Attractors at Weak Gravity

KUL-TF-98/55
hep-th/9812049

Attractors at weak gravity

Frederik Denef

Instituut voor Theoretische Fysica
Universiteit Leuven
B-3001 Leuven, Belgium
frederik.denef@fys.kuleuven.ac.be

Abstract

We study the attractor mechanism in low energy effective $D = 4$, $\mathcal{N} = 2$ Yang-Mills theory weakly coupled to gravity, obtained from the effective action of type IIB string theory compactified on a Calabi-Yau manifold. Using special Kähler geometry, the general form of the leading gravitational correction is derived, and from this the attractor equations in the weak gravity limit. The effective Newton constant turns out to be spacetime-dependent due to QFT loop and nonperturbative effects. We discuss some properties of the attractor solutions, which are gravitationally corrected dyons, and their relation with the BPS spectrum of quantum Yang-Mills theory. Along the way, we obtain a satisfying description of Strominger's massless black holes, moving at the speed of light, free of pathologies encountered in earlier proposals.

December 1998

1 Introduction

From the point of view of physics, the central issue in string theory is what can be extracted from it at "low" energies in four dimensions. At least for models with sufficient supersymmetry, this turned out to be surprisingly much, often even *more* than could be found from the more conventional formulation of four dimensional theories. This is mainly due to the beautiful, unifying geometric picture string theory provides. In particular for four dimensional flat space quantum field theories, many striking results have been obtained [1, 2, 3]. In this context, the main focus till now has been on theories completely decoupled from gravity. However, string theory should be at its strongest in questions *involving* gravity, so investigating gravitational corrections to those results might be very fruitful, and possibly even of some phenomenological importance.

Another area of physics in which much is expected from string theory is black hole quantum mechanics. Interesting black hole models with still enough supersymmetry to make their analysis manageable can be found as BPS solutions of the low energy $D = 4$, $\mathcal{N} = 2$ supergravity theory obtained from IIB strings compactified on a Calabi-Yau manifold. These black holes have the remarkable property of being *attractors* for the scalars in the vectormultiplets: at the horizon, the scalars always have the same value, only determined by the charge of the black hole, and insensitive to the variations of the scalar values at infinity. This phenomenon was discovered by Ferrara, Kallosh and Strominger in [4] (excellent reviews can be found in [5, 6]), and turned out to have a wide range of applications, also far beyond four dimensional physics. In particular, in a fascinating paper [6] (summarized in [7]), Moore uncovered a number of astonishing links between number theory and attractors. Many of the proposals in that paper are still speculative, in part because it is pretty hard to find tractable examples of attractors. So finding and investigating such examples might be very fruitful, and possibly even of some number theoretical importance.

In this paper, we try to combine both projects. We investigate the attractor mechanism close to (but not at) a point in the moduli space of the IIB/CY compactification where an effective $4D$ quantum field theory decoupled from gravity can be isolated [1, 8]. The leading gravitational correction to the field theory has a universal form, as we will show. We thus obtain attractors in an effective quantum gauge theory weakly coupled to gravity. It turns out that as long as their single particle mass is smaller than the Planck mass, the attractors are no longer black holes, but QFT and gravitationally corrected dyons, without event horizon (even for multiparticle states). Nevertheless, the attractor property still holds, but now the attractive moduli values are reached at the boundary of a core at finite distance from the origin. The attractive points in moduli space are precisely the singular points where the charge under consideration becomes massless, which could intuitively be expected from the wrapped 3-brane picture for the BPS states. If we let the moduli at infinity approach these attractor points and at the same time boost

the solution to the speed of light, we find a perfectly satisfactory description[1] of Strominger's massless black holes.

We furthermore find that there exist no (BPS) solutions for some charges (for the ansatz of a spherical symmetric solution centered around a point charge). This seems to be related to the absence of those states from the corresponding quantum Yang-Mills BPS spectrum, though establishing a precise correspondence will require a relaxation of the ansatz, presumably closely related to the appearance of 3-pronged strings in the 3/7-brane picture of $\mathcal{N} = 2$ super Yang-Mills [9, 10, 11].

Along the way, we show that the effective Newton constant in these theories is actually necessarily spacetime dependent. We derive an explicit formula for the space dependence in the presence of an attractor, and find that this variation of the Newton constant is a pure QFT loop and nonperturbative effect, in the sense that it is absent in (electric or magnetic) weak coupling limits. This might be of interest (as a toy model perhaps) in the light of recent claims on measured spacetime variations of the constants of nature [12].

The structure of the paper is as follows. In section 2 we review and clarify some basic aspects of the attractor mechanism, including multicenter solutions. In section 3, we review the emergence of low energy Yang-Mills theories from type IIB Calabi-Yau compactifications, and derive the universal form of the leading gravitational correction. Section 4 combines the two previous sections to obtain the weak gravity attractor equations and their solutions. We conclude with some comments in section 5.

An extensive review on classical dyons coupled to gravity can be found in [13], while quantum dyons without gravity are studied e.g. in [14, 15, 16, 17]. The principle of *dynamical relaxation* of BPS mass which underlies the attractor mechanism, the parallel with supersymmetric Yang-Mills theory, and quantum corrections to supersymmetric black holes in heterotic compactifications are discussed in depth in [21]. Black holes at conifold points in moduli space have been considered before in [6], [22] and in particular in [23].

2 Attractor mechanism

2.1 Invariant formalism

We will follow the manifestly duality invariant formalism of [6]. Consider type IIB string theory compactified on a Calabi-Yau manifold X. The four dimensional low energy theory is $\mathcal{N} = 2$ supergravity coupled to $n_v = h^{1,2}$ abelian vectormultiplets and $n_h = h^{1,1} + 1$ hypermultiplets, where the $h^{i,j}$ are the Hodge numbers of X. The

[1] The solutions we propose are different from the related "conifold solutions" discussed in the literature [22, 23], in part due to the fact that we only insist on continuous differentiability of a set of *good coordinates* on moduli space, and not necessarily of all *periods* (which are not always good coordinates).

hypermultiplets will play no role in the following and are set to arbitrary constant values.

The vectormultiplet scalars are identified with the complex structure moduli of X, and the lattice of electric and magnetic charges with $H^3(X,\mathbb{Z})$, the lattice of integral harmonic 3-forms on X. The total field strength $\mathcal{F}$, which in this case can be identified with the type IIB anti-self-dual five-form field strength, has values in $\Omega^2(M_4)\otimes H^3(X,\mathbb{Z})$, where $\Omega^2(M_4)$ denotes the space of 2-forms on the four dimensional spacetime M_4. The usual components of the field strength are obtained by picking a symplectic basis A^I, B_I of $H^3(X,\mathbb{Z})$:

$$\mathcal{F} = F^I B_I + G_I A^I. \tag{2.1}$$

The total field strength satisfies the anti-self-duality constraint:

$$\mathcal{F} = -*_{10}\mathcal{F}, \tag{2.2}$$

where $*_{10}$ is the Hodge star operator on the ten-dimensional space time, which factorises according to the compactification as $*_{10} = *_4 \otimes *_X$. This constraint relates the F and G components in (2.1).

