Dynamics on Blowups of the Projective Plane

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Dynamics on blowups of the projective plane

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Contents

1 Introduction

In this paper we give a systematic construction of automorphisms of rational surfaces with positive entropy, and investigate their dynamics. Specific cases yield:

1. Surface automorphisms with the minimum possible positive entropy;
2. Attracting basins of full measure, and Julia sets of measure zero; and
3. The first examples of automorphisms of projective algebraic varieties with Siegel disks.

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Surface automorphisms. Let $F : S \to S$ be a holomorphic automorphism of a compact complex surface. By [Ca1], if the topological entropy $h(F)$ is positive, then a minimal model for S is either a K3 surface, an Enriques surface, a complex torus or a rational surface. While constructions of automorphisms with positive entropy are well-known in the first three cases, rather few are known for rational surfaces.

One can aim to construct a rational surface automorphism $F : S \to S$ with a prescribed action on the middle-dimensional cohomology. To make this precise, let

$$\pi : S \to \mathbb{P}^2$$

be a rational surface presented as the blowup of the projective plane at n distinct points $(p_1, \ldots, p_n)$. Let $\mathbb{Z}^{1,n}$ denote the lattice $\mathbb{Z}^{n+1}$ with the Minkowski inner product

$$(x \cdot x) = x^2 = x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2,$$

let $H \subset S$ be the preimage of a generic line in the plane, and let E_i be the exceptional curve lying over p_i. Then there is a natural *marking* isomorphism

$$\phi : \mathbb{Z}^{1,n} \to H^2(S, \mathbb{Z}),$$

defined on the standard basis by $\phi(e_0) = [H]$ and $\phi(e_i) = [E_i]$, $i = 1, \ldots, n$. This marking sends the Minkowski inner product to the intersection pairing on $H^2(S, \mathbb{Z})$.

Any vector $\alpha \in \mathbb{Z}^{1,n}$ with $\alpha^2 = -2$ determines a reflection $\rho : \mathbb{Z}^{1,n} \to \mathbb{Z}^{1,n}$ by $x \mapsto x + (x, \alpha)\alpha$. The *Weyl group* $W_n \subset O(\mathbb{Z}^{1,n})$ is the group generated by the reflections $(s_i)_0^{n-1}$ through the vectors

$$\begin{aligned} \alpha_0 &= e_0 - e_1 - e_2 - e_3 \quad \text{and} \\ \alpha_i &= e_i - e_{i+1}, \quad i = 1, \ldots, n-1. \end{aligned}$$

The *roots* $\Phi_n = \bigcup W_n(\alpha_i)$ of W_n are the orbits of the *simple roots* $(\alpha_0, \ldots, \alpha_{n-1})$; the latter form an integral basis for the *root lattice* $L_n = \bigoplus \mathbb{Z}\alpha_i \subset \mathbb{Z}^{1,n}$.

The Weyl groups for $3 \le n \le 8$ are isomorphic to the finite Coxeter groups $A_1 \times A_2$, A_4, D_5, E_6, E_7 and E_8, and are associated with classical del Pezzo surfaces. The Weyl groups for $n \ge 9$ are infinite, and for $n \ge 10$ they contain elements with spectral radius $\sigma(w) > 1$.

By a theorem of Nagata, if F is an automorphism of S, there is a unique element $w \in W_n$ making the diagram

$$\begin{array}{ccc} \mathbb{Z}^{1,n} & \xrightarrow{w} & \mathbb{Z}^{1,n} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ H^2(S,\mathbb{Z}) & \xrightarrow{F_*} & H^2(S,\mathbb{Z}) \end{array}$$

commute (§5). In this case we say w is *realized* by the automorphism F. By theorems of Gromov and Yomdin, the entropy of F is given by $h(F) = \log \sigma(w)$.

Coxeter elements. The product of the generators $(s_0, \ldots, s_{n-1})$, taken one at a time in any order, yields a *Coxeter element* $w \in W_n$. All Coxeter elements are conjugate, so the spectral radius $\lambda_n = \sigma(w)$ is well-defined.

We can now state our first result on automorphisms of positive entropy.

Theorem 1.1 *For $n \geq 10$, every Coxeter element $w \in W_n$ can be realized by a rational surface automorphism with entropy $h(F_n) = \log \lambda_n > 0$.*

In fact, the automorphism $F_n : S_n \to S_n$ can be chosen to have the following additional properties.

1. The surface S_n is the blowup of n distinct points $(p_i)_1^n$ lying on a cuspidal cubic curve $X \subset \mathbb{P}^2$ (§7).
2. There is a nowhere vanishing meromorphic 2-form η on S_n with a simple pole along the proper transform Y of X.
3. The automorphism satisfies $F_n^*(\eta) = \lambda_n \cdot \eta$, and thus it expands the volume element $\eta \wedge \overline{\eta}$.
4. The Julia set $J^+(F_n)$ has measure zero, and every $z \in S_n - J^+(F_n)$ converges under iteration to the unique singular point $p \in Y$ (§9).
5. The surface S_n equipped with the $\mathbb{Z}$-action generated by F_n is G-minimal in the sense of Manin (§12).

The first three properties determine F_n uniquely. The points $(p_i)_1^n$ admit a simple description in terms of an eigenvector satisfying $w(v) = \lambda_n^{-1} \cdot v$, namely one can take $p_i = (x_i, x_i^3) \in \mathbb{C}^2$, where $x_i = v_i + v_0/3$. This leads to concrete formulas for F_n (§11).

Lehmer's automorphism. The smallest known Salem number $\lambda_{\text{Lehmer}} \approx 1.17628081$ is a root of Lehmer's polynomial

$$L(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1. \tag{1.1}$$

In the Appendix we will show:

Theorem 1.2 *If $F : S \to S$ is an automorphism of a compact complex surface with positive entropy, then $h(F) \geq \log \lambda_{Lehmer}$.*

It is easy to verify that $\lambda_{\text{Lehmer}} = \lambda_{10}$, and therefore:

Corollary 1.3 *The map $F_{10} : S_{10} \to S_{10}$ is a surface automorphism with the smallest possible positive entropy.*

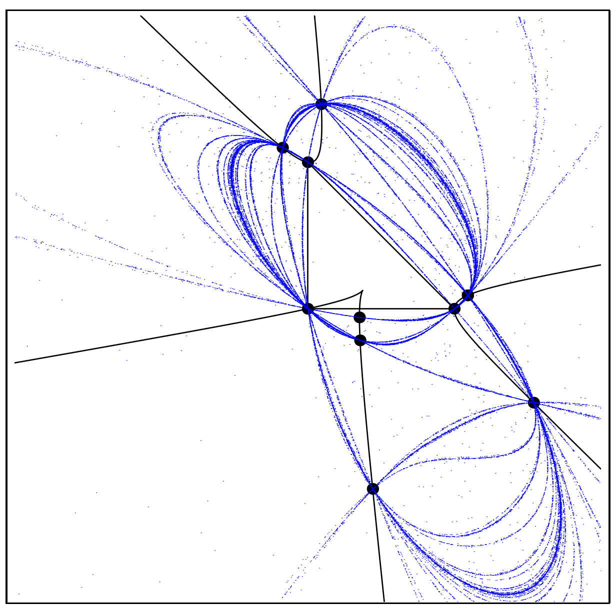

Figure 1. The expanding map F_{10} associated to Lehmer's number.

