Moduli of Galois Representations

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Abstract

The theme of this thesis is the study of moduli stacks of representations of an associative algebra, with an eye toward continuous representations of profinite groups such as Galois groups. The central object of study is the geometry of the map $\bar{\psi}$ from the moduli stack of representations to the moduli scheme of pseudorepresentations. The first chapter culminates in showing that $\bar{\psi}$ is very close to an adequate moduli space of Alper [Alp10]. In particular, $\bar{\psi}$ is universally closed. The second chapter refines the results of the first chapter. In particular, certain projective subschemes of the fibers of $\bar{\psi}$ are identified, generalizing a suggestion of Kisin [Kis09a, Remark 3.2.7]. The third chapter applies the results of the first two chapters to moduli groupoids of continuous representations and pseudorepresentations of profinite algebras. In this context, the moduli formal scheme of pseudorepresentations is semi-local, with each component $\operatorname{Spf} B_{\bar{D}}$ being the moduli of deformations of a given finite field-valued pseudorepresentation $\bar{D}$. Under a finiteness condition, it is shown that $\bar{\psi}$ is not only formally finite type over $\operatorname{Spf} B_{\bar{D}}$, but arises as the completion of a finite type algebraic stack over $\operatorname{Spec} B_{\bar{D}}$. Finally, the fourth chapter extends Kisin's construction [Kis08] of loci of coefficient spaces for p-adic local Galois representations cut out by conditions from p-adic Hodge theory. The result is extended from the case that the coefficient ring is a complete Noetherian local ring to the more general case that the coefficient space is a Noetherian formal scheme.

Contents

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CHAPTER 1

Pseudorepresentations and Representations

In this chapter, we develop the notion of a pseudorepresentation following Chenevier [Che11], who calls them determinants. Essentially, a pseudorepresentation in the data of the characteristic polynomials of a representation. We make it a goal in §1.1 is to give a thorough exposition of the theory of pseudorepresentations. We emphasize that much of this content is due to Chenever; he, in turn, synthesizes and builds on work of Roby, Vaccarino, Donkin, Zubkov, Procesi, and others. We will make these attributions clear. In particular, §§1.1-1.3 are due in large part to Chenevier, and our presentation follows his, adding detail to the sources he draws on. Our original contributions in these sections consist of the application of polynomial identity ring theory, which we begin to discuss in §1.2.2.

Starting in §1.4, we define and study moduli stacks of representations. A representation induces a pseudorepresentation, so that there is a natural morphism from moduli stacks of representations to the moduli scheme of pseudorepresentation. Our task is to study the geometry of this morphism, which we call ψ. Our main result in this chapter, Theorem 1.5.4.2, is that ψ is very nearly an adequate moduli space. Adequate moduli spaces, a notion due to Alper [Alp10, Alp08], are introduced in §1.5.1. They are meant to generalize a situation commonly arising in geometric invariant theory (GIT); they are basically "isomorphisms minus representability," having important properties of both proper morphisms and affine morphisms (cf. Remark 1.5.1.5). Indeed, a more precise way of stating our result is that the moduli space of pseudorepresentations differs by at most a finite universal homeomorphism from the GIT quotient of the moduli scheme of framed representations by the natural action of conjugation.

The controlling idea is that the moduli scheme of pseudorepresentations is a concrete replacement for the GIT quotient of the moduli scheme of representations by the action

of conjugation. As pseudorepresentations have a sensible functorial definition, the moduli problem of pseudorepresentations is representable by an affine scheme. This is what we mean by "concrete." On the other hand, the GIT quotient of a moduli scheme by a reductive group of natural automorphisms has a priori a moduli-theoretic interpretation only for its functor of *geometric* points, even though it is a scheme (cf. Theorem 1.5.1.4(2), Remarks 1.5.1.6 and 1.5.2.3). We lack a functor of points because the universal property of a quotient addresses moprhisms *out* of the quotient instead of morphisms to the quotient. The moduli space of pseudorepresentations is useful because it nearly attains both universal properties.

Later, in Chapter 2, we will improve Theorem 1.5.4.2, identifying certain loci in the base over which ψ is an adequate moduli space. However, we expect that it is always an adequate moduli space; Corollary 2.3.3.9 will provide some evidence.

In this chapter, we also begin to see what the concreteness of the moduli problem of pseudorepresentations affords to us. The main thing we achieve in this chapter is the demonstration of some finiteness properties of representations that are visible when one studies moduli spaces of representations relative to the moduli space of pseudorepresentations. We accomplish this by applying the notion of a Cayley-Hamilton pseudorepresentation and using results from polynomial identity ring theory to study the category of Cayley-Hamilton representations (see §1.2). In particular, if a non-commutative algebra is finitely generated over a commutative Noetherian ring, we show in Theorem 1.4.3.1 that all of its d-dimensional representations canonically factor through a single algebra that is finite as a module over its center. This line of thought will be followed in Chapter 3, when we approach our main goal of studying moduli spaces of representations and pseudorepresentations of profinite groups and algebras in the category of formal schemes. In particular, we show that under a suitable finiteness condition on a profinite algebra, its moduli stacks of representations are algebraizable over the moduli space of pseudorepresentations (cf. Corollary 3.2.4.3).

Notation and Conventions. Throughout this chapter, we will consider representations and pseudorepresentations of non-commutative algebras over commutative rings. All rings and algebras are associative, and they are unital except in some discussion of nil-algebras

in §1.2.2. Generally, we will use A for a commutative base ring of coefficients, R as a A-algebra whose representations we study, and B for a commutative A-algebra of coefficients. When a fixed base is needed to study functors and groupoids of representations and pseudorepresentations, we will use a commutative ring A, so that B are commutative A-algebras. Sometimes, we will also assume that A is Noetherian and that R is finitely generated in order that the schemes paramterizing the representation and pseudorepresentation functors will be finite type over $\operatorname{Spec} A$. We may also use S as a base coefficient scheme. We will use Γ for a group, often finitely generated, when we want to study group representations. Except for some study of the moduli of group scheme valued representations in §1.4.4, we study the representations of Γ by studying the representations and pseudorepresentations of $R = A[\Gamma]$.

1.1. Pseudorepresentations

In this section, we give an introduction to pseudorepresentations, our goal being to provide a thorough exposition of background material on pseudorepresentations. All of this material is due to Roby [Rob63, Rob80][1] and Chenevier [Che11, §§1-2]. Chenevier emphasizes that the main theorems (Theorem 1.3.1.1 and Theorem 2.1.3.3) "should not be considered as original, as they could probably be deduced from earlier works of Procesi via the relation between determinants and generic matrices established by Vaccarino, Donkin, and Zubkov" [Che11, p. 4]. We will give these attributions as they appear. We note that we call "pseudorepresentation" what Chenevier calls a "determinant."

1.1.1. Introduction to Pseudorepresentations. Let A be a commutative ring and R be an A-algebra. We will give a preliminary (but accurate) definition of a pseudorepresentation that we will provide more theoretical context for in the sequel.

[1] A helpful summary of Roby's work appears in [BO78, Appendix A].

DEFINITION 1.1.1.1. A d-dimensional pseudorepresentation D of R over A is the structure of an A-algebra on R and an association to each commutative A-algebra B to the map

$$D_B : R \otimes_A B \longrightarrow B$$

satisfying the following conditions:

(1) D_B is multiplicative and unit-preserving (but not necessarily additive),

(2) D_B is homogenous of degree d, i.e.

$$\forall\ b \in B, \forall\ x \in R \otimes_A B, D_B(bx) = b^d D_B(x),$$

(3) D is functorial on A-algebras, i.e. for any map $B \to B'$ of commutative A-algebras, the diagram

$$\begin{array}{ccc} R \otimes_A B & \xrightarrow{D_B} & B \\ \downarrow & & \downarrow \\ R \otimes_A B' & \xrightarrow{D_{B'}} & B' \end{array}$$

commutes.

We write $D : R \to A$ for such data, although this data is much more than a map from R to A.

Notation. If R is an A-algebra and B is a commutative A-algebra, we denote by $\mathrm{PsR}_R^d(B)$ the set of d-dimensional pseudorepresentations of $R \otimes_A B$ over B. We will soon see that there is a natural structure of a functor on this map from A-algebras to sets.

The main interest in pseudorepresentations comes from their relation to representations through the characteristic polynomial of a representation. Indeed, a d-dimensional representation $\rho : R \to M_d(A)$ of R over A induces a d-dimensional pseudorepresentation D of R over A as follows. For any commutative A-algebra B, let D_B be the composition

$$D_B : R \otimes_A B \xrightarrow{\rho \otimes_A B} M_d(A) \otimes_A B \xrightarrow{\sim} M_d(B) \xrightarrow{\det} B \tag{1.1.1.2}$$

of the representation itself with the determinant map. We observe that

(1) D_B is multiplicative, because ρ and det are multiplicative,

(2) D_B is homogenous of degree d because ρ is linear (homogenous of degree 1) and det is homogenous of degree d, and

(3) one can check that the functionality condition (3) holds.

This map from representations to pseudorepresentations is a bit abstract until one considers the characteristic polynomial associated to a pseudorepresentation. Of course, this characteristic polynomial exists even if the pseudorepresentation does not come from a representation.

DEFINITION 1.1.1.3. Given a d-dimensional pseudorepresentation $D : R \longrightarrow A$ of R over A, its characteristic polynomial function is

$$\chi(\cdot, t) : R \longrightarrow A[t]$$
$$r \mapsto D_{A[t]}(t - r) = t^d - \Lambda_1(r)t^{d-1} + \cdots + (-1)^d\Lambda_d(r)$$

where Λ_i are maps $R \to A$. We will sometimes write these maps as Λ_i^D when specificity is required.

When we have a representation $\rho : R \to M_d(A)$ and the induced pseudorepresentation $D = D(\rho)$, a look at the definitions allows us to see that the characteristic polynomial of $r \in R$ under the representation ρ, that is, the polynomial

$$\det(t \cdot I_{d\times d} - \rho(r)) \in A[t],$$

is identical to the characteristic polynomial of the pseudorepresentation D. Therefore, a pseudorepresentation retains at least as much information as the characteristic polynomial of a representation. Later, we will see that it has exactly this much information, i.e. a pseudorepresentation is characterized by its characteristic polynomial coefficients Λ_i (Corollary 1.1.9.15). It is useful to think of a pseudorepresentation as the data of characteristic

polynomial coefficient functions $\{\Lambda_i\}$ on R satisfying the relations imposed by being the characteristic polynomial of a representation. However, we never have need to these relations explicit.

REMARK 1.1.1.4. The construction of (1.1.1.2) works just as well when $M_d(A)$ is replaced by R, an Azumaya A-algebra of rank d^2 over its center A, and det is replaced by the reduced norm $R \to A$. This includes the case of the Azumaya algebra $\mathrm{End}_A(V)$ for a projective rank d A-module V. These three notions of representation – a homomorphism into a d-by-d matrix algebra, a linear action on a rank d projective module, and a homomorphism into an Azumaya algebra – are defined in Definition 1.4.1.1 and explored in §1.4. Note that each notion of representation listed includes the previous notions in the list. Profinite topological analogues of these representations are given in Definition 3.2.1.1 and studied in Chapter 2.

It is well known that a semisimple representation of an algebra over an algebraically closed field is characterized up to isomorphism by its characteristic polynomial, and therefore also by its associated pseudorepresentation. This leads us to ask when we can reverse the map from representations to pseudorepresentations, and make a representation from a pseudorepresentation. In fact, we describe in Theorem 1.3.1.1 a result of Chenevier: given an algebraically closed A-field $\bar{k}$, the induced map

$$\{\text{semisimple } d\text{-dimensional representations of } R \otimes_A \bar{k}\}/\sim$$

$$\downarrow$$

$$\mathrm{PsR}_R^d(\bar{k})$$

is a bijection! This is an important fact, suggesting that "pseudorepresentations" deserve their name: over an algebraically closed field, they are realizable as the determinant of a representation.

Let us overview the content of this section. Pseudorepresentations are, in fact, particular cases of *multiplicative polynomial laws*, which is a notion due to Roby. Roby's work [Rob63] on polynomial laws is reviewed for several of the next paragraphs, and we discuss

his work on *multiplicative* polynomial laws starting in §1.1.6. Then, we follow Chenevier: pseudorepresentations are defined in §1.1.7, and a universal pseudorepresentation and the resulting moduli scheme are identified in Theorem 1.1.7.4. Then, in §§1.1.8-1.1.9, we explore properties of pseudorepresentations through their characteristic polynomials. In particular, "Amitsur's formula" shows that a pseudorepresentation is characterized by its characteristic polynomial, and a certain "Cayley-Hamilton identity" holds for pseudorepresentations. Work of Vaccarino [Vac08] critical for these identities is reviewed in §1.1.10, along with contributions of Donkin and others that he build upon. The only original part of this section is §1.1.11, where we give a direct sum operation on pseudorepresentations that decategorifies the direct sum operation on representations. This makes the moduli space of pseudorepresentations of all dimensions a "scheme in commutative monoids." We conclude this section in §1.1.12 with a discussion of an alternate notion of pseudorepresentation which we call "pseudocharacters" in order to indicate that they retain only the data of a trace function, while a pseudorepresentation consists of the entire characteristic polynomial. This notion of pseudorepresentation is due to Taylor [Tay91] following a definition of Wiles [Wil88].

1.1.2. Polynomial Laws. A d-dimensional determinant of R over A is a particular case of a polynomial law that is homogenous of degree d and multiplicative. We will define these terms and make a preliminary study of them. This content is originally due to Roby [Rob63, Rob80], and we are indebted to the exposition of Chenevier [Che11, §1]. The main result, Theorem 1.1.7.4, is that the functor of d-dimensional pseudorepresentations PsR_R^d on A-algebras is representable. In particular, there exists a universal pseudorepresentation.

First we define a polynomial law.

Definition 1.1.2.1 ([Rob63, §1.2]).] Let A be a commutative ring and let M, N be A-modules. A *polynomial law* $P : M \to N$ is the association to each commutative A-algebra B a set-theoretic map

$$P_B : M \otimes_A B \to N \times_A B$$

such that for any A-algebras $B \to B'$, the diagram

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{P_B} & N \otimes_A B \\ \downarrow & & \downarrow \\ M \otimes_A B' & \xrightarrow{P_{B'}} & N \otimes_A B' \end{array} \tag{1.1.2.2}$$

commutes. The set of polynomial laws from M to N is denoted $\mathcal{P}_A(M, N)$.

Remark 1.1.2.3. A more sophisticated way of saying this is that if $\underline{M}$ is the quasi-coherent sheaf on the big Zariski site $Sch/\operatorname{Spec} A$ associated to M, then a polynomial law $P : M \to N$ is just a morphism $(\underline{M})_{\mathrm{Set}} \to (\underline{N})_{\mathrm{Set}}$ of the underlying sheaves of sets.

Example 1.1.2.4. Let $M \longrightarrow N$ be a homomorphism of A-modules. Applying $\otimes_A B$ to this homomorphism for each A-algebra B defines a polynomial law. This is a linear polynomial law. The set of linear polynomial laws in $\mathcal{P}_A(M, N)$ is the image of $\mathrm{Hom}_A(M, N)$ under the mapping we have just described.

It is possible to apply to polynomial laws many of the notions applicable to conventional polynomial functions. For example, given A-modules M, N, P, we may compose pairs in $\mathcal{P}_A(M, N) \times \mathcal{P}_A(N, P)$. Also, $\mathcal{P}_A(M, N)$ is naturally an A-module: if $P, P' \in \mathcal{P}_A(M, N)$, then setting $(P + P')_B$ to be $P_B + P'_B$ for each A-algebra B defines a valid polynomial law. Likewise, for $a \in A$, $P \in \mathcal{P}_A(M, N)$, composing P_B with the action of a on $N \otimes_A B$ for each A-algebra B gives an A-module structure on $\mathcal{P}_A(M, N)$.

Polynomials are sums of their homogenous components of each degree. In the same way, we can assign a degree to a polynomial law and define homogenous polynomial laws.

Definition 1.1.2.5. Let $P \in \mathcal{P}_A(M, N)$ be a polynomial law between A-modules M, N and let $d \geq 0$. We call P *homogenous of degree* d if for all A-algebras B, all $b \in B$, and all $x \in M \otimes_A B$,

$$P(bx) = b^d P(x).$$

We write $\mathcal{P}^d_A(M, N)$ for the set of polynomial laws of degree d.

We observe that $\mathcal{P}_A^d(M,N)$ is a sub-A-module of $\mathcal{P}_A(M,N)$: if $P, P' \in \mathcal{P}_A^d(M,N)$, then their sum is homogenous of degree d, as is a scalar multiple. In fact, just as for conventional polynomials, a polynomial law can be decomposed into its homogenous components of each degree. Following [BO78, §A4], take $P \in \mathcal{P}_A(M,N)$ and consider for any A-algebra B the map $P_{B[t]} : M \otimes_A B[t] \to N \otimes_A B[t]$. For any $x \in M \otimes_A B$, define $P_B^d(x) \in N \otimes_A B$ according to the formula

$$P_{B[t]}(x \otimes t) = \sum_{d \geq 0} P^d(x) t^d.$$

Since this is an element of $N \otimes_A B[t]$, $P^d(x) = 0$ for sufficiently large d, i.e. this is a "polynomial" in t. One can then check that the P_B^d are functorial in B, so that they define a $P^d \in \mathcal{P}_A^d(M,N)$. Then by using the functionality of P under the map $B[t] \to B, t \mapsto 1$ for each A-algebra B, we see that $P_B(x) = \sum_{d \geq 0} P_B^d(x)$ for all A-algebras B and all $x \in M \otimes_A B$. This decomposition into homogenous components is locally finite, in the sense that for any given element of $M \otimes_A B$, the result is a polynomial. However, as B and x vary, the polynomial "degree" may grow.

Remark 1.1.2.6. As noted in [BO78, Appendix A], there is a sort of analogy between homogenous polynomial laws (of a given degree) and modular functions (of a given weight).

Example 1.1.2.7 ([Rob63, Proposition I.5]). When $P \in \mathcal{P}_A(M,N)$ is homogenous of degree 0, this is a "constant" polynomial law, and amounts to $n \in N$ such that $P_B(x) = n \otimes 1 \in N \otimes_A B$ for all $x \in M \otimes_A B$.

Example 1.1.2.8 ([Rob63, §I.4]). Any linear polynomial law is homogenous of degree 1. Using the Yoneda Lemma, one can see that the map $\mathrm{Hom}_A(M,N) \to \mathcal{P}_A(M,N)$ defines an isomorphism of A-modules $\mathrm{Hom}_A(M,N) \xrightarrow{\sim} \mathcal{P}_A^1(M,N)$.

Example 1.1.2.9 (cf. [Rob63, §II.3]). Let $P \in \mathcal{P}_A^2(M,N)$ be a homogenous degree 2 polynomial law. This gives rise to a bilinear form $B_P : M \oplus M \to N$ by setting

$$B_P(m,m') = P_A(m+m') - P_A(m) - P_A(m').$$

One can check that this is bilinear by observing that B_P is the t_1t_2-coefficient of

$$P_{A[t_1,t_2]}(m_1 \otimes t_1 + m_2 \otimes t_2);$$

while this is just the standard association of a bilinear form to a quadratic form, see Definition 1.1.2.14 to set this in the context of polynomial laws. In fact, P is characterized by P_A, and is characterized by B_P if 2 is not a zero divisor in A. Conversely, if $Q : M \to N$ is a quadratic map, i.e. $Q(am) = a^2Q(m)$ for all $a \in A, m \in M$, and if B_Q constructed as above is bilinear, then there exists a unique polynomial law $\tilde{Q} \in \mathcal{P}_A^2(M, N)$ such that $\tilde{Q}_A = Q$. This is proved in [Rob63, Proposition II.1].

The example above shows that a polynomial law of degree two $P \in P^2(M, N)$ is actually determined by P_A, i.e. P_A determines P_B for all A-algebras B, cf. [Rob63, Proposition II.1]. However, this is not necessarily the case in general, as the following example shows.

Example 1.1.2.10 ([Che11, Example 1.2(iii)]). The Frobenius automorphism of the field $A = \mathbb{F}_2$ of 2 elements can be used to find a polynomial law of degree 3 $P \in \mathcal{P}_A^3(M, N)$ not determined by P_A. Let M be a two-dimensional A-vector space and let $N = A$. Choose a basis $\{X, Y\}$ of $\mathrm{Hom}_A(M, A)$. Then the A-polynomial law $P : M \to N$ defined by $XY^2 - X^2Y$ is homogenous of degree $q + 1$. Clearly $P \neq 0$, but P_A is the zero map.

On the other hand, there are general conditions for a polynomial law to be determined by its restriction $P_A : M \to N$ to M.

Proposition 1.1.2.11 ([Rob63, Proposition I.8]). *If the ring A is an infinite cardinality domain and if for every $0 \neq x \in N$ there exists some A-linear map $\beta : N \to A$ such that $\beta(x) \neq 0$, then any $P \in \mathcal{P}_A(M, N)$ is determined by P_A.*

For example, any free A-module N satisfies the condition on N in the statement.

One can isolate a single "coefficient" of a polynomial through the following procedure.

Definition 1.1.2.12. Let $P \in \mathcal{P}_A(M, N)$ and choose integers $p \geq 1$ and let $\alpha = (\alpha_1, \ldots, \alpha_p)$ be a p-tuple of non-negative integers. Then if $A[t_1, \ldots, t_p]$ is the free polynomial

algebra over A in p variables $t_1, \ldots, t_p$, write $P^{[\alpha]} : M^{\oplus p}$ to N as the following function. For $(m_1, \ldots, m_p) \in M^p$, $P^{[\alpha]}(m_1, \ldots, m_p)$ is the coefficient of $t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_p^{\alpha_p}$ in

$$P_{A[t_1,\ldots,t_p]} \left(\sum_{i=1}^{p} m_i t_i \right).$$

Note that by applying the homogeneity condition, we may quickly see that if $P \in \mathcal{P}_A^d(M, N)$, then $P^{[\alpha]} \not\equiv 0 \implies \sum_{i=1}^p \alpha_i = d$.

Remark 1.1.2.13 (cf. [Rob63, §II.2]). If $P \in \mathcal{P}_A^d(M, N)$ and $p = d$, i.e. if α is the d-tuple $(1, \ldots, 1)$, then $P^{[\alpha]}$ is a multilinear function $M^{\otimes d} \to N$. Roby calls this the "complete polarization" of a homogenous polynomial law. For $d = 2$, we have already seen this in Example 1.1.2.9 above. We readily observe that this multilinear function is symmetric, i.e. commutes with the action of S_d. When $d!$ is invertible in A, this defines a bijection between homogenous polynomial laws of degree d and symmetric multilinear functions from $M^{\otimes d}$ to N. This means that when $\mathcal{P}_A^d(M, -)$ is representable by $\mathrm{Sym}^d M$, which we note is a *quotient* of $M^{\otimes d}$. The universal object (corresponding to $\mathrm{Sym}^d M \xrightarrow{\mathrm{id}} \mathrm{Sym}^d M$) is the d-dimensional homogenous polynomial law

$$m \mapsto m \otimes \cdots \otimes m/p! \in \mathcal{P}_A^d(M, \mathrm{Sym}^d M).$$

However, since the main goal of introducing pseudorepresentations in place of pseudocharacters[2] is to allow the characteristic of coefficient rings of representations to be arbitrary, we want a theory without this weakness. In fact, there is a universal object for $\mathcal{P}_A^d(M, -)$ for arbitrary A and M, as we will see in the sequel.

We are interested in a simple, explicit set of *functions* that characterize a polynomial law. These are the "coefficients" of the polynomial law.

Definition 1.1.2.14. Let $P \in \mathcal{P}_A(M, N)$. Then for any choice of positive integer $n \geq 1$, any choice of $m_1, \ldots, m_n \in M$, and any ordered n-tuple of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$, set

[2]In short, pseudocharacters keep track of the trace function of a representation, while pseudorepresentations keep track of the entire characteristic polynomial. See §1.1.12 for a discussion of pseudocharacters and the history of notions of pseudorepresentations.

$P^{[\alpha]} : M^n \longrightarrow N$ by

$$P_{A[t_1,\ldots,t_n]}(\sum_{i=1}^{n} m_i \otimes t_i) = \sum_{\alpha} P^{[\alpha]}(m_1, \ldots, m_n)t^{\alpha},$$

where $t^{\alpha} := \prod_{i=1}^{n} t^{\alpha_i}$.

DEFINITION 1.1.2.15. Let I_n^d denote the set of tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that their sum is d. We will use the notation I^d when the size n of the tuple is clear from the context.

Often, we will use I^d will refer to d-tuples, since at most d entries of an element of I_n^d are non-zero and the data, such as $P^{[\alpha]}$, labeled by a choice of an element of I_n^d are completely determined by the data labeled by elements of I_d^d.

The following proposition shows that the coefficients of a polynomial law share analogous properties to the coefficients of a polynomial.

PROPOSITION 1.1.2.16 (cf. [Rob63, Theorem I.1]). *With P, $\alpha \in I_n^d$, and $P^{[\alpha]}$ as in the definition above, then*

(1) For any $m_1, \ldots, m_n \in M^n$, there are only finitely many n-tuples of non-negative integers α such that $P^{[\alpha]}(m_1, \ldots, m_n) \neq 0$.

(2) If $m_1, \ldots, m_n$ generates M, then $P^{[\alpha]}(m_1, \ldots, m_n)$ characterizes P, as α varies over all n-tuples of non-negative integers.

(3) If P is homogenous of degree $d \geq 0$, then P is characterized by the degree d homogenous functions $\{P^{[\alpha]} \mid \alpha \in I_d^d\}$, and $P^{[\alpha]} \equiv 0$ if $\sum_{\alpha} \alpha_i \neq d$.

(4) If P is homogenous of degree d, then P is characterized by the degree d homogenous function $P_{A[t_1,\ldots,t_d]} : R \otimes_A A[t_1, \ldots, t_d] \to A[t_1, \ldots, t_d]$.

PROOF. Parts (1) and (2) are proved in [Rob63, Theorem I.1]. Part (4) clearly follows from (2) and (3).

Let us prove (3). By part (2), it will suffice to show that if $\alpha \in I_{d'}$ where $d \neq d'$ and $P \in \mathcal{P}_A^d(M, N)$, then $P^{[\alpha]} = 0$. Say that we have a counterexample $\alpha \in I_{d'}^n$, so that

there exists some n-tuple $(x_1, \ldots, x_n) \in M^n$ such that $P^{[\alpha]}(x_1, \ldots, x_n) \neq 0$. Then if t is an indeterminant,

$$\begin{aligned} t^d \cdot P(\sum_{i=1}^{n} m_i t_i) &= P(t \cdot \sum_{i=1}^{n} m_i t_i) \\ &= P(\sum_{i=1}^{n} m_i (t t_i) \\ &= \sum_{D \geq 0} \sum_{\alpha \in I_D^n} P^{[\alpha]}(m_1, \ldots, m_n) \cdot t^D \cdot t^\alpha \end{aligned} \tag{1.1.2.17}$$

where we recall that t^α stands for $\prod_{i=1}^n t_i^{\alpha_i}$. Setting $D = d'$ and $(m_1, \ldots, m_n) = (x_1, \ldots, x_n)$ and $\alpha = \alpha$, and expanding the expression above in terms of monomials in the variables $t, t_1, \ldots, t_n$, we have that there is a nonzero term in the final line of (1.1.2.17) where the degree of t is not d. This violates the homogeneity expressed in the first line of (1.1.2.17). $\square$

1.1.3. Representability. Now we assemble notions needed to specify the A-module representing the functors $N \mapsto \mathcal{P}_A(M, N)$ and $N \mapsto \mathcal{P}_A^d(M, N)$ for $d \geq 0$. Firstly we define the divided power A-algebra $\Gamma_A(M)$ of M along with its graded component A-modules $\Gamma_A^d(M)$, and construct a degree d homogenous polynomial law

$$L^d : M \longrightarrow \Gamma_A^d(M).$$

This will turn out to be the representing A-module and universal object for $\mathcal{P}_A^d(M, -)$.

Definition 1.1.3.1. Let M be an A-module. The commutative A-algebra $\Gamma_A(M)$ is the quotient algebra of the polynomial algebra generated by the symbols $m^{[i]}$ where $m \in M$ and $i \geq 0$, subject to the relations

(1) $m^{[0]} = 1$ for $m \in M$,
(2) $(am)^{[i]} = a^i m^{[i]}$ for $a \in A, m \in M, i \geq 0$,
(3) $m^{[i]} m^{[j]} = \frac{i!j!}{(i+j)!} m^{[i+j]}$ for $m \in M, i, j \geq 0$, and
(4) $(m + m')^{[i]} = \sum_{p+q=i} m^{[p]} m'^{[q]}$.

Assigning to $m^{[i]}$ the degree i, we see that as the relations form a homogenous ideal, $\Gamma_A(M)$ is a graded A-algebra under this grading, and we denote by $\Gamma_A^d(M)$ the dth graded piece, so

$$\Gamma_A(M) \cong \bigoplus_{d \geq 0} \Gamma_A^d(M).$$

Definition 1.1.3.2. We define the universal degree d homogenous polynomial law $L^d \in P^d(M, \Gamma_A^d(M))$ by the maps

$$L_B^d : M \otimes_A B \longrightarrow \Gamma_A(M) \otimes_A B \cong \Gamma_B(M \otimes_A B)$$
$$m \otimes b \mapsto m^{[i]} \otimes b^i \cong (bm)^{[i]}.$$

The relations above show that this map is well-defined, functorial in B, and degree d homogenous.

We think of the element $m^{[i]}$ as "$m^i/i!$," even though $i!$ may not be invertible in A.

Remark 1.1.3.3. This graded algebra is called the divided power algebra of M because the ideal $\Gamma_A^+(M) \subset \Gamma_A(M)$ of positive degree elements is a divided power ideal for $\Gamma_A(M)$ with divided power structure $\gamma = (\gamma_i)$ characterized by the property $\gamma_i(m^{[1]}) = m^{[i]}$ for all $m \in M, i \geq 0$ [BO78, Theorem A9]. This algebra has a special universal property among A-algebras with a divided power ideal [BO78, Theorem 3.9].

The universality of L^d among degree d homogenous polynomial laws out of M is summarized by this theorem.

Theorem 1.1.3.4 ([Rob63, Theorem IV.1]). *Let M, N be two A-modules and let $d \geq 0$. There is a canonical isomorphism*

$$\mathrm{Hom}_A(\Gamma_A^d(M), N) \xrightarrow{\sim} \mathcal{P}_A^d(M, N) \tag{1.1.3.5}$$

given by sending $f \in \mathrm{Hom}_A(\Gamma_A^d(M), N)$ to its composition $f \circ L_d$ with the universal degree d homogenous polynomial law $L_d : M \to \Gamma_A^d(M)$.

We can already see that the map (1.1.3.5) exists, by composing L^d with a linear map $\Gamma_A^d(M) \to N$. In order to show it is a bijection as the theorem claims, we need to introduce two notions: a natural functor that the A-algebra $\Gamma_A(M)$ represents, and the notion of the derivative of a polynomial law. In this we follow [BO78, Appendix A].

In order to prove Theorem 1.1.3.4, we introduce two tools. The first is the exp functor.

Definition 1.1.3.6. Let B be a commutative A-algebra. Let $\exp(B)$ be the following B-module, a subgroup of the abelian group of units $f \in B[\![t]\!]^\times$ such that

(1) $f(0) = 1$,

(2) $f(t_1 + t_2) = f(t_1)f(t_2)$ for free commutative variables t_1, t_2,

with B-module structure given by $(b \cdot f)(t) = f(bt)$.

As remarked in [BO78, p. 1], the following property of Γ_A is "in a way, a multiplicative version of Sym_A" as Sym_A is the left-adjoint of the forgetful functor from commutative A-algebras to A-modules.

Proposition 1.1.3.7 ([BO78, Proposition A1], [Rob63, Theorem III.1]). *For B a commutative A-algebra, there is a canonical bijection*

$$\mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma_A(M), B) \xrightarrow{\sim} \mathrm{Hom}_{A-\mathrm{mod}}(M, \exp(B)) \tag{1.1.3.8}$$

given by the relation, for $f \in \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma_A(M), B)$, $g \in \mathrm{Hom}_{A-\mathrm{mod}}(M, \exp(B))$, and $m \in M$,

$$g(m) = \sum_{n=0}^{\infty} f(m^{[n]})t^n, \tag{1.1.3.9}$$

i.e. Γ_A is left-adjoint to exp.

Proof. Write $G_A(M)$ for the free polynomial algebra over A generated by the symbols $m^{[i]}$ for $m \in M$ and $i \geq 0$ and $I_A(M)$ for the ideal of relations of conditions (1) to (4) of Definition 1.1.3.1, so that $G_A(M)/I_A(M) \xrightarrow{\sim} \Gamma_A(M)$.

Given any map of sets $g : M \to B[\![t]\!]$, we have coefficient functions $b_g = b : M \times \mathbb{N} \to B$ so that $g(m) = \sum_{i=0}^{\infty} b(m, i)t^i$. We observe that this defines a map $b : G_A(M) \to B$, and

that the associations $g \mapsto b_g \mapsto b \in \mathrm{Hom}_A(G_A(M), B)$ are bijective. All that we need to do is to show that $b(I_A(M)) = 0$ if and only if the image of g lies in $\exp(B) \subset B[\![t]\!]$ and g is a map of A-modules.

We will progress through the generators of $I_A(M)$ given by conditions (1) to (4) of Definition 1.1.3.1 in sequence.

(1) We see that b kills $(m, 0) - 1$ for all $m \in M$ if and only if the leading coefficient of $g(m)$ is 1 for all $m \in M$, i.e. $g(m)(0) = 1$, which is condition (1) of Definition 1.1.3.6 that $g(m)$ must satisfy in order that $g(m) \in \exp(B) \subset B[\![t]\!]$.

(2) We see that b kills $(am, i) - a^i(m, i)$ for all $m \in M, i \geq 0, a \in A$ if and only if $g(m)(at) = g(am)(t)$; by the A-module structure on $\exp(B)$, this means that g satisfies the $a \cdot g(m) = g(am)$ condition on morphisms of A-modules.

(3) We see that b kills

$$(m, i)(m, j) - \frac{(i + j)!}{i!j!}(m, i + j)$$

for all $m \in M, i, j \geq 0$ if and only if

$$\begin{aligned} g(m)(t_1 + t_2) &= \sum_{k=0}^{\infty} b(m, k)(t_1 + t_2)^k \\ &= \sum_{k=0}^{\infty} \sum_{i+j=k} \left(b(m, i)b(m, j)\frac{i!j!}{(i + j)!} \right) \binom{i + j}{i} t_1^i t_2^j \\ &= \left(\sum_{i=0}^{\infty} b(m, i)t_1^i \right) \left(\sum_{j=0}^{\infty} b(m, j)t_2^j \right) \\ &= g(m)(t_1) \cdot g(m)(t_2), \end{aligned}$$

which is condition (2) of Definition 1.1.3.6 for $g(m)$ to lie in $\exp(B) \subset B[\![t]\!]$.

(4) We see that b kills $(m+m',i)-\sum_{p+q=i}(m,p)(m',q)$ for all $m,m'\in M$, $i\geq 0$ if and only if

$$\begin{aligned} g(m)\cdot g(m') &= \left(\sum_{i=0}^{\infty} b(m,i)t^i\right)\left(\sum_{j=0}^{\infty} b(m',j)t^j\right) \\ &= \sum_{k=0}^{\infty}\sum_{i+j=k} b(m,i)b(m',j)t^{i+j} \\ &= \sum_{k=0}^{\infty} b(m+m',k)t^k \\ &= g(m+m'), \end{aligned}$$

as required for $g: M\to \exp(B)$ to obey the property $g(m+m')=g(m)+g(m')$ of A-module homomorphisms. □

The following corollaries will be very useful.

Corollary 1.1.3.10 ([BO78, Proposition A2], [Rob63, Theorems III.3 and III.4]). *Let A be a commutative ring.*

(1) If B is a commutative A-algebra and M is an A-module, $\Gamma_A(M)\otimes_A B \xrightarrow{\sim} \Gamma_B(M\otimes_A B)$, by sending

$$m^{[i]}\otimes 1 \mapsto (m\otimes 1)^{[i]}.$$

(2) If $M=\varinjlim_\lambda M_\lambda$ is a colimit of A-modules, then

$$\varinjlim \Gamma_A(M_\lambda) \cong \Gamma_A(\varinjlim M_\lambda).$$

(3) If M_1, M_2 are A-modules, then there is a canonical isomorphism

$$\Gamma_A(M_1\oplus M_2) \xrightarrow{\sim} \Gamma_A(M_1)\otimes_A \Gamma_A(M_2)$$

$$(m_1,m_2)^{[i]} \mapsto \sum_{p+q=i} m_1^{[p]} m_2^{[q]}$$

such that the grading on the left corresponds to the "sum" of the bi-grading on the right, i.e. there is an induced isomorphism of A-modules

$$\Gamma_A^d(M_1 \oplus M_2) \xrightarrow{\sim} \bigoplus_{d_1+d_2=d} \Gamma_A^{d_1}(M_1) \otimes_A \Gamma_A^{d_2}(M_2)$$

We supply a proof along the lines of [BO78].

Proof. To show (1) we simply note that (1.1.3.8) can be composed with the standard canonical isomorphisms

$$\mathrm{Hom}_{B-\mathrm{alg}}(\Gamma_A(M) \otimes_A B, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma_A(M), B),$$

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, \exp(B)) \xrightarrow{\sim} \mathrm{Hom}_{B-\mathrm{mod}}(M \otimes_A B, \exp(B))$$

to establish (1).

Since we can state Proposition 1.1.3.7 by saying that $\Gamma_A : A-\mathrm{mod} \longrightarrow \mathrm{Alg}_A$ is left-adjoint to $\exp : \mathrm{Alg}_A \longrightarrow A-\mathrm{mod}$ and left-adjoint functors preserve colimits, we have (2).

As $\otimes_A$ is the coproduct in the category of commutative A-algebras and $\oplus$ is the coproduct in the category of A-modules, we directly derive (3) from Proposition 1.1.3.7. The explicit form of the map may be deduced from relation (4) of Definition 1.1.3.1. □

Now we can concretely describe $\Gamma_A^d(M)$ in the case that M is a free A-module.

Corollary 1.1.3.11 ([BO78, Proposition A3], compare [Rob63, Theorem IV.2]). *If the A-module M is free with basis $\{e_i\}_{i\in I}$, then for $d \geq 0$, $\Gamma_A^d(M)$ is free with basis $\{\prod_{i\in I} e_i^{[k_i]} \mid \sum_I k_i = d\}$, where the k_i are non-negative integers.*

Proof. Corollary 1.1.3.10(2) allows us to confine ourselves to finitely generated A-modules M. Corollary 1.1.3.10(3) allows us to reduce to the case that M is free of rank 1! Finally, Corollary 1.1.3.10(1) allows us to reduce to the case that $A = \mathbb{Z}$. If we write $\{e\}$ for the basis of M, then the definition of the divided power algebra and its grading show that $e^{[d]}$ is a generator for $\Gamma_{\mathbb{Z}}^d(M)$. Therefore it will suffice to show that for all non-negative integers a, $a \cdot e^{[d]} \neq 0$.

Now clearly the Taylor expansion $\exp(t) \in \mathbb{Q}[\![t]\!]$ of e^t lies in $\exp(\mathbb{Q})$. As M is free, there exists a map $M \to \exp(\mathbb{Q})$ sending $x \mapsto \exp(t)$. Then by Proposition 1.1.3.7, there is a canonical map $\Gamma_{\mathbb{Z}}(M) \to \mathbb{Q}$ sending $e^{[d]}$ to the coefficient $1/d!$ of t^d in $\exp(t)$. Clearly $a/d! \neq 0$ when $a \neq 0$, so $a \cdot e^{[d]} \neq 0$ as well. □

Now we add a second tool toward proving Theorem 1.1.3.4 in addition to the exp functor: the derivative operators.

Definition/Lemma 1.1.3.12. Let M, N be A-modules, let $m \in M$, and $i \geq 0$. Then we define the *derivative operator* ∂_m^i on $\mathcal{P}_A(M,N)$ as the A-module endomorphism of $\mathcal{P}_A(M,N)$ given by following notions.

(1) For $P \in \mathcal{P}_A(M,N)$, a commutative A-algebra B, and an element $x \in M \otimes_A B$, then then *Taylor expansion of P at x with respect to m* is

$$S_m(P)_B(x) := P_{B[t]}(m \otimes t + x).$$

(2) We observe that the $S_m(P)_B$ are functorial in B, and thereby defines a polynomial law

$$S_m(P) \in \mathcal{P}_A(M, N \otimes_A A[t]).$$

(3) Decompose $S_m(P)$ into coefficient polynomial laws $\partial_m^i(P)$ such that for all commutative A-algebras B and all $x \in M \otimes_A B$,

$$S_m(P)_B(x) = \sum_{i=0}^{\infty} \partial_m^i(P)_B(x) t^i$$

(4) For $m \in M$ and $i \geq 0$, write ∂_m^i for the (A-linear) endomorphism of $\mathcal{P}_A(M,N)$ defined by the association $P \mapsto \partial_m^i(P)$, and write

$$S_m := \sum_{i=0}^{\infty} \partial_m^i t^i$$

for the resulting formal power series with coefficients in $\mathrm{End}_A(\mathcal{P}_A(M,N))$.

(5) The A-subalgebra $\mathcal{D} \subset \mathrm{End}_A(\mathcal{P}_A(M,N))$ generated by $\{\partial_m^i \mid m \in M, i \geq 0\}$ is *commutative*.

PROOF. For (2) we observe that for any morphism of A-algebras $B \to B'$, any $x \in M \otimes_A B$, the diagram

$$\begin{array}{ccc} x & \longrightarrow & P_{B[t]}(m \otimes t + x) \\ \downarrow & & \downarrow \\ x \otimes_B 1_{B'} & \longrightarrow & P_{B'[t]}(m \otimes t + x \otimes 1) \end{array}$$

commutes.

For (3) we observe that the composition of the polynomial law $S_m(P)$ with the linear polynomial law induced by the homomorphism of A-modules $N \otimes_A A[t] \to N, \sum_i n_i t^i \mapsto n_i$, remains a polynomial law. This composition is the coefficient polynomial law $\partial_m^i(P)$.

For (4) we must show that $\partial_m^i : \mathcal{P}_A(M,N) \to \mathcal{P}_A(M,N)$ is A-linear. This is straightforward:

$$\begin{aligned} S_m(P+P')_B(x) := (P+P')_{B[t]}(m \otimes t + x) &= P_{B[t]}(m \otimes t + x) + P'_{B[t]}(m \otimes t + x) \\ &= S_m(P)_B(x) + S_m(P')_B(x), \end{aligned}$$

and $S_m(a \cdot P) = a \cdot S_m(P)$ for $a \in A$ follows similarly.

The remaining claim is (5), that the operators

$$\{\partial_m^i \mid m \in M, i \geq 0\} \subset \mathrm{End}_A(\mathcal{P}_A(M,N))$$

commute. We deduce this by composing the operation S_m with itself, repeating the argument of [BO78]: for $B \in \mathrm{Alg}_A$, $m_1, m_2 \in M, x \in M \otimes_A B$, and indeterminates t_1, t_2,

$$P_{B[t_1,t_2]}(m_1 t_1 + m_2 t_2 + x) = P_{B[t_1,t_2]}(m_2 t_2 + m_1 t_1 + x)$$

$$S_{m_1}(P)_{B[t_2]}(m_2 t_2 + x) = S_{m_2}(P)_{B[t_1]}(m_1 t_1 + x)$$

$$S_{m_2}(S_{m_1}(P))_B(x) = S_{m_1}(S_{m_2}(P))_B(x),$$

so comparing terms of the polynomial coefficients in t_1, t_2, we find that

$$\partial^i_{m_2}\partial^j_{m_1} t_2^i t_1^j = \partial^j_{m_1}\partial^i_{m_2} t_1^j t_2^i$$

for all $i, j \in \mathbb{N}$. $\square$

The key role that these derivatives will play in showing that $\Gamma^d_A(M)$ represents the functor $\mathcal{P}^d_A(M, -)$ starts to become apparent with this

LEMMA 1.1.3.13. *The map $S := S_{(\cdot)} : M \to \mathcal{D}[\![t]\!]$ defines an A-linear map*

$$S : M \longrightarrow \exp(\mathcal{D}).$$

PROOF. First we show that the image of $S_{(\cdot)} = \sum_{i=0}^{\infty} \partial^i_{(\cdot)} t^i$ lies in $\exp(\mathcal{D}) \subset \mathcal{D}[\![t]\!]$. Since the coefficient of t^0 in $S_m(P)_B(x) = P_{B[t]}(m \otimes t + x)$ is given by $\partial^0_m(P)_B(x) = P_{B[t]}(x) = P_B(x)$ for $x \in M \otimes_A B$, we see that $\partial^0_m(P) = P$, i.e. the coefficient is $1 \in \mathcal{D}$ as desired.

The remaining condition to verify in order to see that $S_m \in \exp(\mathcal{D}) \subset \mathcal{D}[\![t]\!]$ is that

$$S_m(t_1 + t_2) = S_m(t_1) \cdot S_m(t_2) \in \mathcal{D}[\![t_1, t_2]\!].$$

We now write $S_m(t_i)$ in place of S_m to specify the variable put in the place of t in the original definition of S. We check that the required identity is satisfied by calculating that for all $m \in M, P \in \mathcal{P}_A(M, N), B \in \mathrm{Alg}_A, x \in M \otimes_A B$, we have

$$\begin{aligned} S_m(t_1 + t_2)(P)_B(x) &= P_{B[t_1,t_2]}(m \otimes t_1 + m \otimes t_2 + x) \\ &= S_m(t_1)(P)_{B[t_2]}(m \otimes t_2 + x) \\ &= \big(S_m(t_1)S_m(t_2)\big)(P)_B(x), \end{aligned}$$

so $S_m(t_1 + t_2) = S_m(t_1) \cdot S_m(t_2)$ as desired.

It remains to show that $S_{(.)}$ is a homomorphism of A-modules. This is a simple calculation: for $m_1, m_2 \in M$ and all P, B, and x as above,

$$\begin{aligned} S_{m_1+m_2}(P)_B(x) &= P_{B[t]}(m_1 \otimes t + m_2 \otimes t + x) \\ &= S_{m_1}(P)_{B[t]}(m_2 \otimes t + x) \\ &= S_{m_1}(S_{m_2}(P))_B(x). \end{aligned}$$

This is commutative also, and we have $S_{m_1+m_2} = S_{m_1} \cdot S_{m_2} = S_{m_2} \cdot S_{m_1}$. Finally, for $a \in A, m \in M$, and P, B, x as above,

$$\begin{aligned} S_{am}(t)(P)_B(x) &= P_{B[t]}(am \otimes t + x) \\ &= P_{B[t]}(m \otimes (at) + x) \\ &= S_m(at)(P)_B(x), \end{aligned}$$

so $S_{am}(t) = S_m(at) = a \cdot S_m(t)$ as desired. □

Now we prove Theorem 1.1.3.4, which we recall here. Given an A-module M and $d \geq 0$, we have constructed the universal homogenous degree d polynomial law

$$L^d : M \to \Gamma_A^d(M),$$

and want to prove that it deserves its name, i.e. for any A-module N, the natural map $\mathrm{Hom}_A(\Gamma_A^d(M), N) \overset{(1.1.3.5)}{\longrightarrow} \mathcal{P}_A^d(M, N)$ given by composing with L^d is *bijective*.

Proof. (Theorem 1.1.3.4) To show the injectivity of (1.1.3.5), choose $f \in \mathrm{Hom}_A(\Gamma_A^d(M), N)$ and let $P = f \circ L^d$. This is a homogenous degree d polynomial law $P : M \to N$. For $\alpha \in I_d^d$ recall the coefficients $P^{[\alpha]}$ of P (Definition 1.1.2.14). For $m_1, \ldots, m_d \in M$, we have, by definition of the $P^{[\alpha]}$,

$$P_{A[t_1,\ldots,t_d]}(m_1 \otimes t_1 + \cdots + m_d \otimes t_d) = \sum_{\alpha \in I_d^d} P^{[\alpha]}(m_1, \ldots, m_d) t^\alpha.$$

On the other hand, because $\Gamma^d_{A[t_1,\dots,t_d]}(M\otimes_A A[t_1,\dots,t_d]) \cong \Gamma^d_A(M)\otimes_A A[t_1,\dots,t_d]$ (Corollary 1.1.3.10(1)), we see that

$$\begin{aligned} L^d_{A[t_1,\dots,t_d]}(m_1\otimes t_1+\cdots+m_d\otimes t_d) &= (m_1\otimes t_1+\cdots+m_d\otimes t_d)^{[d]} \\ &= \sum_{\alpha\in I^d_d}\prod_{1\le i\le d} m_i^{[\alpha_i]}t_i^{\alpha_i}. \end{aligned}$$

Comparing coefficients, this shows that for each $\alpha\in I^d_d$,

$$f(m_1^{[\alpha_1]}\cdots m_d^{[\alpha_d]}) = P^{[\alpha]}(m_1,\dots,m_d),\ \forall (m_1,\dots,m_d)\in M^d.$$

As $\{\prod_1^d m_i^{[\alpha_i]} \mid \alpha\in I^d_d, (m_i)\in M^d\}$ spans $\Gamma^d_A(M)$ as a module by its construction (Definition 1.1.3.1), this shows that $P = f\circ L^d$ determines f, i.e. (1.1.3.5) is injective.

Now we show that (1.1.3.5) is surjective. Let $P\in\mathcal{P}^d_A(M,N)$. We need to produce a linear map $f:\Gamma^d_A(M)\to N$ such that

$$f:\prod_{i=1}^d m_i^{[\alpha_i]}\mapsto P^{[\alpha]}(m_1,\dots,m_d),\ \forall (m_i)\in M^d, \alpha\in I^d_d.$$

For brevity we write $\underline{m}$ for $(m_i)=(m_1,\dots,m_d)$ and $\underline{m}^{[\alpha]}$ for $\prod_{i=1}^d m_i^{[\alpha_i]}$.

We have $S: M\to \exp(\mathcal{D})$ where $\mathcal{D}\subset \mathrm{End}_A(\mathcal{P}_A(M,N))$ is a commutative subalgebra (Definition/Lemma 1.1.3.12(5)), and therefore by Proposition 1.1.3.7, an induced homomorphism of commutative A-algebras $\tilde{S}:\Gamma_A(M)\to\mathcal{D}$. One can verify from the relation (1.1.3.9) that just as $S_m=\sum_0^\infty \partial_m^i t^i$, so also $\tilde{S}(m^{[i]})=\partial_m^i$. Therefore, for any $\underline{m}^{[\alpha]}\in\Gamma_A(M)$, $\tilde{S}(\underline{m}^{[\alpha]}) = \partial^\alpha_{\underline{m}} := \prod_i \partial^{\alpha_i}_{m_i}$.

Apply these constructions to $P\in\mathcal{P}^d_A(M,N)$ by "evaluating at zero" the derivative of P by some $\partial\in\mathcal{D}$, so that for each such P we have an A-linear map

$$\begin{aligned} \mathrm{ev}_P : \mathcal{D} &\to N \\ \partial &\mapsto \partial(P)_A(0) \end{aligned}$$

Composing ev_P with $\tilde{S}$, we have an A-linear map $f : \Gamma_A(M) \to N$, mapping

$$f : \underline{m}^{[\alpha]} \mapsto \partial^\alpha_{\underline{m}}(P)_A(0).$$

We find these quantities as coefficients of t^α by expanding the definition of the Taylor series: if we restrict α to have cardinality n, then for all B and $x \in M \otimes_A B$,

$$\sum_\alpha \partial^\alpha_{\underline{m}}(P)_B(x)t^\alpha = P_{B[t_1,\dots,t_n]}(m_1 \otimes t_1 + \cdots + m_n \otimes t_n + x),$$

and specializing to $x = 0 \in M$, we have

$$\sum_\alpha \partial^\alpha_{\underline{m}}(P)_A(0)t^\alpha = P_{A[t_1,\dots,t_n]}(m_1 \otimes t_1 + \cdots + m_n \otimes t_n).$$

The coefficient of t^α is $f(\underline{m}^{[\alpha]}) \in N$ in this series, but it is also equal to $P^{[\alpha]}(m_1, \dots, m_n)$ by Definition 1.1.2.14. This is what we wanted to prove. □

We derive a useful corollary of Theorem 1.1.3.4 regarding the functorial behavior of Γ^d_A.

Corollary 1.1.3.14 ([BO78, Corollary A6]). *Let*

$$M' \rightrightarrows M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules. For $d \geq 1$ and an A-module N, this induces an exact sequence of modules of A-polynomial laws

$$0 \longrightarrow \mathcal{P}^d_A(M'', N) \longrightarrow \mathcal{P}^d_A(M, N) \rightrightarrows \mathcal{P}^d_A(M', N),$$

and an exact sequence

$$\Gamma^d_A(M') \rightrightarrows \Gamma^d_A(M) \longrightarrow \Gamma^d_A(M'') \longrightarrow 0.$$

Proof. For any commutative A-algebra B,

$$M' \otimes_A B \rightrightarrows M \otimes_A B \longrightarrow M'' \otimes_A B \longrightarrow 0$$

is also exact. This is what we need in order to see the first exact sequence. The second exact sequence then follows from Theorem 1.1.3.4. □

1.1.4. Symmetric Tensor Algebras. In the case that M is a free A-module, there is an isomorphism of A-modules between the symmetric tensors of M of degree d and the dth graded component of the divided power algebra for M. This will be helpful later in understanding the multiplication law on the algebra representing the functor of pseudorepresentations.

First we define the A-algebra of symmetric tensors.

DEFINITION 1.1.4.1 ([Rob63, §III.5]). Let $\mathrm{TS}_A^d(M)$ be the submodule of $M^{\otimes d}$ of elements invariant under the natural action of the symmetric group S_d. Let $\mathrm{TS}_A(M) := \oplus_{d\geq 0}\mathrm{TS}_A^d(M)$ with the following structure of a graded algebra. For $x \in \mathrm{TS}_A^d(M), x' \in \mathrm{TS}_A^{d'}(M)$, let $D = d + d'$ and consider $x \otimes x' \in M^{\otimes_A^D}$. Then $x \otimes x'$ is invariant under the subgroup $S_d \times S_{d'} \hookrightarrow S_D$, where the injections are defined by the ordering of the coordinates of $x \otimes x'$. Let (σ) be a set of representatives of the left cosets of $S_d \times S_{d'}$. Define multiplication $* : \mathrm{TS}_A(M) \times \mathrm{TS}_A(M) \to \mathrm{TS}_A(M)$ by extending the multiplication on monomials

$$x * x' := \sum_{\sigma} \sigma(x \otimes x'). \tag{1.1.4.2}$$

This multiplication is clearly bilinear so factors through $\mathrm{TS}_A(M) \otimes_A \mathrm{TS}(M)$, but showing that it is associative takes a bit more work. For a full presentation of the commutativity and associativity, see [Rob63, §III.5].

We observe that the map

$$M \longrightarrow M^{\otimes d}$$

$$m \mapsto m \otimes \cdots \otimes m =: m^{\otimes_A^d}$$

is compatible with $\otimes_A B$ for commutative A-algebras B and therefore defines a polynomial law that is homogenous of degree d. Therefore, by Theorem 1.1.3.4, we have a canonical

map

$$\Gamma_A^d(M) \longrightarrow \mathrm{TS}_A^d(M). \tag{1.1.4.3}$$

This map is characterized by the property that $m^{[d]} \mapsto m^{\otimes_A^d}$ for all $m \in M$ [Rob63, Proposition III.1]. Using this property, one can see that the relations of Definition 1.1.3.1 defining $\Gamma_A(M)$ as a quotient of the free commutative algebra on $m^{[i]}$ $(m \in M, i \geq 0)$ are sent to zero under the map from this free commutative algebra to $\mathrm{TS}(M)$ defined by $m^{[i]} \mapsto m^{\otimes_A^i}$, so that we have a canonical map

$$\Gamma_A(M) \longrightarrow \mathrm{TS}_A(M). \tag{1.1.4.4}$$

See [Rob63, Proposition III.1] for further detail.

This map is often an isomorphism!

Proposition 1.1.4.5 ([Rob63, Proposition IV.5]). *When M is either free or is projective of finite rank as an A-module, then* (1.1.4.4) *is an isomorphism of A-modules, induced by an isomorphism of graded A-algebras*

$$\Gamma_A(M) \xrightarrow{\sim} \mathrm{TS}_A(M).$$

Proof. When M is projective of finite rank, we reduce to the case that M is free, as all of the arguments below commute with localization.

Let M be a free A-module and choose a basis $(e_i)_{i \in I}$. Choose a total ordering on the index set I for the basis. Consider the set of simple monomials $e_{i_1} \otimes \ldots \otimes e_{i_d} \in M^{\otimes^d}$ and the equivalence classes under the action of S_d. The ordering gives us a unique representative of each class with the property that

$$i_1 \leq i_2 \leq \cdots \leq i_d.$$

Let $\mathcal{K}$ be the set of equivalence classes, and for $K \in \mathcal{K}$ let $\mathbf{e}_K$ represent the sum of the elements of the equivalence class and let $\tilde{e}_K$ represent the unique representative specified

above of the class K. This representative $\tilde{e}_K$ may be uniquely written (with a new choice of indices $i_j \in I$) in the form

$$\tilde{e}_K = e_{i_1}^{\otimes k_1} \otimes \cdots \otimes e_{i_h}^{\otimes k_h} \quad \text{where } i_1 < i_2 < \cdots < i_h, \sum_{j=1}^{h} k_j = d. \tag{1.1.4.6}$$

We note that

$$k_1! \cdots k_h! \cdot \mathbf{e}_K = \sum_{\sigma \in S_d} \sigma(\tilde{e}_K).$$

Choose some $x \in \mathrm{TS}^d(M)$, which may be uniquely written as

$$x = \sum_{i_1,\ldots,i_d \in I} \lambda_{i_1,\ldots,i_d} e_{i_1} \otimes \ldots \otimes e_{i_d}.$$

Because this is a symmetric tensor, for all $\sigma \in S_d$ we have $\lambda_{i_{\sigma(1)},\ldots,i_{\sigma(d)}} = \lambda_{i_1,\ldots,i_d}$. This means that we can write x uniquely as

$$x = \sum_{K \in \mathcal{K}} \lambda_K \mathbf{e}_K,$$

where we can set $\lambda_K = \lambda_{i_1,\ldots,i_d}$ for any $(i_1, \ldots, i_d)$ such that $e_{i_1} \otimes \ldots, \otimes e_{i_d} \in K$. This shows that $\{\mathbf{e}_K\}_{K \in \mathcal{K}}$ is a basis for $\mathrm{TS}_A^d(M)$. Now we will show that this basis is the image of a basis for $\Gamma_A^d(M)$.

We recall from Corollary 1.1.3.11 that $\Gamma_A^d(M)$ is free with the set

$$\{\prod_{j=1}^{h} e_{i_j}^{[k_j]} \mid \text{conditions of (1.1.4.6) on } i_j, k_j\} \subset \Gamma_A^d(M) \tag{1.1.4.7}$$

being a basis over A. This basis is in natural bijective correspondence with $\mathcal{K}$. We will be done if we can show that the map $\Gamma_A^d(M) \to \mathrm{TS}_A^d(M)$ preserves the correspondence between their respective bases and $\mathcal{K}$. Because $m^{[i]} \mapsto m^{\otimes i}$ for all $m \in M, i \geq 0$, the image of $e_{i_1}^{[k_1]} \cdots e_{i_h}^{[k_h]} \in \Gamma_A^d(M)$ is

$$e_{i_1}^{\otimes k_1} * \cdots * e_{i_h}^{\otimes k_h}.$$

Using (1.1.4.2), we find that this product is precisely the sum over the permutations of the orderings of factors $e_{i_j}^{\otimes k_j}$, which is the basis element $\mathbf{e}_K$ $\square$

1.1.5. Faithful Polynomial Laws. We introduce the notion of a kernel of a polynomial law $P : M \to N$, which is the kernel of the surjection from M onto the smallest quotient A-module of M through which a polynomial law $P \in \mathcal{P}_A(M, N)$ factors.

Definition 1.1.5.1 ([Che11, §1.17]). Let $P \in P(M, N)$. Then $\ker(P) \subset M$ is the subset of elements $m \in M$ such that

$$\forall B \in \mathrm{Alg}_A, \forall b \in B, \forall x \in M \otimes_A B,\ P(m \otimes b + x) = P(x),$$

which we immediately observe is a A-submodule of M. By Proposition 1.1.2.16, m is in $\ker(P)$ if and only if for all $n \geq 1$ and $m_1, \ldots, m_n \in M$,

$$P(tm + t_1 m_1 + \cdots + t_n m_n) \in N[t, t_1, \ldots, t_n]$$

lies in $N \otimes_A A[t_1, \ldots, t_n]$, i.e. it is independent of t.

When $\ker(P) = 0$, we say that P is *faithful*.

This lemma shows that the kernel deserves its name.

Lemma 1.1.5.2 ([Che11, Lemma 1.18]). *Let $P \in \mathcal{P}_A(M, N)$.*

(1) $\ker(P)$ *is the biggest A-submodule $K \subset M$ such that P admits a factorization $P = \widetilde{P} \circ \pi$ where π is the canonical A-linear surjection $M \longrightarrow M/K$.*

(2) $\widetilde{P} : M/\ker(P) \longrightarrow N$ *is a faithful polynomial law, and if P is homogenous of degree d, so is $\widetilde{P}$.*

(3) If B is a commutative A-algebra, then the image of

$$\ker(P) \otimes_A B \longrightarrow M \otimes_A B$$

is contained in $\ker(P \otimes_A B)$.

Proof. Assertion (3) follows from transitivity of the tensor product $- \otimes_A B \otimes_B C$ involved in the "restriction" of P to B-algebras C through the morphism $A \to B$.

Clearly if P factors through $M \to M/K$ for some A-submodule K of M, then K is in the kernel of P by definition. Now we check the converse: say $K \subset \ker(P)$. We need to factor P through a polynomial law $\widetilde{P} : M/K \longrightarrow N$.

For any commutative A-algebra B, set

$$K_B := \Im(K \otimes_A B \longrightarrow M \otimes_A B).$$

Then $\iota : (M/K) \otimes_A B \xrightarrow{\sim} (M \otimes_A B)/K_B$ by the right-exactness of the tensor product functor $- \otimes_A B$, and $K_B \subset \ker(P \otimes_A B)$ by part (3). Now, applying the assumption that $K \subset \ker(P)$ and the definition of the kernel, we observe that the map $P_B : M \otimes_A B \to N \otimes_A B$ satisfies $P_B(k + m) = P_B(m)$ for any $m \in M \otimes_A B, k \in K_B$, so that P_B factors through $\pi_B : M \otimes_A B \to (M \otimes_A B)/K_B$. Composing this map with the isomorphism above, we have a map which is well-defined by the relation

$$\begin{aligned} \widetilde{P}_B &: (M/K) \otimes_A B \to N \otimes_A B \\ \widetilde{P}_B(\iota^{-1} \circ \pi_B(M)) &:= P_B(m) \end{aligned} \tag{1.1.5.3}$$

because $\iota^{-1} \circ \pi_B$ is surjective onto $(M/K) \otimes_A B$. As all of the maps defining this relation are functorial in B, we have defined an polynomial law $\widetilde{P} \in \mathcal{P}_A(M/K, N)$.

Because of the relation (1.1.5.3), we see that $\ker(\widetilde{P}) = \ker(P)/K$. This is the first part of (2), and the second part of (2) also follows from examining (1.1.5.3). □

1.1.6. Multiplicative Polynomial Laws. Now we consider polynomial laws between A-algebras. These are polynomial laws between the underlying A-modules with multiplicativity imposed. It is possible to define these laws when A is neither associative nor unital.

Definition 1.1.6.1. Let R, S have the structure of A-algebras (not necessarily commutative) and $P \in \mathcal{P}_A^d(R, S)$. Then P is called *multiplicative* provided that

(1) $P(1) = 1$, i.e. $P_A(1_R) = 1_S$, from which it follows that $P_B(1_{R \otimes_A B}) = 1_{S \otimes_A B}$ for all commutative A-algebras B, and

(2) P_B is multiplicative for all commutative A-algebras B, i.e. for all B and all $x, y \in R \otimes_A B$, $P_B(xy) = P_B(x)P_B(y)$.

For $d \geq 0$, we denote by $\mathcal{M}_A^d(R,S) \subset \mathcal{P}_A^d(R,S)$ the set of degree d homogenous multiplicative polynomial laws from R to S over A.

Remark 1.1.6.2. A multiplicative polynomial law of degree 0 must be constant, and so must send every element to the multiplicative identity.

We now ask if there exists a universal object for the functor $\mathcal{M}_A^d(R,-)$ on the category of A-algebras. Of course, an element of $\mathcal{M}_A^d(R,S)$ induces an element of $\mathcal{P}_A^d(R,S)$ by the forgetful functor from A-algebras to A-modules, so that $P \in \mathcal{M}_A^d(R,S)$ induces a morphism of A-modules $\Gamma_A^d(R) \to S$. The composite

$$R \to \Gamma_A^d(R) \to S$$

is multiplicative. In fact, there exists a A-algebra structure on $\Gamma_A^d(R)$ such that the first map of the composite is multiplicative, and the multiplicativity of the composite depends on the multiplicativity of the second map. This reasoning, due to Roby [Rob80], makes the first map a universal homogenous degree d multiplicative polynomial law. This is what we will now explain.

Let $M, N \in A-\text{mod}$, and for $d \geq 0$ write L_M^d for the universal homogenous degree d polynomial law $L_M^d : M \to \Gamma_A^d(M)$. By the universal property of the tensor product of modules, $M \otimes_A N$ is universal for bilinear maps out of $M \times N$. The universal map $M \oplus N \to M \otimes_A N$ is manifestly homogenous of degree 2 and compatible with $- \otimes_A B$ for commutative A-algebras B. Therefore we have a degree 2 homogenous polynomial law from $M \oplus N$ to $M \otimes_A N$, which we will denote by $\beta_{M,N} \in \mathcal{P}_A^2(M \oplus N, M \otimes_A N)$.

The composition $L_{M\otimes_A N}^d \circ \beta_{M,N}$ defines a degree $2d$ polynomial law in $\mathcal{P}_A^{2d}(M \oplus N, M \otimes_A N)$, so that by Theorem 1.1.3.4 there exists a canonical A-linear homomorphism

$$\eta_{M,N} : \Gamma_A^{2d}(M \oplus N) \longrightarrow \Gamma_A^d(M \otimes_A N)$$

such that $L^d_{M\otimes_A N}\circ\beta_{M,N} = \eta_{M,N}\circ L^{2d}_{M\oplus N}$ is an equality of polynomial laws. Recall from Corollary 1.1.3.10(2) that there is a canonical isomorphism

$$\bigoplus_{p+q=2d}\Gamma^p_A(M)\otimes_A\Gamma^q_A(N)\xrightarrow{\sim}\Gamma^{2d}_A(M\oplus N),$$

and we will also write $\eta_{M,N}$ for its restriction to $\Gamma^p_A(M)\otimes_A\Gamma^q_A(N)$ for $p+q=2d$, considered as a submodule of $\Gamma^{2n}_A(M\oplus N)$.

Sublemma 1.1.6.3 ([Rob80, p. 869]). *With $M, N, d, \beta_{M,N}$, and $\eta_{M,N}$ as above, let*

$$\underline{m}^{[\alpha]} := m_1^{\alpha_1}\cdots m_r^{\alpha_r},\quad \textit{where } \sum_{i=1}^r\alpha_i = p$$

$$\underline{n}^{[\alpha']} := n_1^{\alpha'_1}\cdots n_s^{\alpha'_s},\quad \textit{where } \sum_{j=1}^s\alpha'_j = q$$

be representative elements of $\Gamma^p_A(M)$ and $\Gamma^q_A(N)$, respectively. Then

$$\eta_{M,N}(\underline{m}^{[\alpha]}\otimes\underline{n}^{[\alpha']}) = \sum_{\gamma_{ij}}\prod_{\substack{1\le i\le r\\ 1\le j\le s}}(m_i\otimes n_j)^{[\gamma_{ij}]},$$

where $\alpha_i = \sum_j\gamma_{ij}$ and $\alpha'_j = \sum_i\gamma_{ij}$. In particular, $\eta_{M,N}$ kills $\Gamma^p_A(M)\otimes_A\Gamma^q_A(N)\subset\Gamma^{2d}_A(M\otimes_A N)$ if $p\neq q$.

Now we replace M and N with an A-algebra R, and write η_R (resp. β_R) for $\eta_{M,N}$ (resp. $\beta_{M,N}$). Let $\theta : R\otimes_A R\to R$ be the linear multiplication structure map for R. As Γ_A is a functor, the data of θ as a map of A-modules induces a morphism of A-algebras

$$\Gamma(\theta):\Gamma_A(R\otimes_A R)\longrightarrow\Gamma_A(R),$$

which restricts to its graded components $\Gamma^n(\theta)$. Now write θ_d for the composition

$$\theta_d:\Gamma^d_A(R)\otimes\Gamma^d_A(R)\xrightarrow{\eta_R}\Gamma^d_A(R\otimes_A R)\xrightarrow{\Gamma^d(\theta)}\Gamma^d_A(R).$$

Lemma 1.1.6.4 ([Rob80, p. 870]). *The A-linear map θ_d defines the structure of an A-algebra on $\Gamma^d_A(R)$. If R is*

(1) unital,

(2) associative, or

(3) commutative,

then for all $d \geq 0$, $\Gamma^d_A(R)$ is as well.

Now we know that an associative unital A-algebra R gives rise to an associative unital A-algebra $\Gamma^d_A(R)$. We can also check that the universal polynomial law $L^d_R : R \to \Gamma^d_A(R)$ of Theorem 1.1.3.4 is multiplicative with respect to this structure. In fact, this the multiplicative polynomial law L^d_R is universal for multiplicative homogenous degree d polynomial laws out of R.

THEOREM 1.1.6.5 ([Rob80, Théorèm]). *For A-algebras R, S, there is a canonical bijection*

$$\mathcal{M}^d_A(R, S) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma^d_A(R), S)$$

with universal object $(L^d_R : R \longrightarrow \Gamma^d_A(R)) \in \mathcal{M}^d_A(R, \Gamma^d_A(R))$.

Recall that the kernel of a polynomial law $P \in \mathcal{P}_A(M, N)$ is kernel of the factor map to the smallest quotient A-module of M through which P factors. Naturally, in the multiplicative case, we would like this quotient to be a quotient ring and the kernel to be a two-sided ideal of a multiplicative polynomial law $P \in \mathcal{M}^d_A(R, S)$. The following lemma proves this and provides a simplification of the description of $\ker(P)$ for the multiplicative case relative to the module-theoretic case.

LEMMA 1.1.6.6 ([Che11, Lemma 1.19]). *Let R, S be a A-algebras and let $P \in \mathcal{M}^d_A(R, S)$. Then*

(1) The submodule $\ker(P) \subset R$ defined in Definition 1.1.5.1 satisfies

$$\ker(P) = \{r \in R \mid \forall B, \forall r' \in R \otimes_A B,\ P(1 + rr') = 1\},$$

and the same equality holds on replacing the condition $P(1 + rr') = 1$ with $P(1 + r'r) = 1$.

(2) $\ker(P) \subset R$ *is a two-sided ideal of* R*. It is proper if* $d > 0$*, and it is the biggest two-sided ideal* $K \subset R$ *such that* P *admits a factorization* $P = \widetilde{P} \circ \pi$ *where* π *is the standard surjection* $\pi : R \to R/K$ *and* $\widetilde{P} \in \mathcal{M}_A^d(R/K, S)$*.*

There is sometimes an even more concise description of the kernel, which we leave to Lemma 1.1.7.2 below.

PROOF. Write $J_1(P)$ for the right and side of the equality in (1), and write $J_2(P)$ for the same set with the condition $P(1 + r'r) = 1$ in place of $P(1 + rr') = 1$. First, we show that $\ker(P) \subseteq J_1(P)$. The same argument will show that $\ker(P) \subseteq J_2(P)$. Choose $r \in \ker(P)$, a commutative A-algebra B, and $r' = 1+h \in R \otimes_A B$. It will suffice to show that $P(1+r(1+th))$ is the unit polynomial 1 in $S \otimes_A B[t]$; we can then deduce the desired identity by specializing t to $t = 1$. Since this polynomial has degree at most d in t, it will suffice to check that this holds in $S \otimes_A B[t]/(t^{d+1})$. Notice that $1+th$ is invertible in $R \otimes_A B[t]/(t^{d+1})$. Applying multiplicativity and the definition of the kernel (Definition 1.1.5.1), we have

$$P(1 + r(1 + th)) = P((1 + th)^{-1} + r)P(1 + th) = P((1 + th)^{-1})P(1 + th) = P(1) = 1.$$

Therefore, $J_1(P) \subseteq \ker(P)$, and the analogous calculation with $P(1 + (1 + th)r)$ shows that $J_2(P) \subseteq \ker(P)$.

A similar argument shows that $\ker(P) \supseteq J_1(P)$: choose B, r, and $r' = 1 + h$ as above. Using the fact that $r \in J_1(P)$, and calculating in $S \otimes_A B[t]/(t^{d+1})$, we have

$$\begin{aligned} P(rt + (1 + h)) = P((1 + rt) + h) &= P(1 + (1 + rt)^{-1}h)P(1 + rt) \\ &= P(1 + (1 + rt)^{-1}h) = P(1 + h + r(\cdots)) = P(1 + h), \end{aligned}$$

and the lack of dependence on t shows that $r \in \ker(P)$ by definition. The same argument shows that $\ker(P) \subseteq J_2(P)$ as well.

By part (1), $\ker(P)$ is visibly a two-sided ideal of R. The rest of part (2) follows directly from the calculations in the proof of Lemma 1.1.5.2, in particular (1.1.5.3). □

1.1.7. Definition of Pseudorepresentations and Representability of the Pseudorepresentation Functor. With the background above in place, we can restate the definition of pseudorepresentations in terms of polynomial laws, and immediately make several conclusions based on the theory of multiplicative homogenous polynomial laws outlined above.

Definition 1.1.7.1 (Reprising Definition 1.1.1.1). Let A be a commutative ring, let R be an A-algebra, and let $d \geq 0$. A *d-dimensional pseudorepresentation* D of R over A is a degree d homogenous multiplicative polynomial law

$$D : R \longrightarrow A.$$

The set of d-dimensional pseudorepresentations of R over A is denoted $\mathrm{PsR}_R^d(A)$. When B is a commutative A-algebra, we use $\mathrm{PsR}_R(B)$ to denote the set of d-dimensional pseudorepresentations of $R \otimes_A B$ over B, and we observe that PsR_R^d is naturally a functor under the tensor product. Following Remark 1.1.6.2, we note that there is always a unique degree 0 pseudorepresentation sending everything to the multiplicative identity. We will formally set this to be the determinant of the unique "zero-dimensional representation."

We will freely use the notions associated to multiplicative polynomial laws to describe pseudorepresentations. One particular notion that we will use heavily of is the kernel of a pseudorepresentation $D : R \to A$, written $\ker(D)$. This is a two-sided ideal of R, which is the kernel of the surjection of A-algebras $R \twoheadrightarrow R/\ker(D)$ with the special property that this is the smallest quotient of R through which D factors (cf. Lemma 1.1.6.6). The following lemma is special to the case that a multiplicative A-polynomial law $P : R \to S$ is a pseudorepresentation (i.e. $S = A$).

Lemma 1.1.7.2. *Let A be an infinite cardinality commutative domain and let $D : R \to A$ be a d-dimensional pseudorepresentation. Then*

$$\ker(D) = \{r \in R \mid \forall r' \in R, D(1 + rr') = 1\}.$$

PROOF. Combine Proposition 1.1.2.11 and Lemma 1.1.6.6. □

REMARK 1.1.7.3. In order to call a homogenous multiplicative polynomial law a pseudorepresentation of R over A, it is occasionally important to be precise about the stipulation that the target is A and the source R is an A-algebra. It is common and reasonable to depart from this precision in the following case: if B is a commutative A-algebra, a degree d homogenous multiplicative A-polynomial law

$$R \longrightarrow B$$

is not, strictly speaking, a pseudorepresentation of R into B or over B (there is no such thing because R is not a B-algebra). However, the distinction is not vast, because this data induces a degree d homogenous multiplicative polynomial law

$$R \otimes_A B \longrightarrow B$$

which is a d-dimensional pseudorepresentation of $R \otimes_A B$ over B. By Corollary 1.1.3.10, this induction of a pseudorepresentation from a multiplicative polynomial law is a bijection. Therefore, we call the degree d homogenous multiplicative polynomial law $R \longrightarrow B$ a *d-dimensional pseudorepresentation of R valued in B*.

The results of Roby on homogenous multiplicative polynomial laws immediately imply important facts about pseudorepresentations. To state these, we recall that the abelianization R^{ab} of an algebra R is its quotient by its two-sided ideal generated by $xy - yx$ for $x, y \in R$. Obviously this quotient has the universal property expected of the abelianization.

THEOREM 1.1.7.4 ([Che11, Proposition 1.6]). *Let R be an A-algebra and $d \geq 1$. The functor $\mathrm{PsR}_R^d : \mathrm{Alg}_A \to Set$ is representable by the commutative A-algebra*

$$\Gamma_A^d(R)^{\mathrm{ab}},$$

with universal pseudorepresentation

$$D^u : R \longrightarrow \Gamma_A^d(R)^{\mathrm{ab}}$$

$$r \mapsto r^{[d]}$$

Moreover, for any commutative A-algebra B,

(1) There is an isomorphism of functors on B-algebras

$$\mathrm{PsR}_R \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, B \xrightarrow{\sim} \mathrm{PsR}_{R\otimes_A B},$$

corresponding to the canonical isomorphism of B-algebras

$$\Gamma_A^d(R)^{\mathrm{ab}} \otimes_A B \xrightarrow{\sim} \Gamma_B^d(R \otimes_A B)^{\mathrm{ab}}.$$

(2) When $D : R \to B$ is a homogenous multiplicative polynomial law (of unspecified and possibly non-existent degree), the degree is constant on connected components of $\mathrm{Spec}\, B$*. In particular, if* $\mathrm{Spec}\, B$ *is connected, then D is a pseudorepresentation of some dimension.*

(3) If R is free as an A-module, then $\mathrm{PsR}_R(-)$ *is represented by the commutative A-algebra* $\mathrm{TS}_A^d(R)^{\mathrm{ab}}$ *with universal pseudorepresentation*

$$R \longrightarrow \mathrm{TS}_A^d(R)$$

$$r \mapsto r^{\otimes_A^d}$$

(4) If R is the group algebra over A of the group (or monoid) Γ, then

$$\mathrm{PsR}_R^d \xrightarrow{\sim} \mathrm{Spec}(\mathrm{TS}_{\mathbb{Z}}^d(\mathbb{Z}[\Gamma]))^{\mathrm{ab}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, A.$$

(5) If R is finite as an A-module, for example the group algebra of a finite group, then $\Gamma_A^d(R)^{\mathrm{ab}}$ is finite as an A-module.

(6) A d-dimensional B-valued representation of R, i.e.

$$R \otimes_A B \longrightarrow \mathcal{E},$$

where $\mathcal{E}$ is a rank d^2 Azumaya B-algebra, induces a pseudorepresentation by composition with the reduced norm $\mathcal{E} \to B$.

For the definition of an Azumaya algebra, see Definition 1.4.1.5.

Before proving this theorem, which summarizes our knowledge up to this point, we note the most glaringly missing basic fact about the pseudorepresentation functor: we do not know if finite generation (or some other condition other than finiteness of R as an A-module) of R over A implies finite generation of $\Gamma_A^d(R)^{\mathrm{ab}}$. This is true (it is Theorem 1.1.10.15), but will require the study of pseudorepresentations on freely generated (non-commutative) A-algebras in §1.1.9, and the application of invariant theory.

Remark 1.1.7.5. Recall that there is a unique 0-dimensional pseudorepresentation which sends everything to the multiplicative identity. This corresponds to the fact that $\Gamma_A^0(R) \cong A$, i.e. $\mathrm{PsR}_R^0 = \operatorname{Spec} A$.

Proof. The main theorem statement follows closely from the representability statement for homogenous multiplicative polynomial laws, Theorem 1.1.6.5. Indeed, we know that for a commutative A-algebra B, the association

$$\mathcal{M}_A^d(R, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Alg}_A}(\Gamma_A^d(R), B)$$

is an isomorphism, and the right hand side is canonically isomorphic to $\mathrm{Hom}_A(\Gamma_A^d(R)^{\mathrm{ab}}, B)$ since B is commutative. As the association

$$\mathcal{M}_A^d(R, B) \xrightarrow{\sim} \mathrm{PsR}_R^d(B)$$

$$D \mapsto D \otimes_A B$$

discussed in Remark 1.1.7.3, following Corollary 1.1.3.10, is bijective, we have the theorem. In addition, part (1) follows directly from Corollary 1.1.3.10 and the main theorem.

Forgetting the algebra structure on R and B, the polynomial law D induces a map of modules $\Gamma_A(R) \to B$ by Theorem 1.1.3.4. By the proof of Theorem 1.1.6.5, the multiplicativity of D implies that $\Gamma_A(R) \to B$ is multiplicative, where this multiplication operation is

not the multiplication of the divided power algebra, but the multiplication on each graded component $\Gamma_A^d(R)$, $d \geq 0$. As B is commutative, this amounts to an A-algebra homomorphism

$$\prod_{d \geq 0} \Gamma_A^d(R)^{\mathrm{ab}} \longrightarrow B.$$

As B has no nontrivial idempotents, this map factors through one of the factors, say of degree d, so that D is a homogenous multiplicative polynomial law of degree d. This establishes (2).

Part (3) follows directly from Proposition 1.1.4.5. Part (4) follows from part (3) and the fact that a group algebra $A[\Gamma]$ is equal to $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} A$. Part (5) is quickly checkable, say with the explicit generators for $\Gamma_A^d(R)$ given in Corollary 1.1.3.11, along with Corollary 1.1.3.14.

Let M_d be the ring-scheme over $\operatorname{Spec} \mathbb{Z}$, the d-by-d matrix algebra. Each coefficient of the characteristic polynomial defines a regular function $M_d \to \mathbb{A}^1_{\mathbb{Z}}$ which is equivariant under the adjoint action of PGL_d on M_d and the trivial action on $\mathbb{A}^1$. Each Azumaya algebra $\mathcal{E}$ is a form of M_d twisted by this action (cf. [Gro68, Corollary 5.11]); therefore, the characteristic polynomial function descends from $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_U \cong M_d(\mathcal{O}_U)$ to $\mathcal{E}$ over $\mathcal{O}_X$ [Gro68, 5.13]. As PsR_R^d is an étale sheaf (it is representable by a scheme), the formation of a pseudorepresentation by taking the determinant of a representation into a matrix algebra descends to the case of a representation into an Azumaya algebra. This establishes (6). □

Remark 1.1.7.6. For any A-scheme X, we may extend the definition of a pseudorepresentation to allow for a $\mathcal{O}_X$-valued d-dimensional pseudorepresentation of R. This functor is still represented by the affine A-scheme PsR_R^d.

Example 1.1.7.7. Let $R = A[X]$. Then R is free as an A-module, so by Theorem 1.1.7.4(4), PsR_R^d is represented by the d graded piece of the symmetric tensor algebra

$$\mathrm{TS}_A^d(A[X])^{\mathrm{ab}} = A[X_1, \ldots, X_d]^{S_d} = A[\Sigma_1, \ldots, \Sigma_d],$$

where (Σ_i) are the standard symmetric polynomials on d variables. The universal pseudorepresentation $A[X] \to A[\Sigma_1, \ldots, \Sigma_d], X \mapsto \Sigma_d$ is realized as the associated pseudorepresentation of a d-dimensional representation: let $A[X]$ act on the rank d free $A[\Sigma_1, \ldots, \Sigma_d]$-module

$$A[\Sigma_1, \ldots, \Sigma_d][Y]/(Y^d - \Sigma_1 Y^{d-1} + \ldots + (-1)^d \Sigma_d)$$

by $X \mapsto Y$. The characteristic polynomial of X is the standard one, i.e. the generator of the ideal in the line above.

Example 1.1.7.8. Let $A = \mathbb{Z}$ and let $\Gamma = \mathbb{Z}$. Letting X represent a generator of Γ, we write $R = A[X, X^{-1}]$. As in the previous example, the dth graded component of the symmetric tensor algebra represents PsR^d_R. We observe that $R^{\otimes^d_{\mathbb{Z}}}$ is a standard presentation of the coordinate ring of the split rank d torus $\mathbb{G}^d_m / \operatorname{Spec} \mathbb{Z}$, and that its subring $\mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}[X, X^{-1}])$ is the subring of invariants of the action of the Weyl group. Via the Chevalley isomorphism, the geometric points of $\operatorname{Spec} \mathrm{TS}^d_{\mathbb{Z}}(R)$ are in natural bijective correspondence with the semisimple geometric points of GL_d, up to conjugation. This latter set is clearly in natural bijective correspondence with d-dimensional semisimple representations of $\mathbb{Z}$ up to isomorphism.

Example 1.1.7.9. As noted in the theorem, for a finite group Γ, $\mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])$ is a finite $\mathbb{Z}$-module. We observe that $\mathbb{Z}[\Gamma]^{\otimes^d} \cong \mathbb{Z}[\Gamma^{\times^d}]$ is generated (as a module, even) by elements of finite multiplicative order, i.e. $\gamma^n = \mathrm{id}$ for some $n \geq 1$. Therefore its subquotient $\mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])^{\mathrm{ab}}$ consists of sums with coefficients in $\mathbb{Z}$ of elements of finite multiplicative order.

Now fix a d-dimensional complex representation of Γ. By the discussion of §1.1.1, we may associate to this representation a $\mathbb{C}$-valued pseudorepresentation, and therefore a map $\mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])^{\mathrm{ab}} \to \mathbb{C}$. The property above shows that the image must be contained in $\varinjlim_n \mathbb{Z}[\mu_n] \subset \mathbb{C}$, where μ_n is a primitive nth root of unity. Since the image of the map is generated by characteristic polynomial coefficients of elements of Γ, we observe the well-known fact that characteristic polynomial coefficients of a given representation generate a cyclotomic integral extension of $\mathbb{Z}$.

Such observations also may be used to observe the Brauer character theory of positive characteristic representations of Γ.

We now investigate the pseudorepresentations of a rank n^2 Azumaya algebra R over a commutative ring A. This includes the case that $R = M_n(A)$, as Azumaya algebras are étale locally matrix algebras (see Definition 1.4.1.5). Since the representation theory of a matrix algebra over a field consists of direct sums of the identity representations, we expect this to be reflected in its pseudorepresentations. This is what we record in the following proposition, due to Ziplies [Zip86].

PROPOSITION 1.1.7.10 ([Zip86], see also [Che11, Exercise 2.5]). *Let R be an Azumaya A-algebra of rank n^2. Then the pseudorepresentations $D : R \to A$ consist precisely of powers of the reduced norm $\det_R : R \to A$. In other words, the reduced norm induces an isomorphism, for each $d \geq 0$ divisible by n,*

$$\Gamma_A^d(R)^{\mathrm{ab}} \xrightarrow{\sim} \Gamma_A^{d/n}(A) \cong A,$$

and for $n \nmid d$ there are no d-dimensional pseudorepresentations of R.

REMARK 1.1.7.11. Proposition 1.1.7.10 reflects the fact that the basic algebra corresponding to e.g. $M_d(\mathbb{C})$ is $\mathbb{C}$, and that $M_d(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent. See Definition 2.2.2.1 for the notion of a basic algebra, see Definition 2.2.2.8 for the notion of a basic algebra associated to an algebra, and Theorem 2.2.2.10 for the fact that their representation categories are equivalent.

We record for future reference some important qualities of the functor ab sending A-algebras to commutative A-algebras.

LEMMA 1.1.7.12 (cf. [Vac08, Lemma 5.14]). *If $f : R \twoheadrightarrow S$ is a surjection of A-algebras, then*

(1) The induced homomorphism of commutative A-algebras $f^{\mathrm{ab}} : R^{\mathrm{ab}} \to S^{\mathrm{ab}}$ is also surjective, and

(2) $\ker f^{\mathrm{ab}} \cong \mathrm{ab}_A(\ker f)$*, where* $\mathrm{ab}_A : A \twoheadrightarrow A^{\mathrm{ab}}$ *is the canonical homomorphism.*

PROOF. Let $I^{\mathrm{ab}}(A)$ be the kernel of ab_A. Then the lemma follows from the snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & I^{\mathrm{ab}}(A) & \longrightarrow & I^{\mathrm{ab}}(B) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{ab}_A} & & \downarrow{\scriptstyle \mathrm{ab}_A} & & \downarrow{\scriptstyle \mathrm{ab}_B} & & \\
0 & \longrightarrow & \ker f^{\mathrm{ab}} & \longrightarrow & A^{\mathrm{ab}} & \xrightarrow{f^{\mathrm{ab}}} & B^{\mathrm{ab}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

□

1.1.8. The Characteristic Polynomial. Let R be an A-algebra and A be a commutative ring as usual, and fix for this subsection a d-dimensional pseudorepresentation $D : R \to A$. As described in the introduction §1.1.1, D induces a characteristic polynomial function $\chi^D(\cdot, t) : R \to A[t]$ according to the following definition, repeated from the introduction.

DEFINITION 1.1.8.1. Given a d-dimensional pseudorepresentation $D : R \longrightarrow A$ of R over A, its characteristic polynomial function is

$$\begin{aligned}
&\chi^D(\cdot, t) : R \longrightarrow A[t] \\
&\qquad r \mapsto D_{A[t]}(t - r) = t^d - \Lambda_1(r)t^{d-1} + \cdots + (-1)^d \Lambda_d(r)
\end{aligned}$$

where Λ_i are maps $R \to A$.

In fact, by taking B-valued pseudorepresentaitons of R for commutative A-algebras B, the characteristic polynomial coefficients are homogenous polynomial laws extending the functions $\Lambda_i : R \to A$ described above.

DEFINITION/LEMMA 1.1.8.2. With D, R, A as above and a commutative A-algebra B and $i \geq 0$, let the function $\Lambda_{i,B} : R \otimes_A B \to B$ be defined by the formula, for $r \in R \otimes_A B$,

$$\chi^D(r,t) := D_{B[t]}(t - r) = \sum_{i=0}^{d} (-1)^i \Lambda_{i,B}(r) t^{d-i}.$$

is a homogenous A-polynomial law of degree i from R to A. We also have *characteristic polynomial coefficient* polynomial laws Λ_i, where $\Lambda_0 \equiv 1$ and $\Lambda_d = D$ are multiplicative, and for $i \geq d+1$ we set $\Lambda_i \equiv 0$. We call Λ_1 the trace.

We note that the data of the polynomial law Λ_1 is characterized by the A-linear map $R \to A$ (cf. Example 1.1.2.8).