The equation of motion and the Bianchi identity of the electromagnetic field are combined in the equation

$$\mathrm{d}\mathcal{F} = 0. \tag{2.3}$$

Electric and magnetic charges inside a region bounded by a surface S are found as

$$\Gamma_S = n_I A^I + m^I B_I = \int_S \mathcal{F} \tag{2.4}$$

Choosing a space/time decomposition and denoting the spatial components of $\mathcal{F}$ as $\mathcal{F}_s$, the electromagnetic energy density is simply given by:

$$\mathcal{H}_{em}\mathrm{d}t = \frac{1}{4}\int_X \mathcal{F}_s \wedge *_{10}\mathcal{F}_s \tag{2.5}$$

2.2 Special geometry

The geometry of the scalar manifold of the vector multiplets, parametrized with n_v coordinates z^a, is special Kähler [18]. The metric

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K} \tag{2.6}$$

is derived from the Kähler potential

$$\mathcal{K} = -\ln(\mathrm{i}\int_X \Omega \wedge \bar{\Omega}), \tag{2.7}$$

where Ω is the holomorphic 3-form on X.

Any harmonic 3-form Γ on X can be decomposed according to

$$H^3(X,\mathbb{C}) = H^{3,0}(X) + H^{2,1}(X) + H^{1,2}(X) + H^{0,3}(X) \tag{2.8}$$

as

$$\Gamma = \mathrm{i}\, e^{\mathcal{K}}\, \Omega \int_X \Gamma \wedge \bar{\Omega} \;-\; \mathrm{i}\, e^{\mathcal{K}}\, D_a\Omega\, g^{a\bar{b}} \int_X \Gamma \wedge \bar{D}_{\bar{b}}\bar{\Omega} \;+\; \text{c.c.}, \tag{2.9}$$

where $D_a\Omega \equiv (\partial_a + \partial_a\mathcal{K})\Omega$. This decomposition is useful because it diagonalizes the Hodge star operator:

$$*_X\, \Gamma^{p,3-p}(X) = (-1)^p\, \mathrm{i}\, \Gamma^{p,3-p}(X) \tag{2.10}$$

This can be used to find a more explicit expression for (2.5). Denote

$$\eta = e^{\mathcal{K}/2} \int_X \mathcal{F}_s \wedge \Omega, \tag{2.11}$$

then:

$$\mathcal{H}_{em}\mathrm{d}t = \eta \wedge *_4\bar{\eta} \;+\; g^{a\bar{b}} D_a\eta \wedge *_4\bar{D}_{\bar{b}}\bar{\eta}, \tag{2.12}$$

with $D_a\eta \equiv (\partial_a + \frac{1}{2}\partial_a\mathcal{K})\eta$.

2.3 Static spherical symmetric configurations

We take a general[2] static spherical symmetric ansatz for the metric:

$$ds^2 = e^{2U(r)}dt^2 - e^{-2U(r)}(\frac{1}{g(r)^2}dr^2 + r^2 d\Omega_2^2), \tag{2.13}$$

or, changing variables $r = c/\sinh c\tau$, $g(r) = h(\tau)\cosh c\tau$:

$$ds^2 = e^{2U}dt^2 - e^{-2U}(\frac{1}{h^2}\frac{c^4}{\sinh^4 c\tau}d\tau^2 + \frac{c^2}{\sinh^2 c\tau}d\Omega_2^2\,). \tag{2.14}$$

We furthermore assume the moduli and electromagnetic fields to be spherical symmetric and produced by a source with charge Γ at $r = 0$, that is

$$\mathcal{F} = \omega \otimes \Gamma - *_4\omega \otimes *_X\Gamma, \tag{2.15}$$

where $\omega = \frac{1}{4\pi}\sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi = *_4(\frac{e^{2U}}{4\pi h}\, \mathrm{d}\tau \wedge \mathrm{d}t)$. In the IIB string theory, this corresponds to a three brane wrapped around the 3-cycle $\hat{\Gamma}$ Poincaré dual to Γ, at the origin. Putting all fermionic fields to zero, we find for the reduced effective action[3]

[2] This ansatz is more general than the one used in [5]. However, it reduces to the latter by the equations of motion, as we show here.

[3] The action $S \sim \int F^I \wedge G_I + \cdots$ is only determined up to symplectic duality transformations (i.e. up to choice of symplectic basis (A^I, B_I) in (2.1)). However, to get a manifestly consistent reduced action principle at fixed field strength $\mathcal{F}$, $F^I = \mathrm{d}A^I$ should only appear in the action such that the action is not varied via changes of F^I when the other fields are varied. With the ansatz we use, this is the case when we choose our basis such that $\Gamma\cdot B_I = 0$, and then the electromagnetic part of the action is equal to the electromagnetic energy (2.12), leading to the above expression for S.

(modulo boundary terms), in Planck units:

$$S = \frac{1}{2}\int_0^\infty \mathrm{d}\tau \ \left\{ h(\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} - c^2) + \frac{1}{h}e^{2U}V(z) - \frac{c^2}{\sinh^2 c\tau}(h + \frac{1}{h} - 2)\right\}, \quad (2.16)$$

where $\dot{f} \equiv \frac{df}{d\tau}$. The "scalar potential" $V(z)$ is derived from (2.12) and (2.15):

$$V(z) = |Z|^2 + g^{a\bar{b}}D_a Z \bar{D}_{\bar{b}}\bar{Z} = |Z|^2 + 4g^{a\bar{b}}\partial_a|Z|\bar{\partial}_{\bar{b}}|Z| \quad (2.17)$$

with the (position dependent) "central charge" Z defined as

$$Z = e^{\mathcal{K}/2}\int_X \Gamma \wedge \Omega = e^{\mathcal{K}/2}\int_{\hat{\Gamma}} \Omega. \quad (2.18)$$

Now the equation of motion for $h(\tau)$ obtained from variations of (2.16) corresponding to radial diffeomorphisms (*i.e.* $\delta f(\tau) = f(\tau) + \epsilon(\tau)\dot{f}(\tau)$), acting on the fields U and z^a only, actually implies $h = 1 + \frac{k}{c^2}\tanh^2 c\tau$, and we can use the 1-parameter diffeomorphism freedom from the introduction of the constant c to put $k = 0$, $h = 1$. Varying h on the other hand implies the constraint (at $h = 1$):

$$\dot{U}^2 + g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}} - e^{2U}V(z) = c^2. \quad (2.19)$$

However, since at $h = 1$, by τ-translational invariance, the left hand side is a conserved quantity along τ-translations anyway, this just determines the value of c (from the boundary conditions), leaving no nontrivial constraint independent of the other equations of motion. Bearing this in mind, we can simply put $h \equiv 1$ and rewrite (2.16) as

$$S = \pm e^U|Z|\,\Big|_{\tau=\infty}^{\tau=0} + \frac{1}{2}\int_0^\infty \mathrm{d}\tau \ \left\{(\dot{U} \pm e^U|Z|)^2 + \|\dot{z}^a \pm 2e^U g^{a\bar{b}}\bar{\partial}_{\bar{b}}|Z|\|^2 - c^2\right\}. \quad (2.20)$$

2.4 BPS solutions: the attractor equations

From (2.20), it is clear that the reduced action (and the energy) at fixed values of c and the boundary moduli, has a minimum at

$$\dot{U} = -e^U|Z| \quad (2.21)$$

$$\dot{z}^a = -2e^U g^{a\bar{b}}\bar{\partial}_{\bar{b}}|Z|, \quad (2.22)$$

(if these equations have a solution). Equation (2.19) then implies $c = 0$, so $r = 1/\tau$ and

$$ds^2 = e^{2U}dt^2 - e^{-2U}d\vec{x}^2. \quad (2.23)$$

Assuming asymptotic flatness, *i.e.* $U \to 0$ at spatial infinity, these solutions saturate the BPS bound

$$E = |Z(\tau = 0)|. \quad (2.24)$$

Here we have dropped the $\tau = \infty$ boundary term since (2.21) and (2.22) imply that both e^U and $|Z|$ are monotonously decreasing functions (see also equation (2.25) below) satisfying the estimate $e^U|Z| \leq \min\{|Z(0)|/(1+|Z(\infty)|\tau), |Z|\}$, and hence $e^U|Z| \to 0$ when $\tau \to \infty$. Note that this estimate also implies that when $Z(\infty) \neq 0$, the solution is a black hole with horizon at $r = 0$. A slightly more detailed analysis [5] gives for the horizon area $A = 4\pi|Z(\infty)|^2$.