Picture in $\mathbb{RP}^2$. When suitably normalized, the projection $\pi : S_{10} \to \mathbb{P}^2$ transports F_{10} to a birational automorphism of the plane of the form $f_{10}(x, y) = (a, b) + (y, y/x)$. The geometry of this map is depicted in Figure 1: it blows up the vertices (p_1, p_2, p_3) of the central triangle, and blows its edges down to (p_2, p_3, p_4), where $p_4 = (a, b)$ (see §11). The remaining dots indicate the forward orbit $p_{4+i} = f^i(p_4)$, up to $p_{11} = p_1$. As shown, the points $(p_i)_1^{10}$ lie on a cuspidal cubic; blowing them up yields the surface S_{10} on which F_{10} acts. The scatter plot is an approximation to the Julia set $J^+(F_{10})$, obtained by backwards iteration of random points. As noted above, every $z \not\in J^+(F_{10})$ converges under forward iteration to the cusp inside the central triangle.

Siegel disks. A linear automorphism $R(z_1, z_2) = (\alpha z_1, \beta z_2)$ of $\mathbb{C}^2$ is an *irrational rotation* if $|\alpha| = |\beta| = 1$ and F has dense orbits on $S^1 \times S^1$. A domain $U \subset S$ is a *Siegel disk* for F if $F(U) = U$ and $F|U$ is analytically conjugate to $R|\Delta^2$ for some irrational rotation R. (Here $\Delta = \{z : |z| < 1\}$.)

It is easy to see an automorphism of a complex torus can never have a Siegel disk. A Siegel disk is possible on a K3 surface, but only when S is nonprojective [Mc2]. On the other hand, for rational surfaces we find (§10):

Theorem 1.4 *There are infinitely many n such that the standard Coxeter element $w \in W_n$ can be realized on a blowup of $\mathbb{P}^2$ by an automorphism with a Siegel disk.*

These maps arise from Galois conjugates of λ_n that lie on the unit circle. They preserve the natural volume form $\eta \wedge \overline{\eta}$, and also have positive entropy. Explicit examples, with $n = 11$ and 12, are given in §11.

Cubic curves. The automorphisms above are all constructed using marked blowups $\pi : S \to \mathbb{P}^2$ whose basepoints $(p_i)_1^n$ lie along a *cubic curve* $X \subset \mathbb{P}^2$.

Cubics play a distinguished role because the proper transform Y of X is then an anticanonical curve on S (an element of the linear system $|-K_S|$). This facilitates the construction of useful invariants in the spirit of Hodge theory.

More precisely, by restricting line bundles from S to Y, we obtain a map

$$\rho : \mathbb{Z}^{1,n} \xrightarrow{\phi} H^2(S, \mathbb{Z}) \cong \mathrm{Pic}(S) \to \mathrm{Pic}(Y) \cong \mathrm{Pic}(X)$$

which we regard as a marking of X. The basepoints are determined by the condition $\rho(e_i) = [p_i] \in \mathrm{Pic}(X)$, so the marked pair (S, Y, ϕ) can be reconstructed from (X, ρ). Provided X is irreducible, the marking ρ is essentially determined by its restriction to the root lattice

$$\rho_0 : L_n \to \mathrm{Pic}_0(X).$$

Here the target $\mathrm{Pic}_0(X)$ is a complex torus, $\mathbb{C}^*$ or $\mathbb{C}$ depending on whether X is a smooth, nodal or cuspidal cubic.

Fixed point formulation. To indicate the construction of automorphisms, first suppose $w \in W_n$ is already realized by a map $F \in \mathrm{Aut}(S)$ preserving Y. Then F covers a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, stabilizing X and inducing an automorphism $f_* : \mathrm{Pic}_0(X) \to \mathrm{Pic}_0(X)$ satisfying

$$\rho_0 \circ w = f_* \circ \rho_0.$$

In other words, $[\rho_0]$ is a fixed point for the natural action of w on the moduli space of markings

$$M_n(X) \subset \mathrm{Hom}(L_n, \mathrm{Pic}_0(X))/\,\mathrm{Aut}(X).$$

Conversely, to realize a given element $w \in W_n$ by a surface automorphism, we begin by locating a fixed point $[\rho_0] \in M_n(X)$. The marking determines basepoints $(p_i)_1^n$ on the cubic curve $X \subset \mathbb{P}^2$. Blowing them up, we obtain a rational surface $\pi : S \to \mathbb{P}^2$ marked by ϕ, with a distinguished anticanonical curve Y.

Now suppose $\rho_0(\alpha) \neq 0$ for all roots $\alpha \in \Phi_n$ (a generic condition). Then the basepoints $(p_i)_1^n$ satisfy no nodal relation (no two are coincident, no three are on a line, no six are on a conic, etc.). By a theorem of Nagata, this implies there is a second projection $\pi' : S \to \mathbb{P}^2$ corresponding to the marking $\phi \circ w$.

Let X' denote the cubic curve $\pi'(Y) \subset \mathbb{P}^2$. By the assumption that $[\rho_0]$ is a fixed point for w, the marked cubic $(X', \rho \circ w)$ is isomorphic to (X, ρ). This implies the marked blowups (S, ϕ) and $(S, \phi \circ w)$ are also isomorphic. But an isomorphism between these two marked blowups is exactly an automorphism $F : S \to S$ satisfying $F_* \circ \phi = \phi \circ w$, as desired.