PROOF. We must prove the implied claim that Λ_i is a homogenous polynomial law of degree i. First we observe that Λ_i is a polynomial law, since the formula for $\Lambda_{i,B}$ given a commutative A-algebra B can be checked to be functorial in B. For $b \in B$ and $x \in R \otimes_A B$, $\Lambda_{i,B}(bx)$ is the coefficient of t^{d-i} in $D_{B[t]}(t - bx)$. If we write $t = t_1$ and let t_2 be an indeterminant, then the functorality of polynomial laws shows that Λ_i^D is the specialization of $D_{B[t_1,t_2]}(t_1 + t_2(-x))$ via $B[t_1, t_2] \to B[t], t_1 \mapsto t, t_2 \mapsto b$. By Proposition 1.1.2.16, the only nonzero coefficient of $D_{B[t_1,t_2]}$ where t_1 appears to the $(d-i)$th power also has t_2 to the ith power. Therefore $\Lambda_{i,B}(bx) = b^i \cdot \Lambda_{i,B}(x)$ as desired. □

The characteristic polynomial coefficient polynomial laws allow for another description of the kernel of $D : R \to A$:

$$\ker(D) = \{r \in R \mid \forall B, \forall r' \in R \otimes_A B, \forall i \geq 1, \Lambda_{i,B}(rr') = 0\}$$

One can, therefore, give a description of the kernel of D as the set of elements of R such that any multiple of r in $R \otimes_A B$ for all B has the characteristic polynomial t^d. When A is an infinite domain, this criteria still works when applied only to R (see Lemma 1.1.7.2).

EXAMPLE 1.1.8.3. Let $T_d(A) \subset M_d(A)$ be the A-subalgebra of upper triangular matrices, with the pseudorepresentation $D : T_d(A) \to A$ induced from the determinant on $M_d(A)$.

Then $\ker(D)$ is the ideal of strictly upper triangular matrices, and D factors through the diagonal subalgebra $T_d(A)/\ker(D)$ of $M_d(A)$.

Implicit in the example above is the Cayley-Hamilton theorem: When R is a matrix algebra $M_d(A)$ and $D = \det$ is induced by the standard determinant, this defines a degree d polynomial law, and the characteristic polynomial χ^D is the same as the standard characteristic polynomial. It is very important that each element $r \in R$ satisfies its own characteristic polynomial! That is, $\chi(r, r) = 0$. This is the Cayley-Hamilton theorem. For a general A, R, D, this may not be the case, and the following polynomial laws measure this failure.

Definition 1.1.8.4. With A, R, D, and χ as above, let $\chi : R \longrightarrow R$ be the homogenous degree d A-polynomial law

$$\chi(r) = \chi^D(r, r) = r^d - \Lambda_1(r)r^{d-1} + \Lambda_2(r)r^{d-2} + \cdots + (-1)^d\Lambda_d(r).$$

For any $n \geq 1$ and $\alpha \in I_n^n$ (the set of n-tuples of non-negative integers $(\alpha_1, \ldots, \alpha_n)$ with sum n), we recall that $\chi^{[\alpha]}$ of Definition 1.1.2.14 are the coefficient functions of χ, defined by the relation

$$\chi_{R[t_1,\ldots,t_n]}(r_1t_1 + \cdots + r_nt_n) = \sum_{\alpha \in I_n^n} \chi^{[\alpha]}(r_1, \ldots, r_n)t^\alpha,$$

where $t^\alpha = \prod_1^n t_i^{\alpha_i}$.

We recall Proposition 1.1.2.16(3): because χ is homogenous of degree d, $\chi^{[\alpha]} \not\equiv 0$ only when $n = d$, and χ is characterized by the functions $\{\chi^{[\alpha]} \mid \alpha \in I_d^d\}$. Therefore given A, R, D, d as usual, every element $r \in R$ "satisfies its characteristic polynomial" if and only if $\chi \equiv 0$ as a polynomial law if and only if $\chi^{[\alpha]} \equiv 0$ for all $\alpha \in I_d^d$.

This equivalence results in the notion of a Cayley-Hamilton pseudorepresentation.

Definition 1.1.8.5 (cf. [Che11, p. 17]). Let R be an A-algebra and let D be a d-dimension pseudorepresentation. Let $\mathrm{CH}(D) \subset R$ be the two-sided ideal generated by $\chi^{[\alpha]}(r_1, \ldots, r_d)$ as (r_i) varies over all d-tuples in R and α varies over I_d^d. We say that D

is *Cayley-Hamilton* if $\mathrm{CH}(D) = 0$. Equivalently, $\chi \equiv 0$ as a polynomial law. We also say that (R, D) is a *Cayley-Hamilton A-algebra of degree d.*

Of course, $R/\mathrm{CH}(D)$ is a Cayley-Hamilton A-algebra.

The following observation will be very important in the sequel (see e.g. Proposition 1.2.4.3).

LEMMA 1.1.8.6. *The Cayley-Hamilton property of a pseudorepresentation $D : R \to A$ is stable under base changes $\otimes_A B$, i.e. if (R, D) is a Cayley-Hamilton A-algebra, then $(R \otimes_A B, D \otimes_A B)$ is as well. In particular, if D is an arbitrary d-dimensional pseudorepresentation, then there is a natural isomorphism*

$$R/\mathrm{CH}(D) \otimes_A B \xrightarrow{\sim} (R \otimes_A B)/\mathrm{CH}(D \otimes_A B). \tag{1.1.8.7}$$

PROOF. The Cayley-Hamilton property and the Cayley-Hamilton ideal $\mathrm{CH}(D)$ are functorial under base change because they are defined by the image of the A-polynomial law $\chi : R \to R$, and the functions χ_B as B varies over the category of commutative A-algebras is functorial (1.1.2.2). Therefore, if (R, D) is Cayley-Hamilton, then $\chi = \chi^D$ is equal to 0 as an A-polynomial law; therefore χ_B, being simply a restriction of χ from the category of A-algebras to the category of B-algebras via the map $A \to B$, is still 0. This proves the first part of the statement of the lemma.

By the definition of $\mathrm{CH}(D)$ as the ideal of R generated by the images of an A-polynomial law, we see that there exists a map

$$\mathrm{CH}(D) \otimes_A B \to \mathrm{CH}(D \otimes_A B) \subset R \otimes_A B \tag{1.1.8.8}$$

and that this map is an surjection. This proves that the A-homomorphism (1.1.8.7) exists and is injective. Using the canonical isomorphism $R/\mathrm{CH}(D) \otimes_A B \xrightarrow{\sim} (R \otimes_A B)/(\mathrm{CH}(D) \otimes_A B)$, we see that (1.1.8.7) is surjective, completing the proof. □

1.1.9. Universal Polynomial Identities. We remarked in the introduction that a pseudorepresentation of an A-algebra R over A amounts to the data of a characteristic

polynomial for each element of r. We will substantiate this comment in Corollary 1.1.9.15, showing that the characteristic polynomial coefficient functions Λ_i characterize a pseudorepresentation. Conversely, given a characteristic polynomial function $\chi(\cdot, t) : R \to A[t]$, one would have to impose a great deal of identities upon this function in order to "call it a pseudorepresentation." While we will not find a complete list of identities, in this section we will prove that the characteristic polynomial of a pseudorepresentation "satisfies all of the identities that one would expect form the characteristic polynomials of a representation" (see (1.1.9.5)), even though it may not be induced by an actual Azumaya/matrix algebra-valued representation. After that, we will deduce a few particular, useful identities from this collection (Proposition 1.1.9.11).

DEFINITION 1.1.9.1. Given a set X, the d-dimensional *generic matrices representation* involves the following data:

(1) The free $\mathbb{Z}$-algebra $\mathbb{Z}\{X\}$ on the set X;

(2) The coefficient ring $F_X(d) = \mathbb{Z}[x_{ij}]$, the free polynomial ring on generators x_{ij} for $x \in X$ and $1 \leq i, j \leq d$;

(3) The representation

$$\rho^{\text{univ}} : \mathbb{Z}\{X\} \longrightarrow M_d(F_X(d))$$

$$x \mapsto (x_{ij})_{ij}$$

(4) We also define the subring $E_X(d) \subset F_X(d)$ generated by characteristic polynomial coefficients of ρ^{univ}, i.e. by $\Lambda_{i,\mathbb{Z}}(x)$ for $x \in \mathbb{Z}\{X\}$ and for $\Lambda_{i,\mathbb{Z}} : \mathbb{Z}\{X\} \to F_X(d)$ for $1 \leq i \leq d$ the characteristic polynomial coefficient functions of the d-dimensional pseudorepresentation $\det \circ \rho^{\text{univ}} : \mathbb{Z}\{X\} \longrightarrow F_X(d)$.

REMARK 1.1.9.2. It remains to be shown that the pseudorepresentation $\det \circ \rho^{\text{univ}}$ factors through $E_X(d) \hookrightarrow F_X(d)$. This will come along with the proof that a pseudorepresentation is determined by its characteristic polynomial coefficient functions.

THEOREM 1.1.9.3 (Vaccarino [Vac08]). *With notation as in Definition 1.1.9.1, the canonical map* $\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}} \to F_X(d)$ *associated to the* d*-dimensional pseudorepresentation*

$$\det \circ \rho^{\mathrm{univ}} : \mathbb{Z}\{X\} \to F_X(d)$$

induces a canonical isomorphism

$$\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}} \xrightarrow{\sim} E_X(d). \tag{1.1.9.4}$$

We postpone to §1.1.10 the the discussion of the results of Donkin, Zubkov, and Vaccarino that are summarized in Theorem 1.1.9.3. Here we discuss the implications of this theorem for pseudorepresentations.

Let $D : R \longrightarrow A$ be a d-dimensional pseudorepresentation of an A-algebra R. Let X be a set of generators for R over $\mathbb{Z}$, e.g. $X = R$, so that there exists a surjection $\pi : \mathbb{Z}\{X\} \twoheadrightarrow R$. Theorem 1.1.9.3, along with the representability Theorem 1.1.6.5, shows that there is a unique ring homomorphism $f_X : E_X(d) \to A$ such that

$$\begin{array}{ccc} \mathbb{Z}\{X\} & \xrightarrow{\det \circ \rho^{\mathrm{univ}}} & E_X(d) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f_X} \\ R & \xrightarrow{D} & A \end{array}$$

is a commutative diagram of homogenous polynomial laws over $\mathbb{Z}$, where the horizontal maps have degree d and the vertical maps have degree 1 (they are ring homomorphisms).

Using $E_X(d) \xrightarrow{f_X} A \longrightarrow R$ where the second map is the structure map, we consider R as an $E_X(d)$-algebra, so that D is a homogenous multiplicative $E_X(d)$-polynomial law of degree d. Therefore we have a diagram of homogenous multiplicative $E_X(d)$-polynomial laws

$$\begin{array}{ccccc} \mathbb{Z}\{X\} \otimes_{\mathbb{Z}} E_X(d) & \xrightarrow{\rho^{\mathrm{univ}} \otimes 1} & M_d(F_X(d)) & \xrightarrow{\det} & E_X(d) \\ \downarrow{\scriptstyle \pi \otimes f_X} & & & & \downarrow{\scriptstyle f_X} \\ R & & \xrightarrow{\qquad\qquad D \qquad\qquad} & & A \end{array} \tag{1.1.9.5}$$

As the top row factors through a matrix algebra, we can use this diagram to show that identities in a matrix algebra, for instance, the Cayley-Hamilton identity, give rise to identities in arbitrary homogenous multiplicative polynomial laws. One of these identities, Amitsur's formula, requires some initial explanation.

DEFINITION 1.1.9.6. Let X be a totally ordered finite set (alphabet), and let X^+ be the monoid of words with letters in this set, with the induced total lexicographic ordering.

(1) A word $w \in X^+$ is called a *Lyndon word* if w is less than or equal to any of its rotations, or, equivalently, if $w = xw'$, then $w \leq w'$. The set of Lyndon words of an alphabet X is denoted $\mathcal{L}_X$.

(2) By the Chen-Fox-Lyndon theorem [CFL58, §1], any word $w \in X^+$ may be uniquely factored into a *Lyndon decomposition* $w = w_1 \cdots w_n$, where $w_1 \geq w_2 \geq \cdots \geq w_n$, $w_i \in \mathcal{L}_X$. We also present the Lyndon decomposition as

$$w = w_1^{l_1} w_2^{l_2} \cdots w_s^{l_s}, \quad \text{where } w_1 > \cdots > w_s, w_i \in \mathcal{L}_X.$$

(3) There is a unique map $\epsilon : X^+ \to \{\pm 1\}$, multiplicative on Lyndon words, given by sending w to 1 if the length of its Lyndon decomposition is even, and -1 otherwise. We can write $\epsilon(w) = (-1)^n$ or $\epsilon(w) = \prod_1^s (-1)^{l_i}$.

With the notion of Lyndon words, we can explain Amitsur's formula.

DEFINITION 1.1.9.7. We say that characteristic polynomial functions $\Lambda_{i,A} : R \to A$ satisfy *Amitsur's formula* when for any finite subset $X = \{r_1, \ldots, r_n\} \subset R$, totally ordered by the indices, we have

$$\Lambda_{i,A}(r_1 + \cdots + r_n) = \sum_{\ell(w)=i} \epsilon(w)\Lambda(w), \tag{1.1.9.8}$$

where $\ell : X^+ \to \mathbb{N}$ is the length of w in terms of the letters X, and

$$\Lambda(w) := \Lambda_{l_s}(w_s) \cdots \Lambda_{l_2}(w_2)\Lambda_{l_1}(w_1).$$

Amitsur's formula applies just as well to the polynomial laws $\Lambda_i : R \to A$ associated to a d-dimensional pseudorepresentation $D : R \to A$ by applying the condition to $\Lambda_{i,B} : R \otimes_A B \to B$ for every commutative A-algebra B. This gives us a notion of when D satisfies Amitsur's formula.

DEFINITION 1.1.9.9 (cf. [Ami80]). For A, R, d as usual, let $D : R \to B$ be a homogenous degree d polynomial law into a commutative A-algebra B. Let $X = \{r_1 \otimes t_1, \ldots, r_n \otimes t_n\} \subset R \otimes_A A[t_1, \ldots, t_n]$ with the standard lexicographic ordering, and preserve the notation of Definition 1.1.9.7 otherwise. We say that D satisfies *Amitsur's formula* if

$$D\left(1 - \sum_{j=1}^{n} r_j t_j\right) = \prod_{w \in \mathcal{L}_X} \left(\sum_{i=0}^{d} (-1)^i \Lambda_i(w)\right), \tag{1.1.9.10}$$

where the product is taken over Lyndon words with length bounded by d, ordered decreasingly. Equivalently, the homogenous of degree i component of this identity holds for all $1 \leq i \leq d$:

$$\Lambda_i(r_1 t_1 + \cdots + r_n t_n) = \sum_{\ell(w)=i} \epsilon(w)\Lambda(w),$$

where the letters in the words on the right hand side are now taken to be the n monomials "$r_i t_i$."

PROPOSITION 1.1.9.11 ([Che11, Lemma 1.12]). *For A, R, d as usual, let $D : R \to B$ be a homogenous degree d polynomial law into a commutative A-algebra B. Let $\Lambda_{i,B} : R \to B$ be the induced characteristic polynomial coefficient polynomial laws (homogenous of degree i), and in case $B = A$, let $\chi^D : R \to R$ be the degree d polynomial law given by evaluation of the characteristic polynomial. Then the following identities hold.*

(1) (commutativity of determinant) For all $r, r' \in R$,

$$D(1 + rr') = D(1 + r'r).$$

(2) (Amitsur's formula) For all $r_1, \ldots, r_n \in R$, Amitsur's relations (1.1.9.8) *on Λ_i are satisfied.*

(3) (Pseudocharacter identity) The "trace function" $\mathrm{Tr} = \Lambda_1 : R \to B$ *satisfies the* d*-dimensional pseudocharacter identity* (1.1.12.2)*.*

(4) (Cayley-Hamilton identity) If $B = A$ *(in which case* D *is a pseudorepresentation), then for all* $\alpha \in I^d$*,* $(r_1, \ldots, r_d) \in R^d$*, and* $r \in R$*,*

$$D(1 + \chi^{[\alpha]}(r_1, \ldots, r_d) \cdot r) = 1.$$

Identity (1) is basic, reflecting the fact that the characteristic polynomial coefficients are central functions. The remaining identities have a particular, prominent use. Amitsur's formula (2) often reduces the study of multiplicative polynomial laws to the study of their characteristic polynomial coefficient functions. For example, we will use it to show that the characteristic polynomial functions characterize a pseudorepresentation. The pseudocharacter identity (3) on Λ_1 will allow us to compare pseudorepresentations to pseudocharacters. And the Cayley-Hamilton identity (4) will be most prominent in the new material of this thesis, and will play a prominent role in relating pseudorepresentations to representations.

Proof. Our strategy is to use the relation (1.1.9.5) between, on the one hand, the universal d-dimensional pseudorepresentation induced by the determinant function $D^{\mathrm{univ}} = \det \circ \rho^{\mathrm{univ}}$ of the universal, generic matrices representation of the free algebra $\mathbb{Z}\{X\}$ with $X = R$, and, on the other hand, the degree d homogenous polynomial law $D : R \to B$. We will show that these identities hold for the universal pseudorepresentation D^{univ} because it is the determinant of a d-dimensional representation, and we will remark on any difficulties in deducing the same identity for D.

We know that the characteristic polynomial functions of a representation are central functions. Therefore, for $r, r' \in \mathbb{Z}\{X\} \otimes_{\mathbb{Z}} E_X(d)$, $\chi_{D^{\mathrm{univ}}}(rr', t) = \chi_{D^{\mathrm{univ}}}(r'r, t)$. Specializing to $t = -1$, we deduce that $D^{\mathrm{univ}}(1 + rr') = D^{\mathrm{univ}}(1 + r'r)$, proving (1).

Part (2) is precisely [Ami80, Theorem B]: the relation (1.1.9.10) is proved for the determinant of an arbitrary matrix algebra-valued representation.

The pseudocharacter identity is given in (1.1.12.2). The fact that the trace function on the multiplicative monoid of a matrix algebra satisfies the identity (1.1.12.2) is originally

due to Frobenius [Fro96, §3, 21]. For a modern source, see e.g. [Tay91, Theorem 1(1)]. Alternatively, taking Chenevier's approach, the identity may be deduced as a particular case of Amitsur's formula: simply let the homogenous degree i in the homogenous form (1.1.9.8) of Amitsur's formula be 1. We immediately observe that this is identical to the pseudocharacter condition (1.1.12.2).

To prove (4), we may replace R by $R \otimes_A A[t_1, \ldots, t_d]$ and recall Definition 1.1.8.4 to see that it will suffice to show that $\Lambda_i(\chi(r)r') = 0$ for all $r, r' \in R$, $1 \leq i \leq d$. Applying this to $R' := \mathbb{Z}\{X\} \otimes_{\mathbb{Z}} E_X(d)$, we see that $\rho^{\text{univ}} \circ \chi(r) = 0$ in $M_d(F_X(d))$ for all $r \in R'$, since $\chi(r)$ is the substitution of r into its own characteristic polynomial, which vanishes in $M_d(F_X(d))$ by the Cayley-Hamilton theorem. Now as Λ_i factors through ρ^{univ}, we have the result. $\square$

Remark 1.1.9.12 (cf. [Che11, Remark 1.13]). Proposition 1.1.9.11(2) (Amitsur's formula) may be proved for homogenous multiplicative polynomial laws into arbitrary associative A-algebras S in the place of commutative A-algebras B. That is, these identities in the case of determinants of representations are due to Amitsur [Ami80], but they are are particular instances of facts known to hold in more generality! In particular, an arbitrary homogenous multiplicative polynomial law is determined by its "characteristic polynomial coefficients." These identities are established in this generality by Chenevier in [Che11, Lemma 1.12] (following [RS87]). Here, we have confined our proof to the case that B is commutative. We refer to Chenevier for the general case.

Remark 1.1.9.13. In contrast to the previous remark, the Cayley-Hamilton identity is special not merely to the case that the target of a multiplicative polynomial law is commutative, but actually only makes sense in the case of pseudorepresentations (i.e. $B = A$).

Remark 1.1.9.14. In Proposition 1.1.2.16(3), we showed that certain functions $P^{[\alpha]} : R \to S$ characterize a polynomial law $P \in \mathcal{P}_A^d(R, S)$. It is quite convenient that when a polynomial law is multiplicative, it can be characterized by what is apparently less data: the d characteristic polynomial coefficient functions $\Lambda_{i,A} : R \to S$ on R alone. Amitsur's formula uses multiplicativity to express $P^{[\alpha]}$ in terms of $\Lambda_{i,A}$.

Now we use Amitsur's formula to show that a pseudorepresentation $D : R \to A$ is characterized by its characteristic polynomial functions on R, i.e. the function $\Lambda_{i,A} : R \to A$ contained in the polynomial law $\Lambda_i : R \to A$. In fact, these notions (characteristic polynomial coefficient polynomial laws Λ_i, and Amitsur's formula) make sense even when $D : R \to S$ is a homogenous multiplicative A-polynomial law into a non-commutative A-algebra S (see Remark 1.1.9.12 below), and we prove this fact in this generality.

COROLLARY 1.1.9.15 ([Che11, Corollary 1.14]). *Let A be a commutative ring, and let R and S be* possibly non-commutative *A-algebras. Let $D : R \to S \in \mathcal{M}_A^d(R, S)$ be a degree d homogenous multiplicative polynomial law. Then characteristic polynomial* functions *$\Lambda_i = \Lambda_{i,A} : R \to S$ of D characterize D. In particular, characteristic polynomial coefficient functions characterize D when D is a pseudorepresentation (i.e. $A = S$).*

PROOF. We know from Proposition 1.1.2.16(4) that the multiplicative polynomial law D is characterized by the *function*

$$D_{A[t_1,\dots,t_d]} : R \otimes_A A[t_1, \dots, t_d] \to S \otimes_A A[t_1, \dots, t_d].$$

We know from the discussion in Remark 1.1.9.12 regarding Chenevier's proof of Amitsur's formula that D satisfies Amitsur's formula. Now Amitsur's formula (1.1.9.10) allows us to express

$$D_{A[t_1,\dots,t_d]}(r_1 t_1 + \cdots + r_d), \quad (r_1, \dots, r_d) \in R^d$$

as a sum of monomials in $\Lambda_{i,A}(w)$ and t_i with prescribed coefficients 1 and -1, where w is a word in the letters $r_1, \dots, r_d$. Therefore the characteristic polynomial functions $\Lambda_{i,A} : R \to S$ characterize D, as desired. □

Because of its importance, Corollary 1.1.9.15 has been stated succinctly and solitarily above. However, there are other consequences of its proof (e.g. consequences of Amitsur's formula) which are significant. We list them here.

COROLLARY 1.1.9.16. *Let $A, R, S, D,$ and d be as in the previous corollary. Let $C \subset S$ be the sub-A-algebra of S generated by the coefficients $\Lambda_i(r)$ of $\chi(w, t)$ for all $r \in R$, $1 \leq i \leq d$.*

(1) Then D factors through a (unique) C-valued degree d multiplicative polynomial law $D_\Lambda : R \to C \subset S$.

(2) The S-valued d-dimensional pseudorepresentation $D \otimes_A B : R \otimes_A B \to B$ induced by D is induced by the C-valued d-dimensional pseudorepresentation $D_\Lambda : R \otimes_A C \to C$ induced by D_Λ.

(3) If R is generated over A by some monoid Γ, i.e. $R = A\{\Gamma\}$, and $\Lambda_{i,A}(\gamma)$ lie in a sub-A-algebra $C \subset B$ for all $\gamma \in \Gamma, 1 \leq i \leq d$, then the conclusion of part (1) holds.

PROOF. Part (1) follows from the comment in the proof above that the only factors in the coefficients other than $\Lambda_i(w)$ and t_i are 1 and -1. Part (2) follows directly from the proof above, along with the equivalence between multiplicative polynomial laws from R to B and pseudorepresentations from $R \otimes_A B$ to B that follows from Corollary 1.1.3.10. Part (3) is a special case of part (1). □

1.1.10. Work of Vaccarino, Donkin, Zubkov, and Procesi. In this paragraph we describe work leading up Vaccarino's proof of Theorem 1.1.9.3. We also deduce that if R is a finitely generated A-algebra, then PsR_R^d is finite type as an affine A-scheme.

The fundamental idea behind the proof of Theorem 1.1.9.3 is the generalization of the ring of symmetric functions Λ, where Λ is to the singleton set as generalizations of Λ are to other sets. This idea goes at least back to [Don93].

First we review the theory of Λ, corresponding to a singleton set X. Then $\mathrm{TS}_{\mathbb{Z}}^d(\mathbb{Z}\{X\}) \cong \mathbb{Z}[\Sigma_1, \ldots, \Sigma_d] =: \Lambda_d$ is the ring of symmetric polynomials on d coefficients (cf. Example 1.1.7.7). The ring of symmetric functions is $\Lambda := \varprojlim_d \Lambda_d$, where the maps are given by

$$l_d : \Lambda_d \to \Lambda_{d-1}, (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{d-1}).$$

One key fact about this limit presentation is its behavior under the filtration by homogenous polynomial degree, which we will denote by n here. Let Λ_d^n denote $\mathrm{gr}^n \Lambda_d$ for $n \geq 0$. Then for $d \geq n$, the composition of the l_i for $n+1 \leq i \leq d$ induces an isomorphism $\Lambda_d^n \xrightarrow{\sim} \Lambda_n^n$. For example, the "trace" $\Sigma_1 = x_1 + \cdots + x_d \in \Lambda_d^1$ is a generator for Λ_d^1 for all $d \geq 1$.

Now we generalize this construction of Λ, starting with a finite set X. Write X^+ for the associated monoid of words with letters in X. As Vaccarino proves these results for an arbitrary commutative base ring A, we will replace $\mathbb{Z}$ above with A. We can assign each element of X degree 1, which induces a degree, which we will index by $n \geq 0$, on $\mathrm{TS}_A^d(A\{X\})$ for any d. Write the nth graded piece as $\mathrm{TS}_A^d(A\{X\})_n$. With the analogous maps of graded (non-commutative) A-algebras $l_d : \mathrm{TS}_A^d(A\{X\}) \to \mathrm{TS}_A^{d-1}(A\{X\})$, the inverse limit

$$\mathrm{TS}_A(A\{X\}) := \varprojlim_d \mathrm{TS}_A^d(A\{X\})$$

stabilizes on each graded piece, so that (cf. [Vac08, Corollary 5.5])

$$\mathrm{TS}_A(A\{X\})_n := \varprojlim_d \mathrm{TS}_A^d(A\{X\})_n \cong \mathrm{TS}_A^n(A\{X\})_n.$$

Therefore $\mathrm{TS}_A(A\{X\})$ is a graded A-algebra with each homogenous summand being finitely generated as an A-module. Moreover, all of these objects are free A-modules with an explicit basis that we do not require here [Vac08, Propostion 3.12]. It will be useful to have a set of generators of $\mathrm{TS}_A^d(A\{X\})$ as a A-algebra, however. Recall the notation of the proof of Proposition 1.1.4.5, in particular the basis element $\mathbf{e}_K$ for $\mathrm{TS}_A^d(A\{X\})$, where $K = K(w,i)$ is the equivalence class of tensors including

$$\tilde{e}_{K(w,i)} := w^{\otimes^i} \otimes 1^{\otimes^{d-i}} \in \mathrm{TS}_A^d(A\{X\})$$

as its special representative for some $w \in X^+$, $i \leq d$. For future reference, it will be helpful to record that if $\mathbf{e}_{K(w,i)}^{(d)} \in \mathrm{TS}_A^d(A\{X\})$, where we make the degree d of the basis element explicit, then $l_d : \mathrm{TS}_A^d(A\{X\}) \to \mathrm{TS}_A^{d-1}(A\{X\})$ is given by the formula

$$l_d : \mathbf{e}_{K(w,i)}^{(d)} \mapsto \begin{cases} \mathbf{e}_{K(w,i)}^{(d-1)} & \text{if } i < d, \\ 0 \text{ if } i = d \end{cases}, \tag{1.1.10.1}$$

which is directly analogous to the maps in the theory of ring of symmetric functions. As an A-algebra, $\mathrm{TS}_A^d(A\{X\})$ is generated by $\mathbf{e}_{K(w,i)}$ as i varies over positive integers less than d and w varies over elements of X^+ that are "primitive," i.e. not proper powers of another word

[Vac08, Theorem 4.10]. By the stabilization of the grading discussed above, these $\mathbf{e}_{K(w,i)}$ have a canonical *preimage* in $\mathrm{TS}_A(A\{X\})$, and these preimages generate the A-algebra as f primitive and i ranges over all positive integers. We summarize our knowledge in this proposition

PROPOSITION 1.1.10.2. *The graded A-algebra $\mathrm{TS}_A(A\{X\})$ is free as an A-module and generated as an A-algebra by $\mathbf{e}_{K(w,i)}$ as w ranges over primitive words in X^+ and i ranges over positive integers.*

All of these statements hold true after replacing each of these A-algebras with their abelianizations,[3] and although this is non-trivial, in fact even more is true: $\mathrm{TS}_A(A\{X\})^{\mathrm{ab}}$ is a polynomial ring over A! We record this result in Theorem 1.1.10.8 below, but first we explain the proof, as we will accomplish our main task of proving Theorem 1.1.9.3 along the way.

Recall the generic matrices representations $\rho_d^{\mathrm{univ}} : A\{X\} \to M_d(F_X(d)_A)$ for each $d \geq 1$, where $F_X(d)_A$ denotes $F_X(d) \otimes_{\mathbb{Z}} A$. The determinant of ρ_d^{univ} is a d-dimensional pseudorepresentation of $A\{X\}$, inducing a canonical ring homomorphism

$$\delta_d : \mathrm{TS}_A^d(A\{X\}) \longrightarrow F_X(d)_A, \tag{1.1.10.3}$$

using $\mathrm{TS}_A^d(A\{X\})$ in place of $\Gamma_A^d(A\{X\})$ in light of Proposition 1.1.4.5. He then observes in [Vac08, Proposition 5.19] that

$$\delta_d(\mathbf{e}_{K(w,i)}) = \Lambda_i(\rho^{\mathrm{univ}}(w)) \tag{1.1.10.4}$$

for all primitive w and $1 \leq i \leq d$, where $\Lambda_i : M_d(F_X(d)_A) \to E_X(d)_A$ is the ith coefficient of the standard characteristic polynomial on the matrix algebra and, recall, $E_X(d)_A$ is the sub-A-algebra of $F_X(d)_A$ generated by coefficients of characteristic polynomials of the image of ρ_d^{univ}. As the $\mathbf{e}_{K(w,i)}$ generate $\mathrm{TS}_A^d(A\{X\})$, this shows that the characteristic polynomial

[3]Indeed, it is the freeness of $\Gamma_{\mathbb{Z}}^d(\mathbb{Z}\{X\})^{\mathrm{ab}}$ as a $\mathbb{Z}$-module that is the fundamental input from the work of Vaccarino et. al. that Chenevier needs to establish the Cayley-Hamilton identity. But this comes part-and-parcel with the rest of these results, cf. [Che11, Remark 1.16].

coefficient functions generate the image of $\mathrm{TS}_A^d(A\{X\})$, i.e. the image of δ_d is precisely $E_X(d)_A$. Since the image of δ_d is a commutative algebra, we have a surjective induced map

$$\delta_d^{\mathrm{ab}} : \mathrm{TS}_A^d(A\{X\})^{\mathrm{ab}} \twoheadrightarrow E_X(d)_A. \tag{1.1.10.5}$$

Remark 1.1.10.6. The line of argument that we have just concluded is sufficient to prove Corollary 1.1.9.15.

Following [Vac08, §5.1.3], we extend this representation and the maps δ_d to the limit as $d \to \infty$. First we filter $F_X(d)_A$ and $E_X(d)_A$ by degree denoted n, where the generators x_{ij} for $x \in X, 1 \leq i, j \leq d$ are given degree 1. With the notation of Definition 1.1.9.1, let $\omega_d : F_X(d)_A \twoheadrightarrow F_X(d-1)_A$ via

$$x_{ij} \mapsto \begin{cases} x_{ij} & \text{if } i, j < d \\ 0 & \text{if } i = d \text{ or } j = d. \end{cases}$$

This induces a map $(\omega_d)_d : M_d(F_X(d)_A) \to M_d(F_X(d-1)_A)$ such that

$$(\omega_d)_d \circ \rho_d^{\mathrm{univ}} = \begin{pmatrix} \rho_{d-1}^{\mathrm{univ}} & \mathbf{0}_{d-1\times 1} \\ \mathbf{0}_{1\times d-1} & 0 \end{pmatrix}.$$

We observe that $\Lambda_i^{(d)} \circ (\omega_d)_d \circ \rho_d^{\mathrm{univ}} = \Lambda_i^{(d-1)} \circ \rho_{d-1}^{\mathrm{univ}}$ for $d \geq 1$, where the superscript on the characteristic polynomial coefficient function indicates the dimension of the matrix algebra on which it is defined. As a result, since the image of $\Lambda_i^{(d)} \circ \rho_d^{\mathrm{univ}}$ generates $E_X(d)_A$, we have a well-defined induced map $\omega_d : E_X(d)_A \to E_X(d-1_A)$ on the A-subalgebra $E_X(d)_A \subset F_X(d)_A$. Therefore the maps ω_d induce limits of graded A-algebras

$$F_{X,A} := \varprojlim_d F_X(d)_A \quad \supset \quad E_{X,A} := \varprojlim_d E_X(d)_A$$

with the same stabilization properties for the filtration by degree as discussed above for the limit defining $\mathrm{TS}_A(A\{X\})$. In particular, for any $w \in X^+$, there is a well defined characteristic polynomial coefficient $\Lambda_i(\rho^{\mathrm{univ}}(w)) \in E_{X,A}$, where $\Lambda_i(\rho^{\mathrm{univ}}(w))$ has bounded degree $i \cdot \ell(w)$ where $\ell(w)$ is the length of w. Strictly speaking, $\Lambda_i \circ \rho^{\mathrm{univ}} := \varprojlim_d \Lambda_i^{(d)} \circ \rho_d^{\mathrm{univ}}$.

Define $\delta : \mathrm{TS}_A(A\{X\}) \twoheadrightarrow E_{X,A}$ by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{TS}_A^d(A\{X\}) & \xrightarrow{l_d} & \mathrm{TS}_A^{d-1}(A\{X\}) & \xrightarrow{l_{d-1}} & \cdots \\ & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^{d-1}} & & \\ \cdots & \longrightarrow & E_X(d)_A & \xrightarrow{\omega_d} & E_X(d-1)_A & \xrightarrow{\omega_{d-1}} & \cdots \end{array}$$

where the fact that δ is a surjection must be deduced from the fact that each δ_d is a surjection by definition, along with a study of the gradings ([Vac08, Lemma 5.22]). The generating set $\mathbf{e}_{K(w,i)}$ for $\mathrm{TS}_A(A\{X\})$ of Proposition 1.1.10.2 and the calculation of (1.1.10.4) shows that the characteristic polynomial coefficients $\Lambda_i(\rho^{\mathrm{univ}}(w))$ generate $E_{X,A}$, where i varies over positive integers and $w \in X^+$ vary over primitive words. Of course, δ factors through

$$\mathrm{TS}_A(A\{X\}) \longrightarrow \mathrm{TS}_A(A\{X\})^{\mathrm{ab}},$$

and our goal is to show that

$$\delta^{\mathrm{ab}} : \mathrm{TS}_A(A\{X\})^{\mathrm{ab}} \longrightarrow E_{X,A}$$

is an isomorphism. This will follow from this result of Donkin:

THEOREM 1.1.10.7 ([Don93, §3(10)]). *The ring $E_{X,A}$ is a polynomial ring over A with free generators $\Lambda_i(\rho^{\mathrm{univ}}(w))$, where w varies over a set Ψ representatives of equivalence classes of primitive words, where the equivalence relation is cyclic permutation.*

Now we can prove that δ^{ab} is an isomorphism.

THEOREM 1.1.10.8 ([Vac08, Theorem 5.23]). *The map of graded A-algebras*

$$\delta^{\mathrm{ab}} : \mathrm{TS}_A(A\{X\})^{\mathrm{ab}} \longrightarrow E_{X,A}$$

is an isomorphism, and, consequently, the commutative A-algebra $\mathrm{TS}_A(A\{X\})^{\mathrm{ab}}$ is a polynomial ring over A with generators $\mathbf{e}_{K(f,i)}$ where $i \geq 1$ and f varies over Ψ.

PROOF. We know that δ^{ab} is a surjection. As $E_{X,A}$ is a free polynomial A-algebra, there exists a section $s : E_{X,A} \to \mathrm{TS}_A(A\{X\})^{\mathrm{ab}}$ sending $\Lambda_i(\rho^{\mathrm{univ}}(w))$ to the image of $\mathbf{e}_{K(w,i)}$ in the abelianization, where $i \geq 1$ and w varies over the representatives of the equivalence classes mentioned in Theorem 1.1.10.7. By [Vac08, Corollary 5.12], $\mathbf{e}_{Kw,i}$, $w \in \Psi$ are sufficient to generate $\mathrm{TS}_A(A\{X\})^{\mathrm{ab}}$. Therefore s is surjective, and δ^{ab} is an isomorphism. □

Now, our goal is to deduce from Theorem 1.1.10.8 that δ^d is an isomorphism as well. Here, Vaccarino's remaining work is to apply work of Procesi, Razmyslov, and Zubkov, whose background we now explain.

The issue we must confront is the determination ideal of relations that the free generators $\Lambda_i(w) := \Lambda_i(\rho^{\mathrm{univ}}(w))$ of $E_{X,A}$ satisfy when they are projected to $\Lambda_i^{(d)}(w) \in E_X(d)_A$. Clearly if $i > d$, then $\Lambda_i(w) \equiv 0 \in E_X(d)_A$, and the $\Lambda_i^{(d)}(w)$ generate $E_X(d)_A$. But are there further relations? And are there more relations among $\Lambda_i^{(d)}(w) \in E_X(d)_A$ than among $\mathbf{e}_{K(w,i)} \in \mathrm{TS}_A^d(A\{X\})$?

When A is an algebraically closed field of characteristic zero, this question was answered by Procesi [Pro76, Theorem 4.6(a)] and Razmyslov [Raz74]: the kernel of $E_{X,A} \twoheadrightarrow E_X(d)_A$ is generated as an ideal by $\Lambda_{d+1}(w)$ as w varies over representatives of equivalence classes of primitive words. For an arbitrary infinite field A, it was shown by Zubkov [Zub96, Main Theorem] that the kernel of $E_{X,A} \twoheadrightarrow E_X(d)_A$ is the ideal generated by

$$\{\Lambda_i(w) \mid i > d, w \text{ primitive}\}. \tag{1.1.10.9}$$

The answers to the analogous questions for $\mathrm{TS}_A(A\{X\}) \twoheadrightarrow \mathrm{TS}_A^d(A\{X\})$ are easier, and A may be an arbitrary commutative ring: we know that $\mathrm{TS}_A(A\{X\})$ is a graded polynomial algebra in the variables $\{\mathbf{e}_{K(w,i)} \mid w \in \Psi, i \geq 1\}$. By examining the explicit presentation of the maps l_d composing the limit defining $A\{X\}$ in (1.1.10.1), we see that

$$0 \longrightarrow (\mathbf{e}_{K(w,i)} : w \text{ primitive}, i > d) \longrightarrow \mathrm{TS}_A(A\{X\}) \longrightarrow \mathrm{TS}_A^d(A\{X\}) \longrightarrow 0$$

is exact. Applying Lemma 1.1.7.12, the sequence

$$0 \to \mathrm{ab}_{\mathrm{TS}}(\mathbf{e}_{K(w,i)} : w \text{ primitive}, i > d) \to \mathrm{TS}_A(A\{X\})^{\mathrm{ab}} \to \mathrm{TS}_A^d(A\{X\})^{\mathrm{ab}} \to 0 \tag{1.1.10.10}$$

is still exact.

Therefore, when A is an infinite field, using Zubkov's result in (1.1.10.9) along with (1.1.10.10) and the isomorphism of Theorem 1.1.10.8, we have that

$$\begin{aligned} \mathrm{TS}_A^d(A\{X\})^{\mathrm{ab}} &\cong A[\mathbf{e}_{K(w,i)} : i \geq 1, w \in \Psi]/\mathrm{ab}_{\mathrm{TS}}(\mathbf{e}_{K(w,i)} : i > d, w \text{ primitive}\} \\ &\cong E_{X,A}/(\Lambda_i(w) \mid i > d, w \text{ primitive}) \qquad (1.1.10.11) \\ &\cong E_X(d)_A. \end{aligned}$$

Vaccarino's final task is to show that this isomorphism over infinite fields A implies that the isomorphism holds in the case $A = \mathbb{Z}$. To explain this last step, we introduce some more background on the interest in these objects, culminating in a result over $\mathbb{Z}$ that we will need to finish the proof of Theorem 1.1.9.3.

Recall the universal representation

$$\rho^{\mathrm{univ}} = \rho_d^{\mathrm{univ}} : A\{X\} \longrightarrow M_d(F_X(d)_A)$$

from Definition 1.1.9.1. The adjoint action of $\mathrm{PGL}_d(A)$ on $M_d(A)$ for all commutative rings A induces an action of the group scheme $\mathrm{PGL}_d/\operatorname{Spec}\mathbb{Z}$ on $\operatorname{Spec} F_X(d) = \operatorname{Spec} F_X(d)_{\mathbb{Z}}$, with $g \in \mathrm{PGL}_d(A)$ sending x_{ij} for $x \in X, 1 \leq i, j \leq d$ to the element of $F_X(d)_A$ appearing in the (i,j)-coordinate after conjugation by g. Clearly $E_X(d)_A \subset F_X(d)_A^{\mathrm{PGL}_d(A)}$ for all A, because characteristic polynomial coefficients are invariant under conjugation. Is this map an isomorphism?

This question was first investigated for A an algebraically closed field of characteristic zero, and then for positive characteristic algebraically closed fields and $A = \mathbb{Z}$. The motivating question was to describe the invariant theory of n-tuples of $d \times d$-matrices $(m_1, \ldots, m_n)$. That is, what is the subring of regular functions on the affine variety $M_d^n = M_d \times \cdots \times M_d$ invariant under the diagonal action of PGL_d by conjugation on $M_d \times \cdots \times M_d$? M. Artin

conjectured[4] that the subring of conjugation-invariant regular functions were generated by traces of products of these n matrices, i.e. for some finite word w in the alphabet $\{1, \ldots, n\}$ with letters w_i, the regular function

$$\mathrm{Tr}(m_{w_1} \cdot m_{w_2} \cdot \cdots \cdot m_{w_n})$$

on M_d^n. In positive characteristic, one conjectures that such functions will generate the invariant subring once other characteristic polynomial coefficients $\Lambda_i, 1 \leq i \leq d$ are also allowed. In other words, the conjecture is that $E_X(d) = F_X(d)^{\mathrm{PGL}_d}$. This conjecture can be extended over arbitrary bases. To be clear, over the base ring A, $F_X(d)_A^{\mathrm{PGL}_d}$ denotes the co-invariants of the co-action of the coordinate ring of $\mathrm{PGL}_{d/A}$

$$F_X(d)_A \longrightarrow F_X(d)_A \otimes_A A[\mathrm{PGL}_d],$$

i.e. those $f \in F_X(d)$ such that its image is $f \otimes 1$. Also, set $F_X(d)^{\mathrm{PGL}_d} := F_X(d)_{\mathbb{Z}}^{\mathrm{PGL}_d}$.

This is the main result of Donkin [Don92, §3] over arbitrary algebraically closed fields and over $\mathbb{Z}$ (depending on his integrality result [Don93]); this was also proved by Zubkov [Zub94]. This followed a proof by Procesi [Pro67] and, independently, Sibirski [Sib67], of Artin's conjecture in the characteristic zero case. Here is the key result of Donkin's work for our purposes.

THEOREM 1.1.10.12 ([Don92, §3.1]). *For $A = \mathbb{Z}$ and $d \geq 1$, the map $E_X(d) \longrightarrow F_X(d)^{\mathrm{PGL}_d}$ is an isomorphism, and, for every algebraically closed field $\bar{k}$, induces an isomorphism*

$$E_X(d) \otimes_{\mathbb{Z}} \bar{k} \xrightarrow{\sim} F_X(d)_{\bar{k}}^{\mathrm{PGL}_d}.$$

Vaccarino uses this theorem along with the following argument (cf. [Vac08, Theorem 6.1]) to complete the proof of Theorem 1.1.9.3.

PROOF. (Theorem 1.1.9.3) Let A be a commutative ring. By Corollary 1.1.3.10(1), we have an isomorphism $\mathrm{TS}_A^d(A\{X\}) \xrightarrow{\sim} \mathrm{TS}_{\mathbb{Z}}^d(\mathbb{Z}\{X\}) \otimes_{\mathbb{Z}} A$. By the universal property of

[4]For the attribution of this conjecture, see [Pro76, Introduction].

abelianization, the map

$$\mathrm{ab}_{\mathrm{TS}_{\mathbb{Z}}} \otimes 1_A : \mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}\{X\}) \otimes_{\mathbb{Z}} A \longrightarrow \mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}} \otimes_{\mathbb{Z}} A$$

can be factored through the abelianization $\mathrm{TS}^d_A(A\{X\}) \to \mathrm{TS}^d_A(A\{X\})^{\mathrm{ab}}$, making the commutative diagram

$$\begin{array}{ccc} \mathrm{TS}^d_A(A\{X\}) & \xrightarrow{\sim} & \mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}\{X\}) \otimes_{\mathbb{Z}} A \\ \downarrow{\scriptstyle \mathrm{ab}_{\mathrm{TS}_A}} & & \downarrow{\scriptstyle \mathrm{ab}_{\mathrm{TS}_{\mathbb{Z}}} \otimes 1_A} \\ \mathrm{TS}^d_A(A\{X\})^{\mathrm{ab}} & \longrightarrow & \mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}} \otimes_{\mathbb{Z}} A \end{array} \tag{1.1.10.13}$$

where the bottom horizontal arrow is surjective. Letting $A = \bar{k}$, an algebraically closed field, this bottom horizontal arrow is the top arrow in the commutative diagram

$$\begin{array}{ccc} \mathrm{TS}^d_{\bar{k}}(\bar{k}\{X\})^{\mathrm{ab}} & \longrightarrow & \mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}} \otimes_{\mathbb{Z}} \bar{k} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \delta_d^{\mathrm{ab}} \otimes 1_{\bar{k}}} \\ (F_X(d)_{\bar{k}})^{\mathrm{PGL}_d} & \xrightarrow{\cong} & E_X(d) \otimes_{\mathbb{Z}} \bar{k} \end{array}$$

where the composite map from the top left to the bottom right is known to be an isomorphism by (1.1.10.11) and the bottom horizontal arrow is known to be an isomorphism by Theorem 1.1.10.12. Since we know from (1.1.10.13) that the top horizontal arrow is surjective, and the right vertical arrow is surjective since it is obtained by $\otimes_{\mathbb{Z}}\bar{k}$ from the surjective map δ_d^{ab} of (1.1.10.5), all of the maps in the diagram are isomorphisms.

Therefore we have a surjective map of graded rings

$$\mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}} \longrightarrow E_X(d) \tag{1.1.10.14}$$

that becomes an isomorphism after tensoring by any algebraically closed field. Each graded component $\mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}}_n$, $E_X(d)_n$ of each ring is a finite $\mathbb{Z}$-module, and these finite $\mathbb{Z}$-modules are free because they are submodules of the polynomial algebra E_X, and therefore torsion-free. As these finite free $\mathbb{Z}$-modules become isomorphic after tensoring by any

algebraically closed field, they must be of the same rank and therefore (1.1.10.14) is an isomorphism. □

Now we discuss the finite generation of $\Gamma_A^d(R)^{\mathrm{ab}}$ over A. The invariant theoretic content above will be very useful for this. We can now show the that $\Gamma_{\mathbb{Z}}^d(\mathbb{Z}\{X\})^{\mathrm{ab}}$ is finitely generated over $\mathbb{Z}$ when X is finite, from which we can deduce that PsR_R^d is finitely type as an affine A-scheme when R is a finitely generated A-algebra. We follow Chenevier, using the invariant theoretic content above with the input of geometric invariant theory.

THEOREM 1.1.10.15 ([Che11, Proposition 2.38]). *Let A be a commutative Noetherian ring, let R be a finitely generated A-algebra, and let $d \geq 0$. Then $\Gamma_A^d(R)^{\mathrm{ab}}$ is finitely generated as an A-algebra.*

PROOF. Let X be a finite set an let $m = |X|$. As R is finitely generated over A, there exists a surjective A-algebra homomorphism

$$A\{X\} \twoheadrightarrow R,$$

and therefore also a surjective A-algebra homomorphism $\Gamma_A^d(A\{X\})^{\mathrm{ab}} \twoheadrightarrow \Gamma_A^d(R)^{\mathrm{ab}}$, where the surjectivity follows from Corollary 1.1.3.14 and Lemma 1.1.7.12. Therefore we are reduced to the case that $R = A\{X\}$. As $\Gamma_A^d(A\{X\})^{\mathrm{ab}} \cong \Gamma_{\mathbb{Z}}^d(\mathbb{Z}\{X\})^{\mathrm{ab}} \otimes_{\mathbb{Z}} A$ by Corollary 1.1.3.10, we further reduce to the case $A = \mathbb{Z}$.

Our main achievement of this section, Theorem 1.1.9.3, shows that the determinant of ρ^{univ} is a pseudorepresentation inducing an isomorphism $\Gamma_{\mathbb{Z}}^d(\mathbb{Z}\{X\})^{\mathrm{ab}} \xrightarrow{\sim} E_X(d)$. By Theorem 1.1.10.12, $E_X(d) \cong F_X(d)^{\mathrm{PGL}_d}$. By the main theorems of geometric invariant theory (see for example [Alp10, Main Theorem, (4)] or the original source [Ses77, Theorem 2]), the fact that $F_X(d)$ is finitely generated over $\mathbb{Z}$ implies that $F_X(d)^{\mathrm{PGL}_d}$ is finitely generated over $\mathbb{Z}$ as well. □

REMARK 1.1.10.16. We will often assume the assumptions of Theorem 1.1.10.15, so that PsR_R^d is an affine Noetherian A-scheme. Later, we will see that these assumptions are also necessary in order to know moduli spaces of representations and are finite type over $\operatorname{Spec} A$.

1.1.11. A Direct Sum Operation on Pseudorepresentations. Given two representations of an A-algebra R, one can form a representation out of their direct sum. In this paragraph, we study the analogy of this construction for pseudorepresentations, and verify that this operation behaves well with respect to dimension.

Let R_1, R_2 be A-algebras, and let B be a commutative A-algebra. We know from Corollary 1.1.3.10(3) along with Theorem 1.1.6.5 that we have an isomorphism of A-algebras

$$\Gamma_A^d(R_1 \times R_2)^{\mathrm{ab}} \xrightarrow{\sim} \prod_{d_1+d_2=d} \Gamma_A^{d_1}(R_1)^{\mathrm{ab}} \otimes_A \Gamma_A^{d_2}(R_2)^{\mathrm{ab}}. \tag{1.1.11.1}$$

By representability, this corresponds to a binary operation, associating two multiplicative A-polynomial laws

$$D_1 : R_1 \longrightarrow A, \quad D_2 : R_2 \longrightarrow A,$$

which are homogenous of degree d_i respectively, to their product, which is a multiplicative A-polynomial law

$$D_1 \cdot D_2 : R_1 \times R_2 \longrightarrow A$$

of degree $d = d_1 + d_2$. One can check that the construction is

$$\begin{aligned} D_1 \oplus D_2 : R_1 \times R_2 &\longrightarrow A \\ (r_1, r_2) &\mapsto D_1(r_1) \cdot D_2(r_2), \end{aligned} \tag{1.1.11.2}$$

which is compatible with $\otimes_A B$, and thereby a polynomial law. One can also quickly see that this polynomial law is multiplicative of degree $d = d_1 + d_2$ (cf. 1.1.11.5).

Remark 1.1.11.3. It is important to notice that the case of degree 0 homogenous multiplicative polynomial laws play an important role in the isomorphisms above: for example, in (1.1.11.1), the d_1, d_2 must vary over all *non-negative* integers such that $d_1 + d_2 = d$. We also see the importance of zero-dimensional pseudorepresentations having constant value 1.

It is natural, since pseudorepresentations are sometimes constructed by taking determinants, to think of this operation as a product. However, we will call it a sum, either by analogy to the data of the trace function that a pseudorepresentation holds, or by observing

that if we have two representations

$$R_1 \longrightarrow M_{d_1}(A), \quad R_2 \longrightarrow M_{d_2}(A),$$

then there is a direct sum representation

$$R_1 \times R_2 \longrightarrow M_{d_1}(A) \times M_{d_2}(A) \hookrightarrow M_{d_1+d_2}(A)$$

which is compatible with the construction above by taking the pseudorepresentations induced by the determinants of the three representations.

Remark 1.1.11.4. We also choose to call this operation a sum $\oplus$ on pseudorepresentations because some preliminary calculations suggest that if R has the structure of a (cocommutative) Hopf algebra, there is a (commutative) tensor product operation $\otimes$ on pseudorepresentations which decategorifies the tensor product of representations of R.

We summarize our discussion about the sum in this proposition, also adding the basic fact that the degree of a homogenous polynomial law into a commutative ring is locally constant; then we know that we are not making any restriction by studying homogenous multiplicative polynomial laws of a given degree.

Proposition 1.1.11.5 (Following [Che11, Lemma 2.2]). *With R_1, R_2 being A-algebras, let B be a commutative A-algebra and let $D_i : R_i \to B$ be a multiplicative A-polynomial laws. If* Spec B *is connected, then*

(1) D_1 (resp. D_2) is homogenous of some degree $d \geq 0$.

(2) any degree d homogenous multiplicative A-polynomial law $D : R_1 \times R_2 \to B$ is the sum, $D_1 \oplus D_2$, of two unique multiplicative homogenous polynomial laws $D_i : R_i \to A$ of degree d_i, with $d_1 + d_2 = d$.

Part (2) is also proved in [Che11, Lemma 2.2(iii)].

Proof. Part (1) is precisely Theorem 1.1.7.4(2).

To prove (2), simply observe that as $\operatorname{Spec} B$ is connected, its image in $\mathrm{PsR}^d_{R_1\times R_2}$ must be confined to one of the elements of the disjoint union

$$\mathrm{PsR}^d_{R_1\times R_2} \xrightarrow{\sim} \coprod_{d_1+d_2=d} \mathrm{PsR}^{d_1}_{R_1} \times_{\operatorname{Spec} A} \mathrm{PsR}^{d_2}_{R_2}$$

induced by (1.1.11.1). □

Now we set $R_1 = R_2 = R$, so that the work above amounts to the analogue in the category of pseudorepresentations of the construction of the $R \times R$-module $M \oplus N$ out of two R-modules M, N, where the first copy of R acts on N trivially and the second copy of R acts on M trivially. To construct from this $R \times R$ module the direct sum R-module $M \oplus N$, we simply compose with the diagonal embedding

$$R \xrightarrow{\Delta} R \times R.$$

This construction inspires the construction of the direct sum of pseudorepresentations.

Definition 1.1.11.6. Let R be an A-algebra, and let D_1, D_2 be pseudorepresentations of dimension d_1, d_2 of R over A. Set $d = d_1 + d_2$. Then the *direct sum pseudorepresentation* $D := D_1 \oplus D_2$ of R over A is given by the d-dimensional homogenous polynomial law such that for each commutative A-algebra B,

$$D_B(x) = D_{1,B}(x) \cdot D_{2,B}(x) \quad \forall x \in R \otimes_A B.$$

We take note of the basic properties of this operation.

Lemma 1.1.11.7. *Let R be an A-algebra, and let d_1, d_2, and d be non-negative integers such that $d_1 + d_2 = d$. Then*

(1) The operation

$$\oplus : \mathrm{PsR}^{d_1}_R \times_{\operatorname{Spec} A} \mathrm{PsR}^{d_2}_R \longrightarrow \mathrm{PsR}^d_R$$

is a morphism in the category of affine A-schemes, corresponding to the homomorphism of commutative A-algebras

$$\Gamma_A^d(R)^{\mathrm{ab}} \xrightarrow{\Gamma^d(\Delta)} \Gamma_A^d(R \times R)^{\mathrm{ab}} \overset{(1.1.11.1)}{\twoheadrightarrow} \Gamma_A^{d_1}(R)^{\mathrm{ab}} \otimes_A \Gamma_A^{d_2}(R_2)^{\mathrm{ab}}.$$

(2) If $D_1 \in \mathrm{PsR}_R^{d_1}(B)$ and $\mathrm{PsR}_R^{d_2}(B)$ are induced from d-dimensional B-valued representations of R, ρ_1 of dimension d_1 and ρ_2 of dimension d_2 respectively, then the $\det \circ(\rho_1 \oplus \rho_2) \cong D_1 \oplus D_2$. In other words, the direct sum operations on representations and pseudorepresentations commute with the map det *from representations to pseudorepresentations.*

Proof. For (1), simply compose (1.1.11.2) and Δ, and note that this is the same as the direct sum given in Definition 1.1.11.6.

To prove (2), we note that the determinant of a direct sum of representations is equal to the product of the determinants of the representations. □

The structures above induce a commutative monoid structure on the functor of all pseudorepresentations.

Definition 1.1.11.8. Let R be an A-algebra. Then write PsR_R^+ for the $\operatorname{Spec} A$-scheme in commutative monoids

$$\mathrm{PsR}_R^+ := \coprod_{d \geq 0} \mathrm{PsR}_R^d,$$

where the group operation is

$$\oplus : \mathrm{PsR}_R^+ \times_{\operatorname{Spec} A} \mathrm{PsR}_R^+ \longrightarrow \mathrm{PsR}_R^+$$

and the identity section is

$$\operatorname{Spec} A \cong \mathrm{PsR}_R^0 \hookrightarrow \mathrm{PsR}_R^+.$$

Later in Theorem 1.3.1.1, we will see that when $A \cong \bar{k}$ is an algebraically closed field, the commutative monoid $\mathrm{PsR}_R^+(\bar{k})$ will be the Grothendieck semigroup of the category of representations of R.

1.1.12. Relation to Pseudocharacters. In this paragraph, we describe a previous version of a pseudorepresentation, which is also commonly known as a pseudorepresentation. This is a pseudocharacter, which is a function on an algebra or multiplicative monoid satisfying the identities one expects of the trace function of a matrix algebra.

DEFINITION 1.1.12.1 (cf. [Tay91, §1.1], [Nys96, Rou96]). Let Γ be a monoid and let A be a commutative ring. Let R be an A-algebra. A *pseudocharacter* of Γ over A of dimension d is the data of a function $T : \Gamma \to A$ such that

(1) $T(1) = d$,

(2) T is central, i.e. $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$, and

(3) the d-dimensional pseudocharacter identity holds:

$$\sum_{\sigma \in S_{d+1}} \operatorname{sgn}(\sigma) T_\sigma(\gamma_1, \ldots, \gamma_{d+1}) \quad \text{for all } \gamma_1, \ldots, \gamma_{d+1} \in \Gamma, \tag{1.1.12.2}$$

where S_{d+1} is the symmetric group on $d+1$ letters and T_σ is the function given by

$$T_\sigma : \Gamma^{d+1} \longrightarrow A$$

$$(\gamma_1, \ldots, \gamma_{d+1}) \mapsto \prod_{j=1}^{s} T(\gamma_{i_1^{(j)}} \cdots \gamma_{i_{r_j}^{(j)}}),$$

where σ has cycle decomposition

$$\sigma = (i_1^{(1)} \ldots i_{r_1}^{(1)})(i_1^{(2)} \ldots i_{r_2}^{(2)}) \ldots (i_1^{(s)} \ldots i_{r_s}^{(s)}).$$

The definition of a pseudocharacter for R is identical, using the multiplicative monoid of R, except that we impose the additional condition that T be A-linear.

Taylor [Tay91] gave the definition of pseudorepresentation above, following on Wiles' definition for two-dimensional representations [Wil88]. That the identity (1.1.12.2) is satisfied by a trace function of a representation is due to Frobenius [Fro96], and Procesi [Pro76, Theorem 1.2] showed that this is the only identity that a central function needs to satisfy

in order to correspond to an invariant (by the adjoint action) function on a space of representations. Taylor used this result to show that pseudorepresentations over an algebraically closed field of characteristic zero are in natural bijection with semisimple characteristic zero representations up to isomorphism. Rouquier [Rou96] extended this to the case that the characteristic of the field is either 0 or greater than the dimension of the pseudocharacter. We will give Chenevier's [Che11] extension of this theorem to arbitrary characteristic, which is achieved by replacing pseudocharacters with pseudorepresentations, in Theorem 1.3.1.1.

Carayol [Car94] showed that the deformations of an absolutely irreducible representation over a field are determined by the induced deformation of its pseudocharacter, where this deformation is given by the trace function of the representation. Nyssen [Nys96, Theorem 1] and Rouquier [Rou96, Theorem 5.1] proved a converse, showing that deformations of a pseudocharacter to a henselian local ring determine a unique (up to isomorphism) deformation of its associated semisimple representation. Definition 1.3.4.1 describes absolutely irreducible pseudorepresentations, and we give Chenevier's an analogous result for pseudorepresentations to the result of Cayayol, Nyssen, and Rouquier's work in the in Theorem 2.1.3.3.

To what extent are pseudocharacters and pseudorepresentations comparable? This proposition, due to Chenevier, gives the state of knowledge on this question.

PROPOSITION 1.1.12.3 ([Che11, Propositions 1.27 and 1.29]). *Let A be a commutative ring and let R be an A-algebra. To each d-dimensional pseudorepresentation $D : R \to A$, we associate to D its trace function function $T = \Lambda_{1,A} : R \to A$ via Definition 1.1.8.2.*

(1) T is a pseudocharacter of dimension d; in particular, it satisfies (1.1.12.2).

(2) The association of determinants to pseudocharacters is injective.

(3) If $(2d)! \in A^\times$, then the association is bijective.

PROOF. The first part is precisely Proposition 1.1.9.11(3). See [Che11, Propositions 1.27 and 1.29] for parts (2) and (3). □

We will use the theory of pseudocharacters in §2.3 in order to apply Bellaïche-Chenevier's definition of generalized matrix algebra. We propose a notion of generalized matrix algebra with respect to pseudorepresentations instead of pseudocharacters in Remark 2.3.3.6. However, when we do this, we will restrict ourselves to the case that $(2d)!$ is invertible in our coefficient rings, so that we can join our theory of pseudorepresentations with the theory of generalized matrix algebras. Proposition 1.1.12.3(3) shows that this is sensible.

1.2. Cayley-Hamilton Pseudorepresentations

Recall from Definition 1.1.8.5 that a pseudorepresentation $D : R \longrightarrow A$ is called Cayley-Hamilton if the homogenous degree d pseudorepresentation

$$\chi = \chi^D : r \mapsto r^d - \Lambda_1(r)r^{d-1} + \Lambda_2(r)r^{d-2} + \cdots + (-1)^d\Lambda_d(r)$$

vanishes identically, i.e. every element of R satisfies its own characteristic polynomial, just as if R were a matrix algebra. We also say that (R, D) is a Cayley-Hamilton A-algebra. Cayley-Hamilton algebras have several special properties which we will explore here. We are motivated by exploring to what extent R has similarities to matrix algebras. For example, Procesi [Pro87] proved that in characteristic zero, a Cayley-Hamilton A-algebra admits an embedding into a matrix algebra $M_d(B)$ for some commutative A-algebra B.

While the material of this section is mostly due to Chenevier [Che11], our main new contribution is the application of polynomial invariant ring (PI ring) theory to show that Cayley-Hamilton algebras are finite over their pseudorepresentation algebra $\mathcal{O}_{\mathrm{PsR}^d_R}$, and in particular finite over their center. This allows us to strengthen one of Chenevier's results.

1.2.1. Properties of Cayley-Hamilton Algebras. We freely use the notation of Definition 1.1.8.5. One of the most basic properties of the two-sided ideal $\mathrm{CH}(D) \subset R$ is the following lemma, showing that any pseudorepresentation factors through a Cayley-Hamilton algebra and that faithful pseudorepresentations are Cayley-Hamilton.

Lemma 1.2.1.1 ([Che11, Lemma 1.21]). *With A, R, D, and d as usual,* $\ker(D)$ *contains* $\mathrm{CH}(D)$. *In particular, if D is faithful, then (R, D) is Cayley-Hamilton.*

PROOF. We have proved the "Cayley-Hamilton identity for pseudorepresentations" in Proposition 1.1.9.11(4), namely

$$D(1 + \chi^{[\alpha]}(r_1, \ldots, r_d) \cdot r) = 1$$

for any $\alpha \in I_d^d$, $(r_1, \ldots, r_d) \in R^d$, and $r \in R$. It remains to show that this holds true after replacing r with an element of $R \otimes_A B$ for B any commutative A-algebra. This follows from the fact that, for a given α, the functions $\chi_B^{[\alpha]} : (R \otimes_A B)^d \to R$ associated to the pseudorepresentation $D \otimes_A B : R \otimes_A B \to B$ belong to the commutative diagram

$$\begin{array}{ccc} R^d & \xrightarrow{\chi^{[\alpha]}} & R \\ \downarrow & & \downarrow \\ (R \otimes_A B)^d & \xrightarrow{\chi_B^{[\alpha]}} & R \otimes_A B \end{array}$$

□

EXAMPLE 1.2.1.2 ([Che11, Example 1.20]). Consider a matrix algebra $M_d(A)$ over a commutative ring A, with its standard d-dimensional pseudorepresentation det coming from the determinant $M_d(A) \to A$. Of course, this pseudorepresentation is Cayley-Hamilton, as every matrix satisfies its characteristic polynomial by the Cayley-Hamilton theorem. It is also faithful, since for any $0 \neq r \in M_d(A)$ there exists $r' \in M_d(A)$ such that the characteristic polynomial of rr' is not t^d. Consider now the restriction $D : T_d(A) \to A$ of det to the A-subalgebra $T_d(A) \subset M_d(A)$ of upper triangular matrices. We see that D is still Cayley-Hamilton, illustrating the general fact that the restriction of a Cayley-Hamilton pseudorepresentation to a subalgebra remains Cayley-Hamilton. However, this example also illustrates that the "faithful" property of a pseudorepresentation is not stable under restriction to a subalgebra. For det is faithful, but the kernel of D is precisely the two-sided ideal of strictly upper triangular matrices in $T_d(A)$.

We record the following lemma on the decomposition of a pseudorepresentation by idempotents. Recall that an idempotent $e \in R$ induces a decomposition $eRe \oplus (1 - e)R(1 - e)$

which is a A-subalgebra of R isomorphic to $eRe \times (1-e)R(1-e)$ via the natural map

$$x \mapsto (ex, (1-e)x). \tag{1.2.1.3}$$

Also recall that a set of idempotents is called orthogonal provided that the product of any pair of distinct elements of the set is zero. Note that not all of this lemma depends on D being Cayley-Hamilton.

LEMMA 1.2.1.4 ([Che11, Lemma 2.4]). *Assume that* $\operatorname{Spec} A$ *is connected and let* $e \in R$ *be an idempotent element. Let* $D : R \to A$ *be a* d*-dimensional pseudorepresentation.*

(1) The polynomial law $D_e : eRe \to A$ *defined by* $r \mapsto D(r + 1 - e)$ *is a pseudorepresentation whose dimension* $r(e)$ *satisfies* $r(e) \leq d$.

(2) We have $r(e) + r(1-e) = d$*, and the restriction of* D *to the* A*-subalgebra* $eRe \oplus (1-e)R(1-e)$ *is the direct sum pseudorepresentation* $D_e D_{1-e}$ *of* (1.1.11.2).

(3) If D *is Cayley-Hamilton (resp. faithful), then so is* D_e.

(4) Assume that D *is Cayley-Hamilton. Then* $e = 1$ *(resp.* $e = 0$*) if and only if* $D(e) = 1$ *(resp.* $r(e) = 0$*). Let* $e_1, \ldots, e_s$ *be a family of nonzero orthogonal idempotents of* R*. Then* $s \leq d$*, and we have an inequality* $\sum_{i=1}^{s} r(e_i) \leq d$*, which is an equality if and only if* $e_1 + e_2 + \cdots + e_s = 1$.

PROOF. Write $S_1 = eRe$ and $S_2 = (1-e)R(1-e)$. Let S be the A-subalgebra $S = S_1 \oplus S_2 \subset R$. As noted above, (1.2.1.3) induces an isomorphism with $S_1 \times S_2$. Now parts (1) and (2) follow directly from Proposition 1.1.11.5.

Assume that D is faithful. Note that for any commutative A-algebra B, the B-algebra $eRe \otimes_A B$ is naturally isomorphic to a direct summand $e(R \otimes_A B)e$ of $R \otimes_A B$. Choose $r \in \ker(D_e) \subset eRe \subset R$. Using the characterization of the kernel in Lemma 1.1.6.6, we have for any $r' \in R \otimes_A B$ that

$$D(1 + rr') = D(1 + erer') = D(1 - e + e + erer') = D_e(e + erer') = 1.$$

Therefore $r \in \ker(D)$ so $r = 0$ by assumption.

Assume that D is Cayley-Hamilton. For $r \in R \otimes_A B$, we have the Cayley-Hamilton identity $\chi^D(r,r) = 0$. From part (2), we know that

$$\chi^D(r,t) = \chi^{D_e}(er,t)\chi^{D_{1-e}}((1-e)r,t) \in B[t].$$

For $r \in e(R \otimes_A B)e$, we apply the Cayley-Hamilton identity for χ^D to r (resp. $r+1-e$) to find that

$$\chi^{D_e}(er,r)r^{d_2} = 0, \text{ resp. } \chi^{D_e}(e(r+1-e),r+1-e)(r-1)^{d_2} = 0.$$

As the ideal of $B[t]$ generated by t^{d_2} and $(t-1)^{d_2}$ is $B[t]$, we get $D_e(r,r) = 0$, showing that D_e is Cayley-Hamilton.

Let us show part (4). It is always the case that $\chi^D(e,e) - D(e) \in Ae \subset R$. If D is Cayley-Hamilton and $D(e) = 1$, then e is a unit in A (see (1.2.3.3) for this fact) and therefore $e = 1$. If $r(e) = 0$, then $D_e(\cdot) = D(\cdot + 1 - e)$ is a determinant of degree 0 on eRe, and is therefore constant and equal to 1. In particular, $D(1-e) = 1$ so $e = 0$ by the argument above. For the last claim of part (4), set $e_{s+1} = 1 - (e_1 + \cdots + e_s)$. Note that $1 \leq r(e_i) \leq d$ for each e_i, since $e_i \neq 0$ and therefore $r(e_i) \neq 0$ for each i. However, $\sum_1^{s+1} r(e_i) = d$ by applying part (2) s times. This proves the last claim in (4). □

LEMMA 1.2.1.5 ([Che11, Lemma 2.6]). *Let $D : R \to A$ be a 1-dimensional Cayley-Hamilton pseudorepresentation. Then $R = A$ and D is the identity map.*

PROOF. For each $r \in R$, $\chi(r,t) = t - D(r)$. As r satisfies its characteristic polynomial and D is A-linear, the lemma follows. □

1.2.2. Background in PI Ring Theory. Our main aim in this paragraph is to apply the theory of polynomial identity rings to prove that a Cayley-Hamilton A-algebra (R, D) is often finite as a module over A. One implication of this is that all representations of an arbitrary finitely generated A-algebra R of a fixed dimension d simultaneously factor through an algebra that is finite over its center.

We begin with a short review of the theory of polynomial invariant algebras over a commutative ring A, following Procesi's book [Pro73]. We will use the notation $A\{x_s\}$ to denote the free (non-commutative) A-algebra on a set X.

DEFINITION 1.2.2.1. Let R be an A-algebra.

(1) An ideal $I \subset A\{x_s\}$ is called a T-ideal if, for any endomorphism $\varphi : A\{x_s\} \to A\{x_s\}$, we have $\varphi(I) \subseteq I$.

(2) The set

$$I = \{f(x_s) \in A\{x_s\} \mid f(r_s) = 0 \text{ for all } r_s \in R\}$$

is called the *T-ideal of polynomial identities of R*.

(3) A T-ideal $I \subset A\{x_s\}$ is called a *proper* T-ideal provided that it is not contained in $J\{x_s\}$ for any ideal $J \neq A$ of A.

(4) We call R a *polynomial ideal algebra* or *PI-algebra* if the T-ideal I of polynomial identities of R is proper.

Since every element of a degree d Cayley-Hamilton A-algebra satisfies its own degree d characteristic polynomial, which is monic and degree d in $A[t]$, it is a PI A-algebra because of the following fact.

PROPOSITION 1.2.2.2 ([Pro73, Proposition 3.22]). *Let d be a positive integer. Then there exists a proper polynomial identity such that for any commutative ring A and any A-algebra R, R satisfies this polynomial identity if every element of R is integral over A of degree bounded by d. In particular, such an algebra R is a PI A-algebra.*

PROOF. We will specify this polynomial identity and leave it to the reader to complete the proof or look up the reference. Let P_n for $n \geq 1$ be the polynomial in the (noncommutative) free algebra over $\mathbb{Z}$ generated by n indeterminates $x_1, \ldots, x_n$ given by

$$P_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)},$$

where S_n is the symmetric group on n letters and sgn is the signature character $\mathrm{sgn} : S_n \to \{\pm 1\}$. Define $f(x, y)$ in the (noncommutative) free algebra over $\mathbb{Z}$ by

$$f(x, y) = P_{d+1}(y^d x, y^{d-1} x, y^{d-2} x, \ldots, yx, x).$$

Then f is a proper polynomial identity whose existence is asserted in the statement of the proposition. □

When A is Noetherian and R is finitely generated a A-algebra and Cayley-Hamilton, the following fact will allow us to conclude immediately that R is finite as an A-module.

THEOREM 1.2.2.3 ([Pro73, Theorem 2.7]). *Let R be a finitely generated PI algebra over a commutative Noetherian ring A. Then if R is integral over A, it is also finite as a module over A.*

However, in some particular cases relevant to our investigation of Cayley-Hamilton algebras, we will be able to establish module finiteness of R over A when R satisfies weaker conditions than the conditions of Theorem 1.2.2.3.[5] In order to accomplish this, it will be particularly important to show that if R is a nil algebra (i.e. every element is nilpotent; in particular, a nil algebra does not have a unit) of bounded nil degree over a field k, then R is finite dimensional as a k-vector space. The following theorems will be very useful to this end.

The first important theorem is known as the Nagata-Higman theorem.

THEOREM 1.2.2.4 (Dubnov-Ivanov [DI43]). *Let k be a field and let R be a nil k-algebra such that there exists a positive integer d with the property that $r^d = 0$ for every $r \in R$. Then if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > d$, there exists some $N \leq 2^d - 1$ such that $R^N = 0$.*

There exist examples showing that $2^d - 1$ is the best possible such bound.

[5]The most important example will be Theorem 3.2.3.2. Here we are working over a fixed pseudorepresentation into a complete local ring, and the condition $\mathbf{\Phi}_{\bar{D}}$ is the finitude of the vector space of self-extensions of the semisimple representation corresponding to the pseudorepresentation of the special fiber $\bar{D}$. This shows that the complete local ring can be taken to be Noetherian (Theorem 3.1.5.3), but finite generation of the algebra over this base is not required. The condition on the self-extensions suffices.

REMARK 1.2.2.5. This theorem is known as the Nagata-Higman theorem, since it was discovered in the western mathematical community by Nagata [Nag52] in characteristic zero, and then generalized to large enough positive characteristic by Higman [Hig56]. It was first discovered Dubnov and Ivanov [DI43] but overlooked in the west. For a few further remarks on the history and context of these works, see [For90].

The following theorem is more in the spirit of Shirshov's height theorem [Šir57], and fulfills Chenevier's suspicion [Che11, Remark 2.29] that there exists some such result which will allow one to show the nilpotence of the kernel of a Cayley-Hamilton pseudorepresentation over a field, even when the characteristic is too small to apply the Nagata-Higman theorem. For further comments on Shirshov's height theorem, see [Kem09].

THEOREM 1.2.2.6 (Samoilov [Sam09]). *Let R be an associative PI algebra over a field k of characteristic $p > 0$. If R is generated by a set X and every word in the elements of x is nilpotent of degree not exceeding d, then R is a nilalgebra, i.e. there exists a positive integer N such that $R^N = 0$. Here N depends on p, the particular polynomial identity it satisfies, and on d, but it does not depend on the cardinality of X.*

For future reference, let us record a particular integer $N = N(p, d)$.

DEFINITION 1.2.2.7. Let p be a prime number and let d be a positive integer. Let $N(p, d)$ be the integer determined by Theorem 1.2.2.6, where, in the notation of the theorem statement, p is the characteristic of the field k, d is the bound on the nil-degree of the elements of X, and the polynomial identity is x^d. Let $N(d)$ be the integer specified in Corollary 1.2.2.8 below.

For a fixed d, Theorems 1.2.2.4 and 1.2.2.6 combine to form the following result.

COROLLARY 1.2.2.8. *There exists an integer $N(d) \geq 0$ dependent only on d with the following property: for any associative, non-unital algebra R over a characteristic $p \geq 0$ field k such that every element of R satisfies the identity x^d where $d \geq 1$, R is nilpotent of degree*

no more than $N(d)$, i.e. $R^{N(d)} = 0$. The integer $N(d,p)$ also has this property over such algebras R where k has characteristic p.

PROOF. Let $N(d)$ be the maximum of the finite collection of integers

$$\{N(p,d) : \text{ prime } p \leq d\} \cup \{2^d - 1\}.$$

Then by Theorems 1.2.2.4 and 1.2.2.6, $R^{N(d)} = 0$. □

While we will prove stronger results later, let us now list some immediate corollaries, applying the results from PI theory above to Cayley-Hamilton A-algebras.

COROLLARY 1.2.2.9. *Let (R, D) be a finitely generated Cayley-Hamilton A-algebra of degree d, where A is a commutative Noetherian ring. Then R is finite as a module over A.*

PROOF. As (R, D) is a Cayey-Hamilton A-algebra, each element $r \in R$ satisfies its characteristic polynomial, which is a degree d monic polynomial equation $\chi(r,t)$ with coefficients in A. Proposition 1.2.2.2 implies that R is a PI A-algebra, and then Theorem 1.2.2.3 implies that R is finite as a module over A. □

There are several more very useful consequences of this finiteness, which we now discuss.

COROLLARY 1.2.2.10. *Let (R, D) be a d-dimensional finitely generated Cayley-Hamilton A-algebra, where A is a commutative Noetherian ring.*

(1) R is finite as an A-module; in particular, it is finite over its center and is a Noetherian ring.

(2) $\ker(D) \subset R$ *is a nilpotent two-sided ideal.*

(3) If A is a Jacobson ring (e.g. a field), then R is a Jacobson ring as well, and $J(R) = N(R)$ is an equality of nilpotent ideals.

(4) If A is an Artinian ring, then R is as well.

PROOF. The first statement in (1) repeats Corollary 1.2.2.9. When A is a commutative Noetherian ring, then an A-algebra which is finite as an A-module via the structure map is also Noetherian (see e.g. [MR01, Lemma 1.1.3]). This proves (1).

Because (R, D) is Cayley-Hamilton, each element $r \in \ker(D)$ satisfies its characteristic polynomial $\chi(r, t) = t^d$. Therefore the kernel is a nil two-sided ideal. Since R is Noetherian, the nilradical of R contains $\ker(D)$ and is nilpotent (see Remark 1.2.2.11 below). Hence $\ker(D)$ is nilpotent as well.

If A is a Jacobson ring, then R is a Jacobson ring as well, as it is finite as a module over A [MR01, §9.1.3] (see also [MR01, Theorem 13.10.4(iii)]). Therefore its nilradical is the same as its Jacobson radical. As R is Noetherian, both are nilpotent (see the Remark immediately below). This proves (3).

Taking R as an A-module, it is the descending chain condition holds on sub-A-modules of R because it is a finitely generated module over an Artinian ring. As ideals of R are certain sub-A-modules of R, the descending chain condition also holds for ideals, proving (4). □

Remark 1.2.2.11. There are several notions of nilradical which coincide for Noetherian rings. Here are the notions for a general noncommutative ring R.

(1) The lower nilradical is the intersection of all prime ideals in a ring, where an ideal $I \subset R$ is prime if for any ideals A, B such that $A \cdot B \subseteq I$, then either $A \subseteq I$ or $B \subseteq I$.
(2) The Levitsky radical is the largest locally nilpotent ideal, where an ideal is called locally nilpotent if any finitely generated sub-ideal is nilpotent.
(3) The upper nilradical is the ideal generated by all nil ideals in R, where an ideal is called nil if every element in it is nil. Note that the ideal generated by nilpotent elements may not be nil in the noncommutative case; this definition of upper radical is chosen so that it the upper radical is a nil ideal.

In general there is an inclusion

$$\text{lower nilradical} \subseteq \text{Levitsky radical} \subseteq \text{upper nilradical},$$

but one can check that these definitions coincide when R is Noetherian, so that one can speak of "the nilradical of R." In particular, in the Noetherian case, its follows from this equivalence that the nilradical is a nilpotent ideal.

The Jacobson radical always contains the (upper) nilradical, and is equal to the nilradical when R is Jacobson and Noetherian. For more information see e.g. [GW04].

1.2.3. The Jacobson Radical of a Cayley-Hamilton Algebra. We write $J(R)$ for the Jacobson radical of an algebra R.

The following lemma is a strengthening of a lemma of Chenevier [Che11, Lemma 2.7-2.8], beginning an exploration of the extent to which the kernel of a d-dimensional Cayley-Hamilton pseudorepresentation behaves like a nilpotent subalgebra (without unit) of a matrix algebra. The addition to and partial simplification of Chenevier's arguments comes from PI ring theory.

LEMMA 1.2.3.1 (Following [Che11, Lemma 2.7]). *Let $D : R \longrightarrow A$ be a Cayley-Hamilton pseudorepresentation, where A is a commutative ring and R is an A-algebra.*

(1) $J(R)$ is the largest two-sided ideal $J \subset R$ such that $D(1 + J) \subset A^\times$.

(2) For any $r \in \ker(D)$, we have $(rr')^d = 0$ for all $r' \in R$. In particular, $\ker(D)$ is a nil ideal and is contained in the upper nilradical of R, and therefore also contained in $J(R)$.

Now assume that A is a field.

(3) $r \in R$ is nilpotent if and only if $D(t - r) = t^d$. Moreover, $J(R)$ consists of nilpotent elements.

(4) $\ker(D)$ and $J(R)$ are nilpotent ideals, with degree of nilpotence bounded by the integer $N(d)$ of Definition 1.2.2.7, which depends only on the integer d. However, if $d!$ is invertible in A, then the bound $2^d - 1$ suffices.

(5) $\ker(D) = J(R)$.

(6) If $I \subset R$ is a two-sided ideal such that $I^n = 0$ for some $n \geq 1$, then $I \subset \ker(D)$ (here it is not necessary to assume that D is Cayley-Hamilton).

REMARK 1.2.3.2. This lemma and its proof is based on Chenevier's lemma [Che11, Lemma 2.7]. It is due to him, except for (3), which comes from our use of PI ring theory.

PROOF. Without applying a Cayley-Hamilton assumption, if $r \in R$ is invertible, then $D(r)$ is invertible since D is multiplicative and preserves units. Assuming the Cayley-Hamilton property, the converse is true: if $D(r) = a$ is invertible in A, then the multiplicative inverse of r is given by manipulating its characteristic polynomial.

$$(r^{d-1} - \Lambda_1(r)r^{d-2} + \cdots + (-1)^{d-1}\Lambda_{d-1}(r)) \cdot r = -a. \tag{1.2.3.3}$$

Since the Jacobson radical $J(R)$ of R is the set of quasiregular elements, i.e. $r \in R$ such that $1 - r$ is a unit in R, we see that $r \in J(R)$ if and only if $D(1 - r)$ is a unit, proving (1).

Now we will prove (2). If $r \in \ker(D)$ and $r' \in R$, then $\Lambda_i(rr') = 0$ for $1 \leq i \leq d$. Then r must satisfy the characteristic polynomial $\chi(r, t) = t^d$. This shows that $\ker(D)$ is a nil ideal of bounded nil-degree d. Therefore $\ker(D) \subseteq N(R)$.

Now let A be a field k. If $r \in R$ is nilpotent, then $1 + tr \in R \otimes_k k[t]$ is invertible. Therefore $D(1 + tr)$ is invertible in $k[t]$, hence $D(1 + tr)$ is in $k^\times$. Using the homogenous multiplicativity of D on $B = A[t, t^{-1}]$, we see that

$$t^{-d} \cdot D_B(1 - tr) = D_B(t^{-1} - r) = \chi(r, t^{-1}),$$

so that $\chi(r, t) = t^d$ and $D(1 + tr) = 1$. Therefore $r^d = 0$, proving one direction of part (3). For the converse, we simply use the Cayley-Hamilton property. Now choose $x \in J(R)$. For all $y \in k[x]$, $1 + yx$ is invertible in $k[x]$, so that $D(1 + yx) \in k^\times$. Then, as in the proof of part (1), we know that $1 + yx$ is invertible in $k[x]$. This means that $x \in J(k[x])$. This only happens when $k[x]$ is finite dimensional as a k-vector space. Since any element of the Jacobson radical of a finite dimensional algebra over a field is nilpotent, we conclude that x is nilpotent as desired. This concludes (3).

Parts (2) and (3) have shown that $\ker(D)$ and $J(R)$ are nil-ideals of bounded nil-degree d, i.e. all of their elements are nilpotent of degree d. Part (4) follows directly from this fact, upon applying Corollary 1.2.2.8.

To prove (5), let us first assume that k is an infinite field. We know from part (3) that $J(R)$ consists of nilpotent elements, and that $D(1+r)=1$ for all $r \in J(R)$. Since k is an infinite domain and $J(R)$ is a two-sided ideal, we may apply Lemma 1.1.7.2, which tells us that

$$\ker(D) = \{r \in R \mid \forall r' \in R, D(1+rr') = 1\}.$$

This shows that $J(R) \subseteq \ker(D)$. The opposite inclusion is part (2). It remains only to reduce to the case that k is an infinite field; this is accomplished in Lemma 1.2.3.5 below. This completes our proof that $J(R) = \ker(D)$ when A is a field.

For part (6), let I be a nilpotent ideal of R and choose $r \in I$. Then for any $y \in R \otimes_A A[t_1, \cdots, m]$ for any m, ry is nilpotent. Therefore $D(1+try)$ is invertible, hence equal to 1 by the logic above. Therefore $r \in \ker(D)$ by definition. □

Remark 1.2.3.4. The nilpotence of the nilradical of a finitely generated PI algebra over a commutative Noetherian ring was first proved by Braun [Bra84]. We proved this more simply because, in our case of concern, R is integral of bounded degree over A and therefore finite as an A-module.

Lemma 1.2.3.5 ([Che11, Lemma 2.8]). *Let k be a field and let $D : R \to k$ be a d-dimensional pseudorepresentation. Then for any separable algebraic extension K/k, the natural injection $R \otimes_k K$ induces isomorphisms*

$$J(R) \otimes_k K \xrightarrow{\sim} J(R \otimes_k K), \quad \ker(D) \otimes_k K \xrightarrow{\sim} \ker(D \otimes_k K).$$

This proof is due to Chenevier.

Proof. By Lemma 1.1.5.2, we have an injection

$$\ker(D) \otimes_k K \longrightarrow \ker(D \otimes_k K).$$

We need to show that this map is surjective. Enlarge K if necessary, so that K/k is normal with Galois group Γ. Consider the natural semilinear action of Γ on $R \otimes_k K$. By Hilbert's Theorem 90, each Γ-stable K-subvector space of V of $R \otimes_k K$ has the form $V^\Gamma \otimes_k K$, where

$V^\Gamma \subset R$ is the k-vector space of fixed points. We claim that $\ker(D \otimes_k K)$ is Γ-stable. Observe that Γ has a natural semilinear action on any K-algebra B. As the characteristic polynomial coefficient functions of $D \otimes_k K$ are defined over k, we have for any K-algebra B, any $r \in R \otimes_k B$, and any $\gamma \in \Gamma$ that D is Γ-equivariant, i.e.

$$D(\gamma(r)) = \gamma(D(r)).$$

The claim now follows upon examining the definition of the kernel: if $r \in \ker(D \otimes_k K)$, then $D(1+rr') = 1$ for all K-algebras B and $r' \in R \otimes_k B$, and this will remain true after replacing r with $\sigma(r)$. Now the desired surjectivity follows from the fact that $\ker(D \otimes_k K)^\Gamma \subset \ker(D)$. This also follows from the Γ-equivariance of D. □

1.2.4. The Universal Cayley-Hamilton Algebra. This paragraph discusses a trivial generalization of [Che11, 1.22-1.23], introducing the category of "Cayley-Hamilton representations" of a given A-algebra R. We may think of this as a generalization of the universal Azumaya-algebra valued representation of R discussed in §1.4 below.

We start with the usual data of an algebra R over a commutative ring A. From Theorem 1.1.7.4, we have the universal pseudorepresentation

$$D^u : R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}} \longrightarrow \Gamma_A^d(R)^{\mathrm{ab}}$$

of R over $\Gamma_A^d(R)^{\mathrm{ab}}$. Now we apply the notion of a Cayley-Hamilton algebra to this universal pseudorepresentation.

Definition 1.2.4.1. Let R, A, and D^u be as above. Let B a commutative A-algebra.

(1) A *Cayley-Hamilton B-representation of R of dimension d over B* is a triple

$$(B, (E, D), \rho)$$

where (E, D) is a Cayley-Hamilton A-algebra relative to the pseudorepresentation $D : E \to B$, and $\rho : R \otimes_A B \to E$ is a homomorphism of B-algebras.

(2) The *universal Cayley-Hamilton representation* of R is

$$(\Gamma_A^d(R)^{\mathrm{ab}}, (E(R,d), D^u|_E), \rho^u),$$

where $E(R,d)$ is the $\Gamma_A^d(R)^{\mathrm{ab}}$-algebra

$$E(R,d) := (R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}})/\mathrm{CH}(D^u)$$

receiving the canonical quotient homomorphism $\rho^u : R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}} \to E(R,d)$, and $D^u|_E : E(R,d) \to \Gamma_A^d(R)^{\mathrm{ab}}$ is the factorization of D^u through ρ^u.

Of course, the factorization $D^u|_E$ exists, in view of Lemma 1.1.6.6(2) and Lemma 1.2.1.1.

Remark 1.2.4.2. Cayley-Hamilton representations are direct generalizations of the usual notion of a representation. With R, A as usual, let $R \otimes_A B \to M_d(B)$ be a B-valued d-dimensional representation of R. Then

$$(B, (M_d(B), \det), \rho)$$

is a d-dimensional Cayley-Hamitlon representation of R over B, where det is the standard determinant map $\det : M_d(B) \to B$.