Choosing the other sign possibility in (2.20), does not give an acceptable[4] solution: now e^U and $|Z|$ are *increasing* functions, satisfying the estimate $e^U|Z| \geq |Z(\tau_*)|/(1-|Z(\tau_*)|\tau)$ for any fixed τ_*, hence any nontrivial solution develops a singularity at finite distance from the origin, and has infinite action and energy. Furthermore, these solutions would be gravitationally repulsive and leave the weak gravity region of moduli space. Note however that in a *finite region* of spacetime, preventing τ to run to infinity, such solutions might be acceptable and possibly important.[5]

Equations (2.21) and (2.22) are called the *attractor equations*. This is because their solutions converge to fixed moduli values at $\tau = \infty$, namely to those values for which $|Z|$ is minimal. Indeed, from (2.22), we see:

$$\frac{d}{dt}|Z| = -4e^U g^{a\bar{b}}\, \partial_a|Z|\, \bar{\partial}_{\bar{b}}|Z| < 0, \tag{2.25}$$

so the moduli will flow "down the hill" till a minimal value of $|Z|$ is reached (with vanishing norm of its gradient). This is intuitively clear in the brane picture, since we expect the 3-brane to "pull" the moduli such that its volume is minimized.

The attractor equations can also be obtained from the requirement of conservation of one half of the supersymmetries [4, 20, 21, 22]. Solutions and generalisations have been discussed for example in [20, 21, 22, 23].

2.5 Multicenter case

The previous discussion is readily extended to the extremal multicenter case *with equal charges*[6], by introducing an effective "radial" coordinate

$$\tau \equiv \frac{1}{n}\sum_{i=1}^{n} \tau_i, \tag{2.26}$$

where i runs over the n different centers and τ_i is defined as τ in the previous discussion, relative to the ith center. Surfaces of equal τ can be considered as

[4] at least not for our purposes; in [19], it is discussed in what sense such solutions could still be meaningful.

[5] a comparable situation is perhaps the occurrence of exponential "tunnelling" solutions of Maxwell's equations between two dielectrics.

[6] Multicenter solutions with charges corresponding to different elements of $H^3(X, \mathbb{Z})$, and in particular with mutually nonlocal charges (=nonzero intersection product), are much more difficult to study, and their existence is not clear.

equipotential surfaces for the multi-source configuration. The ansatz for the metric is the extremal

$$ds^2 = e^{2U(\tau)}dt^2 - e^{-2U(\tau)}d\vec{x}^2. \tag{2.27}$$

The electromagnetic field is given by superposition and has exactly the same form as (2.15), with $\omega \equiv \sum_i \omega_i = n *_4 (\frac{e^{2U}}{4\pi} \mathrm{d}\tau \wedge \mathrm{d}t)$ (this is only the case for equal charges). The scalar fields are supposed to be functions of τ only.

Since the complete setup is formally the same as for the spherically symmetric case, so are the attractor equations. Therefore, everything said about the (extremal) spherical symmetric case applies to the (extremal equal charge) multicenter case as well.

3 Weak gravity Yang-Mills limit

Suppose we tune the moduli such that some 3-cycles in X become very small (measured by their periods $Z = e^{\mathcal{K}/2} \int \Omega$). Then the corresponding BPS states (as described above) become very light and one expects the low energy effective theory, with energies restricted to finite values relative to the masses of those light BPS states, to be a certain four dimensional $\mathcal{N} = 2$ supersymmetric quantum field theory, with BPS spectrum given by those light charges. Since a lot is known about the string theory low energy effective action (including quantum corrections), this observation can be used to derive various nontrivial results about quantum field theories. This "geometric engineering" is of course well known and there exists a very extensive literature on the subject [1]. Most of those studies focus on the rigid quantum field theory itself, completely decoupled from gravity. However, in principle, the special geometry setup as described above also gives gravitational corrections which should be taken into account as long as the exact (singular) decoupling limit is not reached. Some examples were studied in [8]. As we will show, the lowest order gravitational correction has a universal form.

We call the regime in which the lowest order gravity corrected theory is accurate the *weak gravity limit*, while *rigid limit* refers to complete decoupling from gravity.

We will focus on the case where the resulting quantum field theory is $\mathcal{N} = 2$ $SU(N)$ Yang-Mills (with possibly some additional $U(1)$ factors), though generalisations are clearly possible. To make things specific, let us consider an algebraic Calabi-Yau manifold X, in a certain patch with affine coordinates (x, y, w, z) given by a polynomial equation $W(x, y, z, w) = 0$. The coefficients of this polynomial are parametrized by the complex structure moduli. Suppose we can choose a modulus Λ such that large Λ corresponds to the $SU(N)$ weak gravity limit. This can be realized by letting X degenerate locally to an A_n singular ALE fibration over a sphere when $\Lambda \to \infty$ [1], that is, for large Λ and small x, y, w:

$$W \approx W_{ALE} = w^2 + y^2 + x^N + \Lambda^{-2/N} c_{N-2}(z) x^{N-2} + \cdots + \Lambda^{-1} c_0(z) \tag{3.28}$$

$$\Omega \approx \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}z}{z} \wedge \frac{\mathrm{d}x \wedge \mathrm{d}y}{w}, \tag{3.29}$$

where the $c_i(z)$ are certain polynomial functions of z and $1/z$, dependent on a set of "rigid" moduli u_i, $i = 1, \ldots, r$, e.g. in the case of pure $SU(N)$ Yang-Mills $c_i(z) = u_i = \text{const.}$ for $i \neq 0$ and $c_0(z) = \frac{1}{2}(z + \frac{1}{z}) + u_0$.

In the appendix, it is shown (under some weak extra technical assumptions) that the universal form of the Kähler potential in such a weak gravity limit is

$$\mathcal{K} = -\ln(a \ln|\Lambda|^2 + b) + \frac{|\Lambda|^{-2/N}}{a \ln|\Lambda|^2 + b} K(u, \bar{u}) + \cdots, \tag{3.30}$$

where a and b are constants *independent of the rigid moduli* u^i, and $K(u, \bar{u})$ is the Seiberg-Witten rigid special Kähler potential for the limiting gauge theory. The dots include u-independent terms of nonzero (positive) order in $\Lambda^{-1/N}$, and u-dependent terms higher than second order in $\Lambda^{-1/N}$.