Cuspidal cubics. The most flexible instance of the construction above arises when X is a cuspidal cubic. In this case we have $\mathrm{Pic}_0(X) \cong \mathbb{C}$, $[\rho_0]$ resides in the projective space

$$\mathrm{Hom}(L_n, \mathbb{C})/\mathbb{C}^* \cong \mathbb{P}^{n-1},$$

and a marking fixed by w is simply an eigenvector $v \in \mathrm{Hom}(L_n, \mathbb{C})$.

This method naturally yields a generalization of Theorem 1.1:

Theorem 1.5 *Suppose $w \in W_n$ has spectral radius $\sigma(w) > 1$ and no periodic roots (every orbit of $w|\Phi_n$ is infinite). Then w is realized by a surface automorphism of positive entropy.*

In fact we obtain many distinct realizations of w, depending on the choice of the eigenvector v (§7). The Siegel disk examples are constructed similarly, using reducible cubics: a conic with a tangent line, or three lines through a point.

It would be interesting to have a more complete classification of realizable elements in the Weyl group, and of the automorphisms they determine.

Notes and references. The relationship of the Weyl group to the birational geometry of the plane is discussed in Kantor's 1895 book [Kan], and has been much developed since then [Cob], [DV], [Nag1], [Nag2], [Giz], [Lo], [Nik], [Hrw], [Ha1], [Ha4], [Ha3], [Zh], [DZ]; see also the texts [Man2] and [DO]. Similar constructions relating surface automorphisms and cubic curves appear in [Hrw] and [Ha1, §4].

The Coble surfaces, obtained by blowing up the 10 double nodes of a rational plane sextic, also admit automorphisms of positive entropy [Cob, §52]. The Lehmer automorphism F_{10} first appears in the Appendix to [BK1]. Another automorphism of positive entropy, residing on $\mathbb{P}^2$ blown up at 15 points, is studied in [HV] and [Tak].

For more on dynamics on K3 surfaces, see [Sil], [Wa], [Ca2] and [Mc2].

I would like to thank I. Coskun and the referee for useful remarks, and E. Bedford for bringing [BK1] to my attention.

2 Coxeter theory

In this section we review properties of the Weyl group W_n from the perspective of Coxeter theory.

The Weyl group. Given $n \geq 3$, let Γ_n denote the graph with vertices $S_n = \{s_0, \ldots, s_{n-1}\}$ shown in Figure 2. Define an $n \times n$ matrix by

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } s_i \text{ is joined to } s_j \text{ in } \Gamma_n, \text{ and} \\ 2 & \text{otherwise.} \end{cases}$$

The *Weyl group* associated to this diagram is the finitely-presented group

$$W_n \quad = \quad \langle s_0, \ldots, s_{n-1} \ : \ (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Note: when $n = 3$ the graph Γ_n has a single edge, joining s_1 to s_2.

Geometric action. Let $V_n = \mathbb{R}^{S_n}$, equipped with the inner product

$$B_n(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$$

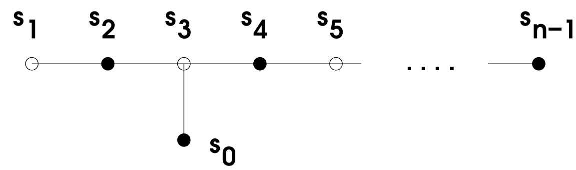

Figure 2. Coxeter graph Γ_n for (W_n, S_n).

on the natural basis (α_i) dual to (s_i). Any element $\alpha \in V_n$ with $B(\alpha, \alpha) = \pm 2$ determines a *reflection*

$$\rho_\alpha(x) = x - \frac{2B(x,\alpha)}{B(\alpha,\alpha)}\alpha \tag{2.1}$$

in the orthogonal group $\mathrm{O}(V_n, B_n)$. The unique homomorphism

$$W_n \to \mathrm{O}(V_n, B_n)$$

sending s_i to ρ_{α_i} defines the *geometric action* of W_n on V_n.

Lattices. Let $L_n \subset V_n$ denote the *root lattice* $\mathbb{Z}^{S_n} \subset \mathbb{R}^{S_n}$. Since the (α_i) form a basis for L_n, and $-2\cos(\pi/m_{ij}) = 2, 0$ or -1, the form $B_n | L_n \times L_n$ assumes integral values. Moreover L_n is invariant under the action of W_n by (2.1).

The lattice L_n is positive-definite for $n \le 8$, semidefinite for $n = 9$ and of signature $(n-1, 1)$ for $n \ge 10$. For $n = 3, \ldots, 8$ we obtain the well-known root-lattices $A_1 \oplus A_2$, A_4, D_5, E_6, E_7 and E_8 respectively. For $n = 10$, $L_n \cong \mathrm{II}_{9,1}$ is the unique even unimodular lattice of signature $(9, 1)$ (see e.g. [CoS, Ch. 27]).

Roots. The basis elements $\alpha_i \in L_n, i = 0, 1, \ldots, n-1$ are the *simple roots* of W_n. Their orbits $\Phi_n = \bigcup_i W_n \cdot \alpha_i$ comprise the *roots* of W_n. A vector $v = \sum c_i \alpha_i \in V_n$ is *positive* if $c_i \ge 0$ for all i. The *positive roots* are denoted Φ_n^+.

Coxeter elements. The products

$$s_{\sigma(0)} s_{\sigma(1)} \cdots s_{\sigma(n-1)}$$

of the generators of W_n, taken one at a time in any order, are the *Coxeter elements* of (W_n, S_n).

General results. We can now state three results which follow from the general theory of Coxeter groups [Bou], [Hum, §5].

1. *The geometric representation of W_n is faithful.*
2. *Any root of W_n is positive or negative; that is, $\Phi_n = \Phi_n^+ \cup (-\Phi_n^+)$.*
3. *All Coxeter elements lie in a single conjugacy class in W_n.*

(The last statement depends on the fact that the Coxeter diagram Γ_n of W_n is a tree.)

Coxeter number. By (3) above, all Coxeter elements $w \in W_n$ have the same order h_n. We have $h_n = 6, 5, 8, 12, 18, 30$ for $n = 3, 4, 5, 6, 7, 8$ and, as we will see below, $h_n = \infty$ for $n \geq 9$.