We want to show that the "universal" d-dimensional Cayley-Hamilton representation of R deserves its name, but first we must define the structure of a category $\mathcal{CH}_d(R)$ where this representation will be initial, following [Che11, §1.22]. The objects are the data of the definition above, and a morphism of Cayley-Hamilton representations of R

$$(B_1, (E_1, D_1), \rho_1) \longrightarrow (B_2, (E_2, D_2), \rho_2)$$

is a pair (f, g) where $f : B_1 \to B_2$ and $g : E_1 \to E_2$ are ring homomorphisms such that if $\iota_i : B_i \to E_i$ is the B_i-algebra structure on E_i, then the diagrams

$$\begin{array}{ccc} B_1 & \xrightarrow{\iota_1} & E_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B_2 & \xrightarrow{\iota_2} & E_2 \end{array} \qquad \begin{array}{ccc} E_1 & \xrightarrow{D_1} & B_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ E_2 & \xrightarrow{D_2} & B_2 \end{array}$$

and

$$\begin{array}{ccc} R \otimes_A B_1 & \xrightarrow{\rho_1} & E_1 \\ \downarrow{\scriptstyle \mathrm{id} \otimes f} & & \downarrow{\scriptstyle g} \\ R \otimes_A B_2 & \xrightarrow{\rho_2} & E_2 \end{array}$$

commute.

Proposition 1.2.4.3 ([Che11, Proposition 1.23]). *The universal d-dimensional Cayley-Hamilton representation*

$$(\Gamma_A^d(R)^{\mathrm{ab}}, (E(R,d), D^u|_E), \rho^u)$$

is the initial object of $\mathcal{CH}_d(R)$.

Proof. Let $(B, (S, D), \eta)$ be a d-dimensional Cayley-Hamilton representation of R. The B-algebra homomorphism $\eta : R \otimes_A B \to S$ induces a d-dimensional B-valued pseudorepresentation of R, namely $D \circ \eta$. This induces an A-algebra homomorphism $f : \Gamma_A^d(R)^{\mathrm{ab}} \to B$. This in turn induces an A-algebra homomorphism

$$R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}} \longrightarrow R \otimes_A B \xrightarrow{\eta} S.$$

Since (S, D) is Cayley-Hamilton, Lemma 1.1.8.6 implies that this map factors through $\rho^u : R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}} \twoheadrightarrow E(R,d)$, with quotient

$$g : E(R,d) \longrightarrow S.$$

We observe that $f \circ D^u|_E = D \circ g$, and that (f, g) has the remaining properties of a morphism in $\mathcal{CH}_d(R)$, as desired. □

Now, assuming that A is Noetherian and R is finitely generated as an A-module, we have a pleasant consequence of the PI theory of §1.2.2. This proposition will be applied in §1.4.3 to show that the representation theory of such an algebra R reduces to the representation theory of a finite-over-center algebra, basically by exploring the consequences of Remark 1.2.4.2.

PROPOSITION 1.2.4.4. *If A is Noetherian and R is finitely generated as an A-algebra, then the universal d-dimensional Cayley-Hamilton algebra of degree d associated to R, namely the $\Gamma_A^d(R)^{\mathrm{ab}}$-algebra $E(R,d)$, is finite as a $\Gamma_A^d(R)^{\mathrm{ab}}$-module. In particular, $E(R,d)$ is a Noetherian ring and is finite as a module over its center.*

PROOF. This is an instance of Corollary 1.2.2.10(1). □

1.3. Pseudorepresentations over Fields

In the current chapter, we are developing the theory of pseudorepresentations and then, starting in §1.4, studying the moduli space of pseudorepresentations relative to the moduli space of representations. The main theorem of this chapter, Theorem 1.5.4.2, depends heavily on the comparison of representations wtih pseudorepresentations over an algebraically closed field. Indeed, it is fair to say that in Chapter 1 we prove what we can about this situation by studying moduli functors through their geometric points, and in Chapter 2 we aim for a closer, local-on-the-base study.

This is our motivation for studying pseudorepresentations over fields. We can find a close relationship between semisimple representations and pseudorepresentations over fields. We will now give the main theorem. This is a critical property of this notion of pseudorepresentation, developed by Chenevier, who calls it a "determinant." Previous notions, which we call "pseudocharacters" here, did not function well in the case that the dimension is greater than or equal to the characteristic of the field (see §1.1.12).

1.3.1. Main Theorem. As usual, let R is an A-algebra. We have seen that representations of R valued in an Azumaya algebra, and in particular a matrix algebra, induce a

pseudorepresentation (Theorem 1.1.7.4(6)). We have also seen that given a pseudorepresentation D of R, the universal Cayley-Hamilton representations of R over D share some similarities with representations of R valued in subalgebras of matrix algebras (cf. Corollary 1.2.2.10). Now we will show that over an algebraically closed field, pseudorepresentations are in natural bijection with representations. Here is our main theorem, due to Chenevier.

THEOREM 1.3.1.1 ([Che11, Theorem 2.12]). *Let $\bar{k}$ be an algebraically closed A-field. There is a bijection between conjugacy classes of semisimple d-dimensional representations ρ of R over $\bar{k}$ and d-dimensional pseudorepresentations of R over $\bar{k}$, given by sending $\rho : R \otimes_A \bar{k} \to M_d(\bar{k})$ to $\det \circ \rho$. In fact, if D is a d-dimensional $\bar{k}$-valued pseudorepresentation of R, then the corresponding semisimple representation may be written as*

$$R \otimes_A \bar{k} \longrightarrow (R \otimes_A \bar{k})/\ker(D) \simeq \prod M_{d_i}(\bar{k}),$$

where $\sum d_i = d$.

We will also find an analogous result over arbitrary A-fields k. Indeed, Theorem 1.3.1.1 follows directly from this more general case. However, it will require that we establish some notions and notation.

The following notion of an "exponent" describes the size of field extensions K/k in a different way than the degree of an extension. Indeed, the exponent may be finite and meaningful even when the degree of the extension is infinite. We also give "determinant" maps from central simple algebras S/K to $k \subset K$ when K/k has finite exponent, generalizing the determinant on a matrix algebra.

DEFINITION 1.3.1.2. Let K/k be a field extension, and let $k' \subset K$ be the maximal separable subextension of K. Assume that k'/k is finite. If the characteristic p of k is positive, let q be the smallest power of p such that $K^q \subseteq k'$, and if $p = 0$ let $q = 1$. Define the *exponent* $(f, q) \in \mathbb{N}^2$ of K/k by $f = [k' : k]$ and q as above. It is possible for both or either of the quantities in the exponent to be infinite.

Now assume that K/k has finite exponent (f, q), and let S be a central simple K-algebra of rank n^2 over K with its reduced norm $N : S \to K$. Let $N_{k'/k} : k' \to k$ be the norm map on finite separable fields, and let $F^q : K \to k'$ be the q-power Frobenius map. Then there is a natural *determinant*

$$\det_S : S \to k$$

of k-homogenous degree nqf defined by $\det_S = N_{k'/k} \circ F^q \circ N$.

We observe that in the case that the exponent of S is trivial $(1, 1)$, $\det_S$ is the standard reduced norm of an Azumaya algebra, such a matrix algebra.

Now we can state the theorem describing pseudorepresentations of an algebra over an arbitrary field

THEOREM 1.3.1.3 ([Che11, Theorem 2.16]). *Let R be a k-algebra. Let $D : R \to k$ be a d-dimensional pseudorepresentation.*

(1) Then there is an isomorphism of k-algebras

$$R/\ker(D) \xrightarrow{\sim} \prod_{i=1}^{s} S_i$$

where S_i is a simple k-algebra which is of finite dimension n_i^2 over its center k_i, and where k_i/k is a with finite exponent (f_i, q_i). In particular, $R/\ker(D)$ is semisimple.

(2) Moreover, under such an isomorphism, D is equal to the sum of determinants

$$D = \bigoplus_{i=1}^{s} \det_{S_i}^{m_i}, \quad d = \sum_{i=1}^{s} m_i n_i q_i f_i,$$

where m_i are certain uniquely determined integers.

(3) The pseudorepresentation D is realizable as the composition of the natural sum of determinants $\det_{S_i}$ with the following product of the natural surjections $R \twoheadrightarrow S_i$, namely

$$R \longrightarrow \prod_{i=1}^{s} \prod_{j=1}^{m_i} S_i,$$

where the integers m_i are as above.

(4) $R/\ker(D)$ *is finite-dimensional as a* k*-vector space and, equivalently, each* k_i *is finite-dimensional if any of the following conditions are satisfied, where* p *is the characteristic of* k*.*

(a) k *is perfect,*

(b) $d < p$*,*

(c) $p > 0$ *and* $[k : k^p] < \infty$*, or*

(d) R *is finitely generated as a* k*-algebra.*

Let us deduce the algebraically closed case from this general case.

Proof. (Theorem 1.3.1.3 implies Theorem 1.3.1.1.) Beginning with the notation of Theorem 1.3.1.1, we let $\bar{k}$ be an algebraically closed A-field and replace R by $R \otimes_A \bar{k}$ and think of R as a $\bar{k}$-algebra and let D be a d-dimensional pseudorepresentation $D : R \to \bar{k}$.

By definition of the exponent, every element of a field extension $K/\bar{k}$ of finite exponent is algebraic over $\bar{k}$. Since $\bar{k}$ is algebraically, closed this means that $K = \bar{k}$ when $K/\bar{k}$ has finite exponent, i.e. the exponent is $(1, 1)$. Now Theorem 1.3.1.3 implies that $R/\ker(D)$ is a product of central simple $\bar{k}$-algebras, which are therefore matrix algebras because $\bar{k}$ is algebraically closed. We write

$$R/\ker(D) \xrightarrow{\sim} \prod_{i=1}^{s} M_{d_i}(\bar{k}).$$

If we write $\det_i$ for the determinant function on $M_{d_i}(\bar{k})$, Theorem 1.3.1.3 tells us that

$$D \cong \bigoplus_{i=1}^{s} \det_i^{m_i}, \quad \text{where } \sum_{i=1}^{s} m_i d_i = d,$$

and where $\oplus$ refers to the direct sum of (1.1.11.2). If V_i a the d_i-dimensional representation of R corresponding to $R \twoheadrightarrow M_{d_i}(\bar{k})$, then clearly the representation

$$\bigoplus_{i=1}^{s} V_i^{\oplus m_i}$$

realizes D as its determinant. Since, by the Brauer-Nesbitt theorem, a semisimple representation over an algebraically closed field is determined up to isomorphism by its characteristic polynomials, this semisimple representation is unique up to isomorphism. Conversely, a pseudorepresentation is determined by its characteristic polynomial functions (Corollary 1.1.9.15). Therefore the correspondence is bijective. □

We will prove Theorem 1.3.1.3 in the next paragraph.

1.3.2. Semisimple k-algebras. Now we work toward proving Theorem 1.3.1.3. Firstly, we will note that our existing knowledge allows us to conclude immediately that $R/\ker(D)$ is semisimple and track the number of orthogonal idempotents.

Recall that R is a k-algebra with a d-dimensional pseudorepresentation $D : R \to k$. Let p be the characteristic of k.

Because Lemma 1.2.1.1 tells us that $(R/\ker(D), D)$ is a Cayley-Hamilton k-algebra, we can apply our study of Cayley-Hamilton algebras from §1.2. Let us review the facts that we can deduce directly from this study.

- Every element of R is integral (i.e. algebraic) of bounded degree d over k: each element satisfies its own characteristic polynomial.
- By Proposition 1.2.2.2 R is a PI-k-algebra.
- By Lemma 1.2.3.1(5), $\ker(D)$ is the Jacobson radical $J(R)$ of R, so $R/\ker(D)$ is semisimple.
- By Lemma 1.2.1.4, the largest possible cardinality of a family of pairwise orthogonal idempotents of $R/\ker(D)$ is d.

Also, Corollary 1.2.2.9, if R is finitely generated as a k-algebra, $R/\ker(D)$ is a finite dimensional k-algebra. However, we are not currently assuming that R is finitely generated as a k-algebra.

All that we need to do is to control the exponent of the centers of the simple factors (Lemma 1.3.2.1 below) and control the possible pseudorepresentations out of simple k-algebras (Lemma 1.3.2.3 below).

The following lemma describes field extensions of k satisfying the first property of the bullet list above; these are the possible fields that can appear as the center of a k-algebra satisfying all of the properties of the bullet list.

LEMMA 1.3.2.1 ([Che11, Lemma 2.14]). *If S is a k-algebra satisfying the properties in the bullet list above, then*

$$S \xrightarrow{\sim} \prod_{i=1}^{s} M_{n_i}(E_i)$$

where E_i is a division k-algebra, finite dimensional over its center k_i, and $s \leq d$. In particular, S is semisimple. The center k_i of E_i is a finite separable extension of k, unless k has positive characteristic p, in which case $k[k_i^q]$ is separable, where q is the greatest power of p less then n. Moreover, S is finite dimensional over k if any of the following conditions are satisfied:

(1) k is perfect,

(2) $p > d$,

(3) $p > 0$ and $[k : k^p] < \infty$, or

(4) R is finitely generated over k.

We record some of the proof here for reference, following the proof of [Che11, Lemma 2.14].

PROOF. Let A be a commutative k-algebra satisfying the properties in the bullet list. If $p > 0$, define q as in the statement of the lemma, and set $q = 1$ otherwise. The bound on the number of idempotents implies that

$$A \xrightarrow{\sim} \prod_{i=1}^{s} A_i$$

where $s \leq d$, and where A_i is an algebraic field extension of k. Since A_i/k has bounded algebraic degree d, its maximal separable subextension $A_i^{\text{ét}}$ is finite dimensional over k. As the center $Z(S)$ of S has the properties of A, we have established the conditions of the last part of the lemma are sufficient to imply that the center is finite dimensional over k.

Now we show that S is semisimple. Let M be a simple S-module, and let E be the division algebra $\mathrm{End}_S(M)$. First, we claim that M is finite dimensional over E. Indeed, Jacobson's density theorem[6] implies that either M is finite dimensional over E and $S \to \mathrm{End}_E(M) \simeq M_s(E^{\mathrm{op}})$ is surjective, or for each $j \geq 1$ there is a k-subalgebra $R_j \subset S$ and a surjective k-algebra homomorphism $R_j \twoheadrightarrow M_j(E^{\mathrm{op}})$, but the second option is not possible since the elements of S are algebraic of bounded degree over k.

Now we claim that there are finitely many simple S-modules $M_1, \ldots, M_s$ up to isomorphism. This will complete the proof that S is semisimple, for in this case the fact that $J(S) = 0$ implies that

$$S \longrightarrow \prod_{i=1}^{s} M_{n_i}(E_i^{\mathrm{op}}), \quad \text{where } E_i := \mathrm{End}_S(M_i) \tag{1.3.2.2}$$

is injective, and the fact that the M_i are pairwise non-isomorphic implies that it is surjective.

It remains to check the claim. For this, we refer the reader to the remainder of the proof, found in [Che11, Lemma 2.14]. □

Now we must describe the possible pseudorepresentations out of a simple k-algebra S whose center K is a finite exponent extension of k. Let us first recall that we have already given such a result in the case that the center of S is k, so that S is an Azumaya algebra over k. This is Proposition 1.1.7.10, due to Ziplies [Zip86], which states that all of the pseudorepresentations out of an Azumaya algebra are induced by integral powers of the reduced norm.

Having described the Azumaya algebra case, we proceed to the general case.

Lemma 1.3.2.3 ([Che11, Lemma 2.17]). *Let K/k be a field extension with finite exponent (f, q), and let S be a central simple finite dimensional K-algebra. Then any pseudorepresentation $D : S \to k$ has the form $\det_S^m$ for some unique integer $m \geq 0$.*

[6]Jacobson's density theorem states that for any simple left module N of a ring R, any $\mathrm{End}_R(N)$-linear transformation η of N, and any finite set of elements $\{x_i\}$ of N, there exists $r \in R$ such that $\eta(x_i) = r \cdot x_i$ for all i. See e.g. [Her68, Theorem 2.1.2].

PROOF. Let $D : S \to k$ be a d-dimensional pseudorepresentation, and define $n^2 := \dim_K(S)$. Note that if $D = \det_S^m$, then we must have $d = fmnq$ since $\det_S$ is homogenous of degree fnq by definition; this shows that m is unique if it exists.

We will use the following fact below: if two d-dimensional pseudorepresentations $D_1, D_2 : R \to A$ are such that $D_1 \otimes_A B \cong D_2 \otimes_A B$ for some commutative A-algebra B with $A \to B$ injective, then $D_1 \cong D_2$. This follows directly from the representability of the moduli space of pseudorepresentations, Theorem 1.1.7.4.

Assume for the moment that k is separably closed, so that K is as well. The Noether-Jacobson theorem implies that S is isomorphic to some matrix algebra $M_n(K)$, $n \geq 1$. Set $A := K \otimes_k K$, and denote by I the kernel of the natural split surjection $A \to K$. We see that I is generated as an A-module by elements of the form $x \otimes 1 - 1 \otimes x$, which are nilpotent of index $\leq q$. In, particular, I is a nil ideal, and any finite type A-submodule of I is nilpotent as an ideal. Now Lemma 1.2.3.1(6) implies that for any pseudorepresentation $D : M_n(A) \to K$, D factors through $\pi : M_n(A) \twoheadrightarrow M_n(A/I) = M_n(K)$. Applying this to

$$D \otimes_k K : S \otimes_k K \simeq M_n(K) \otimes_k K \cong M_n(A) \longrightarrow K,$$

we get a pseudorepresentation $M_n(K) \to K$, which is an integral power of the usual determinant by Proposition 1.1.7.10, say $D \otimes_k K \cong \det^s_{M_n(K)} \circ \pi$ and $d = ns$. Now recall that the restriction of $D \otimes_k K$ to $M_n(K) \otimes 1 \subset M_n(A)$ must be valued in k, since D is valued in l. This means that $\det^s(M_n(K)) \subset k$. Therefore q must divide s, and we observe that $\det_S^{s/q} \otimes_k K \cong D \otimes_k K$. Now by the fact mentioned above, this implies that $D \cong \det_S^{s/q}$.

Now we reduce to the case that k is separably closed. We have

$$K \otimes_k k^{\mathrm{sep}} \xrightarrow{\sim} \prod_{i=1}^{f} K_i,$$

where $K_i = K \cdot k^{\mathrm{sep}}$ is a separable algebraic closure of K such that $K_i^q \subset k^{\mathrm{sep}}$ for each i (q is minimal for this property) and $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ permutes transitively the K_i. Recall that f is

the (finite) separable degree of K over k. Likewise,

$$S \otimes_k k^{\mathrm{sep}} \cong S \otimes_K (K \otimes_k k^{\mathrm{sep}}) \xrightarrow{\sim} \prod_{i=1}^{f} S_i,$$

where $S_i = S \otimes_K K_i$ is central simple of rank n^2 over K_i. By Proposition 1.1.11.5(2), each $D \otimes_k k^{\mathrm{sep}}$ is the product of determinants $S_i \xrightarrow{\sim} M_n(K_i) \to k^{\mathrm{sep}}$, which have the form $\det_{S_i}^{m_i}$ by the previous step above, and $d = n(\sum_{i=1}^{f} m_i)$. As $D \otimes_k k^{\mathrm{sep}}$ is $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-equivariant, this implies that m_i is independent of i, i.e. $m_i = m$ for each i. Therefore, $m = d/nf$, and we observe that $D \otimes_k k^{\mathrm{sep}} \cong \det_S^m \otimes_k k^{\mathrm{sep}}$. Now by the fact mentioned above, this implies that $D \cong \det_S^m$, as desired. □

Now we complete the proof of Theorem 1.3.1.3.

Proof. (Theorem 1.3.1.3) By Lemma 1.3.2.1, we know that $R/\ker(D)$ is isomorphic to a product of $s \leq d$ simple k-algebras S_i whose centers k_i are finite exponent extensions of k. This is part (1). Write (f_i, q_i) for the exponent of k_i.

By Proposition 1.1.11.5(2), any pseudorepresentation out of $R/\ker(D)$ is the sum[7] of pseudorepresentations D_i, one out of each S_i. Indeed, $\operatorname{Spec} k$ is connected, so that the conditions of Proposition 1.1.11.5 are satisfied. Lemma 1.3.2.3 implies that each D_i is a power $\det_{S_i}^{m_i}$ of $\det_{S_i}$. As Proposition 1.1.11.5 tells us that the degree of a sum of pseudorepresentations is the sum of the degrees, and $\det_{S_i}$ has degree $n_i q_i f_i$, the formula for the degree follows; this is part (2).

For part (3), we are simply combining part (1) with Lemma 1.1.6.6(2). Part (4) follows directly from Lemma 1.3.2.1. □

Corollary 1.3.2.4. *Let $D : R \to k$ be a d-dimensional pseudorepresentation of a k-algebra R.*

[7]Recall that this sum is defined in (1.1.11.2).

(1) There exists a field extension L/k such that $D \otimes_k L$ is realizable as the determinant of a matrix algebra-valued representation

$$R \otimes_k K \to M_d(L).$$

If R/k is finitely generated, then L/k may be chosen to be a finite extension.

(2) When the centers k_i/k, $1 \leq i \leq s$, of exponent (f_i, q_i), the simple factors S_i of $R/\ker(D)$ of Theorem 1.3.1.3 are separable extensions, e.g. when k is perfect, then there exists a finite separable extension K of degree bounded by $\prod_{i=1}^{s} f_i$ such that

$$R/\ker(D) \otimes_k K$$

is a product of matrix algebras and the natural map from $R \otimes_k K$ to this algebra is a d-dimensional representation whose determinant induces $D \otimes_k K$.

PROOF. We begin with the case that the integer s from Theorem 1.3.1.3 is 1, i.e. the k-algebra $R/\ker(D)$ is a central simple n^2-dimensional L-algebra S where L/k is a field extension of finite exponent (f, q) such that $d = n \cdot f \cdot q$. Its maximal separable subextension L'/k has degree f. Because universal homeomorphisms such as inseparable extensions induce equivalences of étale topoi and Brauer groups classify central simple algebras over a field, the L' algebra $S \otimes_k L'$ is isomorphic to $M_n(L)$. We then observe that the product L-algebra

$$\prod_{i=1}^{q} \prod_{\sigma \in \mathrm{Gal}(L'/k)} \sigma M_n(L)$$

is naturally embeddable in $M_d(L)$. The pseudorepresentation resulting from

$$R/\ker(D) \otimes_k L \longrightarrow \prod_{i=1}^{q} \prod_{\sigma \in \mathrm{Gal}(L'/k)} \sigma M_n(L) \longrightarrow M_d(L) \xrightarrow{\det} L$$

is then equal to $D \otimes_k L$, upon examining the "determinant" $\det_S$ of S defined in Definition 1.3.1.2 and the conclusion of Theorem 1.3.1.3.

The general result (1) follows by applying this to each of the simple factors S_i of $R/\ker(D)$, taking the sum of the resulting pseudorepresentations on the product of these factors, and tensoring D by the composite field of the extensions L' above of each factor.

The claim that the finite generation of R/k implies the finitude of L/k follows from Theorem 1.3.1.3(4d).

Part (2) follows from part (1) and its proof when $q_i = 1$ for each i. □

1.3.3. Finite-Dimensional Cayley-Hamilton Algebras. In this paragraph, we find conditions under which a Cayley-Hamilton algebra (R, D) over a field k is finite-dimensional. There are three basic ingredients. Results from PI ring theory from §1.2.2 culminated in the fact that the Jacobson radical of a Cayley-Hamilton algebra over a field is nilpotent, with degree of nilpotence bounded in terms of the degree of the pseudorepresentation (Lemma 1.2.3.1). The next ingredient is the conditions we have given above for the maximal semisimple quotient of R to be finite-dimensional. Finally, we require some basic lemma, which we now give. This translates the condition that $\ker(D)/\ker(D)^2$ is a finite dimensional vector space, which is the last fact we require, into a condition on the deformations to $k[\varepsilon]/(\varepsilon^2)$ of the semisimple representation ρ associated to D.

LEMMA 1.3.3.1. *Let A be a commutative ring, R an A-algebra. Let I be a two-sided ideal of R. There is a natural A-module isomorphism*

$$\mathrm{Hom}_R(I/I^2, R/I) \xrightarrow{\sim} \mathrm{Ext}^1_R(R/I, R/I).$$

PROOF. Apply $\mathrm{Hom}_R(-, R/I)$ to the exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

and use that $\mathrm{Ext}^1_R(R, -) = 0$. □

We continue to work with deformations of a fixed d-dimensional pseudorepresentation $\bar{D} : R \to k$. Now let us restrict to the case that $S := R/\ker(\bar{D})$ is finite dimensional as a k-vector space.

THEOREM 1.3.3.2. *Let k be a field of characteristic $p \geq 0$ and let R be a k-algebra equipped with a Cayley-Hamilton d-dimensional pseudorepresentation $D : R \to k$. Assume that $S := R/\ker(D)$ is finite-dimensional over k. If $\mathrm{Ext}^1_R(S, S)$ is finite-dimensional as a k-vector space, where S is treated as an R-module here, then R is finite-dimensional k-algebra.*

Recall that sufficient conditions for S to be finite dimensional over k are given in Theorem 1.3.1.3(4).

PROOF. Apply Lemma 1.3.3.1, so that the assumption that $\dim_k \mathrm{Ext}^1_R(S, S) < \infty$ implies that $\dim_k \mathrm{Hom}_S(\ker(D)/\ker(D)^2, S) < \infty$. This means that $\ker(D)/\ker(D)^2$ is a finite sum of simple representations of S, but this in turn implies that $\ker(D)/\ker(D)^2$ is finite-dimensional as a k-vector space.

Because there are natural surjections

$$(I/I^2)^{\otimes_k^n} \twoheadrightarrow I^n/I^{n+1}$$

for any ideal $I \subset R$, this means that $R/\ker(\bar{D})^{2n}$ is also finite dimensional over k for any positive integer n. Since (R, D) is Cayley-Hamilton, Lemma 1.2.3.1(4) implies that $\ker(D)$ is nilpotent of index bounded by $N(d)$ (or by $N(d, p)$), where $N(d)$ is the integer of Definition 1.2.2.7. This completes the proof. □

1.3.4. Composition Factors of Field-Valued Pseudorepresentations. We conclude this section with some discussion of the simple factor algebras of $R/\ker(D)$, where we continue to let R be a k-algebra where k is a field. Equivalently (almost), we discuss the Jordan-Hölder factors that appear in representations of R arising from pseudorepresentations according to Theorem 1.3.1.1. We mostly follow Chenevier's discussion of [Che11, §2], and introduce some notions – the Grothendieck group of R and dimension vectors of representations – that will be useful in §2.2. These notions will be heavily used when we discussion deformation theory of pseudorepresentations in Chapter 2.

DEFINITION/LEMMA 1.3.4.1 ([Che11, Defn.-Prop. 2.18]). Let $D : R \to k$ be a d-dimensional pseudorepresentation over a field k. We call D *absolutely irreducible* provided that one of the following equivalent conditions is true.

(1) The semisimple representation $\rho_D : R \otimes_k \bar{k} \to M_d(\bar{k})$ with determinant equal to the pseudorepresentation D, which exists and is unique up to isomorphism by Theorem 1.3.1.1, is irreducible,
(2) $(R \otimes_k \bar{k})/\ker(D \otimes_k \bar{k}) \simeq M_d(\bar{k})$,
(3) $R/\ker(D)$ is a central simple k-algebra of rank d^2,
(4) $R/\mathrm{CH}(D)$ is a central simple k-algebra of rank d^2,
(5) for some (resp. all) subset $X \subset R$ generating R as a k-vector space, there exists $x_1, x_2, \ldots, x_{d^2} \in X$ such that the abstract $d^2 \times d^2$ matrix $((\Lambda_1(x_i x_j))_{i,j}$ belongs to $\mathrm{GL}_{d^2}(k)$.

If they are satisfied, then $\mathrm{CH}(D) = \ker(D) = \{x \in R, \forall y \in R, \Lambda_1(xy) = 0\}$.

PROOF. Since we know from Proposition 1.1.7.10 that any pseudorepresentation out of a matrix algebra is a power of the determinant, and a pseudorepresentation factors through the quotient by its kernel, (2) implies (1) since we know that D has dimension d. Conversely, if $\rho : R \otimes_k \bar{k} \to M_d(\bar{k})$ is as in (1), then Wedderburn's theorem tells us that ρ is surjective. We see that $\ker(\rho) \subset \ker(D)$, since the pseudorepresentation $\det \circ \rho$ is invariant under multiplication by $\ker(\rho)$. Therefore (2) follows from Theorem 1.3.1.3. Also, (5) (for any subset $X \subset R$ satisfying the conditions above) follows from (1) or (2) by the nondegeneracy of the trace pairing on $M_d(\bar{k})$. Conversely, if $X \subset R$ satisfies (5), then

$$\dim_{\bar{k}}((R \otimes_k \bar{k})/\ker(D \otimes_k \bar{k})) \geq d^2,$$

and now (5) implies (2) by Theorem 1.3.1.3, since positive integers $n_i, 1 \leq i \leq s$, such that $\sum_1^s n_i = d$ also satisfy $\sum n_i^2 = d^2$ if and only if $s = 1$ and $d = n$.

We have shown that (1), (2), and (5) are equivalent. Because the quotient $R/\mathrm{CH}(D)$ commutes with arbitrary base changes (this is Lemma 1.1.8.6), and a k-algebra R is central

simple of rank d^2 if and only if $R \otimes_k \bar{k}$ is a rank d^2 matrix algebra, we see that (4) $\iff$ (2). Now recall from Lemma 1.2.3.1 that the kernel of the natural surjection

$$R/\mathrm{CH}(D) \longrightarrow R/\ker(D)$$

is nilpotent and equal to the Jacobson radical of $R/\mathrm{CH}(D)$. Clearly (4) implies (3), since the kernel $\ker(D)$ is non-trivial by Lemma 1.1.6.6(2). To complete the proof, we show that (3) implies (5). Since the kernel is stable under separable extensions by Lemma 1.2.3.5 and the central simple algebra $R/\ker(D)$ of finite rank is split by a finite separable extension k'/k, we have that $M_d(k') \cong R/\ker(D) \otimes_k k' \cong (R \otimes_k k')/\ker(D \otimes_k k')$. We can choose $x_1, \ldots, x_{d^2}$ in R to be lifts of a k-basis for $R/\ker(D)$; as this k-basis is also a k'-basis for $(R \otimes_k k')/\ker(D \otimes_k k') \cong M_d(k')$ and $(D \otimes_k k')(t - x_i \otimes 1) = D(t - x_i)$, (5) follows from the nondegeneracy of the trace pairing on $M_d(k')$. $\square$

We can derive from these equivalences the fact that the locus of absolutely irreducible pseudorepresentations is open. First we give a definition.

Definition 1.3.4.2. We write $\mathrm{PsIrr}_R^d \subset \mathrm{PsR}_R^d$ for the subfunctor of PsR_R^d cut out by the following condition: for $B \in \mathrm{Alg}_A$ and $D \in \mathrm{PsR}_R^d(B)$, we say that $D \in \mathrm{PsIrr}_R^d$ provided that for every B-field k, $D \otimes_B k : R \otimes_B k \to k$ is an absolutely irreducible pseudorepresentation.

Corollary 1.3.4.3 (cf. [Che11, Example 2.20]). *The subfunctor* $\mathrm{PsIrr}_R^d \subset \mathrm{PsR}_R^d$ *is Zariski open and therefore representable.*

Proof. We use condition (5) of Definition/Lemma 1.3.4.1: choose $r_1, \ldots, r_{d^2}$ such that (5) holds. This defines a morphism of affine $\mathrm{Spec}\, A$-schemes

$$\mathrm{PsR}_R^d \longrightarrow M_{d^2}$$

$$D \mapsto (\Lambda_1(r_i r_j))_{i,j},$$

and Definition/Lemma 1.3.4.1 tells us that the absolutely irreducible locus is the inverse image of the open subscheme $\mathrm{GL}_{d^2} \subset M_{d^2}$, which is therefore an open subscheme. $\square$

As we will discuss in §2.1.3, the deformation theory of absolutely irreducible pseudorepresentations is especially nice. It amounts to deforming the absolutely irreducible representation associated to it by Theorem 1.3.1.1; this is already suggested by Corollary 1.3.4.3

The next most tractable case for the deformation theory of pseudorepresentations (which we will discuss in §2.1) is the multiplicity free case, which we now define. While "multiplicity free" is defined over any field k by using the base change to the algebraic closure, just like the case for "absolutely irreducible," we will sometimes require that the pseudorepresentation be realizable as the determinant of a matrix algebra-valued representation *over* k. We define the term *split* for this purpose.

Definition/Lemma 1.3.4.4 ([Che11, Definition 2.19]). Given a d-dimensional pseudorepresentation $D : R \to k$, we say that $D : R \to k$ is *multiplicity free* provided that $D \otimes_k \bar{k}$ is the determinant of a direct sum of pairwise non-isomorphic irreducible $\bar{k}$-linear representations. In the notation of Theorem 1.3.1.3, it is equivalent to say that $m_i = q_i = 1$ for each i.

Call D *split* provided that it is induced by the determinant of a representation $R \to M_d(k)$. Equivalently, D is split if and only if $R/\ker(D)$ is a finite product of matrix algebras over k.

Proof. We will prove the equivalence of the definitions of "split." If $R/\ker(D)$ is a finite product of matrix algebras $\prod_1^s M_{n_i}(k)$, then by Proposition 1.1.11.5(2) and Proposition 1.1.7.10, D is a product of powers of the determinants of each $M_{n_i}(k)$, say $D = \oplus \det_{M_{n_i}}^{m_i}$, where $\sum_1^s n_i m_i = d$. If M_i is the representation of R corresponding to $R \to M_{n_i}(k)$, then we can recover D as the determinant of the d-dimensional representation $\oplus M_i^{\oplus m_i}$.

Conversely, assume that $R/\ker(D)$ is not a finite product of matrix algebras. In this case, $R/\ker(D)$ is nonetheless semisimple with additional properties prescribed by Theorem 1.3.1.3: it is a product of simple k-algebras S_i, each of which is of finite dimension n_i^2 over its center k_i, where k_i/k has exponent (f_i, q_i). The k-valued pseudorepresentations of S_i are described in Lemma 1.3.2.3. Using Proposition 1.1.11.5(2) and Lemma 1.3.2.3 in the same way as above, $D = \oplus \det_{S_i}^{m_i}$ for some non-negative integers m_i, and $d = \sum_1^s f_i q_i m_i n_i$. We

note that k_i has separable degree f_i over k, and inseparable degree at least q_i over k. We note that any representation of S_i has dimension at least $f_i q_i n_i$ over k, and this is achieved if and only if S_i is a matrix algebra over k_i. Since at least one S_i is not a matrix algebra by assumption, we see that D cannot possibly be realized as the determinant of a *d-dimensional* sum of representations of the S_i. $\square$

Write $\mathrm{Rep}_R(k)$ for the abelian category of finite-dimensional representations of the k-algebra R over k. To be precise, an object of this category is a finite-dimensional k-vector space V with a k-linear action of R. We give the following definitions in the context of representations of algebras; the second term comes from the theory of quiver representations.

Definition 1.3.4.5. Let $\mathcal{C}$ be an abelian category.

(1) The *Grothendieck group* of $\mathcal{C}$, denoted $K_0(\mathcal{C})$, is the quotient of the free abelian group on the objects of $\mathcal{C}$ by the subgroup generated by exact sequences, i.e. by $[M'] - [M] + [M'']$ where

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence in $\mathcal{C}$.

(2) Assuming that any object of $\mathcal{C}$ has a unique composition series, the *Grothendieck semi-group* is the set of isomorphism classes of semisimple objects of $\mathcal{C}$, with the operation coming from the direct sum of objects.

(3) The *dimension vector* of an object of $\mathcal{C}$ is its image in the Grothendieck group $K_0(\mathcal{C})$.

(4) If any element ρ of $\mathcal{C}$ has a composition series, we consider the dimension vector β_ρ to be a vector with respect to the basis of $K_0(\mathcal{C})$ given by simple objects.

In the case that $\mathcal{C}$ is $\mathrm{Rep}_R(k)$, the finite-dimensional restriction shows that any object has a composition series (the representation factors through a subalgebra of $\mathrm{End}_k(V)$; apply the Hopkins-Levitsky theorem). From this, we deduce that $K_0(\mathrm{Rep}_R(k))$ is generated by the

simple finite-dimensional representations of R over k. We can think of the dimension vector of a representation (or its semisimplification) as a vector with respect to this basis.

Using this basis for $K_0(\mathrm{Rep}_R(\bar{k}))$, one can say that a pseudorepresentation $D : R \to k$ is absolutely irreducible when the associated element of $K_0(\mathrm{Rep}_R(\bar{k}))$ has a single non-zero entry, which is 1. The pseudopresentation is multiplicity free when the corresponding representation has dimension vector with coordinates consisting of 0 and 1.

1.4. Moduli Spaces of Representations

Let S be an affine Noetherian scheme and let R be a finitely generated, not necessarily commutative quasi-coherent $\mathcal{O}_S$-algebra, which amounts to a finitely generated $\Gamma(\mathcal{O}_{\mathrm{Spec}\, S})$-algebra. We consider moduli spaces of representations of R over S-schemes. The Noetherian hypothesis on S will allow for the moduli spaces of representations of R that we will describe below to be Noetherian as well (also cf. Remark 1.1.10.16). We will conclude this section by drawing a morphism from these moduli spaces of representations to their induced pseudorepresentation.

1.4.1. Moduli Schemes and Algebraic Stacks. The following definitions describe the functors and groupoids of representations of R that we will study.

Definition 1.4.1.1. With S and R as above and a positive integer d, define the following S-functors and S-groupoids of d-dimensional representations over an S-scheme X.

(1) Define the functor on S-schemes $\mathrm{Rep}_R^{\square,d}$ by

$$X \mapsto \{\mathcal{O}_X\text{-algebra homomorphisms } R \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow M_d(X).$$

(2) Define the S-groupoid Rep_R^d by

$$\mathrm{ob}\,\mathrm{Rep}_R^d(X) = \{V/X \text{ rank } d \text{ vector bundle},$$
$$\mathcal{O}_X\text{-algebra homomorphism } R \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow \mathrm{End}_{\mathcal{O}_X}(V)\}$$

(3) Define the S-groupoid $\overline{\mathrm{Rep}}_R^d$ by

$$\mathrm{ob}\,\overline{\mathrm{Rep}}_R^d(X) = \{\mathcal{E} \text{ a rank } d^2\mathcal{O}_X\text{-Azumaya algebra,}$$
$$\mathcal{O}_X\text{-algebra homomorphism } R \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow \mathcal{E}\}$$

The functor $\mathrm{Rep}_R^{\square,d}$ is of natural interest, but we will often be interested in studying representations of R up to isomorphism, where isomorphisms come from conjugation. Explicitly, we say that $\rho, \rho' \in \mathrm{Rep}_R^{\square,d}(\mathrm{Spec}\,A)$ are equivalent when there exists some $g \in \mathrm{GL}_d(A)$ such that $\rho = g^{-1} \cdot \rho' \cdot g$. We fix this adjoint action of GL_d or PGL_d on $\mathrm{Rep}_R^{\square,d}$, and we desire a scheme that represents the functor of orbits of this action.

However, the functor sending $\mathrm{Spec}\,A$ to the set of such equivalence classes/orbits – we could say that it is the functor sending an S-scheme X to a free module with an action of R, up to isomorphism – is not representable in general. Projective modules must be allowed in order to put equivalence classes of representations of R into families and still retain representability by an algebraic object. There are two possible strategies that have been explored most. One strategy is to find the S-scheme which does the best possible job, by some standard, in representing the moduli problem up to isomorphism. This approach of "geometric invariant theory" will be discussed in the next section §1.5. Here, we will follow the other approach, which is to remember the data of the isomorphisms between objects, resulting in groupoids fibered over the category of S-schemes that are representable by algebraic stacks. As we will see below (Theorem 1.4.1.4), the groupoids described above will naturally arise as the quotient stacks of the adjoint action.

There is a canonical 1-equivalence to the functor (better, S-setoid) $\mathrm{Rep}_R^{\square,d}$ from the S-groupoid whose fiber over an S-scheme X is the data of a free, rank d $\mathcal{O}_X$-module, a basis, and an $\mathcal{O}_X$-linear action of $R \otimes_{\mathcal{O}_S} \mathcal{O}_X$. Having drawn this equivalence, we observe that there are canonical maps

$$\mathrm{Rep}_R^{\square,d} \longrightarrow \mathrm{Rep}_R^d \longrightarrow \overline{\mathrm{Rep}}_R^d \tag{1.4.1.2}$$

where the first arrow is given by forgetting the basis and retaining the free rank d vector bundle with its action, and the second arrow is given by forgetting the vector bundle and retaining the homomorphism from $R \otimes_{\mathcal{O}_S} \mathcal{O}_X$ into its bundle of endomorphisms. We note that the Azumaya algebras in $\overline{\mathrm{Rep}}_R^d$ are not taken up to equivalence[8], so that they corespond up to isomorphism with PGL_d-torsors, not elements of the Brauer group. In other words, we consider non-trivial but locally isotrivial (i.e. Zariski locally trivializable) Azumaya algebras.

THEOREM 1.4.1.3. *Let S, R, and d be as above. Then the functor $\mathrm{Rep}_R^{\square,d}$ is representable by an affine finite type S-scheme.*

PROOF. Choose a set of generators $r_1, \ldots, r_n$ for R over A. Then we have a morphism of functors

$$\mathrm{Rep}_R^{\square,d} \longrightarrow M_d^n$$

$$\rho \mapsto (\rho(r_1), \rho(r_2), \ldots, \rho(r_n)).$$

Let X be the finite set $X = \{x_1, \ldots, x_n\}$ and let F be the non-commutative quasi-coherent S-algebra freely generated by X. We observe that the map above induces an isomorphism $\mathrm{Rep}_F^{\square,d} \xrightarrow{\sim} M_d^n$. There is a canonical map $F \twoheadrightarrow X$ given by sending $x_i \mapsto r_i$ for each $i, 1 \leq i \leq n$, and let $J \subset F$ be its kernel, which is a two-sided ideal of F. For $f \in J$, consider it as a function $f(x_1, \ldots, x_n)$ of the free variables x_i. There exists a morphism $W^f \in \mathrm{Hom}_{S-\mathrm{schemes}}(M_d^n, M_d)$ corresponding to f, given by sending an n-tuples of $d \times d$-dimensional matrices $(m_1, \ldots, m_n)$ to $f(m_1, \ldots, m_n)$. Let $W_{ij}^f \in \Gamma(\mathcal{O}(M_d^n))$ be the regular function obtained from composing W^f with the projection onto the (i,j)th coordinate of $d \times d$-matrices, and let I_J be the ideal of $\Gamma(\mathcal{O}(M_d^n))$ generated by W_{ij}^f as f varies over elements of J and $1 \leq i, j \leq d$.

We claim that the closed subscheme $V(I_J) \subset M_d^n$ represents the functor $\mathrm{Rep}_R^{\square,d}$ is isomorphic to $\mathrm{Rep}_R^{\square,d}$ under the map above. Clearly we have a monomorphism $\mathrm{Rep}_R^{\square,d} \to V(I_J) \subset M_d^n$, because each of the relations $f \in J$ are sent to zero under the representation. For any affine S-scheme $\mathrm{Spec}\, A$, the map of sets $\mathrm{Rep}_R^{\square,d}(A) \to V(I_J)(A)$ is surjective, since for

[8]Azumaya algebras E_1, E_2 over A are called equivalent if there exist finite rank projective modules V_1, V_2 such that there exists an isomorphism of A modules $E_1 \otimes_A \mathrm{End}_A(V_1) \cong E_2 \otimes_A \mathrm{End}_A(V_2)$. The Azumaya-Brauer group classifies Azumaya A-algebras up to equivalence, cf. [Gro68, §2].

$(m_1, \ldots, m_n) \in V(I_J)(A)$, the A-algebra homomorphism $R \otimes_{\Gamma(\mathcal{O}_S)} A \to M_d(A)$ arising from sending r_i to m_i defines a representation which maps to $(m_1, \ldots, m_n)$. $\square$

We recall that GL_d and PGL_d act on $\mathrm{Rep}_R^{\square,d}$ via the adjoint action, conjugating the matrix coefficients of the representations.

THEOREM 1.4.1.4. *The groupoids* Rep_R^d *and* $\overline{\mathrm{Rep}}_R^d$ *are equivalent to algebraic stacks, in particular the quotient algebraic stacks*

$$\mathrm{Rep}_R^d \cong [\mathrm{Rep}_R^{\square,d}/\mathrm{GL}_d], \quad \overline{\mathrm{Rep}}_R^d \cong [\mathrm{Rep}_R^{\square,d}/\mathrm{PGL}_d].$$

The canonical smooth presentation maps of these quotient stacks

$$\mathrm{Rep}_R^{\square,d} \longrightarrow [\mathrm{Rep}^{\square,d}/\mathrm{GL}_d] \longrightarrow [\mathrm{Rep}_R^{\square,d}/\mathrm{PGL}_d]$$

correspond to the natural maps of groupoids (1.4.1.2).

For the reader's convenience, we recall some equivalent definitions of Azumaya algebras.

DEFINITION 1.4.1.5 ([Gro68, Theorem 5.1]). Let X be a scheme, and let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module which has the structure of a $\mathcal{O}_X$-algebra. Then we say that $\mathcal{E}$ is an *Azumaya algebra* if one of the following equivalent conditions are satisfied.

(1) $\mathcal{E}$ is locally free as a $\mathcal{O}_X$-module, and for every $x \in X$, the fiber $\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ is a central simple algebra.
(2) $\mathcal{E}$ is locally free as a $\mathcal{O}_X$-module, and the canonical homomorphism $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\mathrm{op}} \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ is an isomorphism.
(3) There exists an étale covering $U \to X$ such that $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_U \cong M_d(\mathcal{O}_U)$ for some $d \geq 1$.

Now we prove Theorem 1.4.1.4.

PROOF. The quotient $[\mathrm{Rep}_R^{\square,d}/\mathrm{GL}_d]$ parameterizes, by definition, a GL_d-torsor $\mathcal{G}$ along with a GL_d-equivariant map $\mathcal{G} \to \mathrm{Rep}_R^{\square,d}$. This is what we will create from the data of an $\mathrm{Spec}\, A$-point of Rep_R^d, i.e. the data $(V/A, \rho : R \otimes_{\mathcal{O}_S} A \to \mathrm{End}_A(V))$, where V is a rank

d projective A-module. We create a GL_d torsor over $\operatorname{Spec} A$ corresponding to V, setting its functor of points on $\operatorname{Spec} A$-schemes X to be $\mathcal{G}(X) := \mathrm{Isom}_{\mathcal{O}_X}(V \otimes_A \mathcal{O}_X, \mathcal{O}_X^{\oplus d})$. This defines an equivalence of categories between GL_d-torsors up to isomorphism and rank d locally free sheaves up to isomorphism. The identity map in $\mathcal{G}(\mathcal{G}) = \mathrm{Isom}_{\mathcal{O}_\mathcal{G}}(V \otimes_A \mathcal{O}_\mathcal{G}, \mathcal{O}_\mathcal{G}^{\oplus d})$ is a canonical isomorphism $V \otimes_A \mathcal{O}_\mathcal{G} \xrightarrow{\sim} \mathcal{O}_\mathcal{G}^{\oplus d}$. This defines a $\mathcal{O}_\mathcal{G}$-linear action of $R \otimes_{\mathcal{O}_S} \mathcal{O}_\mathcal{G}$ on the free vector bundle $\mathcal{O}_\mathcal{G}^{\oplus d}$ with its canonical basis, so that we have a map $\mathcal{G} \to \mathrm{Rep}_R^{\square,d}$. It remains to show that this map is GL_d-equivariant. The action of GL_d on $\mathcal{G}$ on the right comes from the standard action of $\mathrm{GL}_d(X)$ on $\mathcal{O}_X^{\oplus d}$. This is effectively the basis change action of GL_d on the map $R \otimes_{\mathcal{O}_S} \mathcal{O}_X \to M_d(X)$, which is the adjoint action. This is an A-point of $[\mathrm{Rep}_R^{\square,d}/\mathrm{GL}_d]$, as desired.

For the inverse construction, we take an A-point of $[\mathrm{Rep}_R^{\square,d}/\mathrm{GL}_d]$, i.e. a GL_d-equivariant map $\mathcal{G} \to \mathrm{Rep}_R^{\square,d}$, and create an object of $\mathrm{Rep}_R^d(A)$. We use the equivalence of categories between vector bundles and GL-torsors mentioned above to find a rank d projective A-module such that $\mathcal{G}(X) \cong \mathrm{Isom}_{\mathcal{O}_X}(V \otimes_A \mathcal{O}_X, \mathcal{O}_X^{\oplus d})$ for all A-schemes X. As $V \otimes_A \mathcal{O}_\mathcal{G}$ is a rank d-free module with a canonical basis as discussed above, we can take our initial data of $R \otimes_{\mathcal{O}_S} \mathcal{O}_\mathcal{G} \to M_d(\mathcal{G})$ and compose it with the canonical map $M_d(\mathcal{G}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_\mathcal{G}}(V \otimes_A \mathcal{O}_\mathcal{G})$, to obtain an action of $R \otimes_{\mathcal{O}_S} \mathcal{O}_\mathcal{G}$ on $V \otimes_A \mathcal{O}_\mathcal{G}$. We leave it as an exercise to show that the GL_d-equivariance of $\mathcal{G} \to \mathrm{Rep}_R^{\square,d}$ is then exactly what we need in order to descend this map to $\operatorname{Spec} A$.

The proof that $\overline{\mathrm{Rep}}_R^d \cong [\mathrm{Rep}_R^{\square,d}/\mathrm{PGL}_d]$ goes along the same lines. We choose a $\operatorname{Spec} A$-point of $\overline{\mathrm{Rep}}_R^d$: a map $R \otimes_{\mathcal{O}_S} A \to E$, where E is a rank d^2 Azumaya A-algebra. We can then create a PGL_d-torsor $\mathcal{G}$ whose X-points for an A-scheme X are $\mathcal{G}(X) := \mathrm{Isom}_{\mathcal{O}_X-\mathrm{alg}}(E \otimes_A \mathcal{O}_X, M_d(X))$, and the action of $\mathrm{PGL}_d(X)$ on $\mathcal{G}(X)$ comes from its adjoint action on $M_d(X)$. Then the identity map $\mathrm{id} \in \mathcal{G}(\mathcal{G})$ corresponds to a canonical isomorphism $E \otimes_A \mathcal{O}_\mathcal{G} \xrightarrow{\sim} M_d(\mathcal{G})$ defining a morphism $\mathcal{G} \to \mathrm{Rep}_R^{\square,d}$, and we observe that the adjoint action on both the source and target make this map PGL_d-equivariant, and therefore an A-point of $[\mathrm{Rep}_R^{\square,d}/\mathrm{PGL}_d]$.

For the inverse construction, from an A-point $\mathcal{G} \to \mathrm{Rep}_R^{\square,d}$ of $[\mathrm{Rep}_R^{\square,d}/\mathrm{PGL}_d]$ we construct a rank d^2 Azumaya A-algebra E so that there is a canonical isomorphism of coherent $\mathcal{O}_{\mathcal{G}}$-algebras $E \otimes_A \mathcal{O}_{\mathcal{G}} \xrightarrow{\sim} M_d \otimes \mathcal{O}_{\mathcal{G}}$. Then the map $R \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathcal{G}} \to M_d(\mathcal{G})$ can be composed with $E(\mathcal{G}) \xrightarrow{\sim} M_d(\mathcal{G})$ to get a map $R \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathcal{G}} \to E \otimes_A \mathcal{O}_{\mathcal{G}}$. The PGL_d-equivariance of $\mathcal{G} \to \mathrm{Rep}_R^{\square,d}$ allows us to descend this map from $\mathcal{G}$ to $\operatorname{Spec} A$.

The claim that the forgetful maps from $\mathrm{Rep}_R^{\square,d}$ and the presentation maps commute with the equivalences we have drawn follows from checking that the universal framed representation over $\mathrm{Rep}_R^{\square,d}$ induces a map to Rep_R^d (resp. $\overline{\mathrm{Rep}}_R^d$) compatible with the universal object on the quotient stack via the correspondence that we have written out above. $\square$

1.4.2. Mapping Algebraic Stacks of Representations to the Moduli Scheme of Pseudorepresentations. Let X be an S-scheme. Having defined these moduli spaces of representations of the $\mathcal{O}_S$-algebra R, we know that the association of an X-valued representation of R, that is, the data

$$(\rho : R \otimes_{\mathcal{O}_S} \mathcal{O}_X \longrightarrow M_d(\mathcal{O}_X)) \in \mathrm{Rep}_R^{\square,d}(X)$$

to an X-valued pseudorepresentation by taking the determinant (see Theorem 1.1.7.4(6) and Remark 1.1.7.6)

$$R \otimes_{\mathcal{O}_S} \mathcal{O}_X \xrightarrow{\rho} M_d(\mathcal{O}_X) \xrightarrow{\det} \mathcal{O}_X$$

defines a morphism of S-schemes

$$\psi^{\square} : \mathrm{Rep}_R^{\square,d} \longrightarrow \mathrm{PsR}_R^d. \tag{1.4.2.1}$$

Think "ψ" for ***ps***eudorepresentation.

We will now show that there is also a natural pseudorepresentation associated to objects of $\mathrm{Rep}_R^d(X)$ and $\overline{\mathrm{Rep}}_R^d(X)$ that is constant across isomorphism classes in the groupoid, so that there are morphisms of algebraic stacks

$$\psi : \mathrm{Rep}_R^d \to \mathrm{PsR}_R^d, \quad \bar{\psi} : \overline{\mathrm{Rep}}_R^d \to \mathrm{PsR}_R^d$$

which commute with the canonical maps (1.4.1.2). Then we will have a commutative diagram

(1.4.2.2)
$$\begin{array}{ccccc} \mathrm{Rep}_R^{\square,d} & \xrightarrow{(1.4.1.2)} & \mathrm{Rep}_R^d & \xrightarrow{(1.4.1.2)} & \overline{\mathrm{Rep}}_R^d \\ & \searrow^{\psi^{\square}} & & \searrow^{\psi} & \downarrow{\scriptstyle \bar{\psi}} \\ & & & & \mathrm{PsR}_R^d \end{array}$$

All that we need to do is construct the vertical arrow $\bar{\psi}$, sending, for X an S-scheme, an Azumaya $\mathcal{O}_X$-algebra-valued representation $R \otimes_{\mathcal{O}_S} \mathcal{O}_X \to E$ to an $\mathcal{O}_X$-valued pseudorepresentation. We will achieve this using the reduced norm map out of any Azumaya algebra, and indeed, the rest of the characteristic polynomial coefficients. We construct these coefficient functions as follows. Each coefficient of the characteristic polynomial defines a regular function $M_d \to \mathbb{A}^1$ which is invariant under the adjoint action of PGL_d. Each Azumaya algebra $\mathcal{E}$ is a form of M_d twisted by this action (cf. [Gro68, Corollary 5.11]); therefore, the characteristic polynomial function descends from $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_U \cong M_d(\mathcal{O}_U)$ to $\mathcal{E}$ over $\mathcal{O}_X$ [Gro68, 5.13].

Now there are at least two perspectives we could take on the pseudorepresentation associated to an object $\rho : R \otimes_{\mathcal{O}_S} \mathcal{O}_S \to \mathcal{E}$ of $\overline{\mathrm{Rep}}_R^d(X)$. We can compose this representation with the reduced norm, which we continue to write as "det" as it is equal to det étale-locally:

$$R \otimes_{\mathcal{O}_S} \mathcal{O}_S \xrightarrow{\rho} \mathcal{E} \xrightarrow{\det} \mathcal{O}_X$$

is compatible with base change, making a pseudorepresentation. Alternatively, as PsR_R^d is a scheme, it is a sheaf on the étale site $S_{\grave{\mathrm{E}}\mathrm{t}}$, so that we can choose an étale cover U of X and descend the pseudorepresentation

$$R \otimes_{\mathcal{O}_S} \mathcal{O}_U \xrightarrow{\rho \otimes \mathcal{O}_U} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_U \cong M_d(\mathcal{O}_U) \xrightarrow{\det} \mathcal{O}_U$$

to a pseudorepresentation over $\mathcal{O}_X$.

In any case, we have completed the construction of the diagram (1.4.2.2).

1.4.3. Representations Factor through the Universal Cayley-Hamilton Algebra. In this paragraph, we will show that our basic assumptions – that R is a finitely generated algebra over an affine Noetherian base – are sufficient to show that the d-dimensional universal representation of R factors through an algebra finite over its center. This is a consequence of the theorem below, which shows that any representation of R factors through the universal d-dimensional Cayley-Hamilton representation associated to R,

$$\rho^u : R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}} \longrightarrow E(R,d)$$

Recall the definition of Cayley-Hamilton representations from §1.2.4. In particular, $E(R,d)$ is a $\Gamma_A^d(R)^{\mathrm{ab}}$-algebra, defined to be $(R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}})/\mathrm{CH}(D^u)$.

THEOREM 1.4.3.1. *Any representation in* $\mathrm{Rep}_R^{\square,d}(B)$ *(resp.* $\mathrm{Rep}_R^d(B)$*, resp.* $\overline{\mathrm{Rep}}_R^d(B)$*) of* R *factors uniquely through the universal Cayley-Hamilton representation*

$$\rho^u \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B : R \otimes_A B \longrightarrow E(R,d) \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B.$$

This factorization induces canonical equivalences of PsR_R^d*-schemes (resp. algebraic stacks)*

$$\mathrm{Rep}_R^{\square,d} \xrightarrow{\sim} \mathrm{Rep}_{E(R,d),D^u|_E}^{\square,d},$$

$$\mathrm{Rep}_R^d \xrightarrow{\sim} \mathrm{Rep}_{E(R,d),D^u|_E}^d,$$

$$\overline{\mathrm{Rep}}_R^d \xrightarrow{\sim} \overline{\mathrm{Rep}}_{E(R,d),D^u|_E}^d,$$

where the left hand side algebraic stacks are considered to be PsR_R^d*-stacks through the map* $\psi^{\square}$ *(resp.* ψ*, resp.* $\bar{\psi}$*).*

The implicit map $\Gamma_A^d(R)^{\mathrm{ab}} \to B$ arises from the determinant (or reduced norm) of the representation, along with the representability result Theorem 1.1.7.4.

REMARK 1.4.3.2. While the theorem has an especially nice consequence when A is assumed to be Noetherian and R is assumed to be finite generated over A, the theorem is true with or without these finiteness assumptions.

PROOF. Let $\operatorname{Spec} B$ be an affine PsR_R^d-scheme (and therefore naturally an A-algebra) and write $\zeta : \operatorname{Spec} B \to \mathrm{PsR}_R^d$ for the structure map, i.e. a choice of a B-valued d-dimensional pseudorepresentation of R. Any object of the $\operatorname{Spec} A$-groupoids $\mathrm{Rep}_R^{\square,d}(B)$, $\mathrm{Rep}_R^d(B)$ induces an Azumaya B-algebra-valued representation $\rho : R \otimes_A B \to \mathcal{E} \in \overline{\mathrm{Rep}}_R^d(B)$ by the forgetful maps (1.4.1.2). The question of the factorization of a representation does not depend on the forgotten data, so it will suffice to prove the result for ρ. So we choose $\rho \in \overline{\mathrm{Rep}}_R^d(B)$, such that

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{\rho} & \overline{\mathrm{Rep}}_R^d \\ & \searrow^{\zeta} & \downarrow{\scriptstyle \bar{\psi}} \\ & & \mathrm{PsR}_R^d \end{array}$$

commutes.

Recall Definition 1.2.4.1, which is the notion of a Cayley-Hamilton representation of R. Following Remark 1.2.4.2, we note that a the data of ρ induces a d-dimensional Cayley-Hamilton representation of R over B, namely

$$(B, (\mathcal{E}, \det), \rho),$$

where $\det : \mathcal{E} \to B$ represents the reduced norm map for the Azumaya B-algebra $\mathcal{E}$.

Proposition 1.2.4.3 shows that the universal d-dimensional Cayley-Hamilton representation $(\Gamma_A^d(R)^{\mathrm{ab}}, (E(R,d), D^u|_E), \rho^u)$ is initial in the category $\mathcal{CH}^d(R)$ of Cayley-Hamilton representations of R. Thus there exists a canonical $\mathcal{CH}^d(R)$-morphism

$$(\Gamma_A^d(R)^{\mathrm{ab}}, (E(R,d), D^u|_E), \rho^u) \longrightarrow (B, (\mathcal{E}, \det), \rho).$$

This includes the datum of a A-morphism $\Gamma_A^d(R)^{\mathrm{ab}} \to B$, corresponding to the pseudorepresentation $\det \circ \rho$ by representability and contravariantly equivalent to ζ. There is also a canonical morphism

$$E(R,d) \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B \to \mathcal{E} \in \overline{\mathrm{Rep}}^d_{E(R,d), D^u|_E}(B),$$

factoring ρ through the canonical quotient map $\rho^u : R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}} \to E(R,d)$.

We have therefore exhibited a PsR_R^d-morphism $\overline{\mathrm{Rep}}_R^d \to \overline{\mathrm{Rep}}_{E(R,d),D^u|_E}^d$.

We can derive a quasi-inverse from ρ^u. Define

$$\eta : E(R,d) \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B \to \mathcal{E} \in \overline{\mathrm{Rep}}_{E(R,d),D^u|_E}(B).$$

We get from η a representation of R, $\eta \circ (\rho^u \otimes B) \in \overline{\mathrm{Rep}}_R^d(B)$. □

When A and R satisfy appropriate finiteness conditions, we know that the universal Cayley-Hamilton algebra, the $\Gamma_A^d(R)^{\mathrm{ab}}$-algebra $E(R,d)$, is finite as a $\Gamma_A^d(R)$ and is a Noetherian ring. Therefore we may show that the representation theory of a finitely generated algebra over a commutative Noetherian ring reduces to the theory of Noetherian (non-commutative) rings that are finite over their Noetherian center.

Corollary 1.4.3.3. *Fix a positive integer d. If A is Noetherian and R is finitely generated as an A-algebra, all of the d-dimensional representations of R factor canonically through an algebra which is finite as a module over its center and Noetherian, namely, each representation factors uniquely through*

$$\rho^u : R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}} \longrightarrow E(R,d).$$

Proof. This follows directly from Theorem 1.4.3.1 along with Corollary 1.2.2.10.

We recapitulate Theorem 1.4.3.1 for clarity. A B-valued d-dimensional representation of R amounts to some map $\rho : R \otimes_A B \to \mathcal{E}$ where $\mathcal{E}$ is a rank d^2 B-Azumaya algebra, possibly with some extra data that we can discard. The induced pseudorepresentation $\bar{\psi}(\rho)$ induces a map $\Gamma_A^d(R)^{\mathrm{ab}} \to B$ by the representability of PsR_R^d. This gives us the B-valued representation of $R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}}$. Then Theorem 1.4.3.1 shows that this representation factors through $\rho^u \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B$.

The rest of the statements follow directly from Corollary 1.2.2.10. Since $E(R,d)$ is an $\Gamma_A^d(R)^{\mathrm{ab}}$-algebra, the center of $E(R,d)$ contains the image of $\Gamma_A^d(R)^{\mathrm{ab}}$ in $E(R,d)$, and $E(R,d)$ is finite as a $\Gamma_A^d(R)^{\mathrm{ab}}$-module, it must also be module-finite over its center. As

noted in Corollary 1.2.2.10, these facts along with the Noetherianness of A imply that R is Noetherian as well. $\square$

REMARK 1.4.3.4. We could prove a version of Corollary 1.4.3.3 with A being a field and demanding that a base pseudorepresentation $D : R \to A$ be fixed. Then the functor of all representations lying over this pseudorepresentation via ψ would factor through the Cayley-Hamilton quotient $R/\mathrm{CH}(D)$ of R relative to D, and Theorem 1.3.3.2 gives conditions for this quotient to be finite dimensional. We will use these ideas later, extending Corollary 1.4.3.3 to the case that R is a profinite algebra satisfying an appropriate finiteness condition (see Theorem 3.2.3.2).

1.4.4. Representations of Groups into Affine Group Schemes. In this paragraph we restrict our attention to representations of group algebras as opposed to general associative algebras, and then generalize this case to representations of a group valued in an arbitrary group scheme.

Let Γ be a finitely generated group. Then $R = \mathcal{O}_S[\Gamma]$ is a finitely generated quasi-coherent $\mathcal{O}_S$-algebra, and the formalism of the above can be repeated. We leave the reader to verify the following basic equivalences, which amount to saying for a ring A that A-valued d-dimensional representations $\rho : \Gamma \to \mathrm{GL}_d(A)$ are equivalent to homomorphisms $A[\Gamma] \to M_d(A)$.

PROPOSITION 1.4.4.1. *Let $R = \mathcal{O}_S[\Gamma]$ and let X represent an S-scheme. Then*

(1) $\mathrm{Rep}_R^{\square,d}$ *is naturally equivalent to the functor*

$$X \mapsto \{\Gamma \longrightarrow \mathrm{GL}_d(X)\}.$$

(2) Rep_R^d *is equivalent to the S-groupoid with objects over X being*

$$\{V/X \text{ a rank } d \text{ vector bundle, } \Gamma \longrightarrow \mathrm{Aut}_{\mathcal{O}_X}(V)(X)\}.$$

and morphisms being isomorphisms of these data.

(3) $\overline{\mathrm{Rep}}_R^d$ *is naturally equivalent to the* S*-groupoid with objects over* X *being*

$$\{H/X \text{ an inner form of } \mathrm{GL}_d, \Gamma \longrightarrow H(X)\}.$$

and morphisms being isomorphisms of these data.

Proof. Omitted. □

We also might be interested in representations of Γ that fix certain tensors, for example, representations valued in Sp_d or SO_d. We will simply let G be an arbitrary finite type flat affine S-group scheme and consider the moduli of representations of Γ into G.

Definition 1.4.4.2. For an abstract group Γ and a finite type flat affine S-group scheme G, we define the following functors and S-groupoids.

(1) Let $\mathrm{Rep}_\Gamma^{\square,G}$ denote the functor on S-schemes X

$$X \mapsto \{\text{homomorphisms } \Gamma \longrightarrow G(X).$$

(2) Define the S-groupoid Rep_d^G by

$$\mathrm{ob}\,\mathrm{Rep}_\Gamma^G(X) = \{\mathcal{G} \text{ a right } G\text{-torsor over } X,$$
$$\Gamma \longrightarrow \mathrm{Aut}_X^G(\mathcal{G})(X)\}.$$

Here $\mathrm{Aut}_X^G(\mathcal{G})$ is the X-group scheme of automorphisms of $\mathcal{G}$, where an automorphism of $\mathcal{G}$ over an X-scheme Y is an endomorphism of the Y-scheme $\mathcal{G} \times_X Y$ which is equivariant for the right action of $G \times_S Y$.

(3) Define the S-groupoid $\overline{\mathrm{Rep}}_d^G$ by

$$\mathrm{ob}\,\overline{\mathrm{Rep}}_\Gamma^G(X) = \{\text{an inner form } H \text{ of } G \text{ over } X,$$
$$\Gamma \longrightarrow H(X)\}.$$

We observe that there are natural maps

$$\mathrm{Rep}_\Gamma^{\square,G} \longrightarrow \mathrm{Rep}_\Gamma^G \longrightarrow \overline{\mathrm{Rep}}_\Gamma^G. \tag{1.4.4.3}$$

To construct the first map, choose a trivial G-torsor $\mathcal{G}$ over S so that

$$\mathrm{Aut}_S^G(\mathcal{G}) \xrightarrow{\sim} G, \tag{1.4.4.4}$$

where the isomorphism follows from the fact that the maps $G \to G$ which are equivariant for the right action of G on itself are precisely the left translations of G on itself. Then the first map is given by sending $(\rho : \Gamma \to G(X)) \in \mathrm{Rep}_\Gamma^{\square,G}(X)$ to the composition

$$\Gamma \longrightarrow G(X) \xrightarrow{\sim} \mathrm{Aut}_S^G(\mathcal{G})(X).$$

The second map is given by forgetting the G-torsor $\mathcal{G}$ over X inducing the inner form $H := \mathrm{Aut}_X^G(\mathcal{G})(X)$, where we see that this is an inner form by the isomorphism, for a trivial G-torsor.

THEOREM 1.4.4.5. *Let Γ, G be as above. Then the functor $\mathrm{Rep}_d^{\square,G}$ is representable by an affine finite type S-scheme.*

PROOF. Choose a set of generators $\gamma_1, \ldots, \gamma_n$ for Γ. Then we have a morphism of functors

$$\mathrm{Rep}_\Gamma^{\square,G} \longrightarrow G^n$$

$$\rho \mapsto (\rho(r_1), \rho(r_2), \ldots, \rho(r_n)).$$

Any word w on the n letters $\gamma_1, \ldots, \gamma_n$ induces a map

$$f_w : G^n \longrightarrow G$$

$$(g_1, \ldots, g_n) \mapsto w(g_1, \ldots, g_n).$$

given by substituting g_i for γ_i. We observe that G^n represents $\mathrm{Rep}_{F_n}^G$, where F_n is the free group on n letters. A representation of F_n valued in A corresponding to a morphism $p : \mathrm{Spec}\, A \to G^n$ induces a representation of Γ if and only if, for every word w in the letters (γ_i) such that $w = \mathrm{id} \in \Gamma$, $f_w \circ p \cong \mathrm{id}_G \times_S \mathrm{Spec}\, A$, where id_G is the identity section $\mathrm{id}_G : S \to G$ of the S-group scheme G. Therefore $\mathrm{Rep}_\Gamma^{\square,G}$ is precisely the intersection over

words w such that $w = \mathrm{id} \in \Gamma$ of the closed subschemes G^w of G^n given by the fiber product

$$\begin{array}{ccc} G^w & \longrightarrow & G^n \\ \downarrow & & \downarrow f_w \\ S & \xrightarrow{\mathrm{id}_G} & G. \end{array}$$

As S is Noetherian and G is finite type over S, so is G^w Noetherian and finite type over S. □

Just as GL_d (or PGL_d) acts on group representations $\Gamma \to \mathrm{GL}_d$ via the adjoint action, so does the adjoint group of G, namely $G/Z(G)$, act on itself by the adjoint action. This gives an action of G and $PG := G/Z(G)$ on $\mathrm{Rep}_\Gamma^{\square,G}$. Also, like before, this is a natural notion of equivalence for the points of $\mathrm{Rep}_\Gamma^{\square,G}$, but the functor of equivalence classes is not representable. The following quotient stacks retain the equivariant geometry of $\mathrm{Rep}_\Gamma^{\square,G}$, and are equivalent to the stacks of representations defined above.

THEOREM 1.4.4.6. *The groupoids* Rep_Γ^G *and* $\overline{\mathrm{Rep}}_\Gamma^G$ *are equivalent to algebraic stacks, in particular the quotient algebraic stacks*

$$\mathrm{Rep}_\Gamma^G \cong [\mathrm{Rep}_\Gamma^{\square,G}/G], \quad \overline{\mathrm{Rep}}_\Gamma^G \cong [\mathrm{Rep}_\Gamma^{\square,G}/PG].$$

The canonical flat presentation maps of these quotient stacks

$$\mathrm{Rep}_\Gamma^{\square,G} \longrightarrow [\mathrm{Rep}^{\square,G}/G] \longrightarrow [\mathrm{Rep}_\Gamma^{\square,G}/PG]$$

correspond to the natural maps of groupoids (1.4.4.3).

PROOF. Let $\mathcal{G}^1$ be a right $\mathcal{G}$-torsor over an S-scheme X, equipped with a group homomorphism $\Gamma \to \mathrm{Aut}_X^G(\mathcal{G}^1)(X)$, where we use the superscript to denote various copies of the same G-torsor. We wish to induce from this data a G-equivariant map $\mathcal{G}^2 \to \mathrm{Rep}_\Gamma^{\square,G}$. We know that $\mathcal{G}^2 \to X$ trivializes $\mathcal{G}^1$ via the map

$$\begin{aligned} G \times_X \mathcal{G}^2 &\xrightarrow{\sim} \mathcal{G}^1 \times_X \mathcal{G}^2 \\ (g, x) &\mapsto (xg, x), \end{aligned} \tag{1.4.4.7}$$

so we have a map

$$\Gamma \longrightarrow \mathrm{Aut}^G_{\mathcal{G}^2}(\mathcal{G}^1 \times_X \mathcal{G}^2)(\mathcal{G}^2) \xrightarrow{\sim} G_{\mathcal{G}^2}(\mathcal{G}^2) \tag{1.4.4.8}$$

inducing a map $\mathcal{G}^2 \to \mathrm{Rep}^{\square,G}_\Gamma$. Now we wish to show that (1.4.4.8) is G-equivariant for the standard right action of G on the left and the adjoint action of G on the right.

The right action of $g' \in G$ on $\mathcal{G}^2$ on the right side of (1.4.4.7) sends

$$(xg, x) \mapsto (xg, xg') = (xg'g'^{-1}g, xg'),$$

and therefore acts on the left side of (1.4.4.7) by

$$(g, x) \mapsto (g'^{-1}g, xg').$$

so its action on $G_{\mathcal{G}^2}$ is the right action by multiplication on the left by the inverse. Now we need to consider $G_{\mathcal{G}^2}$ as a trivial right G-torsor and calculate the induced intertwining action on the functor of automorphisms of $G_{\mathcal{G}^2}$ as a torsor. These automorphisms are precisely the left translations by $g'' \in G$, $g \mapsto g''g$. The intertwining action of $g' \in G$ on this map is then

$$(g, x) \mapsto (g'g, xg'^{-1}) \mapsto (g''g'g, xg'^{-1}) \mapsto (g'^{-1}g''g'^{-1}g, x),$$

which is the adjoint action, as desired.

For the inverse construction, we start with a G-equivariant map from a G-torsor $\mathcal{G}^2$ over an S-scheme X with a G-equivariant map $\mathcal{G}^2 \to \mathrm{Rep}^{\square,G}_\Gamma$. By definition of $\mathrm{Rep}^{\square,G}_\Gamma$, there exists a homomorphism

$$\Gamma \longrightarrow G(\mathcal{G}^2).$$

As $\mathcal{G}^2/X$ is trivialized by $\mathcal{G}^1 \to X$, we can fix an isomorphism $\mathrm{Aut}^G_{\mathcal{G}^1}(\mathcal{G}^2 \times_X \mathcal{G}^1) \cong G_{\mathcal{G}_1}$, and replace G with this expression in the homomorphism above. We leave it as an exercise to check that the G-equivariance of $\mathbb{G}^2 \to \mathrm{Rep}^{\square,G}_\Gamma$ is exactly what we need in order to descend the automorphisms of $\mathcal{G}^2 \times_X \mathcal{G}^1$ from $\mathcal{G}^1$ to X. $\square$

1.5. Geometric Invariant Theory of Representations

In the previous section, we defined the affine, finite type S-scheme $\mathrm{Rep}_R^{\square,d}$ of d-dimensional representations of the quasi-coherent, finitely generated $\mathcal{O}_S$-algebra R. After making note of the natural equivalence relation of conjugation, we defined the algebraic quotient S-stacks arising from this action. These algebraic stacks have a clear, explicit description as an S-groupoid. In this section, we will study an alternative approach, using geometric invariant theory to find the "best possible" S-scheme to stand in for a quotient of $\mathrm{Rep}_R^{\square,d}$. Geometric invariant theory (GIT) was originally developed by Mumford (see e.g. [Mum65]). We will first describe Alper's theory of adequate moduli spaces [Alp10], which summarizes and generalizes the results of geometric invariant theory in a way that will be useful for our purposes, as describes nicely the relationship between the quotient stack and the GIT quotient scheme via the canonical projection morphism.

1.5.1. Alper's Theory of Adequate Moduli Spaces. Say that an affine algebraic group G acts on a finite type affine scheme $X = \operatorname{Spec} A$ over a field k. The GIT quotient scheme, which we will denote $X//G$, is the spectrum of the invariant regular functions on X. That is, $X//G := \operatorname{Spec} A^G$, where $A^G \subset A$ is the k-subalgebra of A of co-invariants of the co-action $A \to \mathcal{O}[G] \otimes_k A$. We have a natural map $X \to X//G$. When G is reductive, Mumford's theory implies that $X//G$ is finite type over $\operatorname{Spec} k$ and the map $X \to X//G$ has appropriate universal properties of the quotient of a group action. The finite type property of the quotient is not necessarily true when G is not reductive [Nag60]. Now we turn our interest toward the relationship of the quotient stack $[X/G]$ to the GIT quotient scheme through the canonical morphism $\phi : [X/G] \to X//G$. We write $\mathcal{X} := [X/G]$ for short.

As Alper notes [Alp10, p. 2], ϕ can be checked to have the special properties

(1) For any surjection of quasi-coherent $\mathcal{O}_\mathcal{X}$-algebras $\mathcal{A} \to \mathcal{B}$ and section $t \in \Gamma(\mathcal{X}, \mathcal{B})$, there exists an integer $N > 0$ and a section $s \in \Gamma(\mathcal{X}, \mathcal{A})$ such that $s \mapsto t^N$.

(2) $A^G \to \Gamma(\mathcal{O}_\mathcal{X})$ is an isomorphism.

Slight extension of these properties to apply locally on non-affine spaces give the definitional conditions for ϕ to be an adequate moduli space. As Alper puts it, "it turns out that properties (1) and (2) capture the stack-intrinsic properties of such GIT quotient stacks $[X/G]$ and that these properties alone suffice to show that the quotient $X//G$ inherits nice geometric properties" [Alp10, p. 2].

Definition 1.5.1.1 ([Alp10]). A quasi-compact and quasi-separated morphism $\phi : \mathcal{X} \to Y$ from an algebraic stack to an algebraic space is an *adequate moduli space* if the following two properties are satisfied:

(1) For every surjection of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras $\mathcal{A} \twoheadrightarrow \mathcal{B}$ and every étale morphism $p : U = \operatorname{Spec} A \to Y$ and section $t \in \Gamma(U, p^*\phi_*\mathcal{B})$ there exists $N > 0$ and a section $s \in \Gamma(U, p^*\phi_*\mathcal{A})$ such that $s \mapsto t^N$, and

(2) $\mathcal{O}_Y \to \phi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

The first property is called "adequately affine," and indeed, any quasi-compact, quasi-separated map of algebraic spaces that is adequately affine is affine [Alp10, Theorem 4.3.1], generalizing Serre's criterion for affineness (which is the same condition with $N = 1$). In sum, we require the following notions of adequacy.

Definition/Lemma 1.5.1.2. Let $A \to B$ be a homomorphism of rings. Let $X \to Y$ be a morphism of algebraic spaces.

(1) We call $A \to B$ *adequate* if for all $b \in B$, there exists some $N > 0$ and $a \in A$ such that $a \mapsto b^N$.

(2) We call $A \to B$ *universally adequate* if for all A-algebras A', $A' \to A' \otimes_A B$ is adequate.

(3) We call $X \to Y$ an *adequate homeomorphism* if its is an integral, universal homeomorphism which is a local isomorphism at all points with a residue field of characteristic zero. In particular, $\operatorname{Spec} B \to \operatorname{Spec} A$ is an adequate homeomorphism if and only if

   (a) $\ker(A \to B)$ is locally nilpotent (i.e. every element is nilpotent),

(b) $\ker(A \to B) \otimes \mathbb{Q} = 0$, and

(c) $A \to B$ is universally adequate.

PROOF. The "if and only if" statement is [Alp10, Proposition 3.3.5(2)]. □

We will be interested in adequate moduli spaces that arise from the conventional GIT setting, where a reductive group scheme acts on a scheme.

EXAMPLE 1.5.1.3 ([Alp10, Theorem 9.1.4]). Let S be an affine scheme and let $X = \operatorname{Spec} A$ be an affine S-scheme. Let G be a reductive group S-scheme with an action on X. Then

$$\phi : [X/G] \longrightarrow \operatorname{Spec} A^G$$

is an adequate moduli space.

Here is Alper's main theorem on adequate moduli spaces.

THEOREM 1.5.1.4 ([Alp10, Main Theorem]). *Let $\phi : \mathcal{X} \to Y$ be an adequate moduli space. Then*

(1) ϕ is surjective, universally closed, and universally submersive.

(2) Two geometric points $x_1, x_2 \in \mathcal{X}(\bar{k})$ are identified in Y if and only if their closures $\overline{\{x_1\}}$ and $\overline{\{x_2\}}$ in $\mathcal{X} \times_{\mathbb{Z}} \bar{k}$ intersect.

(3) If $Y' \to Y$ is any morphism of algebraic spaces, then $\mathcal{X} \times_Y Y' \to Y'$ factors as an adequate moduli spaces $\mathcal{X} \times_Y Y' \to \widetilde{Y}$ followed by an adequate homeomorphism $\widetilde{Y} \to Y'$.

(4) Suppose $\mathcal{X}$ is finite type over a Noetherian scheme S. Then Y is finite type over S and for every coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, $\phi_*\mathcal{F}$ is coherent.

(5) ϕ is universal for maps from $\mathcal{X}$ to algebraic spaces which are either locally separated or Zariski-locally have affine diagonal.

REMARK 1.5.1.5. We note that adequate moduli spaces $\phi : \mathcal{X} \to Y$ share particular similarities with both affine morphisms of schemes and proper morphisms of schemes. Indeed, an adequate moduli space is adequately affine (part (1) of Definition 1.5.1.1), and as we noted

above, a quasi-compact, quasi-separated morphism of algebraic spaces is adequately affine if and only if it is affine. On the other hand, ϕ is universally closed and ϕ_* preserves coherent sheaves, which are characteristics of proper morphisms. Since a morphism of schemes that is both affine and proper is finite, we expect ϕ to behave somewhat like a finite morphism and moreover, by part (2) of the Definition 1.5.1.1, like an isomorphism! The obstruction to being an isomorphism is the lack of representability and the accompanying lack of separatedness (fact: a quotient stack of a separated scheme is separated if and only if all stabilizers are finite). This "isomorphism up to lack of representability" property is encapsulated more precisely in part (5) of the theorem (1.5.1.4).

REMARK 1.5.1.6. One important notion from geometric invariant theory that will be used in the sequel is the following two facts about orbits (of geometric points) of the action of a reductive group on a scheme. Working over an algebraically closed field, let a reductive group G act on an variety X, which for simplicity we assume to be affine. Because $X = \operatorname{Spec} A$ is affine, every orbit is semistable. The standard fact from geometric invariant theory is that every semistable orbit contains a unique *closed* semistable orbit (one can get this by combining Example 1.5.1.3 and part (2) of the theorem above). Now, obviously an invariant regular function on X must remain constant along an orbit. Moreover, it must remain constant along an orbit's closure, since regular functions are "continuous." This means that invariant regular functions cannot distinguish orbits whose closures overlap! It turns out that the geometric points of $X/\!/G := \operatorname{Spec} A^G$ are in bijective correspondence with the orbits of G in X modulo the equivalence relation of overlapping closure. This is what part (5) of Theorem 1.5.1.4 expresses.

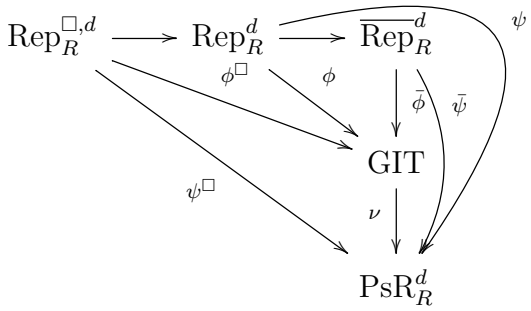
1.5.2. Geometric Invariant Theory on $\mathrm{Rep}^{\square}$. Example 1.5.1.3 shows that in the classical setting of geometric invariant theory, where a reductive group G acts on an affine scheme X, the resulting morphism $[X/G] \to X/\!/G$ is an adequate moduli space. By Theorem 1.4.1.4, the algebraic stacks Rep_R^d (resp. $\overline{\mathrm{Rep}}_R^d$) are quotient stacks for the adjoint action of GL_d (resp. PGL_d) on the finite type affine S-scheme $\mathrm{Rep}_R^{\square,d}$. Therefore, as $\mathrm{Rep}_R^{\square,d}$ is an affine

scheme and $\mathrm{Rep}_R^{\square,d}//\mathrm{GL}_d \cong \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$, each of the morphisms

$$\begin{aligned} \phi &: \mathrm{Rep}_R^d \longrightarrow \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d, \\ \bar{\phi} &: \overline{\mathrm{Rep}}_R^d \longrightarrow \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d \end{aligned} \tag{1.5.2.1}$$

are adequate moduli spaces. Therefore by the universality of the GIT quotient scheme for maps to separated schemes (Theorem 1.5.1.4(5)) we can canonically factor the diagram (1.4.2.2) to get a diagram

(1.5.2.2)



where GIT stands in for the GIT quotient scheme $\mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$. To put these ideas in words, the maps $\psi^{\square}, \psi, \bar{\psi}$ of (1.4.2.2) factor uniquely through the GIT quotient.

Remark 1.5.2.3. One shortcoming of the GIT quotient is that despite the concrete moduli problem that $\mathrm{Rep}_R^{\square,d}$ and the other moduli stacks solve, this does not lend us a complete description of the GIT quotient in terms of a "functor of points." Its one universal property is that of Theorem 1.5.1.4(5), but this characterizes morphisms *out* of it instead of its functor of points. However, we do know the "functor of *geometric* points" of the GIT quotient, following Remark 1.5.1.6: geometric points of a GIT quotient of an affine scheme correspond to closed orbits of geometric points. In the next paragraph, we will discover what these closed orbits in $\mathrm{Rep}_R^{\square,d}$ are in terms of its moduli problem. But we emphasize that this is a property of $\mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$ and not a characterization, since the geometric points of a scheme do not characterize it. For another, related shortcoming of GIT quotients, see Remark 1.5.4.4.

As noted in the introduction, one of the main ideas behind pseudorepresentations is to serve as a concrete (i.e. a moduli problem) replacement for the GIT quotient $\mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$. We will therefore be very interested in the map

$$\nu : \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d \longrightarrow \mathrm{PsR}_R^d, \tag{1.5.2.4}$$

which we expect to be nearly an isomorphism (see Theorem 1.5.4.2).

1.5.3. Work of Kraft, Richardson, et al. on Orbits of the Adjoint Action on Representations. In this paragraph we describe the geometric points of the GIT quotient scheme $\mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$ of the adjoint action of PGL_d on $\mathrm{Rep}_R^{\square,d}$. As we noted in Remark 1.5.1.6 following Theorem 1.5.1.4(2), these geometric points correspond naturally and bijectively to closed orbits in $\mathrm{Rep}_R^{\square,d}$, or, equivalently, closed geometric points in Rep_R^d (resp. $\overline{\mathrm{Rep}}_R^d$). So what are the closed geometric points in Rep_R^d?

Kraft [Kra82] answered this question, proving the following

THEOREM 1.5.3.1 ([Kra82, §II.4.5, Proposition]). *As usual, let R be a finitely generated quasi-coherent $\mathcal{O}_S$-algebra where S is an affine Noetherian scheme. For any algebraically closed S-field $\bar{k}$, the following equivalent statements are true.*

1. *The closed orbits of $\mathrm{PGL}_d(\bar{k})$ in $\mathrm{Rep}_R^{\square,d}(\bar{k})$ are precisely the orbits of semisimple d-dimensional representations of $R \otimes_{\mathcal{O}_S} \bar{k}$.*
2. *The closed geometric points of Rep_R^d (resp. $\overline{\mathrm{Rep}}_R^d$) are in natural bijective correspondence with isomorphism classes of semisimple d-dimensional representations of $R \otimes_{\mathcal{O}_S} \bar{k}$.*
3. *The geometric points of the GIT quotient affine scheme $\mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$ are in natural bijective correspondence with the semisimple d-dimensional representations of $R \otimes_{\mathcal{O}_S} \bar{k}$.*

This result uses the Hilbert-Mumford criterion.

REMARK 1.5.3.2. This theorem implies that the canonical map $\nu : \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d \longrightarrow \mathrm{PsR}_R^d$ of (1.5.2.4) induces a bijection on geometric points! We will take up this point in

the following paragraph, spending this paragraph on the proof of Theorem 1.5.3.1 and its analogue Theorem 1.5.3.7, which addresses $\mathrm{Rep}_\Gamma^{\square,G}$ in place of $\mathrm{Rep}_R^{\square,d}$.

PROOF. Part (1) is due to Kraft [Kra82, §II.4.5, Proposition]. The equivalence of (1) with (2) and (3) follows from Remark 1.5.1.6. □

Richardson [Ric88] answered this question in the case that R is a group algebra; in fact, his proof addresses representations of a finitely generated group Γ into a reductive group G (see the setup for these representation moduli schemes/stacks in §1.4.4), with $G = \mathrm{GL}_d$ as a special case. The techniques of his proof were improved by several people, with notable contributions (for our purposes) of Serre [Ser05] (following [Ser98, Part II]) and Bate-Martin-Röhrle [BMR05]. These are the results that we now overview. They can be summarized in brief by saying that the closed orbits of the adjoint action of G on $\mathrm{Rep}_\Gamma^{\square,G}$ over an algebraically closed field (or, equivalently, the closed geometric points in Rep_Γ^G or $\overline{\mathrm{Rep}}_\Gamma^G$) are in natural bijective correspondence with "semisimple" representations. Of course, we must say what semisimple means in G.

We work over an algebraically closed field $\bar{k}$.

DEFINITION 1.5.3.3 ([Ser05]). A subgroup $H \subset G(\bar{k})$ is called *G-completely reducible* provided that whenever H is contained in some parabolic subgroup P of G, it is contained in a Levi subgroup of P.

This generalizes the familiar case from GL_d: if $H \subset \mathrm{GL}(V)$, then V is a semisimple H-module if and only if H is $\mathrm{GL}(V)$-completely reducible. By the same token, we give a notion of semisimplicity for a reductive group-valued homomorphism.

DEFINITION 1.5.3.4. Let G be a reductive group over an algebraically closed field $\bar{k}$, and let Γ be a group. We say that a homomorphism $\rho : \Gamma \to G(\bar{k})$ is *semisimple* if $\rho(G)$ is G-completely reducible.

One would hope that a result analogous to Theorem 1.5.3.1 can be proved with G in place of GL_d. The basic problem is that in positive characteristic, a reductive subgroup of a

reductive group may not be semisimple. Richardson proved a result to this effect, overcoming problems in positive characteristic, although it was originally proved with the notion of a strongly reductive subgroup in place of a completely reducible subgroup.

Definition 1.5.3.5 ([Ric88, Definition 16.1]). Let G be a reductive group over an algebraically closed field. Let H be a closed subgroup of G and let S be a maximal torus of $C_G(H)$, the centralizer in G of H. We call H a *strongly reductive* subgroup of G provided that H is not contained in any proper parabolic subgroup of $C_G(S)$.

Richardson's definition is set up in order to apply geometric invariant theory – in particular, the Hilbert-Mumford numerical criterion – to show that the closed orbits of the adjoint action on G (or PG) on $\mathrm{Rep}_\Gamma^{\square,G}$ correspond to strongly reductive subgroups. Here, the subgroup in question is the closure of the image of the representation. It was more recently proved that strong reductivity is the same as complete reducibility.

Theorem 1.5.3.6 ([BMR05, Theorem 3.1]). *Let G be a reductive algebraic group over an algebraically closed field $\bar{k}$, and let H be a closed algebraic subgroup. Then H is G-completely reducible if and only if H is strongly reductive in G.*

From this, the desired result follows.