More specifically, there is a Riemann surface Σ fibred over the z-plane, given by

$$x^N + c_{N-2}(z) x^{N-2} + \cdots + c_0(z) = 0, \tag{3.31}$$

endowed with a meromorphic 1-form

$$\lambda = x \frac{\mathrm{d}z}{z}, \tag{3.32}$$

such that for any of the "small" Calabi-Yau 3-cycles Γ, we have a corresponding 1-cycle γ on the Riemann surface for which

$$Z(\Gamma) = \frac{\Lambda^{-1/N}}{\sqrt{a \ln|\Lambda|^2 + b}} Z_0(\gamma) \tag{3.33}$$

with $Z_0(\gamma) = \int_\gamma \lambda$. Then we have

$$K(u, \bar{u}) = Q^{ij} Z_0(\gamma_i) \bar{Z}_0(\gamma_j), \tag{3.34}$$

where $\{\gamma_i\}_i$ is a basis of 1-cycles on Σ and Q^{ij} is the inverse of the intersection form $Q_{ij} = \gamma_i \cdot \gamma_j$.

Consistent weak gravity truncation requires the rigid moduli excitations to be bounded and Λ to remain sufficiently large. Then indeed the local special geometry reduces approximately to rigid special geometry, and the physics is well described by a lowest order gravity corrected quantum field theory (see also below).

4 Attractor equations in the weak gravity limit

4.1 Equations

In the following we will assume there are no other complex structure moduli apart from Λ and u^i. That is, the low energy theory is $\mathcal{N} = 2$ $SU(N)$ Yang-Mills (possibly with matter) without additional $U(1)$ factors.

The weak gravity metric on moduli space, derived from (3.30) is:

$$g_{\Lambda\bar{\Lambda}} \approx \frac{|\Lambda|^{-2/N}}{a\ln|\Lambda|^2+b}\,\frac{a^2}{(a\ln|\Lambda|^2+b)|\Lambda|^{2-2/N}} \tag{4.35}$$

$$g_{i\bar{j}} \approx \frac{|\Lambda|^{-2/N}}{a\ln|\Lambda|^2+b}\,g_{0,i\bar{j}} \tag{4.36}$$

$$g_{i\bar{\Lambda}} \approx -\frac{|\Lambda|^{-2/N}}{a\ln|\Lambda|^2+b}\,\frac{1}{\bar{\Lambda}}\,(\frac{1}{N}+\frac{a}{a\ln|\Lambda|^2+b})\,\partial_i K \tag{4.37}$$

$$\approx -\frac{|\Lambda|^{-2/N}}{a\ln|\Lambda|^2+b}\,\frac{1}{N\bar{\Lambda}}\,\partial_i K. \tag{4.38}$$

where $g_{0,i\bar{j}} \equiv \partial_i\bar{\partial}_{\bar{j}}K$ is the Seiberg-Witten metric on rigid moduli space. For the last approximation, equation (4.38), we have supposed that $\ln|\Lambda|^2+b/a \gg N$, which is of course satisfied when $\Lambda\to\infty$, but which might not be the case if we want to study large but finite Λ (e.g. for phenomenology[7]) or the $N\to\infty$ limit. For now, we will keep (4.37).

Up to a factor $\sim(1+O(|\Lambda|^{-2/N}\ln|\Lambda|))$, the inverse metric is given by

$$g^{\Lambda\bar{\Lambda}} \approx (a\ln|\Lambda|^2+b)|\Lambda|^{2/N}\,\frac{a\ln|\Lambda|^2+b}{a^2}|\Lambda|^{2-2/N} \tag{4.39}$$

$$g^{i\bar{j}} \approx (a\ln|\Lambda|^2+b)|\Lambda|^{2/N}\,g_0^{i\bar{j}} \tag{4.40}$$

$$g^{i\bar{\Lambda}} \approx (a\ln|\Lambda|^2+b)|\Lambda|^{2/N}\,\frac{1}{a^2}g_0^{i\bar{j}}\bar{\Lambda}|\Lambda|^{-2/N}[\frac{1}{N}(a\ln|\Lambda|^2+b)+a]\,g_0^{i\bar{j}}\bar{\partial}_{\bar{j}}K \tag{4.41}$$

We now plug this together with (3.33)[8] in the attractor equations (2.21) and (2.22). For the rigid moduli derivatives, we find (to lowest order):

$$\dot{u}^i = -2\sqrt{a\ln|\Lambda|^2+b}|\Lambda|^{1/N}\,e^U g_0^{i\bar{j}}\bar{\partial}_{\bar{j}}|Z_0|. \tag{4.42}$$

Recall that this is an expression in units such that the Planck mass is one. Let us change to units adapted to the gauge theory by taking the mass of the gauge theory (BPS) particles to be given by

$$M(\gamma) = |Z(\gamma)|M_{Pl} \equiv |Z_0(\gamma)|, \tag{4.43}$$

evaluated at spatial infinity. So, from (3.33), we see that in these units

$$M_{Pl}^2 = (a\ln|\Lambda_\infty|^2+b)|\Lambda_\infty|^{2/N}. \tag{4.44}$$

We can implement this change of units by a change of coordinates $\tau\to\tau'$ defined by

$$\mathrm{d}\tau' = \frac{1}{\sqrt{G(\tau)}}\mathrm{d}\tau, \tag{4.45}$$

[7]For example we consider gravity at typical accelerator energies to be really weak, while $\ln(E_{Planck}/E_{acc})$ is only about 50

[8]Since we want to study the weak gravity limit, we restrict to the small Calabi-Yau 3-cycles, which carry the charges of the rigid quantum field theory.

where

$$G \equiv \frac{|\Lambda|^{-2/N}}{a \ln |\Lambda|^2 + b} \tag{4.46}$$

can be considered as a space-dependent Newton constant (at spatial infinity, $G = 1/M_{Pl}^2$). When G does not vary too much (which will be the case in the weak gravity limit), this transformation is approximately just a rescaling of τ (or r). Denoting derivatives with respect to τ' again with dots, we find for the attractor equations:

$$\dot{u}^i = -2e^U g_0^{i\bar{j}}\, \bar{\partial}_{\bar{j}} |Z_0| \tag{4.47}$$

$$\dot{U} = -Ge^U |Z_0| \tag{4.48}$$

$$\frac{\dot{\Lambda}}{\Lambda} = Ge^U |Z_0|\, (1 - g_0^{i\bar{j}} \partial_i K \bar{\partial}_{\bar{j}} \ln \bar{Z}_0)\, (\ln |\Lambda|^2 + b/a)^2 (\frac{1}{N} + \frac{1}{\ln |\Lambda|^2 + b/a}). \tag{4.49}$$

Here we see clearly that in the (singular) limit $G = 0$, gravity is indeed completely decoupled from the gauge theory degrees of freedom.

The identification of G with the Newton constant can be made more precise by rescaling the metric with a factor $1/G$ (instead of performing the reparametrization (4.45)) and check that the full bosonic $4D$ effective action then indeed becomes (approximately) of the form

$$S = \int_{M_4} d^4x \frac{1}{16\pi G} \sqrt{g}\, R \; + \; S_{YM} + \cdots, \tag{4.50}$$

where S_{YM} is the Seiberg-Witten effective action (in curved spacetime) and the dots denote the remaining part of the action, containing e.g. the kinetic terms for Λ.

Note also that this Newton constant identification is convention (or frame) dependent; we choose the frame in which the Yang-Mills part of the action (or equivalently (4.47)) becomes Λ-independent. A discussion on the relation between different conventions with varying constants of nature can be found in [12].

