphism, $i \xrightarrow{f} j$ in I, the functor $G\,(i) \xrightarrow{F(f)} G\,(j)$ admits a left adjoint[16], then there exists a unique diagram (up to contractible ambiguity)

$$
\begin{array}{c}
I^{\mathrm{op}} \xrightarrow{\ \ F\ \ } \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}} \\[2mm]
i \longmapsto C_i
\end{array}
$$

such that for every morphism, $i \xrightarrow{f} j$, the functor $C_j \xrightarrow{F(f)} C_i$ is left adjoint to $G\,(f)$. Let us call the pair of diagram F and G *adjoint*.

The following lemma appears in [Gai11b, 1.3.3], where it attributed to J. Lurie.

Lemma 8.1.1. *If F and G are adjoint I-diagrams, as above. Then*

(1) There exists an equivalence of stable ∞-categories

$$
\operatorname*{colim}_{i \in I^{\mathrm{op}}} F\,(i) \cong \varprojlim_{i \in I} G\,(i)
$$

(2) Let $C \in \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$, and let $I^{\triangleleft} \xrightarrow{G^{\triangleleft}} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$ be a co-augmentation of C over of G, such that each $C \to C_i$ admits a left adjoint. Then, the natural

---

[16]Not to be confused with the right adjoint it admits by virtue of being a morphism in $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$.

functors

$$\begin{array}{ccc} \operatorname{colim}_{i \in I^{\mathrm{op}}} F(i) & \longrightarrow & C \\ \cong \big\Updownarrow & & \big\| \\ \lim_{i \in I} G(i) & \longleftarrow & C \end{array}$$

become adjoint, after identifying the right hand side via (1).

(3) For every $j \in I$, the natural functors

$$\begin{array}{ccc} C_j & \xrightarrow{\ \rho_j\ } & \operatorname{colim}_{i \in I^{\mathrm{op}}} F(i) \\ \big\| & & \Updownarrow \cong \\ C_j & \xleftarrow[\ \pi_j\ ]{} & \lim_{i \in I} G(i) \end{array}$$

are adjoint, after identifying the left side via 1.

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

(1) The categories $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$ and $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}}$ both admit small limits, and that the inclusion into $\hat{\mathbf{Cat}}_{\infty}$ preserves these ([Lur09, 5.5.3.5,5.5.3.18] and [Lur11a, 1.1.4.4]). Consequently, since the diagram G lands in both categories (viewed as subcategories of $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex}}$), we have an equivalence

$$\lim\left(I \xrightarrow{G} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}\right) \cong \lim\left(I \xrightarrow{G} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}}\right)$$

There exists a duality[17]

$$\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}} \xrightarrow{\cong} \left(\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}\right)^{\mathrm{op}}$$

which is the identity on objects, and carries each functor to its left adjoint. It carries a limit cone $I^{\triangleleft} \xrightarrow{G^{\lim}} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}}$ for G, to a colimit cone $(I^{\mathrm{op}})^{\triangleright} \xrightarrow{F^{\mathrm{colim}}} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$ for F, supported on the same objects. In particular, restricting to the cone point we get an equivalence

$$\lim\left(I \xrightarrow{G} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}}\right) \cong \operatorname{colim}\left(I^{\mathrm{op}} \xrightarrow{F} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}\right)$$

(2) In the limit and colimit diagrams above, the functors $\lim_{i \in I} G(i) \xrightarrow{\pi_j} C_j$ and $\lim_{i \in I} G(i) \xrightarrow{\pi_j} C_j$ correspond under the duality, hence are adjoint.

(3) By the same argument as in the first part of 1., the functor $C \to \lim_{i \in I} G(i)$ can be thought of as a map in $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}}$. Thus, it admits a left adjoint, which is its image under the duality. This dual image corresponds to the the diagram dual to $I^{\triangleleft} \xrightarrow{G^{\triangleleft}} \hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}}$, whence the assertion follows.

---

[17]Thus, $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,L}}$ and $\hat{\mathbf{Cat}}_{\infty}^{\mathrm{Ex,R}}$ admit colimits as well, but these are not (in general) preserved by the inclusion into $\hat{\mathbf{Cat}}_{\infty}$.

8.2. **A comparison lemma for sites.** The following lemma is an analog of the "Comparison lemma" [Joh02, Thm 2.2.3.], which applies to sheaves of sets (cf. [Lur09, Warning 7.1.1.4]).