Spectral radius. Let $A(\Gamma_n) = 2I - B_n$ denote the adjacency matrix of Γ_n, considered as an operator on V_n. We have $A(\Gamma_n)_{ij} = 1$ if s_i is connected to s_j by an edge in Γ_n, and $A(\Gamma_n)_{ij} = 0$ otherwise. It is straightforward to check:

Proposition 2.1 *The spectral radius $\sigma_n = \sigma(A(\Gamma_n))$ is a strictly increasing function of n, with $\sigma_9 = 2$.*

Bipartite theory. A special feature of the Weyl group is that its Coxeter graph Γ_n is *bipartite*; every edge joins an even vertex to an odd vertex. This suggests splitting V_n into the direct sum $V_n^0 \oplus V_n^1$ of the spans of the roots α_i with even and odd indices i, respectively. With respect to this splitting, the adjacency matrix has the form

$$A(\Gamma_n) = \begin{pmatrix} 0 & C_n^t \\ C_n & 0 \end{pmatrix}.$$

Consider the particular Coxeter element

$$w_n = (s_0 s_2 s_4 \cdots) \cdot (s_1 s_3 s_5 \cdots) = w_n^0 \cdot w_n^1.$$

Since generators of the same parity commute, their ordering within each factor of w_n is immaterial. Using the fact that $B_n = 2I - A(\Gamma_n)$, it is easy to see that

$$w_n^0 = \begin{pmatrix} -I & C_n^t \\ 0 & I \end{pmatrix} \quad \text{and} \quad w_n^1 = \begin{pmatrix} I & 0 \\ C_n & -I \end{pmatrix}.$$

Thus the Coxeter element itself is given by

$$w_n = \begin{pmatrix} C_n^t C_n - I & -C_n^t \\ C_n & -I \end{pmatrix}$$

with respect to the splitting $V_n^0 \oplus V_n^1$.

Positivity. By the Perron-Frobenius theorem there is a positive vector $v_n \in V_n$, unique up to scale, such that

$$A(\Gamma_n) \cdot v_n = \sigma_n v_n. \tag{2.2}$$

Let $G_n \subset V_n$ be the 2-dimensional vector space spanned by the even and odd parts of $v_n = v_n^0 + v_n^1$. By (2.2) we have $(C_n \cdot v_n^0, C_n^t \cdot v_n^1) = \sigma_n(v_n^1, v_n^0)$. Thus G_n is invariant under the Coxeter element w_n; indeed, we have

$$w_n|G_n = \begin{pmatrix} \sigma_n^2 - 1 & -\sigma_n \\ \sigma_n & -1 \end{pmatrix} \tag{2.3}$$

with respect to the basis (v_n^0, v_n^1).

Theorem 2.2 *The linear map $w_n|G_n$ is:*

- elliptic*, of order h_n, for $n \leq 8$;*
- parabolic*, of infinite order, for $n = 9$; and*
- hyperbolic*, of infinite order, for $n \geq 10$.*

Proof. By (2.3), $w_n|G_n$ has determinant 1 and trace $\sigma_n^2 - 2$; so it is elliptic when $\sigma_n < 2$, parabolic when $\sigma_n = 2$ and hyperbolic when $\sigma_n > 2$. These alternatives correspond to $n \leq 8$, $n = 9$ and $n \geq 10$ by Proposition 2.1; and in the elliptic case $w_n|G_n$ is actually a rotation by $2\pi/h_n$ [Hum, §3.7]. ∎

Theorem 2.3 *For $n \neq 9$, every root $\alpha \in \Phi_n$ has a nonzero orthogonal projection to $G_n \subset V_n$.*

Proof. Let β be the projection of α to G_n. We may assume $\alpha \in \Phi_n^+$. Then, since both $\alpha = \sum x_i \alpha_i$ and $v_n = \sum y_i \alpha_i$ are positive vectors, we have

$$B(v_n, \beta) = B(v_n, \alpha) = (2 - \sigma_n) \sum x_i y_i \neq 0$$

(using the fact that $\sigma_n \neq 2$ when $n \neq 9$); consequently $\beta \neq 0$. ∎

Corollary 2.4 *Let $w \in W_n$ be a Coxeter element.*

- *For $n < 9$, every orbit of $w|\Phi_n$ consists of h_n elements.*
- *For $n > 9$, every orbit of $w|\Phi_n$ is infinite; that is, w has no periodic roots.*

Remarks. For $n \geq 10$, the Weyl group acts isometrically on the hyperbolic space $\mathbb{H}^{n-1} \subset \mathbb{P}V_n$ determined by the indefinite form B, and G_n corresponds to the unique hyperbolic geodesic $\gamma_n \subset \mathbb{H}^{n-1}$ stabilized by the action of w_n. In geometric terms, Theorem 2.3 states that γ_n is not contained in any of the hyperplanes $H_\alpha \subset \mathbb{H}^{n-1}$ defined by the roots $\alpha \in \Phi_n$.

Even though $h_9 = \infty$, there are roots with periods 2, 3 and 5 under the action of the Coxeter element w_9. For more on periodic roots, see [Par].

Salem and Pisot numbers. An algebraic integer $\lambda > 1$ is a *Pisot number* if its Galois conjugates satisfy $|\lambda'| < 1$; it is a *Salem number* if its conjugates satisfy $|\lambda'| \leq 1$ and at least one conjugate lies on the unit circle. The smallest Pisot number is the root $\lambda_{\text{Pisot}} \approx 1.32471795$ of the polynomial $t^3 - t - 1$. The smallest *known* Salem number is the root $\lambda_{\text{Lehmer}} \approx 1.17628081$ of Lehmer's polynomial (equation (1.1)).

Salem numbers arise naturally as eigenvalues of Coxeter elements. Indeed, the characteristic polynomial of a Coxeter element $w \in W_n$ is given explicitly by

$$P_n(t) = \det(tI - w) = \frac{t^{n-2}(t^3 - t - 1) + (t^3 + t^2 - 1)}{t - 1}. \tag{2.4}$$

Compare [MRS, Lemma 5]. For $n \neq 9$ this polynomial has simple roots, and for $n \geq 10$ it factors as

$$P_n(t) = Q_n(t) R_n(t),$$

where $R_n(t)$ is a product of cyclotomic polynomials and $Q_n(t)$ is a Salem polynomial. The roots $t = \lambda_n^{\pm 1}$ of $Q_n(t)$ are simply the eigenvalues of $w_n|G_n$; the remaining roots lie on the unit circle.

It is easily checked that $Q_{10}(t)$ coincides with Lehmer's polynomial, and thus $\lambda_{10} = \lambda_{\text{Lehmer}}$. Inspection of (2.4) shows that as $n \to \infty$ we have

$$\lambda_n \to \lambda_{\text{Pisot}}.$$

(A similar construction shows that every Pisot number is a limit of Salem numbers [Sa, p.30].)