In the approximation where $\ln G$ can be considered very large as well, we can rewrite the real part of the last equation as

$$\frac{d}{d\tau'} (\frac{1}{(\ln G)^2 G}) \approx e^U |Z_0| (2 - \nabla K \cdot \nabla \ln |Z_0|^2) \tag{4.51}$$

Equation (4.47) is just what one would obtain for BPS field configurations from the abelian Seiberg-Witten effective action (in curved spacetime), and equation (4.48) is the coupling of gravity to the gauge theory fields which one would intuitively expect: it is just (the relativistic generalisation of) Newton's law for the gravitational field of a spherically symmetric energy distribution. The remaining equation however, determining the variation of G, is more intricate. Notice the rigid moduli dependent factor

$$f \equiv 1 - g_0^{i\bar{j}}\, \partial_i K\, \bar{\partial}_{\bar{j}} \ln \bar{Z}_0. \tag{4.52}$$

It is straightforward to check that $f = 0$ when K is quadratic in the moduli and Z_0 linear. Since (for a suitable choice of coordinates in moduli space) this is effectively the case at weak gauge coupling (electric or magnetic), we can conclude that f only differs from zero thanks to QFT loop and nonperturbative effects, and hence that, from this point of view, the spatial variation of G is entirely a quantum effect.

4.2 Solutions

In the following, we drop the primes on τ. A very useful property following from (4.47) is the constantness of the phase of the *rigid* central charge:

$$\frac{d}{d\tau} \arg Z_0 = 0. \tag{4.53}$$

Its absolute value on the other hand is a decreasing function:

$$\frac{d}{d\tau}|Z_0| = -e^U \|\partial Z_0\|^2. \tag{4.54}$$

In the case of pure $SU(2)$ Yang-Mills at $G = 0$, the attractor equations reduce entirely to the abelian truncation[9] of the "quantum corrected monopole equations" studied in [14]. It is therefore not surprising that many of the features discussed there, including the constantness of phase (which indeed, as anticipated in [14], extends to the multicenter case), apply here as well. The inclusion of the nonabelian degrees of freedom affects the exact τ-dependence of the flow in moduli space, though not really drastically. On the other hand, since we do not include the nonabelian excitations, we don't have an immediate obstruction to enter into the "strong coupling" region of moduli space, where the electric nonabelian description becomes inadequate.

Let us study the flows in rigid moduli space in more detail. Since Z_0 is analytic, its absolute value has no minima except at zeros of Z_0. Therefore, in the weak gravity limit, for our ansatz, the central charge Z_0 (and Z) will always flow to zero.

Note that Z_0 can only be zero at points of marginal stability; for example in the pure $SU(2)$ case, $0 = Z_0 = na + ma_D$ implies $a_D/a = -n/m \in \mathbb{R}$. However, if the zero of Z_0 is at a nonsingular point of moduli space, we run into trouble. Indeed, close to such a point, we can choose a coordinate v such that $Z_0 = v$.[10] Equations (4.47) and (4.48) then reduce to

$$\arg v = \text{const.} \tag{4.55}$$

$$\frac{d|v|}{d\tau} = -ke^U \tag{4.56}$$

$$\frac{de^{-U}}{d\tau} = G|v|, \tag{4.57}$$

[9] Which is obtained by sending the free parameter δ in [14] to infinity.

[10] Z_0 cannot have a double zero since this would imply $\partial_i Z_0 = \int_{\hat{\gamma}} \partial_i \lambda = 0$ for all i there, and hence that $\hat{\gamma}$ is a vanishing cycle at this point, which therefore would be singular.

with k a finite positive constant. So U will be approximately constant, and

$$|v| \approx |v_0| - ke^U(\tau - \tau_0) \tag{4.58}$$

where v_0 is the value of v at "initial" $\tau = \tau_0$. At $\tau = \tau_* \equiv \tau_0 + e^{-U_0}|v_0|/k$, $Z_0 = 0$, and the flow breaks down: it is not possible to continue the BPS solution beyond this point, since $d|v|/d\tau$ cannot be negative at $v = 0$. One could try to continue the solution by gluing it to the "reverse BPS flow" obtained by changing the sign of the RHS of the attractor equations, but as discussed above, those solutions break down as well and are not acceptable. So we conclude that there simply are no BPS solutions[11] (satisfying the ansatz) for charges for which Z_0 becomes zero at a regular point of moduli space, and we expect such states to be absent from the quantum BPS spectrum of the theory. Looking back at the pure $SU(2)$ example, and taking into account that $-1 < a_D/a < 1$ on the curve of marginal stability, we see that this happens when $|m| > |n| > 0$, states which are indeed known not to be present in the BPS spectrum. This phenomenon is however not restricted to charges with $|m| > |n| > 0$ at $\tau = 0$. Due to monodromies, charges which asymptotically do not satisfy this condition, can start to do so at finite τ by crossing a cut in moduli space.

Now if the zero of Z_0 occurs at a singular point of moduli space, at a point where the cycle $\hat{\gamma}$ vanishes, as is the case e.g. for an $SU(2)$ monopole at $u = 1$, things change considerably. Taking again $v = Z_0$, we now have for small v, $K \sim |v|^2 \ln |v|^{-2} + \cdots$ [26] and

$$\frac{d|v|}{d\tau} = -e^U \frac{k}{\ln |v|^{-2}}. \tag{4.59}$$

Again, we reach $v = 0$ at a finite $\tau = \tau_*$, but now $d|v|/d\tau$ *vanishes* at this point (as well as the first derivative of the other fields). Therefore we can continue the solution with continuous first derivatives to an "interior" ($\tau > \tau_*$) constant solution given by $v(\tau) = 0$, $U(\tau) = U(\tau_*)$, $G(\tau) = G(\tau_*)$. Only the electromagnetic field strength is nontrivial in this core region — it is still given by (2.15) — though it contains no energy! This gives a physically reasonable solution (see also below), contrary to matching to an inverted flow, which is again not acceptable. Higher order and nonabelian corrections could be important for the precise value of the inverse core radius τ_* (and could even push it all the way to infinity), but presumably this will not alter the essential conclusions. Indeed, in the brane picture, such a state corresponds to a 3-brane wrapped around a vanishing cycle located at the origin, and we expect the brane to minimize it's volume by attracting the moduli to the conifold point.

So we conclude that the attractors at weak gravity are precisely the conifold points in rigid moduli space.

[11] to the leading abelian low energy effective action, however, since the higher derivatives of v vanish when $\tau \to \tau_*$, higher order corrections are not likely to alter qualitatively the conclusions.

This solution can be used to study Strominger's massless black holes [24]. If we take Z_0 to be already zero at spatial infinity, our solution gives simply flat space with constant moduli (but still the — energyless — electromagnetic field of a point charge). We shouldn't expect more of course for a massless particle at rest! However, we can let the rest mass approach zero and simultaneously boost the solution along the x-axis to approach the speed of light, keeping the energy $E = |Z_0(0)|/\sqrt{1-v^2}$ fixed. When $\gamma \equiv 1/\sqrt{1-v^2} \to \infty$, the boosted metric is given by

$$ds^2 = dt^2 - dx^2 + 4\gamma^2 U(dt-dx)^2 - dy^2 - dz^2, \tag{4.60}$$

and, for nonzero $x-t$, $\tau \to \frac{1}{\gamma|x-t|} \equiv \frac{\chi}{\gamma}$. Denoting $V \equiv \gamma^2 U$ and $Y_0 \equiv \gamma Z_0$, we have $|Y_0(0)| = E$, and from the attractor equations

$$G = \text{const.} \tag{4.61}$$

$$\frac{dV}{d\chi} = -G|Y_0| \tag{4.62}$$

$$\frac{d|Y_0|}{d\chi} = -\frac{k}{2\ln(\gamma|Y_0|^{-1})} \to 0, \tag{4.63}$$

which implies $|Y_0| = \text{const.} = E$ and $V = -GE\chi$.