Theorem 1.5.3.7. *As usual, let Γ be a finitely generated group and let G be a reductive S-group scheme. For any algebraically closed S-field $\bar{k}$, the following equivalent statements are true.*

(1) The closed orbits of $G(\bar{k})$ (resp. $PG(\bar{k})$) in $\mathrm{Rep}_\Gamma^{\square,G}(\bar{k})$ are precisely the orbits of semisimple representations $\Gamma \to G(\bar{k})$.

(2) The closed geometric points of Rep_Γ^G (resp. $\overline{\mathrm{Rep}}_\Gamma^G$) are in natural bijective correspondence with isomorphism classes of semisimple representations $\Gamma \to G(\bar{k})$.

(3) The geometric points of the GIT quotient affine scheme $\mathrm{Rep}_\Gamma^{\square,G}//PG$ are in natural bijective correspondence with semisimple representations $\Gamma \to G(\bar{k})$.

PROOF. Richardson proved that the n-tuples of geometric points of G whose orbit under the adjoint action of G or PG is closed are precisely those n-tuples whose generated subgroup of G is strongly reductive [Ric88, Theorem 16.4]. This is equivalent to statement (1) by Theorem 1.5.3.6. The equivalence of (1) with (2) and (3) is clearly, in light of Remark 1.5.1.6. □

1.5.4. The GIT quotient and PsR_R^d are Almost Isomorphic. In this paragraph, we will show that the canonical map $\nu : \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d \to \mathrm{PsR}_R^d$ is a finite universal homeomorphism. Another name for finite universal homeomorphisms is "almost isomorphisms," so we will be able to say that the two schemes are almost isomorphic. This reduces the question of the difference between the GIT quotient and PsR_R^d to a local question; will will take up this question locally in Chapter 2.

From the previous paragraph, we know that the geometric points of the GIT quotient by the adjoint action $\mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$ are in natural bijective correspondence with isomorphism classes of semisimple representations of R. This naturality of this bijection refers to the canonical map $\phi : \mathrm{Rep}_R^d \to \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$ (resp. $\bar{\phi} : \mathrm{Rep}_R^d \to \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$) of (1.5.2.1), as each geometric fiber of ϕ (resp. $\bar{\phi}$) has a unique closed geometric point (cf. Remark 1.5.1.6) corresponding to a semisimple representation.

The map $\psi : \mathrm{Rep}_R^d \to \mathrm{PsR}_R^d$ (resp. $\bar{\psi} : \overline{\mathrm{Rep}}_R^d \to \mathrm{PsR}_R^d$) has a similar property: by Theorem 1.3.1.1, the geometric points of PsR_R^d are in natural bijective correspondence with isomorphism classes of semisimple representations, meaning that there is a unique semisimple point in each geometric fiber of ψ (resp. $\bar{\psi}$). As ψ (resp. $\bar{\psi}$) factors uniquely through $\nu : \mathrm{Rep}_R^d//\mathrm{PGL}_d$ in (1.5.2.2), ν is an isomorphism on geometric points.

From this we know that ν is finite type, radicial, and surjective. We recall this and a few other useful basic definitions and properties.

DEFINITION 1.5.4.1 ([Gro60, Definition 3.5.4]). Let $f : X \to Y$ be a morphism of schemes.

(1) We call f *radicial* or, equivalently, *universally injective*, if for all fields k, the induced map of sets $X(k) \to Y(k)$ is injective. As remarked in [Gro60, §3.5.5], it suffices to verify this property on algebraically closed fields.

(2) We call f a *universal homeomorphism* if after any base change by $Y' \to Y$, $f_{Y'}$ is a homeomorphism. By [Gro67, Corollary 18.12.11], f is a universal homeomorphism if and only if it is integral, radicial, and surjective.

(3) We call f an *almost isomorphism* if f is a finite universal homeomorphism, or, equivalently, if f is a finite type universal homeomorphism.

To verify that ν is a universal homeomorphism, it remains to show that it is integral, or, equivalently, finite. This is what we show in the following

THEOREM 1.5.4.2. *Let S be an affine Noetherian scheme, and let R be a quasi-coherent finitely generated $\mathcal{O}_S$-algebra. The map $\nu : \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d \to \mathrm{PsR}_R^d$ induced by ψ or $\bar{\psi}$ is a finite universal homeomorphism.*

REMARK 1.5.4.3. We must remark that there are well known facts that can be immediately applied to improve this theorem. Indeed, Chenevier has generalized in [Che11, Theorem 2.22(i)] (which we record below in Theorem 2.1.3.3) a result of Nyssen [Nys96] and Rouquier [Rou96], showing that deforming an absolutely irreducible pseudorepresentation (recall Defintiion/Lemma 1.3.4.1) is equivalent to deforming the associated absolutely irreducible representation. This shows that ν is an isomorphism over this locus; this is already visible in the study of the locus that we already did in §1.3.4. However, we are deferring this local study of pseudorepresentations to Chapter 2. These results on absolutely irreducible pseudorepresentations will be discussed in §2.1.3. The full extent of what we prove, which extends the result of Chenevier to the multiplicity free case, may be found in Theorem 2.3.3.7.

We thank Brian Conrad for comments leading to this remark.

REMARK 1.5.4.4. Let us remark on the basic difficulty in making Theorem 1.5.4.2 more precise than it currently is. That is, why is ν hard to control? One major issue is that

GIT quotients are not stable under base change. For example, if one could prove that ν had geometrically reduced fibers, then by [Gro67, Corollaire 18.12.6], ν is a closed immersion. Therefore we might take a geometric fiber of ν and ψ at a point $\bar{D} \in \mathrm{PsR}_R^d(\bar{k})$,

$$\begin{array}{ccc}
\mathrm{Rep}_R^{\square,d} \times_{\mathrm{PsR}_R^d} \operatorname{Spec} \bar{k} & \longrightarrow & \mathrm{Rep}_R^{\square,d} \\
\downarrow & & \downarrow \\
\overline{\mathrm{Rep}}_R^d \times_{\mathrm{PsR}_R^d} \operatorname{Spec} \bar{k} & \longrightarrow & \overline{\mathrm{Rep}}_R^d \\
\downarrow & & \downarrow \\
\mathrm{Rep}_R^{\square,d} /\!/ \mathrm{PGL}_d \times_{\mathrm{PsR}_R^d} \operatorname{Spec} \bar{k} & \longrightarrow & \mathrm{Rep}_R^{\square,d} /\!/ \mathrm{PGL}_d \\
\downarrow & & \downarrow \\
\operatorname{Spec} \bar{k} & \longrightarrow & \mathrm{PsR}_R^d.
\end{array}$$

Then we would want to draw conclusions about this fiber of ν by studying the fiber of ψ or $\psi^{\square}$. We could study the invariants of the action of PGL_d on the upper left entry. However, non-flat base change of an adequate moduli space is no longer an adequate moduli space, but may differ from an adequate moduli space by an adequate homeomorphism [Alp10, Proposition 5.2.9(3)]. Adequate homeomorphisms are not necessarily reduced, so we are unable to conclude anything about the fiber of ν over $\bar{D}$ by considering the fiber of ψ or $\psi^{\square}$ over $\bar{D}$. In other words, the GIT quotient is not stable under base change, and the base change of a GIT quotient may differ from the GIT quotient of a base change by an adequate homeomorphism.

Remark 1.5.4.5. We expect that it follows from Procesi's solution of the "embedding problem" for Cayley-Hamilton algebras[9] in characteristic zero [Pro87] that we can show that ν is not only an finite universal homeomorphism, but an adequate homeomorphism. The additional content required is that ν is an isomorphism in characteristic zero.

[9]The embedding problem for Cayley-Hamilton algebras (R, D) is the problem of finding an *embedding* ρ of R into a matrix algebra M which is compatible with the native pseudorepresentation D, i.e. ρ induces a morphism in $\mathcal{CH}_d(R)$ over D, i.e. $D = \det \circ \rho$.

PROOF. As noted above, Kraft's result (Theorem 1.5.3.1) implies that ν is surjective and radicial. It is also finite type, since the source and target of ν are each finite type over the Noetherian scheme S by Theorems 1.5.1.4(4) and 1.1.10.15, respectively. Then by [Gro67, Corollary 18.12.11], if in addition ν is finite, then ν is a finite universal homeomorphism. We will actually check that ν is universally closed in order to show that it is finite; this will suffice because ν is clearly affine.

To show that ν is proper, we verify the valuative criterion for universal closedness. Using our knowledge that this morphism is separated and finite type between Noetherian schemes, [LMB00, Theorem 7.10] (see also [Gro61a, Remark 7.3.9(i)]) allows us to verify the following valuative criterion on spectra of complete discrete valuation rings B with algebraically closed residue fields: for every diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \operatorname{Rep}_R^{\square,d}//\mathrm{PGL}_d \\ \downarrow & & \downarrow \nu \\ \operatorname{Spec} B & \longrightarrow & \mathrm{PsR}_R^d \end{array}$$

where K is the fraction field of B, there exists a field extension $\operatorname{Spec} K' \to \operatorname{Spec} K$ which is the fraction field of a valuation ring B' with a dominant map $\operatorname{Spec} B' \to \operatorname{Spec} B$ such that there exists a section

$$\begin{array}{ccccc} \operatorname{Spec} K' & \longrightarrow & \operatorname{Spec} K & \longrightarrow & \operatorname{Rep}_R^{\square,d}//\mathrm{PGL}_d \\ \downarrow & \nearrow & \downarrow & & \downarrow \nu \\ \operatorname{Spec} B' & \longrightarrow & \operatorname{Spec} B & \longrightarrow & \mathrm{PsR}_R^d \end{array} \tag{1.5.4.6}$$

In fact, we will achieve this where K'/K is a finite field extension and B' is the integral closure of B in K'.

Let D denote the pseudorepresentation of R over K associated to the K-point of PsR_R^d induced by the B-point induced above, and let D_B denote the underlying B-valued pseudorepresentation, so that $D = D_B \otimes_B K$. Corollary 1.3.2.4(1) and its proof gives us a semisimple representation $\rho : R \otimes_{\mathcal{O}_S} K' \to M_d(K')$ in $\operatorname{Rep}_R^{\square,d}(K')$ where K'/K is a finite extension of

fields, and whose induced pseudorepresentation $D_{K'} := \det \circ \rho$ appears as the base change from K to K' of the pseudorepresentation D. Sending the K'-point $\rho \in \mathrm{Rep}_R^{d,\square}(K')$ via $\phi^\square$ to an K'-point of $\mathrm{Rep}_R^{d,\square}//\mathrm{PGL}_d$, this K'-point lies over the K-point D of $\mathrm{Rep}_R^{d,\square}//\mathrm{PGL}_d$ given in the data of the valuative criterion above. Taking B' to be the integral closure of B in K', we now have all of the maps of (1.5.4.6) except the desired diagonal section.

We claim that ρ is conjugate to the tensor by $\otimes_{B'} K'$ of a representation $\rho_{B'} : R \otimes_{\mathcal{O}_S} B' \to \mathrm{End}_{B'}(L')$, where L' is a rank d projective B'-module. The projection of this B'-point of Rep_R^d to $\mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$ via ϕ is the desired section in (1.5.4.6). Therefore, proving the claim will complete the proof of the theorem.

To prove the claim, first note that the $\mathcal{O}_S$-algebra homomorphism $R \to M_d(K')$ induced by ρ factors through the Cayley-Hamilton algebra $R \to (R \otimes_{\mathcal{O}_S} K)/\mathrm{CH}(D)$ by Proposition 1.2.4.3. Moreover, since D is induced by $\otimes_B K$ from $D_B \in \mathrm{PsR}_R^d(B)$, it factors through $R \to R \otimes_{\mathcal{O}_S} B/\mathrm{CH}(D_B)$, i.e. this map lies in the composite

$$R \longrightarrow (R \otimes_{\mathcal{O}_S} B)/\mathrm{CH}(D_B) \longrightarrow M_d(K').$$

By Corollary 1.2.2.9, the B-algebra $(R \otimes_{\mathcal{O}_S} B)/\mathrm{CH}(D_B)$ is finite as a B-module.

Choose a d-dimensional K'-vector space V' and choose a basis in order to draw an isomorphism $M_d(K') \xrightarrow{\sim} \mathrm{End}_{K'}(V')$. Also choose a rank d B'-lattice $L \subset V'$, where B' is the integral closure of B in the finite extension K'/K. Now let L' be the B'-linear span of the translates of L by $R \otimes_{\mathcal{O}_S} B$. Since this is a finite B-module, L' is a finite projective B'-submodule of V, which is therefore rank d. Its action of $R \otimes_{\mathcal{O}_S} B'$ induces ρ by applying $\otimes_{B'} K'$. Now $R \otimes_{\mathcal{O}_S} B' \to \mathrm{End}_{B'}(L')$ is an object of $\mathrm{Rep}_R^d(B')$ inducing $\rho \in \mathrm{Rep}_R^d(K')$, completing the proof of the claim. $\square$

Here is a nice result of our work: the maps ψ and $\bar{\psi}$ are adequate moduli spaces up to an almost isomorphism. Some of the properties of an adequate moduli space still hold despite this defect.

COROLLARY 1.5.4.7. *With assumptions as in Theorem 1.5.4.2, the morphisms ψ and $\bar{\psi}$ have the properties (1), (2), and (4) proved of adequate moduli spaces in Theorem 1.5.1.4, as well as property (1) defining adequate moduli spaces in Definition 1.5.1.1. Namely, ψ and $\bar{\psi}$ are finite type, universally closed, push forward coherent sheaves to coherent sheaves, and two geometric points in $\mathrm{Rep}_R^d(\bar{k})$ (resp. $\overline{\mathrm{Rep}}_R^d(\bar{k})$) have overlapping closures if and only if their images under ψ (resp. $\bar{\psi}$) are isomorphic.*

PROOF. We have shown that $\nu : \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d \to \mathrm{PsR}_R^d$ is a finite universal homeomorphism. Now we apply Theorem 1.5.1.4: because the canonical map ϕ from Rep_R^d to the GIT quotient is finite type and has the geometric point closure property, the same is true of the composition $\nu \circ \phi = \psi$; because it is universally closed, its composition with the finite and therefore proper map ν is still universally closed. Finally, since push forwards of coherent sheaves are coherent under ϕ and under the finite morphism ν, the same is true of ψ. This all holds for $\bar{\psi}$ on $\overline{\mathrm{Rep}}_R^d$ as well. $\square$

CHAPTER 2

Local Study of Pseudorepresentations

In Chapter 1, we described moduli spaces of representations and pseudorepresentations and proved that the maps $\psi, \bar{\psi}$ sending algebraic stacks of representations to their associated pseudorepresentations are very close to adequate moduli spaces. In particular, they are universally closed. We accomplished this almost entirely through a study of the geometric points of these moduli spaces, the only additional input being the verification that ν satisfies the valuative criterion for properness in Theorem 1.5.4.2. However, as we noted in Remark 1.5.4.4, the study of the defect ν of ψ (resp. $\bar{\psi}$) from being an adequate moduli space is not visible through the fibers of ψ (resp. $\bar{\psi}$). The challenge is that the GIT quotient, which is the base of adequate moduli space, does not admit a good moduli interpretation – only its geometric points have a satisfying moduli interpretation. However, as remarked at the beginning of §1.5.4, we have reduced the study of the defect ν to ψ being an adequate moduli space to a local question on the base. This is one reason why we will now study the moduli space of pseudorepresentaitons locally. For example, in §2.3, we make progress in showing that ν is an isomorphism by adding more linear structure to representations whose induced pseudorepresentation deforms a fixed *multiplicity free* psueodrepresentation. In this case, we will be able to eliminate the defect ν. This result is recorded in Theorem 2.3.3.7.

Of course, there are other reasons to study pseudorepresentations locally. One reason is to study their tangent spaces and deformation theory, which is what we begin with, following Chenevier [Che11]. Our main result here is Proposition 2.1.2.3. This gives a representation theoretic condition for the finitude of the dimension of the tangent space to a field-valued pseudorepresentation. In this, we make an improvement on [Che11, Proposition 2.28] by eliminating the assumption that the characteristic of the field must be larger than

the dimension or must be 0. This follows from the application of PI ring theory described in Chapter 1.

The other major goal in this chapter is to identify some projective subschemes of $\overline{\mathrm{Rep}}_R^d$, locally on the base PsR_R^d of ψ. To accomplish this fiber-wise is to apply one of the results of King [Kin94] (Theorem 2.2.1.12 here), which shows that these projective spaces exist inside geometric fibers of $\bar{\psi}$. Our additional contribution is the deformation of this ample line bundle to henselian neighborhoods of a point, so that the projective subscheme can be deformed to complete local neighborhoods (Theorem 2.2.4.1). To this end, our work here is to carefully identify the ample line bundle implicit in King's result.

Our motivating case of interest for this local study is the moduli of continuous pseudorepresentations and representations of a profinite group or algebra, with a certain finiteness condition. In this case, the results above apply very well, as the moduli formal scheme of continuous pseudorepresentations is *semi-local* (see Corollary 3.1.6.13). Each component is the formal spectrum of a complete local Noetherian ring! We are preparing the results in Chapter 2 with their application to profinite representation theory in Chapter 3 in mind.

2.1. Pseudorepresentations over Local Rings

In this section, we will study pseudorepresentations of an algebra R over a commutative local ring A. In practice, we will often fix a d-dimensional pseudorepresentation

$$D : R \longrightarrow A$$

and draw conclusions about D given some conditions about the data. We will begin with deformation theory of a field-valued pseudorepresentation and then discuss the tangent space of the pseudorepresentation functor at such a point. We will conclude with some facts about Cayley-Hamilton algebras (R, A) over local rings.

2.1.1. Deformation Theory Setup. Our study of the deformations of pseudorepresentations will follow Chenevier [Che11]. As usual, let A be a commutative ring and let R be an A-algebra. We could consider a closed $\operatorname{Spec} A$-subscheme $X \subset \mathrm{PsR}_R^d$ with its reduced

structure, and then study the completion of PsR_R^d at X. However, our purposes do not require this generality; in particular, our work in Chapter 1 shows that we can study the morphism ν locally on the base. Our setting for the study deformations will be a complete Noetherian local base ring A with residue field $\mathbb{F}_A$ of characteristic $p \geq 0$, along with a given d-dimensional $\mathbb{F}_A$-valued pseudorepresentation of R, denoted

$$\bar{D} : R \otimes_A \mathbb{F}_A \longrightarrow \mathbb{F}_A.$$

For example, in this setting, A may be the Cohen ring of $\mathbb{F}_A$, which we denote by W.

We study deformations of $\bar{D}$ to the following rings, writing $\mathbb{F}$ for $\mathbb{F}_A$.

Definition 2.1.1.1. Let $\mathcal{A}_{\mathbb{F}}$ be the category of Artinian local A-algebras with residue field $\mathbb{F}$, where morphisms are local A-algebra homomorphisms.

Let $\hat{\mathcal{A}}_{\mathbb{F}}$ be the category of Noetherian local A-algebras with residue field $\mathbb{F}$, where morphisms are local W-algebra homomorphisms. For $B \in \mathcal{A}_{\mathbb{F}}$ we write $\mathfrak{m}_B$ for its maximal ideal.

The category $\hat{\mathcal{A}}_{\mathbb{F}}$ includes $\mathcal{A}_{\mathbb{F}}$ as a full subcategory, and objects in $\hat{\mathcal{A}}_{\mathbb{F}}$ consist of limits (filtered projective limits with surjective maps) in $\mathcal{A}_{\mathbb{F}}$.

We define the deformation functor $\mathrm{PsR}_{\bar{D}}$ as follows.

Definition 2.1.1.2. With the data $p, A, R, \bar{D}, d$ and $\mathbb{F}$ as above, let $\mathrm{PsR}_{\bar{D}}$ be the covariant functor on $\hat{\mathcal{A}}_{\mathbb{F}}$ associating to each $B \in \mathrm{ob}\,\hat{\mathcal{A}}_{\mathbb{F}}$ the set of d-dimensional pseudorepresentations

$$D : R \otimes_A B \longrightarrow B$$

such that $D \otimes_B \mathbb{F} \longrightarrow \mathbb{F} \cong \bar{D}$. We call such deformations of $\bar{D}$ *pseudodeformations*.

The representability of this deformation functor in the category $\hat{\mathcal{A}}_{\mathbb{F}}$ follows immediately from the representability for the usual pseudorepresentation moduli scheme PsR_R^d over $\mathrm{Spec}\, A$.

PROPOSITION 2.1.1.3. *Given $\mathbb{F}, A, R, d$, and $\bar{D}$ as above, let $\bar{D}$ also denote its associated $\mathbb{F}$-point of PsR_R^d. Then the pseudodeformation functor is representable by the completion $B_{\bar{D}}$ of the local ring $\mathcal{O}_{\mathrm{PsR}_R^d,\bar{D}}$ at its maximal ideal $\mathfrak{m}_{\bar{D}}$, with the universal object $D^u \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B_{\bar{D}}$.*

PROOF. By definition of pseudodeformation, any object of $\mathrm{PsR}_{\bar{D}}$ over $B \in \hat{\mathcal{A}}_{\bar{D}}$ corresponds to a map $\operatorname{Spec} B \to \mathrm{PsR}_R^d$ factoring through the natural map $\operatorname{Spec} B_{\bar{D}} \to \mathrm{PsR}_R^d$. □

COROLLARY 2.1.1.4. *If R is finitely generated as a A-algebra, $B_{\bar{D}}$ is Noetherian.*

PROOF. In this case, PsR_R^d is Noetherian and finitely generated over $\operatorname{Spec} A$ by Theorem 1.1.10.15. Then since $B_{\bar{D}}$ is the completion of a localization of a Noetherian ring, $\mathrm{PsR}_{\bar{D}}$ is Noetherian. □

Since $B_{\bar{D}}$ is a complete local ring, there are several conditions on $B_{\bar{D}}$ equivalent to the Noetherian condition.

LEMMA 2.1.1.5. *Since $B_{\bar{D}}$ is a complete local A-algebra and A is a complete Noetherian local ring, the Cohen structure theorem (see e.g. [MR10, Theorem 3.2.4]) implies that the following properties are equivalent.*

(1) There exists a surjection $W[\![t_1, \ldots, t_n]\!] \twoheadrightarrow B_{\bar{D}}$ for some $n \geq 0$.

(2) There exists a surjection $A[\![t_1, \ldots, t_n]\!] \twoheadrightarrow B_{\bar{D}}$ for some $n \geq 0$.

(3) $B_{\bar{D}}$ topologically finite type[1] as a A-algebra.

(4) $B_{\bar{D}}$ is Noetherian.

(5) $\dim_{\mathbb{F}}(\mathfrak{m}_{\bar{D}}/\mathfrak{m}_{\bar{D}}^2)$ is finite.

(6) The tangent $\mathbb{F}$-vector space $\mathrm{PsR}_{\bar{D}}(\mathbb{F}[\varepsilon]/(\varepsilon^2))$ is finite-dimensional.

This is our motivation to study the tangent space $\mathcal{T}_{\bar{D}} := \mathrm{PsR}_{\bar{D}}(\mathbb{F}[\varepsilon]/(\varepsilon^2))$ of PsR_R^d at $\bar{D}$. We will give a (co)homological condition for the finiteness of the tangent space in the next paragraph.

[1]When we say that B is topologically finite type over A, we mean that it is (or admits a surjection from) the completion of some finite type A-algebra with respect to the powers of some ideal of A.

2.1.2. Tangent Spaces of the Pseudorepresentation Functor. We now describe and give a sufficient condition for the finiteness of the tangent space of the pseudorepresentation functor (or pseudodeformation functor) at a point. We follow Chenevier [Che11, 2.24-2.29] here. We make an improvement on Chenevier's results, generalizing [Che11, Proposition 2.28] to arbitrary characteristic in Proposition 2.1.2.3; the improvement entirely rests on the use of the reference [Sam09] (see Theorem 1.2.2.6 and its use in Lemma 1.2.3.1), and we follow Chenevier's techniques otherwise.

This study is especially useful in preparation for giving sufficient conditions for the Noetherianess of a complete local "pseudodeformation ring" of continuous deformations of a field-valued pseudorepresentation of a profinite algebra (see Theorem 3.1.5.3).

As usual, we have a commutative ring A and an A-algebra R. Let $\bar{D} : R \to A$ be a d-dimensional pseudorepresentation. We will write $A[\epsilon]$ for $A[\epsilon]/(\epsilon^2)$, i.e. $\epsilon^2 = 0$. For any A-module M, we write $M[\epsilon]$ for $M \otimes_A A[\epsilon]$.

Definition 2.1.2.1. Let $\bar{D}$ be a d-dimensional pseudorepresentation $\bar{D} : R \to A$. We call a pseudorepresentation $D : R[\epsilon] \to A[\epsilon]$ a *lift* of $\bar{D}$ when $D \otimes_{A[\epsilon]} A \cong \bar{D}$. Through the canonical identification

$$\mathcal{M}_A^d(R, A[\epsilon]) \xrightarrow{\sim} \mathrm{PsR}_{R[\epsilon]}^d(A[\epsilon]),$$

of Corollary 1.1.3.10, the set of lifts is canonically functorially isomorphic to the set of multiplicative A-polynomial laws

$$P : R \longrightarrow A[\epsilon]$$

such that they map to $\bar{D}$ via composition with the A-algebra homomorphism $\pi = \pi_A : A[\epsilon] \xrightarrow{\epsilon \mapsto 0} A$. We denote this set of multiplicative polynomial laws by $\mathcal{T} = \mathcal{T}_{\bar{D}} \subset \mathcal{M}_A^d(R, A[\epsilon])$, the *tangent space at* $\bar{D}$.

Another way of defining this tangent space is to say that $\mathcal{T}_{\bar{D}} := (\pi^*)^{-1}(D_0)$, where

$$\pi^* : \mathrm{Hom}_{A-alg}(\Gamma_A^d(R)^{\mathrm{ab}}, A[\epsilon]) \longrightarrow \mathrm{Hom}_{A-alg}(\Gamma_A^d(R)^{\mathrm{ab}}, A)$$

$$f \mapsto \pi \circ f.$$

One can check that it has a natural A-module structure.

From now on, in our discussion of lifts of pseudorepresentations, we let A be a field $\mathbb{F}$.

LEMMA 2.1.2.2 ([Che11, Lemma 2.26]). *Let R be a $\mathbb{F}$-algebra and let $\bar{D} : R \to \mathbb{F}$ be a d-dimensional pseudorepresentation. Assume that there exists a positive integer N such that $\ker(\bar{D})^N \subset \mathrm{CH}(\bar{D})$. Then $\mathcal{T} \subset \mathcal{P}^d_{\mathbb{F}}(R/\ker(\bar{D})^{2N}, \mathbb{F})$.*

PROOF. Let $P \in \mathcal{T}_{\bar{D}}$ and let $D : R[\epsilon] \to \mathbb{F}[\epsilon]$ be the associated pseudorepresentation. One can check that $\ker(P) = \ker(D) \cap R$. Therefore want to show that $2N$ such that satisfies the relation $\ker(\bar{D})^{2N} \subseteq \ker(D)$.

Consider the Cayley-Hamilton $\mathbb{F}[\epsilon]$-algebras $S := R/\mathrm{CH}(\bar{D})$ and $S[\epsilon]$, which is canonically isomorphic to $R[\epsilon]/\mathrm{CH}(D)$ by Lemma 1.1.8.6. For $r \in \ker(\bar{D})[\epsilon] \subset R[\epsilon]$, we have by assumption $\Lambda_i(r) \in \epsilon\mathbb{F}$ for all $1 \leq i \leq d$, and therefore $s^d \in \epsilon \cdot \mathbb{F}[s]$ for all $s \in J[\epsilon] \subset S[\epsilon]$. Let $J := \ker(\bar{D})/\mathrm{CH}(\bar{D}) \subset S$, so $J \cong J[\epsilon]/\epsilon J[\epsilon]$. The assumption $\ker(\bar{D})^N \subset \mathrm{CH}(\bar{D})$ implies that $(J[\epsilon]/\epsilon J[\epsilon])^N = 0$, and then $J[\epsilon]^{2N} = 0$. Consequently, $\ker(\bar{D})^{2N} \subset \mathrm{CH}(D) \subset \ker(D)$, where the latter inclusion is the content of Lemma 1.2.1.1. □

We continue to work with deformations of a fixed d-dimensional pseudorepresentation $\bar{D} : R \to \mathbb{F}$. Now let us restrict to the case that $S := R/\ker(\bar{D})$ is finite-dimensional as a $\mathbb{F}$-vector space. By Theorem 1.3.1.3, this condition will hold when $\mathbb{F}$ is perfect, $p = \mathrm{char}(\mathbb{F}) > 0$ and $[\mathbb{F} : \mathbb{F}^p] < \infty$, $R/\mathbb{F}$ is finitely generated, or $d < p$. This lemma improves [Che11, Lemmas 2.26].

PROPOSITION 2.1.2.3 (Following [Che11, Proposition 2.28]). *Let $R, \mathbb{F}, \bar{D} : R \to \mathbb{F}$, and assume that $S = R/\ker(\bar{D})$ is finite-dimensional over $\mathbb{F}$. Then if $\mathrm{Ext}^1_R(S, S)$ is finite-dimensional over $\mathbb{F}$, where S is treated as a R-module here, then $\mathcal{T}_{\bar{D}}$ is also finite-dimensional over $\mathbb{F}$.*

Compare this statement with Theorem 1.3.3.2; the methods of proof also correspond in large part.

PROOF. Choose N such that $\ker(\bar{D}) \subset \mathrm{CH}(\bar{D})$. Such an N exists by Lemma 1.2.3.1(4), and is bounded by the integers $N(d)$ and $N(d,p)$ of Definition 1.2.2.7.

Now Lemma 1.3.3.1 tells us that

$$\mathrm{Hom}_R(I/I^2, S) \cong \mathrm{Hom}_S(I/I^2, S)$$

is also finite-dimensional as a $\mathbb{F}$-vector space. Since S is semisimple and the S-module S contains all simple S-modules as submodules, the finiteness of the dimension of $\mathrm{Hom}_S(I/I^2, S)$ implies that I/I^2 is finite length as a S-module, and therefore also implies that $\dim_{\mathbb{F}} I/I^2 < \infty$.

Because there are natural surjections

$$(I/I^2)^{\otimes_{\mathbb{F}}^n} \twoheadrightarrow I^n/I^{n+1}$$

for any ideal $I \subset R$, this means that $R/\ker(\bar{D})^{2N}$ is also finite-dimensional over $\mathbb{F}$. Using Lemma 2.1.2.2, we know that $\mathcal{T}_{\bar{D}} \subset \mathcal{P}^d_{\mathbb{F}}(R/\ker(\bar{D})^{2N}, \mathbb{F})$. Finally, because the finitude of a general A-module M implies the finitude of $\Gamma^d_A(M)$ as an A-module for any $d \geq 0$, we apply Theorem 1.1.3.4 to conclude. □

We can apply Proposition 2.1.2.3 to give a criterion depending only on $\bar{D}$ for the Noetherianness of the complete local deformation ring $B_{\bar{D}}$ defined in Proposition 2.1.1.3, using the Noetherianness criteria of Lemma 2.1.1.5. For this statement, we resume the language of §2.1.1, also setting $\bar{R} := R \otimes_A \mathbb{F}_A$.

COROLLARY 2.1.2.4. *Let A be a complete Noetherian local ring and let R be an A-algebra. Fix a d-dimensional pseudorepresentation $\bar{D} : \bar{R} \to \mathbb{F}_A$ such that if we take $\bar{S} := \bar{R}/\ker(\bar{D})$ as an R-module, $\mathrm{Ext}^1_R(S,S)$ is finite-dimensional as a $\mathbb{F}_A$-vector space. Then the complete local pseudodeformation ring $B_{\bar{D}}$ of Proposition 2.1.1.3 is Noetherian.*

Since we will be interested in this primarily in the profinite topological case, we will give the proof for the profinite case in Theorem 3.1.5.3. A proof in this case would feature the same techniques without the topological considerations.

2.1.3. Cayley-Hamilton Pseudorepresentations over Local Rings. As we have remarked, a d-dimensional Cayley-Hamilton A-algebra (R, D) shares properties with algebras appearing as subalgebras of $d \times d$ matrix algebras. For example, each element is integral of degree d over A, and if A is a field, we have shown that the Jacobson radical is nilpotent (see Lemma 1.2.3.1 and Corollary 1.2.2.10 for this and other properties of Cayley-Hamilton algebras). When A is a henselian local ring and the semisimple representation corresponding to the special fiber of D is absolutely irreducible and split over the residue field, this correspondence with matrix algebras is exact. This is what we describe in this paragraph.

First we require a lemma on the Jacobson radical of Cayley-Hamilton algebras over local rings. Recall that $J(R)$ denotes the Jacobson radical of the ring R.

LEMMA 2.1.3.1 (Following [Che11, Lemma 2.10]). *Let A be a local ring with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbb{F} = \mathbb{F}_A$. Let R be an A-algebra with a d-dimensional Cayley-Hamilton pseudorepresentation $D : R \to A$ with residual pseudorepresentation $\bar{D} = D \otimes_A \mathbb{F}$.*

(1) The kernel of the canonical surjection $R \twoheadrightarrow (R \otimes_A \mathbb{F})/\ker \bar{D}$ is $J(R)$.

(2) If $\mathfrak{m}_A^s = 0$ for $s \geq 1$ an integer, then $J(R)^{N(d)s} = 0$, where $N(d)$ is the integer of Definition 1.2.2.7, which depends only on d. The possibly lesser integer $N(d, \operatorname{char} \mathbb{F}_A)$ can be used in place of $N(d)$.

Our use of polynomial identity ring theory improves Chenevier's result in the case $d \geq \operatorname{char} \mathbb{F}_A$.

PROOF. Write I for the two-sided ideal named in statement (1). Let us first show that $I \subseteq J(R)$, which will follow from checking that $1 + I \subset R^\times$. By Lemma 1.2.3.1(1), it is equivalent to check that $D(1 + I) \subset A^\times$. But it is clear that $D(1 + I) \subseteq 1 + \mathfrak{m}_A$ by assumption, so we have $I \subseteq J(R)$. To show the reverse inclusion, we first observe that $\mathfrak{m}_A \cdot R \subseteq I \subseteq J(R)$, so it will suffice to prove the reverse inclusion with $A = \mathbb{F}_A$. Now the desired inclusion $J(R) \subseteq I$ is given by Lemma 1.2.3.1(5).

Now we assume that $\mathfrak{m}_A^s = 0$. It is clear that we may replace R by $R/\mathfrak{m}_A \cdot R$, assume that $A = \mathbb{F}_A$, and show that $J(R)^{N(d)} = 0$. This is precisely what we get from Lemma 1.2.3.1(4). □

Recall this essential property of henselian rings. The idempotent lifting is what we require in order to make a comparison with a matrix algebra over all of $\operatorname{Spec} A$, and not just over the closed point.

LEMMA 2.1.3.2 (cf. [BLR90, §2.3, Proposition 4]). *Let A be a local ring with residue field $\mathbb{F}_A$. Then A is Henselian if and only if for any finite A-algebra B, the canonical map on idempotents*

$$\operatorname{Idem}(B) \longrightarrow \operatorname{Idem}(B \otimes_A \mathbb{F}_A)$$

is an isomorphism.

Now we can give the main theorem of this paragraph.

THEOREM 2.1.3.3 ([Che11, Theorem 2.22(i)]). *Assume that D is Cayley-Hamilton and that A is a henselian local ring with residue field $\mathbb{F}_A$. If $\bar{D}$ is split and absolutely irreducible, then there is an A-algebra isomorphism*

$$\rho : R \xrightarrow{\sim} M_d(A)$$

such that $D = \det \circ \rho$.

PROOF. Omitted. □

Recall the representation theoretic moduli spaces of §1.4. The local result Theorem 2.1.3.3 is enough for us to show that the universal Cayley-Hamilton algebra is globally an Azumaya algebra when restricted to the absolutely irreducible locus $\mathrm{PsIrr}_R^d \subset \mathrm{PsR}_R^d$. We also point out that $\bar{\psi}$ is an isomorphism over this locus, which immediately implies that the deformation functor of a chosen absolutely irreducible field-valued representation of R is equivalent to the deformation functor of the representation. This is an improvement of

Chenevier of the results of Nyssen [Nys96] and Rouquier [Rou96], who showed that deforming an absolutely irreducible *pseudocharacter* is equivalent to deforming the associated an absolutely irreducible representation.

COROLLARY 2.1.3.4 ([Che11, Corollary 2.23]). *Let A be a commutative ring and let R be an A-algebra.*

(1) Over the absolutely irreducible locus $\mathrm{PsIrr}_R^d \subset \mathrm{PsR}_R^d$, the restriction of the universal Cayley-Hamilton algebra $E(R,d)$ to PsIrr_R^d is an Azumaya $\mathcal{O}_{\mathrm{PsIrr}_R^d}$-algebra of rank d^2.

(2) Over $\mathrm{PsIrr}_R^d \subset \mathrm{PsR}_R^d$, $\bar{\psi}$ and ν are isomorphisms.

(3) For each split point $\bar{D} \in \mathrm{PsIrr}_R^d$, the $\mathfrak{m}_{\bar{D}}$-adic completion of $\mathcal{O}_{\mathrm{PsIrr}_R^d,\bar{D}}$ is canonically isomorphic to the deformation ring for the representation

$$R \longrightarrow M_d(\kappa(\bar{D})).$$

PROOF. Chose $x \in \mathrm{PsIrr}_R^d$ and let B be the strict henselization of $\mathcal{O}_{\mathrm{PsR}_R^d,x}$. By Lemma 1.1.8.6,

$$E(R,d) \otimes_{\mathcal{O}_{\mathrm{PsR}_R^d}} B \xrightarrow{\sim} (R \otimes_A B)/\mathrm{CH}(D^u \otimes A).$$

Theorem 2.1.3.3 now implies that the right hand side is isomorphic to $M_{d^2}(B)$. Hence $E(R,d) \otimes_{\mathcal{O}_{\mathrm{PsR}_R^d}} \mathcal{O}_{\mathrm{PsR}_R^d,x}$ is an Azumaya algebra of rank d^2 since $\mathcal{O}_{\mathrm{PsR}_R^d,x} \to B$ is faithfully flat (cf. [Sta, Lemma 07QM]). Now we observe that $E(R,d)$ is an Azumaya algebra, as the definition of an Azumaya algebra may be given locally (see Definition 1.4.1.5(1)).

Parts (2) and (3) follow at once, as the Azumaya $\mathcal{O}_{\mathrm{PsIrr}_R^d}$-algebra defines a section to $\bar{\psi}$ over PsIrr_R^d. □

REMARK 2.1.3.5. This generalizes results previously known for pseudocharacters when the characteristic is larger than the dimension, e.g. [Nys96, Rou96, Car94]. In particular,

Carayol [Car94] showed that a deformation of an absolutely irreducible residual representation is characterized by its trace. Nyssen and Rouquier [Nys96, Rou96] showed the "converse," that the deformation of a residual pseudocharacter arising as the trace of an absolutely irreducible representation is realizable as the trace of a deformation of said representation.

We have succeeded in showing that $\bar{\psi}$ is an adequate moduli space over the absolutely irreducible locus, but this is a trivial case since $\bar{\psi}$ is an isomorphism here. We will prove this in a nontrivial case in Corollary 2.3.3.9.

2.2. Fibers of ψ

Recall Theorem 1.4.3.1, where we show that schemes and stacks parameterizing d-dimensional representations of an algebra R are equivalent to the analogous moduli space for representations of the universal Cayley-Hamilton algebra $E(R,d)$ over the universal pseudorepresentation of R. This is a particularly useful result in the case that $E(R,d)$ is finite as a $\mathcal{O}_{\mathrm{PsR}_R^d}$-module. We have shown that this is true when, for example, A is Noetherian and R is finitely generated (Corollary 1.4.3.3).

Assuming that $E(R,d)$ is finite, we study of the fibers of $\psi^{\square}$ (resp. ψ, resp. $\bar{\psi}$). Fix a residue field $\mathbb{F}$ of PsR_R^d and let $\bar{D}$ denote the associated pseudorepresentation

$$\bar{D} = D^u \otimes \mathbb{F} : E(R,d) \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} \mathbb{F} \longrightarrow \mathbb{F}.$$

Recall that by Lemma 1.1.8.6, the formation of the Cayley-Hamilton quotient of R commutes with base change over PsR_R^d. Therefore, when $E(R,d)$ is finite over $\mathcal{O}_{\mathrm{PsR}_R^d}$, the study of the fibers of ψ amounts to the study of representations of a *finite-dimensional* algebra

$$E(R,d) \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} \mathbb{F}$$

over the field $\mathbb{F}$, with the condition that the induced pseudorepresentation of these representations is precisely $\bar{D}$.

Consider also the case where R is an algebra over a field $\mathbb{F}$ that is not finitely generated, but where a pseudorepresentation $\bar{D} : R \to \mathbb{F}$ satisfies the conditions of Theorem 1.3.3.2, so

that the associated Cayley-Hamilton algebra $E := R/\mathrm{CH}(\bar{D})$ is finite-dimensional over $\mathbb{F}$. Then, using the universality of the Cayley-Hamilton algebra (Theorem 1.4.3.1), the representations of R inducing $\bar{D}$ as a determinant amount to the representations of E inducing $\bar{D}|_E$ as a determinant. Once again, we are reduced to the study of the representations of a finite-dimensional algebra. We will also find ourselves remanded to this case when we study representations of profinite topological algebras in Chapter 3, provided that an appropriate finiteness condition is satisfied.

Therefore, for this section we will let E be a finite-dimensional $\mathbb{F}$-algebra with a given Cayley-Hamilton d-dimensional pseudorepresentation

$$\bar{D} : E \to \mathbb{F}.$$

Certainly, this satisfies the conditions of Theorem 1.4.1.3, so that the scheme of framed representations and the algebraic stacks of representations are finite type $\operatorname{Spec}\mathbb{F}$-schemes. We will study the fiber of the representation spaces of E over $\bar{D}$, i.e.

$$\begin{aligned}
\mathrm{Rep}_{\bar{D}}^{\square,\circ} &:= \psi^{\square,-1}(\bar{D}) &&\subset \mathrm{Rep}_E^{\square,d},\\
\mathrm{Rep}_{\bar{D}}^{\circ} &:= \psi^{-1}(\bar{D}) &&\subset \mathrm{Rep}_E^{d},\\
\overline{\mathrm{Rep}}_{\bar{D}}^{\circ} &:= \bar{\psi}^{-1}(\bar{D}) &&\subset \overline{\mathrm{Rep}}_E^{d}.
\end{aligned}$$

Important point. This condition that a representation lie in the fiber of $\bar{D}$ is equivalent, once $\bar{D}$ is split, to the condition that its Jordan-Hölder factors match those of the semisimple representation $\rho_{\bar{D}}^{ss}$ associated to ρ via Theorem 1.3.1.1. Therefore, $\mathrm{Rep}_{\bar{D}}^{\circ}$ is a geometric realization of the category (whose morphisms are isomorphisms) of representations of E with a given semisimplification. Of course, semisimple representations of a finite-dimensional algebra are naturally in bijective correspondence with functions from simple representations to the non-negative integers. This description is known as a *dimension vector*. Therefore, once $\mathbb{F}$ is large enough so that $E/\ker(\bar{D})$ is split, we can speak of $\mathbb{F}$-valued pseudorepresentations as dimension vectors, and vice-versa. This lange will be particularly natural as we introduce representations of quivers.

The main goal of our study is to show that there are projective subspaces of $\overline{\mathrm{Rep}}^{\circ}_{\bar{D}}$ corresponding to certain notions of (semi)stability, formally analogous to the theory of vector bundles over a curve. Basically, we are reviewing a result of A. D. King [Kin94]. Let us begin with a brief summary of his result. We will freely use terminology from §1.3.4.

Given an integer-valued character $\theta : K_0(\mathrm{Rep}_E(\mathbb{F})) \to \mathbb{Z}$ of the Grothendieck group of E, he develops a corresponding notion of semi-stability and stability for representations of E. He then shows that semistability (resp. stability) of a representation $\rho \in \mathrm{Rep}^{\square}_{\bar{D}} \subset \mathrm{Rep}^{\square,d}_{E}$ is equivalent to it lying in a semistable (resp. stable) orbit for a certain action of a certain reductive group and a linearization of $\mathrm{Rep}^{\square}_{E}$ corresponding to θ. Then the GIT quotient of the semistable orbit locus is a projective space which is a coarse moduli space parameterizing θ-semistable representations of E up to S-equivalence. The notion of S-equivalence is analogous to the notion of S-equivalence of vector bundles on curves due to Seshadri [Ses67]. The equivalence relation is better on the stable locus within the GIT quotient: it is a coarse moduli space for θ-stable representations with respect to the *usual* notion of equivalence between E-modules.

We will not pursue these generalities and the notion of S-equivalence. Rather, we will focus on a particular case when we get a *projective, fine* moduli space out of this GIT construction. This case is noted by King [Kin94, Remark 5.4]: θ may be chosen (relative to $\bar{D}$) so that θ-semistability implies θ-stability in $\mathrm{Rep}^{\square}_{\bar{D}}$, and such that the GIT quotient is a *fine* moduli space.[2] This will show that there are large projective subschemes of $\overline{\mathrm{Rep}}^{\circ}_{\bar{D}}(\theta) \subset \overline{\mathrm{Rep}}^{\circ}_{\bar{D}}$ corresponding to θ-(semi)stable representations of E.

Remark 2.2.0.6. This observation generalizes and answers affirmatively a suspicion of Kisin [Kis09a, Remark 3.2.7] on the existence of projective loci (relative to ψ) inside moduli spaces of representations, and adds many more instances to the cases that Kisin pointed out (see Corollary 2.2.2.14). Of course, we must wait until Chapter 3 to see that the case of extensions of continuous representations of a profinite group with finite field coefficients can be reduced to the case that we now work with.

[2]The former condition is the more interesting one.

In terms of the intrinsic study of the special fiber of ψ, King's result is all that we require, and it would suffice to quote his result in order to show which families of representations of E form projective spaces in $\overline{\mathrm{Rep}}^{\circ}_{\bar{D}}$. However, we are also interested in showing that these projective sub-moduli-spaces exist locally on the base in PsR^d_R, or complete-locally in the profinite case $\mathrm{PsR}_{\bar{D}} \subset \mathrm{PsR}^d_R$. Of course, this will follow if we can show that an ample line bundle for King's projective space is the specialization of a locally well-defined line bundle in $\overline{\mathrm{Rep}}^d_R$ to PsR^d_R. We will accomplish this over complete local rings, with our work culminating in Theorem 2.2.4.1. This will take some work, since King's work uses the fact that the category of modules for any finite-dimensional algebra is equivalent to the category of modules for a a quotient algebra of a path algebra for a finite quiver. It is in the category of representations of quivers that these projective spaces are most naturally constructed, and our work is to follow the ample line bundle on a space of representations of a quiver through several equivalences necessary to identify an ample line bundle on a certain space of representations of E.

Assumption. We assume that $\mathbb{F}$ is algebraically closed. This assumption will be in place only for this section.

Remark 2.2.0.7. This assumption is used to ensure that $R/\ker(\bar{D})$, the semisimple algebra associated to the pseudorepresentation, will be *split* in the sense of Definition 1.3.4.4. It is also necessary in order to ensure that statements about points of a GIT quotient are accurate, as GIT only has a good functor of geometric points (cf. Remark 1.5.1.6). The former issue is more serious, as we will need to find as many idempotents as the dimension of an algebra in order to draw comparisons with quivers. In many cases, including those that we will be concerned with for profinite algebras in Chapter 3, this can be achieved with a finite separable extension of a field $\mathbb{F}$. Therefore an assumption that $\bar{D}$ is split over $\mathbb{F}$ will be sufficient to apply the results of this section.

2.2.1. King's Result on Quiver Representation Moduli. We give a brisk introduction to quivers in order to state King's result. For more background on quivers, see for example [ASS06].

DEFINITION 2.2.1.1. A *quiver* Q is an oriented graph $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices, and Q_1 the set of oriented edges, also known as arrows. We define the head and tail functions

$$h, t : Q_1 \to Q_0$$

to be the maps sending an arrow $a \in Q_1$, to the head $h(a)$ of the arrow and the tail $t(a)$ of the arrow. A quiver Q is called *finite* if Q_0 and Q_1 are finite.

DEFINITION 2.2.1.2. Let Q be a quiver.

(1) A *representation* of Q over a field $\mathbb{F}$ is a collection of $\mathbb{F}$-vector spaces W_v for each $v \in Q_0$ and a collection of $\mathbb{F}$-linear maps $\phi_a : W_{t(a)} \to W_{h(a)}$ for each arrow $a \in Q_1$.

(2) A morphism of such representations, $(W_v, \phi_a) \to (U_v, \psi_a)$ is a collection of $\mathbb{F}$-linear maps $f_v : W_v \to U_v$ such that $f_{h(a)} \circ \phi_a = \psi_a \circ f_{t(a)}$ for each $a \in Q_1$.

(3) The *dimension vector* $\beta \in \mathbb{Z}^{Q_0}$ of a representation (W_v, ϕ_a) is the vector of integers $\beta_v = \dim_{\mathbb{F}} W_v$ for each $v \in Q_0$. A representation is called *finite-dimensional* if W_v is finite-dimensional for all v and $\beta_v = 0$ for all but finitely many $v \in Q_0$.

(4) Given a dimension vector β we use $\mathrm{GL}(\beta)$ to denote the group $\times_{v \in Q_0} \mathrm{GL}(W_v)$ of linear automorphisms of (W_v).

(5) $\Delta \subset \mathrm{GL}(\beta)$ denotes the diagonal subgroup of scalars $(t, \ldots, t) \subset \mathrm{GL}(\beta)$, and $\mathrm{PGL}(\beta)$ denotes the quotient.

Note that $\mathrm{PGL}(\beta)$ is not generally the product over $v \in Q_0$ of $\mathrm{PGL}(W_v)$.

Convention. We will work with finite quivers and finite-dimensional representations from now on, without remarking on their finiteness.

Once we define the path algebra $\mathbb{F}Q$ of Q, we will see that under the equivalence of between representations of Q and representations of $\mathbb{F}Q$, dimension vectors correspond to pseudorepresentations of $\mathbb{F}Q$.

First, we note that framed moduli spaces of representations of quivers are affine spaces! Indeed, the set of representations of $\mathbb{F}$ over a given dimension vector $\beta \in \mathbb{Z}^{Q_0}$ corresponding

to the set of vector spaces $(W_v)_{v\in Q_0}$ is

$$\mathrm{Rep}^{\square}_{\beta}(\mathbb{F}) := \bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{F}}(W_{t(a)}, W_{h(a)}).$$

The group $\mathrm{GL}(\beta)(\mathbb{F})$ acts naturally on this set, and one can check that two representations in $\mathrm{Rep}^{\square}_{\beta}(\mathbb{F})$ are isomorphic if and only if they lie in the same orbit of $\mathrm{GL}(\beta)(\mathbb{F})$.

We let $\mathrm{Rep}^{\square}_{\beta}$ represent the functor from $\operatorname{Spec}\mathbb{F}$-schemes to the set of such representations; explicitly, this functor sends a $\operatorname{Spec}\mathbb{F}$-scheme X to the $\mathcal{O}_X$-module

$$\bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathcal{O}_X}(W_{t(a)}\otimes_{\mathbb{F}}\mathcal{O}_X, W_{h(a)}\otimes_{\mathbb{F}}\mathcal{O}_X).$$

Observe that there is a natural isomorphism

$$\mathrm{Rep}^{\square}_{\beta} \xrightarrow{\sim} \operatorname{Spec}\mathrm{Sym}^{*}_{\mathbb{F}}\left(\bigoplus_{a\in Q_1}\mathrm{Hom}_{\mathbb{F}}(W_{t(a)}, W_{h(a)})^{\wedge}\right), \tag{2.2.1.3}$$

and that the algebraic group $\mathrm{GL}(\beta)$ acts naturally on $\mathrm{Rep}^{\square}_{\beta}$, with orbits consisting of isomorphism classes of representations. In addition, $\mathrm{PGL}(\beta)$ acts on $\mathrm{Rep}^{\square}_{\beta}$; it acts on each space $\mathrm{Hom}(W_{t(a)}, W_{h(a)})$ even though it does not have a sensible action on W_v for $v\in Q_0$. In analogy to Definition 1.4.1.1, we define the following groupoids of representations.

Definition 2.2.1.4. Let Q be a quiver. Define groupoids on $\operatorname{Spec}\mathbb{F}$-schemes by mapping an $\operatorname{Spec}\mathbb{F}$-scheme X to the following sets.

$$\mathrm{Rep}_{\beta} := X \mapsto \{\text{For each } v\in Q_0, \text{ a vector bundle } W_v/X \text{ of rank } \beta_v,$$
$$\text{for each } a\in Q_1, \phi_a \in \mathrm{Hom}_{\mathcal{O}_X}(W_{t(a)}, W_{h(a)})\}.$$

The definition of $\overline{\mathrm{Rep}}_\beta$ amounts to tracking the data of the the data of (W_v, ϕ_a) modulo simultaneous twists of (W_v) by a line bundle $\mathcal{L} \in \operatorname{Pic} X$.[3]

$$\begin{aligned}\overline{\mathrm{Rep}}_\beta := X \mapsto \{&\text{For } v, w, x \in Q_0, \text{ a vector bundle } H_{vw}/X \text{ of rank } \beta_v\beta_w,\\ &\mathcal{O}_X\text{-Azumaya algebra structure on } \mathcal{E}_v := H_{vv},\\ &\mathcal{O}_X\text{-module surjections } c_{vwx} : H_{vw} \otimes_{\mathcal{O}_X} H_{wx} \to H_{vx},\\ &\text{and, for } a \in Q_1, \phi_a \in H_{t(a)h(a)}\}\end{aligned}$$

such that the following conditions on c_{vwx} hold (following [BC09, §1.3.2]):

(UNIT) For all $v, w \in Q_0$, $c_{vvw} : \mathcal{E}_v \otimes H_{vw} \to H_{vw}$ (resp. $c_{vww} : H_{vw} \otimes \mathcal{E}_{ww} \to H_{vw}$) is compatible with the Azumaya algebra structure on $\mathcal{E}_v$.

(ASSO) For all $v, w, x, y \in Q_0$, the two natural maps $H_{vw} \otimes H_{wx} \otimes H_{xy} \to H_{vy}$ coincide.

(COM) For all $v, w \in Q_0$, $x \in H_{vw}, y \in H_{wv}$, $c_{vwv}(x \otimes y) = c_{wvw}(y \otimes x)$.

In analogy to Theorem 1.4.1.4, one can check that there are natural equivalences of $\operatorname{Spec}\mathbb{F}$-groupoids

$$\mathrm{Rep}_\beta \xrightarrow{\sim} [\mathrm{Rep}_\beta^\square/\mathrm{GL}(\beta)], \quad \overline{\mathrm{Rep}}_\beta \xrightarrow{\sim} [\mathrm{Rep}_\beta^\square/\mathrm{PGL}(\beta)]. \tag{2.2.1.5}$$

Now choose $\theta \in \mathbb{Z}^{Q_0}$, which we call a *character of* $\mathrm{Rep}(Q)$. In fact, such a character naturally determines a character of the Grothendieck group of $\mathrm{Rep}(Q)$ (cf. Definition 1.3.4.5). Simple Q-representations over $\mathbb{F}$ are in natural bijective correspondence with Q_0, sending $w \in Q_0$ to the representation (W_v, ϕ_a), where we have

$$W_v = \begin{cases} W_v = \mathbb{F} & v = w \\ W_v = \{0\} & v \neq w \end{cases}$$

and we have $\phi_a = \mathrm{id}_{W_v}$ if $h(a) = t(a) = v$, and $\phi_a = 0$ otherwise. This establishes a natural equivalence between characters of $\mathrm{Rep}(Q)$ characters of the Grothendieck group

[3]This corrects a small oversight in [Kin94] – he does not mention the twists of families of representations of Q by a line bundle and the resulting lack of representability of Rep_β.

$K_0(\mathrm{Rep}_Q(\mathbb{F}))$. We call a character of $\mathrm{Rep}(Q)$ *indivisible* if it is not the scalar multiple of another character.

We will define two notions of θ-semistability (resp. θ-stability), one intrinsic to the representation theory, and one being that of the GIT notion of semistability (resp. stability) of a point in $\mathrm{Rep}_\beta^\square$ for the χ_θ-linearized action of $\mathrm{GL}(\beta)$, where χ_θ is a character of $\mathrm{GL}(\beta)$ associated to θ.

First we give the representation theoretic definition for a general $\mathbb{F}$-algebra E, which makes sense for representations of a quiver Q even though we have not yet realized the representations of Q as the representations of its path algebra. As King points out, this definition makes sense for any abelian category; special cases of the notion include Mumford's notion of stability for vector bundles over a curve.

Definition 2.2.1.6 ([Kin94, Definition 1.1]). With $\mathbb{F}, E$, and a character

$$\theta : K_0(\mathrm{Rep}_E) \to \mathbb{Z}$$

as above,

(1) a representation $W \in \mathrm{Rep}_E$ is called θ-*semistable* if $\theta(M) = 0$, and for every subrepresentation $W' \subseteq W$, $\theta(W') \geq 0$.
(2) if $W \in \mathrm{Rep}_E$ is θ-semistable, and if, additionally, it satisfies the property

$$\theta(W') = 0 \implies W' = W \text{ or } W' = 0$$

for all subrepresentations $W' \subseteq W$, then we call W θ-*stable*.

We call two θ-semistable representations S-equivalent if they have identical composition factors in the full abelian subcategory of θ-semistable representations; the stable representations are the simple objects in this subcategory. We will not focus here on S-equivalence except when it coincides with the usual notion of equivalence.

In fact, as we pointed out in the introduction, we are mainly interested in families of representations in which θ-semistability implies θ-stability. As connected families of representations of finite-dimensional algebras have constant residual pseudorepresentation (by Theorem 1.1.7.4(5), for example), this condition is dependent upon the semisimplification of the representation. Semisimple representations amount to non-negative integer-valued linear combinations of simple representations and simple representations are a basis for $K_0(\mathrm{Rep}_E)$. We recall that this is the dimension vector (Definition 1.3.4.5) of the representation. Note that the condition $\theta(W) = 0$ of semisimplicity depends only on its dimension vector θ, and can be expressed in terms of the dot product of the dimension vector with θ, i.e.

$$\langle \beta, \theta \rangle = 0.$$

Using this terminology, we will give a condition such that semistability will imply stability. First, we require the following definition.

Definition 2.2.1.7. A *standard projection operator* on characters of $K_0(\mathrm{Rep}_E)$ sends $\theta \in K_0(\mathrm{Rep}_E)$ to its projection along the submodule spanned by a subset of the simple representations of E. We say that a standard projection operator is non-trivial on the support of $\beta \in K_0(\mathrm{Rep}_E)$ provided that $P\beta \neq \beta$.

Lemma 2.2.1.8. *Let θ be a character of the Grothendieck group $K_0(\mathrm{Rep}_E)$ of Rep_E, and let β be a dimension vector such that $\langle \beta, \theta \rangle = 0$. If for every standard projection P that is non-trivial on the support of β we have a strict inequality $\langle P\beta, P\theta \rangle \neq 0$, then for every representation $W \in \mathrm{Rep}_E$ with dimension vector β, W is θ-semistable if and only if it is θ-stable.*

Definition 2.2.1.9. If β and θ satisfy the conditions of Lemma 2.2.1.8, we say that β is *stabilizing with respect to θ*.

Example 2.2.1.10. Let $\rho_1, \ldots, \rho_n$ be simple representations of E, possibly with multiplicity except that we demand that $\rho_n \not\cong \rho_i$ for $1 \leq i < n$. Let β be the dimension vector

supported on the ρ_i, with these multiplicities. Later in Example 2.2.3.1, we will study this dimension vector relative to the character θ on $K_0(E)$ sending

$$\theta : \begin{array}{ll} \rho_i \mapsto 1 & 1 \leq i < n \\ \rho_n \mapsto -(n-1) & \end{array}$$

We see that β is stabilizing with respect to θ. The only way to get a sum of zero out of a projection to some subset of the isomorphism classes of the ρ_i is to choose the identity projection.

Now we give a character of $\mathrm{GL}(\beta)$ associated to θ.

Definition 2.2.1.11. For each $v \in Q_0$, write $\det_v$ for the determinant of the vth component of $\mathrm{GL}(\beta) \cong \times_{v \in Q_0} \mathrm{GL}(W_v)$. Then set χ_θ to be the character

$$\begin{aligned} \mathrm{GL}(\beta) &\longrightarrow \mathbb{G}_m \\ (g_v) &\mapsto \prod_{v \in Q_0} \det_v(g_v). \end{aligned}$$

This geometric notion of semistability (resp. stability) cuts out a subfunctor of $\mathrm{Rep}_\beta^\square$, which geometric invariant theory implies is open. We write

$$\mathrm{Rep}_\beta^{\square,s}(\theta) \subset \mathrm{Rep}_\beta^{\square,ss}(\theta) \subset \mathrm{Rep}_\beta^\square$$

for these open subschemes. Let

$$\mathrm{Rep}_\beta^{\square,ss}(\theta) \times \mathcal{L}(\eta)$$

denote the total space of the trivial line bundle $\mathcal{L}$ over $\mathrm{Rep}_\beta^{\square,ss}(\theta)$ with an action of $\mathrm{GL}(\beta)$ extended to this space by acting on $\mathcal{L}$ by η^{-1} with the character χ_θ^{-1}. Then standard GIT results give us that a quotient by $\mathrm{GL}(\beta)$ exists. This linearized GIT quotient space is

$$\mathrm{Rep}_\beta^\square /\!/ (\mathrm{GL}(\beta), \chi_\theta) := \mathrm{Proj}_{\mathbb{F}} \left(\bigoplus_{n \geq 0} \mathbb{F}[\mathrm{Rep}_\beta^{\square,ss}(\theta) \times \mathcal{L}(\chi_\theta^n)]^{\mathrm{GL}(\beta)} \right),$$

and it is projective over the GIT quotient

$$\mathrm{Rep}_\beta^\square // \mathrm{GL}(\beta) \cong \mathrm{Spec}\,\mathbb{F}.$$

We write

$$\overline{\mathrm{Rep}}_\beta^{ss}(\theta) := \mathrm{Rep}_\beta^\square // (\mathrm{GL}(\beta), \chi_\theta)$$

for this projective $\mathrm{Spec}\,\mathbb{F}$-scheme, and standard GIT theory gives an open $\mathbb{F}$-subscheme

$$\overline{\mathrm{Rep}}_\beta^{s}(\theta) \subseteq \overline{\mathrm{Rep}}_\beta^{ss}(\theta)$$

image of the χ_θ-stable locus in $\mathrm{Rep}_\beta^\square(\theta)$.

This theorem summarizes the GIT content of King's paper.

THEOREM 2.2.1.12 ([Kin94, Propositions 3.1-3.2, Proposition 5.2-5.3]).

(1) A point in $\mathrm{Rep}_\beta^\square$ *corresponding to a representation* W *of* Q *is* χ_θ*-semistable, i.e. lies in* $\mathrm{Rep}_\beta^{\square,ss}(\theta)$ *(resp.* χ_θ*-stable, i.e. lies in* $\mathrm{Rep}_\beta^{\square,s}(\theta)$*), if and only if* W *is* θ*-semistable (resp.* θ*-stable).*

(2) Two θ*-semistable representations correspond to points in* $\mathrm{Rep}_\beta^{\square,ss}(\theta)$ *with* $\mathrm{GL}(\beta)$*-orbits with overlapping Zariski closures in* $\mathrm{Rep}_\beta^{\square,ss}(\theta)$ *if and only if the representations are S-equivalent with respect to* θ*.*

(3) $\overline{\mathrm{Rep}}_\beta^{ss}(\theta)$ *is a coarse moduli space for families of* θ*-semistable modules up to S-equivalence*

(4) When the dimension vector β *is indivisible, the stable quotient* $\overline{\mathrm{Rep}}_\beta^{s}(\theta)$ *is a fine moduli space for families of* θ*-stable modules.*

Much the content of King's paper works toward proving Theorem 2.2.1.12 in the context of representations of a finite-dimensional $\mathbb{F}$-algebra, using an equivalence between quiver representations with a fixed dimension vector on one hand, and representations of a finite-dimensional $\mathbb{F}$-algebra E with induced pseudorepresentation $\bar{D}$ on the other. However, we need a more concrete realization of this equivalence than King provides. We are staying

within the context of quiver representations for the moment so that we can carefully identify an ample line bundle on $\overline{\mathrm{Rep}}^{ss}_{\beta}(\theta)$. Our goal is to give an explicit translation between representations of Q and representations of E, also translating conditions of semistability, etc., so that under the resulting closed immersion

$$\overline{\mathrm{Rep}}^{ss}_{E,\bar{D}}(\theta) \hookrightarrow \overline{\mathrm{Rep}}^{ss}_{Q,\beta}(\theta),$$

we can identify the pullback of the ample line bundle to the left side in terms of the moduli problem there. Therefore, let us conclude our overview of King's results by identifying this ample line bundle on the right side in terms of the moduli problem there.

An ample line bundle on the linearized GIT quotient

$$\overline{\mathrm{Rep}}^{ss}_{\beta}(\theta) = \mathrm{Proj}_{\mathbb{F}}\left(\bigoplus_{n\geq 0} \mathbb{F}[\mathrm{Rep}^{\square,ss}_{\beta}(\theta) \times \mathcal{L}(\chi^n_\theta)]^{\mathrm{GL}(\beta)}\right)$$

is the standard ample line bundle $\mathcal{O}^{\theta}(1)$ of this Proj construction,[4] which consists of the regular functions on $\mathrm{Rep}^{\square,ss}_{\beta}(\theta) \times \mathcal{L}(\chi^n_\theta)$ such that $\mathrm{GL}(\beta)$ acts by χ_θ. By reviewing the definition of χ_θ and the explicit form of the coordinate ring for $\mathrm{Rep}^{\square}_{\beta}$ in (2.2.1.3), we observe that this line bundle is the descent of the $\mathrm{PGL}(\beta)$-equivariantly linearized line bundle

$$\tilde{\mathcal{O}}^{\theta}(1) := \bigotimes_{v\in Q_0} \det(W_v)^{\otimes\theta(v)} \tag{2.2.1.13}$$

on $\mathrm{Rep}^{\square}_{\beta}$ to $\overline{\mathrm{Rep}}^{ss}_{\beta}(\theta)$. In saying that this bundle is $\mathrm{PGL}(\beta)$-equivariantly linearized, we are using the fact that the $\mathrm{GL}(\beta)$-linearization of $\tilde{\mathcal{O}}^{\theta}(1)$ descends to a $\mathrm{PGL}(\beta)$-linearization (cf. the comments on the action of $\mathrm{PGL}(\beta)$ on $\mathrm{Rep}^{\square}_{\beta}$ at (2.2.1.3)). This is the case because $\chi_\theta(\Delta) = \{1\}$; indeed, this is a condition for (GIT) semistability, which, in the translation between the representation theoretic and GIT notions of semistability for a representation W, corresponds to the condition $\theta(W) = 0$. Alternatively, we know that $\tilde{\mathcal{O}}^{\theta}(1)$ will descend

[4] It is not necessarily very ample.

to $\overline{\mathrm{Rep}}_\beta$ by (2.2.1.5), and one can check that in terms of the intrinsic definition of $\overline{\mathrm{Rep}}_\beta$,

$$\mathcal{O}^\theta(1) \cong \bigotimes_{v,w \in Q_0} (\wedge^{\beta_v \beta_w} H_{vw})^{\otimes n(v,w)} \tag{2.2.1.14}$$

for appropriate integers $n(v,w)$ dependent on $\theta \in \mathbb{Z}^{Q_0}$ and β. We derive these integers from (2.2.1.13) by recalling that the natural association is $H_{vw} = \mathrm{Hom}(W_v, W_w)$, and

$$\wedge^{\beta_v \beta_w} H_{vw} \cong (\wedge^{\beta_v} W_v)^{\otimes -\beta_w} \otimes (\wedge^{\beta_w} W_w)^{\otimes \beta_v}.$$

Indeed, the integers $n(v,w)$ are specified by the following

LEMMA 2.2.1.15. *Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ be indivisible, and let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{Z}^n$ such that the dot product $\beta \cdot \theta$ is zero. Then θ is a $\mathbb{Z}$-linear combination of the vectors $\mathbf{e}^{ij}$ for $1 \leq i < j \leq n$, where $\mathbf{e}^{ij} = (e_1^{ij}, \ldots, e_n^{ij})$ is given by*

$$e_k^{ij} = \begin{cases} +\beta_j & \text{if } k = i \\ -\beta_i & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}.$$

The integers $n(v,w)$ are the coefficients of $\mathbf{e}^{vw}$ in the expression for θ. Of course, since $\mathbf{e}_{vw} = -\mathbf{e}_{wv}$ for any $v, w \in Q_0$, we don't lose anything by restricting to $i < j$. Also, note that we are assuming that $\beta_v > 0$ for each $v \in Q_0$.

PROOF. We write $\mathbb{Z}^n$ for the $\mathbb{Z}$-module of n-tuples of integers. The standard dot product defines a perfect pairing $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, and pairing with β defines a $\mathbb{Z}$-module morphism $\mu = \langle \cdot, \beta \rangle : \mathbb{Z}^n \to \mathbb{Z}$. The indivisibility of β is equivalent to the surjectivity of μ. Therefore if we let M represent the kernel of μ, we have an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z} \to 0$$

which admits a splitting. We observe that M must be free of rank $n-1$ over $\mathbb{Z}$. We want to verify that the sub-$\mathbb{Z}$-module of M generated by $\mathbf{e}_{ij}, 1 \leq i < j \leq n$, which we will denote by N, is in fact equal to M. Let P denote the cokernel of $N \hookrightarrow M$, so that we have an exact

sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0.$$

Since $\mathbf{e}_{1j}, 2 \leq j \leq n$, are linearly independent over $\mathbb{Q}$, N is free of rank $n-1$ and P is finite in cardinality. We will complete the proof by showing that $P \otimes_{\mathbb{Z}} \mathbb{F}_p = 0$ for all rational primes p, and we will do this by showing that $\mathbf{e}_{ij}$ span M modulo p for each p.

Fix a prime p. Because β is indivisible, there exists some $v \in Q_0$ such that $\beta_v \not\equiv 0 \pmod{p}$. Without loss of generality, assume $v = 1$. Then the $n-1$ elements $\mathbf{e}_{1j}$ are linearly independent modulo p. This shows that the image of $N \otimes_{\mathbb{Z}} \mathbb{F}_p$ in $M \otimes_{\mathbb{Z}} \mathbb{F}_p$ is $n-1$-dimensional, and therefore is equal to $M \otimes_{\mathbb{Z}} \mathbb{F}_p$. □

2.2.2. Finite Dimensional Algebras and Path Algebras. Now we prepare background material on finite-dimensional algebras to show that their representations can be expressed as representations of quivers.

One step will be to show that given any algebra, its abelian category of representations is equivalent to that of some basic algebra.

Definition 2.2.2.1. Let E be a finite-dimensional $\mathbb{F}$-algebra.

(1) We call E *basic* provided that it has a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotent such that Ee_i and Ee_j are not isomorphic as $\mathbb{F}$-algebras for all $i \neq j$.
(2) We call E *connected* provided that it cannot be written as a proper product of algebras $E \cong E_1 \times E_2$.

One can show (cf. [ASS06, Proposition I.6.2]) that the simple representations of a basic algebra are all one-dimensional, or, equivalently, that if E is basic, then

$$E/J(E) \cong \mathbb{F}^n, \quad \text{some } n \geq 0. \tag{2.2.2.2}$$

If $\{e_1, \ldots, e_n\}$ are a complete set of primitive orthogonal idempotents for E, then each simple representation into $\mathbb{F}$ is given by sending e_i to 1 for a single i, and $J(E)$ and the remaining $e_j, j \neq i$, to 0. Therefore, pseudorepresentations of a basic algebra E are in

bijective correspondence with n-tuples of non-negative integers, where we have numbered a complete set of n primitive idempotents correspondingly.

Next, we describe the path algebra $\mathbb{F}Q$ of a quiver Q. This is a basic algebra whose abelian category of representations is naturally equivalent to the abelian category of representations of a given quiver Q. We will describe this equivalence below.

Definition 2.2.2.3. Let Q_0 be a finite quiver.

(1) The *path algebra* $\mathbb{F}Q$ of Q is the quotient of the free algebra on the set $Q_0 \cup Q_1$ where we write ε_v for $v \in Q_0$ and α_a for $a \in Q_1$, subject to the relations

$$\varepsilon_v \varepsilon_w = \delta_{vw}\varepsilon_v, \varepsilon_v \alpha_a = \delta_{vt(a)}\alpha_a, \alpha_a \varepsilon_v = \delta_{h(a)v}\alpha_a,$$

$$\alpha_a \alpha_b = 0 \text{ if } h(a) \neq t(b),$$

$$\sum_{v \in Q_0} \varepsilon_v = 1.$$

(2) Let $J(\mathbb{F}Q)$ be the Jacobson radical of $\mathbb{F}Q$, which one can check is generated by the arrows. We call $\mathcal{I} \subset \mathbb{F}Q$ an *admissible ideal* provided that there exists $m \geq 2$ such that $J(Q)^m \subseteq \mathcal{I} \subseteq J(Q)^2$.

We observe that $\mathbb{F}Q$ is a basic algebra.

Remark 2.2.2.4. There is another sensible definition of $\mathbb{F}Q$ when Q is not finite, expressing $\mathbb{F}Q$ as ring graded by the lengths of paths. However, this definition does not have a unit when Q_0 is infinite, and is equivalent to the definition given above when Q is finite.

Now we give a construction of a quiver from a connected basic algebra. This is an inverse construction to the construction of the path algebra.

Definition 2.2.2.5 (cf. [ASS06, Definition II.3.1]). Let E be a basic connected $\mathbb{F}$-algebra. Number off the complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$. Now define the *(ordinary) quiver* $Q_E = (Q_0, Q_1)$ of E by $Q_0 = \{v_1, \ldots, v_n\}$ in correspondence with the idempotents, and each arrow a in Q_1 consists of a head $h(a) = v_i$, tail $t(a) = v_j$, and an element of a fixed $\mathbb{F}$-basis of $e_i(J(E)/J(E)^2)e_j$.

The term "ordinary" is not standard notation, but we mention it as there are other quivers associable to E such as its Auslander-Reiten quiver. We observe that if E is finite-dimensional, then Q is finite. One can check that Q does not depend on the choices made in order to construct it. Also, given a primitive idempotent $e \in E$, we will often simply write v_e for the vertex of Q associated to e.

THEOREM 2.2.2.6 (cf. [ASS06, Theorem II.3.7]). *Let E be a basic connected $\mathbb{F}$-algebra. Then there exists a surjection of $\mathbb{F}$-algebras from the path algebra of a connected quiver, namely, from the path algebra of its ordinary quiver*

$$\mathbb{F}Q_E \longrightarrow E,$$

inducing an isomorphism of E with the quotient $Q_E/\mathcal{I}$ of $\mathbb{F}Q_E$ by an admissible ideal.

PROOF. Map the set Q_0 into $\mathbb{F}Q_E$ by sending v to ε_v. By definition, an arrow $a \in Q_1$ such that $t(a) = v$ and $h(a) = w$ is an element of some basis for $e_w(J(E)/J(E)^2)e_v$. Choose a lift of this basis element to $J(E)$ and map the set Q_1 to $\mathbb{F}Q_E$ according to the choices above. This map is, in fact, surjective with admissible kernel [ASS06, Theorem II.3.7], and we can already see that the kernel is contained in $J(E)^2$. □

Let us explicitly describe an equivalence between representations of $Q = Q_E$ and representations of the path algebra $\mathbb{F}Q$. We will give the construction a representation of $\mathbb{F}Q$ out of a representation of Q in terms of the algebraic stacks

$$\overline{\mathrm{Rep}}_{Q,\beta} \xrightarrow{\sim} \overline{\mathrm{Rep}}_{\mathbb{F}Q,\bar{D}_\beta},$$

since we are interested in keeping track of the line bundle $\mathcal{O}^\theta(1)$ of (2.2.1.14) on $\overline{\mathrm{Rep}}_{Q,\beta}$ after its pullback to $\overline{\mathrm{Rep}}_{\mathbb{F}Q}$ or $\overline{\mathrm{Rep}}_E$. Here $\bar{D}_\beta$ denotes a pseudorepresentation of $\mathbb{F}Q$ that corresponds via Theorem 1.3.1.1 to the direct sum of representations

$$\bigoplus_{v \in Q_0} M_v^{\oplus \beta_v},$$

where M_v is the one-dimensional simple representation on which only e_v acts as 1 and $e_w, w \neq v$ and Q_1 act as 0. This is the semisimple representation of $\mathbb{F}Q$ associated to the $|Q_0|$-tuple $(e_v)_{v\in Q_0}$, i.e. the direct sum with multiplicity $(e_v)_{v\in Q_0}$ over the $|Q_0|$ simple (1-dimensional) representations of $\mathbb{F}Q$ corresponding to e_v, cf. (2.2.2.2). For this construction in the more usual framed setting (i.e. for elements of $\mathrm{Rep}^{\square}_{Q,\beta}$), see e.g. [ASS06, Theorem III.1.6].

Let $(H_{vw}, \phi_a) \in \overline{\mathrm{Rep}}_\beta(X)$ be a representation of Q over $X \in \mathrm{Sch}_{\mathrm{Spec}\,\mathbb{F}}$ of dimension vector β, as in Definition 2.2.1.4. Define

$$\mathcal{E} = \mathcal{E}(W_v, \phi_a) := \bigoplus_{v,w\in Q_0} H_{vw}. \tag{2.2.2.7}$$

The structure maps c_{vwx} of Definition 2.2.1.4 endow $\mathcal{E}$ with the structure of an Azumaya $\mathcal{O}_X$-algebra, where the $\mathcal{O}_X$-algebra structure map induced by the sum over $v \in Q_0$ of the maps $\mathcal{O}_X \to \mathcal{E}_v \cong H_{vv}$. One can readily check that the map of sets

$$\begin{aligned} Q = Q_0 \cup Q_1 &\longrightarrow \mathcal{E} \\ \varepsilon_v &\mapsto \mathrm{id}_v \in \mathcal{E}_v \subset \mathcal{E} \\ \alpha_a &\mapsto \phi_a \in H_{t(a)h(a)} \subset \mathcal{E}. \end{aligned}$$

extends to a homomorphism $\mathbb{F}Q \otimes_{\mathbb{F}} \mathcal{O}_X \to \mathcal{E}$ of $\mathcal{O}_X$-algebras. Here $\mathcal{E}_v$ and $H_{t(a)h(a)}$ are being considered as ($\mathcal{O}_X$-module) summands of $\mathcal{E}$. Let $\bar{D}_\beta : \mathbb{F}Q \to \mathbb{F}$ be the pseudorepresentation associated to the semisimplification $\mathbb{F}Q \to \bigoplus_{v\in Q_0} \mathcal{E}_v$, which one can check from the definitions. We have constructed a map of $\mathrm{Spec}\,\mathbb{F}$-groupoids $\overline{\mathrm{Rep}}_{Q,\beta} \to \overline{\mathrm{Rep}}_{\mathbb{F}Q,\bar{D}_\beta}$; one can examine the inverse construction (cf. [ASS06, Theorem III.1.6]) to see that this is an equivalence, and use e.g. [Kin94, Proposition 5.2] to show that the map is algebraic.

Now let E be a basic connected $\mathbb{F}$-algebra. Choose a (non-canonical) surjection $\mathbb{F}Q_E \twoheadrightarrow E$ described in the proof of Theorem 2.2.2.6. Choose also a Cayley-Hamilton pseudorepresentation $\bar{D}_\beta : E \to \mathbb{F}$, where we let β denote the dimension vector of Q corresponding to the semisimple representation of $\mathbb{F}Q$ induced by the semisimple representation $\rho^{ss}_{\bar{D}_\beta}$. We have a

closed immersion

$$\mathrm{Rep}^{\square}_{E,\bar{D}_\beta} \hookrightarrow \mathrm{Rep}^{\square}_{\mathbb{F}Q,\bar{D}_\beta} \xrightarrow{\sim} \mathrm{Rep}^{\square}_{Q,\beta},$$

where the maps are equivariant for the natural action of $\mathrm{PGL}(\beta)$. Therefore we have a closed immersion of algebraic stacks

$$\overline{\mathrm{Rep}}_{E,\bar{D}_\beta} \hookrightarrow \overline{\mathrm{Rep}}_{\mathbb{F}Q,\bar{D}_\beta} \xrightarrow{\sim} \overline{\mathrm{Rep}}_{Q,\beta},$$

and we observe that the line bundle $\mathcal{O}^\theta(1)$, expressed in terms of the data H_{vw} on $\overline{\mathrm{Rep}}_{Q,\beta}$ in (2.2.1.14), pulls back via the association (2.2.2.7) to a line bundle that can be constructed out of appropriate sub-modules of the universal Azumaya algebra $\mathcal{E}$ on $\overline{\mathrm{Rep}}^{\circ}_{E,\bar{D}_\beta}$ receiving the universal representation of E with pseudorepresentation $\bar{D}_\beta$.

Now, in the case of basic algebras, we have achieved our goal of identifying the line bundle $\mathcal{O}^\theta(1)$. Let us extend this to the general case of a finite-dimensional algebra E.

First we explain how to associate a basic algebra to a general $\mathbb{F}$-algebra.

Definition 2.2.2.8. Let E be a finite-dimensional $\mathbb{F}$-algebra with a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$. Partition these idempoents according to the equivalence relation $e_i \sim e_j$ if and only if there is a $\mathbb{F}$-algebra isomorphism $Ee_i \xrightarrow{\sim} Ee_j$, and choose a representatives $e_{j_1}, \ldots, e_{j_b}$. Write e_E for $e_E = \sum_1^b e_{j_i}$. Then the $\mathbb{F}$-subalgebra

$$E^b := e_E E e_E$$

as a *basic algebra associated to* E.

This subalgebra E^b of E is clearly not canonical, but one can show that the isomorphism class of E^b does not depend on the choices above. For the time being, we fix such choices and the resulting $\mathbb{F}$-subalgebra $E^b \subseteq E$.

Remark 2.2.2.9. For this association to work, we must have a complete set of orthogonal idempotents as the definition above requires. When we apply this construction, we will always work with finite-dimensional algebras E whose semisimple quotient $E/J(E)$ by the Jacobson radical is a product of matrix algebras. Such a set of idempotents clearly exists

for a product of matrix algebras, and we then apply the well known fact that one can (non-canonically) lift idempotents over a nilpotent ideal (the Jacobson radical of a finite dimension algebra is nilpotent).

Here we see that we can reduce the study of representations of finite-dimensional algebras to the study of the representations of basic algebras.

THEOREM 2.2.2.10 (cf. [ASS06, Corollary I.6.10]). *Let E be a finite-dimensional $\mathbb{F}$-algebra. Then E contains a basic $\mathbb{F}$-subalgebra $E^b := e_E E e_E$ as above, and the natural restriction functor*

$$\mathrm{res} : \mathrm{Rep}_E(\mathbb{F}) \longrightarrow \mathrm{Rep}_{E^b}(\mathbb{F})$$

$$V \mapsto e_E V$$

is an equivalence of abelian categories with quasi-inverse $- \otimes_{E^b} E e_E$.

Now let us observe that this equivalence extends from representations with coefficients $\mathbb{F}$ to functors of families of representations over $\operatorname{Spec}\mathbb{F}$-schemes. Fix a choice of idempotents to produce $\iota : E^b \hookrightarrow E$ as above. We observe that the restriction functor res extends naturally to families of representations over $\mathbb{F}$-schemes X, via the natural transformations

$$\mathrm{res} : \mathrm{Rep}_E \longrightarrow \mathrm{Rep}_{E^b},$$

$$V \mapsto e_E V,$$

$$\rho : E \otimes_{\mathbb{F}} \mathcal{O}_X \to \mathrm{End}_{\mathcal{O}_X}(V) \mapsto (x \mapsto \rho(e_E) x \rho(e_E)) \circ \rho \circ (\iota \otimes \mathcal{O}_X).$$

of $\operatorname{Spec}\mathbb{F}$-scheme functors with a quasi-inverse as in the theorem above, so that it is an isomorphism. The induced isomorphic functor $\overline{\mathrm{res}} : \overline{\mathrm{Rep}}_E \to \overline{\mathrm{Rep}}_{E^b}$ is given by sending $\rho : E \otimes_{\mathbb{F}} \mathcal{O}_X \to \mathcal{E}$ to

$$\overline{\mathrm{res}}(\rho) : E^b \otimes_{\mathbb{F}} \mathcal{O}_X \longrightarrow e_E \mathcal{E} e_E. \tag{2.2.2.11}$$

Now that we have given the association of representations, we are able to calculate the line bundle $\tilde{\mathcal{O}}^\theta(1)$ on $\overline{\mathrm{Rep}}_{E,\bar{D}}$ in terms of data native to the moduli problem for $\overline{\mathrm{Rep}}_{E,\bar{D}}$ we summarize this calculation and the choices involved in this

PROPOSITION 2.2.2.12. *Let E be a $\mathbb{F}$-algebra. Fix a choice of*

(1) a set of primitive idempotents $\{e_{j_v}\}$ indexed by a finite set $Q_0, v \in Q_0$, representing the isomorphism classes of Definition 2.2.2.8, and the resulting idempotent $e_E = \sum_{Q_0} e_{j_v}$ and basic subalgebra $E^b = e_E E e_E \subseteq E$,

(2) a $\mathbb{F}$-basis for $e_{j_v}(J(E^b)/J(E^b)^2)e_{j_v}$, which produces the $\mathbb{F}$-algebra homomorphism $\mathbb{F}Q_{E^b} \twoheadrightarrow E^b \hookrightarrow E$ of Theorem 2.2.2.6,

(3) a Cayley-Hamilton pseudorepresentation $\bar{D} : E \to \mathbb{F}$, the associated semisimple representation $\rho_{\bar{D}}^{ss}$ of E, the semisimple representation $e_E\rho_{\bar{D}}^{ss}$ of E^b, and the resulting dimension vector β for Q corresponding to the induced semisimple representation of $\mathbb{F}Q$ via $\mathbb{F}Q \twoheadrightarrow E^b$.

These choices define morphisms

$$\overline{\mathrm{Rep}}_{E,\bar{D}} \xrightarrow{\sim} \overline{\mathrm{Rep}}_{E^b,\bar{D}} \hookrightarrow \overline{\mathrm{Rep}}_{\mathbb{F}Q_E,\bar{D}} \longrightarrow \overline{\mathrm{Rep}}_{Q,\beta}.$$

Let $\theta : K_0(E) \longrightarrow \mathbb{Z}$. Under these maps with the choices above, the line bundle $\mathcal{O}^\theta(1)$ pulls back to

$$\bigotimes_{v,w\in Q_0} \left(\wedge^{\beta_v\beta_w} (e_{j_v}\mathcal{E}e_{j_w})\right)^{n(v,w)}, \tag{2.2.2.13}$$

where $n(v,w)$ are a choice of integers as specified in (2.2.1.14).

Recall that Lemma 2.2.1.15 shows that there exist integers $n(v,w)$ with the properties demanded by (2.2.1.14).

PROOF. On $\overline{\mathrm{Rep}}_{Q,\beta}$, we recall from (2.2.1.14) that we have a natural isomorphism

$$\mathcal{O}^\theta(1) \cong \bigotimes_{v,w\in Q_0} (\wedge^{\beta_v\beta_w} H_{vw})^{\otimes n(v,w)}.$$

We see in (2.2.2.7) that H_{vw} pulls back to $\overline{\mathrm{Rep}}_{\mathbb{F},\bar{D}_\beta}$ as a direct summand of $\mathcal{E}'$, namely $\varepsilon_v\mathcal{E}'\varepsilon_w$, where $\mathcal{E}'$ is the universal Azumaya algebra on $\overline{\mathrm{Rep}}_{\mathbb{F}Q,\bar{D}_\beta}$. These idempotents $\varepsilon_v \in \mathbb{F}Q_E$ correspond to the chosen idempotents $\{e_{j_v}\}_{v\in Q_0}$ of E^b; the homomorphism from $\mathbb{F}Q_E$ to $\mathcal{E}'$ factors through E^b, so that the data of $\mathcal{E}'$ and $\varepsilon_v\mathcal{E}\varepsilon_w$ still make sense on the closed substack

$\overline{\mathrm{Rep}}_{E^\flat,\bar{D}_\beta} \subseteq \mathrm{Rep}_{\mathbb{F}Q,\bar{D}_\beta}$. Finally, if we write $\mathcal{E}$ for the universal Azumaya algebra on $\overline{\mathrm{Rep}}_{E,\bar{D}}$, we see in (2.2.2.11) that $\mathcal{E}' \simeq e_E\mathcal{E}e_E$. So the pullback of H_{vw} to $\mathrm{Rep}_{E,\bar{D}}$ from $\overline{\mathrm{Rep}}_{Q,\beta_{\bar{D}}}$ is expressible in terms of its universal Azumaya algebra $\mathcal{E}$ as

$$e_{j_v}e_E\mathcal{E}e_Ee_{j_w} \cong e_{j_v}\mathcal{E}e_{j_w}. \qquad \square$$

By combining King's Theorem 2.2.1.12 with this calculation, we have the following deduction. Recall the notation of θ-(semi)stability of a representation of E from Definition 2.2.1.6.

Corollary 2.2.2.14. *Let E be a $\mathbb{F}$-algebra. Choose a character $\theta : K_0(E) \to \mathbb{Z}$ and a pseudorepresentation $\bar{D} : E \to \mathbb{F}$ with associated dimension vector $\beta_{\bar{D}} \in \mathbb{Z}^{K_0(E)}$ such that $\langle\beta,\theta\rangle = 0$. If $\beta_{\bar{D}}$ is indivisible and $\langle\beta,\theta\rangle = 0$, then the θ-stable locus of representations of E descends to a quasi-projective subscheme $\overline{\mathrm{Rep}}^s_{E,\bar{D}}(\theta)$ of $\overline{\mathrm{Rep}}^\circ_{E,\bar{D}}$. If, moreover, $\beta_{\bar{D}}$ is stabilizing with respect to θ, then the $\overline{\mathrm{Rep}}^s_{E,\bar{D}}(\theta)$ is a projective subscheme of the algebraic stack $\overline{\mathrm{Rep}}^\circ_{E,\bar{D}}$, with ample line bundle given in terms of the universal Azumaya algebra on $\overline{\mathrm{Rep}}_{E,\bar{D}}$ by* (2.2.2.13).