Since λ_{Pisot} is not itself a Salem number, we have $\deg(Q_n) \to \infty$. By [Bi] (see also [Rum]), this implies:

Theorem 2.5 *As $n \to \infty$, the roots of $Q_n(t)$ other than $\lambda_n^{\pm 1}$ become equidistributed on the unit circle.*

This result will be used in Theorem 10.5 to construct Siegel disks.

Leading eigenvalues. It is convenient to extend the factorization (2.4) to $n \leq 8$ by defining $Q_n(t)$ to be the cyclotomic polynomial for the h_n-th roots of unity, and to $n = 9$ by setting $Q_9(t) = (t - 1)$. With this convention, $Q_n(t)$ is irreducible for all n.

We say $\lambda \in \mathbb{C}$ is a *leading eigenvalue* for w if $Q_n(\lambda) = 0$. The leading eigenvalues are simply the eigenvalues of $w_n|G_n$ and their Galois conjugates. Their associated eigenvectors are *leading eigenvectors.*

Theorem 2.6 *Let $v \in L_n \otimes \mathbb{C}$ be a leading eigenvector for a Coxeter element $w \in W_n$. Then provided $n \neq 9$, we have $v \cdot \alpha \neq 0$ for all roots $\alpha \in \Phi_n$.*

Proof. The conclusion is formulated over $\mathbb{Q}$, so it suffices to prove the assertion when $w(v) = \lambda_n^{\pm 1} v$. In this case v belongs to $G_n \otimes \mathbb{C}$. Since $n \neq 9$, G_n is spanned by v and one of its Galois conjugates v'. If $v \cdot \alpha = 0$, then $v' \cdot \alpha = 0$ as well, so the projection of α to G_n is zero. This contradicts Theorem 2.3. ∎

Similar reasoning shows:

Theorem 2.7 *Suppose $w \in W_n$ has no periodic roots, and $w \cdot v = \lambda v$ where λ is not a root of unity. Then $0 \notin v \cdot \Phi_n$.*

Proof. Let $S \subset V_n$ be the span of the Galois conjugates of v, and suppose $v \cdot \alpha = 0$. Then $\alpha \in S^\perp$. But $(S^\perp, B)$ is positive-definite and w-invariant, so α is periodic. ∎

3 The Minkowski model

Next we discuss a natural action of the Weyl group on Minkowski space.

The Minkowski lattice. Let $\mathbb{R}^{1,n}$ denote $\mathbb{R}^{n+1}$ equipped with the Minkowski inner product

$$(x \cdot x) = x^2 = x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2.$$

The integral points $\mathbb{Z}^{1,n} \subset \mathbb{R}^{1,n}$ are a model for the unique odd unimodular lattice of signature $(1, n)$. Let $(e_0, e_1, \ldots, e_n)$ denote the standard basis in these coordinates.

Let $k_n = (-3, 1, 1, 1, \ldots, 1)$ denote the *canonical vector* in $\mathbb{Z}^{1,n}$, let

$$V_n = k_n^\perp \subset \mathbb{R}^{1,n},$$

and let

$$L_n = V_n \cap \mathbb{Z}^{1,n}.$$

The stabilizer $\mathrm{O}(L_n)$ of k_n in $O(\mathbb{Z}^{1,n})$ acts faithfully on L_n.

Reflections. Any $\alpha \in L_n$ with $\alpha^2 = -2$ determines a reflection in $O(L_n)$ by

$$\rho_\alpha(x) = x + (x \cdot \alpha)\alpha.$$

The simplest such is the *transposition* τ_{ij}, given by reflection in the vector

$$\alpha_{ij} = e_i - e_j \tag{3.1}$$

for distinct indices $i, j \geq 1$; it simply exchanges the basis elements e_i and e_j while fixing the others.

Cremona involutions. The *Cremona involution* $\kappa_{ijk} \in \mathrm{O}(L_n)$ is given by reflection in the vector

$$\alpha_{ijk} = e_0 - e_i - e_j - e_k \tag{3.2}$$

for distinct indices $i, j, k \geq 1$. It acts by

$$\begin{array}{rcl} e_0 & \mapsto & 2e_0 - e_i - e_j - e_k, \\ e_i & \mapsto & e_0 - e_j - e_k, \\ e_j & \mapsto & e_0 - e_i - e_k, \\ e_k & \mapsto & e_0 - e_i - e_j, \quad \text{and} \\ e_l & \mapsto & e_l \quad \text{if } l \notin \{0, i, j, k\}. \end{array}$$

We will see in §5 that κ_{123} arises naturally from the standard quadratic Cremona involution on $\mathbb{P}^2$.

The Weyl group, *reprise*. An integral basis for L_n is given by

$$(\alpha_0, \ldots, \alpha_{n-1}) = (\alpha_{123}, \alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1,n}).$$

In this basis, the Minkowski inner product satisfies

$$\alpha_i \cdot \alpha_j = 2\cos(\pi/m_{ij}) = -B_n(\alpha_i, \alpha_j),$$

and thus $(L_n, -x^2)$ is isometric to the root lattice $(\mathbb{Z}^n, B_n)$ for the Weyl group W_n. Identifying these two lattices, we obtain a representation

$$W_n \subset O(L_n) \subset O(\mathbb{Z}^{1,n})$$

that extends the geometric action from L_n to $\mathbb{Z}^{1,n}$. In this model, W_n is generated by the reflections

$$S_n = \{s_0, \ldots, s_{n-1}\} = \{\kappa_{123}, \tau_{12}, \tau_{23}, \ldots, \tau_{n-1,n}\}$$

through the simple roots $\alpha_0, \ldots, \alpha_{n-1} \in L_n$.

Note that the subgroup generated by $(s_1, \ldots, s_{n-1})$ is a copy of the symmetric group Σ_n, acting by permutations on the basis elements $(e_1, \ldots, e_n)$. The Weyl group also contains all the Cremona involutions, since κ_{ijk} is conjugate to κ_{123} under the action of Σ_n.

Positive roots. The positive roots $\alpha = de_0 - \sum_1^n m_i e_i$ have a convenient form in the Minkowski model.

1. First, the conditions $k_n \cdot \alpha = 0$ and $\alpha^2 = -2$ translate into

$$\sum m_i = 3d \quad \text{and} \quad \sum m_i^2 = d^2 + 2. \tag{3.3}$$

   Positivity implies $d \geq 0$. The positive roots with $d = 0$ are given by $\alpha = \alpha_{ij}$ as in (3.1); those with $d = 1$, by $\alpha = \alpha_{ijk}$ as in (3.2).