So we find a simple but nontrivial solution, with $Z_0 = 0$ everywhere, the electromagnetic field strength of a point particle boosted to the speed of light, and a "shockwave metric" (due to the combined effect of expansion of the core region in the rest frame and longitudinal Lorentz contraction while taking the limit):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{4GE}{|x-t|}(dx-dt)^2. \tag{4.64}$$

This is the Aichelburg-Sexl metric for a massless particle [25]. It is locally flat and can be brought to standard form by changing coordinates as $t' = t \mp 2GE\ln|t-x|$, $x' = x \pm 2GE\ln|t-x|$, where the upper (lower) sign is to be used for $t-x>0$ $(t-x<0)$.

Thus we see the solutions we find are physically quite satisfactory. In particular, there are no undesirable features such as negative ADM mass, gravitational repulsion or unphysical divergences, encountered in some of the earlier proposals for the description of these states (as in [22], or the Seiberg-Witten approximation in [16], where the solutions are essentially continued as inverted flows beyond the attractor point). The main reason for this is the fact that those approaches insist on having continuous first derivatives of *all* periods, while we only require continuous first derivatives of a set of *good* coordinates on moduli space. For example at the boundary of the core of the pure $SU(2)$ monopole, we indeed find a discontinuity for the derivative of the "electric" period a, but we consider this as an artifact of a not being a good coordinate at the attractor point $u=1$.

Note that the above solutions never develop an event horizon as long as the moduli at infinity stay within the weak gravity region, that is, roughly as long as

the mass of the one particle BPS state under consideration is less than the order of the Planck mass. Considering states with n particles to increase the mass will *not* create a horizon, no matter how large n is (at least in this low energy approximation and for our ansatz): from the attractor equations, it is easy to see that the solution f_n for n charges γ on top of each other can be obtained from the corresponding single charge solution f_1 as

$$f_n(\tau) = f_1(n\tau); \tag{4.65}$$

in other words, the n particle solutions are obtained by simply rescaling all distances with a factor n. In particular, the radius of the n particle core region is n times larger than the one particle core radius. As such, there is never enough energy in a sufficiently small region of space to produce a black hole; the particles protect themselves from collapse by the attractor mechanism!

However, this does not at all exclude the existence of regular black hole solutions (with weak gravity moduli at infinity) which are not BPS or which satisfy a different ansatz.

Finally, in view of the above discussion, it seems reasonable to conjecture, in the spirit of [6], that a given charge appears in the BPS spectrum of the (gravity corrected) quantum Yang-Mills theory under consideration, if and only if there exists a solution of the corresponding weak gravity attractor equations. Unfortunately, things are not that simple. Take for example pure $SU(2)$. When one starts with a monopole solution (attracted to $u = 1$), and performs on u at spatial infinity a monodromy about $u = \infty$, one encounters at a certain point a flow containing the other singularity, at $u = -1$. If one tries to continue the monodromy by "pulling" the flow across this singularity, one finds that our ansatz does no longer yield a solution (one gets a crash-flow with $Z_0 = 0$ at a regular point). On the other hand, states obtained by monodromy from states already occurring in the BPS spectrum, without crossing a line of marginal stability, should also be in the BPS spectrum [26]. So the conjectured correspondence between quantum BPS states and attractor solutions can only work if we generalize our ansatz in one way or another. This looks very much like what happens in the type IIB 3/7 brane picture of $\mathcal{N} = 2$ $SU(2)$ gauge theory[12] [9, 10, 11]. The open strings representing the BPS states there follow trajectories given by the attractor equations, and fail to exist exactly at the point where our solutions fail to exist. Interpreting the strings as deformations of the 3-branes, as in [17, 29], it is clear that this is no coincidence. In the 3/7 brane picture, we get a *three-pronged string* instead as BPS representative. Work is in progress to develop the corresponding picture from the field theory point of view.

[12] I am grateful to R. von Unge for pointing out this connection to me.

5 Conclusions

In this paper, we studied attractors at weak gravity. We derived the form of the leading gravitational correction to effective low energy $SU(N)$ quantum field theories, and from this the attractor equations in the weak gravity regime. We investigated the spatial dependence of the effective Newton constant and discussed some properties of the (BPS) attractor solutions and their relation with the BPS spectrum of quantum Yang-Mills theory. To establish a full correspondence, a more general ansatz for the solutions is needed, probably involving a combination of the ideas in [11, 17, 29], presumably allowing multicentered solutions with different charges. Having such a correspondence would be a powerful tool to study the BPS spectrum of (gravitationally corrected) gauge theories, with ramifications for the question of existence of supersymmetric Calabi-Yau cycles [6].

It would be very interesting to consider the generalization to the nonextremal case and investigate possible phase transitions to genuine black holes, as well as connections to the recently discussed non-BPS states in string and Yang-Mills theory [27]. It could also be interesting to explore the phenomenological implications of the gravitational correction.

Unfortunately, the use of these solutions for the number theoretical considerations in [6] is (at first sight) limited, since they don't have horizons and hence no macroscopic entropy to connect to counting problems. However, the generalizations mentioned above could change this.

Finally, we want to point out that some caution is needed in interpreting these results. Everything is done in the low energy approximation, neglecting higher derivatives, nonabelian excitations, and the deformation of the geometry of the Calabi-Yau and the direct product structure of spacetime due to the presence of the brane wrapped around a CY cycle at the origin of space. It would be worthwile to study the modifications in the picture presented here when some of these corrections are taken into account. Some recent work relevant to this problem can be found in [16, 28].

Acknowledgements

I am very grateful to G. Chalmers, G. Moore, M. Rocek and R. von Unge for useful and stimulating correspondence, and to B. Craps, F. Roose, J. Troost, W. Troost and A. Van Proeyen for enlightening discussions. This work was supported by the European Commission TMR program ERBFMRX-CT96-0045. F.D. is Aspirant of the Belgian FWO.

Appendix

In this appendix, we present a general argument leading to the form (3.30) of the Kähler potential. Explicit examples, supporting the general results found here, can be found in [8].

We assume X to be an algebraic Calabi-Yau manifold which in the $\Lambda \to \infty$ limit degenerates in a certain coordinate patch to the product of an infinite cylinder (the complex plane punctured at the origin), parametrized by the coordinate z, and an A_{N-1} singular compact manifold M. More precisely, for $\Lambda \to \infty$, in the patch parametrized by z, w, y, x, the polynomial defining X is, to order Λ^{-1}, given by $W \approx W_{ALE} + W'(w, y, x)$, where W_{ALE} is defined as in (3.28) and W' is a polynomial containing terms of order x^{N+1}, w^3, y^3 and higher, possibly still depending on moduli different from the u^i (which will not be important in what follows). We furthermore assume the holomorphic 3-form Ω, to order Λ^{-1}, to have the following natural form:

$$\Omega \approx \frac{1}{\pi \mathrm{i}} \frac{\mathrm{d}z}{z} \wedge \frac{\mathrm{d}x \wedge \mathrm{d}y}{\partial_w W} \tag{A.1}$$

At large Λ, close to the locus $S : x = y = w = 0$ where the singularity develops, the polynomial W describing X is then given by (3.28), and Ω by (3.29).