Lemma 8.2.1. *Let C be a small category with a Grothendieck topology, and let $C^0 \xrightarrow[\subseteq]{j} C$ be a full subcategory. Assume that:*

(1) *C admits all finite limits.*
(2) *For any fibered product in C, $c_1 \times_c c_2$, if $c_1, c_2 \in C^0$ then $c_1 \times_c c_2 \in C^0$ (we do not assume that $c \in C^0$).*
(3) *C^0 is dense in C. I.e., every object in C has a cover by objects in C^0.*

Then, the restriction functor

$$\mathcal{S}hv\left(C^0\right) \xleftarrow{j_*} \mathcal{S}hv\left(C\right)$$

is an equivalence, where the topology of C^0 is the pullback of the topology of C.

For example, the full subcategory $\mathrm{Dom}_X^{00} \subseteq \mathrm{Dom}_X$ satisfies the assumptions of this lemma.

The proof uses the slice category notation introduced in 5.5.2.

Proof. We will show that the functor given by right Kan extension

$$\mathcal{S}hv\left(C^0\right) \xrightarrow{RKE_j} \mathcal{P}shv\left(C\right)$$

lands in $\mathcal{S}hv\left(C\right)$. We will then show that the resulting adjoint functors (j_*, RKE_j)

$$\mathcal{S}hv\left(C\right) \underset{RKE_j}{\overset{j_*}{\rightleftarrows}} \mathcal{S}hv\left(C^0\right)$$

are inverse equivalences. It is immediate that the co-unit transformation $j_* \circ RKE_j \to 1_{C^0}$ is an equivalence.

We assert that the unit transformation is also an equivalence. Let $\mathcal{F} \in \mathcal{S}hv\left(C\right)$, let $c \in C$, and let us prove that

$$\mathcal{F}(c) \to RKE_j \circ j_*\mathcal{F}(c)$$

is an equivalence of ∞-groupoids (we emphasize that $\mathcal{F}$ is assumed to be a sheaf, and not an arbitrary presheaf). It is a-priori true that this map is an equivalence whenever $c \in C^0$. For general $c \in C$, let $c^0 \xrightarrow{f} c$ be a cover with $c^0 \in C^0$, since C is assumed to admit all limits, $\mathcal{F}(c)$ may be calculated using the Čech complex associated to f. By assumption (2), all the terms in this complex belong to C^0 and the assertion that the unit transformation is an equivalence follows.

It remains to show that for every $\mathcal{F}_0 \in \mathcal{S}hv\left(C^0\right)$, its right Kan extension, $\mathcal{F} := RKE_j\mathcal{F}_0$, is a sheaf on C. Let $c \in C$, and let $\mathcal{S}_c \subseteq C_{/c}$ be a covering sieve. We

must show that

$$(8.1) \qquad \lim \left(\mathcal{S}_c^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Gpd}_\infty \right) \leftarrow \mathcal{F}(c)$$

is an equivalence.

The categories $C_{/c}^0$, and $\mathcal{S}_c$ are both full subcategories of $C_{/c}$, and we denote their intersection

$$\mathcal{S}_c^0 := C_{/c}^0 \cap \mathcal{S}_c$$

The following triangle is a right Kan extension

$$
\begin{array}{ccc}
\left(\mathcal{S}_c^0 \right)^{\mathrm{op}} & \xrightarrow{\ \mathcal{F}_0\ } & \mathbf{Gpd}_\infty \\
{\scriptstyle \subseteq}\Big\downarrow & \nearrow{\scriptstyle \mathcal{F}} & \\
\mathcal{S}_c^{\mathrm{op}} & &
\end{array}
$$

since for every $d \to c \in \mathcal{S}_c$ we have that $C_{/d}^0 \xrightarrow{\cong} \left(\mathcal{S}_c^0 \right)_{/d \to c}$. Thus it suffices to show that

$$\lim \left(\left(\mathcal{S}_c^0 \right)^{\mathrm{op}} \xrightarrow{\mathcal{F}_0} \mathbf{Gpd}_\infty \right) \leftarrow \lim \left(\left(C_{/c}^0 \right)^{\mathrm{op}} \xrightarrow{\mathcal{F}_0} \mathbf{Gpd}_\infty \right) \cong \mathcal{F}(c)$$

is an equivalence. In turn, the latter equivalence will follow if we show that the following triangle is a right Kan extension

$$(8.2) \qquad
\begin{array}{ccc}
\left(\mathcal{S}_c^0 \right)^{\mathrm{op}} & \xrightarrow{\ \mathcal{F}_0\ } & \mathbf{Gpd}_\infty \\
{\scriptstyle \subseteq}\Big\downarrow & \nearrow{\scriptstyle \mathcal{F}_0} & \\
\left(C_{/c}^0 \right)^{\mathrm{op}} & &
\end{array}
$$