Note that while this subscheme is projective, it is not closed in $\overline{\mathrm{Rep}}_{E,\bar{D}}$ in any non-trivial case. The usual geometric situation is just like the standard construction of $\mathbb{P}^n$ as $\mathbb{A}^{n+1}\backslash\{0\}/\mathbb{G}_m$, lying inside $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

Proof. By Theorem 2.2.1.12, the indivisibility of β implies that the θ-stable locus $\overline{\mathrm{Rep}}^s_{E,\bar{D}}(\theta)$ of the χ_θ-linearized GIT quotient of $\mathrm{Rep}^{\square,\circ}_{E,\bar{D}}$ is a fine moduli space. As a result, we have an immersion $\overline{\mathrm{Rep}}^s_{E,\bar{D}}(\theta) \hookrightarrow \overline{\mathrm{Rep}}^\circ_{E,\bar{D}}$. When $\beta_{\bar{D}}$ is stabilizing with respect to θ, then θ-semistability of representations of E is equivalent to θ-stability by Lemma 2.2.1.8. Therefore $\overline{\mathrm{Rep}}^s_{E,\bar{D}}(\theta)$ is projective and a subscheme of $\overline{\mathrm{Rep}}^\circ_{E,\bar{D}}$, and Proposition 2.2.2.12 identifies a line bundle on $\overline{\mathrm{Rep}}^\circ_{E,\bar{D}}$ that is ample on $\overline{\mathrm{Rep}}^s_{E,\bar{D}}(\theta)$. $\square$

2.2.3. Examples of Projective Moduli Spaces. We will give the motivating example, suggested by Kisin, of a moduli space of representations that is projective relative to

the moduli space of pseudorepresentations below. First we give an example of the projective spaces constructed above.

EXAMPLE 2.2.3.1. Let R be a finitely generated algebra over a field $\mathbb{F}$. Choose n simple representations $\rho_1, \ldots, \rho_n$ of R of dimension $d_i < \infty$ over $\mathbb{F}$. We stipulate that $\rho_n \not\cong \rho_i$ for $1 \leq i < n$, but allow multiplicity among the ρ_i otherwise. Let ρ represent the d-dimensional direct sum $\bigoplus_1^n \rho_i$. Let D_ρ be the pseudorepresentation of R associated to ρ, i.e. $D_\rho := \det \circ \rho$. We will illustrate in this example that the moduli space of families of representations of R whose semisimplification is ρ and whose unique simple quotient is ρ_n is in fact projective over the point $D_\rho \in \mathrm{PsR}_R^d(\mathbb{F})$ with residue field $\mathbb{F}_{D_\rho}$.

Let E be the universal Cayley-Hamilton representation of R over D_ρ, i.e. $E := R/\mathrm{CH}(D)$. By abuse of notation, we will write D for the factorization of D through E, and likewise for the representations ρ_i. It is visible that D_ρ is split over $\mathbb{F}$. We know that E is finite-dimensional over $\mathbb{F}$ by Corollary 1.2.2.9, and we know from Theorem 1.4.3.1 that

$$\overline{\mathrm{Rep}}_{R,D} \cong \overline{\mathrm{Rep}}_{E,D}.$$

Consider now a character θ on $K_0(E)$ sending

$$\theta : \begin{array}{ll} \rho_i \mapsto 1 & 1 \leq i < n \\ \rho_n \mapsto -(n-1) & \end{array}$$

Write $\beta = \beta_\rho$ for the dimension vector of ρ, which is essentially the image of ρ in $K_0(E)$. Now let us consider the projective $\operatorname{Spec} \mathbb{F}_{D_\rho}$-scheme $\overline{\mathrm{Rep}}_{E,D}^{ss}(\theta)$. We want to show that the conditions of Corollary 2.2.2.14 are satisfied. Here are the conditions:

- Since $\theta \cdot \beta = 0$, i.e. $\theta(\rho) = 0$, it is possible for this space to be non-empty (the first condition of semisimplicity in Definition 2.2.1.6 is satisfied)
- β is indeed indivisible – this is guaranteed because ρ_n appears with multiplicity 1 as a factor of ρ.
- It is also the case that β is stabilizing with respect to θ (see Definition 2.2.1.9). Simply see Example 2.2.1.10.

With these conditions satisfied, Corollary 2.2.2.14 now tells us that

$$\mathrm{Rep}^{s}_{E,D}(\theta) \xrightarrow{\sim} \overline{\mathrm{Rep}}^{ss}_{E,D}(\theta)$$

is a fine moduli space for θ-semistable (equivalently, θ-stable) representations of E lying over $D \in \mathrm{PsR}^d_E$, i.e. it is naturally a subscheme of $\overline{\mathrm{Rep}}_{E,D}$. We also know from Corollary 2.2.2.14 that the restriction of $\psi : \overline{\mathrm{Rep}}^d_E \to \mathrm{PsR}^d_E$ to $\mathrm{Rep}^{ss}_{E,D}(\theta)$ is projective.

Finally, we give a translation of the last bullet point above: a representation M with dimension vector β is θ-semistable (equivalently, θ-stable) if and only if its unique simple quotient is ρ_n. For if there exists some other simple quotient of M, then there exists a proper subrepresentation M' of M with ρ_n as a Jordan-Hölder factor, implying that $\theta(M') < 0$ and that M is not θ-semistable. Conversely, if M is not θ-semistable, there must exist some subrepresentation $M' \subset M$ such that $\theta(M') < 0$, which implies that ρ_n is a Jordan-Hölder factor of M' and that M/M' (which must have some simple quotient) has a simple quotient not isomorphic to ρ_n.

We were motivated to investigate these projective spaces of representations by an example and suggestion of Kisin [Kis09a, §3.2, esp. Remark 3.2.7]. Kisin gives his construction and suggestion in the context of continuous representations of a profinite group, but we will see later (e.g. Theorem 3.2.4.1) that the continuous representations of a profinite algebra over a fixed finite field-valued pseudorepresentation amounts to the representations of a certain finite-dimensional algebra over the finite field. Therefore these constructions of projective spaces apply to Kisin's context. Then the next paragraph §2.2.4 shows that deformations of these projective spaces are projective, as he suggests.

Definition 2.2.3.2. Let E be a finite-dimensional algebra over $\mathbb{F}$ with pairwise non-isomorphic simple representations $\rho_i, 1 \leq i \leq n$, each of dimension d_i. Write $\rho = \bigoplus_1^n \rho_i$ and let D_ρ (resp. β_ρ) be the corresponding pseudorepresentation (resp. dimension vector). Let $\overline{\mathrm{Rep}}'_{D_\rho} \subset \overline{\mathrm{Rep}}_{D_\rho}$ be the full subgroupoid of families of representations $E \otimes_{\mathbb{F}} \mathcal{O}_X \to \mathcal{E}$ which

locally in the Zariski topology are of the form

$$\rho \simeq \begin{pmatrix} \rho_1 & * & \cdots & \cdots \\ 0 & \rho_2 & * & \cdots \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \rho_n \end{pmatrix} \tag{2.2.3.3}$$

with the additional condition that follows. When we write

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = M$$

for the filtration where $L_i/L_{i-1} \simeq \rho_i$, we stipulate that the extension class of M/L_{i-1} as an extension of M/L_i by ρ_i is non-trivial.[5]

Remark 2.2.3.4. As Kisin notes, this condition guarantees that such representations have no non-trivial automorphisms, making the isomorphism (2.2.3.3) unique. The uniqueness only holds once one considers representations as maps into Azumaya algebras (an object of $\overline{\mathrm{Rep}}$) instead of vector bundles with an action (an object of Rep). In the latter case, the trivial (scalar) automorphisms are taken into account.

We immediately observe that this groupoid is a subgroupoid of the projective $\mathbb{F}_{D_\rho}$-subscheme $\overline{\mathrm{Rep}}^{ss}_{D_\rho}(\theta)$ of $\overline{\mathrm{Rep}}_{D_\rho}$ described in Example 2.2.3.1 above, where θ is the character of $K_0(\mathrm{Rep}_E)$ with $\theta : \rho_i \mapsto 1$ for $1 \leq i < n$ and $\theta(\rho_n) = -(n-1)$. Indeed, ρ_n is the unique simple quotient of any object of $\overline{\mathrm{Rep}}'_{D_\rho}$, and this condition defines $\overline{\mathrm{Rep}}^{ss}_{D_\rho}(\theta)$ in $\overline{\mathrm{Rep}}_{D_\rho}$. We claim that $\overline{\mathrm{Rep}}'_{D_\rho}$ is a closed subscheme of $\overline{\mathrm{Rep}}^{ss}_{D_\rho}(\theta)$, and is therefore projective over $\mathrm{Spec}\,\mathbb{F}_{D_\rho}$ as well.

Proof. First we fix certain idempotents in E. We know from Lemma 1.2.3.1 that the representation ρ has kernel precisely the Jacobson radical $J(E)$ of E, and draws a surjection

$$\rho : E \twoheadrightarrow \prod_{i=1}^{n} M_{d_i}(\mathbb{F}).$$

[5]Actuallly, Kisin uses the dual condition that the extension L_i of ρ_i by L_{i-1} is non-trivial.

Let e_i represent a (non-canonical) lift to E of the idempotent of $E/J(E)$ corresponding to the identity element of $M_{d_i}(\mathbb{F})$ via ρ (see Remark 2.2.2.9). One can quickly check that they remain pairwise orthogonal.

Let $\mathcal{E}^{\text{univ}}$ be the universal Azumaya algebra over $\text{Rep}^{ss}_{D_\rho}(\theta)$, receiving the universal representation η^{univ} from $E \otimes_{\mathbb{F}} \mathcal{O}_{\text{Rep}^{ss}_{D_\rho}(\theta)}$. These idempotents $\eta^{\text{univ}}(e_i)$ along with the standard reduced trace on $\mathcal{E}^{\text{univ}}$ correspond to the additional structure of a generalized matrix algebra of type $(d_1, \ldots, d_n)$ on the Azumaya algebra $\mathcal{E}^{\text{univ}}$; we will use the notation of Lemma 2.3.1.4 describing generalized matrix algebras to offer additional clarity to the following calculations without requiring any additional theory of generalized matrix algebras. We have an isomorphism

$$\mathcal{E}^{\text{univ}} \cong \begin{pmatrix} M_{d_1}(\mathcal{A}_{1,1}) & M_{d_1 \times d_2}(\mathcal{A}_{1,2}) & \cdots & M_{d_1 \times d_n}(\mathcal{A}_{1,n}) \\ M_{d_2 \times d_1}(\mathcal{A}_{2,1}) & M_{d_2}(\mathcal{A}_{2,2}) & \cdots & M_{d_2 \times d_n}(\mathcal{A}_{2,n}) \\ \vdots & \vdots & \vdots & \vdots \\ M_{d_n \times d_1}(\mathcal{A}_{n,1}) & M_{d_2}(\mathcal{A}_{n,2}) & \cdots & M_{d_n}(\mathcal{A}_{n,n}) \end{pmatrix},$$

where the $\mathcal{A}_{ij}$ are line bundles on $\text{Rep}^{ss}_{D_\rho}(\theta)$ (with a canonical trivialization for $\mathcal{A}_{ii}$ for each i) and the algebra structure is determined by canonical isomorphisms

$$M_{d_i d_j}(\mathcal{A}_{ij}) \xrightarrow{\sim} e_i R e_j.$$

Consider a representation $(\eta : E \otimes_{\mathbb{F}} \mathcal{O}_X \to \mathcal{E}) \in \text{ob}\, \overline{\text{Rep}}^{ss}_{D_\rho}(\theta)$, so that $\eta = \eta^{\text{univ}} \otimes_{\overline{\text{Rep}}^{ss}_{D_\rho}(\theta)} \mathcal{O}_X$. It inherits the structure of a generalized matrix algebra from $\mathcal{E}^{\text{univ}}$, which we denote again with $\mathcal{O}_X$-line bundles $\mathcal{A}_{ij}$, abusing notation. The condition that η belongs to the subgroupoid $\overline{\text{Rep}}'_{D_\rho}$ is equivalent to the triviality of the projection of the image of E in $\mathcal{E}$ to $M_{d_i d_j}(\mathcal{A}_{ij})$ via

$$x \mapsto \eta(e_i) \cdot x \cdot \eta(e_j)$$

for all pairs (i, j) such that $1 \leq j < i \leq n$. To illustrate this equivalence, notice that the condition for the pair $(n, n-1)$ is equivalent to the condition (in the language of Definition 2.2.3.2) that the extension M/L_{n-2} of M/L_{n-1} by ρ_{n-1} is non-trivial; following this, the

condition that the extension M/L_{n-3} of M/L_{n-2} by ρ_{n-2} is non-trivial is expressed by the pairs $(n, n-1), (n, n-2), (n-1, n-2)$; and so forth.

Notice in particular that the condition for pairs $(n, j), 1 \leq j < n$, is what defines the subscheme $\overline{\mathrm{Rep}}^{ss}_{D_\rho}(\theta) \subset \overline{\mathrm{Rep}}_{D_\rho}$; this shows that Rep'_{D_ρ} is contained in the θ-semistable locus.

It remains to show that the locus cut out by the condition that

$$\eta(e_i E e_j) \subset M_{d_i d_j}(\mathcal{A}_{ij}) \text{ is trivial for } 1 \leq j < i \leq n.$$

Let $N_{ij} \subset M_{d_i d_j}(\mathcal{A}_{ij})$ be the $\mathcal{O}_X$-submodule of $M_{d_i d_j}(\mathcal{A}_{ij})$ generated by $\eta(e_i x e_j)$ for $x \in E$. This submodule will be trivial over the locus of support for the quotient module $M_{d_i d_j}(\mathcal{A}_{ij})/N_{ij}$, which is a closed subscheme.

Now we can apply these constructions to the universal representation η^{univ} over $\overline{\mathrm{Rep}}^{ss}_{D_\rho}(\theta)$. The intersection of all of these support loci is therefore the closed subscheme $\overline{\mathrm{Rep}}'_{D_\rho} \subset \overline{\mathrm{Rep}}^{ss}_{D_\rho}(\theta)$. Consequently, $\overline{\mathrm{Rep}}'_{D_\rho}$ is projective over $\operatorname{Spec} \mathbb{F}_{D_\rho}$. □

We summarize what we have shown in the following theorem, confirming Kisin's expectation [Kis09a, Remark 3.2.7] that the space Rep'_{D_ρ} is projective.

THEOREM 2.2.3.5. *Let $\rho_i, 1 \leq i \leq n$ be pairwise non-isomorphic simple representations of E with sum ρ, and let θ be a character of $K_0(\mathrm{Rep}_E)$ sending $\rho_i, 1 \leq i < n$ to 1 and ρ_n to $-(n-1)$ as in the example above. The subgroupoid $\overline{\mathrm{Rep}}'_{D_\rho} \subset \overline{\mathrm{Rep}}_{D_\rho}$ defined by the condition* (2.2.3.3) *is a closed sub-$\operatorname{Spec} \mathbb{F}_{D_\rho}$-scheme of $\overline{\mathrm{Rep}}^{ss}_{D_\rho}(\theta)$, and is consequently a projective subscheme of $\overline{\mathrm{Rep}}_{D_\rho}$ with ample line bundle $\mathcal{O}^\theta(1)$.*

2.2.4. Deformation of Ample Line Bundles. We conclude our discussion of the fibers of $\bar{\psi}$ by giving conditions such that the projective subspaces $\overline{\mathrm{Rep}}^{s}_{E,\bar{D}}(\theta) \subset \overline{\mathrm{Rep}}^{\circ}_{E,\bar{D}}$ identified in the previous paragraph are the special fiber of a projective morphism to a local neighborhood on the base moduli space of pseudorepresentations. The question of which condition we must impose has a fairly clear answer in light of the calculation of the ample line bundle in Proposition 2.2.2.12: the idempotents on the fiber must be locally liftable to a

neighborhood. By Lemma 2.1.3.2, this is true precisely for *henselian* local rings. To deform the projectivity condition, we require that A is complete.[6]

THEOREM 2.2.4.1. *Let A be a complete local Noetherian ring with residue field $\mathbb{F}_A$ and maximal ideal $\mathfrak{m}_A$. Let R be an A-algebra that is finite as an A-module, and write E for $R \otimes_A \mathbb{F}_A$. Let R be equipped with a d-dimensional Cayley-Hamilton pseudorepresentation $D : R \to A$ such that its special fiber $\bar{D} : E \to \mathbb{F}_A$ is split. For any indivisible character θ of $K_0(\mathrm{Rep}_E)$, the line bundle $\mathcal{O}^\theta(1)$ on the special fiber $\overline{\mathrm{Rep}}^\circ_{E,\bar{D}}$ is the restriction to the special fiber of a line bundle defined over all of $\overline{\mathrm{Rep}}_{R,D}$.*

In particular, if $\bar{D}$ is stabilizing with respect to θ, the projective fine θ-stable moduli space $\overline{\mathrm{Rep}}^{\circ,s}_{E,\bar{D}}(\theta)$ of Corollary 2.2.2.14 is the special fiber of a projective subscheme $\overline{\mathrm{Rep}}^s_{R,D}(\theta)$ of the moduli stack $\overline{\mathrm{Rep}}_{R,D}$ arising as the algebraization of the completion of $\overline{\mathrm{Rep}}_{R,D}$ along $\overline{\mathrm{Rep}}^{\circ,s}_{E,\bar{D}}(\theta)$.

PROOF. Firstly, we show that the ample line bundle $\mathcal{O}^\theta(1)$ is a specialization of an ample line bundle that exists on all of $\overline{\mathrm{Rep}}_D$. This follows directly from the fact that we can lift the idempotents defining $\mathcal{O}^\theta(1)$ according to (2.2.2.13) from $E \otimes_A \mathbb{F}_A$ to E. We then use the same formula.

Now we apply formal GAGA [Gro61b, Theorem 5.4.5] to draw the conclusion. □

We thank Mark Kisin for comments leading to the following remark.

REMARK 2.2.4.2. In fact, Theorem 2.2.4.1 can be extended to a henselian base. The line bundle ample on the particular subspace certainly exists. Then, in place of the completion of $\mathrm{Rep}_{R,D}$ along $\mathrm{Rep}^{\circ,s}_{E,\bar{D}}(\theta)$, one can consider the henselization along this subscheme.

This theorem is especially significant in the context of continuous representations and pseudorepresentations of a profinite algebra. Of course, it is necessary to show that we can reduce the topological profinite case to the non-topological case under a finiteness condition $\mathbf{\Phi}_{\bar{D}}$, which we do in Chapter 3 (see e.g. Theorem 3.2.4.1). Then, firstly, the moduli space of

[6] See Remark 2.2.4.2.

pseudorepresentations of a profinite algebra is a disjoint union of formal spectral of complete local rings (Corollary 3.1.6.13). This allows Theorem 2.2.4.1 to be applied over the *whole* moduli space of pseudorepresentations! Each component $\operatorname{Spf} B_{\bar{D}}$ arises from the complete local ring $B_{\bar{D}}$, which is Noetherian upon the finiteness assumption $\mathbf{\Phi}_{\bar{D}}$. These notions will be defined and discussed in Chapter 3. For now we discuss the moduli of representations of a Cayley-Hamilton $B_{\bar{D}}$-algebra (R, D) where R is finite as a $B_{\bar{D}}$-module.

This is the context in which Kisin proposed the projectivity of a moduli formal scheme of representations of a profinite group with residually constant, split, multiplicity free pseudorepresentation $\bar{D}$, and a certain ordering of non-trivial residual extensions of the representation given in Definition 2.2.3.2 [Kis09a, Remark 3.2.7]. We verified in Theorem 2.2.3.5 that the special fiber of ψ in this space $\overline{\mathrm{Rep}}'_{E,\bar{D}}$ is projective and is a closed subspace of the larger projective subscheme $\overline{\mathrm{Rep}}^{ss}_{E,\bar{D}}(\theta) \subset \overline{\mathrm{Rep}}_{E,\bar{D}}$. The ample line bundle $\mathcal{O}^{\theta}(1)$ on $\overline{\mathrm{Rep}}^{ss}_{E,\bar{D}}(\theta)$ is therefore also ample on $\overline{\mathrm{Rep}}'_{E,\bar{D}}$. Since this line bundle deforms to $\overline{\mathrm{Rep}}_{R,D}$ as discussed in Theorem 2.2.4.1, formal GAGA implies that the formal completion of $\overline{\mathrm{Rep}}'_{E,\bar{D}}$ in $\overline{\mathrm{Rep}}_{R,D}$ is projective. In particular, it is algebraizable and is a projective $\operatorname{Spec} B_{\bar{D}}$-subscheme of $\overline{\mathrm{Rep}}_{R,D}$, which we denote by $\overline{\mathrm{Rep}}'_{R,D}$. This completes the confirmation of Kisin's suggestion that the space of representations with reduction in $\overline{\mathrm{Rep}}'_{E,\bar{D}}$ is projective. We summarize this in the following

Corollary 2.2.4.3. *Let $\bar{\rho}^{ss}_{\bar{D}} : E \to M_d(\mathbb{F}_A)$ be chosen as in Theorem 2.2.3.5, and choose an ordering of its simple factors in order to define the subgroupoids $\overline{\mathrm{Rep}}'_{R,D} \subset \overline{\mathrm{Rep}}_{R,D}, \overline{\mathrm{Rep}}'_{E,\bar{D}} \subset \overline{\mathrm{Rep}}_{E,\bar{D}}$ as above. Assume that the associated pseudorepresentation satisfies condition $\mathbf{\Phi}_{\bar{D}}$. The formal completion of $\overline{\mathrm{Rep}}_{R,D}$ along the projective subscheme $\overline{\mathrm{Rep}}'_{E,\bar{D}}$ of the special fiber of ψ is projective over $\operatorname{Spf} B_{\bar{D}}$ with ample line bundle $\mathcal{O}^{\theta}(1)$. Consequently, this formal scheme is algebraizable with algebraization $\overline{\mathrm{Rep}}'_{R,D}$, a projective $\operatorname{Spec} B_{\bar{D}}$-subscheme of $\overline{\mathrm{Rep}}_{R,D}$.*

2.3. Multiplicity Free Pseudorepresentations

Chenevier showed that a Cayley-Hamilton algebra (R, D) over a henselian local ring A which is residually split and absolutely irreducible is a matrix algebra (Theorem 2.1.3.3). This corresponds to very tidy results in in the moduli theory of representations and pseudorepresentations that locally satisfy these conditions: the deformations of representations and pseudorepresentations are equivalent (Corollary 2.1.3.4).

Our goal in this section is to generalize these results to the case that a pseudorepresentation is residually multiplicity free (see Definition 1.3.4.4). Here, the moduli of representations and pseudorepresentations are no longer equivalent. For example, over a multiplicity free geometric point $\bar{D}$ of PsR_R^d, non-trivial extensions of the Jordan-Hölder factors M_i of $\rho_{\bar{D}}^{ss}$ may form positive dimensional families of representations lying over this single pseudorepresentation; for example, this often happens if there exists M_i, M_j, $i \neq j$, such that $\dim_{\mathbb{F}_A} \mathrm{Ext}_R^1(M_i, M_j) \geq 2$. What we want to show is that around multiplicity free points in PsR_R^d, ψ (resp. $\bar{\psi}$) is an adequate moduli space. This will mean that the multiplicity free locus of pseudorepresentations is a universal scheme-theoretic quotient for representations of R up to conjugation. This improves the results of GIT (Theorem 1.5.4.2), which only have fine enough resolution to give a satisfactory theory for geometric points.

The main tool to accomplish this will be a generalized matrix algebra. The key result generalizing Theorem 2.1.3.3 is Theorem 2.3.1.2, which shows that a Cayley-Hamilton algebra (R, D) over a henselian local ring is a generalized matrix algebra. The linear structure we can put on the moduli space of representations of a generalized matrix algebra, an "adaptation" of its representations, will allow us to show that the invariant functions on the space of framed representations are exactly the coefficients of the universal pseudorepresentation, as desired (Theorem 2.3.3.7).

Remark 2.3.0.4. Currently, the notion of generalized matrix algebra that we use is meant to work with pseudocharacters as opposed to pseudorepresentations (see §1.1.12). Therefore, we state our main result here in the case that $(2d)!$ is invertible in A (i.e. $\mathrm{char}\, \mathbb{F}_A > 2d$), which is a sufficient condition for pseudocharacters to be equivalent to pseudorepresentations

according to Proposition 1.1.12.3(3). We expect to develop a theory of generalized matrix algebras compatible with pseudorepresentations, which will allow us to remove the condition on the characteristic (see Remark 2.3.3.6).

2.3.1. Generalized Matrix Algebras. We define generalized matrix algebras.

Definition 2.3.1.1 ([BC09, Definition 1.3.1]). Let A be a commutative ring and R an A-algebra. We say that R is a generalized matrix algebra (GMA) of type $(d_1, \dots, d_r)$ if R is equipped with

(1) a family of orthogonal idempotents $e_1, \dots, e_r$ of sum 1

(2) for each i, an A-algebra isomorphism $\psi_i : e_i R e_i \to M_{d_i}(A)$,

such that the trace map $T : R \to A$ defined by

$$T(x) := \sum_{i=1}^{r} \mathrm{Tr}(\psi_i(e_i x e_i)),$$

satisfies $T(xy) = T(yx)$. The "data of idempotents" of the GMA is $\mathcal{E} = \{e_i, \psi_i\}$.

Here is the main result making possible our use of generalized matrix algebras to study $\psi : \overline{\mathrm{Rep}} \to \mathrm{PsR}$: given a henselian local ring A and a Cayley-Hamilton algebra (R, D) over A, R must be a generalized matrix algebra.

Theorem 2.3.1.2 ([Che11, Theorem 2.22(ii)]). *Let $D : R \to A$ be a Cayley-Hamilton pseudorepresentation over a henselian local ring A. If $\bar{D}$ is split and multiplicity free, then (R, T_D) is a generalized matrix algebra.*

We use the GMA structure on $(R, \mathcal{E})$ to establish notation for elements of R analogous to matrices with a single non-zero entry.

Definition 2.3.1.3. Let $E_i^{k,l} \in e_i R e_i$ be the unique element mapping under ψ_i to the matrix in $M_{d_i}(A)$ with 1 in the $(k,l)^{th}$ entry and 0 elsewhere. Write $E_i = E_i^{1,1}$. For $1 \le i, j \le r$ set $\mathcal{A}_{i,j} := E_i R E_j$. For $1 \le i, j, k \le r$ we have $\mathcal{A}_{i,j}\mathcal{A}_{j,k} \subset \mathcal{A}_{i,k}$ so that the product in R induces maps

$$\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}.$$

Bellaïche and Chenevier use these elementary matrix-like elements to exhibit a matrix-like structure on any given GMA $(R, \mathcal{E})$.

LEMMA 2.3.1.4 (Bellaïche-Chenevier, §1.3.2). *There is a canonical isomorphism of A-algebras*

$$R \cong \begin{pmatrix} M_{d_1}(\mathcal{A}_{1,1}) & M_{d_1 \times d_2}(\mathcal{A}_{1,2}) & \cdots & M_{d_1 \times d_r}(\mathcal{A}_{1,r}) \\ M_{d_2 \times d_1}(\mathcal{A}_{2,1}) & M_{d_2}(\mathcal{A}_{2,2}) & \cdots & M_{d_2 \times d_r}(\mathcal{A}_{2,r}) \\ \vdots & \vdots & \vdots & \vdots \\ M_{d_r \times d_1}(\mathcal{A}_{r,1}) & M_{d_2}(\mathcal{A}_{r,2}) & \cdots & M_{d_r}(\mathcal{A}_{r,r}) \end{pmatrix},$$

where the A-algebra structure is determined by canonical isomorphisms

$$M_{d_i d_j}(\mathcal{A}_{ij}) \xrightarrow{\sim} e_i R e_j.$$

In analogy to Definition 2.2.1.4, we take note of the conditions that the maps $\varphi_{i,j,k}$ must satisfy (cf. [BC09, Lemma 1.3.5]) as a result of the construction above. Here we implicitly use a canonical morphism for each i, $\mathcal{A}_{i,i} \xrightarrow{\sim} A$, that arises from the trace T, i.e. $\mathcal{A}_{i,i} \hookrightarrow R \xrightarrow{T} A$ is an isomorphism.

(UNIT) For all i, $\mathcal{A}_{i,i} \cong A$ and for all i, j, $\varphi_{i,i,j} : A \otimes \mathcal{A}_{i,j} \to \mathcal{A}_{i,j}$ (resp. $\varphi_{i,j,j} : \mathcal{A}_{i,j} \otimes A \to \mathcal{A}_{i,j}$) is the A-module structure of $\mathcal{A}_{i,j}$.

(ASSO) For all i, j, k, l, the two natural maps $\mathcal{A}_{i,j} \otimes \mathcal{A}_{j,k} \otimes \mathcal{A}_{k,l} \to \mathcal{A}_{i,l}$ coincide.

(COM) For all i, j and for all $x \in \mathcal{A}_{i,j}, y \in \mathcal{A}_{j,i}$, we have $\varphi_{i,j,i}(x \otimes y) = \varphi_{j,i,j}(y \otimes x)$.

Bellaïche and Chenevier note that specifying the data of A-modules $\mathcal{A}_{i,j}$, $1 \leq i, j \leq r$ with maps $\varphi_{i,j,k}$ as above satisfying (UNIT), (ASSO), and (COM), then $R := M_{d_i,d_j}(\mathcal{A}_{i,j})$ is uniquely a GMA of type $(d_1, \ldots, d_r)$ such that $E_i R E_j \cong \mathcal{A}_{i,j}$ for all i, j. This completes a satisfying structure theory for GMAs.

REMARK 2.3.1.5. Compare these conditions (UNIT), (ASSO), (COM) with the groupoid of families of quiver representations in Definition 2.2.1.4.

2.3.2. Trace Representations and Adapted Representations. Now we define the notion of an adapted representation of a GMA $(R, \mathcal{E})$. Adapted representations have extra

linear structure that makes their moduli easier to handle than the general moduli problem of representations.

Definition 2.3.2.1 ([BC09, Definition 1.3.6]). Let B be a commutative A-algebra and let $(R, \mathcal{E})$ be a generalized matrix A-algebra. A representation $\rho : R \to M_d(B)$ is said to be *adapted to* $\mathcal{E}$ if its restriction to the A-subalgebra $\bigoplus_{i=1}^r e_i R e_i$ is the composite of the representation $\bigoplus_{i=1}^r \psi_i$ by the natural "diagonal" map

$$M_{d_1}(A) \oplus \cdots \oplus M_{d_r}(A) \to M_d(B).$$

We define $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$ to be the functor associating an A-algebra B to the set of adapted representations of $(R, \mathcal{E})$ over B.

We also give a definition of a trace representation. This is nothing more than the analogue, where pseudocharacters replace pseudorepresentations, of the functor of representations lying over a given pseudorepresentation.

Definition 2.3.2.2 ([BC09, §1.3.3]). If R is an A-algebra equipped with a d-dimensional pseudocharacter $T : R \to A$ and B is a commutative A-algebra, we will say that a map of A-algebras $\rho : R \to M_d(B)$ is a *trace representation* if $\mathrm{Tr} \circ \rho(x) = T(x)1_B$ for any $x \in R$. We write $\mathrm{Rep}^{\square}_T$ for the functor of trace representations on Alg_A.

Of course, this definition can be applied to Azumaya algebra valued representations as well, to get a groupoid Rep_T analogous to the definition for pseudocharacters (Definition 1.4.1.1). We will assume that $(2d)!$ is a unit in A so that we can consider pseudocharacters and pseudorepresentations to be the same object (cf. Proposition 1.1.12.3).

The key result is that a trace representation can be made into an adapted representation after base change and conjugation. This is the key result we require in order to compare the moduli problem for adapted representations with our usual moduli problem for representations.

LEMMA 2.3.2.3 ([BC09, Lemma 1.3.7]). *Let B be a commutative A-algebra and $\rho : R \to M_d(B)$ be a trace representation. Then there is a commutative ring C containing B and a $P \in \mathrm{GL}_d(C)$ such that $P\rho P^{-1} : R \to M_d(C)$ is adapted to $\mathcal{E}$. Moreover, if every finite type projective B-module is free, then we can take $C = B$.*

We omit the proof, since we will give a proof more precisely tailored to the situation we are required to address in order to prove Proposition 2.3.3.5 below.

Adapted representations have a very concrete moduli functor.

PROPOSITION 2.3.2.4 ([BC09, Propositions 1.3.9, 1.3.13]). *When $(R, \mathcal{E})$ is a GMA over A, the functor $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E}) : \mathrm{Alg}_A \to Set$ associating a commutative A-algebra B to the set of homomorphisms $R \to M_d(B)$ adapted to $\mathcal{E}$ is representable by a faithful A-algebra B^u with an injective universal adapted homomorphism $R \hookrightarrow M_d(B^u)$.*

PROOF. The proof shows that one can find a ring B^u with inclusions $\mathcal{A}_{i,j} \hookrightarrow B^u$ (where $\mathcal{A}_{i,j}$ are from Lemma 2.3.1.4 above) such that the isomorphism in Lemma 2.3.1.4 is precisely the injection required. B^u is constructed as a quotient of the symmetric power algebra on $\oplus_{i \neq j} \mathcal{A}_{i,j}$. For additional details, see [BC09]. □

It follows from the existence of the universal adapted representation that the trace function on a GMA is Cayley-Hamilton (cf. [BC09, Corollary 1.3.16]), where Cayley-Hamilton is defined for pseudocharacters in analogy with the definition for pseudorepresentations in Definition 1.1.8.5 (see [BC09, §1.2.3]). We give this brief argument: the trace T of the GMA data $(R, \mathcal{E})$ is equal to the composition of the trace function Tr on $M_d(B^u)$ with the universal adapted representation $R \to M_d(B^u)$ given by Proposition 2.3.2.4. Since Tr is Cayley-Hamilton and $R \to M_d(B^u)$ is an algebra homomorphism, so is T Cayley-Hamilton.

2.3.3. Invariant Theory of Adapted Representations. In this paragraph, our goal is to naturally identify the GIT quotient of $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$ with the algebra of traces, which is A. This will allow us to do for adapted representations what we have not yet done for general representations: show that pseudorepresentations are an adequate moduli space for

representations. After completing this paragraph, we will use the comparison of adapted representations with trace representations to show that ψ (resp. $\bar{\psi}$) is an adequate moduli space over the multiplicity free locus of PsR_R^d.

Let $(R, \mathcal{E})$ be a d-dimensional generalized matrix algebra of type $\beta = (d_1, \ldots, d_r)$. We set up the notation for the following group schemes; the group $Z(\beta)$ is made to act naturally on the affine scheme $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$.

Definition 2.3.3.1. In analogy with automorphism groups of quiver representations, define $\mathrm{GL}(\beta) := \mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_r}$ as a subgroup

$$\mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_r} \subset \mathrm{GL}_d,$$

compatible with the maps $\psi_i : M_{d_i} \to M_d$ of Definition 2.3.2.1. Let $Z(\beta)$ denote the center of $\mathrm{GL}(\beta)$. Likewise, let $\mathrm{PGL}(\beta)$ denote the quotient of $\mathrm{GL}(\beta)/\Delta$ of $\mathrm{GL}(\beta)$ by the diagonally embedded central 1-dimentional torus $\Delta \cong \mathbb{G}_m$, and let $PZ := Z(\beta)/\Delta$.

Because $Z(\beta)$ commutes with $\oplus_i M_{d_i}$, its adjoint action preserves the adaptation. Therefore we have a natural action of $Z(\beta)$ on $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$, inducing a natural action of $PZ(\beta)$.

There is a natural map

$$\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E}) \hookrightarrow \mathrm{Rep}^{\square, d}_R \tag{2.3.3.2}$$

given by forgetting the adaptation data. The map $\oplus_i M_{d_i} \hookrightarrow M_d$ induces a canonical injection $Z(\beta) \hookrightarrow \mathrm{GL}_d$ (resp. $PZ(\beta) \hookrightarrow \mathrm{PGL}_d$). In this lemma, we record the fact that (2.3.3.2) is equivariant for the action of $Z(\beta)$ (resp. $PZ(\beta)$). We also calculate the GIT quotient of the action of $Z(\beta)$ on $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$.

Lemma 2.3.3.3. *Given $(R, \mathcal{E})$ a GMA over A of type β. The map* (2.3.3.2) *is equivariant for the action of $Z(\beta)$ (resp. $PZ(\beta)$). The invariant regular functions on $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$ under this action are precisely $A \subset B^u$.*

Proof. The first claim can be checked by each of the embeddings of functors and groups set up above. For the claim on the invariant functions, as we mentioned in the proof of

Proposition 2.3.2.4, B^u is generated over A by $\mathcal{A}_{i,j}$ for $i \neq j$. As $Z(G) \simeq \mathbb{G}_m^r$ acts on B^u (observe the form of the matrices in Lemma 2.3.1.4), it acts on each of $\mathcal{A}_{ij}$ $(i \neq j)$ by a (distinct) non-trivial character, namely, through the roots of GL_d. Since these modules $\mathcal{A}_{ij}$ generate the coordinate ring B^u of $\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$, we see that $(B^u)^{Z(\beta)} \cong A$, i.e.

$$\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})//Z(\beta) \cong \mathrm{Spec}\, A$$

as desired. □

If as usual, we let $(R, \mathcal{E})$ be a d-dimensional generalized matrix A-algebra of type β, the lemma above shows that we have a morphism of stacks

$$\begin{aligned} [\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})/Z(\beta)] &\longrightarrow \mathrm{Rep}_{R,T} \\ [\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})/PZ(\beta)] &\longrightarrow \overline{\mathrm{Rep}}_{R,T}, \end{aligned} \tag{2.3.3.4}$$

because of the equivariance of the adaptation-forgetting map (2.3.3.2) with respect to the embedding $Z(\beta) \hookrightarrow \mathrm{GL}_d$ (resp. $PZ(\beta) \hookrightarrow \mathrm{PGL}_d$).

Now we will show that (2.3.3.4) is an isomorphism when A is a henselian local ring! To do this, we will find a quasi-inverse. We recall here that we are assuming that $(2d)!$ is a unit in A, so that pseudorepresentations and pseudocharacters are identical by Proposition 1.1.12.3, and we can apply our knowledge of pseudorepresentations to this problem.

Proposition 2.3.3.5. *Let A be a henselian local ring and let (R, D) be a d-dimensional Cayley-Hamilton A-algebra, so that $(R, \mathcal{E})$ is a generalized matrix A-algebra with trace function T. Then the natural induced maps of* Spec A*-algebraic stacks* (2.3.3.4) *are isomorphisms.*

Remark 2.3.3.6. We record an alternative notion of generalized matrix algebra, replacing the notion relative to pseudocharacters with one for pseudorepresentations. Using the notation of Definitions 2.3.1.1 and 2.3.1.3, we replace the trace map T with a "determinant map" $D : R \to A$ as follows: let the symmetric group S_d act on the complete set of d

primitive orthogonal idempotents $E_i^{j,j} \in R$.

$$D(r) := \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{1 \leq i \leq r} \prod_{1 \leq j \leq d_i} E_i^{j,j} r \sigma(E_i^{j,j}).$$

This determinant map is compatible with tensor products, and defines a d-dimensional pseudorepresentation. We expect to extend the theory of generalized matrix algebras of [BC09] to this case. This would eliminate the complications with the characteristic of coefficient rings.

PROOF. (Proposition 2.3.3.5) Let X be a $\operatorname{Spec} A$-scheme. Choose $(\rho, V_X) \in \operatorname{Rep}_T(X)$. The idempotents $e_i \in R$ break V_X into a direct sum of projective sub-$\mathcal{O}_X$-modules $V_i := e_i V_B$ of rank d_i,

$$V_X \cong \bigoplus_{i=1}^{r} V_i.$$

Each V_i receives an A-linear action of $e_i R e_i \subset R$, and therefore a $\mathcal{O}_X$-linear action of $e_i R e_i \otimes_A \mathcal{O}_X$. Using the GMA data $\psi_i : e_i R e_i \xrightarrow{\sim} M_{d_i}(A)$, we see that $\operatorname{End}_B(V_i) \cong M_{d_i}(\mathcal{O}_X)$. This means that as a $\mathcal{O}_X$-module, V_i is isomorphic to a twist of a free rank d_i vector bundle F_i by a line bundle $\mathcal{L}_i$.

Let $\mathcal{G}_i := \operatorname{Isom}_{\mathcal{O}_X}(\mathcal{L}_i, \mathcal{O}_X)$ be the $\mathbb{G}_m$-torsor over X corresponding to $\mathcal{L}_i$. Then $\mathcal{G} := \times_{i=1}^{r} \mathcal{G}_i$ is naturally a $Z(\beta)$-torsor. Indeed, the base change of V_X to $\mathcal{G}$ from X is a free rank d $\mathcal{O}_{\mathcal{G}}$-vector bundle with a canonical basis adapted to $(R, \mathcal{E})$. This defines a map $\mathcal{G} \to \operatorname{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})$, equivariant for the action of $Z(\beta)$. We have therefore established a morphism

$$\operatorname{Rep}_T \longrightarrow [\operatorname{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})/Z(\beta)].$$

We observe that this provides a quasi-inverse to (2.3.3.4). □

We now replace pseudorepresentations with pseudocharacters, using the fact that they are equivalent to each other; this is the case because we are assuming that $(2d)!$ is invertible in the base ring A (cf. Proposition 1.1.12.3).

We recall the notation of §1.5. S is an affine Noetherian scheme, and R is a quasi-coherent finitely generated $\mathcal{O}_S$-algebra. The map $\nu : \operatorname{Rep}_R^{\square,d} /\!/ \mathrm{PGL}_d \to \mathrm{PsR}_R^d$ measures the difference

between the GIT quotient of the space of framed d-dimensional representations $\mathrm{Rep}_R^{\square,d}$ by the action of PGL_d and the moduli scheme PsR_R^d of d-dimensional pseudorepresentations. We showed in Theorem 1.5.4.2 that ν is a finite universal homeomorphism, or "almost isomorphism." We showed in Corollary 2.1.3.4 that ν is an isomorphism in the neighborhood of points corresponding to absolutely irreducible representations of R. Now we will extend this result, showing that ν is an isomorphism in the neighborhood of points corresponding to multiplicity free pseudorepresentations.

THEOREM 2.3.3.7. *Let A be a commutative Noetherian ring and let R be a finitely generated A-algebra. There exists a Zariski open subscheme $U \subset \mathrm{PsR}_R^d$ with the following two properties:*

(1) the set U contains all points of residue characteristic greater than $2d$ corresponding to multiplicity free pseudorepresentations of R, and

(2) ν is an isomorphism onto U.

PROOF. We will write $X = \mathrm{Rep}_R^{\square,d}//\mathrm{PGL}_d$ for convenience.

We already know that ν is a finite universal homeomorphism of finite type Spec A-schemes (Theorem 1.5.4.2). Therefore ν is étale in the neighborhood of some point $D \in \mathrm{PsR}_R^d$ if and only if it is an isomorphism in that neighborhood.[7] Since being an isomorphism is a local property on the base, in order to prove the theorem, it will suffice to show that ν is étale in a neighborhood of each of the specified points. Since ν is finite type, it will suffice to show that the induced maps on complete local rings are étale; we will simply show that they are isomorphisms. We may have to make an étale base change in order that the pseudorepresentation may be assumed to be split; this is not a problem, since we can descend the étale property along this morphism.

We apply Theorem 1.4.3.1 to replace $\mathrm{Rep}_R^{\square,d}$ (resp. Rep_R^d, resp. $\overline{\mathrm{Rep}}_R^d$) with $\mathrm{Rep}_{E,D^u}^{\square,d}$ (resp. Rep_{E,D^u}^d, resp. $\overline{\mathrm{Rep}}_{E,D^u}^d$) where $E = E(R,d)$. We will think of ψ (resp. $\bar{\psi}$) as a morphism out of Rep_{E,D^u}^d (resp. $\overline{\mathrm{Rep}}_{E,D^u}^d$).

[7] In fact, any étale universal homeomorphism is an isomorphism.

Let $\bar{D}$ be a point of PsR^d_R of residue characteristic greater than $2d$, and write $x = \nu^{-1}(\bar{D}) \in X$ for the corresponding point of X. We have a canonical map

$$\hat{\mathcal{O}}_{\mathrm{PsR}^d_R,\bar{D}} \longrightarrow \hat{\mathcal{O}}_{X,x}$$

which we wish to show is an isomorphism. Write $\hat{U}_{\bar{D}} := \operatorname{Spec} \hat{\mathcal{O}}_{\mathrm{PsR}^d_R,\bar{D}}$, $\hat{V}_x := \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$. Of course, $\hat{U}_{\bar{D}}$ classifies the pseudodeformations of $\bar{D}$ to Artinian A-algebras with residue field $\mathbb{F}_{\bar{D}}$ and has a universal pseudodeformation $D^u_{\bar{D}}$, so we will just write $\mathrm{PsR}_{D^u_{\bar{D}}}$ in place of $\hat{U}_{\bar{D}}$.

Because X and PsR^d_R are Noetherian, the morphisms $\mathrm{PsR}_{D^u_{\bar{D}}} \to \mathrm{PsR}^d_R$, $\hat{V}_x \to X$ are flat. By the Artin-Rees theorem and the finitude of ν, they form a cartesian square

(2.3.3.8)
$$\begin{array}{ccc} \hat{V}_x & \longrightarrow & X \\ \downarrow & & \downarrow \nu \\ \hat{\mathrm{PsR}}_{D^u_{\bar{D}}} & \longrightarrow & \mathrm{PsR}^d_R. \end{array}$$

Now [Alp10, Proposition 5.2.9(1)] says that the flatness of the completion maps along with the fact that the maps $\phi, \bar{\phi}$ of (1.5.2.2) are adequate moduli spaces (Definition 1.5.1.1) will imply that the maps $\phi, \bar{\phi}$ of

$$\begin{array}{ccc} \mathrm{Rep}^d_{E,D^u} \times_{\mathrm{PsR}^d_R} \mathrm{PsR}_{D^u_{\bar{D}}} & \xrightarrow{\sim} & \mathrm{Rep}^d_{E,D^u} \times_X \hat{V}_x \\ & & \Big\downarrow \phi \\ & \psi & \hat{V}_x \\ & & \Big\downarrow \nu \\ & & \mathrm{PsR}_{D^u_{\bar{D}}} \end{array} \qquad \begin{array}{c} \mathrm{PsR}_{D^u_{\bar{D}}} \\ \Big\uparrow \nu \\ \hat{V}_x \quad \bar{\psi} \\ \Big\uparrow \bar{\phi} \end{array}$$

$$\overline{\mathrm{Rep}}^d_{E,D^u} \times_{\mathrm{PsR}^d_R} \mathrm{PsR}_{D^u_{\bar{D}}} \xrightarrow{\sim} \overline{\mathrm{Rep}}^d_{E,D^u} \times_X \hat{V}_x$$

are also adequate moduli spaces. Since (2.3.3.8) is cartesian, we get an identical picture by replacing $\mathrm{Rep}^d_{E,D^u} \times_X \hat{V}_x$ with $\mathrm{Rep}^d_{E,D^u} \times_{\mathrm{PsR}^d_R} \mathrm{PsR}_{D^u_{\bar{D}}}$, as we have indicated with the horizontal isomorphisms above.

Write $\mathfrak{p}_{\bar{D}}$ for the prime ideal of $\Gamma_A^d(R)^{\mathrm{ab}}$ corresponding to $\bar{D} \in \mathrm{PsR}_R^d$, and write $B_{\bar{D}}$ for the $\mathfrak{p}_{\bar{D}}$-adic completion of $(\Gamma_A^d(R)^{\mathrm{ab}})_{\mathfrak{p}_{\bar{D}}}$. Now, unraveling definitions for the fiber product $\mathrm{Rep}^d_{E,D^u} \times_{\mathrm{PsR}_R^d} \mathrm{PsR}_{D^u_{\bar{D}}}$, using Lemma 1.1.8.6, and noting that the $\mathfrak{p}_{\bar{D}}$-adic completion $\hat{E}_{\bar{D}}$ of $E \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} (\Gamma_A^d(R)^{\mathrm{ab}})_{\mathfrak{p}_{\bar{D}}}$ is isomorphic to $E \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B_{\bar{D}}$, we see that the fiber product is isomorphic to

$$\mathrm{Rep}^d_{\hat{E}_{\bar{D}},(D^u \otimes_\Gamma B_{\bar{D}})},$$

the groupoid of representations of the Cayley-Hamilton $B_{\bar{D}}$-algebra $(\hat{E}_{\bar{D}}, D^u \otimes B_{\bar{D}})$ compatible with its pseudorepresentation. The universal pseudorepresentation also is compatible with these completions and base changes; we write $D^u_{\bar{D}} : E_{\bar{D}} \to B_{\bar{D}}$ in place of $D^u \otimes B_{\bar{D}}$. Of course, the same things can be said with $\overline{\mathrm{Rep}}$ in the place of Rep.

Because $B_{\bar{D}}$ is a henselian ring, we see that we are now in the situation of Proposition 2.3.3.5. Indeed, Theorem 2.3.1.2 implies that $(\hat{E}_{\bar{D}}, T_{\bar{D}})$ is a generalized matrix algebra, where we write $T_{\bar{D}}$ for the trace function $\Lambda_{1,B_{\bar{D}}}^{D^u_{\bar{D}}}$ associated to $D^u_{\bar{D}}$. Then Proposition 2.3.3.5 gives us isomorphisms of algebraic stacks

$$[\mathrm{Rep}^{\square}_{\mathrm{Ad}}(E_{\bar{D}}, T_{\bar{D}})/Z(\beta)] \quad \xrightarrow{\sim} \mathrm{Rep}_{D^u_{\bar{D}}},$$

$$[\mathrm{Rep}^{\square}_{\mathrm{Ad}}(E_{\bar{D}}, T_{\bar{D}})/PZ(\beta)] \quad \xrightarrow{\sim} \overline{\mathrm{Rep}}_{D^u_{\bar{D}}}$$

Lemma 2.3.3.3 tells us that

$$\Gamma(\mathcal{O}(\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})))^{Z(\beta)} \cong B_{\bar{D}}.$$

This means that ψ (resp. $\bar{\psi}$) is an adequate moduli space with source $[\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})/Z(\beta)]$ (resp. $[\mathrm{Rep}^{\square}_{\mathrm{Ad}}(R, \mathcal{E})/PZ(\beta)]$), since this situation from GIT outlined in Example 1.5.1.3 is an example of an adequate moduli space. Theorem 1.5.1.4(5) implies that adequate moduli spaces arising from a reductive group acting on an affine scheme have a unique base. Therefore ν induces an isomorphism $\hat{V}_x \xrightarrow{\sim} \mathrm{PsR}_{D^u_{\bar{D}}}$ as desired. □

Corollary 2.3.3.9. *Over the base locus defined in Theorem 2.3.3.7, ψ (resp. $\bar{\psi}$) is an adequate moduli space.*

This means that the pseudorepresentation scheme consists precisely of the invariant functions of the framed moduli scheme under the action of conjugation. This sort of statement on invariants is made clearly in Lemma 2.3.3.3.

CHAPTER 3

Representations and Pseudorepresentations of Profinite Algebras

In this chapter, we apply the results on moduli spaces of representations of representations and pseudorepresentations to the study of the moduli theory of *continuous* representations of profinite algebras R. Our approach is to develop the topological theory of pseudorepresentations and prove (see e.g. Corollary 3.1.6.13) the representability of their moduli by formal schemes that are disjoint unions of formal spectra $\mathrm{PsR}_{\bar{D}} \cong \operatorname{Spf} B_{\bar{D}}$ of complete local deformation rings $B_{\bar{D}}$ of residual pseudorepresentations $\bar{D}$ (pseudodeformation rings). Up to this point we will have been following Chenevier [Che11]. Then, we give conditions for the Noetherianness of $B_{\bar{D}}$, the most important being known as $\mathbf{\Phi}_{\bar{D}}$ (Definition 3.1.5.1). Then we study the moduli space of representations more simply by studying the connected component over each pseudodeformation spectrum. However, we will hold short of developing moduli formal schemes/algebraic stacks of representations directly. Instead, upon the assumption of $\mathbf{\Phi}_{\bar{D}}$, we show that when the moduli problem of continuous representations is *finitely presented* over the moduli of pseudorepresentations. Then, the moduli formal scheme/stacks of continuous representations on formal schemes arise, over $\operatorname{Spf} B_{\bar{D}}$, as completions of a natural *algebraic*, finite type scheme/algebraic stack of representations.

We accomplish this by showing that under the condition $\mathbf{\Phi}_{\bar{D}}$, the universal Cayley-Hamilton representation $E(R, D^u_{\bar{D}})$ of R over the universal deformation $D^u_{\bar{D}}$ of $\bar{D}$ is finite as a module over $B_{\bar{D}}$. Then we simply observe that over coefficient rings that are separated continuous $B_{\bar{D}}$-algebras, all (non-topological) representations of $E_{\bar{D}}$ lying over $D^u_{\bar{D}}|_E$ are automatically $\mathfrak{m}_{\bar{D}}$-adically continuous. Now, any representation of $E(R, D^u_{\bar{D}})$ is continuous, and we can apply the theory of Chapters 1 and 2 directly to show that the functors of continuous representations on formal schemes over $\operatorname{Spf} B_{\bar{D}}$ are not only representable by adic formal schemes, but are algebraizable over $\operatorname{Spec} B_{\bar{D}}$.

Not only can we apply representability results from the previous chapters, but the results of Chapter 3 overviewed above will allow us to apply all of the results of Chapters 1 and 2 to this profinite topological case (assuming condition $\mathbf{\Phi}_{\bar{D}}$). We present these conclusions in Theorem 3.2.5.1.

3.1. Pseudorepresentations of Profinite Algebras

In this section we introduce continuous pseudorepresentations, due to Chenevier [Che11], and recall some basic topological facts about profinite rings and group algebras of profinite groups. We assume that all topologies are Hausdorff. We will focus on working with profinite rings, and then pro-discrete rings.

Let A be a commutative topological ring and let R be a topological (continuous) A-algebra. We establish notions of continuity for pseudorepresentations of R.

Definition 3.1.0.10. With A, R as above, a d-dimensional pseudorepresentation $D : R \to A$ is said to be *continuous* provided that the following equivalent conditions hold.

(1) for each $n \geq 1, \alpha \in I_n$, the functions $D^{[\alpha]} : R \to A$ of Definition 1.1.2.14 are continuous.
(2) the characteristic polynomial functions $\Lambda_i = \Lambda_i^D : R \to A, 1 \leq i \leq d$ are continuous.
(3) For every commutative continuous A-algebra B, the function $D_B : R \otimes_A B \to B$ is continuous.

We will show that the notions of continuity in the definition are indeed equivalent.

Proof. The equivalence of (1) and (2) is immediate from Amitsur's formula (Proposition 1.1.9.11(2)).

Recalling that a pseudorepresentation $D : R \to A$ consists of a function $D_B : R \otimes_A B \to B$ for every commutative A-algebra B, let us verify that (2) implies (3). When B is any continuous topological A-algebra, this definition does indeed guarantee that each of the induced homogenous functions

$$D_B : R \otimes_A B \to B$$

that make up the polynomial law are continuous. For we can write B as a continuous quotient of a polynomial algebra C, where C is given its natural topology as a free A-module. The $D^{[\alpha]}$ are coefficient functions of D_C by definitiion, and the functions D_B are the composition of D_C with the continuous quotient map from C to B.

Conversely, using the case that B is a polynomial algebra $B = A[t_1, \ldots, t_n]$, we see that (3) implies (1). A polynomial coefficient $D^{[\alpha]}$ for $\alpha \in I_d^d$ is the composition of a continuous map

$$R^n \longrightarrow R \otimes_A A[t_1, \ldots, t_n]$$

$$(r_i) \mapsto \sum_{i=1}^{n} r_i t_i$$

followed by $D_{A[t_1,\ldots,t_n]}$, followed by the continuous function from $A[t_1, \ldots, t_n]$ to A given by taking the α^{th} coefficient. Therefore $D^{[\alpha]}$ is continuous, as desired. □

3.1.1. Pro-discrete Topological Notions. We will be interested exclusively in either discrete or pro-discrete topologies. We begin by recalling some basic notions on profinite topologies on rings, with an eye toward group algebras of profinite groups. We note that rings are unital and associative but not necessarily commutative unless stated.

LEMMA 3.1.1.1. *Let R be a topological ring. The following conditions on R are equivalent.*

(1) R is a profinite ring.

(2) R is Hausdorff and compact.

(3) R is Hausdorff, compact, and totally disconnected.

(4) R is compact and has a fundamental system of neighborhoods of zero consisting of open ideals of R.

(5) There is an inverse system of finite discrete rings with surjective maps such that R is its limit.

PROOF. This is [RZ10, Proposition 5.1.2]. □

We will often denote by I a general open ideal of R.

When A is a profinite (e.g. finite) commutative ring, we will be interested in the studying continuous representations and pseudorepresentations of a profinite group $\hat{\Gamma}$ with coefficients in A or in commutative A-algebras. The group algebra $A[\hat{\Gamma}]$ is clearly not a profinite A-algebra. Therefore we discuss its natural topology and its profinite completion.

The topology on $A[\hat{\Gamma}]$ is defined by the fundamental system of neighborhoods of zero given by the kernels of the canonical surjections

$$\kappa(I,U) := \ker(A[\hat{\Gamma}] \longrightarrow (A/I)[\hat{\Gamma}/U]) \tag{3.1.1.2}$$

where I varies over open ideals of A and U varies over open normal subgroups of $\hat{\Gamma}$. Each of these ideals have finite index in $A[\hat{\Gamma}]$. We then define the *complete group algebra* to be the completion of $A[\hat{\Gamma}]$ with respect to this topology,

$$A[\![\hat{\Gamma}]\!] := \varprojlim (A/I)[\hat{\Gamma}/U].$$

We see that this is a profinite ring, with open ideals

$$\ker(A[\![\hat{\Gamma}]\!] \longrightarrow (A/I)[\hat{\Gamma}/U]),$$

where we abuse notation by writing $\kappa(I,U)$ for these ideals of $A[\![\hat{\Gamma}]\!]$ as well. It is also possible to express the complete group algebra as the limit

$$A[\![\hat{\Gamma}]\!] \cong \varprojlim_{U} A[\hat{\Gamma}/U].$$

Here are some basic facts about this construction.

Lemma 3.1.1.3. *Let A be a commutative profinite ring and let $\hat{\Gamma}$ be a profinite group.*

(1) The intersection of all the ideals of the form (3.1.1.2) *is zero.*

(2) $A[\hat{\Gamma}]$ is densely embedded in $A[\![\hat{\Gamma}]\!]$.

(3) $\hat{\Gamma} \mapsto A[\![\hat{\Gamma}]\!]$ behaves functorially in $\hat{\Gamma}$.

Proof. This is [RZ10, Lemma 5.3.5]. □

One more notion that remains before discussing continuous pseudorepresentations of profinite algebras is that of the topology on the tensor product of pro-discrete algebras. Let us therefore be explicit in explaining this topology on these tensor products and their completions in the primary setting that we will require.

Definition 3.1.1.4. Let A be a profinite commutative ring. Let R be a profinite continuous A-algebra with a fundamental system of finite index ideals (I_λ). Let B be a continuous linearly topologized commutative A-algebra[1] with fundamental system of ideals (J_η). Then a neighborhood of ideals of 0 in $R \otimes_A B$ is given by the ideals

$$\text{Image}(I_\lambda \otimes_A J_\eta \longrightarrow R \otimes_A B)$$

as I_λ, J_η vary over elements of the fundamental systems of ideals mentioned above. The completed tensor product is the limit

$$R\hat{\otimes}_A B := \varprojlim_{\lambda,\eta} R/I_\lambda \otimes_A B/J_\eta.$$

We observe that $R\hat{\otimes}_A B$ is profinite when R and B are profinite. The completed tensor product is, of course, complete, even if B is not complete with respect to its topology. Also, the natural map $B \to R\hat{\otimes}_A B$ factors through the completion $\hat{B}$. See [RZ10, §5.5] for some further discussion in the profinite case.

Remark 3.1.1.5. As discussed in Definition 3.1.0.10 and the proof the equivalence of the definitions of continuity given theret, a continuous pseudorepresentation consists of continuous functions $D_B : R \otimes_A B \to B$ for every A-algebra B. Because all of the topological rings involved are Hausdorff and the targets are complete, D_B will factor uniquely through the completion map $R \otimes_A B \to R\hat{\otimes}_A B$. When we need to distinguish these two cases, we will write D_B for the map out of $R \otimes_A B$ and $\tilde{D}_B$ for the map out of $R\hat{\otimes}_A B$.

Now we would like to discuss continuous pseudorepresentations over profinite algebras. First let us specify the data that we start with.

[1] See [Gro60, 0_{I}, §7.1] for this definition.

Conventions. Our general setup is the following: A is a commutative profinite ring, and R is a profinite continuous A-algebra. It is important to note that if (I_i) are a set of ideals of A forming a fundamental system of neighborhoods around 0 in A, then the induced (I_i)-adic topology on R is not necessarily equivalent to the profinite topology on R. We will always use the native profinite topology on R unless otherwise noted.

We are interested in continuous representations of R. We will generally use B to represent a topological A-algebra of coefficients for the representation, or $\mathfrak{X}$ for a $\mathrm{Spf}(A)$-formal scheme of coefficients. For most of our discussion, we will let B be an admissible A-algebra,[2] where we write $\mathcal{A}\mathrm{dm}_A$ for the category of admissible A-algebras. Sometimes B will be restricted to certain subcategories of admissible A-algebras, such as local Artinian A-algebra with a fixed residue A-field.

Any commutative profinite ring A is canonically a continuous $\hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$-algebra. Since $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ and the functors of representations over this base ring respect this decomposition, we will assume that A is a continuous $\mathbb{Z}_p$-algebra for some rational prime p. This means that p will be topologically nilpotent in the rings and algebras that we will be concerned with.

3.1.2. Continuous Pseudorepresentations of Profinite Algebras. In this paragraph, we provide more characterizations of continuous pseudorepresentations in the case of profinite or prodiscrete topologies, and show that the Cayley-Hamilton ideal $\mathrm{CH}(D)$ of a continuous pseudorepresentation is closed.

The following lemma, due to Chenevier, shows that the conventional notion that a homomorphism from a profinite object to a discrete object is continuous if and only if it has open kernel extends to the case of pseudorepresentations.

Lemma 3.1.2.1 (Following [Che11, Lemma 2.33]). *Let A be a profinite commutative ring and let R be a profinite A-algebra. Let B be a commutative continuous discrete A-algebra, and choose a B-valued d-dimensional pseudorepresentation D of R. Let P_D denote the*

[2]See [Gro60, 0_{I}, §7] for this and other notions for the topological coefficient rings and formal schemes.

corresponding degree d homogenous multiplicative A-polynomial law $P_D \in \mathcal{M}_A^d(R, B)$. *Then the following conditions are equivalent.*

(1) D *is continuous.*

(2) P_D *is continuous.*

(3) $\ker(D)$ *is open.*

(4) $\ker(P_D)$ *is open.*

(5) D *factors through a continuous discrete quotient ring of* $R\hat{\otimes}_A B$.

(6) P_D *factors through a continuous finite quotient ring of* R.

PROOF. The equivalences (3) $\iff$ (5), (4) $\iff$ (6) are clear.

If $\ker(P_D)$ (resp. $\ker(D)$) contains an open ideal, then it is continuous because the characteristic polynomial coefficient functions factor through a discrete space and have target a discrete space. Therefore (3) $\implies$ (1), (4) $\implies$ (2).

Assume that P_D is continuous; this means that each characteristic polynomial function $\Lambda_i : R \to B$ is continuous. Since B is discrete and the topology on R is given by finite index ideals, each Λ_i factors through R/I_i for some open ideal I_i. The intersection of these ideals is open, so P_D factors continuously through a finite discrete quotient. We have shown (2) $\implies$ (4). The proof (1) $\implies$ (3) is identical, except that the ideals topologizing $R\hat{\otimes}_A B$ are not necessarily finite index.

Since P_D factors through D along the natural continuous map $R \to R\hat{\otimes}_A B$, we have (1) $\implies$ (2). Now assume (3); we will prove (4). Since the contraction of $\ker(D)$ along this continuous map is an open (equivalently, finite index) ideal and contained in $\ker(P_D)$, we see that $\ker(P_D)$ is also finite index and therefore open. □

We prove this lemma in a more generality than the profinite case, although we only prove the converse statement in the pro-discrete case.

LEMMA 3.1.2.2. *Let* $D : R \to A$ *be a d-dimensional pseudorepresentation between topological rings. If* D *is continuous, then* $\ker(D) \subset R$ *is closed. If* A, R *are profinite as discussed above and* $B \in \mathcal{A}\mathrm{dm}_A$, *then the converse is also true.*

PROOF. The closure $\overline{\ker(D)}$ of the two-sided ideal $\ker(D)$ is a two-sided ideal. Because the characteristic polynomial functions $\Lambda_i : R \to A$ are constant on cosets of $\ker(D)$ in R and are also continuous, they are also constant on the closure of cosets. This means that D factors through $R \twoheadrightarrow R/\overline{\ker(D)}$. According to Lemma 1.1.6.6, $\ker(D)$ is the largest two-sided ideal $K \subset R$ such that D factors through R/K. Therefore $\ker(D) = \overline{\ker(D)}$.

Now we prove the converse statement, assuming that A, R are profinite and $B \in \mathcal{A}\mathrm{dm}_A$. Present B as a limit of A-algebras $B = \varprojlim_\lambda B_\lambda$, where the system (B_λ) is composed of finite discrete continuous A-algebras and the maps are surjective continuous A-algebra homomorphisms. For each map $\pi_\lambda : B \twoheadrightarrow B_\lambda$, let P_λ denote the induced polynomial law $D_\lambda := \pi_\lambda \circ D$. Lemma 3.1.2.1 tells us that $\ker(D_\lambda) \subset R \otimes_A B$ is open and closed, and therefore $\ker(D) = \bigcap_\lambda \ker(D_\lambda)$ is closed. □

In the non-topological case discussed in Chapters 1 and 2, the notion of a Cayley-Hamilton pseudorepresentation $D : R \to A$ and a Cayley-Hamilton A-algebra (R, D) played a large role. This will be especially true as we consider the moduli of representations of profinite algebras. Therefore the following lemma will be useful, showing that in the case of profinite coefficients, the Cayley-Hamilton ideal $\mathrm{CH}(D)$ is closed, so that the natural surjection $R \twoheadrightarrow R/\mathrm{CH}(D)$ is continuous.

LEMMA 3.1.2.3. *Let A be a complete Noetherian local ring, with finite residue field $\mathbb{F}_A$. Let R be a profinite continuous A-algebra. Then $\mathrm{CH}(D)$ is a closed ideal. Consequently, $R/\mathrm{CH}(D)$ is profinite, the natural map $R \twoheadrightarrow R/\mathrm{CH}(D)$ is continuous, and is a continuous A-algebra.*

PROOF. Freely using the notation of Definition 1.1.8.5, we recall that $\mathrm{CH}(D)$ is the two-sided ideal of R generated by the image of $\chi^{[\alpha]}(r_1, \ldots, r_d)$ where α varies over I_d^d and r_i vary over R. Let $R^{I_d^d}$ have its standard set-theoretic meaning, i.e. the set of tuples of elements of R, each one corresponding to an element of I_d^d. Let R_l, R_r be copies of R distinguished for notational purposes, and let (r_r^α) denote an element of $R_l^{I_d^d}$, and (r_i) denotes an element of

R^d. Now define a function

$$R_l^{I_d^d} \times R^d \times R_r^{I_d^d} \longrightarrow R$$

$$((r_l^\alpha), (r_i), (r_r^\alpha)) \mapsto \sum_{\alpha \in I_d^d} r_l^\alpha \cdot \chi_D^{[\alpha]}((r_i)) \cdot r_r^\alpha.$$

The image of this map is precisely the two-sided ideal generated by the image of the $\chi_D^{[\alpha]}$, i.e. $\mathrm{CH}(D)$.

Because R is profinite, it is compact Hausdorff. And every map in sight is continuous. Therefore the image $\mathrm{CH}(D)$ of the map above is closed by the closed map lemma. □

3.1.3. The $\mathbf{\Phi}_p$ Finiteness Condition on Profinite Groups. When we consider the case that $R = A[\![\hat{\Gamma}]\!]$, we will often want to impose a condition on $\hat{\Gamma}$ weaker than topological finite generation, but strong enough to imply that the various functors of representations are finite in the appropriate manner (e.g. finite type or Noetherian). This is the $\mathbf{\Phi}_p$ condition, developed by Mazur [Maz89].

Definition 3.1.3.1. Let $\hat{\Gamma}$ be a profinite group and let p be a prime number. We say that $\hat{\Gamma}$ satisfies the $\mathbf{\Phi}_p$ finiteness condition when one of the following equivalent conditions holds, for every finite index (and therefore open) subgroup $H \subset \hat{\Gamma}$.

(1) The maximal pro-p quotient of H is topologically finitely generated.
(2) For any finite dimensional $\mathbb{F}_p$-vector space M with a continuous $\mathbb{F}_p$-linear action of H, the continuous cohomology group $H_c^1(H, M)$ is finite dimensional over $\mathbb{F}_p$.
(3) There are only a finite number of continuous homomorphisms from H to the additive group $\mathbb{F}_p$.

Example 3.1.3.2. When $K/\mathbb{Q}_\ell$ is a finite field extension, $\hat{\Gamma} = \mathrm{Gal}(\bar{K}/K)$ satisfies $\mathbf{\Phi}_p$ because $\hat{\Gamma}$ is topologically finitely generated.

Example 3.1.3.3. When $F/\mathbb{Q}$ is a finite field extension and S is a finite set of places of F, let F_S denote the maximal extension of F unramified outside S. Then by Hermite's theorem, $\mathrm{Gal}(F_S/F)$ satisfies $\mathbf{\Phi}_p$.

Given a finite index subgroup $H \subset \hat{\Gamma}$, there exists a maximal quotient $\tilde{\Gamma}$ of $\hat{\Gamma}$ with the property that the image of H in $\tilde{\Gamma}$ is pro-p. If $\hat{\Gamma}$ has property $\mathbf{\Phi}_p$, then one can check that $\tilde{\Gamma}$ (and of course the image of H in $\tilde{\Gamma}$) topologically is finitely generated. This quotient $\tilde{\Gamma}$ is called the *p-completion of $\hat{\Gamma}$ relative to H*. This notion will come up in the following sort of example.

EXAMPLE 3.1.3.4 (cf. [Maz89, p. 389]). Let $\hat{\Gamma}$ satisfy condition Φ_p, and let $\mathbb{F}$ be a finite characteristic p field. Fix a continuous homomorphism $\bar{\rho} : \hat{\Gamma} \to \mathrm{GL}_d(\mathbb{F})$, where $d \geq 0$. Then for any Artinian ring A with residue field $\mathbb{F}$, the kernel of $\mathrm{GL}_d(A) \twoheadrightarrow \mathrm{GL}_d(\mathbb{F})$ is a pro-p group. Therefore the action of $\hat{\Gamma}$ through any deformation of $\bar{\rho}$ from $\mathbb{F}$ to A factors through the p-completion of $\hat{\Gamma}$ relative to $\ker(\bar{\rho})$.

3.1.4. Continuous Deformations of a Finite Field-Valued Pseudorepresentation. Let A be a Noetherian local commutative $\mathbb{Z}_p$-algebra with finite residue field $\mathbb{F}_A$, and let R be a profinite A-algebra. Let $\mathbb{F}$ be a finite A-field (of characteristic p) and let

$$\bar{D} : R \otimes_A \mathbb{F} \longrightarrow \mathbb{F}$$

be a continuous d-dimensional determinant. We are interested in continuous deformations of $\bar{D}$.

It will be useful in the sequel to apply Theorem 1.3.1.1 and write $\rho_{\bar{D}}^{ss}$ for a representative

$$\rho_{\bar{D}}^{ss} : R \otimes_A \mathbb{F} \longrightarrow M_d(\mathbb{F}),$$

assuming that $\mathbb{F}$ is large enough so that $\bar{D}$ is split and therefore $\rho_{\bar{D}}^{ss}$ exists over $\mathbb{F}$.

REMARK 3.1.4.1. Applying Theorem 1.3.1.3 along with the fact that finite fields are perfect, any finite field valued pseudorepresentation of R is *automatically* continuous, as is $\rho_{\bar{D}}^{ss}$, since the kernel of such $\bar{D}$ must be finite index in R.

Artinian A-algebras with residue field $\mathbb{F}$ are the natural context to study deformations of an object, such as $\bar{D}$, defined over $\mathbb{F}$. We reprise Definition 2.1.1.1.

DEFINITION 3.1.4.2. Let $\mathcal{A}_{\mathbb{F}}$ be the category of Artinian local A-algebras with residue field $\mathbb{F}$, where morphisms are local and continuous A-algebra homomorphisms. For $B \in \mathcal{A}_{\mathbb{F}}$ we write $\mathfrak{m}_B$ for its maximal ideal and endow it with the discrete ($\mathfrak{m}_B$-adic) topology.

Let $\hat{\mathcal{A}}_{\mathbb{F}}$ be the category of profinite local A-algebras with residue field $\mathbb{F}$, where morphisms are local continuous A-algebra homomorphisms. For $B \in \mathcal{A}_{\mathbb{F}}$ we write $\mathfrak{m}_B$ for its maximal ideal and endow it with the $\mathfrak{m}_B$-adic topology.

The category $\hat{\mathcal{A}}_{\mathbb{F}}$ includes $\mathcal{A}_{\mathbb{F}}$ as a full subcategory, and objects in $\hat{\mathcal{A}}_{\mathbb{F}}$ consist of limits (filtered projective limits with surjective maps) in $\mathcal{A}_{\mathbb{F}}$.

We define the deformation functor $\mathrm{PsR}_{\bar{D}}$ as follows.

DEFINITION 3.1.4.3. With the data $p, A, R, \bar{D}, d$ and $\mathbb{F}$ as above, let $\mathrm{PsR}_{\bar{D}}$ be the covariant functor on $\hat{\mathcal{A}}_{\mathbb{F}}$ associating to each $B \in \mathrm{ob}\,\hat{\mathcal{A}}_{\mathbb{F}}$ the set of continuous d-dimensional pseudorepresentations

$$D : R\hat{\otimes}_A B \longrightarrow B$$

such that $D\hat{\otimes}_B\mathbb{F} \longrightarrow \mathbb{F} \cong \bar{D}$. We call such deformations of $\bar{D}$ *pseudodeformations*.

REMARK 3.1.4.4. Let us clarify the notation $D\hat{\otimes}_A B$ where $D : R \to A$ is a continuous pseudorepresentation from a profinite A-algebra R to a profinite commutative continuous A-algebra B. Let $D \otimes_A B : R \otimes_A B \to B$ be the non-topological version of the base change of D from A to B. The tensor product has a profinite topology defined by the ideals used in its profinite completion, although it is not complete with respect to this topology. Since the characteristic polynomial coefficient functions of $D \otimes_A B$ are continuous and B is complete, they factor through the completion with respect to this topology – a full argument along these lines (but addressing a slightly different question) may be found in the proof of Lemma 3.1.6.4. We denote this pseudorepresentation by $D\hat{\otimes}_A B$. We will extend this notion when we allow B to be an admissible A-algebra.

We need to show that this definition of a functor is indeed functorial in $\hat{\mathcal{A}}_{\mathbb{F}}$ and respects surjective projective limits. It will suffice to prove functorality on $\mathcal{A}_{\mathbb{F}}$ and that it respects surjective projective limits. This is due to Chenevier.

LEMMA 3.1.4.5 ([Che11, Lemma 3.2]). *The functor* PsR_R^d *on* $\hat{\mathcal{A}}_{\mathbb{F}}$ *is compatible with surjective projective limits.*

PROOF. For a morphism $(B \to B') \in \hat{\mathcal{A}}_{\mathbb{F}}$ and $D \in \mathrm{PsR}_{\hat{D}}(B)$, we observe that $D\hat{\otimes}_B B' \in \mathrm{PsR}_{\hat{D}}(B')$.

If R is any A-algebra, the functor of degree d homogenous multiplicative (not necessarily continuous) A-polynomial laws $\mathcal{M}_A^d(R, -)$ from A-algebras to sets is representable (Theorem 1.1.6.5), and therefore commutes with projective limits. For finite continuous A-algebras B_i, a function $R \to \varprojlim_i B_i$ is continuous if and only if $R \to B_i$ is continuous for every i. Applying this to the characteristic polynomial coefficient functions and recalling the definition of continuity of a pseudorepresentation, we see that the same equivalence applies to pseudorepresentations. This completes the lemma. □

Now we will show that the functor of continuous pseudodeformations of $\bar{D} : R \otimes_A \mathbb{F} \to \mathbb{F}$ is representable.

THEOREM 3.1.4.6 ([Che11, Proposition 3.3]). *The functor* $\mathrm{PsR}_{\bar{D}} : \hat{\mathcal{A}}_{\mathbb{F}} \longrightarrow Sets$ *is representable, i.e. there exists a profinite local* A*-algebra* $B_{\bar{D}}$ *and a continuous* d*-dimensional pseudorepresentation*

$$D_{\bar{D}}^u : R\hat{\otimes}_A B_{\bar{D}} \longrightarrow B_{\bar{D}}$$

such that for any $B \in \hat{\mathcal{A}}_{\mathbb{F}}$ *and any* $D \in \mathrm{PsR}_{\bar{D}}(B)$*, there exists a unique* $\hat{\mathcal{A}}_{\mathbb{F}}$*-morphism* $B_{\bar{D}} \to B$ *such that* $D_{\bar{D}}^u \hat{\otimes}_{B_{\bar{D}}} B \cong D$.

PROOF. We will construct the representing algebra $B_{\bar{D}}$ as the profinite completion of the representing object in the analogous non-topological case. By Theorem 1.1.6.5, there exists a universal degree d multiplicative homogenous A-polynomial law

$$D^u : R \longrightarrow \Gamma_A^d(R)^{\mathrm{ab}}$$

inducing the universal d-dimensional pseudorepresentation of Theorem 1.1.7.4 upon applying $\otimes_A \Gamma_A^d(R)^{\mathrm{ab}}$.

Let $\psi : \Gamma^d_A(R)^{\mathrm{ab}} \to \mathbb{F}$ be the A-algebra homomorphism corresponding to $\bar{D}$. Call an ideal $I \subset \Gamma^d_A(R)^{\mathrm{ab}}$ open if $I \subset \ker(\psi)$, $\Gamma^d_A(R)^{\mathrm{ab}}/I$ is a finite local ring, and the induced degree d multiplicative A-polynomial law (called D_I)

$$D^u \otimes_{\Gamma^d_A(R)^{\mathrm{ab}}} \Gamma^d_A(R)^{\mathrm{ab}}/I = D_I : R \longrightarrow \Gamma^d_A(R)^{\mathrm{ab}}/I$$

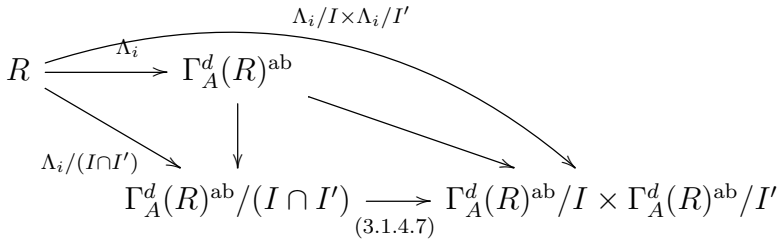
is continuous. We must check that these ideals define a topology on $\Gamma^d_A(R)^{\mathrm{ab}}$. We will call this topology the $\bar{D}$-adic topology on $\Gamma^d_A(R)^{\mathrm{ab}}$.

As a union of ideals of I, I' of this type is a union of translates of $I \cap I'$, it will suffice to show that $I \cap I'$ is open. We consider the canonical A-homomorphism

$$\Gamma^d_A(R)^{\mathrm{ab}}/(I \cap I') \longrightarrow \Gamma^d_A(R)^{\mathrm{ab}}/I \times \Gamma^d_A(R)^{\mathrm{ab}}/I'. \tag{3.1.4.7}$$

It will suffice to show that this map is a homeomorphism onto its image for the $\bar{D}$-adic topology. As this map is injective and induces the diagonal map $\mathbb{F} \to \mathbb{F} \times \mathbb{F}$ after taking the quotient of each of these three rings by their maximal ideals, we see that the properties "finite" and "local" are preserved. It remains to show that the D_J for $J = I, I', I \cap I'$ are topologically compatible with this map (we will specify what this means below).

Recall that the continuity of a multiplicative homogenous polynomial law is defined in terms of its characteristic polynomial coefficients. Write Λ_i/I for the reduction modulo I of the universal characteristic polynomial functions $\Lambda_i : R \to \Gamma^d_A(R)^{\mathrm{ab}}$. By assumption, Λ_i/I and Λ_i/I' are continuous. Now consider the commutative diagram



As (3.1.4.7) is a homeomorphism onto its image for the discrete topology, we have the continuity of $\Lambda_i/(I \cap I')$. The fact that (3.1.4.7) is a homeomorphism onto its image for the discrete topology implies that a quotient of its image will induce a continuous polynomial

law if and only if a quotient of $\Lambda_i/(I \cap I')$ will. Since, as we noted above, the same claim will hold true for the properties "finite" and "local," we have shown that (3.1.4.7) induces a homeomorphism onto its image for the topology on $\Gamma_A^d(R)^{\mathrm{ab}}$ defined above, as desired.

Define $B_{\bar{D}}$ to be the completion of $\Gamma_A^d(R)^{\mathrm{ab}}$ with respect to this topology. This is a profinite A-algebra, by definition of the topology. There is a universal continuous d-dimensional pseudorepresentation which we will call $D^u_{\bar{D}}$,

$$D^u_{\bar{D}} : R\hat{\otimes}_A B_{\bar{D}} \longrightarrow B_{\bar{D}},$$

which we obtain from D^u by the canonical map from $\Gamma_A^d(R)^{\mathrm{ab}}$ to its completion $B_{\bar{D}}$. We verify that this is an object of $\mathrm{PsR}_{\bar{D}}(B_{\bar{D}})$ by applying Lemma 3.1.4.5:

$$\mathrm{PsR}_{\bar{D}}(B_{\bar{D}}) = \varprojlim_I \mathrm{PsR}_{\bar{D}}(\Gamma_A^d(R)^{\mathrm{ab}}/I),$$

and D_I for each $\bar{D}$-adically open ideal I defines a projective system of continuous $\Gamma_A^d(R)^{\mathrm{ab}}/I$-valued homogenous degree d multiplicative A-polynomial laws whose limit is $D^u_{\bar{D}}$.

Having constructed $D^u_{\bar{D}}$, we now verify its universality. Let $B \in \mathcal{A}_{\mathbb{F}}$ and choose $D \in \mathrm{PsR}_{\bar{D}}(B)$. By Theorem 1.1.6.5, there exists a unique A-algebra map $f : \Gamma_A^d(R)^{\mathrm{ab}} \to B$ such that $D = f \circ D^u$ and $f \pmod{\mathfrak{m}_B} \cong \psi$. Therefore $\ker(f) \subset \ker(\psi)$. Also, $\Gamma_A^d(R)^{\mathrm{ab}}/\ker(f) \subset B$ is finite local. Finally, the continuity of D implies that $\ker(f)$ is open in the $\bar{D}$-adic topology on $\Gamma_A^d(R)^{\mathrm{ab}}$. Therefore we have the universality of $(B_{\bar{D}}, D^u_{\bar{D}})$ as a functor on $\mathcal{A}_{\mathbb{F}}$; Lemma 3.1.4.5 implies that it is universal on $\hat{\mathcal{A}}_{\mathbb{F}}$. □

Remark 3.1.4.8. In the case that the profinite A-algebra R arises as the complete group ring $R = A[\![\hat{\Gamma}]\!]$ for some profinite group $\hat{\Gamma}$, one can alternatively form $B_{\bar{D}}$ by replacing R with $A[\hat{\Gamma}]$ and completing $\Gamma_A^d(R)^{\mathrm{ab}}$ with respect to the $\bar{D}$-adic topology described above. This is Chenevier's approach [Che11, Proposition 3.3]. Because $A[\hat{\Gamma}]$ is dense in $A[\![\hat{\Gamma}]\!]$, one can check the functors and constructions amount to the same thing.

Remark 3.1.4.9. We note that just as $\Gamma_A^d(R)^{\mathrm{ab}}$ is generated by $\Lambda_i^{D^u}(r)$ for $r \in R$ and $1 \leq i \leq d$ (this is the non-topological case), so is $B_{\bar{D}}$ topologically generated by $\Lambda_i^{D^u_{\bar{D}}}(r)$ for

$r \in R$ and $1 \leq i \leq d$. This is a consequence of Amitsur's formula (Proposition 1.1.9.11(2)). This statement remains true when the choice of $r \in R$ is restricted to r in a dense subset of R. For example, if $\Gamma \subset \hat{\Gamma}$ is a dense subgroup of $\hat{\Gamma}$, then a pseudodeformation of a $\bar{D} : A[\![\hat{\Gamma}]\!] \hat{\otimes}_A \mathbb{F} \to \mathbb{F}$ is determined by the characteristic polynomial coefficients of the universal pseudodeformation of $\bar{D}$ evaluated at the elements of Γ.

REMARK 3.1.4.10. Following on Corollary 2.1.3.4, the *continuous* deformations of an absolutely irreducible residual pseudorepresentation are equivalent to *continuous* deformations of the associated absolutely irreducible representation. Compare [Nys96, Theorem 2] in the case of pseudocharacters.

3.1.5. Finiteness Condition $\mathbf{\Phi}_{\bar{D}}$. Having defined the universal continuous pseudodeformation of a finite field-valued continuous pseudorepresentation of a profinite A-algebra R, we are interested in finiteness properties of this functor. The main finiteness property of interest for the complete local continuous A-algebra representing this deformation functor is the Noetherian property.

We recall Lemma 2.1.1.5, which gives equivalent conditions under which a complete local ring is Noetherian. As in Chapter 2, we will aim to show the finiteness of the tangent space in order to show that the pseudodeformation ring is Noetherian. Our strategy is to show that the tangent space is finite-dimensional if one assumes $\mathbf{\Phi}_{\bar{D}}$ We have already developed the non-topological notion of tangent spaces to a field-valued pseudorepresentation in §2.1.2, and have given criteria for the finiteness of the dimension for this tangent space in Proposition 2.1.2.3. We will freely use the notation of §2.1.2, and aim to prove a topological version of Proposition 2.1.2.3. As in §2.1.2, these are results of Chenevier, which we extend to arbitrary characteristic.

If D_0 denotes a d-dimensional (possibly non-continuous) pseudorepresentation $D : R \to A$, we recall that $\mathcal{T}_{D_0}$ denotes the non-topological tangent space at $D_0 \in \mathrm{PsR}^d_R(A)$. Assuming now that D_0 is continuous, denote by $\mathcal{T}^c_{D_0}$ the A-submodule of continuous lifts of D_0. We

can write this as a union of A-modules

$$\mathcal{T}_{D_0}^c = \bigcup_I \mathcal{T}_{D_0}^I,$$

where I varies in the set of all open two-sided ideals of R such that $\ker(D_0) \supseteq I$, and $\mathcal{T}_{D_0}^I$ is defined to be the liftings P such that $\ker(P) \supseteq I$.

Now assume that A is a finite field $\mathbb{F}$ and replace D_0 with a continuous d-dimensional pseudorepresentation $\bar{D} : R \to \mathbb{F}$. Then $S_{\bar{D}} := R/\ker(\bar{D})$ is finite dimensional by Theorem 1.3.1.3.

Definition 3.1.5.1. With $\bar{D} : R \to \mathbb{F}$ a continuous pseudorepresentation of a profinite $\mathbb{F}$-algebra R into a finite field $\mathbb{F}$, we say that $\bar{D}$ *satisfies condition* $\mathbf{\Phi}_{\bar{D}}$ or that $\mathbf{\Phi}_{\bar{D}}$ holds when the set of continuous extensions $\mathrm{Ext}^1_R(S_{\bar{D}}, S_{\bar{D}})^c$ is finite dimensional as a $\mathbb{F}$-vector space.

The finiteness condition $\mathbf{\Phi}_{\bar{D}}$ on continuous extension is the finiteness condition we require to give a topological generalization of Proposition 2.1.2.3.

Proposition 3.1.5.2 (Following [Che11, Proposition 2.35]). *Let R be a profinite $\mathbb{F}$-algebra where $\mathbb{F}$ is a finite field. Let $\bar{D} : R \to \mathbb{F}$ be a continuous d-dimensional pseudorepresentation satisfying condition $\mathbf{\Phi}_{\bar{D}}$. Then $\mathcal{T}_{\bar{D}}^c$ is finite dimensional over $\mathbb{F}$.*

Proof. It will suffice to show that there exists a bound on the dimension of $\mathcal{T}_{\bar{D}}^I$ that is independent of choice of finite index two-sided ideal $I \subset R$ such that $I \subset \ker(\bar{D})$. Fix such an ideal I. Also choose N such that $\ker(\bar{D}) \subset \mathrm{CH}(\bar{D})$; such an N exists by Lemma 1.2.3.1(4). By Lemma 2.1.2.2, we know that

$$\mathcal{T}_{\bar{D}}^I \subset \mathcal{P}_{\mathbb{F}}^d(R/(\ker(\bar{D})^{2N} + I), \mathbb{F}).$$

Therefore, it will suffice to show that the right hand side has $\mathbb{F}$-dimension bounded independently of I.

Since $I \subseteq \ker(\bar{D})$, we have for each $n \geq 1$ the natural surjection

$$(\ker(\bar{D})/(\ker(\bar{D})^2 + I))^{\otimes_{\mathbb{F}}^n} \longrightarrow (\ker(\bar{D})^n + I)/(\ker(\bar{D})^{n+1} + I),$$

similar to the proof of Proposition 2.1.2.3. Because $S_{\bar{D}}$ is finite dimensional, it will be enough to show that the $\mathbb{F}$-dimension of $\ker(\bar{D})/(\ker(\bar{D})^2 + I)$ is bounded independently of I.

Because $S_{\bar{D}}$ is a semisimple $\mathbb{F}$-algebra, it will suffice to show that

$$\dim_{\mathbb{F}}(\mathrm{Hom}_R(\ker(\bar{D})/(\ker(\bar{D})^2 + I), S_{\bar{D}}))$$

is bounded independently of I. By noting that the action of R on $\ker(\bar{D})/(\ker(\bar{D})^2 + I)$ factors through R/I and applying Lemma 1.3.3.1, we have

$$\mathrm{Hom}_R(\ker(\bar{D})/(\ker(\bar{D})^2 + I), S_{\bar{D}}) \xrightarrow{\sim} \mathrm{Ext}^1_{R/I}(S_{\bar{D}}, S_{\bar{D}}).$$

The right hand side is a sub-$\mathbb{F}$-vector space of $\mathrm{Ext}^1_R(S_{\bar{D}}, S_{\bar{D}})^c$, i.e. the action of R on the extension is continuous, since the action of R on any of these extensions factors through the finite (cardinality) $\mathbb{F}$-algebra R/I. Because of the assumption that the dimension of $\mathrm{Ext}^1_R(S_{\bar{D}}, S_{\bar{D}})^c$ is constant (and clearly independent of I), we are done. □

Now we are ready to give some sufficient conditions on the continuous d-dimensional pseudorepresentation $\bar{D} : R \otimes_A \mathbb{F} \to \mathbb{F}$ to guarantee that the deformation functor is represented by a Noetherian ring. We recall that with $\bar{D}$ as specified above, $\rho^{ss}_{\bar{D}} : R \otimes_A \bar{\mathbb{F}} \to M_d(\bar{\mathbb{F}})$ denotes a semi-simple representation associated to $\bar{D}$ by Theorem 1.3.1.1, which is continuous because the continuity of $\bar{D}$ implies that $\ker(\bar{D})$ is closed (Lemma 3.1.2.2). We will also use $S_{\bar{D}}$ to denote $S_{\bar{D}} := (R \otimes_A \mathbb{F})/\ker(\bar{D})$, which is finite dimensional over $\mathbb{F}$ by Theorem 1.3.1.3.