To study the *ALE* fibration structure close to S, it is convenient to rescale $w = \Lambda^{-1/2}\tilde{w}$, $y = \Lambda^{-1/2}\tilde{y}$ and $x = \Lambda^{-1/N}\tilde{x}$. "Close to S" is equivalent to finite values of the rescaled variables, and there X is approximately described by

$$\tilde{w}^2 + \tilde{y}^2 + \tilde{x}^N + c_{N-2}(z)\tilde{x}^{N-2} + \cdots + c_0(z) = 0 \tag{A.2}$$

while

$$\Omega \approx \Lambda^{-1/N} \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}z}{z} \wedge \frac{\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}}{\tilde{w}}. \tag{A.3}$$

Now choose a basis $\{\Gamma_I\}_I$ of 3-cycles which are compact on the *ALE* fibration, i.e. finite in the rescaled variables (see e.g. [1, 8] for explicit constructions). Denote the (nondegenerate) intersection matrix by $Q_{IJ} = \Gamma_I \cdot \Gamma_J$. Extend this basis with 3-cycles $\Gamma'_{I'}$ to a basis for $H_3(X, \mathbb{Q})$, in such way that $\Gamma_I \cdot \Gamma'_{I'} = 0$. Since Q_{IJ} is nondegenerate, this is always possible (however, in general it is *not* possible to construct such a basis for $H_3(X, \mathbb{Z})$). Note that since there are no intersections of the Γ cycles with the Γ' cycles, there is no obstruction for deforming the Γ' cycles away from the neighbourhood of S (where the Γ cycles are localized). We will furthermore assume that we can keep the Γ' cycles at finite values of x, y, w when $\Lambda \to \infty$ (which is not a strong assumption). Explicit constructions in specific examples are discussed in [8].

The Γ periods $\int_\Gamma \Omega$ can be seen from (A.3) to be proportional to $\Lambda^{-1/N}$. As indicated in the paper, equation (3.33), they can be reduced to Seiberg-Witten Riemann surface periods [1].

We now turn to the Γ' periods. These can be divergent when $\Lambda \to \infty$, but since the Γ cycles stay finite in x, y, w and away from the singularity locus S (such that

$\partial_w W$ is bounded from below), the only potential source of divergencies is the fact that X factorizes as a direct product (W becomes independent of z) when $\Lambda \to \infty$: consequently, some cycles can (and will) be "stretched" to infinity in the z-plane. Since W is polynomial in z and $1/z$, this will give period integrals of the form

$$\int_{\Lambda^{-q}}^{\Lambda^p} \frac{\mathrm{d}z}{z} \times (\text{finite}), \tag{A.4}$$

which to leading order are proportional to $\ln \Lambda$. So we have

$$\int_{\Gamma'_{I'}} \Omega = a_I \ln \Lambda + b_I + \cdots \tag{A.5}$$

where a_I and b_I could a priori still be dependent on all other moduli. We will now show that the leading order u-dependent term is actually at most proportional to $\Lambda^{-2/N} \ln \Lambda$, so that a_I and b_I must be independent of the rigid moduli u^i. Indeed, using (A.1), the form of W and (3.28), we find

$$\frac{\partial}{\partial u^k} \int_{\Gamma'} \Omega \;=\; -\int_{\Gamma'} \Omega \, \frac{1}{\partial_w W} (\partial_w - \frac{\partial_w^2 W}{\partial_w W}) \frac{\partial W}{\partial u^k} \tag{A.6}$$

$$= \Lambda^{-1+k/N} \int_{\Gamma'} \Omega \, \frac{\partial_w^2 W}{(\partial_w W)^2} \, x^k. \tag{A.7}$$

Again because Γ' stays finite in x, y, w and stays away from the singular locus S (such that $\partial_w W$ stays bounded from below), the integral factor above can be at most proportional to $\ln \Lambda$, and since $k \leq N-2$, the full period at most to $\Lambda^{-2/N} \ln \Lambda$.[13] This is what we wanted to show.

Combining all this to compute the form of the Kähler potential, we find

$$\mathcal{K} \;=\; -\ln(\mathrm{i} \int_X \Omega \wedge \bar{\Omega}) \tag{A.8}$$

$$= \; -\ln(Q^{I'J'} \int_{\Gamma'_{I'}} \Omega \int_{\Gamma'_{J'}} \bar{\Omega} \; + \; Q^{IJ} \int_{\Gamma_I} \Omega \int_{\Gamma_J} \bar{\Omega}) \tag{A.9}$$

$$= \; -\ln(a \ln |\Lambda|^2 + b + |\Lambda|^{-2/N} K(u, \bar{u}) + \cdots), \tag{A.10}$$

where a and b are u-independent constants and the dots include u-independent terms of nonzero order in $\Lambda^{-1/N}$, and u-dependent terms higher than second order in $\Lambda^{-1/N}$. The reason for the absence of terms proportional to $\ln \Lambda \ln \bar{\Lambda}$ or $\ln(\Lambda/\bar{\Lambda})$ is the fact that $e^{-\mathcal{K}} = \mathrm{i} \int_X \Omega \wedge \bar{\Omega}$ must be invariant under the monodromy $\Lambda \to e^{2\pi \mathrm{i} N} \Lambda$. Finally, expanding the logarithm, we find

$$\mathcal{K} \approx -\ln(a \ln |\Lambda|^2 + b) + \frac{|\Lambda|^{-2/N}}{a \ln |\Lambda|^2 + b} \, K(u, \bar{u}). \tag{A.11}$$

[13] Note that this argument fails for the Γ periods because on these $\partial_w W$ is *not* bounded from below, and indeed the u-derivatives of those periods are proportional to $\Lambda^{-1/N}$.

The presence of the divergent term can also be inferred from (A.1) and the fact that X degenerates to the direct product of an infinite cylinder and a compact manifold. The integral over the cylinder (parametrized by z) gives a logarithmically divergent factor. Note that this divergent term would be absent for compactifications on e.g. the direct product of a torus and $K3$ (yielding $\mathcal{N} = 4$ in four dimensions). Such $a = 0$ cases could be discussed along the same lines, though some features will be qualitatively different.

References

[1] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Nonperturbative Results on the Point Particle Limit of N=2 Heterotic String Compactifications*, Nucl. Phys. **B459** (1996) 537, hep-th/9508155;
A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Self-Dual Strings and N=2 Supersymmetric Field Theory*, Nucl. Phys. **B477** (1996) 746, hep-th/9604034;
S. Katz, A. Klemm and C. Vafa, *Geometrical Engineering of Quantum Field Theories*, Nucl. Phys. **B497** (1997) 173, hep-th/9609239;
S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and Exact Solution of 4D N=2 Gauge Theories I*, Adv. Theor. Math. Phys. **1** (1998) 53, hep-th/9706110;
A. Klemm, *On the Geometry behind N=2 Supersymmetric Effective Actions in Four Dimensions*, hep-th/9705131;
W. Lerche, *Introduction to Seiberg-Witten Theory and its Stringy Origin Nucl. Phys. Proc. Suppl.* 55B (1997) 83, *Fortsch. Phys.* 45 (1997) 293, hep-th/9611190.