2. Positive roots with $d > 0$ are invariant under the action of Σ_n, and satisfy $m_i \geq 0$. Thus any such root can be *normalized* so that $m_1 \geq m_2 \geq \cdots \geq m_n$.

3. A normalized positive root with $d > 1$ satisfies $0 \leq m_i$ and $m_1 + m_2 \leq d < m_1 + m_2 + m_3$. See [DO, Prop. 4, p. 74].

The normalized positive roots $\alpha = (d, -m_1, \ldots, -m_n)$ with $1 \leq d \leq 4$ are

$$\begin{array}{ccc} (1, -1^3, 0^{n-3}), & (2, -1^6, 0^{n-6}), & (3, -2^1, -1^7, 0^{n-8}), \\ (4, -2^3, -1^6, 0^{n-10}), & (4, -3, -1^9). & \end{array}$$

Here k^i indicates that k is repeated i times; a given root does not occur for W_n if the exponent of 0 is negative.

4 Marked cubic curves

In this section we introduce marked cubic curves (X, ρ). These objects will form the basis for constructing rational surfaces with given automorphisms.

Cubic curves. A *cubic curve* $X \subset \mathbb{P}^2$ is a reduced curve of degree three. We allow X to be singular or reducible, and denote its smooth points by X^*.

Picard group. The Picard group of X is described by the exact sequence

$$0 \to \mathrm{Pic}_0(X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to 0,$$

where $\mathrm{Pic}_0(X)$ is isomorphic to either:

1. A compact torus $\mathbb{C}/\Lambda$ (when X is a smooth); or
2. The multiplicative group $\mathbb{C}^*$ (when X is a nodal cubic, or a conic with a transverse line, or three lines meeting in three points); or
3. The additive group $\mathbb{C}$ (when X is a cuspidal cubic, a conic with a tangent line or three lines through a single point).

(See e.g. [HM, Chap. 5B].) Every element of $\mathrm{Pic}(X)$ is represented by a divisor $D = \sum n_j p_j$ supported in X^*.

Automorphisms. Let $\mathrm{Aut}(X)$ denote the automorphism group of X as an abstract complex variety. When X is irreducible, it is a familiar fact that $\mathrm{Aut}(X)$ acts transitively on its smooth points X^*. For general cubics, one can easily check:

Proposition 4.1 *Any set E consisting of one point from each component of X^* is equivalent, under* $\mathrm{Aut}(X)$*, to any other such collection.*

The derivative $D(f)$. Let $\Omega(X)$ denote the space of sections of the dualizing sheaf ω_X. (When X is smooth, $\Omega(X)$ is just the space of holomorphic 1-forms on X.) Since X has arithmetic genus one, $\Omega(X)$ is one-dimensional. Thus we have a natural homomorphism

$$D : \mathrm{Aut}(X) \to \mathbb{C}^*$$

characterized by $f^*\omega = D(f)\omega$ for all $\omega \in \Omega(X)$. Equivalently, $D(f)$ is the derivative of f_* on the tangent space to the origin in $\mathrm{Pic}_0(X)$. If X is irreducible, then f acts on the universal cover $\widetilde{X^*} \cong \mathbb{C}$ by $f(z) = D(f)z + c$.

Proposition 4.2 *If* $\mathrm{Pic}_0(X) \not\cong \mathbb{C}$ *then* $D(f)$ *is a* k*th root of unity, where* $k = 1, 2, 3, 4$ *or* 6.

Proof. For any cubic curve we have $\mathrm{Pic}_0(X) \cong \mathbb{C}/\Lambda$ for some discrete group $\Lambda \subset \mathbb{C}$, and $D(f)\Lambda = \Lambda$ for all $f \in \mathrm{Aut}(f)$. Thus $D(f)$ must be a root of unity as above, unless Λ is trivial. ■

Particular cubics. The cubic curves with $\mathrm{Pic}_0(X) \cong \mathbb{C}$ — namely the cuspidal cubic, a conic with a tangent line and three lines through a point — will play a leading role in the sequel. In these cases $f \in \mathrm{Aut}(X)$ acts on $\mathrm{Pic}_0(X)$ by $E \mapsto D(f)E$, where $D(f) \in D(\mathrm{Aut}(X)) = \mathbb{C}^*$. For example, the automorphisms of a cuspidal cubic act on $X^* \cong \mathbb{C}$ by $f(t) = at + b$, with $D(f) = a$. The fact that we can have $|D(f)| > 1$ makes these cubics suitable for the construction of automorphisms with positive entropy.

Marked cubics. A *marked cubic curve* (X, ρ) is an abstract curve X equipped with a homomorphism $\rho : \mathbb{Z}^{1,n} \to \mathrm{Pic}(X)$, such that

1. The sections of the line bundle $\rho(e_0)$ provide an embedding $X \hookrightarrow \mathbb{P}^2$, making X into a cubic curve; and
2. There are distinct *basepoints* $p_i \in X^*$ such that $\rho(e_i) = [p_i]$ for $i = 1, 2, \ldots, n$.

The basepoints p_i are uniquely determined by ρ, since X^* embeds into $\mathrm{Pic}(X)$. Conversely, a cubic embedding $X \hookrightarrow \mathbb{P}^2$ together with a choice of distinct points $p_i \in X^*$ determines a marking of X.

We emphasize that different markings of X can yield different projective embeddings $X \hookrightarrow \mathbb{P}^2$ (e.g. different locations for its flexes); but these embeddings are all equivalent under the action of $\mathrm{Aut}(X)$.

Isomorphism. An *isomorphism* $(X, \rho) \cong (X', \rho')$ is a biholomorphic map $f : X \to X'$ such that $\rho' = f_* \circ \rho$. We define

$$\begin{aligned} W(X,\rho) &= \{w \in W_n \,:\, (X, \rho \circ w) \text{ is a marked cubic}\}, \quad \text{and} \\ \mathrm{Aut}(X,\rho) &= \{w \in W(X,\rho) \,:\, (X,\rho) \cong (X, \rho \circ w)\}. \end{aligned}$$

Action of $\mathrm{Aut}(X)$. It is convenient to break up the marking ρ of X into two pieces: the map

$$\rho_0 : \mathrm{Ker}(\deg \circ \rho) \to \mathrm{Pic}_0(X),$$

and

$$\deg \circ \rho : \mathbb{Z}^{1,n} \to H^2(X, \mathbb{Z}).$$

Proposition 4.1 readily implies:

Theorem 4.3 *The maps ρ_0 and $\deg \circ \rho$ determine (X, ρ) up to isomorphism.*

When X is irreducible, we have $\deg(\rho(u)) = -u \cdot k_n$, and thus:

Corollary 4.4 *An irreducible marked cubic (X, ρ) is determined up to isomorphism by $\rho_0 : L_n \to \mathrm{Pic}_0(X)$.*

Moduli spaces. The moduli space of markings of X is given by:

$$M_n(X) = \{\rho : \mathbb{Z}^{1,n} \to \mathrm{Pic}(X) \ : \ (X, \rho) \text{ is a marked cubic}\} / \mathrm{Aut}(X).$$

A second model for $M_n(X)$ is obtained by fixing an embedding $X \subset \mathbb{P}^2$; then a marking is determined by a choice of basepoints, and we have

$$M_n(X) \cong ((X^*)^n - \Delta) / \mathrm{Aut}(\mathbb{P}^2, X).$$

Here $\Delta = \{(p_i) \ : \ p_j = p_k \text{ for some } j \neq k\}$.