Theorem 3.1.5.3 (Following [Che11, Proposition 3.7]). *Let A be a Noetherian complete local $\mathbb{Z}_p$-algebra and let R be a profinite continuous A-algebra. Let $\bar{D} : R\hat{\otimes}_A\mathbb{F} \to \mathbb{F}$ be a continuous d-dimensional pseudorepresentation, where $\mathbb{F}$ is a finite continuous A-field. Then the complete local profinite continuous A-algebra $B_{\bar{D}}$ is Noetherian if any of the following conditions are true.*

(1) R is topologically finitely generated as an A-algebra.

(2) $\bar{D}$ satisfies condition $\mathbf{\Phi}_{\bar{D}}$.

(3) $\hat{\Gamma}$ *is a profinite group,* $R = A[\![\hat{\Gamma}]\!]$*, and the continuous cohomology* $H^1_c(\hat{\Gamma}, \mathrm{ad}(\rho^{ss}_{\bar{D}}))$ *is finite dimensional over* $\bar{\mathbb{F}}$*.*

(4) $\hat{\Gamma}$ *is a profinite group satisfying Mazur's* $\mathbf{\Phi}_p$*-condition and* $R = A[\![\hat{\Gamma}]\!]$*.*

PROOF. We will show that any of these conditions implies that the tangent space $\mathcal{T}_{\bar{D}}$ (Definition 2.1.2.1) to PsR^d_R at $\bar{D}$ is finite dimensional over $\mathbb{F}$. This tangent space is naturally dual to $\mathfrak{m}_{\bar{D}}/(\mathfrak{m}_A, \mathfrak{m}^2_{\bar{D}})$, which is therefore finite-dimensional. Since A is assumed to be Noetherian, this finiteness in turn implies that $\mathfrak{m}_{\bar{D}}/\mathfrak{m}^2_{\bar{D}}$ is finite-dimensional. Therefore, $\mathrm{PsR}_{\bar{D}}$ is Noetherian by Lemma 2.1.1.5.

That condition (2) is sufficient to prove that $B_{\bar{D}}$ is Noetherian is immediate from Proposition 3.1.5.2. Condition (3) is the same condition as (2) in the case that $R = A[\![\hat{\Gamma}]\!]$, after tensoring by $\otimes_{\mathbb{F}}\bar{\mathbb{F}}$. Condition (4) is sufficient to imply condition (3), as discussed in Definition 3.1.3.1.

Assume condition (1). Let R^{fg} be a finitely generated dense sub-A-algebra of R. Then $\mathrm{PsR}^d_{R^{fg}}$ is a finite type (hence Noetherian) A-scheme by Theorem 1.1.10.15. Upon observing that $\mathrm{PsR}_{\bar{D}}$ is the formal scheme arising from $\mathrm{PsR}^d_{R^{fg}}$ by completion at the maximal ideal of $\mathrm{PsR}^d_{R^{fg}}$ corresponding to $\bar{D}$, we are done. □

3.1.6. Pseudorepresentations valued in Formal Schemes. So far we have discussed pseudorepresentations of a profinite A-algebra R, where A is a complete local Noetherian $\mathbb{Z}_p$-algebra with finite residue field $\mathbb{F}$. We have found that the functor of deformations of a given finite field valued pseudorepresentation is representable (Theorem 3.1.4.6), and have given sufficient conditions for it to be Noetherian (Theorem 3.1.5.3). We have restricted ourselves to profinite coefficient rings, in particular Artinian local commutative rings with finite residue field. However, in order to discuss algebraic families of representations of a profinite algebra, we will need to consider coefficient rings that are not profinite. For example, a one-dimensional family of representations will be valued in a polynomial ring like $\mathbb{F}_p[t]$. Our goal in this paragraph is to investigate the families of continuous pseudorepresentations that arise in these larger coefficient rings.

Our main result, Theorem 3.1.6.11, tells us that the study above is sufficient: even on larger appropriately topologized rings, the universal pseudorepresentations are valued in a complete local profinite ring. The first task must be to specify what exactly these larger coefficient rings are.

EGA contains the basic facts and terminology to describe linearly topologized rings and formal schemes. We will now freely use these terms, providing some references as we go. We will introduce here, however, some terminology that we have not found universal agreement upon, but which is an important distinction for our purposes.

Definition 3.1.6.1. Let A be a commutative adic Noetherian ring with ideal of definition I. Let B be a linearly topologized commutative ring which is a continuous A-algebra.

(1) If B is topologically isomorphic over A to an admissible completion of a finitely generated A-algebra, then we say that B is *topologically finitely generated* as an A-algebra.

(2) If B is topologically isomorphic over A to the I-adic completion of a finitely generated A-algebra, then we say that B is *formally finitely generated* as an A-algebra. Equivalently, B is a (continuous) quotient of a restricted power series over A.

We use this terminology in consonance with terminology established in [Gro60, 0_{I}, §7; 1, §10]. We are allow following the definition of "topologically finitely generated" used in [Che11, §3.9]: a completion of a finite type algebra. In particular, here are the corresponding definitions in the category of formal schemes.

Definition 3.1.6.2 ([Gro60, §0_{I}, §10.13]). Let $\mathfrak{Y}$ be a locally Noetherian formal scheme with ideal of definition $\mathcal{K}$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes. Then if any of the following equivalent conditions are satisfied, we say that f is *formally finite type*.

(1) $\mathfrak{X}$ is locally Noetherian, f is an adic morphism, and if we write $\mathcal{J} := f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$, then the morphism $f_0 : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K})$ induced by f is finite type.

(2) $\mathfrak{X}$ is locally Noetherian and is the inductive adic limit X_n over the inductive limit $Y_n := \mathcal{S}\mathrm{pec}(\mathcal{O}_{\mathfrak{Y}}/\mathcal{K}^n)$ such that $f_0 : X_0 \to Y_0$ is finite type.

(3) Every point of $\mathfrak{Y}$ is continued in an open formal affine Noetherian subschemes $V \subset \mathfrak{Y}$ such that $f^{-1}(V)$ is a finite union of open formal affine Noetherian subschemes U_i, such that the adic Noetherian ring $\Gamma(U_i, \mathcal{O}_{\mathfrak{Y}})$ is formally finitely generated over $\Gamma(V, \mathcal{O}_{\mathfrak{Y}})$.

When the context is clear, we say that such a morphism of formal schemes is simply "finite type."

With these definitions in place, we can now specify the category of topological rings on which we will define the functor of pseudorepresentations, and later the functor and groupoids of representations. We write $\mathcal{A}\mathrm{dm}_A$ for the category of continuous admissible A-algebras.

Definition 3.1.6.3. Let A be a commutative local complete Noetherian $\mathbb{Z}_p$-algebra with finite residue field $\mathbb{F}$ and its adic topology. Let R be a profinite continuous A-algebra, which we assume to be complete and separated as an A-module. Let PsR_R^d denote the functor

$$\mathrm{PsR}_R^d : \mathcal{A}\mathrm{dm}_A \longrightarrow Sets$$

sending B to the set of continuous B-valued d-dimensional pseudorepresentations of R,

$$D : R \otimes_A B \to B.$$

We will often use the equivalent formulation in terms of a continuous homogenous degree d multiplicative A-polynomial law, which we will denote by $P = P_D$, i.e. $P_D : R \to B$ such that the induced multiplicative polynomial law $R\hat{\otimes}_A B \to B$ is equal to D. Let us confirm that these notions are indeed equivalent in this topological setting. We will write $\mathcal{M}_A^d(R, B)^c$ for the set of continuous degree d homogenous multiplicative A-polynomial laws from R to B.

Lemma 3.1.6.4. *With A, R, D, d as above and $B \in \mathcal{A}\mathrm{dm}_A$, the natural association*

$$\mathrm{PsR}_R^d(B) \longrightarrow \mathcal{M}_A^d(R, B)^c$$

$$(D : R \otimes_A B \to B) \mapsto D \circ (R \xrightarrow{\mathrm{id}\otimes 1} R \otimes_A B)$$

is a bijection.

PROOF. Clearly the map exists. We will exhibit a two-sided inverse. For $P \in \mathcal{M}_A^d(R,B)^c$, we have by e.g. Corollary 1.1.3.10(1) an induced determinant

$$D_P : R \otimes_A B \to B.$$

such that $D_P(r \otimes b) = b^d \cdot D_P(r)$. The characteristic polynomial functions $\Lambda_{i,A}^P : R \to B$ are continuous by assumption. Recalling that the characteristic polynomial functions are in fact polynomial laws $\Lambda_i^P : R \to B$, we take the function associated to the A-algebra B,

$$\Lambda_{i,B}^P : R \otimes_A B \longrightarrow B \otimes_A B,$$

which we concatenate with the A-algebra structure map $B \otimes_A B \to B$ to get a characteristic polynomial function. This function is continuous, and it is also identical to $\Lambda_{i,B}^{D_P}$. This shows that D_P is continuous. □

The equivalence of Lemma 3.1.6.4 makes it clear that PsR_R^d is a covariant functor: for a morphism $(\iota : B \to B') \in \mathcal{A}\mathrm{dm}_A$ and $P \in \mathcal{M}_A^d(R,B)^c$, we have $\mathrm{PsR}_R^d(\iota)(P) := \iota \circ P \in \mathrm{PsR}_R^d(B')$.

REMARK 3.1.6.5. The equivalence $P \leftrightarrow D_P$ described in the lemma above shows that the description of $D\hat{\otimes}_A B$ in Remark 3.1.4.4 extends to the case that B an admissible A-algebra, and also for $\mathfrak{X}$ a $\mathrm{Spf}(A)$-formal scheme since pseudorepresentations on $\mathfrak{X}$ will be defined as a Zariski sheaf of algebra homomorphisms. The lemma above shows that one can simply reduce to the underlying multiplicative A-polynomial law out of R in order to test continuity.

EXAMPLE 3.1.6.6. The main example of A, R that we will concern ourselves with is the case that $A = \mathbb{Z}_p$ and $R = \mathbb{Z}_p[\![\hat{\Gamma}]\!]$, where $\hat{\Gamma}$ is a profinite group.

LEMMA 3.1.6.7 (Following [Che11, Lemma 3.10]). *Let $B \in \mathcal{A}\mathrm{dm}_A$, and let $D : R \otimes_A B \to B$ be a continuous d-dimensional pseudorepresentation $D \in \mathrm{PsR}_R^d(B)$. Denote by $C \subset B$ the*

closure of the sub-A-algebra generated by the characteristic polynomial coefficients $\Lambda_i^P(r)$ for $r \in R$ of the associated continuous homogenous multiplicative A-polynomial law $P : R \to B$.

(1) C is an admissible profinite sub-A-algebra of B. In particular, it is a finite product of local A-algebras with finite residue field.

(2) Assume that $\iota : B \to B'$ is a morphism in $\mathcal{A}\mathrm{dm}_A$ and let $D' : R \otimes_A B' \to B'$ be the induced continuous d-dimensional pseudorepresentation. Let $C' \subset B'$ be the sub-A-algebra associated to B' as above. Then ι induces a continuous surjection $C \twoheadrightarrow C'$.

PROOF. Assume that B is discrete, so that admissibility means that $\mathfrak{m}_A^n \cdot B = 0$ for some $n \geq 1$. Let $P : R \to B$ be the associated continuous multiplicative degree d A-polynomial law associated to D as above. By Lemma 3.1.2.1, P factors through some finite index, i.e. open two-sided ideal $I \subset R$ containing $\mathfrak{m}_A^n \cdot R$. In particular, we can consider P to be a polynomial law over the finite cardinality ring $A/\mathfrak{m}_A^n$. Now $\Gamma^d_{A/\mathfrak{m}_A^n}(R/I)^{\mathrm{ab}}$ is a finite cardinality commutative ring, and therefore so is the ring C of the statement of the lemma, since C is, by Amitsur's relations (Proposition 1.1.9.11(2)), the image of the A-algebra map

$$\Gamma^d_{A/\mathfrak{m}_A^n}(R/I) \longrightarrow B$$

canonically associated to P by Theorem 1.1.6.5.

Now we consider the general case. Since B is admissible as an A-algebra, there is a topological A-algebra isomorphism $B \xrightarrow{\sim} \lim B_\lambda$, where B_λ is a discrete A-algebra and the maps of the limit have nilpotent kernel. Write $\pi_\lambda : B \to B_\lambda$ for the natural projection. Let $P : R \to B$ denote the continuous homogenous degree d multiplicative A-polynomial law associated to D. Write P_λ for $\pi_\lambda \circ P$.

Let $C \subset B$ be the sub-A-algebra defined in the statement of the lemma. By the discrete case above, the image $C_\lambda \subset B_\lambda$ of $C \subset B$ in B_λ is of finite cardinality, and therefore

$$C \xrightarrow{\sim} \varprojlim C_\lambda$$

is a profinite admissible A-subalgebra. The Jacobson radical of a profinite admissible ring must include all ideals of definition. Therefore $C/J(C)$ is finite, and part (1) follows.

For part (2), we simply note that the ring $C \subset B$ is simply the closure of the induced canonical map $\Gamma_A^d(R) \to B$; this is functorial for $\iota : B \to B'$. □

With this lemma controlling the characteristic polynomial coefficients of pseudorepresentations of R in place, we are going to show that the functor $\mathrm{PsR}_R^d(A)$ of all continuous d-dimensional pseudorepresentations of R into admissible A-algebras is represented by the disjoint union of deformation functors of finite field valued pseudorepresentations. We now establish the notation necessary to describe this result.

Definition 3.1.6.8. Denote by $\mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$ the set of closed points of $\mathrm{Spec}(\Gamma_A^d(R)^{\mathrm{ab}})$ with finite residue field. We denote the associated pseudorepresentation by $\bar{D}$ and the point of PsR_R^d by $\mathrm{Spec}\,\mathbb{F}_{\bar{D}}$.

By Remark 3.1.4.1, such pseudorepresentations and their associated semisimple representations are automatically continuous.

In the case that C is a local ring instead of being merely semi-local, then we know that the B-valued continuous pseudorepresentation $D : R\hat{\otimes}_A B \to B$ induces a $C/\mathfrak{m}_C$-valued pseudorepresentation. This pseudorepresentation corresponds via representability of the (non-topological) pseudorepresentation functor to the canonical surjective map

$$\Gamma_A^d(R)^{\mathrm{ab}} \twoheadrightarrow C_0 \twoheadrightarrow C_0/\mathfrak{m}_{C_0} \cong C/\mathfrak{m}_C,$$

so $C/\mathfrak{m}_C$ is canonically isomorphic to $\mathbb{F}_{\bar{D}}$ for some $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$.

Definition 3.1.6.9. Let A, R, B, D, and $C \subset B$ be as in the statement of Lemma 3.1.6.7. If C is local, and $C/\mathfrak{m}_C$ is canonically isomorphic to $\mathbb{F}_{\bar{D}}$ as $\Gamma_A^d(R)^{\mathrm{ab}}$-algebras as per the discussion above, we call D *residually constant*, and say that it is *residually equal* to $\bar{D}$.

Now we define subfunctors of PsR_R^d on $\mathcal{A}\mathrm{dm}_A$ which are residually constant.

Definition 3.1.6.10. Let $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$. Let $\mathrm{PsR}_{\bar{D}}$ be the subfunctor of PsR_R^d on $\mathcal{A}\mathrm{dm}_A$ defined by the following relation. For each $B \in \mathcal{A}\mathrm{dm}_A$, let $\mathrm{PsR}_{\bar{D}}(B) \subset \mathrm{PsR}_R^d(B)$ be the subset of d-dimensional pseudorepresentations that are residually constant and residually equal to $\bar{D}$.

Lemma 3.1.6.7(2) shows that $\mathrm{PsR}_{\bar{D}}$ is indeed (covariantly) functorial in morphisms $(B \to B') \in \mathcal{A}\mathrm{dm}_A$.

We have now defined two functors which we call $\mathrm{PsR}_{\bar{D}}$. We will temporarily distinguish these functors in order to show that they correspond in a natural way. Write $\mathrm{PsR}_{\bar{D}}^{\mathcal{A}\mathrm{dm}_A}$ for the functor of residually constant pseudorepresentations of Definition 3.1.6.10. Write $\mathrm{PsR}_{\bar{D}}^{\hat{\mathcal{A}}_{\mathbb{F}_{\bar{D}}}}$ for the deformation functor of the residual pseudorepresentation $\bar{D} : R \otimes_A \mathbb{F}_{\bar{D}} \to \mathbb{F}_{\bar{D}}$ defined in Definition 3.1.4.3.

Theorem 3.1.6.11 (Following [Che11, Proposition 3.13]). *Let A be a complete Noetherian local $\mathbb{Z}_p$-algebra with finite residue field, and let R be a profinite continuous A-algebra. Let $\bar{D} \in \mathrm{PsR}_R^{d,\mathcal{A}\mathrm{dm}_A}(\bar{\mathbb{F}}_A)$. Then $\mathrm{PsR}_{\bar{D}}^{\mathcal{A}\mathrm{dm}_A}$ is representable by a local admissible A-algebra $\tilde{B}_{\bar{D}} \in \mathrm{ob}\,\mathcal{A}\mathrm{dm}_A$ whose residue field is canonically isomorphic to $\mathbb{F}_{\bar{D}}$. Moreover,*

(1) The $W(\mathbb{F}_{\bar{D}})$-algebra $B_{\bar{D}}$ representing $\mathrm{PsR}_{\bar{D}}^{\hat{\mathcal{A}}_{\mathbb{F}_{\bar{D}}}}$ is canonically topologically isomorphic to $\tilde{B}_{\bar{D}}$.

(2) If $\mathbf{\Phi}_{\bar{D}}$ holds, then $\tilde{B}_{\bar{D}}$ is topologically finite type over A and Noetherian, and therefore topologically finite type over $\mathbb{Z}_p$ as well.

Proof. Lemma 3.1.6.7 implies that for any $B \in \mathrm{ob}\,\mathcal{A}\mathrm{dm}_A$ and any $(P : R \to B) \in \mathrm{PsR}_R^{d,\mathcal{A}\mathrm{dm}_A}(B)$, P is the composite of a continuous multiplicative polynomial law $P' : R \to C$ with $C \to B$, where $C \in \mathcal{A}\mathrm{dm}_A$ is semi-local. If $P \in \mathrm{PsR}_{\bar{D}}^{\mathcal{A}\mathrm{dm}_A}(B)$, then by definition of the subfunctor, C is canonically a complete local A-algebra with residue field canonical A-isomorphic to $\mathbb{F}_{\bar{D}}$, i.e. C is canonically an object of $\hat{\mathcal{A}}_{\mathbb{F}_{\bar{D}}}$. Consequently, P' is naturally an element of $\mathrm{PsR}_{\bar{D}}^{\hat{\mathcal{A}}_{\mathbb{F}_{\bar{D}}}}(C)$.

Now Theorem 3.1.4.6 gives rise to a canonical continuous A-algebra homomorphism $B_{\bar{D}} \to C$ corresponding to P', and whose universal pseudodeformation of $\bar{D}$ induces P'.

Composing this map with $C \to B$, we have the representability result, as well as the canonical isomorphism $B_{\bar{D}} \xrightarrow{\sim} \tilde{B}_{\bar{D}}$.

Part (1) following directly from the arguments above along with Theorem 3.1.5.3; (2) follows directly from definitions. □

It remains to address the representability of PsR_R^d. This is best done over the category $\mathcal{FS}_A$ of $\mathrm{Spf}(A)$-formal schemes.

Definition 3.1.6.12. Let $\mathrm{PsR}_R^d = \mathrm{PsR}_R^{d,\mathcal{FS}_A}$ denote the contravariant functor sending a $\mathrm{Spf}(A)$-formal scheme $\mathfrak{X}$ to the set of continuous d-dimensional pseudorepresentations

$$R \otimes_{\mathrm{Spf}\, A} \mathcal{O}(\mathfrak{X}) \to \mathcal{O}(\mathfrak{X}).$$

Likewise, for any $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$, let $\mathrm{PsR}_{\bar{D}}^{\mathcal{FS}_A}(\mathfrak{X}) \subset \mathrm{PsR}_R^{d,\mathcal{FS}_A}(\mathfrak{X})$ define a subfunctor cut out by the condition on $D \in \mathrm{PsR}_R^{d,\mathcal{FS}_A}(\mathfrak{X})$ that for any open affine $U \subset \mathfrak{X}$, the restriction of D to $\mathrm{PsR}_R^{d,\mathcal{A}\mathrm{dm}_A}(\Gamma(\mathcal{O}_U))$ belongs to $\mathrm{PsR}_{\bar{D}}^{\mathcal{A}\mathrm{dm}_A}$.

Clearly the restriction of $\mathrm{PsR}_R^{d,\mathcal{FS}_A}$ to $(\mathcal{A}\mathrm{dm}_A)^{\mathrm{op}}$ coincides with the opposite functor of $\mathrm{PsR}_R^{d,\mathcal{A}\mathrm{dm}_A}$. Note that the $D \in \mathrm{PsR}_R^{d,\mathcal{A}\mathrm{dm}_A}(B)$ belongs to $\mathrm{PsR}_{\bar{D}}^{\mathcal{A}\mathrm{dm}_A}(B)$ if and only if, for any affine covering $\mathrm{Spf}(B) = \bigcup_i (U_i)$, the image $D_i \in \mathrm{PsR}_R^{d,\mathcal{FS}_A}(V_i)$ of D belongs to $\mathrm{PsR}_{\bar{D}}^{\mathcal{FS}_A}$ for all i; this follows directly from Lemma 3.1.6.7. Now the same statement can be made of $\mathrm{PsR}_{\bar{D}}^{\mathcal{FS}_A}$: its restriction to $(\mathcal{A}\mathrm{dm}_A)^{\mathrm{op}}$ coincides with the opposite functor of $\mathrm{PsR}_{\bar{D}}^{\mathcal{A}\mathrm{dm}_A}$.

Corollary 3.1.6.13 ([Che11, Corollary 3.14]). *Assume that condition $\mathbf{\Phi}_{\bar{D}}$ holds for all $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$. Then $\mathrm{PsR}_{\bar{D}}^{d,\mathcal{FS}_A}$ is representable by the formal scheme*

$$\coprod_{\bar{D} \in \bar{\mathbb{F}}_A} \mathrm{Spf}(B_{\bar{D}}).$$

In particular, the functor PsR_R^d of continuous d-dimensional pseudorepresentations is locally Noetherian and semi-local with local Noetherian component decomposition

$$\mathrm{PsR}_R^d \cong \coprod_{\bar{D} \in \bar{\mathbb{F}}_A} \mathrm{PsR}_{\bar{D}}.$$

As a result of the Theorem and Corollary, we will not bother to distinguish between $\mathrm{PsR}_{\bar{D}}^{\hat{\mathcal{A}}_{\mathbb{F}_{\bar{D}}}}$, $\mathrm{PsR}_{\bar{D}}^{\mathcal{A}\mathrm{dm}_A}$, and $\mathrm{PsR}_{\bar{D}}^{\mathcal{FS}_A}$, and will simply denote these by $\mathrm{PsR}_{\bar{D}}$ and make the source of the functor clear. Generally, it will be the category of admissible continuous A-algebras $\mathcal{A}\mathrm{dm}_A$ or the category of $\mathrm{Spf}(A)$-formal schemes $\mathcal{FS}_A$. We will also denote the object of $\mathcal{A}\mathrm{dm}_A$ representing $\mathrm{PsR}_{\bar{D}}$ by $B_{\bar{D}}$, or by $\mathrm{Spf}(B_{\bar{D}}) \in \mathrm{ob}\,\mathcal{FS}_A$.

3.2. Moduli of Representations of a Profinite Algebra

In analogy to §1.4 in the non-profinite case, we will introduce moduli spaces of topological representations of the profinite A-algebra R. While we could proceed along the same lines as §1.4, defining functors and groupoids of representations fibered over the category of $\mathrm{Spf}(A)$-formal schemes, then proving representability, etc., we will follow a different strategy. Under the assumption of $\mathbf{\Phi}_{\bar{D}}$, we will show that the universal Cayley-Hamilton representation $E(R, D^u_{\bar{D}})$ of R over the universal pseudodeformation $D^u_{\bar{D}}$ of $\bar{D}$ is finite as a module over $B_{\bar{D}}$ and that its native profinite topology is equivalent to its $\mathfrak{m}_{\bar{D}}$-adic topology. This will allow us to deduce that the natural functor of continuous representations of R with constant residual pseudorepresentation $\bar{D}$ over $\mathrm{Spf}(A)$-formal schemes can be found as the $\mathfrak{m}_{\bar{D}}$-adic completion of a *finite type* $\mathrm{Spec}\, B_{\bar{D}}$*-scheme/algebraic stack* of (non a priori continuous) representations of $E(R, D^u_{\bar{D}})$.

Throughout this section, A represents a complete Noetherian local ring with finite residue field $\mathbb{F}_A$ and maximal ideal $\mathfrak{m}_A$. We write R for a profinite continuous A-algebra, not necessarily commutative. Of course, the topology on R is not necessarily the $\mathfrak{m}_A$-adic topology.

3.2.1. Groupoids of Representations. Here are the functors and groupoids of representations of R that we will study on the category of $\mathrm{Spf}(A)$-formal schemes $\mathcal{FS}_A$.

Definition 3.2.1.1. Let A and R be as specified above, and let d be a positive integer.

(1) Define the functor $\mathcal{R}\mathrm{ep}_R^{\square,d}$ on $\mathcal{FS}_A$ by

$$\mathfrak{X} \mapsto \{\text{continuous } \mathcal{O}_{\mathfrak{X}}\text{-algebra homomorphisms } R \otimes_A \mathcal{O}_{\mathfrak{X}} \longrightarrow M_d(\mathfrak{X})\}.$$

(2) Define the groupoid $\mathcal{R}\mathrm{ep}_R^d$, fibered over $\mathcal{FS}_A$, by

$$\mathrm{ob}\,\mathcal{R}\mathrm{ep}_R^d(\mathfrak{X}) = \{V/\mathfrak{X} \text{ rank } d \text{ vector bundle,}$$
$$\text{continuous } \mathcal{O}_\mathfrak{X}\text{-algebra homomorphism } R \otimes_A \mathcal{O}_\mathfrak{X} \longrightarrow \mathrm{End}_{\mathcal{O}_\mathfrak{X}}(V)\}$$

and morphisms being isomorphisms of this data.

(3) Define the groupoid $\overline{\mathcal{R}\mathrm{ep}}_R^d$, fibered over $\mathcal{FS}_A$, by

$$\mathrm{ob}\,\overline{\mathcal{R}\mathrm{ep}}_R^d(\mathfrak{X}) = \{\mathcal{E} \text{ a rank } d^2 \ \mathcal{O}_\mathfrak{X}\text{-Azumaya algebra, with a}$$
$$\text{continuous } \mathcal{O}_\mathfrak{X}\text{-algebra homomorphism } R \otimes_A \mathcal{O}_\mathfrak{X} \longrightarrow \mathcal{E}\}$$

and morphisms being isomorphisms of this data.

The basic initial observations regarding these groupoids and the natural maps to PsR_R^d hold in direct analogy to the non-topological case discussed in §1.4, although we hold off on discussing representability of these groupoids until §3.2.4. Namely, there are canonical maps

$$\mathcal{R}\mathrm{ep}_R^{\square,d} \longrightarrow \mathcal{R}\mathrm{ep}_R^d \longrightarrow \overline{\mathcal{R}\mathrm{ep}}_R^d \tag{3.2.1.2}$$

in direct analogy to (1.4.1.2). Following §1.4.2, the reduced norm on Azumaya algebras, which is étale locally the determinant of a matrix algebra, allows us to associate to any object of these groupoids a d-dimensional continuous pseudorepresentation. We write these maps as

$$\psi^\square : \mathcal{R}\mathrm{ep}_R^{\square,d} \longrightarrow \mathrm{PsR}_R^d,$$
$$\psi : \mathcal{R}\mathrm{ep}_R^d \to \mathrm{PsR}_R^d, \quad \bar{\psi} : \overline{\mathcal{R}\mathrm{ep}}_R^d \to \mathrm{PsR}_R^d.$$

Indeed, a pseudorepresentation induced by a continuous representation of R is continuous (see Definition 3.1.0.10) because the characteristic polynomial coefficient functions $\Lambda_i : \mathcal{E} \to B$ on an Azumaya B-algebra $\mathcal{E}$ are continuous. This shows that the maps $\psi^\square, \psi, \bar{\psi}$ are well defined.

Just like (1.4.2.2), the canonical maps above form a commutative diagram

$$\begin{array}{ccccc} \mathcal{R}\mathrm{ep}_R^{\square,d} & \xrightarrow{(1.4.1.2)} & \mathcal{R}\mathrm{ep}_R^d & \xrightarrow{(1.4.1.2)} & \overline{\mathcal{R}\mathrm{ep}}_R^d \\ & \searrow^{\psi^\square} & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar{\psi}} \\ & & & & \mathrm{PsR}_R^d \end{array} \tag{3.2.1.3}$$

This allows us to consider the $\mathrm{Spf}(A)$-formal groupoids of representations as PsR_R^d-formal groupoids. Now we establish notation to decompose the representation groupoids into the fiber of $\psi^\square$ (resp. ψ, resp. $\bar{\psi}$) over each component $\mathrm{PsR}_{\bar{D}} \subset \mathrm{PsR}_R^d$, $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$. Indeed, an object of any of the representation groupoids over $B \in \mathcal{A}\mathrm{dm}_A$ induces a map $\mathrm{Spf}(B) \to \mathrm{PsR}_R^d$ via the appropriate ψ-map, and the condition that this map correspond to a residually constant pseudorepresentation will define a PsR_R^d-sub-fibered-groupoid, since we observe that the residually constant condition is stable under pullbacks in the category of PsR_R^d-formal schemes (cf. Corollary 3.1.6.13).

Definition 3.2.1.4. For any $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$, we write $\mathcal{R}\mathrm{ep}_{\bar{D}}^\square$ (resp. $\mathcal{R}\mathrm{ep}_{\bar{D}}$, resp. $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}$) for the fiber of $\psi^\square$ (resp. ψ, resp. $\bar{\psi}$) over the component $\mathrm{PsR}_{\bar{D}} \subset \mathrm{PsR}_R^d$.

Our next goal is to show that $\mathcal{R}\mathrm{ep}_{\bar{D}}^\square$ is representable by a $\mathrm{Spf}(A)$-formal scheme, and, moreover, that condition $\mathbf{\Phi}_{\bar{D}}$ implies that $\mathcal{R}\mathrm{ep}_{\bar{D}}^\square$ is formally finite type over $\mathrm{PsR}_{\bar{D}}$, i.e. that $\mathcal{R}\mathrm{ep}_{\bar{D}}^\square$ is a formally finite type $\mathrm{Spf}(B_{\bar{D}})$-formal scheme. While this may be shown rather directly, we will deduce it from the finiteness result of the next paragraph.

3.2.2. Finiteness Results. In this paragraph, our goal is to prove Proposition 3.2.2.1. This proposition gives us the module-finiteness of the universal Cayley-Hamilton algebra associated to R, whose definition we will recall below. This module-finiteness is the key result we require to prove the algebraizability of the representation functors on $\mathcal{A}\mathrm{dm}_A$.

Proposition 3.2.2.1. *Let B be a admissible A-algebra and let $D : R \otimes_A B \to B$ be a continuous d-dimensional residually constant pseudorepresentation $\bar{D}$. Assume that $\bar{D}$ satisfies $\mathbf{\Phi}_{\bar{D}}$. Then*

(1) The B-algebra $(R \otimes_A B)/\mathrm{CH}(D)$ is finitely presented as a B-module.

(2) The native pro-discrete topology on $(R \otimes_A B)/\mathrm{CH}(D)$ given by open ideals is equivalent to the topology induced by a fundamental system of ideals for B.

First we require some lemmas.

LEMMA 3.2.2.2. *Let $\mathbb{F}$ be a finite characteristic p field, let R be a profinite $\mathbb{F}$-algebra, and let $\bar{D} : R \to \mathbb{F}$ be a continuous d-dimensional pseudorepresentation satisfying $\mathbf{\Phi}_{\bar{D}}$. Then $R/\mathrm{CH}(\bar{D})$ is finite dimensional as a $\mathbb{F}$-vector space and, equivalently, $\mathrm{CH}(\bar{D})$ is open as a two-sided ideal of R.*

PROOF. We first note that the equivalence of the conclusions is immediate from R having the profinite topology.

Replace R with $R/\mathrm{CH}(\bar{D})$, so that $(R, \bar{D})$ is a Cayley-Hamilton $\mathbb{F}$-algebra. Let $S := R/\ker(\bar{D})$, which we know from Theorem 1.3.1.3 to be a finite dimensional semisimple $\mathbb{F}$-algebra. It is naturally a quotient of R by Lemma 1.2.1.1. The proof of Proposition 2.1.2.3 shows that the assumption $\mathbf{\Phi}_{\bar{D}}$ is sufficient to imply that $R/\ker(\bar{D})^n$ is finite dimensional as a $\mathbb{F}$-vector space for any $n \geq 0$. Now by Lemma 1.2.3.1, $\ker(\bar{D})$ is nilpotent. This completes the proof that R is finite dimensional. □

REMARK 3.2.2.3. We emphasize that in the proof above, we do not assume that $\mathrm{CH}(\bar{D})$ is a closed ideal of R, nor, equivalently, that the natural surjection $R/\mathrm{CH}(\bar{D})$ is continuous. This fact is a consequence of the proof.

LEMMA 3.2.2.4 (Infinite Nakayama Lemma). *Let A be a complete Noetherian local ring with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbb{F}_A$ and let M be an A-module. Assume that M is $\mathfrak{m}_A$-adically separated, i.e. $\bigcap_{i\geq 0} \mathfrak{m}_A^i \cdot M = 0$, and assume that $M/\mathfrak{m}_A \cdot M$ is finite dimensional as a $\mathbb{F}_A$-vector space. Then M is a finite A-module generated by any set of lifts for a generating set for $M/\mathfrak{m}_A \cdot M$. In particular, one can apply the (standard) Nakayama Lemma to M.*

PROOF. Choose a basis $\bar{m}_1, \ldots, \bar{m}_n$ for $M/\mathfrak{m}_A \cdot M$, and let $m_1, \ldots, m_n$ be a choice of lifts to M for the basis. Choose $0 \neq x \in M$, and let $k \geq 0$ be the greatest integer such that

$x \in \mathfrak{m}_A^k \cdot M$; write x_k for x. Because $(\bar{m}_i)$ is a basis and the Noetherianness of A implies that $\mathfrak{m}_A^b/\mathfrak{m}_A^{b+1}$ is finite dimensional over $\mathbb{F}_A$ for all b, there exists an A-linear combination $\sum_1^n a_{ik}m_i$ such that $a_{ik} \in \mathfrak{m}_A^k$ and

$$x_k - \sum_{i=1}^{n} a_{ik}m_i \in \mathfrak{m}_A^{k+1} \cdot M \tag{3.2.2.5}$$

Now set x_{k+1} to this difference, and choose $a_{i,(k+1)} \in \mathfrak{m}_A^{k+1}, 1 \leq i \leq n$ such that (3.2.2.5) is satisfied with $k+1$ in place of k; iterate this process for all $j \geq k$, generating x_j, a_{ij} for $j \geq k, 1 \leq i \leq n$.

Now set, for each $i, 1 \leq i \leq n$,

$$a_i := \sum_{j=k}^{\infty} a_{ij} \in A,$$

where the sum is convergent because A is $\mathfrak{m}_A$-adically complete and $a_{ij} \in \mathfrak{m}_A^j$ for any $j \geq k$.

Observe that

$$x - \sum_{i=1}^{n} a_i m_i \in \mathfrak{m}_A^j \cdot M$$

for any $j \geq k$. Therefore, by the separation hypothesis on M, $x = \sum_1^n a_i m_i$. This shows that (m_i) is an A-basis for M, as desired. □

Now we can prove Proposition 3.2.2.1

Proof. First, we will prove the result when B is discrete. We already know that $\mathrm{CH}(D)$ is a two-sided ideal of $R \otimes_A B$, so we must show that it is open.

By Lemma 3.1.6.7 and the definition of residual constancy of D (Defintion 3.1.6.10), D factors through a finite cardinality Artinian local sub-A-algebra $C \subset B$ (C is the image of the canonical continuous homomorphism $B_{\bar{D}} \to B$) with residue field $\mathbb{F}_{\bar{D}}$, i.e. there exists a continuous deformation

$$D_C : R \otimes_A C \to C$$

of $\bar{D}$ inducing D upon $\otimes_C B$.

Consider the Cayley-Hamilton quotient $(R \otimes_A C)/\mathrm{CH}(D_C)$. Using the canonical surjection $C \to C/\mathfrak{m}_C \xrightarrow{\sim} \mathbb{F}_{\bar{D}}$, we tensor D_C by $\otimes_C \mathbb{F}_{\bar{D}}$. Now Lemma 1.1.8.6 implies that we have an isomorphism

$$(R \otimes_A C)/\mathrm{CH}(D_C) \otimes_C \mathbb{F}_{\bar{D}} \xrightarrow{\sim} (R \otimes_A \mathbb{F}_{\bar{D}})/\mathrm{CH}(\bar{D}).$$

Applying our assumption that $\mathbf{\Phi}_{\bar{D}}$ holds, Lemma 3.2.2.2 tells us that the right hand side is a finite dimensional $\mathbb{F}$-vector space. Since the C-algebra $(R \otimes_A C)/\mathrm{CH}(D_C)$ is trivially $\mathfrak{m}_C$-adically separated because C is Artinian, the "infinite Nakayama lemma" (Lemma 3.2.2.4) implies that it is also finite as a C-module. Since all of the involved rings are profinite and the maps factor through profinite completions, we may apply Lemma 3.1.2.3 so that we know that the factor map

$$(R \otimes_A C) \longrightarrow (R \otimes_A C)/\mathrm{CH}(D_C)$$

is continuous and $\mathrm{CH}(D_C)$ is closed in $R \otimes_A C$. The target is also finite cardinality, showing that $\mathrm{CH}(D_C)$ is also an open ideal of $R \otimes_A C$. We have now completed the proof in the case that B is a finite Artinian ring.

Now we deduce the general discrete case over B from the local discrete case completed for C above. The natural map

$$(R \otimes_A C)/\mathrm{CH}(D_C) \otimes_C B \longrightarrow (R \otimes_A B)/\mathrm{CH}(D)$$

exists and is an isomorphism by Lemma 1.1.8.6; it is continuous and $\mathrm{CH}(D) \subset (R \otimes_A B)$ is open, as the natural topology on both sides is discrete, proving part (2). Since the left hand side is finitely presented as a B-module by the arguments above, so is the right hand side proving part (1). This completes the argument.

Now we no longer assume that B is discrete. We may write B as a limit of discrete continuous A-algebras $B = \varprojlim_\lambda B_\lambda$ where the maps are surjective with nilpotent kernel, and write $C_\lambda \subset B_\lambda$ for the algebra C in the discrete case above. Then $C = \varprojlim_\lambda C_\lambda$ is a complete local Noetherian sub-A-algebra of B with residue field $\mathbb{F}_{\bar{D}}$, since we are assuming condition $\mathbf{\Phi}_{\bar{D}}$ and may apply Theorem 3.1.5.3. Write $\pi_\lambda : B \to B_\lambda$ for the natural surjections. Write

D_λ for $\pi_\lambda \circ D$; it is a continuous d-dimensional pseudorepresentation

$$D_\lambda : R \otimes_A B_\lambda \to B_\lambda$$

that satisfies the conditions of the discrete pseudorepresentation called "D" above. Likewise, we write D_{C_λ} the pseudorepresentations called "D_C" above.

Consider the C_λ-algebra homomorphism

$$(R \otimes_A C_\lambda)/\mathrm{CH}(D_{C_\lambda}) \longrightarrow (R \otimes_A C_{\lambda'})/\mathrm{CH}(D_{C'_\lambda}).$$

By Lemma 1.1.8.6, it becomes an isomorphism after applying $\otimes_{C_\lambda} C_{\lambda'}$ to the left side. Therefore, it is continuous and surjective; it has nilpotent kernel since the maps $C_\lambda \to C_{\lambda'}$ do too. By the same reasoning, for every λ there is a canonical surjection

$$(R \otimes_A C)/\mathrm{CH}(D_C) \longrightarrow (R \otimes_A C_\lambda)/\mathrm{CH}(D_{C_\lambda}).$$

Therefore, the image of the natural map

$$(R \otimes_A C)/\mathrm{CH}(D_C) \longrightarrow (R\hat{\otimes}_A C)/\mathrm{CH}(\tilde{D}_C) \tag{3.2.2.6}$$

is dense, since the right hand side surjects onto each $(R \otimes_A C_\lambda)/CH(D_{C_\lambda})$ as well. The map (3.2.2.6) is also injective, since $\mathrm{CH}(D_C)$ is dense in $\mathrm{CH}(\tilde{D}_C)$. We now aim to show that it is an isomorphism.

We know that $\mathrm{CH}(\tilde{D}_C) \subset R\hat{\otimes}_A C$ is closed by Lemma 3.1.2.3, so we have

$$(R\hat{\otimes}_A C)/\mathrm{CH}(D_C) \xrightarrow{\sim} \varprojlim_\lambda (R\hat{\otimes}_A C)/\mathrm{CH}(D_{C_\lambda} \circ \pi_\lambda).$$

Now formal GAGA [Gro61b, Theorem 5.1.4] and the finiteness of $(R\hat{\otimes}_A C)/\mathrm{CH}(D_{C_\lambda})$ as C_λ-modules proved in the discrete case above, $(R\hat{\otimes}_A C)/\mathrm{CH}(D_C)$ is a finitely generated C-module. Alternatively, we can apply the infinite Nakayama lemma again. We note that the $\mathfrak{m}_C$-adic topology on $(R\hat{\otimes}_A C)/\mathrm{CH}(\tilde{D}_C)$ is equivalent to its profinite topology arising from the complete tensor product.

Now we observe that (3.2.2.6) is an isomorphism of finite C-modules. Indeed, the image is a dense sub-C-module of a finite C-module. We have now completed the proof in the case that B was a Noetherian complete local ring.

We now deduce the general case from what we have done. Lemma 1.1.8.6 shows us that we have a natural isomorphism

$$(R \otimes_A C)/\mathrm{CH}(D_C) \otimes_C B \xrightarrow{\sim} (R \otimes_A B)/\mathrm{CH}(D), \tag{3.2.2.7}$$

since D arises from D_C by $\otimes_C B$. We conclude that the right hand side is finitely presented as a B-module, since C is Noetherian, proving (1). The compatibility of (3.2.2.7) with $\otimes_B B_\lambda$, yielding the isomorphism of discrete algebras

$$(R \otimes_A C_\lambda)/\mathrm{CH}(D_{C_\lambda}) \otimes_{C_\lambda} B_\lambda \longrightarrow (R \otimes_A B_\lambda)/\mathrm{CH}(D_\lambda),$$

shows us that (2) is true. □

3.2.3. Universality Results. Recall the (non-topological) notion of universal Cayley-Hamilton representation of R (§1.2.4). This is a $\Gamma_A^d(R)^{\mathrm{ab}}$-algebra

$$E(R,d) := (R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}})/\mathrm{CH}(D^u),$$

with the data of the universal pseudorepresentation $D^u\ |_E$: $E(R,d) \to \Gamma_A^d(R)^{\mathrm{ab}}$ and the canonical quotient map from $R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}}$. We have shown in Theorem 1.4.3.1 that moduli spaces of d-dimensional representations of R are equivalent to their counterpart moduli spaces of d-dimensional representations of $E(R,d)$. Our goal in this paragraph is to prove this result in the profinite topological setting of this chapter.

We will carry out this task over each component $\mathrm{PsR}_{\bar{D}}$ of PsR_R^d. There is no significant loss of generality in doing this. Let us establish the notation for these universal Cayley-Hamilton algebras.

DEFINITION 3.2.3.1. Let $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$. The universal Cayley-Hamilton representation over $\mathrm{PsR}_{\bar{D}}$, denoted $E(R, D_{\bar{D}}^u)$, is the $B_{\bar{D}}$-algebra

$$E(R, D_{\bar{D}}^u) := (R \otimes_A B_{\bar{D}})/\mathrm{CH}(D_{\bar{D}}^u),$$

often considered with its canonical factor map $\rho_{\bar{D}}^u : R \otimes_A B_{\bar{D}} \to E(R, D_{\bar{D}}^u)$. We establish notation for the completed case as well,

$$\tilde{E}(R, \tilde{D}_{\bar{D}}^u) := (R\hat{\otimes}_A B_{\bar{D}})/\mathrm{CH}(\tilde{D}_{\bar{D}}^u),$$

with the canonical factor map $\tilde{\rho}_{\bar{D}}^u : R\hat{\otimes}_A B_{\bar{D}} \to \tilde{E}(R, \tilde{D}_{\bar{D}}^u)$.

Before proving the universality theorem for the Cayley-Hamilton algebra $E(R, D_{\bar{D}}^u)$, we point out the consequences of $\mathbf{\Phi}_{\bar{D}}$ for this algebra. This theorem follows directly from Proposition 3.2.2.1, and the last part from Corollary 1.2.2.10.

THEOREM 3.2.3.2. *Assume that $\bar{D} \in \mathrm{PsR}_R^d(\bar{\mathbb{F}}_A)$ satisfies $\mathbf{\Phi}_{\bar{D}}$. Then*

(1) The natural profinite completion map

$$E(R, D_{\bar{D}}^u) \longrightarrow \tilde{E}(R, \tilde{D}_{\bar{D}}^u)$$

is an isomorphism.

(2) $E(R, D_{\bar{D}}^u)$ is finite as a $B_{\bar{D}}$-module

(3) The native topology on $E(D_{\bar{D}}^u)$ is equivalent to the its $\mathfrak{m}_{\bar{D}}$-adic topology as a $B_{\bar{D}}$-module.

(4) $E(D_{\bar{D}}^u)$ is finite as a module over its center and is a Noetherian ring.

Now we prove an analogous result in our profinite topological setting to the universality of the Cayley-Hamilton algebra (Proposition 1.2.4.3) and the resulting equivalence of representation categories between R and $E(R, d)$ (Theorem 1.4.3.1). The non-topological results produce universal maps, and we check that they are continuous.

We require some notation. Following the convention that $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}}$ denotes the fiber of $\mathcal{R}\mathrm{ep}^{\square,d}_{R}$ over $\mathrm{PsR}_{\bar{D}}$, denote by $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}|_E}$ the fiber of $\mathcal{R}\mathrm{ep}^{\square,d}_{E(R,D^u_{\bar{D}})}$ over $\mathrm{PsR}_{\bar{D}|_E} \subset \mathrm{PsR}^d_{E(R,D^u_{\bar{D}})}$.

THEOREM 3.2.3.3. *Let $\mathfrak{X}$ be a $\mathrm{Spf}(A)$-formal scheme. Any representation in the formal groupoids $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}}(\mathfrak{X}), \mathcal{R}\mathrm{ep}_{\bar{D}}(\mathfrak{X}), \overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}(\mathfrak{X})$ factors uniquely continuously through the universal Cayley-Hamilton representation $\rho^u_{\bar{D}} \otimes_{B_{\bar{D}}} \mathcal{O}(\mathfrak{X})$. This factorization induces equivalences of $\mathrm{PsR}_{\bar{D}}$-formal groupoids*

$$\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}} \xrightarrow{\sim} \mathcal{R}\mathrm{ep}^{\square}_{\bar{D}|_E},$$

$$\mathcal{R}\mathrm{ep}_{\bar{D}} \xrightarrow{\sim} \mathcal{R}\mathrm{ep}_{\bar{D}|_E},$$

$$\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}} \xrightarrow{\sim} \overline{\mathcal{R}\mathrm{ep}}_{\bar{D}|_E}.$$

PROOF. It will suffice to work formally Zariski-locally on $\mathfrak{X}$, so we may replace $\mathcal{O}_{\mathfrak{X}}$ with an admissible A-algebra B. As in the proof of Theorem 1.4.3.1, it will suffice to work with a continuous B-algebra homomorphism $\rho : R\hat{\otimes}_A B \to \mathcal{E}$ in $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}(B)$, since objects of the other groupoids amount to additional data on top of the rank d^2 Azumaya B-algebra $\mathcal{E}$ and the map ρ.

Recall Definition 1.2.4.1, which is the notion of a Cayley-Hamilton representation of R. Following Remark 1.2.4.2, we note that a the data of ρ induces a d-dimensional Cayley-Hamilton representation of R over B, namely

$$(B, (\mathcal{E}, \det), \rho),$$

where $\det : \mathcal{E} \to B$ represents the reduced norm map for the Azumaya B-algebra $\mathcal{E}$.

Proposition 1.2.4.3 shows that the universal d-dimensional Cayley-Hamilton representation $(\Gamma^d_A(R)^{\mathrm{ab}}, (E(R,d), D^u|_E), \rho^u)$ is initial in the category $\mathcal{CH}^d(R)$ of Cayley-Hamilton representations of R. Thus there exists a canonical $\mathcal{CH}^d(R)$-morphism

$$(\Gamma^d_A(R)^{\mathrm{ab}}, (E(R,d), D^u|_E), \rho^u) \longrightarrow (B, (\mathcal{E}, \det), \rho).$$

We know that the map $\Gamma^d_A(R)^{\mathrm{ab}} \to B$ included in this data is continuous with respect to the topology on $\Gamma^d_A(R)^{\mathrm{ab}}$ defined in Theorem 3.1.4.6 for the choice of $\bar{D} \in \mathrm{PsR}^d_R(\bar{\mathbb{F}}_A)$, since B has

residually constant pseudorepresentation $\bar{D}$. The completion with respect to this topology is $B_{\bar{D}}$ and B is complete, so that we have a map $B_{\bar{D}} \to B$ factoring $\Gamma_A^d(R)^{\mathrm{ab}} \to B$.

Therefore the continuous B-algebra homomorphism $E(R,d)\otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B \to \mathcal{E}$ which is part of the data of the morphism in $\mathcal{CH}^d(R)$ factors through $E(R,d) \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B_{\bar{D}}$. Recalling that $E(R,d) := (R \otimes_A \Gamma_A^d(R)^{\mathrm{ab}})/\mathrm{CH}(D^u)$, we have by Lemma 1.1.8.6 a canonical isomorphism

$$E(R,d) \otimes_{\Gamma_A^d(R)^{\mathrm{ab}}} B_{\bar{D}} \xrightarrow{\sim} (R \otimes_A B_{\bar{D}})/\mathrm{CH}(D^u_{\bar{D}}) \cong E(R, D^u_{\bar{D}}),$$

so that we now have a canonical continuous map $E(R, D^u_{\bar{D}}) \otimes_{B_{\bar{D}}} B \to \mathcal{E}$ factoring ρ.

We have now exhibited a $\mathrm{PsR}_{\bar{D}}$-groupoid morphism $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}} \to \overline{\mathcal{R}\mathrm{ep}}^d_{E(R,D^u_{\bar{D}})}$. We observe that this lies in $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}|_E}$ because $\Gamma_A^d(R)^{\mathrm{ab}} \to B$ factors through $B_{\bar{D}}$. The map $\rho^u_{\bar{D}} : R \otimes_A B_{\bar{D}} \to E(R, D^u_{\bar{D}})$ induces an inverse morphism by composition. □

Here is an interesting consequence of this universality. Once we show that the groupoids are representable by formal algebraic stacks, this corollary says, essentially, that $\psi^{\square}, \psi, \bar{\psi}$ are adic morphisms.

Corollary 3.2.3.4. *As usual, let A be a commutative Noetherian local profinite ring, let R be a profinite A-algebra, and choose $\bar{D} \in \mathrm{PsR}^d_R(\bar{\mathbb{F}}_A)$ satisfying $\mathbf{\Phi}_{\bar{D}}$. Choose an admissible A-algebra B along with a continuous d-dimensional representation $\rho : R \otimes_A B \to \mathcal{E}$ of residually constant pseudorepresentation $\bar{D}$. Then ρ is still continuous with respect to the finer $\mathfrak{m}_{\bar{D}}$-adic topology on B.*

Proof. Let B, ρ be as in the statement of the corollary. Theorem 3.2.3.3 implies that we have a continuous map $B_{\bar{D}} \to B$ and a canonical continuous factorization of ρ, and a continuous $B_{\bar{D}}$-algebra map $E_{\bar{D}} \to \mathcal{E}$, through which $R \to \mathcal{E}$ factors. The fact that $B_{\bar{D}} \to B$ is continuous means that the $\mathfrak{m}_{\bar{D}}$-adic topology on B is (not necessarily strictly) stronger than its native topology. Clearly if we topologize $\mathcal{E}$ with respect to $\mathfrak{m}_{\bar{D}}$, the map $E_{\bar{D}} \to \mathcal{E}$ will remain continuous. □

3.2.4. Representability Results. Now we will work toward showing that the formal groupoids $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}}$, $\mathcal{R}\mathrm{ep}_{\bar{D}}$, $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}$ are representable by $\mathrm{PsR}_{\bar{D}} = \mathrm{Spf}(B_{\bar{D}})$-formal schemes.

In fact, we will show much more, using the universality (Theorem 3.2.3.3) and finiteness (Theorem 3.2.3.2) of the universal Cayley-Hamilton representation of R of residually constant pseudorepresentation $\bar{D}$. We will show that the d-dimensional representation groupoid $\mathcal{R}\mathrm{ep}^{\square,d}_{E(R,D^u_{\bar{D}})}$ (resp. $\mathcal{R}\mathrm{ep}^{d}_{E(R,D^u_{\bar{D}})}$, resp. $\overline{\mathcal{R}\mathrm{ep}}^{d}_{E(R,D^u_{\bar{D}})}$) for $E(R,D^u_{\bar{D}})$ is formally finite type over $\mathrm{PsR}_{\bar{D}}$ and will prove this by showing that it is algebraizable with finite type algebraization $\mathrm{Rep}^{\square,d}_{E(R,D^u_{\bar{D}})}$ (resp. $\mathrm{Rep}^{d}_{E(R,D^u_{\bar{D}})}$, resp. $\overline{\mathrm{Rep}}^{d}_{E(R,D^u_{\bar{D}})}$). This will show, by Theorem 3.2.3.3, that the representation groupoids above are topologically finite type, Noetherian formal schemes over $\mathrm{Spf}(A)$ that are formally finite type over $\mathrm{PsR}_{\bar{D}} = \mathrm{Spf}(B_{\bar{D}})$.

In order to prove algebraization of the formal $\mathrm{PsR}_{\bar{D}}$-groupoids of representations of $E(R,D^u_{\bar{D}})$, we need to find algebraic groupoids of continuous representations. In fact, what we will show is that, after applying a natural topology to groupoids of non-topological representations such as $\mathrm{Rep}^{d}_{E(R,D^u_{\bar{D}})}$, this non-topological groupoid of representations consists entirely of continuous representations. This result depends critically on the finiteness condition $\mathbf{\Phi}_{\bar{D}}$ and the work done in the previous paragraphs.

THEOREM 3.2.4.1. *With $A, R, d, \bar{D}$, and $E(R,D^u_{\bar{D}})$ as above, assume that $\mathbf{\Phi}_{\bar{D}}$ is true. Then the restrictions to admissible $B_{\bar{D}}$-algebras of the non-topological* $\mathrm{Spec}\, B_{\bar{D}}$*-groupoids of representations* $\mathrm{Rep}^{\square}_{E(R,D^u_{\bar{D}}),D^u_{\bar{D}}}$, $\mathrm{Rep}_{E(R,D^u_{\bar{D}}),D^u_{\bar{D}}}$, *and* $\overline{\mathrm{Rep}}_{E(R,D^u_{\bar{D}}),D^u_{\bar{D}}}$ *of $E(R,D^u_{\bar{D}})$ lying over $\mathrm{PsR}_{\bar{D}}$ are equivalent to their counterparts $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}}$, $\mathcal{R}\mathrm{ep}_{\bar{D}}$, and $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}$.*

PROOF. Because $\mathbf{\Phi}_{\bar{D}}$ is satisfied, Theorem 3.2.3.2 gives us that the $B_{\bar{D}}$-algebra $E(R,D^u_{\bar{D}})$ is module-finite and its native topology as a quotient of $R\hat{\otimes}_A B_{\bar{D}}$ is identical to its $\mathfrak{m}_{\bar{D}}$-adic topology. Choose $B \in \mathrm{Alg}_{B_{\bar{D}}}$ that is an admissible $B_{\bar{D}}$-algebra, and choose a non-topological representation $\rho \in \overline{\mathrm{Rep}}^{d}_{E(R,D^u_{\bar{D}}),D^u_{\bar{D}}}(B)$. This is the data

$$\rho : E(R,D^u_{\bar{D}}) \otimes_{B_{\bar{D}}} B \longrightarrow \mathcal{E}$$

where $\mathcal{E}$ is a rank d^2 B-Azumaya algebra and $\det \circ \rho = (B_{\bar{D}} \to B) \circ D^u_{\bar{D}}$. Rembering the topology on B, we have a topology on both the source and the target of ρ, and we observe that

ρ is a map of finitely presented B-modules, and is therefore continuous under the $\mathfrak{m}_{\bar{D}}$-adic topology. Therefore $\rho \in \overline{\mathrm{Rep}}^{\mathrm{cont}}_{E(R,D^u_{\bar{D}}),D^u_{\bar{D}}}(B)$.

Likewise, one can start with $\rho \in \overline{\mathcal{R}\mathrm{ep}}^{d}_{E(R,D^u_{\bar{D}}),D^u_{\bar{D}}}(B)$, and observe that forgetting the topology gives us an object of $\overline{\mathrm{Rep}}^{d}_{E(R,D^u_{\bar{D}}),D^u_{\bar{D}}}(B)$, providing a quasi-inverse morphism. $\square$

Remark 3.2.4.2. Let us note what may go wrong when $\mathbf{\Phi}_{\bar{D}}$ is not satisfied. The Cayley-Hamilton ideal $\mathrm{CH}(D^u_{\bar{D}}) \subset R \otimes_A B_{\bar{D}}$ is still closed by Lemma 3.1.2.3, so that $E(R, D^u_{\bar{D}})$ is a profinite $B_{\bar{D}}$-algebra. However, $E(R, D^u_{\bar{D}})/\mathfrak{m}_{\bar{D}}E(R, D^u_{\bar{D}}) \cong (R \otimes_A \mathbb{F}_{\bar{D}})/\mathrm{CH}(\bar{D})$ is not necessarily a finite $\mathbb{F}_{\bar{D}}$-vector space, and does not necessarily carry the discrete topology. The former fact suggests that a non-topological moduli space of representations may not be finite type over $\operatorname{Spec} B_{\bar{D}}$, and the latter fact implies that a non-topological moduli space of representations may not correspond to continuous representations.

Let $(-)^{\wedge}_{\bar{D}}$ denote $\mathfrak{m}_{\bar{D}}$-adic completion of a $B_{\bar{D}}$-scheme. This is the formal completion of a $B_{\bar{D}}$-scheme X at the subscheme $X^{\circ} := X \times_{\operatorname{Spec} B_{\bar{D}}} \operatorname{Spec} \mathbb{F}_{\bar{D}}$.

Corollary 3.2.4.3. *Assume $\mathbf{\Phi}_{\bar{D}}$. The formal $\mathrm{Spf}(A)$-groupoid of representations $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}}$ (resp. $\mathcal{R}\mathrm{ep}_{\bar{D}}$, resp. $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}$) is naturally isomorphic over $\mathrm{PsR}_{\bar{D}}$ to the $\mathfrak{m}_{\bar{D}}$-adic completion of $\mathrm{Rep}^{\square}_{\bar{D}|_E}$ (resp. $\mathrm{Rep}_{\bar{D}|_E}$, resp. $\overline{\mathrm{Rep}}_{\bar{D}|_E}$). In particular, it is a formally finite type, Notherian $\mathrm{PsR}_{\bar{D}}$-formal scheme (resp. a formally finite type Notherian $\mathrm{PsR}_{\bar{D}}$-formal algebraic stack). Additionally, the map*

$$\psi : \mathcal{R}\mathrm{ep}_{\bar{D}} \longrightarrow \mathrm{PsR}_{\bar{D}}, (\textit{resp. } \bar{\psi} : \overline{\mathcal{R}\mathrm{ep}}_{\bar{D}} \to \mathrm{PsR}_{\bar{D}})$$

pushes forward coherent sheaves to coherent sheaves and is universally closed.

Remark 3.2.4.4. Note that once we know that the $\mathcal{R}$ep groupoids are representable by formal schemes/algebraic stacks, Corollary 3.2.3.4 can be used to deduce that they are adic over $\mathrm{Spf}(B_{\bar{D}})$.

Proof. The isomorphism between these formal groupoids follows directly from Theorem 3.2.4.1.

For the rest of the proof, we recall Corollary 1.5.4.7, which describes the properties of $\bar{\psi}$ in the non-topological case. The properties in the statement of the corollary are stable under adic completion. The finiteness of the pushforward of coherent sheaves involves a bit of work. Modulo each power $\mathfrak{m}_{\bar{D}}^n$ of the maximal ideal of $B_{\bar{D}}$, the map from $\overline{\mathrm{Rep}}_{\bar{D}|_E} \times \operatorname{Spec} B_{\bar{D}}/\mathfrak{m}_{\bar{D}}^n$ to $\operatorname{Spec} B_{\bar{D}}/\mathfrak{m}_{\bar{D}}^n$ is an adequate moduli space following by a finite morphism by [Alp10, Proposition 5.2.9(3)]. Therefore the pushforward of a coherent sheaf is coherent. The pushforward of a coherent sheaf on the whole formal scheme consists of the inverse limit at each of these finite levels, and this is a finite $B_{\bar{D}}$ module by e.g. Lemma 3.2.2.4. □

Remark 3.2.4.5. The result on coherent sheaves would be more straightforward if we knew that formal GAGA holds over adequate moduli spaces. It has been recently proved in the slightly narrower case of good moduli spaces [GZB12].

3.2.5. Consequences of Algebraization. We conclude our work on pseudorepresentations by applying our best results from Chapter 1 and Chapter 2 to the moduli of continuous representations and pseudorepresentations of a profinite algebra R over a Noetherian profinite local ring A. In particular, we find pleasant conclusions as corollaries of

(1) the projective subspaces of θ-stable representations in fibers of $\bar{\psi}$, and complete local projective deformations of these spaces (Theorem 2.2.4.1), and
(2) the adequacy of ψ and $\bar{\psi}$ in the neighborhood of residually multiplicity free pseudorepresentations (Corollary 2.3.3.9).
(3) the projectivity over complete local pseudodeformation rings of moduli spaces of representations which have a certain ordering of extensions (Corollary 2.2.4.3), verifying a proposal of Kisin [Kis09a, Remark 3.2.7].

Theorem 3.2.5.1. *Let A be a Noetherian profinite local ring with residue field $\mathbb{F}_A$ and let R be a continuous A-algebra. Choose a residual pseudorepresentation $\bar{D} \in \mathrm{PsR}_R^d(\mathbb{F}_A)$ satisfying finiteness condition $\mathbf{\Phi}_{\bar{D}}$. Then*

(1) All continuous representations of R over admissible A-algebras factor uniquely continuously through the Cayley-Hamilton algebra $E(R, D^u_{\bar{D}})$, which is an algebra finite as a module over the complete Notherian local A-algebra $B_{\bar{D}}$.

(2) The $\mathrm{Spf}(A)$-formal scheme (resp. formal algebraic stacks) of representations $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}}$ (resp. $\mathcal{R}\mathrm{ep}_{\bar{D}}$, resp. $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}$) are the $\mathfrak{m}_{\bar{D}}$-adic completion of the finite type, non-topological $\mathrm{Spec}\, B_{\bar{D}}$-scheme (resp. algebraic stack) of representations $\mathrm{Rep}^{\square}_{\bar{D}|_E}$ (resp. $\mathrm{Rep}_{\bar{D}|_E}$, resp. $\overline{\mathrm{Rep}}_{\bar{D}|_E}$), which are also continuous representations when restricted to admissible $B_{\bar{D}}$-algebras. Consequently, $\mathcal{R}\mathrm{ep}^{\square}_{\bar{D}}$ (resp. $\mathcal{R}\mathrm{ep}_{\bar{D}}$, resp. $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}$) are finite type over $\mathrm{Spf}\, B_{\bar{D}}$.

(3) If the residual pseudorepresentation $\bar{D}$ is split over $\mathbb{F}_A$ and is stabilizing relative to a character θ of the Grothendieck group of the abelian category of representations of the finite dimensional $\mathbb{F}_A$-algebra $E(R, D^u_{\bar{D}}) \otimes_{B_{\bar{D}}} \mathbb{F}_A$, there is a $\mathrm{PsR}_{\bar{D}}$-projective subscheme $\overline{\mathrm{Rep}}^s_{\bar{D}}(\theta)$ of $\overline{\mathrm{Rep}}_{\bar{D}}$ parameterizing representations whose reduction modulo $\mathfrak{m}_{\bar{D}}$ is θ-stable.

(4) Assuming that $\bar{D}$ is split and multiplicity free over $\mathbb{F}_A$, given an ordering of the non-isomorphic simple representations $\bar{\rho}_i, 1 \leq i \leq n$ of R over $\mathbb{F}_A$ such that $\bar{D} = \det \circ (\oplus_1^n \bar{\rho}_i)$, there exists a $\mathrm{PsR}_{\bar{D}}$-projective subscheme $\overline{\mathrm{Rep}}'_{\bar{D}} \subset \overline{\mathrm{Rep}}_{\bar{D}}$ of representations which are residually a certain ordering of extensions given in Definition 2.2.3.2.

(5) If a d-dimensional residual representation $\bar{D}$ of R is split and multiplicity free and of characteristic greater than $2d$, then ψ (resp. $\bar{\psi}$ is an adequate moduli space. In particular, this means that $\mathrm{PsR}_{\bar{D}}$ is precisely the GIT quotient of $\mathrm{Rep}^{\square}_{\bar{D}}$ with respect to the adjoint action.

Proof. Part (1) is Theorem 3.2.3.3. Part (2) is Corollary 3.2.4.3. For part (3), we apply Theorem 2.2.4.1 to $\overline{\mathrm{Rep}}_{\bar{D}}$, using the fact that the base $B_{\bar{D}}$ is complete. Part (4) is an application of Corollary 2.2.4.3. Part (5) is Corollary 2.3.3.9, where we use the fact that $B_{\bar{D}}$ is complete and therefore henselian. □

In particular, this theorem can take A to be the universal deformation ring $B_{\bar{D}}$ of a residual pseudorepresentation of R over $\mathbb{F}_A$ satisfying $\mathbf{\Phi}_{\bar{D}}$.

CHAPTER 4

p-adic Hodge Theory in Group-Theoretic Families

This chapter is a generalization of Mark Kisin's *Potentially Semistable Deformation Rings* [Kis08, §§1-2]. We also provide some additional expository content as we do this. There, the constructions start given a continuous representation of the absolute Galois group $\hat{\Gamma} = \hat{\Gamma}_K := \mathrm{Gal}(\bar{K}/K)$ of K, a finite extension of $\mathbb{Q}_p$, on a free module over a complete Noetherian local $\mathbb{Z}_p$-algebra A with finite residue field. Then loci of $\mathrm{Spec}\, A[1/p]$ such that the associated Galois representation satisfies conditions from p-adic Hodge theory are determined. Our goal is to generalize the arguments and constructions of [Kis08] to the case that A is formally finitely generated over a complete Noetherian local ring R with finite residue field, i.e. the quotient of a restricted power series ring $R\langle z_1, \ldots, z_a\rangle$. We know from Corollary 3.2.4.3 that the moduli spaces of representations $\mathrm{Rep}_{\bar{D}}$ of $\hat{\Gamma}$ with a residually constant d-dimensional pseudorepresentation $\bar{D} : \mathbb{F}[\![\hat{\Gamma}]\!] \to \mathbb{F}$ are formally finite type over the complete local Noetherian pseudodeformation ring $R_{\bar{D}}$ with residue field $\mathbb{F}$. Because the whole moduli space of d-dimensional pseudorepresentations is semi-local with local components $\mathrm{Spf}\, R_{\bar{D}}$ in bijective correspondence with $\overline{\mathbb{F}}_p$-valued d-dimensional pseudorepresentations of $\hat{\Gamma}$ (Corollary 3.1.6.13), the results of this chapter apply to the whole moduli space of representations of $\hat{\Gamma}$. This means, for example, that given the condition "semistable with Hodge-Tate weights in $[0, h]$," there exists a Zariski closed subspace of $\overline{\mathrm{Rep}}^d_{\hat{\Gamma}}[1/p]$ parameterizing precisely these representations. See Theorem 4.12.12 for the p-adic Hodge theoretic conditions for which we prove such a result.

As a concrete example of the application of this theorem, consider two crystalline representations ρ_1, ρ_2 of $\hat{\Gamma}_K$ over $\mathbb{Q}_p$. It is well known that the subset of the vector space of

extensions of the form

$$\begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}$$

that are crystalline form a sub-vector space of $\mathrm{Ext}_{\mathbb{Q}_p[\hat{\Gamma}]}(\rho_2, \rho_1)$. Our results show that for a much wider set of conditions – e.g. potentially semi-stable of a certain Galois type, and prescribed Hodge type – the locus of extensions fulfilling this representation will be Zariski closed. This is a proper generalization of the results of Kisin [Kis08], since there is not necessarily one finite field valued representation of $\hat{\Gamma}$ such that the entire family of extensions reduces to it. An example of such a case is when the mod p reductions $\bar{\rho}_i$ are absolutely irreducible and $\dim_{\mathbb{F}_p} \mathrm{Ext}_{\mathbb{F}_p[\hat{\Gamma}]}(\bar{\rho}_2, \bar{\rho}_1) > 1$.

We will not reference the moduli spaces and pseudorepresentations in what follows, but will simply assume that A is a formally finite type R-algebra, where R is a complete Noetherian local $\mathbb{Z}_p$-algebra with finite residue field $\mathbb{F}$ and maximal ideal $\mathfrak{m}$. We will sometimes use α for an Artinian ring $R/\mathfrak{m}^n$.

4.1. Changes in Notation from [Kis08]

For the reader familiar with the notation of [Kis08], we remark that we follow the notation there with the following exceptions. For the most part, the changes come from generalizing the coefficient ring of the representation, as described above.

(1) We use Γ for the Galois group denoted as G_K in [Kis08] and Γ_∞ for G_{K_∞}.

(2) For the portions of [Kis08] where A represents an Artinian local ring with residue field $\mathbb{F}$, and V_A is a free A-module with an A-linear continuous action of Γ_∞, we let A be a finitely generated over an Artinian local ring α. Here α stands in for $R/\mathfrak{m}^n$ for some $n > 0$. The topology on A is the discrete ($\mathfrak{m}_\alpha$-adic) topology. In particular, this means that A is finite type over $\mathbb{Z}$. We let V_A be a projective rank d A-module with an$A$-linear action of $\hat{\Gamma}_\infty$ with open kernel.

(3) When in [Kis08] A represents a complete Noetherian local ring with finite residue field $\mathbb{F}$, in the analogous sections of our work A will represent a formally finite type R-algebra, i.e. a quotient of a finitely generated restricted power series ring over

R. The ring R is a complete Noetherian local ring with finite residue field $\mathbb{F}$. This makes A a topologically finite type $\mathbb{Z}_p$-algebra, where the topology on A is $\mathfrak{m}_R \cdot A$-adic. We then use V_A to denote a projective rank d A-module with an action of $\hat{\Gamma}$.

(4) When [Kis08] changes notation and uses A° in place of A, and then $A = A^\circ[1/p]$, we do the same. We also require that A° be p-torsion free along with this transition, i.e. A° is a flat continuous topologically of finite type $\mathbb{Z}_p$-algebra.

4.2. Background for Representations of Bounded E-height (§§4.3-4.5)

Let k be a finite field of characteristic $p > 0$ and $W := W(k)$ its ring of p-typical Witt vectors. W is the ring of integers of a finite unramified extension $K_0 := W(k)[1/p]$ of $\mathbb{Q}_p$. Let K/K_0 be a totally ramified extension of degree e. Fix an algebraic closure $\bar{K}$ of K, and a completion $\mathbb{C}_p$ of $\bar{K}$ and let $\hat{\Gamma} := \hat{\Gamma}_K = \mathrm{Gal}(\bar{K}/K)$.

Our entire aim is to study the moduli of representations of $\hat{\Gamma}$ with p-adic Hodge theoretic properties. We recall the definitions of some p-adic period rings.

Let $\mathcal{O}_{\bar{K}}$ be the ring of integers of $\bar{K}$ and $\mathcal{O}_{\mathbb{C}_p}$ the ring of integers of $\mathbb{C}_p$. Let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$, where each transition map is the Frobenius endomorphism of the characteristic p ring $\mathcal{O}_{\bar{K}}/p$. This is a complete valuation ring which is perfect of characteristic p and whose residue field is $\bar{k}$ and is also canonically a $\bar{k}$-algebra [FO, Proposition 4.6]. The fraction field $\mathrm{Fr}R$ of R is a complete nonarchimedean algebraically closed characteristic p field. The elements x of R are in natural bijection with sequences of elements $(x_{(n)})_{n\geq 0}$ of $\mathcal{O}_{\mathbb{C}_p}$ such that $x_{(n+1)}^p = x_{(n)}$ for all $n \geq 0$. A canonical valuation on R is given by taking the valuation v on $\mathbb{C}_p$ normalized so that $v(p) = 1$ and setting $v_R((x_{(n)})_{n\geq 0}) = v(x_{(0)})$. Frobenius φ acts on R by the pth power map also, or, equivalently, a single shift in the limit defining R or, in terms of the presentation $x = (x_{(n)})_{n\geq 0}$, $\varphi(x) = (x_{(n)}^p)_{n\geq 0}$.

Consider the ring $W(R)$, and write an element of $W(R)$ as $(x_0, x_1, \dots, x_n, \dots)$. There is a unique continuous surjective W-algebra map

$$\theta : W(R) \longrightarrow \mathcal{O}_{\mathbb{C}_p}$$
$$(x_0, x_1, \dots) \mapsto \sum_{n=0}^{\infty} p^n x_{n,(n)}$$

lifting the projection to the first factor $R \to \mathcal{O}_{\bar{K}}/p$ onto the 0th truncation $W_0(R)$ of the limit of truncated Witt vectors defining $W(R)$ (cf. [FO, Remark 5.10]). There is a Frobenius action on the perfect, characteristic p ring R, and therefore also a Frobenius map φ on $W(R)$ which sends $(x_0, x_1, \dots)$ to $(x_0^p, x_1^p, \dots)$.

We fix the notation $\mathfrak{S} := W[\![u]\!]$, the power series ring in the variable u. We equip $\mathfrak{S}$ with a Frobenius map denoted φ, which acts by the usual Frobenius map on W and sends u to u^p. We think of these as the functions bounded by 1 on the open analytic unit disk over K_0, and $\mathfrak{S}[1/p]$ as the ring of bounded functions on the open unit disk. Fix a uniformizer $\pi \in K$, and elements ${\pi_n}$[1] for $n \geq 0$ such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Write $E(u) \in W[u]$ for the minimal, Eisenstein polynomial of π. We note that $\varphi^n(E(u))$ is a minimal, Eisenstein polynomial for π_n for $n \geq 0$.

Write $\underline{\pi} := (\pi_n)_{n \geq 0} \in R$, and let $[\underline{\pi}] \in W(R)$ be its Teichmüller lift $(\underline{\pi}, 0, 0, \dots)$. Because the R is canonically a $\bar{k}$-algebra, we have a canonical embedding $W \hookrightarrow W(\bar{k}) \hookrightarrow W(R)$. We consider $W(R)$ as a $W[u]$-algebra by sending u to $[\underline{\pi}]$. Since $\theta([\underline{\pi}]) = \pi$, this embedding extends to an embedding of $\mathfrak{S}$ into $W(R)$ (cf. the formulation of $W(R)$ in [FO, §5.2.1]), and we will consider $W(R)$ and rings derived from $W(R)$ as $\mathfrak{S}$-algebras via this map from now on. From the discussion above, this map is visibly φ-equivariant.

We define another important element $[\underline{\varepsilon}] \in W(R)$. Firstly define a sequence of p^nth roots of unity

$$\varepsilon_0 = 1, \varepsilon_1 \neq 1, \text{ and } \varepsilon_{n+1}^p = \varepsilon_n \quad \forall n \geq 0. \tag{4.2.1}$$

[1]In the notation above, these would be $\pi_{(n)}$.

This sequence defines an element $\underline{\varepsilon}$ in R. Let $[\underline{\varepsilon}] \in W(R)$ be its Teichmülller lift. Notice that $\theta([\underline{\varepsilon}] - 1) = 0$.

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with residue field $k(\!(u)\!)$ and maximal ideal generated by p. Write $\mathcal{E}$ for its fraction field $\mathrm{Fr}\mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}[1/p]$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Fr}R)$, since $\underline{\pi} \in \mathrm{Fr}R$ and $W(\mathrm{Fr}R)$ is p-adically complete. Let $\mathcal{E}^{\mathrm{ur}} \subset W(\mathrm{Fr}R)[1/p]$ denote the maximal unramified extension of $\mathcal{E}$ contained in $W(\mathrm{Fr}R)[1/p]$, and $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ its ring of integers. Since $\mathrm{Fr}R$ is algebraically closed, the residue field $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ is a separable closure of $k(\!(u)\!)$. If $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is the p-adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$, or, equivalently, the closure of $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ in $W(\mathrm{Fr}R)$ with respect to its p-adic topology, set $\mathfrak{S}^{\mathrm{ur}} := \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cap W(R) \subset W(\mathrm{Fr}R)$. All of these rings are subrings of $W(\mathrm{Fr}R)[1/p]$, and are equipped with a Frobenius operator coming from $W(\mathrm{Fr}R)[1/p]$.

For $n \geq 0$ let $K_{n+1} := K(\pi_n)$, and let $K_\infty = \cup_{n\geq 0} K_n$ and $\hat{\Gamma}_\infty := \mathrm{Gal}(\bar{K}/K_\infty)$. Clearly the action of $\hat{\Gamma}_\infty$ on $W(R)$ fixes the subring $\mathfrak{S}$, since it fixes both W and π_n $\forall n \geq 0$. Therefore $\hat{\Gamma}_\infty$ has an action on $\mathfrak{S}^{\mathrm{ur}}$ and $\mathcal{E}^{\mathrm{ur}}$.

The discussion above provides the needed background and definitions for §§4.3-4.5, where "representations of E-height $\leq h$" are discussed. Background and definitions for the rest of the chapter are given in §4.6.

4.3. Families of Étale φ-modules

In this section, let A denote an algebra of finite type over an Artinian local ring α with finite residue field $\mathbb{F}$ of characteristic p. Let V_A be a finite projective constant rank A-module with an A-linear action of Γ_∞ with open kernel, i.e. an object of the additive exact tensor category

$$\mathrm{Rep}_{\hat{\Gamma}_\infty}(A).$$

Write $\mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$ for the category of finitely generated A-modules with an action of Γ_∞ with open kernel.

Let $\mathcal{O}_{\mathcal{E},A}$ denote $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$, with an A-linear extension φ of the Frobenius on $\mathcal{O}_{\mathcal{E}}$. We note that this is a Noetherian ring, as $\mathcal{O}_{\mathcal{E}}$ is Noetherian and A is finitely generated over $\mathbb{Z}_p$.

Write $\Phi'_M(A)$ for the category of $\mathcal{O}_{\mathcal{E},A}$-modules M with an isomorphism $\varphi^*(M) \xrightarrow{\sim} M$, and write $\Phi_M(A)$ for the full subcategory of projective constant rank modules. These are known as étale φ-modules over A. This is also a additive exact rigid tensor category.

Much of p-adic Hodge theory has to do with equivalences between categories of Galois representations and categories of linear algebraic data. We wish to prove an equivalence of this sort between the categories above.

In the case that $A = \alpha = \mathbb{Z}_p$, this is due to Fontaine [Fon90, A.1.2.6], who proved that the following functors are quasi-inverse and therefore define an equivalence of categories:

$$\begin{aligned} M : \mathrm{Mod}_{\hat{\Gamma}_\infty}(\mathbb{Z}_p) &\longrightarrow \Phi'_M(\mathbb{Z}_p) \\ V_{\mathbb{Z}_p} &\mapsto (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V^*_{\mathbb{Z}_p})^{\hat{\Gamma}_\infty} \\ V : \Phi'_M(\mathbb{Z}_p) &\longrightarrow \mathrm{Mod}(\hat{\Gamma}_\infty, A) \\ M_{\mathbb{Z}_p} &\mapsto (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{O}_\mathcal{E}} M_{\mathbb{Z}_p})^{\varphi=1} \end{aligned}$$

By adding A-structure for A Artinian with residue field $\mathbb{F}$ (say $A = \alpha$), it is immediate that M, V extend to mutually quasi-inverse functors on the analogous abelian categories with A-linear structure, $\mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$ and $\Phi'_M(A)$. It is shown in [Kis09c, Lemma 1.2.7(4)] that this equivalence of categories restricts to an equivalence of the respective additive exact subcategories of projective, finite, constant rank objects, $\mathrm{Rep}_{\hat{\Gamma}_\infty} \cong \Phi_M(A)$.

Our goal in this section is to extend this theorem to the case that A is finite type over α. We make the following definitions in order to accomplish this, also reviewing the definitions we made at the beginning of this section.

Definition 4.3.1. Let A be a finite type α-algebra, where α is a local Artinian ring with residue field $\mathbb{F}$.

(1) Let $\mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$ be the category of finite A-modules with a A-linear action of $\hat{\Gamma}_\infty$ with open kernel. Let $\mathrm{Rep}^A_{\hat{\Gamma}_\infty}$ be the full subcategory whose objects are finite, projective, and constant rank as A-modules.

(2) Let $\Phi'_M(A)$ be the category of finite $\mathcal{O}_{\mathcal{E},A}$-modules M equipped with an A-linear isomorphism $\varphi^*(M) \xrightarrow{\sim} M$. Let $\Phi_M(A)$ be the full subcategory whose objects are finite, projective, and constant rank as $\mathcal{O}_{\mathcal{E},A}$-modules.

(3) Let M be the functor

$$M : \mathrm{Mod}_{\hat{\Gamma}_\infty}(A) \longrightarrow \Phi'_M(A)$$
$$V_A \mapsto (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V_A^*)^{\hat{\Gamma}_\infty}.$$

(4) Let $\Phi_M^{'\mathrm{Gal}}(A)$ be the essential image of M_A in $\Phi'_M(A)$, and let $\Phi_M^{\mathrm{Gal}}(A)$ be the essential image of M in $\Phi_M(A)$.

(5) Let V be the functor

$$V : \Phi_M^{'\mathrm{Gal}}(A) \longrightarrow \mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$$
$$M_A \mapsto (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{O}_\mathcal{E}} M_A)^{\varphi=1}.$$

Of course, it remains to be confirmed that the definition above is valid, e.g. that $M(V_A)$ is finite as a $\mathcal{O}_{\mathcal{E},A}$-module when $V_A \in \mathrm{ob}\,\mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$

We note that V makes sense on all of $\Phi'_M(A)$, but we only confirm after restricting it to the full subcategory $\Phi_M^{'\mathrm{Gal}}(A)$ of $\Phi'_M(A)$ that it yields an object of $\mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$. There, we confirm that it defines a quasi-inverse to M, making M fully faithful and exact. Therefore it will define equivalences of categories

$$\mathrm{Mod}_{\hat{\Gamma}_\infty}(A) \xrightarrow{\sim} \Phi_M^{'\mathrm{Gal}}(A),$$
$$\mathrm{Rep}_{\hat{\Gamma}_\infty}(A) \xrightarrow{\sim} \Phi_M^{\mathrm{Gal}}(A).$$

In summary, this is what we want to prove.

Proposition 4.3.2 (Generalizing [Kis09c, Lemma 1.2.7]).

(1) The functor $M : \mathrm{Mod}_{\hat{\Gamma}_\infty}(A) \to \Phi'_M(A)$ is exact and fully faithful, and is an equivalence onto the full subcategory $\Phi_M^{'\mathrm{Gal}}(A)$ with quasi-inverse V.

(2) If A' is a finite A-algebra, then there is a functor $\Phi_M^{\mathrm{Gal}}(A) \to \Phi_M^{\mathrm{Gal}}(A')$ induced by $- \otimes_A A'$.

(3) For W a finite A-module and $V_A \in \mathrm{Rep}_{\hat{\Gamma}_\infty}(A)$, there is a natural isomorphism

$$M(V_A \otimes_A W) \cong M(V_A) \otimes_A W.$$

(4) M restricts to an equivalence of categories

$$M : \mathrm{Rep}_{\hat{\Gamma}_\infty} \xrightarrow{\sim} \Phi_M^{\mathrm{Gal}}(A).$$

In particular, this means that

(a) if V_A is projective as an A-module of constant rank d, then M_A is a projective $\mathcal{O}_{\mathcal{E},A}$-module of constant rank d, equipped with an isomorphism $\varphi^* M_A \xrightarrow{\sim} M_A$.

(b) if V_A is free as an A-module with rank d, then M_A is a free rank d $\mathcal{O}_{\mathcal{E},A}$-module.

Remark 4.3.3. In the proof of this proposition, we will see that the obstruction to proving that M and V are mutually quasi-inverse on all of $\Phi'_M(A)$ is that there may not be a filtration of $M_A \in \Phi'_M(A)$ into finite α-submodules M_i such that the structure $\varphi^*(M_A) \xrightarrow{\sim} M_A$ is the limit of such maps on M_i. The analogous filtration always exists in $\mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$ because we demand that the action of $\hat{\Gamma}_\infty$ factors through a finite quotient.

First we assemble these facts on limits. We will append $(-)^\infty$ to various categories to indicate that the A-module finiteness condition has been dropped; however, it is important that we do not drop the condition that the action of $\hat{\Gamma}_\infty$ has open kernel.

Fact 4.3.4. In a category of modules, tensor products commute with direct limits, since tensor product operations are left-adjoint functors and therefore commute with colimits.

Lemma 4.3.5. *If the maps of a filtered direct limit of finite modules in $\mathrm{Mod}^\infty_{\hat{\Gamma}_\infty}(\alpha)$ (resp. in $\Phi'^\infty_M(\alpha)$) are all injective, then the functor $(-)^{\hat{\Gamma}_\infty}$ (resp. $(-)^{\varphi=1}$) commutes with this direct limit.*

Lemma 4.3.6. *With A as specified above, both A and $\mathcal{O}_{\mathcal{E},A}$ are commutative Jacobson Noetherian rings.*

PROOF. We know that A is a Noetherian ring because it is finitely generated over an Artinian ring α with a finite residue field, and it is Jacobson because it is finitely generated as a $\mathbb{Z}$-algebra. We observe that if $p^i = 0$ in A, then

$$\mathcal{O}_{\mathcal{E},A} \cong (W(k)/p^i W(k))[\![u]\!][1/u] \otimes_{\mathbb{Z}_p} A.$$

Since the left factor of the tensor product is a Noetherian ring and the right factor is finitely generated over $\mathbb{Z}_p$, $\mathcal{O}_{\mathcal{E},A}$ is Noetherian.

A commutative Noetherian ring B is Jacobson if and only if there are no primes $\mathfrak{p}$ such that $B/\mathfrak{p}$ is 1-dimensional and semi-local (see Sublemma 4.5.8). Let $\mathfrak{p}$ be a 1-dimensional prime of $\mathcal{O}_{\mathcal{E},A}$. The factor map to $\mathcal{O}_{\mathcal{E},A}/\mathfrak{p}$ factors through the quotient ring $k(\!(u)\!) \otimes_{\mathbb{Z}_p} A$. This induces a prime $\mathfrak{p}^c$ of A by contraction along the map $A \to k(\!(u)\!) \otimes_{\mathbb{F}_p} A/pA$, and we observe that since $\mathcal{O}_{\mathcal{E},A}/\mathfrak{p}$ is 1-dimensional, so is $A/\mathfrak{p}^c$. Since $A/\mathfrak{p}^c$ is not semi-local and injects into $\mathcal{O}_{\mathcal{E},A}/\mathfrak{p}$, neither is $\mathcal{O}_{\mathcal{E},A}/\mathfrak{p}$ semi-local. □

We also record this fact, which will be of use later.

FACT 4.3.7. Inverse limits in $\mathrm{Mod}_{\hat{\Gamma}_\infty}(\alpha)$ (resp. $\Phi'_M(\alpha)$) commute with the invariant functor $(-)^{\hat{\Gamma}_\infty}$ (resp. $(-)^{\varphi=1}$), since an invariant functor is a right-adjoint functor and therefore commutes with limits.

In order to prove the proposition above, our basic strategy will be to forget the A-linear structure and write the objects of the categories above as direct limits of finite α-submodules with the respective additional structure of φ or a group action.

PROOF (PROPOSITION 4.3.2). Let $V_A \in \mathrm{ob}\,\mathrm{Mod}_{\hat{\Gamma}_\infty}(A)$. Because the action of $\hat{\Gamma}_\infty$ has a finite index kernel, we have a canonical isomorphism as $\alpha[\hat{\Gamma}_\infty]$-modules of V_A with $\varinjlim_i V_i$, where $(V_i)_{i\in I} \in \mathrm{ob}\,\mathrm{Mod}_{\hat{\Gamma}_\infty}(\alpha)$ are the α-module-finite $\alpha[\hat{\Gamma}_\infty]$-submodules of V_A. We note that the functor M (resp. V) commutes with injective direct limits in $\mathrm{Mod}_{\hat{\Gamma}_\infty}(\alpha)$ (resp. $\Phi'_M(\alpha)$), using the fact and lemma above and the fact that the tensor product $\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ (resp. $\otimes_{\mathcal{O}_\mathcal{E}}\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$) preserve injective maps.

Therefore there are canonical isomorphisms of colimits of objects of $\Phi'_M(\alpha)$,

$$M(V_A) = M(\varinjlim_i V_i) = \varinjlim_i M(V_i),$$

and the fact that M is an equivalence of categories out of $\mathrm{Mod}_{\hat{\Gamma}_\infty}(\alpha)$ commuting with the necessary colimits implies that there is a canonical isomorphism respecting all structures

$$V_A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \xrightarrow[\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}},\alpha,\hat{\Gamma}_\infty,\varphi]{\sim} M(V_A) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \tag{4.3.8}$$

The A-linear structure on the left hand side then provides a canonical A-linear structure on the right hand side, commuting with the action of $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}, \hat{\Gamma}_\infty$, and φ.