[2] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three dimensional gauge dynamics*, Nucl. Phys. **B492** (1997) 152, hep-th/9611230;
E. Witten *Solutions of Four-dimensional Field Theories via M Theory*, Nucl. Phys. **B500** (1997) 3, hep-th/9703166;
M. Douglas and M. Li, *D-brane realization of $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions*, hep-th/9604041;
P. Howe, N. Lambert and P. West, *Classical M-fivebrane dynamics and quantum N=2 Yang-Mills*, Phys. Lett. **B418** (1998) 85, hep-th/9710034;
N. Lambert and P. West, *Gauge fields and M-fivebrane dynamics*, Nucl. Phys. **B524** (1998) 141, hep-th/9712040;
N. Lambert and P. West, *Brane dynamics and four-dimensional quantum field theory*, hep-th/9811177.

[3] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998)231, hep-th/9711200;
S. Gubser, I. Klebanov and A. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105, hep-th/9802109

E. Witten, *Anti de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[4] S. Ferrara, R. Kallosh and A. Strominger, *N=2 Extremal Black Holes*, Phys. Rev. **D52** (1995) 5412, hep-th/9508072.

[5] S. Ferrara, G. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, Nucl. Phys. **B500** (1997) 75, hep-th/9702103.

[6] G. Moore, *Arithmetic and attractors*, hep-th/9807087

[7] G. Moore, *Attractors and arithmetic*, hep-th/9807056

[8] M. Billó, F. Denef, P. Frè, I. Pesando, W. Troost, A. Van Proeyen and D. Zanon, *The rigid limit in Special Kahler geometry*, Class. Quant. Grav. **15** (1998) 2083, hep-th/9803228;
M. Billó, F. Denef, P. Frè, I. Pesando, W. Troost, A. Van Proeyen and D. Zanon, *Special geometry of Calabi-Yau compactifications near a rigid limit*, hep-th/9801140;
S. Cacciatori and D. Zanon, *Gravitational corrections to N=2 supersymmetric Lagrangians*, hep-th/9809202.

[9] T. Banks, M. Douglas and N. Seiberg, *Probing F-theory with branes*, Phys. Lett. **B387** (1996) 278 hep-th/9605199.

[10] A. Sen, *BPS states on a three brane probe*, Phys. Rev. **D55** (1997) 2501, hep-th/9608005.

[11] M.R. Gaberdiel, T. Hauer and B. Zwiebach, *Open string-String junction transitions*, Nucl. Phys. **B525** (1998) 117, hep-th/9801205;
O. Bergman and A. Fayyazuddin, *String junctions and BPS states in Seiberg-Witten theory*, Nucl. Phys. **B531** (1998) 108, hep-th/9802033;
A. Mikhailov, N. Nekrasov and S. Sethi, *Geometric realizations of BPS states in N=2 theories* Nucl. Phys. **B531** (1998) 345, hep-th/9803142;
O. DeWolfe, T. Hauer, A. Iqbal, B. Zwiebach, *Constraints on the BPS spectrum of N=2, D=4 theories with A-D-E flavor symmetry*, Nucl. Phys. **B534** (1998) 261, hep-th/9805220.

[12] J. Barrow, *Varying G and other constants*, gr-qc/9711084;
J. Webb, V. Flambaum, C. Curchill, M. Drinkwater and J. Barrow, *Evidence for time variation of the fine structure constant*, astro-ph/9803165.

[13] M. Volkov and D. Gal'tsov, *Gravitating non-abelian solitons and black holes with Yang-Mills fields*, hep-th/9810070.

[14] G. Chalmers, M. Rocek, R. von Unge, *Monopoles in quantum corrected N=2 super Yang-Mills theory*, hep-th/9612195.

[15] J. Gauntlett, Jq. Gomis and P. Townsend, *BPS bounds for world volume branes*, J. High Energy Phys. (1998) 01:003, hep-th/9711205.

[16] N. Lambert and P. West, *Monopole dynamics from the M-fivebrane*, hep-th/9811025.

[17] C. Callan and J. Maldacena, *Brane dynamics from the Born-Infeld action*, Nucl. Phys. **B513** (1998) 198;
A. Hashimoto, *The shape of branes pulled by strings*, Phys. Rev. **D57** (1998) 6441, hep-th/9711097.

[18] B. de Wit, P. Lauwers and A. Van Proeyen *Lagrangians of N=2 supergravity-matter systems*, Nucl. Phys. **B255** (1985) 569;
B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is special geometry?*, Nucl. Phys. **B503** (1997) 565, hep-th/9703082

[19] I. Gaida, H. Hollmann and J. Stewart, *Classical and quantum analysis of repulsive singularities in four dimensional extended supergravity*, hep-th/9811032.

[20] A. Strominger, *Macroscopic entropy of N=2 extremal black holes*, Phys. Lett. **B383** (1996) 39, hep-th/9602111;
S. Ferrara and R. Kallosh *Universality of supersymmetric attractors*, Phys. Rev. **D54** (1996) 1525, hep-th/9603090;
G. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, Phys. Rev. Lett. **77** (1996) 4992, hep-th/9607108;
R. Kallosh and A. Linde, *Black hole superpartners and fixed scalars*, Phys. Rev. **D56** (1997) 3509, hep-th/9611161;
P. Fré, *Supersymmetry and first order equations for extremal states: monoploes, hyperinstantons, black holes and p-branes*, Nucl. Phys. Proc. Suppl. 57 (1997) 52, hep-th/9701054.

[21] S.J. Rey, *Classical and quantum aspects of BPS black holes in N=2, D=4 heterotic string compactifications*, Nucl. Phys. **B508** (1997) 569, hep-th/9610157.

[22] K. Behrndt, G. Lopes Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, *Classical and quantum N=2 supersymmetric black holes*, Nucl. Phys. **B488** (1997) 236, hep-th/9610105;
M. Shmakova, *Calabi-Yau Black Holes*, Phys. Rev. **D56** (1997) 540, hep-th/9612076;
W. Sabra, *General static N=2 black holes*, Mod. Phys. Lett. **A12** (1997) 2585, hep-th/9703101;
W. Sabra, *Black holes in N=2 supergravity theories and harmonic functions*, Nucl. Phys. **B510** (1998) 247, hep-th/9704147;
K. Behrndt, D. Lüst and W. Sabra, *Stationary solutions of N=2 supergravity* Nucl. Phys. **B510** (1998) 264, hep-th/9705169;

[23] K. Behrndt, D. Lüst and W. Sabra, *Moving moduli, Calabi-Yau phase transitions and massless BPS configurations in type II superstrings*, Phys. Lett. **B418** (1998) 303, hep-th/9708065.

[24] A. Strominger, *Massless black holes and conifolds in string theory*, Nucl. Phys. **B451** (1995) 96, hep-th/9504090.

[25] G. 't Hooft, *The scattering matrix approach for the quantum black hole: an overview*, Int. J. Mod. Phys. A11 (1996) 4623-4688, gr-qc/9607022;
P. Aichelburg and R. Sexl, Gen. Rel. and Gravitation 2 (1971) 303, W. Bonner, Commun. Math. Phys. **13** (1969) 163.

[26] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl. Phys. **B426** (1994) 19, hep-th/9407087.

[27] O. Bergman, *Stable non-BPS dyons in N=2 SYM*, hep-th/9811064;
A. Sen, *D-branes on non-supersymmetric cycles*, hep-th/9812031.

[28] R. Koch and R. Tatar, *Higher derivative terms from threebranes in F theory*, hep-th/9811128.

[29] K. Hashimoto, H. Hata and N. Sasakura, *Multi-pronged strings and BPS saturated solutions in SU(n) supersymmetric Yang-Mills theory*, hep-th/9804164;
K. Lee and P. Li, *Dyons in N=4 supersymmetric theories and three-pronged strings*, hep-th/9804174.