In the irreducible case, another rather explicit model is given by

$$M_n(X) \cong \{\rho_0 : L_n \to \mathrm{Pic}_0(X) \ : \ \rho_0(e_i - e_j) \neq 0 \ \forall i > j \geq 1\} / \mathrm{Aut}(X).$$

When $\mathrm{Pic}_0(X) \cong \mathbb{C}$, this model exhibits $M_n(X)$ as the complement of finitely many hyperplanes in the projective space

$$\mathrm{Hom}(L_n, \mathbb{C}) / \mathbb{C}^* \cong \mathbb{C}^n / \mathbb{C}^* = \mathbb{P}^{n-1}.$$

5 Marked blowups

In this section we summarize the connection between the Weyl group W_n and the blowups of the projective plane at n points. See also [Man2], [Ha1] and [DO].

Marked blowups. A *marked blowup* (S, ϕ) is a smooth projective surface S equipped with an isomorphism

$$\phi : \mathbb{Z}^{1,n} \to H^2(S, \mathbb{Z})$$

such that:

1. The marking ϕ sends the Minkowski inner product to the intersection pairing;
2. There exists a birational morphism $\pi : S \to \mathbb{P}^2$, presenting S as the blowup of the projective plane at n distinct *basepoints* $p_1, \ldots, p_n$; and

3. The marking satisfies $\phi(e_0) = [H]$ and $\phi(e_i) = [E_i]$, $i = 1, \ldots, n$, where $H = \pi^{-1}(L)$ is the preimage of a generic line in $\mathbb{P}^2$ and $E_i \subset S$ is the exceptional curve $\pi^{-1}(p_i)$.

The marking determines $\pi : S \to \mathbb{P}^2$ up to post-composition with an automorphism of $\mathbb{P}^2$. Note that the canonical class of S is given by

$$K_S = \left[-3H + \sum E_i\right] = \phi(k_n).$$

As in the preceding sections, we assume $n \geq 3$.

Moduli. An *isomorphism* $(S, \phi) \cong (S', \phi')$ is given by a biholomorphic map $F : S \to S'$ such that the diagram

$$\begin{tikzcd} \mathbb{Z}^{1,n} \arrow[r, equal] \arrow[d, "\phi"] & \mathbb{Z}^{1,n} \arrow[d, "\phi'"] \\ H^2(S, \mathbb{Z}) \arrow[r, "F_*"] & H^2(S', \mathbb{Z}) \end{tikzcd} \tag{5.1}$$

commutes. In this case $p'_i = g(p_i)$ for some $g \in \mathrm{PGL}(3, \mathbb{C}) \cong \mathrm{Aut}(\mathbb{P}^2)$. Thus the moduli space of marked blowups is given by the configuration space

$$\mathcal{P}_n = ((\mathbb{P}^2)^n - \Delta) / \mathrm{PGL}_3(\mathbb{C}),$$

where $\Delta = \{(p_i) \,:\, p_j = p_k \text{ for some } j \neq k\}$.

Role of the Weyl group. Now suppose there are two birational morphisms $\pi, \pi' : S \to \mathbb{P}^2$, exhibiting S as the blowup of $\mathbb{P}^2$ at (p_i) and (p'_i) respectively. Then there is a birational map f making the diagram

$$\begin{tikzcd} S \arrow[r, equal] \arrow[d, "\pi"] & S \arrow[d, "\pi'"] \\ \mathbb{P}^2 \arrow[r, dashed, "f"] & \mathbb{P}^2 \end{tikzcd} \tag{5.2}$$

commute; and the corresponding markings are related by $\phi' = \phi \circ w$ for a unique $w \in \mathrm{O}(\mathbb{Z}^{1,n})$.

The following signal result shows these different blowups are related by the action of the Weyl group.

Theorem 5.1 (Nagata) *Let (S, ϕ) be a marked blowup, and $w \in O(\mathbb{Z}^{1,n})$. If $(S, \phi \circ w)$ is also a marked blowup, then $w \in W_n$.*

See [Nag2, p.283], [DO, p.90, Thm. 2].

Cremona involutions. Let

$$W(S,\phi) = \{w \in W_n \ : \ (S, \phi \circ w) \text{ is a marked blowup}\}.$$

The right action of the symmetric group simply reorders the basepoints of a blowup, so we have

$$\Sigma_n \subset W(S,\phi). \tag{5.3}$$

To obtain more elements in $W(S,\phi)$, consider the standard quadratic *Cremona involution* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by

$$f[x,y,z] = [yz, xz, xy]. \tag{5.4}$$

This map blows up the three points $(q_1, q_2, q_3) = ([1,0,0],[0,1,0],[0,0,1])$ and blows down the lines between them.

Theorem 5.2 *If $p_k \notin \overline{p_i p_j}$ for all distinct indices with $i, j \in \{1,2,3\}$, then $(S, \phi \circ \kappa_{123})$ is a marked blowup.*

Proof. Choose coordinates so that $p_i = q_i$ for $i = 1,2,3$; then $\pi' = f \circ \pi : S \to \mathbb{P}^2$ is a birational morphism, presenting $(S, \phi \circ \kappa_{123})$ as a marked blowup with basepoints p_1, p_2, p_3 and $f(p_i)$, $i \geq 4$. These points are distinct so long as $(p_4, \ldots, p_n)$ lie outside the lines blown down by f. ∎

Nodal roots. We say $\alpha \in \Phi_n$ is a *nodal root* for (S,ϕ) if $\phi(\alpha) \in H^2(S,\mathbb{Z})$ is represented by an effective divisor D. In this case D projects to a curve of degree $d > 0$ on $\mathbb{P}^2$, and hence $\alpha = de_0 - \sum m_i e_i$ is a positive root with $d > 0$.