Let H be the open kernel of the action of $\hat{\Gamma}_\infty$ on V_A. Since H acts trivially on V_A, the canonical isomorphism above induces a canonical isomorphism

$$V_A \otimes_{\mathbb{Z}_p} (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^H \xrightarrow{\sim} M(V_A) \otimes_{\mathcal{O}_\mathcal{E}} (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^H. \tag{4.3.9}$$

Since $\hat{\Gamma}_\infty/H$ is finite and $(\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^{\hat{\Gamma}_\infty} = \mathcal{O}_\mathcal{E}$, we know that $(\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^H$ is finite as a $\mathcal{O}_\mathcal{E}$-module. Therefore the left hand side is finite as a $\mathcal{O}_{\mathcal{E},A}$-module, so that the right hand side is as well. As $M(V_A)$ is a $\mathcal{O}_{\mathcal{E},A}$-submodule of the right hand side and $\mathcal{O}_{\mathcal{E},A}$ is a Noetherian ring, $M(V_A)$ is finite as a $\mathcal{O}_{\mathcal{E},A}$-module. This confirms that the target of M can be taken to be $\Phi'_M(A)$. Since V commutes with the same limits as M does, we observe that V defines a quasi-inverse on the essential image $\Phi'^{\mathrm{Gal}}_M(A)$ of M. This establishes (1). In particular, M is exact, since $\varinjlim$ over a direct limit with injective maps is an exact functor on a category of modules.

Part (2) is clear from the fact that (2) holds when A is replaced by α (cf. [Kis09c, Lemma 1.2.7(2)]), along with the compatibility of tensor products with direct limits of modules.

For part (3), observe that this is clear for free A-modules W and then use the exactness of M on a presentation for a general finite module W.

For part (4), first observe that the exactness of M implies that $M(V_A)$ is flat over $\mathcal{O}_{\mathcal{E},A}$ if and only if V_A is flat over A. As these modules are finite over Noetherian rings, they are projective. Therefore, it remains only to verify that the ranks are constant, as claimed.

Since both V_A and $M(V_A)$ are flat, the rank function is locally constant. At a maximal ideal $\mathfrak{m}$, we know that the ranks $\dim_{A/\mathfrak{m}} V_A \otimes_A A/\mathfrak{m}$ and $\mathrm{rk}_{\mathcal{O}_{\mathcal{E},A/\mathfrak{m}}} M(V_A \otimes_A A/\mathfrak{m})$ are the same since, by (2),

$$M(V_A) \otimes_A A/\mathfrak{m} \cong M(V_A \otimes_A A/\mathfrak{m})$$

and since $A/\mathfrak{m}$ is a finite field, [Fon90, A.1.2.4(i)] tells us that the $\mathcal{O}_{\mathcal{E},A/\mathfrak{m}}$-rank of $M(V_A/\mathfrak{m})$ is constant and is the same as the $A/\mathfrak{m}$-dimension of $V_{A/\mathfrak{m}}$. Any maximal ideal I of $\mathcal{O}_{\mathcal{E},A}$ contains the kernel of the factor map $\mathcal{O}_{\mathcal{E},A} \to \mathcal{O}_{\mathcal{E},A/\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ of A. Therefore the rank of $M(V_A)$ is constant and equal to the A-rank of V_A at all maximal ideals. Since $\mathcal{O}_{\mathcal{E},A}$ is Jacobson and Noetherian by Lemma 4.3.6, this means that maximal ideals are dense in $\operatorname{Spec} \mathcal{O}_{\mathcal{E},A}$ and $M(V_A)$ has constant rank. We conclude that $M(V_A)$ is a finite, projective, constant rank $\mathcal{O}_{\mathcal{E},A}$-module with rank equal to $\mathrm{rk}_A(V_A)$.

We conclude by proving (4b): $M(V_A)$ is free when V_A is free. The isomorphism (4.3.9) shows that both $V_A \otimes_{\mathbb{Z}_p} (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^H$ and $M(V_A) \otimes_{\mathcal{O}_{\mathcal{E}}} (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^H$ are free $(\mathcal{O}_{\mathcal{E}^{\mathrm{ur}},A})^H$-modules. But $\operatorname{Spec}(\mathcal{O}_{\mathcal{E}^{\mathrm{ur}},A})^H \to \operatorname{Spec} \mathcal{O}_{\mathcal{E},A}$ is a finite surjective étale morphism. Because vector bundles are locally isotrivial (i.e. Hilbert theorem 90, or GL_d is special in the sense of Serre [Ser58, Exposé 1]), $M(V_A)$ must be free. □

4.4. Functors of Lattices and Affine Grassmanians

We recall that A denotes a discrete commutative ring, finitely generated over an Artinian commutative ring α with finite residue field $\mathbb{F}$. Also, V_A denotes a rank d projective A-module with an A-linear action of $\hat{\Gamma}$ with open kernel.

In the previous section, we established an equivalence between representations V_A of $\hat{\Gamma}_\infty$ over A and certain $\mathcal{O}_{\mathcal{E},A}$-modules $M(V_A)$ with a Frobenius semi-linear endomorphism. Since p is nilpotent in A (say $p^i = 0$ in A), $\mathcal{O}_{\mathcal{E},A} \cong (\mathbb{Z}/p^i\mathbb{Z})[[u]][1/u] \otimes_{\mathbb{Z}_p} A$. Therefore $\mathfrak{S}_A[1/u] \cong \mathcal{O}_{\mathcal{E},A}$, and we may consider $\mathfrak{S}_A$-lattices within $M(V_A)$ with a Frobenius semi-linear endomorphism inducing that on $M(V_A)$. The functor of such $\mathfrak{S}_B$-sublattices of $M(V_A) \otimes_A B$, for B a commutative A-algebra, is represented by an affine Grassmannian, as we will see

below. An affine Grassmannian is an Ind-projective scheme, but a condition called "finite E-height," which we will describe below, cuts out a closed subscheme that turns out to be finite type over A. It will turn out that the condition "E-height $\leq h$" corresponds to the condition "Hodge-Tate weights in $[0, h]$" for representations of $\hat{\Gamma}$. These lattices are generalizations of the functor of finite flat group scheme models for V_A in the case that $h = 1$. This was the case studied initially in [Kis09c].

Recall that when R is a complete local ring and B is an R-algebra, R_B denotes the $\mathfrak{m}_R$-adic completion of the tensor product $R \otimes_{\mathbb{Z}_p} B$ (so this completion will be discrete in this section). Note also that the assumptions on A imply that A is a finitely generated $\mathbb{Z}_p$-algebra. This implies that if R (for example $R = \mathfrak{S}$) is Noetherian and B is a finitely generated A algebra, then R_B is Noetherian as well. In particular, $\mathfrak{S}_A$ is Noetherian. We write $\hat{\mathfrak{S}}_B$ for the u-adic completion of $\mathfrak{S}_B$, which is also Noetherian.

Definition 4.4.1. Where $R \hookrightarrow S$ is an injection of rings, we mean by a R-sublattice of a projective rank d S-module M a R-submodule N of M that is projective rank d as an R-module and spans M, i.e. the natural map $N \otimes_R S \to M$ is surjective.

Affine Grassmannians for inner forms of GL_d are functors of sublattices of projective constant rank modules. The local affine Grassmanian parameterizes these vector bundles over the formal one-dimensional disk $\mathcal{D}$ which are trivialized on the punctured disk. The global affine Grassmanian parameterizes these vector bundles over the affine line $\mathbb{A}^1$ which are trivialized on the punctured line.

Definition 4.4.2. Let M be a projective rank d A-module. Then the affine Grassmanians are the following functors.

(1) The *local affine Grassmanian* $\mathrm{Gr}^{loc}_{\mathrm{GL}(V_A)}$ for $\mathrm{GL}(V_A)$ is the functor associating to a A-algebra B the set of pairs $(P_{\mathcal{D}}, \eta)$ where $P_{\mathcal{D}}$ is a projective rank d $B[\![t]\!]$-module and η is an isomorphism

$$P_{\mathcal{D}} \otimes_{B[\![t]\!]} B(\!(t)\!) \xrightarrow{\sim} M \otimes_A B(\!(t)\!).$$

(2) The *global affine Grassmannian* $\mathrm{Gr}^{glob}_{\mathrm{GL}(V_A)}$ for $\mathrm{GL}(V_A)$ is the functor assigning to an A-algebra B the set of pairs $(P_{\mathbb{A}^1}, \eta)$, where $P_{\mathbb{A}^1}$ is a projective rank d $B[t]$-module and η is an isomorphism

$$P_{\mathbb{A}^1} \otimes_{B[t]} B[t][1/t] \xrightarrow{\sim} M \otimes_A B[t][1/t].$$

We observe that there is a natural functor

$$\mathrm{Gr}^{glob}_{\mathrm{GL}(V_A)} \longrightarrow \mathrm{Gr}^{loc}_{\mathrm{GL}(V_A)} \tag{4.4.3}$$

given by restriction from a line to the disc. Remarkably,

THEOREM 4.4.4 (Beauville-Laszlo [BL95]). *The functor* (4.4.3) *is an isomorphism.*

Therefore we can call the Ind-projective scheme which represents these functors "the" affine Grassmanian. Let us overview this Ind-projective structure, and namely its canonical ample line bundle, using the local affine Grassmanian for a free module.

Recalling the definition of the local affine Grassmannian, its B-points when V_A is the free module $A^{\oplus d}$ amounts to the set of projective rank d $B[\![t]\!]$-submodules L of $B(\!(t)\!)^{\oplus d}$ which are sublattices. For any such L, there exists some $n \geq 0$ such that

$$t^n \cdot B[\![t]\!]^{\oplus d} \subseteq L \subseteq t^{-n} \cdot B[\![t]\!]^{\oplus d}. \tag{4.4.5}$$

We call these lattices t^i for short. Now let $\bar{L}$ be the image of L in the finite free rank $2dn$ B-module t^{-n}/t^n. Therefore $\bar{L}$ defines a point in some (conventional) projective Grassmannian parameterizing submodules of t^{-n}/t^n:

$$\bar{L} \in \coprod_{k=0}^{2dn} \mathbb{P}\mathrm{Gr}(k, t^{-n}/t^n)(B),$$

where we write $\mathbb{P}\mathrm{Gr}(k, t^{-n}/t^n)$ for the Grassmannian $\mathbb{P}\mathrm{Gr}(k, 2dn)$ of rank k projective submodules of a free rank $2dn$ B-module, identifying the lattice t^{-n}/t^n that the Grassmannian is constructed from. The sublattice $\bar{L}$ is a t-stable submodule, i.e. it is closed under the natural action of t. The t-stability condition is a Zariski closed condition in this disjoint

union of Grassmannians; we denote the resulting projective Spec A-scheme by

$$\mathbb{P}\mathrm{Gr}(t^{-n}/t^n)^{\mathrm{stable}}. \tag{4.4.6}$$

It turns out that any t-stable submodule $\bar{L}$ of t^{-n}/t^n lifts to a well-defined sublattice $L \subset B(\!(t)\!)$, so that we can canonically identify $\mathbb{P}\mathrm{Gr}(t^{-n}/t^n)^{\mathrm{stable}}$ as a subfunctor of the local affine Grassmannian for $A^{\oplus d}$. That is, we have for each n a canonical embedding

$$\mathbb{P}\mathrm{Gr}(t^{-n}/t^n)^{\mathrm{stable}} \hookrightarrow \mathrm{Gr}^{loc}_{\mathrm{GL}(A^{\oplus d})} \cong \mathrm{Gr}^{loc}_{\mathrm{GL}_d}.$$

There are also natural closed immersions

$$\mathbb{P}\mathrm{Gr}(t^{-n}/t^n)^{\mathrm{stable}} \hookrightarrow \mathbb{P}\mathrm{Gr}(t^{-n'}/t^{n'})^{\mathrm{stable}} \tag{4.4.7}$$

for all $n' \geq n$. Since, as we noted above, any $L \in \mathrm{Gr}^{loc}_{\mathrm{GL}(A^{\oplus d})}$ belongs to one of these $\mathbb{P}\mathrm{Gr}(t^{-n}/t^n)^{\mathrm{stable}}$, we have written the local affine Grassmannian as an Ind-projective A-scheme.

There is a canonical line bundle on $\mathrm{Gr}^{loc}_{\mathrm{GL}(A^{\oplus d})}$ which is very ample on every one of the projective subschemes $\mathbb{P}\mathrm{Gr}(t^{-n}/t^n)^{\mathrm{stable}}$, and this is the determinant line bundle $\wedge^d_{\mathcal{O}(\mathrm{Gr}^{loc})}(L) = \det L$ of the universal lattice L. Strictly speaking, the canonical line bundle is the quotient of the determinant by the determinant of the standard lattice which is, in the construction above, for any A-algebra B, the lattice $B[\![t]\!]^{\oplus d} \subset B(\!(t)\!)^{\oplus d}$. One can check that this line bundle is compatible with the maps (4.4.7), and that its restriction of $\det L$ to each of the conventional Grassmannians $\mathcal{P}\mathrm{Gr}(k, t^{-n}/t^n)^{\mathrm{stable}}$ is canonically isomorphic to the the restriction of the standard very ample line bundle on $\mathcal{P}\mathrm{Gr}(k, 2dn)$ to the t-stable locus.

As a result of the overview above, we can identify Ind-projective scheme $\mathrm{Gr}_{\mathrm{GL}(V_A)}$ and the canonical very ample line bundle on $\mathrm{Gr}_{\mathrm{GL}(V_A)}$ even when V_A is merely finite projective and not free. Of course, this could be done directly, but we will accomplish this by gluing. We may choose a Zariski cover of Spec A trivializing V_A and then follow the construction of the Ind-projective scheme representing the affine Grassmannian for $\mathrm{GL}(V_A)$ on this cover, along with its very ample line bundle. Since the very ample line bundle is canonical, it can

be glued together along with the Ind-scheme. Projectivity of a morphism is local on the base when the base is locally Noetherian and the very ample line bundle is considered to be part of the data of a projective morphism. This fact, and a discussion of notions of projectivity of morphisms, are discussed in Appendix A.

We summarize our discussion in the following

THEOREM 4.4.8. *Let S be a locally Noetherian scheme, and let V be a projective, coherent, constant rank $\mathcal{O}_S$-module. Then the affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}(V)}$ is an Ind-projective S-scheme with a canonical very ample invertible sheaf arising from the determinant of the universal lattice.*

REMARK 4.4.9. For a discussion of the universal very ample determinant line bundle for the affine Grassmannian for SL_d, see [Fal03, p. 42]).

In preparation to apply the Beauville-Laszlo theorem and the affine Grassmannian to the functor of $\mathfrak{S}_A$-sublattices of $M(V_A)$, we give the following proposition, which says that the functors of sublattices that arise in our study are sandwiched between the global and local affine Grassmanians via (4.4.3), and therefore are all isomorphic to the affine Grassmanian. Some of these functors will not arise in the study below, but this proposition shows that considering those functors would amount to the same thing.

PROPOSITION 4.4.10. *If V_A is an object of $\mathrm{Rep}_{\hat{\Gamma}_\infty}(A)$, projective of rank d, and $M_A := M(V_A)$ is the corresponding $\mathcal{O}_{\mathcal{E},A}$-module in $\Phi_M^{\mathrm{Gal}}(A)$, then there exist equivalences between the following functors on A-algebras.*

1. *The global affine Grassmanian $\mathrm{Gr}^{glob}_{\mathrm{Res}_{W/\mathbb{Z}_p}\mathrm{GL}(V_A)}$ for $\mathrm{Res}_{W/\mathbb{Z}_p}\mathrm{GL}(V_A)/A$.*
2. *The functor associating to a finitely generated A-algebra B the $\mathfrak{S}_B$-sublattices of $M_B := M_A \otimes_A B$*
3. *The functor associating to a finitely generated A-algebra B the $\hat{\mathfrak{S}}_A \otimes_A B$-sublattices of $(M_A \otimes_{\mathfrak{S}_A} \hat{\mathfrak{S}}_A) \otimes_A B$.*
4. *The functor associating to a finitely generated A-algebra B the $\hat{\mathfrak{S}}_B$-sublattices of $\hat{M}_B := M_B \otimes_{\mathfrak{S}_B} \hat{\mathfrak{S}}_B$.*

(5) The local affine Grassmanian $\mathrm{Res}_{W/\mathbb{Z}_p}\mathrm{Gr}^{loc}_{\mathrm{GL}(V_A)}$ *for* $\mathrm{Res}_{W/\mathbb{Z}_p}\mathrm{GL}(V_A)/A$.

REMARK 4.4.11. We will see in the proof that the equivalence between the affine Grassmannians and the functors (2), (3), (4) is not canonical. This is not a new phenomenon that arises when A is no longer Artinian as it was in [Kis08]. There, both V_A and M_A were free modules whenever they were projective of constant rank since their respective base rings A and $\mathcal{O}_{\mathcal{E},A}$ were semi-local, and the isomorphism between the functor of lattices of M_A and the affine Grassmannian for GL_d rested on choosing non-canonical isomorphisms with a standard free module, e.g. $V_A \cong A^{\oplus d}$.

PROOF. First let us assume that V_A is free of rank d, so that M_A is as well, by Proposition 4.3.2(4). For simplicity we assume that $W = \mathbb{Z}_p$. Let B be a finitely generated A-algebra. Since p is nilpotent in A (say $p^i = 0$) we observe that $\mathcal{O}_{\mathcal{E},B} \cong \mathbb{Z}/p^i\mathbb{Z}[[u]][1/u] \otimes_{\mathbb{Z}_p} B$ and

$$B[u][1/u] \subset \mathcal{O}_{\mathcal{E},B} \subseteq \mathcal{O}_{\mathcal{E},B} \otimes_{\mathfrak{S}_B} (\hat{\mathfrak{S}}_A \otimes_A B) \subseteq \mathcal{O}_{\mathcal{E},B} \otimes_{\mathfrak{S}_B} \hat{\mathfrak{S}}_B \subseteq B[[u]][1/u].$$

Likewise, we observe that

$$B[u] \subset \mathfrak{S}_B \subseteq \hat{\mathfrak{S}}_A \otimes_A B \subseteq \hat{\mathfrak{S}}_B \subseteq B[[u]],$$

and that each member of the row above is obtained by adjoining $[\frac{1}{u}]$ to the corresponding member of the row below.

Let $\bar{M}_A$ be a free $A[u][1/u]$-module and choose an isomorphism $\bar{M}_A \otimes_{A[u][1/u]} \mathcal{O}_{\mathcal{E},A} \xrightarrow{\sim} M_A$. The functor associating to a finitely generated A-algebra B the set of $B[u]$-sublattices of $\bar{M}_A \otimes_A B$ is naturally equivalent to the global affine Grassmannian $Gr^{glob}_{\mathrm{GL}_d}$. The local affine Grassmannian is naturally equivalent to the functor (4). The natural transformations between the functors (1) to (4) by tensoring factor the usual natural transformation from the global affine Grassmannian to the local affine Grassmanian. Since this transformation is known to be an equivalence by Theorem 4.4.4, all of the functors are equivalent. We note that the choice of basis makes the equivalence between the affine Grassmannian and the functors of lattices non-canonical.

In the case that V_A is a projective, rank d $\mathcal{O}_{\mathcal{E},A}$-module trivialized by a Zariski cover $\operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$, then Proposition 4.3.2(4) implies that the same cover trivializes M_A. Then one can apply descent (gluing) to the equivalences above to produce an Ind-projective scheme parameterizing these lattices. Since finite étale morphisms induce equivalences of categories of locally free coherent sheaves (Hilbert Theorem 90), the isomorphism (4.3.9) shows that this Ind-projective scheme is isomorphic to $\mathrm{Gr}_{\mathrm{GL}(V_A)}$. □

The functor of $\mathfrak{S}$-lattices of E-height at most h for V_A is defined on the category of A-algebras as follows. Recall that $M_A := M(V_A)$.

Definition 4.4.12. For B an A-algebra, let $M_B = M_A \otimes_A B$; M_B admits an extension of φ by linearity. Choose a positive integer h. A $\mathfrak{S}_B$-lattice of E-height $\leq h$ is a $\mathfrak{S}_B$-submodule $\mathfrak{M}_B \subset M_B$ such that

(1) $\mathfrak{M}_B$ is a finite projective $\mathfrak{S}_B$-module of rank d which generates M_B as a $\mathcal{O}_{\mathcal{E},B}$-module, i.e. it is a sublattice.

(2) $\mathfrak{M}_B$ is stable by φ and the cokernel of $\varphi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ is killed by $E(u)^h$.

We write $\mathcal{L}_{V_A}^{\leq h}(B)$ for the set of $\mathfrak{S}_B$-lattices of E-height at most h in $M_A = M(V_A)$.

Proposition 4.4.13 (Following [Kis09c, Proposition 2.1.7]). *The functor $L_{V_A}^{\leq h}$ sending a commutative B algebra to the set of $\mathfrak{S}_B$-lattices of M_B of E-height at most h is represented by a projective A-scheme $\mathcal{L}_{V_A}^{\leq h}$. If $A \to A'$ is a finite map and $V_{A'} = V_A \otimes_A A'$, then there is a canonical isomorphism $\mathcal{L}_{V_A}^{\leq h} \otimes_A A' \xrightarrow{\sim} \mathcal{L}_{V_A}^{\leq h}$. Moreover $\mathcal{L}_{V_A}^{\leq h}$ is equipped with a canonical (functorial in A) very ample line bundle.*

The proof is just the same as [Kis09c, Proposition 2.1.7], except that we need Proposition 4.4.10 to see that the set of $\mathfrak{S}_B$-sublattices of $\mathfrak{M}_B \subset M_B$ is parameterized by the affine Grassmannian for $\mathrm{Res}_{W/\mathbb{Z}_p} \mathrm{GL}(M_A)$ over A.

Proof. To show that $L_{V_A}^{\leq h}$ is represented by an Ind-projective $\operatorname{Spec} A$-scheme, we note that it is naturally, by Proposition 4.4.10, a subfunctor of $\mathrm{Gr}_{\mathrm{Res}_{W/\mathbb{Z}_p} \mathrm{GL}(V_A)}$. The $\mathfrak{S}_B$-sublattice $\mathfrak{M}_B \subset M_B$ is an object listed under (2) in Proposition 4.4.10, and therefore these sublattices

define points of the affine Grassmannian. The affine Grassmannian is an Ind-projective $\operatorname{Spec} A$-scheme by Theorem 4.4.8. One can check that this subfunctor is Zariski closed, the condition coming from the finite E-height, and therefore $L^{\leq h}_{\widetilde{V}_A}$ is a Ind-projective scheme. It remains to show that this scheme is in fact finite type over A.

Choose a $\mathfrak{S}_A$-sublattice $\mathfrak{N}_A \subset M_A$ (with no φ-structure). In direct analogy with the construction of the Ind-projective model for $\mathrm{Gr}^{loc}_{\mathrm{GL}_d}$ out of projective subschemes (4.4.6), the condition

$$u^n \mathfrak{N}_B \subset \mathfrak{M}_B \subset u^{-n} \mathfrak{N}_B$$

(which is analogous to (4.4.5)) is a projective subscheme of $\mathrm{Gr}_{\mathrm{Res}_{W/\mathbb{Z}_p} \mathrm{GL}(V_A)}$. We will complete the proof by showing that there exists an n such that all $\mathfrak{M}_B$ of E-height $\leq h$ satisfy this condition.

In this we follow the proof of [Kis09c, Proposition 2.1.7] directly. The only modification we need to make is to remark that we can reduce to the case that V_A and M_A are free by replacing $\operatorname{Spec} A$ with a Zariski cover. This reduction is possible because the affine Grassmannian can be canonical glued together, cf. the discussion immediately before Theorem 4.4.8. Let B be a finitely generated commutative A-algebra and choose $\mathfrak{M}_B \in L^{\leq h}_{\widetilde{V}_A}(B)$. Let r be the least integer such that

$$u^r \mathfrak{N}_B \subset (1 \otimes \varphi) \varphi^*(\mathfrak{N}_B) \subset u^{-r},$$

and let i be the least integer such that $\mathfrak{N}_B \subset u^{-iB} \mathfrak{M}_B$. Consider a matrix which transforms a $\mathfrak{S}_B$-basis of $\mathfrak{N}_B$ into a $\mathfrak{S}_B$-basis of $\mathfrak{M}_B$. From this we see that, as $\varphi(u) = u^p$, the least integer j such that

$$(1 \otimes \varphi) \varphi^*(\mathfrak{N}_B) \subset u^{-j} (1 \otimes \varphi) \varphi^*(\mathfrak{M}_B) \tag{4.4.14}$$

is equal to ip. Therefore we have inclusions

$$(1 \otimes \varphi) \varphi^*(\mathfrak{N}_B) \subset u^{-r} \mathfrak{N}_B \subset u^{-i-r} \mathfrak{M}_B = E(u)^{-h} u^{-i-r} (E(u)^h \mathfrak{M}_B). \tag{4.4.15}$$

Recall that e denotes the degree of E and let s be the least integer such that $p^s = 0$ in A. Now because $E(u) = u^e + pf(u)$ for some $f(u) \in W[u]$ of degree $e-1$,

$$E(u)^{-1} = \frac{1}{u^e + pf(u)} = \frac{u^{-e}}{1 + u^{-e}pf(u)} = u^{-e}(1 - pu^{-e}f(u) + \cdots + (-1)^{s-1}u^{-e(s-1)}f(u)^{s-1}).$$

Therefore $E(u)^{-h} \cdot N \subset u^{-ehs}N$ for any $\mathfrak{S}$-lattice N. We also know that $E(u)^h \cdot \mathfrak{M}_B \subset (1 \otimes \varphi)\varphi^*(\mathfrak{M}_B)$ because $\mathfrak{M}_B$ has E-height $\leq h$, by definition. Therefore (4.4.15) extends to an inclusion

$$(1 \otimes \varphi)\varphi^*(\mathfrak{N}_B) \overset{(4.4.15)}{\subset} E(u)^{-h}u^{-i-r}(E(u)^h\mathfrak{M}_B) \subset u^{-i-r-ehs}(1 \otimes \varphi)\varphi^*(\mathfrak{M}_B).$$

Combining this inclusion with the fact that ip is the least integer satisfying (4.4.14) means that

$$ip \leq ehs + i + r, \text{ i.e. } i \leq \frac{ehs + r}{p-1}.$$

On the other hand, if i is the least integer such that $\mathfrak{M}_B \subset u^{-i}(1 \otimes \varphi)\varphi^*(\mathfrak{N}_B)$, then

$$(1 \otimes \varphi)\varphi^*(\mathfrak{M}_B) \subset \mathfrak{M}_B \subset u^{-i}\mathfrak{N}_B \subset u^{-i-r}(1 \otimes \varphi)\varphi^*(\mathfrak{N}_B),$$

by definition of r. Then since ip is the least integer satisfying (4.4.14), we have

$$ip \leq i + r, \text{ i.e. } i \leq \frac{r}{p-1}.$$

To summarize, we first showed that if we set $n = \lfloor \frac{ehs+r}{p-1} \rfloor$, then $u^n\mathfrak{N}_B \subset \mathfrak{M}_B$. Then we showed that n is large enough so that $\mathfrak{M}_B \subset u^{-n}\mathfrak{N}_B$, and in fact the lesser number $\lfloor r/(p-1) \rfloor$ would work in place of n. Therefore n has the desired property that for any lattice $\mathfrak{M}_B$ of E-height $\leq h$ in M_B, $u^n\mathfrak{N}_B \subset \mathfrak{M}_B \subset u^{-i}\mathfrak{N}_B$. This shows that $\mathcal{L}^{\leq h}_{\vec{V}_A}$ is finite type and projective, as desired.

To get the equivalence $\mathcal{L}^{\leq h}_{\vec{V}_A} \otimes_A A' \xrightarrow{\sim} \mathcal{L}^{\leq h}_{\vec{V}_A}$, firstly we recall Proposition 4.3.2(2-3), which implies that $M(V \otimes_A A') \cong M(V_A) \otimes_A A'$. Then the fact that the affine Grassmannian is compatible with base change, i.e. $\mathrm{Gr}_{\mathrm{GL}(V_A)} \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A' \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{GL}(V_A \otimes_A A')}$, completes the proof.

Finally, the canonical very ample line bundle on $\mathcal{L}_{V_A}^{\leq h}$ arises by restriction from the canonical very ample line bundle on the affine Grassmannian, cf. Theorem 4.4.8 and the discussion preceding it. □

Write Θ_A for the projective map $\Theta_A : \mathcal{L}_{V_A}^{\leq h} \to \operatorname{Spec} A$. Write $\underline{\mathfrak{M}}$ for the universal sheaf of $\Theta_A^*(\mathfrak{S}_A)$-modules on $\mathcal{L}_{V_A}^{\leq h}$ and $\hat{\underline{\mathfrak{M}}}$ for its u-adic completion.

Now we prove a generalization of [Kis08, Lemma 1.4.1], showing that the global sections of the universal $\mathfrak{S}_A$-lattice in M_A, with its Frobenius semi-linear structure, can recover V_A in a similar fashion to the correspondence between V_A and $M_A = M(V_A)$ in Proposition 4.3.2, but without simply recovering M_A from its $\mathfrak{S}$-sublattice and using Proposition 4.3.2.

Lemma 4.4.16 (Following [Kis08, Lemma 1.4.1]). *Set* $\tilde{A} := \Theta_{A*}(\mathcal{O}_{\mathcal{L}_{V_A}^{\leq h}})$. *There is a canonical* $\tilde{A}$*-linear* $\hat{\Gamma}_\infty$*-equivariant isomorphism*

$$V_{\tilde{A}} := V_A \otimes_A \tilde{A} \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\Theta_{A*}(\underline{\mathfrak{M}}), \mathfrak{S}_{\tilde{A}}^{\mathrm{ur}}). \tag{4.4.17}$$

In the case that A is Artinian, this was proved in [Kis08, Lemma 1.4.1]. An important input to this argument is the result of Fontaine [Fon90, B.1.8.4], showing that if $\mathfrak{N}$ is a finite $\mathfrak{S}$-module with a Frobenius semi-linear isomorphism of bounded E-height, then the natural $\mathbb{Z}_p[\hat{\Gamma}_\infty]$-linear map

$$\operatorname{Hom}_{\mathfrak{S},\varphi}(\mathfrak{N}, \mathfrak{S}^{\mathrm{ur}}) \longrightarrow \operatorname{Hom}_{\mathfrak{S},\varphi}(\mathfrak{N}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}). \tag{4.4.18}$$

induced by the inclusion $\mathfrak{S}^{\mathrm{ur}} \hookrightarrow \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ is an isomorphism. When $\mathfrak{N}$ has A-linear structure then taking A-linear maps induces a canonical $A[\hat{\Gamma}_\infty]$-linear isomorphism

$$\operatorname{Hom}_{\mathfrak{S}_A,\varphi}(\mathfrak{N}, \mathfrak{S}_A^{\mathrm{ur}}) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{S}_A,\varphi}(\mathfrak{N}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},A}). \tag{4.4.19}$$

When A is Artinian with finite residue field and $\mathfrak{N}$ is finite as a $\mathfrak{S}_A$ module, then $\mathfrak{N}$ is finite as a $\mathfrak{S}$-module. Our contribution is to generalize the argument of [Kis08, Lemma 1.4.1] when this is not the case.

PROOF. Let M_A^* denote the $\mathcal{O}_{\mathcal{E},A}$-dual of $M_A := M(V_A)$, equipped with the induced structure of an object of $\Phi_M^{\mathrm{Gal}}(A)$. Using the canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},\tilde{A}}}(M_{\tilde{A}}^*, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}) \cong M_{\tilde{A}} \otimes_{\mathcal{O}_{\mathcal{E},\tilde{A}}} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}$$

and applying $(-)^{\varphi=1}$ to the canonical isomorphism (4.3.8), we have a canonical isomorphism.

$$V_{\tilde{A}} \xrightarrow{\sim} (M_{\tilde{A}}^* \otimes_{\mathcal{O}_{\mathcal{E},\tilde{A}}} \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}})^{\varphi=1} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},\tilde{A}},\varphi}(M_{\tilde{A}}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}) \tag{4.4.20}$$

We want to show that the rightmost factor of (4.4.20) and the rightmost factor of (4.4.17) are canonically $\hat{\Gamma}_\infty$-equivariantly isomorphic.

Note that $\Theta_{A*}(\underline{\mathfrak{M}})$ is a finite $\Theta_{A*}\Theta_A^*(\mathfrak{S}_A) = \mathfrak{S} \otimes_A \tilde{A}$-module, and the φ-semi-linear $\mathcal{O}_{\mathcal{L}_{V_A}^{\leq h}}$-linear endomorphism of $\underline{\mathfrak{M}}$ descends to $\Theta_{A*}(\underline{\mathfrak{M}})$ with E-height $\leq h$: for upon applying Θ_{A*} to a linear endomorphism $\varphi^*(\underline{\mathfrak{M}}) \to \underline{\mathfrak{M}}$ we have a $\mathfrak{S}_A$-linear map

$$\Theta_{A*}(\varphi^*(\underline{\mathfrak{M}}) \longrightarrow \Theta_{A*}(\underline{\mathfrak{M}}).$$

By pre-composing this map with the natural map $\varphi^*(\Theta_{A*}(\underline{\mathfrak{M}})) \to \Theta_{A*}(\varphi^*(\underline{\mathfrak{M}}))$ (which is an isomorphism because φ is finite and flat as a morphism $\mathrm{Spec}\,\mathfrak{S} \to \mathrm{Spec}\,\mathfrak{S}$), we have the required structure

$$\varphi^*(\Theta_{A*}(\underline{\mathfrak{M}})) \xrightarrow{\sim} \Theta_{A*}(\varphi^*(\underline{\mathfrak{M}})) \longrightarrow \Theta_{A*}(\underline{\mathfrak{M}}).$$

By the projection formula, we have

$$\Theta_{A*}(\underline{\mathfrak{M}}) \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \xrightarrow{\sim} \Theta_{A*}(\Theta_A^*(M_A)) \xrightarrow{\sim} M_A \otimes_A \tilde{A} = M_{\tilde{A}}$$

The $\mathfrak{S}_A$-linear map $\Theta_{A*}(\underline{\mathfrak{M}}) \to M_A, x \mapsto x \otimes 1$ is injective because it is the global sections of the canonical injection $\underline{\mathfrak{M}} \hookrightarrow M_A \otimes_A \mathcal{O}_{\mathcal{L}_{V_A}^{\leq h}}$.

Choose now some $V_i \subset V_{\tilde{A}}$, an $\alpha[\hat{\Gamma}_\infty]$-submodule, finite as an α-module (i.e. an object of $\mathrm{Mod}_{\hat{\Gamma}_\infty}(\alpha)$), such that the natural map $V_i \otimes_\alpha \tilde{A} \to V_{\tilde{A}}$ is an surjection. Clearly such a V_i exists, since V_A is finitely generated as an A-module and the action of $\hat{\Gamma}_\infty$ factors through a finite quotient. Let $M_i = M(V_i) \subset M(V_{\tilde{A}}) = M_{\tilde{A}}$ be the corresponding $\mathcal{O}_{\mathcal{E},\alpha}$-submodule,

an object of $\Phi'^{\mathrm{Gal}}_M(\alpha)$; by Proposition 4.3.2, this is naturally a submodule and the canonical $\Phi'^{\mathrm{Gal}}_M(A)$-morphism $M_i \otimes_\alpha \tilde{A} \to M_{\tilde{A}}$ is surjective. Let $\mathfrak{N}$ be the intersection

$$\mathfrak{N} := \Theta_{A*}(\underline{\mathfrak{M}}) \cap M_i \subset M_{\tilde{A}},$$

which we observe is a $\mathfrak{S}_\alpha$-submodule of $M_{\tilde{A}}$. We have the natural surjection $\mathfrak{N} \otimes_\alpha \tilde{A} \twoheadrightarrow \Theta_{A*}(\underline{\mathfrak{M}})$.

Now the result of Fontaine (4.4.18) discussed above makes for the isomorphism (4.4.19), which we repeat here:

$$\mathrm{Hom}_{\mathfrak{S}_\alpha,\varphi}(\mathfrak{N}, \mathfrak{S}^{\mathrm{ur}}_\alpha) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_\alpha,\varphi}(\mathfrak{N}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\alpha}).$$

Thinking of $\tilde{A}$ as an α-module for a moment, applying $\otimes_\alpha \tilde{A}$ to this isomorphism induces an isomorphism

$$\mathrm{Hom}_{\mathfrak{S}_\alpha,\varphi}(\mathfrak{N}, \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_\alpha,\varphi}(\mathfrak{N}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}).$$

Then tensor-Hom adjunction results in an isomorphism

$$\mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\mathfrak{N} \otimes_\alpha \tilde{A}, \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\mathfrak{N} \otimes_\alpha \tilde{A}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}).$$

Finally, because the map $\mathfrak{S}^{\mathrm{ur}}_{\tilde{A}} \to \mathcal{O}_{\mathcal{E},\tilde{A}}$ inducing this isomorphism may be checked to be an injection, an element of the left hand side factors through the quotient $\Theta_{A*}(\underline{\mathfrak{M}})$ if and only if its image on the right hand side factors through $\Theta_{A*}(\underline{\mathfrak{M}})$. As all of the maps in this construction were canonical, this completes the construction of the desired canonical $\tilde{A}[\hat{\Gamma}_\infty]$-linear isomorphism

$$\mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\Theta_{A*}(\underline{\mathfrak{M}}), \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{E},\tilde{A}},\varphi}(M_{\tilde{A}}, \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},\tilde{A}}). \qquad \square$$

Now, extending the results of [Kis08, §1.5] where A is taken to be a complete local ring with finite residue field, we extend the above situation to $\mathfrak{m}_R$-adic limits. Let A be a finite type (in the sense of formal schemes) R-algebra, compatibly with the representation V_A and its induced determinant. This means that A is complete and separated with respect to the

$\mathfrak{m}_R A$-adic topology. As in [Kis08], for any $\mathbb{Z}_p$-algebra R we denote by R_A for the $\mathfrak{m}_R$-adic completion of $R \otimes_{\mathbb{Z}_p} A$.

The functor M generalizes to this setting naturally from the above, since

$$M_A = (\mathcal{O}_{\mathcal{E}^{ur}} \hat{\otimes}_{\mathbb{Z}_p} V_A^*)^{\hat{\Gamma}\infty} \xrightarrow{\sim} \varprojlim (\mathcal{O}_{\mathcal{E}^{ur}} \otimes_{\mathbb{Z}_p} V_A^* \otimes_A A/(\mathfrak{m}_R A)^i)^{\hat{\Gamma}\infty} \tag{4.4.21}$$

by Fact 4.3.7. This isomorphism follows from the fact that inverse limits commute with invariant functors, and the ideal $(p \otimes A) + (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes \mathfrak{m}_R A)$ (with which the left side is completed) is equal to $\mathcal{O}_{\mathcal{E}^{ur}} \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{m}_R A$ (with which the right side is completed). This means that M_A is a projective $\mathcal{O}_{\mathcal{E},A}$-module of rank d as expected.

For B an A-algebra such that $\mathfrak{m}_R^i \cdot B = 0$ for some $i \geq 1$, set $L^{\leq h}_{V_A}(B) = L^{\leq h}_{V_{A/(\mathfrak{m}_R A)^i}}(B)$.

Corollary 4.4.22. *The functor $L^{\leq h}_{V_A}$ on A-algebras B such that $\mathfrak{m}_R^i \cdot B = 0$ for some $i \geq 1$ is represented by a projective A-scheme $\mathcal{L}^{\leq h}_{V_A}$.*

Proof. By Proposition 4.4.13, this functor is represented by a projective formal scheme with a very ample line bundle compatible with its limit structure. By applying formal GAGA (perhaps locally and gluing) we conclude that $L^{\leq h}_{V_A}$ is the $\mathfrak{m}_R$-adic completion of a projective A-scheme. □

It will be useful later to know that $\mathfrak{S}_A$ is Noetherian. This is the main technical use of the condition that A is finite type (in the sense of formal schemes) over R.

Lemma 4.4.23. *The formal scheme* $\mathrm{Spf}(\mathfrak{S}_A)$ *is Noetherian.*

Proof. Firstly, we claim that $\mathfrak{S}_R$ is Noetherian. This is the case because of two facts: the standard complete tensor product $- \hat{\otimes}_{\mathbb{Z}_p} -$ is the tensor product in the category of complete Noetherian local rings with finite residue fields. Therefore $\mathfrak{S} \hat{\otimes}_{\mathbb{Z}_p} R$ is Noetherian (see e.g. [Gro64, 0_{IV}, Lemme 19.7.1.2]). It is also isomorphic as a ring (though not necessarily as a topological ring) to $\mathfrak{S}_R$, the $\mathfrak{m}_R$-adic completion of $\mathfrak{S} \otimes_{\mathbb{Z}_p} R$, because the residue fields of both $\mathfrak{S}$ and R are finite, and therefore $\mathfrak{S}_R$ is Noetherian.

Now note that $\mathfrak{S}_A$, defined to be the $\mathfrak{m}_R$-adic completion of $\mathfrak{S} \otimes_{\mathbb{Z}_p} A$, is isomorphic to $\mathfrak{S}_R \hat{\otimes}_R A$, where this completed tensor product is taken in the category of adic R-algebras,

i.e. this is the categorical dual of the fiber product of formal $\mathrm{Spf}(R)$ schemes. Because $\mathrm{Spf}(A)/\mathrm{Spf}(R)$ is finite type and $\mathrm{Spf}(\mathfrak{S}_R)$ is Noetherian, [Gro60, Proposition 10.13.5(ii)] implies that $\mathrm{Spf}(\mathfrak{S}_A)$ is Noetherian. □

4.5. Universality of $\underline{\mathfrak{M}}$ in Characteristic 0

We now study the image of the map $\Theta_A : \mathcal{L}_A^{\leq h} \to \operatorname{Spec} A$ in characteristic 0, i.e. after inverting p, following [Kis08, §1.6]. We will study these properties through their points in finite local $W(\mathbb{F})[1/p]$-algebras B, and therefore will need to study $\mathfrak{S}$-modules or $\mathcal{O}_\mathcal{E}$-modules with coefficients in such rings B. Therefore very little new is needed in addition to [Kis08] to accomplish this. The main new content is Lemma 4.5.6, which is needed in order to draw conclusions about $\Theta_A[1/p]$ by its behavior on *finite* $W(\mathbb{F})[1/p]$-algebras alone.

Remark 4.5.1. We are venturing outside the realm of linearly topologized rings in considering $W(\mathbb{F})[1/p]$-algebras. For example, there is no filtered set of ideals giving a basis of the p-adic topology on $\mathbb{Q}_p$ around 0 since all ideals are trivial!

However, even big rings like $A[1/p]$ are still Noetherian. For A is the quotient of $\mathbb{Z}_p[\![t_1, \ldots, t_a]\!]\langle z_1, \ldots, z_b\rangle$ for some $a, b \geq 0$ by the Cohen structure theorem (see e.g. [MR10, Theorem 3.2.4]), and this ring is Noetherian since is is the $(p, t_1, \ldots, t_a)$-adic completion of $\mathbb{Z}[t_1, \ldots, t_a, z_1, \ldots, z_b]$. Then $A[1/p]$ is Noetherian by the Hilbert basis theorem.

The preparatory Lemmas 4.5.2, 4.5.3, and 4.5.4 require no modification from [Kis08].

Lemma 4.5.2 ([Kis08, Lemma 1.6.1]). *Let B be a finite $\mathbb{Q}_p$-algebra, and $\mathfrak{M}_B$ a finite $\mathfrak{S}_B \cong \mathfrak{S} \otimes_{\mathbb{Z}_p} B$-module, which is flat over $\mathfrak{S}[1/p]$ and equipped with a map $\varphi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ whose cokernel is killed by $E(u)^h$. Suppose that $\mathcal{E} \otimes_{\mathfrak{S}[1/p]} \mathfrak{M}_B$ is finite free over $\mathcal{E} \otimes_{\mathbb{Q}_p} B$. Then $\mathfrak{M}_B$ is a finite projective $\mathfrak{S}_B$-module.*

The statement proof is identical to that of [Kis08], so we have omitted the proof. The same is true of the next two results.

LEMMA 4.5.3 ([Kis08, Lemma 1.6.2]). *Let B be a finite $\mathbb{Q}_p$-algebra, and $J \subset K_0[\![u]\!]_B = K_0[\![u]\!] \otimes_{\mathbb{Q}_p} B$ be an ideal such that $\varphi(J)K_0[\![u]\!]_B = J$, where φ acts B-linearly. Then J is induced by an ideal of B.*

COROLLARY 4.5.4 ([Kis08, Corollary 1.6.3]). *Let A be a finite flat $\mathbb{Z}_p$-algebra, and V_A a finite free A-module equipped with a continuous action of $\hat{\Gamma}_\infty$. Set $M_A := (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \hat{\otimes}_{\mathbb{Z}_p} V_A^*)^{\hat{\Gamma}_\infty}$. Suppose that V_A, considered as a $\mathbb{Z}_p[\hat{\Gamma}_\infty]$-module, is of E-height $\leq h$, and let $\mathfrak{M}_A \subset M_A$ be the unique $\mathfrak{S}$-lattice of E-height $\leq h$.*

Then $\mathfrak{M}_A$ is a $\mathfrak{S}_A$-submodule of M_A, and $\mathfrak{M}_A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is finite projective over $\mathfrak{S}_A[1/p]$.

In the following proposition, we we work in characteristic zero, translating the uniqueness of $\mathfrak{S}$-lattices of E-height $\leq h$ into a statement about Θ_A (part (1)) and showing that the scheme theoretic image of Θ_A has the property we expect (part (2)).

PROPOSITION 4.5.5 (Following [Kis08, Proposition 1.6.4]). *Let A and V_A be as specified above. Then*

(1) The map $\Theta_A : \mathcal{L}_{V_A}^{\leq h} \to \operatorname{Spec} A$ is a closed immersion after inverting p.

(2) If $A^{\leq h}$ is the quotient of A corresponding to the scheme-theoretic image of Θ_A, then for any finite $W(\mathbb{F})[1/p]$-algebra B, a continuous $A \to B$ factors through $A^{\leq h}$ if and only if $V_B = V_A \otimes_A B$ is of E-height $\leq h$.

PROOF. Omitted. All of the elements of the proof of Proposition 4.5.5 are entirely local, but depend on the fact that $A[1/p]$ is Jacobson with residue fields of closed points finite over $\mathbb{Q}_p$, and that the image of A lies in the ring of integers of the residue fields. This property of $S[1/p]$ when S is a complete Noetherian local $\mathbb{Z}_p$-algebra, and this is what is used in [Kis08]. Lemma 4.5.6 will show that $A[1/p]$ has this property even though A is no longer local. Otherwise the proof requires no modification from that of [Kis08], so we omit it. □

The following lemma is the main new content needed to generalize [Kis08, Proposition 1.6.4] to Proposition 4.5.5. It is well-known, and quoted and deduced from [Gro66, §§10.4-10.5] in what follows.

LEMMA 4.5.6. *Let A be a finite type (in the sense of formal schemes) R-algebra, where R is a complete Noetherian local $\mathbb{Z}_p$-algebra. Then*

(1) $A[1/p]$ is Jacobson and Noetherian,

(2) all residue fields of maximal ideals are finite extension of $\mathbb{Q}_p$, and

(3) the image of A in any such residue field is contained in its ring of integers.

We develop some notation that will be used in the proof of Lemma 4.5.6 and record a few basic facts about these notions in Sublemma 4.5.8.

DEFINITION 4.5.7. Let R be a commutative ring.

(1) If R is a domain, we call it a Goldman domain if its fraction field is finitely generated over itself.

(2) A prime ideal $\mathfrak{p} \subset R$ is called a Goldman prime ideal provided that $R/\mathfrak{p}$ is a Goldman domain, i.e. provided that the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ is finitely generated over $R/\mathfrak{p}$.

(3) We call R a Hilbert ring provided that every Goldman prime ideal is maximal.

The following facts will be useful in proving Lemma 4.5.6.

SUBLEMMA 4.5.8. *Let R be a commutative ring.*

(1) R is Jacobson if and only if R is Hilbert.

(2) A Noetherian Goldman domain that is not a field must be of height 1 and have finitely many prime ideals.

(3) The fraction field of a Goldman domain R can be generated by one element over R.

PROOF. Parts (1), (2), and (3) are proved in [Gro66, Proposition 10.4.5], [Gro66, Proposition 10.5.1], and [Gro66, Proposition 10.4.4] respectively. □

PROOF. (Lemma 4.5.6) Invoking the Cohen structure theorem, we can write R as a continuous quotient of $\mathbb{Z}_p[\![t_1, \ldots, t_a]\!]$. Then, as A is finite type over R, we can find a surjection from $\mathbb{Z}_p[\![t_1, \ldots, t_a]\!]\langle z_1, \ldots, z_b\rangle$ to A. We will replace A with $\mathbb{Z}_p[\![t_1, \ldots, t_a]\!]\langle z_1, \ldots, z_b\rangle$ and show that it has the desired property.

First we show that all residue fields of all maximal ideals are finite extensions of $\mathbb{Q}_p$. Let $B = [\![t_1, \ldots, t_a]\!] \subset A$. First we address the case that $A = \mathbb{Z}_p[\![t_1, \ldots, t_a]\!]$. Choose a maximal ideal $\mathfrak{m} \subset A[1/p]$, and let $F = A[1/p]/\mathfrak{m}$ be the residue field with ω representing the quotient map. Since A is complete with respect to the I-adic topology where $I = (p, t_1, \ldots, t_a)$, the maximal ideals of A are contractions of the maximal ideals of $A/I \cong \mathbb{F}_p[z_1, \ldots, z_b]$. In particular, these maximal ideals have finite, characteristic p residue fields. But since F has characteristic zero, the image of $\omega(A) \subset F$ must not be a field and the canonical map $\mathbb{Z}_p \to \omega(A)$ is injective. This means that F is generated by $1/p$ over $\omega(A)$, so that $\omega(A)$ is a Goldman domain by definition. Therefore, by Sublemma 4.5.8(2), $\omega(A)$ is a Krull dimension one Noetherian domain containing $\mathbb{Z}_p$ and having finitely many primes. This domain is also complete with respect to $\omega(I)$. As $\omega(I)$ is not (0) and $\omega(A)$ has dimension 1, its radical $r(\omega(I))$ must be maximal. Therefore $\omega(A)$ is a $r(\omega(I))$-adically complete local Noetherian domain of dimension 1. By Noether's normalization lemma for complete Noetherian mixed-characteristic local rings [MR10, Theorem 3.2.4], $\omega(A)$ is finite as a module over a subring isomorphic to $\mathbb{Z}_p[\![s_1, \ldots, s_d - 1]\!]$ where $d = \dim \omega(A)$. Therefore $\omega(A)$ is finite over $\mathbb{Z}_p$ and is the ring of integers of F, a finite extension of $\mathbb{Q}_p$. This proves parts (2) and (3) of the lemma.

Now we show that $A[1/p]$ is Jacobson and Noetherian. Indeed, $A[1/p]$ is Noetherian (cf. Remark 4.5.1. Now, because p is in the Jacobson radical of A and A is Noetherian, we may directly apply [Gro66, Corollaire 10.5.8] to say that $A[1/p]$ is Jacobson. □

Now we replicate [Kis08, Corollary 1.7]. Much of the work in [Kis08] goes through in the same way, except the construction of $\mathfrak{M}_{A^{\leq h}}$.

Proposition 4.5.9 (Following [Kis08, Corollary 1.7]). *There exists a finite $\mathfrak{S}_{A^{\leq h}}$-module $\mathfrak{M}_{A^{\leq h}}$ such that*

(1) $\mathfrak{M}_{A^{\leq h}}$ is equipped with a map $\varphi^*(\mathfrak{M}_{A^{\leq h}}) \to \mathfrak{M}_{A^{\leq h}}$ whose cokernel is killed by $E(u)^h$.

(2) $\mathfrak{M}_{A^{\leq h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a projective $\mathfrak{S}_{A^{\leq h}}[1/p]$-module.

(3) For any finite $W(\mathbb{F})[1/p]$-algebra B, any map $h : A^{\leq h} \to B$ and any $C \in \mathrm{Int}_B$ through which h factors, there is a canonical, φ-compatible isomorphism of $\mathfrak{S} \otimes_{\mathbb{Z}_p} B$-modules

$$\mathfrak{M}_{A^{\leq h}} \otimes_{A^{\leq h}} B \xrightarrow{\sim} \mathfrak{M}_C \otimes_C B.$$

(4) There is a canonical isomorphism

$$V_{A^{\leq h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{A^{\leq h}}[1/p],\varphi}(\mathfrak{M}_{A^{\leq h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathfrak{S}^{\mathrm{ur}}_{A^{\leq h}}[1/p]).$$

Proof. Recall that $\mathfrak{S}_A$ is the $\mathfrak{m}_R$-adic completion of $\mathfrak{S} \otimes_{\mathbb{Z}_p} A$ and is Noetherian by Lemma 4.4.23. Let $\hat{\mathcal{L}}^{\leq h}_{V_A}$ be the $\mathfrak{m}_R$-adic completion of $\mathcal{L}^{\leq h}_{V_A}$. Then

$$\hat{\Theta}_{\mathfrak{S}_A} : \hat{\mathcal{L}}^{\leq h}_{V_A} \times_{\mathrm{Spf}\, A} \mathrm{Spf}\, \mathfrak{S}_A \to \mathrm{Spf}\, \mathfrak{S}_A$$

is a projective morphism of $\mathrm{Spf}(A)$-formal schemes over a Noetherian formal scheme. The $\mathfrak{m}_R$-adic completion $\hat{\underline{\mathfrak{M}}}$ of $\underline{\mathfrak{M}}$ may be regarded as a formal coherent (further, locally free) sheaf on $\hat{\mathcal{L}}^{\leq h}_{V_A} \times_{\mathrm{Spf}\, A} \mathrm{Spf}\, \mathfrak{S}_A$. Applying formal GAGA to $\Theta_{\mathfrak{S}_A}$ (which requires that $\mathfrak{S}_A$ be Noetherian), $\hat{\underline{\mathfrak{M}}}$ is the completion of a coherent (further, locally free) sheaf $\underline{\mathfrak{M}}$ on the projective $\mathfrak{S}_A$-scheme

$$\Theta_{\mathfrak{S}_A} : \mathrm{Spec}\, \mathcal{L}^{\leq h}_{V_A} \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, \mathfrak{S}_A \to \mathrm{Spec}\, \mathfrak{S}_A.$$

The scheme theoretic image of $\Theta_{\mathfrak{S}_A}$ is $\mathfrak{S}_{A^{\leq h}}$. We set

$$\mathfrak{M}_{A^{\leq h}} := \Theta_{\mathfrak{S}_A *}(\underline{\mathfrak{M}}).$$

With this work done, the proofs of part (1), (2), (3) and (4) may be repeated from [Kis08, Corollary 1.7].

Part (1) results from the fact that $\underline{\mathfrak{M}}$ has the desired map $\varphi^*(\mathfrak{M}_{A^{\leq h}}) \to \mathfrak{M}_{A^{\leq h}}$, and that, as φ is a flat map on $\mathfrak{S}$, φ^* commutes with direct images. Part (2) follows from the fact that Θ_A is the identity operator after p is inverted, and $\underline{\mathfrak{M}}$ is a locally free coherent sheaf on $\mathcal{L}^{\leq h}_A \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, \mathfrak{S}_A$. Part (3) builds on Proposition 4.5.5(2) and its proof. The statement of Proposition 4.5.5(2) tells us that $V_C = V_A \otimes_A C$ is of E-height $\leq h$, which means that

$M_C = M(V_C)$ contains a unique $\mathfrak{S}$-lattice of E-height $\leq h$, $\mathfrak{M}_C$. Now consider the image of $\mathfrak{M}_{A^{\leq h}} \otimes_{A^{\leq h}} C$ in M_C: this is a torsion free, φ-stable $\mathfrak{S}_C$-submodule of M_C such that the cokernel of $\varphi^*(\mathfrak{M}'_C) \to \mathfrak{M}'_C$ is killed by $E(u)^h$. Following the proof of Proposition 4.5.5(2), this implies that $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}'_C \cap \mathfrak{M}'_C[1/p]$ is a $\mathfrak{S}$-lattice of E-height $\leq h$ in $\mathfrak{M}_C$, and therefore is equal to $\mathfrak{M}_C$. This shows that $\mathfrak{M}'_C \otimes_C C[1/p] \cong \mathfrak{M}_C \otimes_C C[1/p]$, from which the statement of part (3) follows.

For part (4), we use Lemma 4.4.16: Let $\tilde{A} := \hat{\Theta}_{\mathfrak{S}_{A^*}}(\mathcal{O}_{\hat{\mathcal{L}}^{\leq h}_{V_A}})$. We observe that there is a canonical isomorphism

$$V_{\tilde{A}} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\hat{\Theta}_{\mathfrak{S}_{A^*}}(\underline{\hat{\mathfrak{M}}}), \mathfrak{S}^{ur}_{\tilde{A}})$$

by combining Lemma 4.4.16, which implies this statement for $\tilde{A}$ replaced by $\tilde{A}/\mathfrak{m}_R^i$, and the theorem on formal functions. Applying formal GAGA, inverting p, and noting that Proposition 4.5.5 implies that $A^{\leq h}[1/p] \xrightarrow{\sim} \tilde{A}[1/p]$, we get a canonical isomorphism

$$V_{A^{\leq h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_{\tilde{A}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_{\tilde{A}},\varphi}(\mathfrak{M}_{A^{\leq h}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \mathfrak{S}^{\mathrm{ur}}_{\tilde{A}}[1/p]).$$

Since the map $A^{\leq h} \to \tilde{A}$ has p-torsion kernel and cokernel, the same is true of $\mathfrak{S}^{ur}_{A^{\leq h}} \to \mathfrak{S}^{ur}_{\tilde{A}}$ and $\mathfrak{S}_{A^{\leq h}} \to \mathfrak{S}_{\tilde{A}}$, completing the proof. □

4.6. Background for Families of Filtered (φ, N)-modules (§§4.7-4.12)

Following [Kis08, §2], we change notation, now denoting with A° the adic R-algebra A from above. We now assume that A° is p-torsion free, i.e. flat over $\mathbb{Z}_p$, and write A for $A^\circ[1/p]$, which (Lemma 4.5.6) is Jacobson with residue fields of maximal ideals finite over $\mathbb{Q}_p$.

For R a $\mathbb{Z}_p$-algebra we write $R_A := R_{A^\circ}[1/p]$, where we recall that R_{A° is the $\mathfrak{m}_R A$-adic completion of $R \otimes_{\mathbb{Z}_p} A^\circ$. We extend φ to an A-linear endomorphism of $\mathfrak{S}_A$. We will use the canonical isomorphism $\mathfrak{S}_A/u\mathfrak{S}_A \xrightarrow{\sim} W_A \cong W[1/p] \otimes_{\mathbb{Q}_p} A$.

Let $\mathcal{O} := \varprojlim_n (W[\![u, u^n/p]\!][1/p])$, which we may think of as the ring of rigid analytic functions on the open disk of radius 1, including $\mathfrak{S}[1/p]$ the dense subring of bounded

functions. The Frobenius endomorphism φ has a unique continuous extension from $\mathfrak{S}[1/p]$ to each ring $W[\![u, u^n/p]\!][1/p]$, and therefore to $\mathcal{O}$ as well.

Let $c_0 = E(0)$ be the constant coefficient of the Eisenstein polynomial for π, and set

$$\lambda := \prod_{n=0}^{\infty} \varphi^n(E(u)/c_0) \in \mathcal{O}$$

Denote by $\hat{\mathfrak{S}}_0$ the completion of $K_0[u]$ at the ideal $(E(u))$.

In order to study families over A of φ-modules over $\mathcal{O}$, we need to define the correct notion of the ring of coefficients. In fact, two candidate definitions end up being the same:

$$\mathcal{O}_A := \varprojlim_n (W[\![u, u^n/p]\!]_A) \cong \varprojlim_n (W_{A^\circ}[\![u, u^n/p]\!][1/p].$$

While it is clear that these rings are isomorphic when A° is local, we prove the isomorphism here in the general case.

LEMMA 4.6.1. *The natural inclusions*

$$W[\![u, u^n/p]\!]_A \hookrightarrow W_{A^\circ}[\![u, u^n/p]\!][1/p]$$

induce an isomorphism

$$\mathcal{O}_A := \varprojlim_n (W[\![u, u^n/p]\!]_A) \xrightarrow{\sim} \varprojlim_n (W_{A^\circ}[\![u, u^n/p]\!][1/p]).$$

PROOF. Write $B_n := W[\![u, u^n/p]\!]_A$ and $C_n := W_{A^\circ}[\![u, u^n/p]\!][1/p]$, with the canonical map $B_n \hookrightarrow C_n$ that we get from considering an element of B_n as a power series in u. Since the maps making up these limits are injective, it will suffice to show for $f \in C_{2n}$ that its image in C_n under the inclusions making up the limit lies in the image of B_n in C_n. With $f \in C_{2n}$ chosen, write it as

$$f = \sum_{m \geq 0} f_m \frac{u^m}{p^{\lfloor m/2n \rfloor}}$$

where $f_m \in p^{-N}A^\circ \subseteq A$ for some fixed $N \geq 0$. This expression also denotes the natural image of f in C_n under inclusion. We rewrite it as

$$f = \sum_{m \geq 0} f_m p^{\lfloor m/n \rfloor - \lfloor m/2n \rfloor} \frac{u^m}{p^{\lfloor m/n \rfloor}}.$$

We want to show that f lies in the image of B_n in C_n. This is the case because the coefficient $f_m p^{\lfloor m/n \rfloor - \lfloor m/2n \rfloor}$ of $u^n/p^{\lfloor m/n \rfloor}$ lies in $\mathfrak{m}_{A^\circ}^{i(n)}$ where $\lim_{n \to +\infty} i(n) = +\infty$; this is the case because $p \in \mathfrak{m}_{A^\circ}$. □

We observe that $\mathfrak{S}_A \hookrightarrow \mathcal{O}_A$, and we extend φ to an A-linear endomorphism of $\mathcal{O}_A$ as it was for $\mathcal{O}$ above. Write $\widehat{\mathfrak{S}}_{0,A}$ for the completion of $K_0[u] \otimes_{\mathbb{Q}_p} A$ at the ideal $(E(u))$.

Now, following [Kis08, §2.3], we consider period rings over the base A. Firstly, we recall the period rings themselves (the basic case $A^\circ = \mathbb{Z}_p$). Let A_{cris} be the p-adic completion of the divided power envelope of $W(R)$ (see §4.2) with respect to $\ker(\theta)$, and let $B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p]$. The action of φ on $W(R)$ extends to an action on A_{cris} [FO, §6.1.2]. The map $\mathfrak{S}[1/p] \hookrightarrow B^+_{\mathrm{cris}}$ extends uniquely to an continuous inclusion $\mathcal{O} \hookrightarrow B^+_{\mathrm{cris}}$, because $\mathfrak{S}[1/p]$ is dense in $\mathcal{O}$, and the eth power of the image $[\underline{\pi}]$ of u in $W(R)$ is in the divided power ideal $(\ker\theta, p)$ for A_{cris} [FO, Proposition 6.5] (for more detail on the radius of rigid analytic functions appearing in B^+_{cris}, see Lemma 4.6.6).

Define B^+_{dR} to be the $\ker(\theta)$-adic completion of $W(R)[1/p]$, where θ is extended to a map $\theta : W(R)[1/p] \twoheadrightarrow \mathbb{C}_p$, and let B_{dR} be its fraction field. This is a discrete valuation ring with residue field $\mathbb{C}_p$ and maximal ideal $\ker\theta$ and B_{dR} is its valuation field, but the topology of B^+_{dR} as a (complete) discrete valuation ring does not agree with its topology induced by the constructions we have made so far, and we use the latter topology. General theory of characteristic zero complete local rings implies that θ has a section, but there is no choice of section that is $\hat{\Gamma}$-equivariant, nor is there a section which is continuous. The rings $A_{\mathrm{cris}}, B^+_{\mathrm{cris}}$ are canonically subrings of B^+_{dR}.

Recall from §4.2 the definition of $[\underline{\pi}]$, the image of u in $W(R)$, and $[\underline{\varepsilon}]$ of (4.2.1). The "logarithms" of these elements are important elements of B^+_{dR}, which we now define.

Write $\ell_u \in B_{\mathrm{dR}}^+$ for

$$\ell_u = \log [\underline{\pi}] := \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \left(\frac{[\underline{\pi}] - \pi}{\pi} \right)^i .$$

This series converges in B_{dR}^+ because $\theta([\underline{\pi}] - \pi) = \pi - \pi = 0$, and B_{dR}^+ is by definition the $\ker \theta$-adic completion of $W(R)[1/p]$. Because $\theta([\underline{\varepsilon}] - 1) = 0$ and B_{dR}^+ is the $\ker(\theta)$-adic completion of $W(R)[1/p]$, the series

$$t = \log[\underline{\varepsilon}] := \sum_{i=1}^{\infty} (-1)^{i+1} \frac{([\underline{\varepsilon}] - 1)^i}{i}$$

converges to an element in B_{dR}^+, which we call t. In fact, because the denominators in this series are sufficiently bounded in terms of the powers of $([\underline{\varepsilon}] - 1)$, one can show that $t \in A_{\mathrm{cris}}$ and $t^{p-1} \in pA_{\mathrm{cris}}$ [FO, Proposition 6.6].

Using ℓ_u and t, we can define several more period rings: $B_{\mathrm{cris}} := B_{\mathrm{cris}}^+[1/t] \subset B_{\mathrm{dR}}$, $B_{\mathrm{st}}^+ := B_{\mathrm{cris}}^+[\ell_u] \subset B_{\mathrm{dR}}^+$, and $B_{\mathrm{st}} := B_{\mathrm{cris}}[\ell_u]$, which we can think of as a polynomial ring because ℓ_u is transcendental over the fraction field of B_{cris} [FO, Proposition 6.11]. As both ℓ_u and t are "logarithms," it is not hard to see that $\varphi(\ell_u) = p\ell_u$ and $\varphi(t) = pt$, so we extend φ to these rings according to those rules.

Equip B_{st}^+ with an endomorphism N, by formal differentiation of the variable ℓ_u with coefficients in B_{cris}^+, i.e. so that $N(B_{\mathrm{cris}}^+) = 0$. Extend φ to B_{st}^+ as well, with $\varphi(\ell_u) = p\ell_u$. We note that φ and N define endomorphisms of the polynomial subring $K_0[\ell_u] \subset B_{\mathrm{st}}^+$, with N again acting by formal differentiation with respect to the variable ℓ_u. We observe that $p\varphi N = N\varphi$ on B_{st}. Neither φ nor N extend continuously to an action of B_{dR}^+ (cf. [FO, Remark 5.18(3)]); only the filtrations that we will now describe come from B_{dR}^+.

There is an exhaustive, decreasing filtration on each of $A_{\mathrm{cris}}, B_{\mathrm{cris}}^+$, written

$$\mathrm{Fil}^i A_{\mathrm{cris}}, \mathrm{Fil}^i B_{\mathrm{cris}}^+$$

where $\mathrm{Fil}^0 A_{\mathrm{cris}} = A_{\mathrm{cris}}$ (resp. $\mathrm{Fil}^0 B^+_{\mathrm{cris}} = B^+_{\mathrm{cris}}$) induced by their inclusion in the filtered ring B^+_{dR}. The filtration on B^+_{dR} is given by

$$\mathrm{Fil}^i B^+_{\mathrm{dR}} := (\ker \theta)^i, \quad i \geq 0.$$

In fact, $t \in \mathrm{Fil}^1 B^+_{\mathrm{dR}}$ and $t \notin \mathrm{Fil}^2 B^+_{\mathrm{dR}}$ [FO, Proposition 5.19], so also $t \in \mathrm{Fil}^1 A_{\mathrm{cris}}$, and t is a generator for the maximal ideal of B^+_{dR}. We note that the associated graded ring of B^+_{dR} may be represented as $\mathbb{C}_p[t]$, and, when we naturally extend the filtration to $B_{\mathrm{dR}} \cong B^+_{\mathrm{dR}}[1/t]$, $\mathrm{gr} B_{\mathrm{dR}} \cong \mathbb{C}_p[t, 1/t]$. We also note that

$$\mathrm{Fil}^i A_{\mathrm{cris}} \cdot \mathrm{Fil}^j A_{\mathrm{cris}} \subseteq \mathrm{Fil}^{i+j} A_{\mathrm{cris}}, \tag{4.6.2}$$

and similarly for B^+_{cris}.

Now we discuss the action of $\hat{\Gamma}$ on these period rings. The action arises from the action of $\hat{\Gamma}$ on $\mathcal{O}_{\bar{K}}/p$, and extends from there to continuous actions on R, $W(R)$, and all of the subrings of B_{dR} defined above. That B^+_{st} is stable under $\hat{\Gamma}$ follows from Lemma 4.6.4 below, where we calculate the action of $\hat{\Gamma}$ on ℓ_u, finding that for $\sigma \in \hat{\Gamma}$ that $\sigma(\ell_u)$ differs from ℓ_u by a product of an element of $\mathbb{Z}_p$ and t. It will also be useful to know the action of $\hat{\Gamma}$ on t: we see that

$$\sigma(\varepsilon_n) = \varepsilon_n^{\chi_n(\sigma)},$$

where $\chi_n : \hat{\Gamma} \to \mathbb{Z}/p^n\mathbb{Z}$ is the reduction modulo p^n of the p-adic cyclotomic character $\mathbb{Z}_p(1) = \chi : \hat{\Gamma} \to \mathbb{Z}_p$. We find that $\sigma(\underline{\varepsilon}) = \underline{\varepsilon}^{\chi(\sigma)}$, and a calculation with the "logarithm" defining $t = \log [\underline{\varepsilon}]$ tells us that

$$\sigma(t) = \chi(\sigma) \cdot t. \tag{4.6.3}$$

Now we calculate the Galois action on ℓ_u. For $\sigma \in \hat{\Gamma}$, define $\beta(\sigma) \in B^+_{\mathrm{cris}}$ as

$$\beta(\sigma) := \sigma(\ell_u) - \ell_u.$$

LEMMA 4.6.4. *The map β is a 1-cocycle with respect to the cyclotomic character, belonging to the cohomology class associated to π by Kummer theory. When $\beta(\sigma) \neq 0$, it generates the maximal ideal of B_{dR}^{+}.*

PROOF. For $\sigma \in \hat{\Gamma}$ and $n \geq 1$, define $\eta_n(\sigma) \in \mathbb{Z}/p^n\mathbb{Z}$ by the relation

$$\sigma(\pi_n) = \varepsilon_n^{\eta_n(\sigma)} \cdot \pi_n.$$

As $\pi_{n+1}^p = \pi_n$ and $\varepsilon_{n+1}^p = \varepsilon_n$, we see that $\eta_{n+1}(\sigma) \equiv \eta_n(\sigma) \pmod{p^n}$ and we have a well defined map $\eta : \hat{\Gamma} \to \mathbb{Z}_p$. We observe that η is a cocycle for the cyclotomic character $\chi : \hat{\Gamma} \to \mathbb{Z}_p^{\times}$, because

$$\varepsilon_n^{\eta_n(\tau\sigma)} = \frac{\tau\sigma(\pi_n)}{\pi_n} = \frac{\tau\big(\varepsilon_n^{\eta_n(\sigma)} \cdot \pi_n\big)}{\pi_n} = \frac{\varepsilon_n^{\chi_n(\tau)\eta_n(\sigma)} \cdot \varepsilon_n^{\eta_n(\tau)}}{\pi_n} = \varepsilon_n^{\chi_n(\tau)\eta_n(\sigma)+\eta_n(\tau)}.$$

A change in the choice of roots of unity (ε_n) amounts to a change in η by a coboundary for χ; the same is true for a new choice of p^nth roots (π_n) of π. This η is the definition of the "Kummer cocycle" in $H^1(\hat{\Gamma}, \mathbb{Z}_p(1))$ induced by π under the map

$$K^{\times}/(K^{\times})^{p^n} \xrightarrow{\sim} H^1(\hat{\Gamma}, \mathbb{Z}_p/p^n\mathbb{Z}_p(1)), \quad n \geq 1$$

coming from the long exact sequence in Galois cohomology, which is an isomorphism by Hilbert's Theorem 90.

We now see that $\sigma(\underline{\pi}) = \underline{\pi} \cdot \underline{\varepsilon}^{\eta(\sigma)}$. Therefore $\sigma([\underline{\pi}]) = [\underline{\pi}] \cdot [\underline{\varepsilon}]^{\eta(\sigma)}$, and one can quickly verify that even though $\ell_u = \log[\underline{\pi}], t = \log[\underline{\varepsilon}]$ are not standard logs, we still have the expected identity

$$\sigma(\log[\underline{\pi}]) = \log[\underline{\pi}] + \eta(\sigma) \cdot \log[\underline{\pi}].$$

Now β is given in terms of η:

$$\beta(\sigma) = \sigma(\ell_u) - \ell_u = \log[\underline{\pi}] + \eta(\sigma) \cdot \log[\underline{\pi}] - \log[\underline{\pi}] = \eta(\sigma) \cdot t.$$

It is also clear that $\beta(\sigma) = 0$ if and only if $\sigma \in \hat{\Gamma}_\infty$. Therefore we see that because $t \in A_{\mathrm{cris}}$, we have for any $\sigma \in \hat{\Gamma}$ that $\beta(\sigma) \in A_{\mathrm{cris}}$, and when $\beta(\sigma) \neq 0$, then $\beta(\sigma)$ generates the maximal ideal $\ker\theta$ of B^+_{dR} because t is a generator and $\ker\theta/(\ker\theta)^2 \cong \mathbb{C}_p$ as a $\mathbb{Z}_p$-module. $\square$

As this maximal ideal generates the filtration on B^+_{dR}, if $\beta(\sigma) \neq 0$ then

$$\beta(\sigma) \in \mathrm{Fil}^1 B^+_{\mathrm{cris}}, \beta(\sigma) \notin \mathrm{Fil}^2 B^+_{\mathrm{cris}}. \tag{4.6.5}$$

Having completed our summary of the period rings we will need, we now explain the construction of period rings with coefficients in A.

Define $B^+_{\mathrm{cris},A} := A_{\mathrm{cris},A^\circ}[1/p]$, where $A_{\mathrm{cris},A^\circ}$ is, as usual, the $\mathfrak{m}_R A^\circ$-adic completion of $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$. For any A-algebra B, we write $B^+_{\mathrm{cris},B}$ for $B^+_{\mathrm{cris},A} \otimes_A B$. Set $B^+_{\mathrm{st},A} := B^+_{\mathrm{cris},A}[\ell_u]$ and $B^+_{\mathrm{st},B} := B^+_{\mathrm{st},A} \otimes_A B$. The map φ extends to each of these rings B-linearly, with N again acting as formal differentiation with respect to ℓ_u here. In particular, $N(B^+_{\mathrm{cris},B}) = 0$. Analogous notation is used for the elements of the filtration on these rings: denote by $\mathrm{Fil}^i A_{\mathrm{cris},A^\circ}$ the $\mathfrak{m}_R A^\circ$-adic completion of $\mathrm{Fil}^i A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$, and for any A-algebra B let $\mathrm{Fil}^i B^+_{\mathrm{cris},B} := \mathrm{Fil}^i A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} B$. Basic properties over $\mathbb{Z}_p$, mainly (faithful) flatness of both period rings and graded pieces of their filtrations, are extended to these period rings and filtrations with coefficients in Lemmas 4.8.1 and 4.8.2.

It will be important to know in the construction of (4.9.4) that there is a canonical inclusion $\mathcal{O}_A \hookrightarrow B^+_{\mathrm{cris},A}$ extending the map $\mathcal{O} \hookrightarrow B^+_{\mathrm{cris}}$ discussed above, and also a map $\mathfrak{S}^{\mathrm{ur}}_A \hookrightarrow B^+_{\mathrm{cris},A}$. By Lemma 4.6.1, it will suffice to show that for large enough n,

$$W[\![u, u^n/p]\!]_{A^\circ} \hookrightarrow A_{\mathrm{cris},A^\circ}.$$

In order to construct the map, it will suffice to draw, for sufficiently large n, maps

$$W[\![u, u^n/p]\!] \otimes_{\mathbb{Z}_p} A^\circ/(\mathfrak{m}_R A^\circ)^j \hookrightarrow A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/(\mathfrak{m}_R A^\circ)^j$$

for each $j \geq 1$. We will get such maps if we show, for large enough n, the existence of maps

$$W[\![u, u^n/p]\!] \hookrightarrow A_{\mathrm{cris}}.$$

Then Lemma 4.8.1 implies that this map will remain injective after tensoring with A° and completing with respect to the $\mathfrak{m}_R A^\circ$-adic topology. This same construction gives us a canonical map $\mathfrak{S}_A^{\mathrm{ur}} \hookrightarrow B^+_{\mathrm{cris},A}$.

In fact we will show much more than this, which will be useful later (e.g. the proof of Lemma 4.9.9).

LEMMA 4.6.6. *For $r \in (0,1)$ let $\mathcal{O}_r$ denote the coordinate ring of the open rigid analytic disk over K_0 with radius r. Then for any $r > (e(p-1))^{-1}$, $\mathcal{O} \hookrightarrow B^+_{cris}$ factors through $\mathcal{O}_r$. In particular, $W[\![u, u^n/p]\!] \hookrightarrow A_{cris}$ for $n > e(p-1)$.*

PROOF. Recall that u is sent to $[\underline{\pi}]$ in A_{cris}, and A_{cris} is the p-adic completion of the divided power envelope of $W(R)$ with respect to $\ker\theta$. In fact, the kernel of $\bar{\theta}$, which is defined as the composition

$$A_{\mathrm{cris}} \xrightarrow{\theta} \mathcal{O}_{\mathbb{C}_p} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}/p,$$

is also a divided power ideal of A_{cris} [FO, Proposition 6.5]. Recall that $\theta([\underline{\pi}]) = \pi$ and that π is the uniformizer of an extension K of $\mathbb{Q}_p$ with ramification degree e, so that $[\underline{\pi}]^e$ is in the divided power ideal. Since denominators $m!$ may accompany the image of powers of u as small as u^{em}, and the p-adic valuation of $m!$ satisfies $v_p(m!) \sim m/(p-1)$ as $m \to +\infty$, we see that A_{cris} is a $W[\![u, u^a/p^b]\!]$-algebra whenever $a/b > e(p-1)$. □

4.7. Families of (φ, N)-modules over the Open Unit Disk

Following the conclusion of §4.5, we assume that we are given a finite projective $\mathfrak{S}_A$-module $\mathfrak{M}_A$ of constant rank d with a φ-semilinear, A-linear endomorphism $\varphi : \mathfrak{M}_A \to \mathfrak{M}_A$ such that the induced $\mathfrak{S}_A$-linear $\varphi^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ has cokernel killed by $E(u)^h$. We write

$$\mathcal{M}_A := \mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathcal{O}_A, \quad D_A := \mathcal{M}_A/u\mathcal{M}_A,$$

each of which have a natural induced action of φ.

LEMMA 4.7.1 ([Kis08], Lemma 2.2). *There is a unique, φ-compatible, W_A-linear map $\xi : D_A \to \mathcal{M}_A$, whose reduction modulo u is the identity on D_A.*

The induced map $D_A \otimes_{W_A} \mathcal{O}_A \to \mathcal{M}_A$ has cokernel killed by λ^h, and the image of the map $D_A \otimes_{W_A} \widehat{\mathfrak{S}}_{0,A} \to \mathcal{M}_A \otimes_{\mathcal{O}_A} \widehat{\mathfrak{S}}_{0,A}$ is equal to that of

$$\varphi^*(\mathcal{M}_A) \otimes_{\mathcal{O}_A} \widehat{\mathfrak{S}}_{0,A} \to \mathcal{M}_A \otimes_{\mathcal{O}_A} \widehat{\mathfrak{S}}_{0,A}.$$

The proofs of [Kis08, Lemma 2.2 and Lemma 2.2.1] go through verbatim. None of it depends on A° being local. They are a generalization of [Kis06, Lemma 1.2.6], where $A^\circ = \mathbb{Z}_p$.

PROOF. Let $s_0 : D_A \to \mathcal{M}_A$ be any W_A-linear section of the projection $\mathcal{M}_A \to D_A$. Our candidate for the map ξ is the sum

$$s = s_0 + \sum_{i=0}^{\infty} (\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i}).$$

We claim that s converges to a well-defined W_A-linear map from D_A to $\mathcal{M}_A$. Accepting this, we see immediately that s is equivalent to the identity modulo u and that $\varphi \circ s = s \circ \varphi$, as desired.

Let $D_A^\circ \subset D_A$ be a finitely generated W_{A°-submodule which spans D_A. Similarly, we choose a finitely generated $\mathfrak{S}_{A^\circ}$-submodule $\mathfrak{M}_A^\circ \subset \mathfrak{M}_A$ which spans $\mathfrak{M}_A$. We may choose $\mathfrak{M}_A^\circ$ so that

$$\varphi \circ s_0 \circ \varphi^{-1} - s_0 : D_A \longrightarrow u\mathfrak{M}_A$$

takes D_A° into $u\mathfrak{M}_A^\circ$. Acting on this map by φ, we find that

$$\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i} : D_A \to u^{p^i}\mathfrak{M}_A$$

as well. Choose $j \geq 0$ with the property that φ induces a map $\mathfrak{M}_A^\circ \to p^{-j}\mathfrak{M}_A^\circ$ and φ^{-1} induces a map $D_A^\circ \to p^{-j}D_A$. Then for each $i \geq 0$, we have

$$\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i} : D_A^\circ \to p^{-2ij}u^{p^i}\mathfrak{M}_A^\circ.$$

Because the exponential outraces the polynomial, for arbitrarily large $n \geq 0$ this series converges to a well defined map

$$D_A^\circ \longrightarrow \mathfrak{M}_A^\circ \otimes_{\mathfrak{S}_{A^\circ}} W_{A^\circ} W[\![u, u^{p^n}/p]\!]_A,$$

and therefore to a map $\xi : D_A \to \mathcal{M}_A$.

To see the uniqueness of ξ, we argue following [Kis06, Lemma 1.2.6]. Given another map ξ' satisfying the stipulations of the statement of the lemma, we consider the image $X = (\xi - \xi')(D_A)$ of $(\xi - \xi')$. The image lies in $u\mathcal{M}_A$ because both maps are sections of the projection onto D_A. Then because φ is an automorphism of D_A, $X \subset \varphi^j(u\mathcal{M}_A) \subseteq u^{p^j}\mathcal{M}$ for all $j \geq 0$. Therefore $X \cong 0$ as desired.

Note that ξ reduces to the identity modulo u and that its determinant, being an element of $\mathcal{O}_A$, may be safely said to belong to $W[\![u/p]\!]_A$. As a result, Lemma 4.7.4 tells us that for sufficiently large $n \geq 0$, ξ induces an isomorphism

$$D_A \otimes_{W_A} W[\![u/p^n]\!]_A \xrightarrow{\sim} \mathcal{M}_A \otimes_{\mathcal{O}_A} W[\![u/p^n]\!]_A. \tag{4.7.2}$$

Denote by ξ_s the map

$$D_A \otimes_{W_A} W[\![u^{p^s}/p^n]\!]_A \longrightarrow \mathcal{M}_A \otimes_{\mathcal{O}_A} W[\![u^{p^s}/p^n]\!]_A.$$

induced by ξ.

Consider the commutative diagram

$$\begin{array}{ccc} \varphi^*(D_A \otimes_{W_A} \mathcal{O}_A) & \xrightarrow{\varphi^*\xi} & \varphi^*\mathcal{M}_A \\ \downarrow \sim & & \downarrow 1\otimes\varphi \\ D_A \otimes_{W_A} \mathcal{O}_A & \xrightarrow{\xi} & \mathcal{M}_A \end{array}$$

Let r be the least integer such that $e < p^r/n$. Applying $\otimes_{\mathcal{O}_A} W[\![u, u^{p^s}/p^n]\!]_A$ to the diagram above for $s = 0, \ldots, r-1$ yields a diagram where the right vertical arrow is an isomorphism, because $\pi \in K^\times$ lies outside the radius of convergence of some of the elements of $W[\![u, u^{p^s}/p^n]\!]$ when $e \geq p^s/n$, and the cokernel of this map is supported at π. Using the fact that the

vertical arrows send u to u^p and ξ is φ-equivariant, we see that if we tensor the top row by $W[\![u^{p^s}/p^n]\!]_A$ to get ξ_s, we may tensor the lower row by $W[\![u^{p^{s+1}}/p^n]\!]_A$ to get ξ_{s+1}:

$$\begin{array}{ccc} D_A \otimes_{W_A} W[\![u^{p^s}/p^n]\!]_A & \xrightarrow{\xi_s} & \mathcal{M}_A \otimes_{\mathcal{O}_A} W[\![u^{p^s}/p^n]\!]_A \\ \downarrow \sim & & \downarrow 1\otimes\varphi \\ D_A \otimes_{W_A} W[\![u^{p^{s+1}}/p^n]\!]_A & \xrightarrow{\xi_{s+1}} & \mathcal{M}_A \otimes_{\mathcal{O}_A} W[\![u^{p^{s+1}}/p^n]\!]_A \end{array} \tag{4.7.3}$$

We now have a visible argument by induction with base case (4.7.2) that ξ_s is an isomorphism for $s = 0, 1, \ldots, r-1$.

Now consider the case for $s = r-1$, where the radius of convergence will contain π but not $\pi^{1/p}$. Now the top row of (4.7.3) will be an isomorphism, but the right side vertical arrow will may have non-trivial cokernel killed by $E(u)^h$. We also see the final claim of the lemma, which is that the image of the right vertical arrow coincides with the image of the lower horizontal arrow formally locally around π.

Repeating the argument as above shows, for any $s \geq 0$, that the cokernel of ξ_s is killed by $\prod_{i=r}^{s} \varphi^{s-r}((E(u)/c_0)^h)$. Therefore, recalling the definition of $\mathcal{O}_A$, we see that λ^h kills the cokernel of $\xi \otimes_{W_A} \mathcal{O}_A$. □

Lemma 4.7.4 ([Kis08, Lemma 2.2.1]). *Let $I \subset W[\![u]\!]$ be a finitely generated ideal such that $IW[\![u/p]\!]_A/uW[\![u/p]\!]_A$ is the unit ideal. Then for n sufficiently large, $IW[\![u/p^n]\!]_A$ is the unit ideal.*

Proof. Suppose first that I is principal, equal to (f) for $f \in W[\![u/p]\!]_A$. The assumptions imply that the image of f in $W[\![u/p]\!]_A/uW[\![u/p]\!]_A \cong W_A$ is a unit. Because there is a natural injection $W_A \hookrightarrow W[\![u/p]\!]_A$, f may be multiplied by a unit in $W[\![u/p]\!]_A$ so that its image in $W[\![u/p]\!]/uW[\![u/p]\!]_A = W_A$ is 1. In particular, $f \in 1 + uW[\![u/p]\!]_A$. There exists some $j \geq 0$ such that $f \in 1 + p^{-j}W[\![u/p]\!]_A^\circ$ since $W[\![u/p]\!]_A = W[\![u]\!]_{A^\circ}[1/p]$ by definition. Therefore f has an inverse in $W[\![u/p^{j+1}]\!]_{A^\circ}$. This is the desired n of the statement of the lemma.

In general, write $I = (f_1, \ldots, f_r)$ for $f_1, \ldots, f_r \in W[\![u/p]\!]_A$. Write $\bar{f}_i$ for the image of f_i in $W[\![u/p]\!]_A/uW[\![u/p]\!]_A$. Then $1 = \sum_{i+1}^{r} \bar{g}_i \bar{f}_i$ for some $\bar{g}_i \in W[\![u/p]\!]_A/u$. Choose lifts

$g_i \in W[\![u/p]\!]_A$ of the $\bar{g}_i$. Then by the first part, $\sum_{i=1}^r g_i f_i$ is a unit in $W[\![u/p^n]\!]$ for some sufficiently large n. $\square$

4.8. Period Rings in Families

We will now record some lemmata to ensure that the large rings B^+_{cris}, A_{cris}, and so forth behave well in families. This will allow us to show later that, for example, (φ, N)-modules in families behave as one would expect (cf. Theorem 4.10.9).

LEMMA 4.8.1 ([Kis08], Lemma 2.3.1). *For any A°-module M, denote by $\widehat{M}$ its $\mathfrak{m}_R$-adic completion. If M is a flat A°-module, then*

(1) For any finite A°-module N, the natural map

$$N \otimes_{A^\circ} \widehat{M} \to \widehat{N \otimes_{A^\circ} M}$$

is an isomorphism.

(2) $\widehat{M}$ is flat over A°. If M is faithfully flat over A°, then so is $\widehat{M}$.

(3) The functor $M \mapsto \widehat{M}$ preserves short exact sequences of flat A°-modules.

For completeness, we elaborate on the proof in [Kis08].

PROOF. First we claim that the functor on finite A°-modules $N \mapsto \widehat{N \otimes_{A^\circ} M}$ is exact. Say that we have an exact sequence of finite A°-modules

$$0 \to N_1 \to N_2 \to N_3 \to 0.$$

The Artin-Rees Lemma implies that the filtrations $(\mathfrak{m}_R A^\circ)^n \cdot N_i$ have bounded difference, i.e. there exists k such that for all $n \geq k$,

$$(\mathfrak{m}_R A^\circ)^n \cdot N_1 \subseteq ((\mathfrak{m}_R A^\circ)^n \cdot N_2) \cap N_1 \subseteq (\mathfrak{m}_R A^\circ)^{n-k} N_1,$$

and the filtrations on N_2 and N_3 are easily seen to be equal. This implies that the native $\mathfrak{m}_R A^\circ$-adic topologies on N_1 and N_3 are equivalent to the topologies induced by the $\mathfrak{m}_R A^\circ$-adic topology on N_2. Therefore completion with respect to these topologies maintains

exactness (AM Cor. 10.3). We claim that since M is flat, tensoring this exact sequence and these adic filtrations by M preserves the bounded difference of the filtrations. Indeed, because M is flat, for any ideal I of A° and A°-module N, $(I \cdot N) \otimes_{A^\circ} M \cong I \cdot (N \otimes_{A^\circ} M)$. Therefore the composition of the $- \otimes_{A^\circ} M$-functor with the $(\mathfrak{m}_R A^\circ)$-adic completion functor is exact as desired.

To see (1), let N be a finite A° module and observe that (1) is obvious when N is free. For N a general finite A°-module, we take a presentation by free modules $F_\alpha \hookrightarrow F_\beta$ and find

$$\begin{array}{ccccccccc} & & F_\alpha \otimes_{A^\circ} \widehat{M} & \longrightarrow & F_\beta \otimes_{A^\circ} \widehat{M} & \longrightarrow & N \otimes_{A^\circ} \widehat{M} & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr \\ 0 & \longrightarrow & \widehat{F_\alpha \otimes_{A^\circ} M} & \longrightarrow & \widehat{F_\beta \otimes_{A^\circ} M} & \longrightarrow & \widehat{N \otimes_{A^\circ} M} & \longrightarrow & 0 \end{array}$$

The five-lemma shows that $N \otimes_{A^\circ} \widehat{M} \xrightarrow{\sim} \widehat{N \otimes_{A^\circ} M}$ as desired.

The first part of (2) follows from the fact that the injectivity of the map $I \otimes_{A^\circ} M \to M$ is preserved under $\mathfrak{m}_R A^\circ$-adic completion. For the second part, if M is faithfully flat over A° if and only if $M_{\mathfrak{m}} \neq 0$ for all maximal ideals of the (formal scheme corresponding to) A°. This property is clearly preserved under completion.