We now use our assumptions on the relation between C and C^0. Let $c^0 \xrightarrow{f} c$ where $c^0 \in C^0$. Using hypothesis (3), conclude that $\mathcal{S}_c^0$ generates a covering sieve, over c, in C. It is always true that the maps

$$\left\{ c_i \times_c c^0 \to c^0 \ : \ (c_i \to c) \in \mathcal{S}_c^0 \right\}$$

generate a covering sieve, over c^0, in C. However, according to hypothesis (2), each of the fiber products belongs to C^0, so that the latter maps also generate a covering sieve in C^0 (over c_0), which is simply the fibered product

$$
\begin{array}{ccc}
\left(\mathcal{S}_c^0 \right)_{/(c^0 \to c)} & \longrightarrow & \mathcal{S}_c^0 \\
{\scriptstyle \subseteq}\Big\downarrow & & \Big\downarrow{\scriptstyle \subseteq} \\
C_{/c^0}^0 & \xrightarrow{\ \circ f\ } & C_{/c}^0
\end{array}
$$

Finally, since $\mathcal{F}_0$ is a sheaf on C^0 we have an equivalence

$$\lim\left(\left(\left(\mathcal{S}_c^0\right)_{/(c^0\to c)}\right)^{\mathrm{op}} \xrightarrow{\ \mathcal{F}_0\ } \mathbf{Gpd}_\infty\right) \overset{\cong}{\Leftarrow} \mathcal{F}_0\left(c^0\right)$$

implying that 8.2 is a right Kan extension. Tracing back, we conclude that $RKE_j\mathcal{F}_0 \in \mathcal{P}shv\left(C\right)$ is a sheaf. $\square$

8.3. The Ran space.

Proposition 8.3.1. *The functor of points* $\mathfrak{Aff}^{\mathrm{op}} \xrightarrow{\ \mathcal{R}\mathrm{an}_X\ } \mathbf{Gpd}_\infty$ *takes values in sets. Namely, for every* $S \in \mathfrak{Aff}$.

$$\mathcal{R}\mathrm{an}_X\left(S\right) = \{F \subseteq \mathrm{Hom}\left(S, X\right)\ :\ F\ \textit{finite, non-empty}\}$$

Proof. Consider the augmented $\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}}$ diagram

$$\left(\mathfrak{Fin}_{\mathrm{sur}} \cup \{\emptyset\}\right)^{\mathrm{op}} \longrightarrow \mathbf{Set} \subseteq \mathbf{Gpd}_\infty$$

given by

(the circular arrows represent the action of the respective symmetric groups on n elements, S_n). By definition, $\mathcal{R}\mathrm{an}_X\left(S\right)$ is the colimit in $\mathbf{Gpd}_\infty$[18] of the top row, so we must show that diagram is a colimit diagram.

It suffices to prove that for every $F \in \{F \subseteq \mathrm{Hom}\left(S, X\right)\ :\ F\ \mathrm{finite}\}$, the following homotopy fiber is contractible

Since colimits in $\mathbf{Gpd}_\infty$ are universal, this fiber is the colimit of the $\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}}$ diagram in $\mathbf{Gpd}_\infty$

---

[18]We emphasize that we want to show this diagram is a homotopy colimit. That this is a colimit diagram in sets is obvious.

where $\{n\}$ denotes a finite set with n elements and $\mathrm{Surj}\left(\{n\}, F\right)$ is the set of surjections $\{n\} \twoheadrightarrow F$. We prove that this colimit is contractible. Applying the Grothendieck un-straightening construction, we get the Cartesian fibration

$$(\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}})_{/F}$$

$$\downarrow$$

$$\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}}$$

(where F is now considered as an abstract finite set). The homotopy type we are after is the the weak homotopy type of the total space, $(\mathfrak{Fin}_{\mathrm{sur}}^{\mathrm{op}})_{/F}$, which is evidently contractible since it has a terminal element. $\square$

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