(3) follows from the same considerations on filtrations discussed above. If $N_1 \subseteq N_2$ are flat A°-modules, then the native $\mathfrak{m}_R A^\circ$-adic filtration on N_1 and the filtration induced from the native filtration on N_2 are equal. □

Lemma 4.8.2 (Following [Kis08, Lemma 2.3.2]).

(1) For $i \geq 0$, the ideal $\mathrm{Fil}^i A_{cris,A^\circ}$ of A_{cris,A° is a faithfully flat A°-module.

(2) For $i \geq 0$, $\mathrm{Fil}^i A_{cris,A^\circ}$ is a faithfully flat A°-module, which is isomorphic to the $\mathfrak{m}_R$-adic completion of $(\mathrm{Fil}^i A_{cris} / \mathrm{Fil}^{i+1} A_{cris}) \otimes_{\mathbb{Z}_p} A^\circ$.

(3) For any A-algebra B, $i \geq 1$, and $\sigma \in \hat{\Gamma}$, $B^+_{\mathrm{cris},B} / (\beta(\sigma) B + \mathrm{Fil}^i B^+_{\mathrm{cris},B})$ is a flat B-module. If $\beta(\sigma) \notin \mathrm{Fil}^i B^+_{cris}$, then $\beta(\sigma) \notin \mathrm{Fil}^i B^+_{\mathrm{cris},B}$.

(4) Let B° be a finite continuous A°-algebra with ideal of definition I. Then the I-adic completion of $A_{cris} \otimes_{\mathbb{Z}_p} B^\circ$ is canonically isomorphic to $A_{cris,A^\circ} \otimes_{A^\circ} B^\circ$.

(5) The map

$$A_{cris,A^\circ} \to \prod A_{\mathrm{cris},A^\circ/\mathfrak{q}}$$

is injective, where $\mathfrak{q}$ runs over ideals of A° such that $A^\circ/\mathfrak{q}$ is a finite flat $\mathbb{Z}_p$-algebra.

(6) If $0 \neq f \in A_{cris}$, then f is not a zero divisor in A_{cris,A°.

PROOF. For part (1), since $\mathrm{Fil}^i A_{\mathrm{cris}}$ is a faithfully flat $\mathbb{Z}_p$-module, $\mathrm{Fil}^i A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$ is a faithfully flat A°-module. Then Lemma 4.8.1(2) implies that $\mathrm{Fil}^i A_{\mathrm{cris},A^\circ}$ is a faithfully flat A°-module.

To demonstrate part (2), take the exact sequence of faithfully flat $\mathbb{Z}_p$-modules

$$0 \to \mathrm{Fil}^{i+1} A_{\mathrm{cris}} \to \mathrm{Fil}^i A_{\mathrm{cris}} \to \mathrm{Fil}^i A_{\mathrm{cris}}/\mathrm{Fil}^{i+1} A_{\mathrm{cris}} \to 0,$$

apply the exact functor $\otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$, and the $\mathfrak{m}_R A^\circ$-adic completion functor. The latter functor is exact by Lemma 4.8.1(3) and preserves the condition "fully faithful" by 4.8.1(3). The desired result is then visible in the resulting exact sequence of faithfully flat A°-modules.

To prove part (3), first consider the case $\beta(\sigma) = 0$. Using the logic of part (2) and applying it by induction on i to $A_{\mathrm{cris}}/\mathrm{Fil}^i A_{\mathrm{cris}}$, thinking of it as an extension of $A_{\mathrm{cris}}/\mathrm{Fil}^{i-1} A_{\mathrm{cris}}$ by $\mathrm{Fil}^{i-1} A_{\mathrm{cris}}/\mathrm{Fil}^i A_{\mathrm{cris}}$, we find that $A_{\mathrm{cris},A^\circ}/\mathrm{Fil}^i A_{\mathrm{cris},A^\circ}$ is a flat A°-module. We may then apply $\otimes_{A^\circ} B$ for any A-algebra B to the exact sequence

$$0 \to \mathrm{Fil}^i A_{\mathrm{cris},A^\circ} \to A_{\mathrm{cris},A^\circ} \to A_{\mathrm{cris},A^\circ}/\mathrm{Fil}^i A_{\mathrm{cris},A^\circ} \to 0$$

to get the result.

Now allow $\beta(\sigma) \neq 0$. Let j be the largest integer such that $\beta(\sigma)/p^j \in A_{\mathrm{cris}}/\mathrm{Fil}^i A_{\mathrm{cris}} \subset (A_{\mathrm{cris}}/\mathrm{Fil}^i A_{\mathrm{cris}})[1/p] \cong B^+_{\mathrm{cris}}/\mathrm{Fil}^i B^+_{\mathrm{cris}}$. Then $A_{\mathrm{cris}}/(\beta(\sigma)/p^j \cdot \mathbb{Z}_p + \mathrm{Fil}^i A_{\mathrm{cris}})$ will be $\mathbb{Z}_p$-flat (cf. the argument after (4.12.4)). We then apply the reasoning of the proof of part (2) and the first case of (3) found above to the exact sequence of flat $\mathbb{Z}_p$-modules

$$0 \to \mathbb{Z}_p \xrightarrow{\cdot\beta(\sigma)/p^j} A_{\mathrm{cris}}/\mathrm{Fil}^i A_{\mathrm{cris}} \to A_{\mathrm{cris}}/(\beta(\sigma)/p^j \cdot \mathbb{Z}_p + \mathrm{Fil}^i A_{\mathrm{cris}}) \to 0$$

in order to conclude that $B^+_{\mathrm{cris},B}/(\beta(\sigma) \cdot B + \mathrm{Fil}^i B^+_{\mathrm{cris},B})$ is a flat B-module.

It remains to prove the final statement in part (3). Because tensor products and direct limits commute, we may assume that B is a finitely generated A-algebra. Then there exists a map $B \to B'$, where B' is a finite $W(\mathbb{F})[1/p]$-algebra. We show the contrapositive: If $\beta(\sigma) \in \mathrm{Fil}^i B^+_{\mathrm{cris},B}$, then $\beta(\sigma) \in \mathrm{Fil}^i B^+_{\mathrm{cris},B'}$ as well, which implies that $\beta(\sigma) \in \mathrm{Fil}^i B^+_{\mathrm{cris}}$ (because B' is trivially a flat $\mathbb{Q}_p$-algebra).

For part (4), the fact that $f : A^\circ \to B^\circ$ is finite and continuous implies that f is adic, i.e. that $f(\mathfrak{m}_R A^\circ) \cdot B^\circ$, like the ideal I of B°,, is an ideal of definition for B°. So we simply take I to be $f(\mathfrak{m}_R A) \cdot B^\circ$. Then because B° is a finite A°-algebra, the $\mathfrak{m}_R A^\circ$-adic topology on $A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} B^\circ$ is equivalent to its I-adic topology by the Artin-Rees lemma. Now we apply Lemma 4.8.1(1) to conclude that the natural map

$$A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} B^\circ \to \varprojlim_n (A_{\mathrm{cris}} \otimes_{A^\circ} B^\circ) \otimes_{B^\circ} B^\circ / I^n$$

is an isomorphism as desired.

Now we prove (5). Let M be the set of maximal ideals of $\mathrm{Spf}(A^\circ)$ as a $\mathrm{Spf}(\mathbb{Z}_p)$-formal scheme, corresponding to maximal ideals of $A^\circ/\mathfrak{m}_R A^\circ$. For $\mathfrak{m} \in M$, let $A_{\mathrm{cris},A^\circ_{\mathfrak{m}}}$ denote the completion of $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$ at $1 \otimes \mathfrak{m}$. First, we will reduce to the case that A° is a complete local Noetherian ring by showing that the natural map

$$A_{\mathrm{cris},A^\circ} \longrightarrow \prod_{\mathfrak{m} \in M} A_{\mathrm{cris},A^\circ_{\mathfrak{m}}} \tag{4.8.3}$$

is injective. Therefore we may assume that the connected components of $\mathrm{Spec}\, A^\circ/\mathfrak{m}_R A^\circ$ are positive-dimensional.

The map

$$A^\circ \to \prod_{\mathfrak{m} \in M} A^\circ_{\mathfrak{m}} \tag{4.8.4}$$

is injective and flat, since A° is Noetherian and $A^\circ/\mathfrak{m}_R A^\circ$ is Jacobson since it is finitely generated over $\mathbb{Z}$. Since A_{cris} is a flat $\mathbb{Z}_p$-module, Lemma 4.8.1(2) implies that that the map

$$A_{\mathrm{cris},A^\circ} \longrightarrow A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} \prod_{\mathfrak{m}\in M} A^\circ_{\mathfrak{m}}$$

is injective. To complete the proof that the map (4.8.3) is injective, we will show that for any $0 \neq f \in A_{\mathrm{cris},A^\circ}$, there exists a finite subset $M_f \subset M$ such that the image of f in $\prod_{\mathfrak{m}\in M_f} A_{\mathrm{cris},A^\circ_{\mathfrak{m}}}$ is not 0.

Having chosen a nonzero $f \in A_{\mathrm{cris},A^\circ}$, there exists some $n \geq 1$ such that the image of f, call it $\bar{f}$, in $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$ is not 0. Write out $\bar{f}$ in tensor form as

$$\bar{f} = \sum_j \bar{g}_j \otimes \bar{h}_j$$

where $\bar{g}_j \in A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}$ (resp. $\bar{h}_j \in A^\circ/(\mathfrak{m}_R A^\circ)^n$) are $\mathbb{Z}_p$-linearly independent in $A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}$ (resp. $\bar{h}_j \in A^\circ/(\mathfrak{m}_R A^\circ)^n$), i.e. such that this tensor product cannot be reduced. Let $H \subset A^\circ/(\mathfrak{m}_R A^\circ)^n$ be the $\mathbb{Z}_p$-linear span of $\{\bar{h}_j\}$. Because (4.8.4) is injective and we are assuming that the connected components of $\operatorname{Spec} A^\circ/\mathfrak{m}_R A^\circ$ are positive dimensional, for each nonzero $\bar{h} \in H$, there exist infinitely many $\mathfrak{m} \in M$ such that there exists a power $k = k(\mathfrak{m}, \bar{h})$ of $\mathfrak{m}$ such that the image of $\bar{h}$ in $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/((\mathfrak{m}_R A)^n + \mathfrak{m}^{k(\mathfrak{m},\bar{h})})$ is not zero. Since there are only finitely many $\bar{h}_j$, this means that there are finitely many powers of maximal ideals $\mathfrak{m}^{k_\mathfrak{m}}$ such that the images of $\{\bar{h}_j\}$ in $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/((\mathfrak{m}_R A^\circ)^n + \cap \mathfrak{m}^{k_\mathfrak{m}})$ remain $\mathbb{Z}_p$-linearly independent. We set M_f to be this finite set of maximal ideals. Since A_{cris} is p-adically complete, the corresponding statement for the $\bar{g}_j$ is trivial: we need only make sure that p^{n-1} remains nonzero, which we can do by increasing each $k_\mathfrak{m}$ to at least n. Now we have shown that the reduced tensor form of $\bar{f}$ in $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/((\mathfrak{m}_R A)^n + \cap \mathfrak{m}^{k_\mathfrak{m}})$ remains reduced, so that it cannot vanish. Therefore the image of f in $\prod_{M_f} A_{\mathrm{cris},A^\circ_{\mathfrak{m}}}$ has nonzero image in $\prod_{M_f} A_{\mathrm{cris},A^\circ_{\mathfrak{m}}}/(\mathfrak{m} A_{\mathrm{cris},A^\circ_{\mathfrak{m}}})^{k_\mathfrak{m}}$, and is nonzero as desired.

Now we prove (5) in the case that A° is a complete local Noetherian $\mathbb{Z}_p$-algebra with maximal ideal $\mathfrak{m}$ and finite residue field, simply adding more detail to the proof in [Kis08], Lemma 2.3.2. First we deduce that for any $n \geq 1$, there exists $\mathfrak{q} \subset A^\circ$ such that $A^\circ/\mathfrak{q}$ is

a finite flat $\mathbb{Z}_p$-algebra and $\mathfrak{q} \subseteq \mathfrak{m}^n$. For since $A^\circ[1/p]$ is Jacobson with maximal ideals $\mathfrak{a}$ and residue fields $K_\mathfrak{a}$ being finite extensions of $\mathbb{Q}_p$ with rings of integers $\mathcal{O}_\mathfrak{a}$, there exists an injective map

$$A^\circ \longrightarrow \prod_\mathfrak{a} \mathcal{O}_\mathfrak{a}$$

such that the kernel consists of nilpotents (recall the standing assumption that A° is a flat as a $\mathbb{Z}_p$-algebra). Let $\mathfrak{N}$ be the nilradical of A°. Choose a finite set S of maximal ideals of $A^\circ[1/p]$ such that the induced map

$$(A^\circ/\mathfrak{N}) \longrightarrow \prod_{\mathfrak{a}\in S} \mathcal{O}_\mathfrak{a}$$

is injective after tensoring with $(A^\circ/\mathfrak{N})/((\mathfrak{m}^n+\mathfrak{N})/\mathfrak{N})$. Such an S exists because of the finite length of $(A^\circ/\mathfrak{N})/((\mathfrak{m}^n + \mathfrak{N})/\mathfrak{N})$. Now choose representatives in A° of the finite cardinality set $\mathfrak{N}/(\mathfrak{N} \cap \mathfrak{m}^n)$. Then there exists a finite set of powers $\mathfrak{a}^{k_\mathfrak{a}}$ of maximal ideals $\mathfrak{a}$ of $A^\circ[1/p]$ such that the image of these representatives in $\prod_\mathfrak{a} A^\circ[1/p]/\mathfrak{a}^{k_\mathfrak{a}}$ does not vanish. Now we observe that

$$A^\circ \longrightarrow \prod_{\mathfrak{a}\in S} A^\circ/(A^\circ \cap \mathfrak{a}^{k_\mathfrak{a}})$$

is injective after tensoring with $A^\circ/\mathfrak{m}^n$, showing that the ideal

$$\mathfrak{q} := A^\circ \cap \bigcap_{\mathfrak{a}\in S} \mathfrak{a}^{k_\mathfrak{a}}$$

of A° satisfies the desired condition: the quotient $A^\circ/\mathfrak{q}$ is a finite flat $\mathbb{Z}_p$-algebra, and $\mathfrak{q} \subseteq \mathfrak{m}^n$.

Now choose $0 \neq f \in A_{\mathrm{cris},A^\circ}$. Then there exists $n \geq 1$ such that the image of f in $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/\mathfrak{m}^n$ is not zero. Using the ideal $\mathfrak{q}$ constructed above, we see that the image of f in $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ/\mathfrak{m}^n$ is nonzero as desired.

To show (6), we use (5) to reduce to the case that A° is a finite flat $\mathbb{Z}_p$-algebra. Then $A_{\mathrm{cris},A^\circ} \cong A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$ because both rings have the p-adic topology. Now (6) follows from the flatness of A° over $\mathbb{Z}_p$, considering that the injective map from A_{cris} to itself by multiplication by $0 \neq f \in A_{\mathrm{cris}}$ remains injective after tensoring with A°. $\square$

LEMMA 4.8.5 (Generalization of [Kis08, Lemma 2.3.3]). *Let M be an A°-module and $x \in A_{cris,A^\circ} \otimes_{A^\circ} M$. The set of A°-submodules $N \subset M$ such that $x \in A_{cris,A^\circ} \otimes_{A^\circ} N$ has a smallest element $N(x)$.*

Here we cannot repeat the proof of [Kis08], for that proof only covers the case that A° is local.

PROOF. Assume that $\mathfrak{m}_R^n \cdot M = 0$ for some $n \geq 1$. Therefore there is a natural isomorphism of A°-modules

$$A_{\mathrm{cris},R} \otimes_R M \xrightarrow{\sim} A_{\mathrm{cris},A^\circ} \otimes_{A^\circ} M.$$

Applying [Kis08, Lemma 2.3.3] to the right hand side, there exists a smallest R-submodule P of M such that $x \in A_{\mathrm{cris},R} \otimes_R P$. We claim that the image N of the natural map

$$P \otimes_R A^\circ \to M$$

is the smallest A°-submodule of M with the required property. Clearly it contains x. If there were a A° submodule N' with the property, then $N' \supset P$ since N' is also a R-module with the property. But then N' must contain N, which is the A°-span of P. This shows that N is the smallest A°-submodule of M with the property.

Now we allow M to be any finite A°-module. Now the proof may be completed according to [Kis08, Lemma 2.3.3]. The same is true for the generalization to an arbitrary A° module M. □

4.9. Period Maps

Following [Kis08, §2.4], recall our situation: A° is an adic and finite type R-algebra, where the structure map is compatible with the action of $\hat{\Gamma}$ on a projective, rank d A°-module V_{A°. Suppose that $(A^\circ)^{\leq h} = A^\circ$. Write $\mathfrak{M}_{A^\circ}$ for the finite projective $\mathfrak{S}_{A^\circ}$-module of given by Proposition 4.5.9, with a map $\varphi^*(\mathfrak{M}_{A^\circ}) \to \mathfrak{M}_{A^\circ}$ with cokernel killed by $E(u)^h$. Also set $\mathfrak{M}_A := \mathfrak{M}_{A^\circ} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Write $V_A := V_{A^\circ} \otimes_{A^\circ} A$, so that Proposition 4.5.9(4) provides

a canonical, $\hat{\Gamma}_\infty$-equivariant isomorphism

$$\iota : V_A \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_A,\varphi}(\mathfrak{M}_A, \mathfrak{S}_A^{\mathrm{ur}}). \tag{4.9.1}$$

We will follow [Kis08, §2.4] in deducing a period map from the data above. This gives us a $\hat{\Gamma}_\infty$-equivariant comparison with coefficient ring A between the Galois representation V_A and the periods of $\mathfrak{M}_A$. We will then discuss additional data needed in order to extend this to a $\hat{\Gamma}$-equivariant map, although this additional data ends up simply being a restriction (Proposition 4.9.11). In what follows, B is an arbitrary A-algebra.

We deduce from the map ι a $\mathfrak{S}_A$-linear, φ-equivariant map

$$\mathfrak{M}_A \to \mathrm{Hom}_A(V_A, \mathfrak{S}_A^{\mathrm{ur}}); \quad m \mapsto (v \mapsto \langle m, \iota(v)\rangle). \tag{4.9.2}$$

Tensoring this map by $\otimes_{\mathfrak{S}_A}\mathcal{O}_A$ and using the map $\xi : D_A \to \mathcal{M}_A$ from Lemma 4.7.1, we have a φ-equivariant map

$$D_A \xrightarrow{\xi} \mathcal{M}_A \to \mathrm{Hom}_A(V_A, \mathfrak{S}_A^{\mathrm{ur}}) \otimes_{\mathfrak{S}_A} \mathcal{O}_A \to \mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A}). \tag{4.9.3}$$

Tensoring the composition of these maps by $\otimes_A B$ for B our chosen A-algebra, there is a $B^+_{\mathrm{cris},B}$-linear map

$$D_B \otimes_{W_B} B^+_{\mathrm{cris},B} \to \mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A}) \otimes_A B \cong \mathrm{Hom}_B(V_B, B^+_{\mathrm{cris},B}). \tag{4.9.4}$$

We see that the right hand side has an action of $\hat{\Gamma}$, and the left hand side has an action of $\hat{\Gamma}_\infty$ through the action on $B^+_{\mathrm{cris},B}$. This map is $\hat{\Gamma}_\infty$-equivariant because $\hat{\Gamma}_\infty$ acts equivariantly on the inclusions $\mathfrak{S} \hookrightarrow \mathcal{O} \hookrightarrow B^+_{\mathrm{cris}}$ and that ι above is $\hat{\Gamma}_\infty$-equivariant. In order to extend the action of $\hat{\Gamma}_\infty$ on the left hand side of (4.9.4) to an action of $\hat{\Gamma}$, we suppose that there is a W_B-linear map

$$N : D_B \to D_B$$

which satisfies the identity $p\varphi N = N\varphi$. Then the action of $\hat{\Gamma}$ on $D_B \otimes_{W_B} B^+_{\mathrm{cris},B}$ is

$$\sigma(d \otimes b) = \left(\sum_{i=0}^{\infty} \frac{N^i(d) \otimes \beta(\sigma)^i}{i!} \right) \sigma(b) = \exp(N \otimes \beta(\sigma)) \cdot d \otimes \sigma(b) \tag{4.9.5}$$

for $\sigma \in \hat{\Gamma}$. Then we observe that such an action of $\hat{\Gamma}$ commutes with φ, using the fact that $\varphi(\beta(\sigma)) = p\beta(\sigma)$. Here is the calculation:

$$\begin{aligned}
\sigma\varphi(d \otimes b) &= \exp(N \otimes \beta(\sigma))\varphi(d) \otimes \sigma(\varphi(b)) \\
&= \left(\sum_{i=0}^{\infty} \frac{N^i(\varphi(d)) \otimes \beta(\sigma)^i}{i!} \right) \varphi(\sigma(b)) \\
&= \left(\sum_{i=0}^{\infty} \frac{p^i\varphi(N^i(d)) \otimes \beta(\sigma)^i}{i!} \right) \varphi(\sigma(b)) \\
&= \left(\sum_{i=0}^{\infty} \frac{\varphi(N^i(d)) \otimes \varphi(\beta(\sigma)^i)}{i!} \right) \varphi(\sigma(b)) \\
&= \varphi\sigma(d \otimes b).
\end{aligned}$$

Now we set up a theory for semistable representations, recalling that we adjoin ℓ_u to $B^+_{\mathrm{cris},B}$ to get $B^+_{\mathrm{st},B} = B^+_{\mathrm{cris},B} \otimes_{K_0} K_0[\ell_u]$, naturally extending the actions of φ and N to the tensor product, with N acting as $\frac{d}{d\ell_u}$ on $K_0[\ell_u]$. Consider the composite of the isomorphisms

$$D_B \otimes_{K_0} K_0[\ell_u] \xrightarrow{\sim} (D_B \otimes_{K_0} K_0[\ell_u])^{N=0} \otimes_{K_0} K_0[\ell_u] \xrightarrow{(\ell_u \mapsto 0) \otimes 1} D_B \otimes_{K_0} K_0[\ell_u] \tag{4.9.6}$$

where the first map is the inverse to the natural isomorphism

$$(D_B \otimes_{K_0} K_0[\ell_u])^{N=0} \otimes_{K_0} K_0[\ell_u] \xrightarrow{\sim} D_B \otimes_{K_0} K_0[\ell_u]$$

induced by polynomial multiplication in $K_0[\ell_u]$. Tensoring (4.9.6) by $B^+_{\mathrm{cris},B}$ over W_B and tensoring (4.9.4) by $K_0[\ell_u]$ over K_0, we obtain the composite map

$$D_B \otimes_{W_B} B^+_{\mathrm{st},B} \xrightarrow{(4.9.6)} D_B \otimes_{W_B} B^+_{\mathrm{st},B} \xrightarrow{(4.9.4)} \mathrm{Hom}_B(V_B, B) \otimes_B B^+_{\mathrm{st},B}. \tag{4.9.7}$$

We claim that (4.9.4) is $\hat{\Gamma}$-equivariant if and only if (4.9.7) is equivariant when $\hat{\Gamma}$ is regarded as acting trivially on D_B. A key observation is that the an inverse to the bijection $(D \otimes_{K_0} K[\ell_u])^{N=0} \xrightarrow{\ell_u \mapsto 0} D$ is given by $d \mapsto \exp(-N \otimes \ell_u) \cdot d$. We give the calculations in the form of this lemma.

LEMMA 4.9.8. *The map*

$$D_B \otimes_{W_B} B^+_{\mathrm{st},B} \longrightarrow D_B \otimes_{W_B} B^+_{\mathrm{st},B}$$

given by tensoring (4.9.6) *by* $\otimes_{W_B} B^+_{\mathrm{cris},B}$ *is* $\hat{\Gamma}$*-equivariant, when* $\hat{\Gamma}$ *is given the trivial action on* D_B *on the left side and the action of* (4.9.5) *on the right.*

PROOF. We write f, g for the isomorphisms

$$\begin{aligned} g: \quad & D \longrightarrow (D \otimes_{K_0} K_0[\ell_u])^{N=0} \\ & d \mapsto \exp(-N \otimes \ell_u) \cdot d. \\ f: \quad & (D \otimes_{K_0} K[\ell_u])^{N=0} \longrightarrow D \\ & \ell_u \mapsto 0 \end{aligned}$$

To see that g is an inverse to f note that any element of $(D \otimes_{K_0} K[\ell_u])^{N=0}$ has the form

$$\sum_{i \geq 0} d_i \ell_u^i, \quad \text{where } i d_i + N(d_{i-1}) = 0 \ \forall \ i \geq 1,$$

so that it is determined by the coefficient d_0, i.e. f is an injection. Now observe that

$$g(d_0) = \sum_{j \geq 0} \frac{(-1)^j N^i(d_0) \ell_u^j}{j!},$$

which certainly has the correct form with constant coefficient, so that f and g are inverse.

Now we wish to show that $g \otimes_{K_0} B^+_{\mathrm{st},B}$ is $\hat{\Gamma}$-equivariant with $\hat{\Gamma}$-actions defined in the statement of the lemma. Since this map is clearly equivariant on elements of $B^+_{\mathrm{st},B}$, it will suffice to show for $d_0 \in D$ (letting $B = \mathbb{Q}_p$ for simplicity) that

$$g \circ \sigma(d_0) = \sigma \circ g(d_0),$$

where σ acts according to (4.9.5)on the left side and with the *trivial* action on the right side.

First we calculate the left hand side. We will use the fact that that $N(\beta(\sigma)) = 0$ for all $\sigma \in \hat{\Gamma}$, since $\beta(\sigma) \in B^+_{\mathrm{cris}}$. Or one can use the chain rule: since N commutes with the action of $\hat{\Gamma}$,

$$N((\sigma(\ell_u) - \ell_u)^i) = i(\sigma(\ell_u) - \ell_u)^{i-1}(\sigma(N(\ell_u)) - N(\ell_u)) = 0$$

since $N(\ell_u) = 1$. Now here is the calculation:

$$\begin{aligned} g \circ \sigma(d_0) &:= g\left(\sum_{i\geq 0} \frac{N^i(d_0) \otimes \beta(\sigma)^i}{i!}\right) \\ &= \sum_{i,j\geq 0} \frac{(-1)^j N^j(N^i(d_0)) \otimes \beta(\sigma)^i \ell_u^j}{i!j!}, \end{aligned}$$

which is sent by f to d_0, since f kills ℓ_u and $\beta(\sigma) = \sigma(\ell_u) - \ell_u$. We observe that this result, namely $f \circ g \circ \sigma(d_0) = d_0$, is exactly the same as $f \circ \sigma \circ g(d_0)$, where σ is given the trivial action in this second expression. Since f and g are mutually inverse, we have completed the proof. □

Lemma 4.9.9 (Following [Kis08, Lemma 2.4.6]). *For each A-algebra B, the maps* (4.9.4) *and* (4.9.7) *are injective, and their cokernels are flat B-modules.*

As usual our proof follows the proof in [Kis08], adding some additional exposition and making the points of generalization clear.

Proof. First we note that it suffices to prove the assertions only for (4.9.4), and for $B = A$. For if there is an exact sequence of A-modules $0 \to N' \to N \to N'' \to 0$ and N'' is flat, then this sequence remains exact after applying $- \otimes_A B$ for any A-algebra B. It also suffices to prove the statement for (4.9.4) alone, since (4.9.6) is an isomorphism.

Recall Lemma 4.8.2(5), which states that

$$A_{\mathrm{cris},A^\circ} \to \prod_{\mathfrak{q}} A_{\mathrm{cris},A^\circ/\mathfrak{q}}$$

is injective, where $\mathfrak{q}$ varies over ideals of A° such that $A^\circ/\mathfrak{q}$ is a finite flat $\mathbb{Z}_p$-algebra. Applying this lemma, we may confine ourselves to the case that A° is a finite flat $\mathbb{Z}_p$-algebra.

We are remanded to the case that A° is local, so we can repeat the proof of injectivity from [Kis08, Lemma 2.4.6]. Recall that (4.9.4) as defined in (4.9.3): it is the composition of the map ξ constructed in Lemma 4.7.1 with the map (4.9.2), tensored up through the maps $\mathfrak{S}_A \hookrightarrow \mathcal{O}_A$ and $\mathcal{O}_A \hookrightarrow B^+_{\mathrm{cris},A}$.

Consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{S}_A^{\mathrm{ur}} \otimes_{\mathfrak{S}_A} \mathfrak{M}_A & \longrightarrow & \mathrm{Hom}_A(V_A, \mathfrak{S}_A^{\mathrm{ur}}) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{E}^{\mathrm{ur}},A^\circ}[1/p] \otimes_{\mathfrak{S}_A} \mathfrak{M}_A & \longrightarrow & \mathrm{Hom}_A(V_A, \mathcal{O}^{\mathrm{ur}}_{\mathcal{E},A^\circ}[1/p]) \end{array}$$

where the top map is obtained from (4.9.2). From (4.4.21), we know that the bottom row is an isomorphism. The left arrow is injective, because the finiteness of A° over $\mathbb{Z}_p$ gives us that this arrow is obtained by applying $- \otimes_{\mathfrak{S}} \mathfrak{M}_A$ to the canonical inclusion $\mathfrak{S}^{\mathrm{ur}} \hookrightarrow \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$, and $\mathfrak{M}_A$ is flat as a $\mathfrak{S}$-module. This means that the top map must be injective. Furthermore, it is an injective map of finite free $\mathfrak{S}_A^{\mathrm{ur}}$-modules of equal rank and remains injective after tensoring by $- \otimes_{\mathfrak{S}^{\mathrm{ur}}} B^+_{\mathrm{cris}}$. Therefore the map

$$\mathcal{M}_A \otimes_{\mathcal{O}_A} B^+_{\mathrm{cris},A} \xrightarrow{\sim} \mathfrak{M}_A \otimes_{\mathfrak{S}_A} B^+_{\mathrm{cris},A} \to \mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A})$$

obtained from (4.9.2) by tensoring up to B^+_{cris} is injective.

It remains to address the map $\xi : D_A \to \mathcal{M}_A$, which induces the first factor in (4.9.4). From the first part of Lemma 4.7.1, we know that the determinant of

$$D_A \otimes_{W_A} B^+_{\mathrm{cris},A} \xrightarrow{\xi \otimes 1} \mathcal{M}_A \otimes_{\mathcal{O}_A} B^+_{\mathrm{cris},A} \tag{4.9.10}$$

is a divisor of λ^s for some positive integer s. However, the image in B^+_{cris} of each of the factors $\varphi^n(E(u)/c_0$, $n \geq 1$ of λ are units in B^+_{cris} because the p-adic radius of convergence of these functions with respect to u is $(e(p-1))^{-1}$ (Lemma 4.6.6). Since the zeros of $\varphi^n(E(u)/c_0)$ have p-adic valuation $(ep^n)^{-1}$, they lie outside this radius of convergence for $n \geq 1$. Therefore the determinant of (4.9.10) is supported at the ideal $(E([\underline{\pi}]))$ is a divisor of $E([\underline{\pi}])^s$. Now as

$E([\underline{\pi}])$ is not a zero divisor in A_{cris}, neither is it a zero divisor in $B^+_{\mathrm{cris},A}$. This completes our proof of the injectivity statement.

It remains to show that the cokernel of (4.9.4), appearing in the exact sequence

$$0 \to D_A \otimes_{W_A} B^+_{\mathrm{cris},A} \overset{(4.9.4)}{\longrightarrow} \mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A}) \longrightarrow \mathrm{coker} \to 0,$$

is flat as an A-module. Because an A-module M is flat if and only if $\mathrm{Tor}_1^A(A/I, M) = 0$ for all ideals I of A and $\mathrm{Tor}_1^R(A/I, \mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A})) = 0$ for all finitely generated ideals $I \subset A$ because $\mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A})$ is flat, the cokernel will be flat if and only if (4.9.4) remains injective after tensoring with A/I for any finitely generated ideal I of A. This is what we will now prove.

If we had started our proof with A/I in the place of A, we would still have the injectivity statement for A/I, just as we proved it for A above. This *almost* completes our proof, for we want to show that

$$D_A \otimes_{W_A} B^+_{\mathrm{cris},A} \otimes_A A/I \overset{(4.9.4)\otimes_A A/I}{\longrightarrow} \mathrm{Hom}_A(V_A, B^+_{\mathrm{cris},A}) \otimes_A A/I$$

is injective, and we know that

$$D_{A/I} \otimes_{W_{A/I}} B^+_{\mathrm{cris},A/I} \overset{(4.9.4)}{\longrightarrow} \mathrm{Hom}_{A/I}(V_{A/I}, B^+_{\mathrm{cris},A/I})$$

is injective.

One can check that there is a natural isomorphism $D_A \otimes_A A/I \xrightarrow{\sim} D_{A/I}$, with an implicit choice of $I^\circ \subset A^\circ$ such that $A^\circ/I^\circ[1/p] \cong A/I$ needed to draw the map. It remains to show that that natural map $B^+_{\mathrm{cris},A} \otimes_A A/I \to B^+_{\mathrm{cris},A/I}$ is an isomorphism. This is precisely what Lemma 4.8.2(4) tells us, completing the proof. □

Proposition 4.9.11 (Following [Kis08, Proposition 2.4.7]). *The functor which to an A-algebra B assigns the collection of W_B-linear maps $N : D_B \to D_B$ which satisfy $p\varphi N = N\varphi$ and such that* (4.9.4) *is compatible with the action of $\hat{\Gamma}$ is representable by a quotient A^{st} of A.*

This proof requires very little modification from that of [Kis08].

PROOF. Since we will require a basis for V_A, we replace A with the coordinate ring of a trivialization of V_A if necessary, and note that the constructions to produce A^{st} can be glued together.

We consider the functor sending an A-algebra B to the set of W_B-linear maps

$$N : D_B \to D_B$$

satisfying $p\varphi N = N\varphi$. This is a closed condition on the representable functor $B \to \mathrm{End}_B(D_A \otimes_A B)$. Write A^N for the representing A-algebra, which is of finite presentation over A.

Write ψ_B for the map of (4.9.4). Let $B = A^N$ with the universal map $N : D_{A^N} \to D_{A^N}$, and let $\hat{\Gamma}$ act on $D_{A^N} \otimes_{W_{A^N}} B^+_{\mathrm{cris},A^N}$ according to (4.9.5). For any $d \in D_{A^N}$ and $\sigma \in \hat{\Gamma}$, let $\delta_\sigma(d)$ measure the failure of ψ_{A^N} to commute with the action of $\hat{\Gamma}$ as follows:

$$\delta_\sigma(d) := \psi_{A^N}(\sigma(d)) - \sigma(\psi_{A^N}(d)).$$

Then $\delta_\sigma(d) \in Q := \mathrm{Hom}_{A^N}(V_{A^N}, B^+_{\mathrm{cris},A^N})$. Fix a B^+_{cris,A^N}-basis for Q, and let $x_1, \ldots, x_r$ denote the coordinates of $\delta_\sigma(d)$ with respect to this basis. Applying Lemma 4.8.5 with $M = A^N$ and $x = x_i$ for $i = 1, \ldots, r$, we obtain A° submodules $N(x_i) \subset A^N$. Let $I_{\sigma,d} \subset A^N$ be the ideal generated by the $N(x_i)$, so that $I_{\sigma,d}$ is the smallest ideal $I \subset A^N$ such that $\delta_\sigma(d) \in IQ$. We take

$$A^{\mathrm{st}} := A^N / \sum_{\sigma,d} I_{\sigma,d},$$

where d runs over all elements of D_{A^N} and σ over $\hat{\Gamma}$. Clearly if $B = A^{\mathrm{st}}$ with the induced N from A^N, then $\psi_{A^{\mathrm{st}}}$ is Galois -equivariant. We must show that this property holds functorially on A-algebras.

If B is an A-algebra, then a map $A^N \to B$ factors through A^{st} if and only if the kernel K contains $I_{\sigma,d}$ for each $\sigma \in \hat{\Gamma}, d \in D_{A^N}$. Since Q is faithfully flat over A^N by Lemma 4.8.1(2), it is the same to ask that for all σ, d that $I_{\sigma,d}Q \subset KQ$, or equivalently that $\delta_\sigma(d) \in KQ$ for

all σ, d. As noted above, this last condition amounts to saying that ψ_B is compatible with the action of $\hat{\Gamma}$. Hence A^{st} represents the functor as claimed in the statement of the proposition. Now we must show that $\operatorname{Spec} A^{\mathrm{st}}$ is a closed subscheme of $\operatorname{Spec} A$, which we will accomplish by showing that it is a proper monomorphism. This remaining work is no different than what was done in [Kis08, Proposition 2.4.7], so we include it only for convenience.

To show that $\operatorname{Spec} A^{\mathrm{st}} \to \operatorname{Spec} A$ is a monomorphism, we show, given two maps $N, N' : D_B \to D_B$ satisfying the conditions of the proposition, that they are equal. Since Lemma 4.9.9 provides a canonical injection into a Galois module, the Galois action on $D_B \otimes_{W_B} B^+_{\mathrm{cris},B}$ induced by N in (4.9.5) is identical to that of N'. Thus for $\sigma \in \hat{\Gamma}, d \in D_B$, we have an equality

$$d = \exp(N \otimes \beta(\sigma)) \cdot \exp(-N' \otimes \beta(\sigma)) = \exp((N - N') \otimes \beta(\sigma))d$$

of elements of $D_B \otimes_{W_B} B^+_{\mathrm{cris},B}$. Recall from (4.6.5) that $\beta(\sigma) \in \operatorname{Fil}^1 B^+_{\mathrm{cris},B}$. Recalling also the multiplicativity of the filtration (4.6.2), we see that

$$(N - N')(d)\beta(\sigma) \cong 0 \text{ modulo } \operatorname{Fil}^2 B^+_{\mathrm{cris},B}.$$

Then by (4.6.5) and the second part of Lemma 4.8.2, $\beta(\sigma) \notin \operatorname{Fil}^2 B^+_{\mathrm{cris},B}$ whenever $\sigma \notin \hat{\Gamma}_\infty$, so $N = N'$ as desired.

Now we will check the valuative criterion of properness. Suppose that the A-algebra B is a discrete valuation ring with uniformizer π_B. Let $N : D_{B[1/\pi_B]} \to D_{B[1/\pi_B]}$ be an endomorphism satisfying the conditions of the proposition. Let $\sigma \in \hat{\Gamma}$ be such that $\beta(\sigma) \neq 0$. If $d \in D_B$, then

$$\exp(N \otimes \beta(\sigma)) \cdot d \in D_B \otimes_{W_B} B^+_{\mathrm{cris},B}[1/\pi_B] \cap \operatorname{Hom}_B(V_B, B^+_{\mathrm{cris},B}), \tag{4.9.12}$$

using (4.9.4) to consider both $D_B \otimes_{W_B} B^+_{\mathrm{cris},B}[1/\pi_B]$ and $\operatorname{Hom}_B(V_B, B^+_{\mathrm{cris},B})$ as a subset of $\operatorname{Hom}_{B[1/\pi_B]}(V_{B[1/\pi_B]}, B^+_{\mathrm{cris},B[1/\pi_B]})$. Since the cokernel of (4.9.4) is flat over B by Lemma 4.9.9, it has no π_B-torsion so that the intersection in (4.9.12) is the isomorphic image of $D_B \otimes_{W_B} B^+_{\mathrm{cris},B}$.

Therefore, modulo the ideal $\operatorname{Fil}^2 B^+_{\mathrm{cris},B} \subset B^+_{\mathrm{cris},B}$,

$$d - \exp(N \otimes \beta(\sigma))d \cong -N(d) \otimes \beta(\sigma) \in D_B \otimes_{W_B} B^+_{\mathrm{cris},B}/\operatorname{Fil}^2 B^+_{\mathrm{cris},B}.$$

The first part of Lemma 4.8.2(3) may be used to show that $N(d) \in D_B$ as follows: in this diagram, (where we use Fil^2 as an abbreviation for $\operatorname{Fil}^2 B^+_{\mathrm{cris},B}$ or $\operatorname{Fil}^2 B^+_{\mathrm{cris},B[1/\pi_B]}$ as appropriate, write $F = B[1/\pi_B]$, and we assume $\sigma \notin \hat{\Gamma}_\infty$)

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
D_B & \longrightarrow & D_F \\
\downarrow{\scriptstyle \cdot\beta(\sigma)} & & \downarrow{\scriptstyle \cdot\beta(\sigma)} \\
D_B \otimes_{W_B} B^+_{\mathrm{cris},B}/\operatorname{Fil}^2 & \longrightarrow & D_F \otimes_{W_F} B^+_{\mathrm{cris},F}/\operatorname{Fil}^2 \\
\downarrow & & \downarrow \\
D_B \otimes_{W_B} B^+_{\mathrm{cris},B}/(\beta(\sigma)\cdot B + \operatorname{Fil}^2) & \longrightarrow & D_F \otimes_{W_F} B^+_{\mathrm{cris},F}/(\beta(\sigma)\cdot F + \operatorname{Fil}^2) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where both columns are exact and all of the horizontal maps are injective, with Lemma 4.8.2(3) being used to show that the lowest horizontal map is injective. Now we know that $N(d) \otimes \beta(\sigma)$ lies in the image of the middle horizontal map. Since all of the horizontal maps are injective, $N(d) \in D_B$. Therefore we see that N induces a map $N : D_B \to D_B$ as desired. This endomorphism will satisfy the conditions of the proposition because it does so after extending scalars to $F = B[1/\pi_B]$. □

4.10. Moduli Space of Semistable Representations

Because this section deals entirely with coefficient rings being finite $\mathbb{Q}_p$-algebras, there is no fundamentally new content. We simply reprise [Kis08, §2.5].

Our goal here is to show that A^{st} is the maximal quotient of A over which the representation V_A is semistable with Hodge-Tate weights in $[0, h]$, in the sense that for any A-algebra

B which is finite as a $\mathbb{Q}_p$-algebra, the representation $V_A \otimes_A B$ is semi-stable if and only if $A \to B$ factors through A^{st}. In order to prove this, we recall the following relations between weakly admissible filtered (φ, N)-modules and $\mathfrak{S}$-lattices of finite E-height, due to Kisin [Kis06].

THEOREM 4.10.1 ([Kis06], e.g. Corollary 1.3.15). *Let D be a weakly admissible filtered (φ, N)-module with $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^{h+1} D = 0$. Then there is a finite free $\mathfrak{S}[1/p]$-module $\mathfrak{M}$ and a map $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ whose cokernel is killed by $E(u)^h$ such that*

(1) There is a canonical φ-equivariant isomorphism $\mathfrak{M}/u\mathfrak{M} \xrightarrow{\sim} D$.

(2) If $\mathcal{M} := \mathfrak{M} \otimes_{\mathfrak{S}[1/p]} \mathcal{O}$, then $\mathcal{M}$ admits a unique logarithmic connection

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}}[1/u\lambda]$$

such that $\nabla \circ \varphi = \varphi \circ \nabla$ and induces a differential operator

$$N_\nabla : \mathcal{M} \to cM, \quad m \mapsto \langle \nabla(m), -u\lambda \frac{d}{du} \rangle$$

such that $N_\nabla|_{u=0} = N$.

(3) $\mathfrak{M}$ admits a lattice, i.e. there exists a finite free $\mathfrak{S}$-module $\mathfrak{M}^\circ$ which spans $\mathfrak{M}$ and such that the cokernel of $1 \otimes \varphi : \varphi^*(\mathfrak{M}^\circ) \to \mathfrak{M}^\circ$ is killed by $E(u)^h$.

There is also a $\hat{\Gamma}_\infty$-equivariant isomorphism

$$\mathrm{Hom}_{\mathfrak{S}[1/p],\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi,N}(D, B^+_{\mathrm{st}}) \tag{4.10.2}$$

which is constructed using some maps that have already appeared in this text (see [Kis08, p. 18] for the recipe). There is also an isomorphism

$$\mathrm{Hom}_{B^+_{\mathrm{cris}},\mathrm{Fil},\varphi}(D \otimes_{K_0} B^+_{\mathrm{cris}}, B^+_{\mathrm{cris}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi}(D, B^+_{\mathrm{cris}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi,N}(D, B^+_{\mathrm{st}}) \tag{4.10.3}$$

and an isomorphism

$$\mathrm{Hom}_{\mathfrak{S}[1/p],\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi}(D, B^+_{\mathrm{cris}}) \tag{4.10.4}$$

constructed as in [Kis08, p. 18].

We now show that the candidate A^{st} of Proposition 4.9.11 for the moduli space of semi-simple representations satisfies this property.

PROPOSITION 4.10.5 ([Kis08, Proposition 2.5.4]). *Assume that $A^\circ = (A^\circ)^{\leq h}$. Let B be a finite $\mathbb{Q}_p$-algebra, $\zeta : A \to B$ a map of $\mathbb{Q}_p$-algebras, and $V_B := V_A \otimes_A B$. Then ζ factors through A^{st} if and only if V_B is semistable as a representation of $\hat{\Gamma}$ over $\mathbb{Q}_p$.*

PROOF. Suppose that ζ factors through A^{st}. Then Proposition 4.9.11 implies that (4.9.7) is a $\hat{\Gamma}$-equivariant map

$$D_B \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}} \cong D_B \otimes_{W_B} B^+_{\mathrm{st},B} \longrightarrow V_B^* \otimes B^+_{\mathrm{st},B} \cong V_B \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}}.$$

which is injective according to Lemma 4.9.9. We get an injection of Galois invariants

$$D_B \hookrightarrow (V_B \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}})^{\hat{\Gamma}}$$

so that the dimension of the right side as a K_0-vector space is at least as much as that of the left side. But $\dim_{K_0} D_B = \dim_{\mathbb{Q}_p} V_B$. Therefore $\dim_{\mathbb{Q}_p} V_B \geq \dim_{K_0}(V_B \otimes_{\mathbb{Q}_p} B^+_{\mathrm{st}})^{\hat{\Gamma}}$, so that V_B is semistable.

Now suppose that V_B is semistable. Let

$$\tilde{D}_B := \mathrm{Hom}_{B[\hat{\Gamma}]}(V_B, B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} B)$$

be the weakly admissible filtered (φ, N)-module associated to V_B^*. Denote by $\tilde{\mathfrak{M}}_B$ the $\mathfrak{S}[1/p]$-module attached to $\tilde{D}_B$ according to the discussion at the beginning of §4.10. Let $\mathfrak{M}_B := \mathfrak{M}_A \otimes_A B$ as usual, where $\mathfrak{M}_{A^\circ}$ was produced in Proposition 4.5.9 and $\mathfrak{M}_A := \mathfrak{M}_{A^\circ} \otimes_{A^\circ} A$. Composing the map $\iota^{-1} \otimes_A B$ of (4.9.1) with (4.10.2) and taking B-linear maps, we find that

$$\mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\mathfrak{M}_B, \mathfrak{S}_B^{\mathrm{ur}}) \xrightarrow{\sim} V_B \xrightarrow{\sim} \mathrm{Hom}_{B,\mathrm{Fil},\varphi,N}(\tilde{D}_B, B^+_{\mathrm{st},B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\tilde{\mathfrak{M}}_B, \mathfrak{S}_B^{\mathrm{ur}}). \tag{4.10.6}$$

Because $\mathfrak{S}$-lattices of height $\leq h$ are unique in $M_B = (\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V_B^*)^{\hat{\Gamma}}$ by [Kis06, Proposition 2.1.12], we may identify $\mathfrak{M}_B$ and $\tilde{\mathfrak{M}}_B$. Using this identification and the maps in

(4.10.6), Theorem 4.10.1 allows us to identify $\tilde{D}_B$ with $D_B = \mathfrak{M}_B/u\mathfrak{M}_B$ (W_B-linearly and φ-equivariantly), and then endow D_B with an operator N coming from the operator on $\tilde{D}_B$.

We then have a commutative diagram, where the top right horizontal arrow comes from (4.10.4).

(4.10.7)

$$\begin{array}{ccccc} V_B & \xrightarrow{\sim} & \mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\tilde{\mathfrak{M}}_B, \mathfrak{S}_B^{\mathrm{ur}}[1/p]) & \xrightarrow{\sim} & \mathrm{Hom}_{B^+_{\mathrm{cris},B},\mathrm{Fil},\varphi}(\tilde{D}_B \otimes_{W_B} B^+_{\mathrm{cris},B}, B^+_{\mathrm{cris},B}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sim} & & \downarrow \\ V_B & \xrightarrow{\sim} & \mathrm{Hom}_{\mathfrak{S}_B,\varphi}(\mathfrak{M}_B, \mathfrak{S}_B^{\mathrm{ur}}[1/p]) & \longrightarrow & \mathrm{Hom}_{B^+_{\mathrm{cris},B},\mathrm{Fil},\varphi}(D_B \otimes_{W_B} B^+_{\mathrm{cris},B}, B^+_{\mathrm{cris},B}) \end{array}$$

The maps in the top horizontal row are $\hat{\Gamma}$-equivariant. Observing the diagram, it follows that the same holds for the maps in the bottom row. The composite of the bottom row maps induces a $\hat{\Gamma}$-equivariant map

$$D_B \otimes_{W_B} B^+_{\mathrm{cris},B} \to \mathrm{Hom}_B(V_B, B^+_{\mathrm{cris},B}). \tag{4.10.8}$$

We claim that this map is identical to that of (4.9.4). Since (4.9.4) is $\hat{\Gamma}$-equivariant, it follows by Proposition 4.9.11 that ζ factors through A^{st}. □

THEOREM 4.10.9. *As is standard in this section, let A° be an algebra formally finitely generated over R, with a continuous action of $\hat{\Gamma}$ on a projective rank d A° module V_{A°. If h is a non-negative integer, then there exists a quotient $A^{\mathrm{st},h}$ of A such that*

1. *If B is a finite $\mathbb{Q}_p$-algebra, and $\zeta : A \to B$ a map of $\mathbb{Q}_p$-algebras, then ζ factors through $A^{\mathrm{st},h}$ if an only if $V_B = V_A \otimes_A B$ is semistable with Hodge-Tate weights in $[0, h]$.*
2. *There is a projective $W_{A^{\mathrm{st},h}}$-module $D_{A^{\mathrm{st},h}}$ of rank d equipped with a Frobenius semi-linear automorphism φ and with a $W_{A^{\mathrm{st},h}}$-linear automorphism N such that for all $\zeta : A \to B$ factoring through $A^{\mathrm{st},h}$, there is a canonical isomorphism*

$$D_B = D_{A^{\mathrm{st},h}} \otimes_{A^{\mathrm{st},h}} B \xrightarrow{\sim} \mathrm{Hom}_{B[\hat{\Gamma}]}(V_B, B^+_{st} \otimes_{\mathbb{Q}_p} B)$$

respecting the action of φ and N.

PROOF. Assume that V_B is semistable with Hodge-Tate weights in $[0,h]$. Then V_B is of E-height $\leq h$: for we call V_B of E-height $\leq h$ when the cokernel associated map of $\mathfrak{S}_B$-modules $\varphi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ of Proposition 4.5.9(1) is killed by $E(u)^h$. The proof of Proposition 4.10.5 identifies this $\mathfrak{S}$-module with another one, denoted $\tilde{\mathfrak{M}}_B$, created from the (φ, N) module associated to V_B. Now [Kis06, Lemma 1.2.2] associates the Hodge-Tate weights of the (φ, N)-module with the cokernel in the way that we require.

Because V_B is of E-height $\leq h$, we know that $A \to B$ factors through $(A^\circ)^{\leq h}[1/p]$. Therefore we may replace A° by the quotient $(A^\circ)^{\leq h}$ defined in Proposition 4.5.5. Let $A^{\mathrm{st},h}$ be the ring A^{st} of Proposition 4.9.11 and set $D_{A^{\mathrm{st}}} := \mathfrak{M}_A/u\mathfrak{M}_A \otimes_A A^{\mathrm{st}}$. If V_B is semistable then ζ factors through A^{st} by Proposition 4.10.5.

Conversely, if ζ factors through A^{st} then Proposition 4.10.5 implies that V_B is semistable of E-height $\leq h$. If $\tilde{D}_B := (V_B^* \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}^+)^{\hat{\Gamma}}$ and $\tilde{\mathfrak{M}}_B$ is the $\mathfrak{S}[1/p]$-module associated to $\tilde{D}_B$ as in the proof of Proposition 4.10.5, then the uniqueness of $\mathfrak{S}$-lattices of finite E-height [Kis06, Proposition 2.1.12] implies that $\tilde{\mathfrak{M}}_B$ has E-height $\leq h$, and the claim about Hodge-Tate weights once more follows from [Kis06, Lemma 1.2.2].

To see (2), concatenate the $\hat{\Gamma}$-equivariant isomorphisms

$$V_B \xrightarrow{\sim} \mathrm{Hom}_{B^+_{\mathrm{cris},B},\mathrm{Fil},\varphi}(\tilde{D}_B \otimes_{W_B} B^+_{\mathrm{cris},B}, B^+_{\mathrm{cris},B}) \xrightarrow{\sim} \mathrm{Hom}_{B,\mathrm{Fil},\varphi,N}(D_B, B^+_{\mathrm{st},B}), \tag{4.10.10}$$

where the first isomorphism appears on the top line of (4.10.7) and is deduced from in (4.10.3) by taking B-linear maps. Now (2) follows from applying $\mathrm{Hom}_{B[\hat{\Gamma}]}(-, B^+_{\mathrm{st},B})$ and the fact that this functor is inverse to $\mathrm{Hom}_{B,\mathrm{Fil},N,\varphi}(-, B^+_{\mathrm{st},B})$. □

4.11. Hodge Type

In this section we follow [Kis08, §2.6] and the erratum [Kis09b, §A.4] to construct a quotient of $A^{\mathrm{st},h}$ corresponding to semistable representations with a specified p-adic Hodge type. First we recall the notion of p-adic Hodge type. For this, we fix an finite extension field E of $\mathbb{Q}_p$ and suppose that A admits the structure of an E-algebra.

DEFINITION 4.11.1. Suppose we are given a finite dimensional E-vector space D_E with a filtration of $D_{E,K} := D_E \otimes_{\mathbb{Q}_p} K$ by $E \otimes_{\mathbb{Q}_p} K$-submodules such that the associated graded is concentrated in degrees $[0, h]$. Let $\mathbf{v} := \{D_E, \mathrm{Fil}^i D_{E,K}, i = 0, \ldots, h\}$.

If B is a finite E-algebra and $V_B \in \mathrm{Rep}^d_{\hat{\Gamma}}(B)$ such that V_B is a de Rham representation, then we say that V_B is of p-adic Hodge type $\mathbf{v}$ if the Hodge filtration on the associated filtered (φ, N)-module has the same graded degrees as $\mathbf{v}$. That is, V_B has all its Hodge-Tate weights in $[0, h]$ and for $i = 0, \ldots, h$ there is an isomorphism of $B \otimes_{\mathbb{Q}_p} K$-modules

$$\mathrm{gr}^i \mathrm{Hom}_{B[\hat{\Gamma}]}(V_B, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} B) \xrightarrow{\sim} \mathrm{gr}^i D_{E,K} \otimes_E B.$$

THEOREM 4.11.2 ([Kis08, Corollary 2.6.2]). *With* $\mathbf{v}$ *as above, there exists a quotient* $A^{\mathrm{st},\mathbf{v}}$ *of* A^{st} *corresponding to a union of connected components of* $\mathrm{Spec}\, A^{\mathrm{st}}$ *with the following property. If* B *is a finite* E*-algebra and* $\zeta : A \to B$ *is a map of* E*-algebras, then* ζ *factors through* $A^{\mathrm{st},\mathbf{v}}$ *if and only if* $V_B = V_A \otimes_A B$ *is semistable of* p*-adic Hodge type* $\mathbf{v}$.

PROOF. To begin with, we establish some notation. Let

$$\mathrm{Fil}^i \varphi^*(\mathfrak{M}_A) = (1 \otimes \varphi)^{-1}(E(u)^i \mathfrak{M}_A) \subset \varphi^*(\mathfrak{M}_A). \tag{4.11.3}$$

The second part of Lemma 4.7.1 identifies $D_B \otimes_{W_B} \mathcal{O}_B$ and $\varphi^*(\mathfrak{M}_B \otimes_{\mathfrak{S}_B} \mathcal{O}_B$ in a formal neighborhood of the ideal $(E(u)) \subset \mathcal{O}_B$. In particular, we have a φ-compatible identification

$$D_B \otimes_{K_0} K \xrightarrow{\sim} \varphi^*(\mathfrak{M}_B)/E(u)\varphi^*(\mathfrak{M}_B).$$

From here we equip $D_B \otimes_{K_0} K$ with a filtration, setting

$$\mathrm{Fil}^i(D_B \otimes_{K_0} K) := \mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)) \otimes_A B. \tag{4.11.4}$$

Next we check that this filtration on $D_B \otimes_{K_0} K$ coincides with the one induced by the isomorphism of Theorem 4.10.9(2), namely

$$D_B \xrightarrow{\sim} \mathrm{Hom}_{B[\hat{\Gamma}]}(V_B, B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} B.$$

Write $\tilde{D}_B$ for D_B equipped with the filtration from Theorem 4.10.9(2). This is the standard weakly admissible (φ, N) module over B attached to V_B^*. Now let $\tilde{\mathfrak{M}}_B$ be the $\mathfrak{S}[1/p]$-module attached to $\tilde{D}_B$ as summarized in Theorem 4.10.1. The uniqueness of lattices of E-height $\leq h$ implies that $\mathfrak{M}_B$ may be identified with $\mathfrak{M}_A \otimes_A B$. We conclude the proof by recalling [Kis06, 1.2.6-1.2.7], which reconstructs the filtration on $\tilde{D}_B \otimes_{K_0} K$ from $\tilde{\mathfrak{M}}_B$ as the preimage filtration on $D_B \otimes_{W_B} \mathcal{O}_B$ the filtration defined in (4.11.3) (with $\tilde{\mathfrak{M}}_B$ in the place of $\mathfrak{M}_A$) under the map ξ of Lemma 4.7.1, specialized at $\mathcal{O}_B/E(u)\mathcal{O}_B$. This is precisely the same as the filtration of (4.11.4), as desired.

Notice that we have identified $\mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)) \otimes_A B$, which was originally used to define the filtration on $D_B \otimes_{K_0} K$, with the a priori projective $W_B \otimes_{K_0} K$-module $\mathrm{Fil}^i(\tilde{D}_B \otimes_{K_0} K)$. Since this is valid for all finite local E-algebras B, this implies that $\mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i \varphi^*(\mathfrak{M}_A))$ is a projective A-module. Moreover, the discussion above shows that, over A-algebras B that are finite E-algebras, the graded components of these modules determine the Hodge type of the associated Galois representation $V_B = V_A \otimes_A B$. Since projective modules have locally constant rank, this shows that the points of $\operatorname{Spec} A$ corresponding to a given hodge type $\mathbf{v}$ form a union of connected components, whose coordinate ring we will denote by $A^{\mathbf{v}}$. Namely, these are points $\mathfrak{p}$ of $\operatorname{Spec} A$ such that for $i = 0, 1, \ldots, h$, there is an isomorphism of $W_{A_\mathfrak{p}} \otimes_{K_0} K$-modules

$$\mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i \varphi^*(\mathfrak{M}_A)) \otimes_A A_\mathfrak{p} \xrightarrow{\sim} \mathrm{Fil}^i D_{E,K} \otimes_E A_\mathfrak{p}.$$

Let $A^{\mathrm{st},\mathbf{v}} := A^{\mathrm{st},h} \otimes_A A^{\mathbf{v}}$. □

4.12. Galois Type

In this section we further stipulate that B is local with residue field E, so that it is a finite, local E-algebra with residue field E. Let $V_B \in \mathrm{Rep}^d_{\hat{\Gamma}}(B)$. Following [Fon94], set

$$D^*_{\mathrm{pst}}(V_B) = \varinjlim_{K'} \mathrm{Hom}_{B[\hat{\Gamma}_{K'}]}(V_B, B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} B),$$

where K' runs over finite field extensions of K.

Let $\bar{K}_0 \subset \bar{K}$ denote the maximal unramified extension of K_0, and let $\hat{\Gamma}_0 \subset \hat{\Gamma}$ be the inertia group of $\hat{\Gamma}$. Then $D^*_{\mathrm{pst}}(V_B)$ is a $B \otimes_{\mathbb{Q}_p} \bar{K}_0$-module with a Frobenius semi-linear Frobenius automorphism φ, a nilpotent endomorphsm N such that $p\varphi N = N\varphi$, and a $B \otimes_{\mathbb{Q}_p} \bar{K}_0$-linear action of $\hat{\Gamma}_0$ which has open kernel and commutes with φ and N.

We claim that $D^*_{\mathrm{pst}}(V_B)$ is finite and free as a $B \otimes_{\mathbb{Q}_p} \bar{K}_0$-module. We will show this using the fact that φ is an automorphism and following the line of reasoning of [Kis09c, Lemma 1.2.2(4)]. Firstly, we know that this module is finite and flat over $B \otimes_{\mathbb{Q}_p} \bar{K}_0$ by definition of the functor. To check that $D^*_{\mathrm{pst}}(V_B)$ is free, we need only check that the fibers over the residue fields, i.e. the points of $\operatorname{Spec} E \otimes_{\mathbb{Q}_p} \bar{K}_0$, are of constant rank. This module arises by $\otimes_{K'_0} \bar{K}_0$ from a free, rank d (φ, N)-module D over $E \otimes_{\mathbb{Q}_p} K'_0$, where K'/K is a finite extension making V_B semistable as a representation of $\hat{\Gamma}_{K'}$. Since an unramified base change cannot make "potentially semistable" into "semistable," we may assume that $K'_0 = K_0$. For any unramified extension L_0/K_0, now let K'_0 (resp. L'_0) denote $K_0 \cap E$ (resp. $L_0 \cap E$), and also let E_0 be the maximal subfield of E unramified over $\mathbb{Q}_p$. We observe that φ permutes the factors labeled by μ of the decomposition

$$E \otimes_{\mathbb{Q}_p} L_0 \cong \prod_{\mu} E \otimes_{\mathbb{Q}_p} K_0,$$

where μ runs over the set of embeddings $\{\mu : E_0 \hookrightarrow L'_0 \text{ fixing } K'_0\}$. This shows that φ will permute the factors of $D \otimes_{K_0} L_0$ under this decomposition by $\{\mu\}$, and each of these factors is free of rank d. Therefore $D^*_{\mathrm{pst}}(V_B)$ is free of rank d as a $B \otimes_{\mathbb{Q}_p} \bar{K}_0$-module, as desired.

Since the action of $\hat{\Gamma}_0$ commutes with the action φ, the traces of elements of $\hat{\Gamma}_0$ are contained in B, and D^*_{pst} descends to a representation of $\hat{\Gamma}_0$ on a finite free B-module $\tilde{P}_B$. Because characteristic zero representations of finite groups are rigid, i.e. the deformations of an action of a finite group on a vector space over a characteristic zero field E to artinian E-algebras arise by extension of scalars, this representation must be an extension of scalars from a representation P_B of $\hat{\Gamma}_0$ over E.

We have associated to a potentially semistable d-dimensional representation V_B of $\hat{\Gamma}$ over B a representation of the inertia group of K over E which reflects the failure of V_B to be semistable. We will call this the "Galois type" of V_B, as follows.

Fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

DEFINITION 4.12.1. Let $T : \hat{\Gamma}_0 \to \mathrm{GL}_d(\bar{\mathbb{Q}}_p)$ be a representation with open kernel. We say that V_B is *potetially semistable of type* T provided that P_B defined above is isomorphic to T.

Because we are working over a characteristic 0 field, it is equivalent to say that for any $\gamma \in \hat{\Gamma}_0$, the trace of $T(\gamma)$ is equal to the trace of γ on $D^*_{\mathrm{pst}}(V_B)$.

Our goal is to find a moduli space for Galois representations that are both potentially semistable and have Galois type T. Before we give a supporting lemma, we recall that the element $t \in A_{\mathrm{cris}} \subset B^+_{\mathrm{dR}}$, which generates the maximal ideal of B^+_{dR}, is used in the definitions $B_{\mathrm{st}} = B^+_{\mathrm{st}}[1/t]$ and $B_{\mathrm{st},A} = B^+_{\mathrm{st},A}[1/t]$ (see §4.6).

LEMMA 4.12.2 (Following [Kis08, Lemma 2.7.1]). *For $i \geq 0$ there is an isomorphism*

$$W_A \cdot t^i \xrightarrow{\sim} \mathrm{Hom}_{A[\hat{\Gamma}]}(A(i), B^+_{\mathrm{st},A})$$

induced by multiplication by p^{-r_i} for r_i a positive integer defined below, where $A(i)$ denotes A with $\hat{\Gamma}$ acting via the ith power of the p-adic cyclotomic character χ. In particular, if $B_{\mathrm{st},A} := B^+_{\mathrm{st},A}[1/t]$, then $B^{\hat{\Gamma}}_{\mathrm{st},A} = W_A$.

This proof was done for local A° in [Kis08], based on the well known case when A° is a finite flat $\mathbb{Z}_p$-algebra. The general case requires additional notions, much along the lines of Lemma 4.8.2(5).

PROOF. First we will show, following [Kis08], that any element $x \in B^+_{\mathrm{st},A}$ such that $\hat{\Gamma}$ acts on x via χ^i lies in $B^+_{\mathrm{cris},A}$. We may represent x as $x = \sum_{i=0}^n a_i \ell_u^i$ where $a_i \in B^+_{\mathrm{cris},A}$. As the lemma is well known for A° finite over $\mathbb{Z}_p$, one can apply Lemma 4.8.2(5) to conclude that $a_i = 0$ for $i > 0$. Therefore, replacing x with a multiple of itself by a power of p, we see

that it suffices to prove that $x \in A_{\mathrm{cris},A^\circ}$ such that $\hat{\Gamma}$ acts on it via χ^i lies in

$$W_{A^\circ} \cdot \frac{t^i}{p^{r_i}} \subset A_{\mathrm{cris},A^\circ}, \tag{4.12.3}$$

where r_i is the largest non-negative integer such that $t^i/p^{r_i} \in B^+_{\mathrm{cris}}$ lies in A_{cris}.

In the case that A° is a complete local Noetherian $\mathbb{Z}_p$-algebra with finite residue field, this Lemma was proved in [Kis08]. We will reduce the proof to this case, and then recapitulate the proof from [Kis08] afterwards.

First we note that if r_i is the integer defined for (4.12.3), then the cokernel of the map of $\mathbb{Z}_p$-modules

$$\begin{aligned} W &\longrightarrow A_{\mathrm{cris}} \\ x &\mapsto x \cdot \frac{t^i}{p^{r_i}} \end{aligned} \tag{4.12.4}$$

is torsion-free, and therefore also flat as a $\mathbb{Z}_p$-module. Since r_i is chosen to saturate the sub-$\mathbb{Z}_p$-module $W \cdot t^i \subset A_{\mathrm{cris}}$, this is clear enough: choose a representative $y \in A_{\mathrm{cris}}$ of a nonzero element of the cokernel of this map. If y is a torsion element of the cokernel, then there exists some positive integer n such that $p^n \cdot y \cong x \cdot \frac{t^i}{p^{r_i}}$ for some $x \in W$ but $p^{n-1} \cdot y$ does not lie in $W \cdot \frac{t^i}{p^{r_i}}$. But since A_{cris} is a flat $\mathbb{Z}_p$-module, this would imply that x is a unit in W, and without loss of generality $x = 1$. But then $p^{n-1}y \cong \frac{t^i}{p^{r_i+1}} \in A_{\mathrm{cris}}$, a contradiction.

Secondly, we note that the image of (4.12.3) lies in the submodule $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ$ of $A_{\mathrm{cris},A^\circ}$. Recall that since A_{cris} is p-adically complete, the natural map $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ \to A_{\mathrm{cris},A^\circ}$ is indeed an inclusion. Also observe that the natural map $W \otimes_{\mathbb{Z}_p} A^\circ \to W_{A^\circ}$ is an isomorphism: since $W/\mathbb{Z}_p$ is finite, $W \otimes_{\mathbb{Z}_p} A^\circ$ is $\mathfrak{m}_R A$-adically complete. Therefore we can factor the inclusion (4.12.3) as the composition of natural inclusions

$$\begin{aligned} W \otimes_{\mathbb{Z}_p} A^\circ &\hookrightarrow A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ \\ x \otimes y &\mapsto x \cdot \frac{t^i}{p^{r_i}} \otimes y \end{aligned} \tag{4.12.5}$$

followed by the inclusion

$$A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ \hookrightarrow A_{\mathrm{cris},A^\circ}.$$

Recall from the proof of Lemma 4.8.2(5) the following notions: let M be the set of maximal ideals of $\mathrm{Spf}(A^\circ)$ as a $\mathrm{Spf}(\mathbb{Z}_p)$-formal scheme, corresponding to maximal ideals of $A^\circ/\mathfrak{m}_R A^\circ$. In the proof, we showed that the natural maps

$$A^\circ \to \prod_{\mathfrak{m}\in M} A^\circ_{\mathfrak{m}}, \qquad A_{\mathrm{cris},A^\circ} \to \prod_{\mathfrak{m}\in M} A_{\mathrm{cris},A^\circ_{\mathfrak{m}}} \tag{4.12.6}$$

are injective. Of course, everything in the discussion about (4.12.5) applies to $A^\circ_{\mathfrak{m}}$ in the place of A°, so that for each $\mathfrak{m} \in M$ there are maps

$$W \otimes_{\mathbb{Z}_p} A^\circ_{\mathfrak{m}} \hookrightarrow A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ_{\mathfrak{m}} \tag{4.12.7}$$

factoring $W_{A^\circ_{\mathfrak{m}}} \cong W \otimes_{\mathbb{Z}_p} A^\circ_{\mathfrak{m}} \hookrightarrow A_{\mathrm{cris},A^\circ_{\mathfrak{m}}}$. Therefore, assuming the result of this lemma when A° is a complete local ring with finite residue field, we can deduce the general case (i.e. A° not necessarily local) from the truth of the result over complete local rings $A^\circ_{\mathfrak{m}}$ of A° as follows.

Consider the inclusions

$$\begin{array}{ccc} W \otimes_{\mathbb{Z}_p} A^\circ & \longrightarrow & \prod_{\mathfrak{m}\in M} W \otimes_{\mathbb{Z}_p} A^\circ_{\mathfrak{m}} \\ \Big\downarrow{\scriptstyle (4.12.5)} & & \Big\downarrow{\scriptstyle (4.12.7)} \\ A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ & \xrightarrow{(4.12.6)} & \prod_{\mathfrak{m}\in M} A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} A^\circ_{\mathfrak{m}} \end{array}$$

We will be done if we can show that the image of $W \otimes_{\mathbb{Z}_p} A^\circ$ in the bottom right is the intersection of the images of (4.12.7) and (4.12.6). Using the fact that the cokernel of (4.12.4) (and therefore the cokernel of the vertical maps as well) is flat, we apply Lemma 4.12.8 to draw this conclusion and finish the proof.

We have finished the deduction of the proof of the lemma in the general case from the proof in the case that A° is local. It remains to prove the case that A° is local, reprising [Kis08].

Let A° be a complete local Noetherian $\mathbb{Z}_p$-algebra with finite residue field $\mathbb{F}$ and maximal ideal $\mathfrak{m}$. Let $x \in B^+_{\mathrm{st},A}$ such that $\hat{\Gamma}$ acts on it by χ^i. The beginning of this proof has reduced our remaining work to the case that $x \in A_{\mathrm{cris},A^\circ}$; we must show that x is in the image of (4.12.3).

Let $\mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots$ be a decreasing sequence of ideals of A such that $\cap_{j=1}^\infty \mathfrak{q}_j = \{0\}$ and $A/\mathfrak{q}_j$ is a finite $W(\mathbb{F})[1/p]$-algebra. Let $\mathfrak{q}_j^\circ := A^\circ \cap \mathfrak{q}_j$. Then for each $m \geq 0$, it follows that we have $\mathfrak{q}_j^\circ \subset \mathfrak{m}_{A^\circ}^m$ for large enough j. Since A° is $\mathfrak{m}$-adically complete, $A^\circ \xrightarrow{\sim} \varprojlim_j A^\circ/\mathfrak{q}_j^\circ$. Moreover, $(\mathfrak{q}_j^\circ)$ is a sequence of ideals of definition for the topology on A°. Therefore

$$A_{\mathrm{cris},A^\circ} \xrightarrow{\sim} \varprojlim_j A_{\mathrm{cris},A^\circ/\mathfrak{q}_j^\circ}.$$

The same is true with W in place of A_{cris}.

Using the integer r_i defined for (4.12.3), then for all $j \geq 1$, the image of x in $A_{\mathrm{cris},A^\circ/\mathfrak{q}_j^\circ}$ is contained in the image of $W_{A^\circ} \cdot \frac{t^i}{p^{r_i}}$ in $A_{\mathrm{cris},A^\circ/\mathfrak{q}_j^\circ}$ because the lemma is known for A° finite over $\mathbb{Z}_p$. This property is stable under taking the inverse limit indexed by j, so that we conclude that $x \in W_{A^\circ} \cdot \frac{t^i}{p^{r_i}}$ as desired. □

Lemma 4.12.8. *Let R be a commutative ring and let M, N, S, and T be flat R-modules. Fix injective maps $M \hookrightarrow N$, $S \hookrightarrow T$ such that M is a pure submodule of N, i.e. the cokernel of the inclusion is flat. Consider $N \otimes_R S$, $M \otimes_R T$, and $M \otimes_R S$ as submodules of $N \otimes_R T$ under the natural inclusion maps. Then*

$$N \otimes_R S \cap M \otimes_R T = M \otimes_R S.$$

PROOF. Let L be the cokernel of $M \hookrightarrow N$, which is a flat R-module. Let U be the cokernel of $S \hookrightarrow T$. Then we have the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M \otimes_R S & \longrightarrow & M \otimes_R T & \longrightarrow & M \otimes_R U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N \otimes_R S & \longrightarrow & N \otimes_R T & \longrightarrow & N \otimes_R U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L \otimes_R S & \longrightarrow & L \otimes_R T & \longrightarrow & L \otimes_R U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let x be an element of $N \otimes_R S$ which is in the intersection described in the statement of the lemma. Then the image of x in $L \otimes_R T$ is 0, so that the image of x in $L \otimes_R S$ is also 0. Therefore x is in the image of $M \otimes_R S$ in $N \otimes_R S$, as desired. □

Now we show that the theory of (φ, N)-modules with coefficients functions as expected.

PROPOSITION 4.12.9 (Following [Kis08, Proposition 2.7.2]). *Suppose that $A = A^{\mathrm{st},h}$. Then the map*

$$D_A \otimes_{W_A} B_{\mathrm{st},A} \longrightarrow \mathrm{Hom}_A(V_A, B_{\mathrm{st},A}) \tag{4.12.10}$$

induced from (4.9.7) *by setting $B = A$ and tensoring by $\otimes_{B^+_{\mathrm{st},A}} B_{\mathrm{st},A}$ is an isomorphism. In particular,*

$$D_A \xrightarrow{\sim} \mathrm{Hom}_{A[\hat{\Gamma}]}(V_A, B_{\mathrm{st},A}). \tag{4.12.11}$$

This proof requires small modifications from that of [Kis08].

PROOF. Lemma 4.9.9 tells us that (4.12.10) is an injection. Furthermore, because $A = A^{\mathrm{st},h}$, Theorem 4.10.9 tells us that the right hand side and left hand side of (4.12.10) are

finite free $B_{\mathrm{st},A}$-modules of the same rank. Therefore it will suffice to show that this map induces an isomorphism on top exterior powers, and we freely restrict ourselves to the case that V_{A° is free of rank 1 over A°. We note that in either of these cases, V_{A° arises by extension of scalars from a complete local Noetherian ring. This is the case because the universal moduli space of 1-dimension representations of $\hat{\Gamma}$ is semi-local, and in particular, the underlying scheme is the disjoint union of spectra of finite fields. Therefore, once we show that (4.12.10) is stable under extension of coefficients in a sense we will define in a moment, we can resort with no concern to the arguments of [Kis08], which are working in the case that A° is a complete Noetherian local ring with finite residue field.

We will now show that the property that (4.12.10) is an isomorphism is preserved by extension of coefficients which are adic R-algebras, in a sense we now define. This extends an observation made at the beginning of the proof of [Kis08, Proposition 2.7.2]. Let $A^\circ \to A'^\circ$ be a map in the category of adic R-algebras, and write $A' := A'^\circ[1/p]$ as usual. We claim that the map (4.12.10) for $V_{A'}$, i.e. the representation arising by extension of scalars from $V_{A'^\circ} := V_{A^\circ} \otimes_{A^\circ} A'^\circ$, is obtained from (4.12.10) by extending scalars by $\otimes_{B_{\mathrm{st},A}} B_{\mathrm{st},A'}$. To see this, we observe that each of the factors of the map (namely (4.9.6), (4.9.2), which arises from Proposition 4.5.9, and the map ξ of Lemma 4.7.1) are compatible with the scalar extension process.

1-dimensional semistable representations are crystalline and crystalline characters are the product of an unramified character and a Lubin-Tate character[2] determined by the Hodge filtration. Therefore, $V_A|_{\hat{\Gamma}_0}$ is locally constant on $\operatorname{Spec} A$ because, according to Theorem 4.11.2, the Hodge type is constant on connected components of $\operatorname{Spec} A$. Replacing $\operatorname{Spec} A$ with one of its connected components, we may assume that $V_A|_{\hat{\Gamma}_0} \sim \theta|_{\hat{\Gamma}_0}$, where θ is the product of conjugates of Lubin-Tate characters. It will suffice to prove the proposition in two cases, $V_A \sim \theta$ and V_A an unramified character. This is the case because we may tensor the factors in (4.12.10) for $V_A \sim \theta$ with the factors for V_A unramified to derive the general case.

[2]See e.g. [Ser68, III.A.4] for a discussion of Lubin-Tate characters.

If $V_A \sim \theta$, then V_A arises by extension of scalars from a representation valued in the ring of integers of a finite extension of $\mathbb{Q}_p$. The observation about extension of scalars given above now allows us to assume that A° is such a ring of integers. Therefore this case follows directly from Theorem 4.10.9(2), as A is finite as a $\mathbb{Q}_p$-algebra.

Now for the unramified case, the filtration on D_A is trivial so $h = 0$. Let $\bar{k}$ be the residue field of $\bar{K}$. As a result (cf. [FO, §7.2.2]), the slope of D_A is zero so that (4.9.4) arises by extension of scalars $\otimes_A B^+_{\mathrm{cris},A}$ from an isomorphism

$$D_A \xrightarrow{\sim} \mathrm{Hom}_{A[\hat{\Gamma}]}(V_A, W(\bar{k})_A),$$

and therefore is an isomorphism as well.

Now we come to the second statement of the proposition. Lemma 4.12.2 gives us that $B^{\hat{\Gamma}}_{\mathrm{st},A} = W_A$, so that the usual regular G-ring formalism (e.g. [FO, §2]) will apply, and allow us to conclude that

$$D_A \xrightarrow{\sim} \mathrm{Hom}_{A[\hat{\Gamma}]}(V_A, B_{\mathrm{st},A}).$$

As elements of the image of D_A under (4.12.10) have image in $B^+_{\mathrm{st},A} \subset B_{\mathrm{st},A}$, (4.12.11) follows. □

Recall that we have fixed E as a finite extension of $\mathbb{Q}_p$ such that A admits the structure of a E-algebra. Let $\mathbf{v}$ be a p-adic Hodge type as in Definition 4.11.1. We fix a representation

$$T : \hat{\Gamma}_0 \to \mathrm{End}_E(D_E) \xrightarrow{\sim} \mathrm{GL}_d(E).$$

THEOREM 4.12.12 ([Kis08, Theorem 2.7.6]). *There exists a quotient $A^{T,\mathbf{v}}$ of A such that for any finite E-algebra B, a map of E-algebras $\zeta : A \to B$ factors through $A^{T,\mathbf{v}}$ if and only if $V_B = V_A \otimes_A B$ is potentially semistable of Galois type T and p-adic Hodge type $\mathbf{v}$.*

PROOF. Let L/K be a finite Galois extension such that the inertia subgroup $I_L \subset I_K$ is contained in $\ker T$. The group change map along with Theorem 4.11.2 give us a quotient $A^{\mathrm{pst},\mathbf{v}}$ of A such that ζ factors through $A^{\mathrm{pst},\mathbf{v}}$ if and only if $V_B|_{\hat{\Gamma}_L}$ is semistable of p-adic Hodge type $\mathbf{v}$. Assume $A = A^{\mathrm{pst},\mathbf{v}}$ from now on.

Let W_L denote the ring of integers of L_0, the maximal unramified subfield of L. Set $W_{L,A} := (W_L)_A$. Proposition 4.12.9 gives us an isomorphism of finite free $W_{L,A}$-modules

$$D_A \xrightarrow{\sim} \mathrm{Hom}_{A[\hat{\Gamma}_L]}(V_A, B^+_{\mathrm{st},A})$$

that is compatible with the natural action of φ. The Galois group $\mathrm{Gal}(L/K)$ acts L_0-semi-linearly on $\mathrm{Hom}_{A[\hat{\Gamma}_L]}(V_A, B^+_{\mathrm{st},A})$, and the inertia group $I_{L/K} \subset \mathrm{Gal}(L/K)$ acts L_0-linearly (cf. [FO, Prop. 6.58]). Since the action of $\mathrm{Gal}(L/K)$ commutes with φ, if $\sigma \in I_{L/K}$, then the trace $\mathrm{Tr}(\sigma)$ is in $(W_{L,A})^{\varphi=1} = A$. Because characteristic 0 representations of finite groups are rigid, $\mathrm{Tr}(\sigma)$ is a locally constant function on $\mathrm{Spec}\, A$. Denote by $A^{T,\mathbf{v}}$ the quotient of A corresponding to the union of components of $\mathrm{Spec}\, A$ where $\mathrm{Tr}(\sigma) = \mathrm{Tr}(T(\sigma))$ for all $\sigma \in \hat{\Gamma}_0$. □

Corollary 4.12.13 ([Kis08, Corollary 2.7.7]). *There exists a quotient $A^{T,\mathbf{v}}_{\mathrm{cr}}$ of A such that for any finite E-algebra B, a map of E-algebras $\zeta : A \to B$ factors through $A^{T,\mathbf{v}}_{\mathrm{cr}}$ if and only if $V_B = V_A \otimes_A B$ is potentially crystalline of Galois type T and p-adic Hodge type $\mathbf{v}$.*

Proof. Theorem 4.10.9 provides for us a finite projective $W_{A^{T,\mathbf{v}}}$-module $D_{A^{T,\mathbf{v}}}$ equipped with a linear endomorphism N. We know that V_B is potentially crystalline if and only if the specialization of N by ζ to B vanishes. Therefore we may take $A^{T,\mathbf{v}}_{\mathrm{cr}}$ to be the quotient of $A^{T,\mathbf{v}}$ defined by the relation $N = 0$. □

4.13. Final Remarks

Combining the results of Chapter 3 (see Theorem 3.2.5.1) with Chapter 4, we have several the following results. Let $\hat{\Gamma}$ is the absolute Galois group of a finite field extension K of $\mathbb{Q}_p$, and choose a residual pseudorepresentation $\bar{D}$. Recall that $\mathrm{PsR}_{\bar{D}} = \mathrm{Spf}\, B_{\bar{D}}$ is the deformation space of $\bar{D}$, which is Noetherian since $\hat{\Gamma}$ is finitely generated. Also recall that $\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}$ denotes the groupoid of Azumaya algebra-valued continuous representations of $\hat{\Gamma}$ with constant residual pseudorepresentation $\bar{D}$. The natural map

$$\bar{\psi} : \overline{\mathcal{R}\mathrm{ep}}_{\bar{D}} \to \mathrm{PsR}_{\bar{D}}$$

is universally closed, formally of finite type, and is the algebraization of a finite type Spec $B_{\bar{D}}$-algebraic stack $\overline{\mathrm{Rep}}_{E(R,D^u_{\bar{D}}),\bar{D}}$, where $E(R, D^u_{\bar{D}})$ is the universal Cayley-Hamilton $B_{\bar{D}}$-algebra

$$E(R, D^u_{\bar{D}}) := (\mathbb{Z}_p[\![\hat{\Gamma}]\!] \otimes_{\mathbb{Z}_p} B_{\bar{D}})/\mathrm{CH}(D^u_{\bar{D}}),$$

which is finitely generated as a $B_{\bar{D}}$-module.

Here are a few observations regarding the implications of what we have proved.

Observation 4.13.1. Combining the algebraicity of $\overline{\mathcal{R}\mathrm{ep}}_{\hat{\Gamma}}$ over $\mathrm{PsR}_{\hat{\Gamma}}$ with the projectivity of the moduli of Kisin modules $\mathcal{L}^{\leq h}$ over $\mathcal{R}\mathrm{ep}_{\hat{\Gamma}}$, the moduli of Kisin modules is algebraizable over $\mathrm{PsR}_{\hat{\Gamma}}$ and universally closed, with projective $\mathrm{PsR}_{\hat{\Gamma}}$-subschemes.

Observation 4.13.2. Let A° denote the admissible coefficient $\mathbb{Z}_p$-algebra of a continuous A°-Azumaya algebra-valued representation

$$\rho : \mathbb{Z}_p[\![\hat{\Gamma}]\!] \otimes_{\mathbb{Z}_p} A^\circ \longrightarrow \mathcal{E}$$

of $\hat{\Gamma}$ with constant residual pseudorepresentation $\bar{D}$, which we can assume to be formally finitely generated over $B_{\bar{D}}$ (i.e. the quotient of a restricted power series over $B_{\bar{D}}$ in finitely many indeterminates). For example, one can think of A° as the universal coefficient sheaf of rings $\mathcal{O}_{\overline{\mathcal{R}\mathrm{ep}}_{\bar{D}}}$. Chapter 4 constructs closed subspaces $X^{\mathrm{cond}} = \mathrm{Spec}(A^\circ[1/p])/I^{\mathrm{cond}}$ of the Noetherian Jacobson scheme $X := \mathrm{Spec}\, A^\circ[1/p]$, which are precisely the loci of specializations of ρ to A-algebras, finite as $\mathbb{Q}_p$-algebras, satisfying certain conditions from p-adic Hodge theory. Now consider the algebraization statement over $\mathrm{PsR}_{\bar{D}}$. It implies that there exists a universal finite type Spec $B_{\bar{D}}$-algebraic stack $\overline{\mathrm{Rep}}_{E(R,D^u_{\bar{D}}),\bar{D}}$ such that ρ arises from the universal representation of $E(R, D^u_{\bar{D}})$ over $\overline{\mathrm{Rep}}_{E(R,D^u_{\bar{D}}),\bar{D}}$ by pullback along some morphism

$$f_\rho : \mathrm{Spec}\, A^\circ \longrightarrow \overline{\mathrm{Rep}}_{E(R,D^u_{\bar{D}}),\bar{D}}.$$

(We have set $R := \mathbb{Z}_p[\![R]\!]$.) Therefore there is a finitely generated $B_{\bar{D}}$-subalgebra A^{fg} of A° with a ($B_{\bar{D}}$-typically) continuous representation

$$\rho^{fg} : E(R, D^u_{\bar{D}}) \otimes_{B_{\bar{D}}} A^{fg} \to \mathcal{E}^{fg}$$

such that $\rho \simeq \rho^{fg} \otimes_{A^{fg}} A^\circ$. We can now consider the closed subscheme of $\operatorname{Spec} A^{fg}[1/p]$ corresponding to the condition "cond" from p-adic Hodge theory: it is cut out by the ideal that is the quotient of the composite map

$$A^{fg}[1/p] \longrightarrow A^\circ[1/p] \longrightarrow A^\circ[1/p]/I^{\mathrm{cond}}.$$

This is an example of finite type $\operatorname{Spec} B_{\bar{D}}[1/p]$-schemes which are universal moduli spaces for representations of the module finite $B_{\bar{D}}[1/p]$-algebra $E(R, D^u_{\bar{D}})[1/p]$ which are required to satisfy a p-adic Hodge theoretic condition.

Observation 4.13.3. One can make sense of the notion of a K-valued pseudorepresentation D of $\hat{\Gamma}$ satisfying or not satisfying certain conditions from p-adic Hodge theory, where K is a finite field extension of $\mathbb{Q}_p$. After a finite extension of K, D is realizable as the determinant of a semisimple K-valued representation ρ^{ss}_D. Then one can say that D has a p-adic Hodge theoretic property if ρ^{ss}_D does. Of course, this does not imply that all representations of $\hat{\Gamma}$ with semisimplification ρ^{ss}_D (i.e. representations in the fiber of $\bar{\psi}$ over D) have this property.

Observation 4.13.4. Since $\bar{\phi}$ is universally closed, the constructions above give a closed subspace of $\mathrm{PsR}_{\bar{D}}[1/p]$ corresponding to certain p-adic Hodge theory conditions, say cut out by an ideal $I^{\mathrm{cond}}_{\mathrm{PsR}} \subset B_{\bar{D}}[1/p]$. As a result, one can construct a quotient

$$E(R, D^u_{\bar{D}})^{\mathrm{cond}} := E(R, D^u_{\bar{D}})[1/p] \otimes_{B_{\bar{D}}[1/p]} B_{\bar{D}}[1/p]/I^{\mathrm{cond}}_{\mathrm{PsR}}$$

through which all representations satisfying this condition must factor. Conversely, it seems that its semisimple, p-adic field-valued representations must satisfying the condition, as long as they induce pseudorepresentations parameterized by $\operatorname{Spec} B_{\bar{D}}[1/p]$. This construction may

even be able to be refined if representations satisfying this condition are shown to cut out appropriately linear subspaces of the projective spaces of extensions described in §2.2. In this case, there should exist a quotient algebra of $E(R, D^u_{\bar{D}})^{\mathrm{cond}}$ whose representations (given that they are parameterized by $\operatorname{Spec} B_{\bar{D}}[1/p]$) are precisely those satisfying the condition.

We are curious if there is any useful application of these observations.

APPENDIX A

A Remark on Projective Morphisms

There are several notions of projectivity of a morphism of schemes. We will use the following terminology.

DEFINITION A.1 ([Sta, Definition 01W8]). Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *projective* if X is isomorphic as an S-scheme to a closed subscheme of a projective bundle $\mathbb{P}(\mathcal{E})$ for some quasi-coherent finite type $\mathcal{O}_S$-module $\mathcal{E}$.

(2) We say that f is *H-projective* if there exists an integer n and a closed immersion $X \to \mathbb{P}^n_S$ over S.

(3) We say that f is *locally projective* if there exists an open cover of S such that the restriction of f to each element of the cover is projective.

EXAMPLE A.2. A finite morphism is always projective, but is not always H-projective.

Local projectiveness and local H-projectiveness are equivalent. Though H-projectivity is preserved under composition using the Segre embedding ([Sta, Lemma 01WE]), this property of projectivity requires quasi-compactness of the base [Vak12, Exercise 18.3.B]. Projectivity is not a local property on the base. However, given a (very) ample line bundle for a projective morphism, one can check projectivity locally when the base is locally Noetherian [Vak12, Exercise 18.3.G].

We will use "projective" morphisms so that we can prove projectivity of a morphism locally on the base (as long as we can glue together the ample line bundle).

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