A nodal root is *geometric* if we can take D to be a sum of smooth rational curves.

Example. To see the distinction, let E be a smooth elliptic curve through points $(p_i)_1^9$ in general position, subject only to the condition that (p_1, p_2, p_3) lie on a line L. Identifying these curves with their strict transforms on the blowup with basepoints $(p_i)_1^9$, we obtain a nodal root

$$\alpha = \left(e_0 - \sum_1^3 e_i\right) + \left(3e_0 - \sum_1^9 e_i\right) = (4, -2^3, -1^6)$$

represented uniquely by the effective divisor $L + E$. Since E is irrational, α is not geometric.

Theorem 5.3 *If (p_i) has three collinear points, then (S, ϕ) has a geometric nodal root.*

Proof. Let L be a line passing through three or more of the basepoints (p_i). After reordering, we can assume the points on L are $(p_1, \ldots, p_k)$. Then the strict transform C of L gives a smooth rational curve on S with $[C] = [H - \sum_1^k E_i]$, and thus $[C + \sum_4^k E_i] = \phi(\alpha_{123})$. ■

Theorem 5.4 *If (S, ϕ) has no geometric nodal roots, then $W(S, \phi) = W_n$.*

Proof. If (S, ϕ) has no geometric nodal roots and $w \in W(S, \phi)$, then $(S, \phi \circ w)$ also has no geometric nodal roots. Thus it suffices to show the generators of W_n belong to $W(S, \phi)$. This is immediate by equation (5.3) for the transpositions $\tau_{12}, \tau_{23}, \ldots$, and for κ_{123} it follows from the preceding two results. ■

Corollary 5.5 *A marked surface has a nodal root iff it has a geometric nodal root.*

Proof. Suppose (S, ϕ) has no geometric nodal roots, then $(S, \phi \circ w)$ is a marked blowup for all $w \in W_n$. If $[D] = \phi(\alpha)$ is a nodal root, then D maps to a plane curve of positive degree under each corresponding projection $S \to \mathbb{P}^2$, and thus

$$0 < w(e_0) \cdot \alpha = e_0 \cdot w^{-1}(\alpha)$$

for all $w \in W_n$. Taking w to be reflection through α yields a contradiction. ■

Realization. Let

$$\mathrm{Aut}(S, \phi) = \{w \in W(S, \phi) \ : \ (S, \phi) \cong (S, \phi \circ w)\}.$$

There is a natural surjection $\mathrm{Aut}(S) \to \mathrm{Aut}(S, \phi)$. Shifting focus, we say $w \in W_n$ is *realized* on (S, ϕ) if $w \in \mathrm{Aut}(S, \phi)$. In this case there is an $F \in \mathrm{Aut}(S)$ making the diagram

$$\begin{array}{ccc} \mathbb{Z}^{1,n} & \xrightarrow{w} & \mathbb{Z}^{1,n} \\ \downarrow \phi & & \downarrow \phi \\ H^2(S, \mathbb{Z}) & \xrightarrow{F_*} & H^2(S, \mathbb{Z}) \end{array}$$

commute. The map F is unique so long there are 4 points in general position among the $(p_i)_1^n$.

Note that $F : S \to S$ covers a unique birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, yielding a commutative diagram

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[d, "\pi"] & S \arrow[r, phantom, "="] \arrow[d, "\pi'"'] & S \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[r, phantom, "="] & \mathbb{P}^2 \arrow[r, dashed, "f"] & \mathbb{P}^2. \end{tikzcd} \tag{5.5}$$

Here π and π' correspond to the markings ϕ and $\phi \circ w$. This diagram is a combination of (5.1) and (5.2); the right square indicates that ϕ and $\phi \circ w$ are both marked blowups, and the left square indicates they are isomorphic.

Examples.

1. The whole Weyl group W_4 is realized by automorphisms when the basepoints $(p_i)_1^4$ are in general position (because $\operatorname{Aut} \mathbb{P}^2$ acts transitively on such configurations of points).

2. On the other hand, there is no realization of $\kappa_{123} \in W_8$. Indeed, any realization would give a quadratic Cremona involution $f : \mathbb{P}^2 \to \mathbb{P}^2$ based at (p_1, p_2, p_3) and fixing $(p_4, \dots, p_8)$; but a quadratic involution has only 4 fixed points.

3. Let $K_9 \subset W_9$ be the kernel of the natural map to $W_8/(\pm I)$, and let S be the elliptic surface obtained by blowing up the 9 basepoints of a generic pencil of cubics. Then every element $w \in K_9$ is realized on S; indeed, we have $K_9 \cong \operatorname{Aut}(S) \cong \mathbb{Z}^8 \ltimes (\mathbb{Z}/2)$. See [Cob, §52], [Giz], [DO, p.106].

4. Let $W_{10}(2)$ denote the congruence subgroup of W_{10} that acts trivially on $L_{10}/2L_{10}$, and let S be a surface obtained by blowing up the 10 nodes of a generic rational sextic in the plane. Then every element of $W_{10}(2)$ is realized on S [Cob, §52].

Fixed point formulation. Let $\mathcal{P}_n^* \subset \mathcal{P}_n$ denote the moduli space of marked blowups without nodal roots. Then W_n acts on $\mathcal{P}_n^*$ by Theorem 5.4, and:

> *The fixed points of $w \in W_n$ in $\mathcal{P}_n^*$ correspond to the surfaces on which it is realized.*

In particular the question of whether or not w can be realized by a surface automorphism depends only on its conjugacy class, provided we restrict attention to surfaces without nodal roots.

6 Synthesis

In this section we observe that a marked cubic curve determines a marked blowup S with a distinguished anticanonical curve Y. Their automorphism groups are related by the following two results.

Theorem 6.1 *Let (S,Y,ϕ) be the marked pair obtained by blowing up (X,ρ). Then we have* $\mathrm{Aut}(S,Y,\phi)=\mathrm{Aut}(X,\rho)\cap W(S,\phi)$.

Theorem 6.2 *If X is irreducible and $0\notin\rho(\Phi_n)$, then (S,ϕ) has no nodal roots and*

$$\mathrm{Aut}(S,\phi)\supset\mathrm{Aut}(X,\rho).$$

If $n\geq 10$ then equality holds.

These results *linearize* the problem of constructing surface automorphisms, by reducing it to the study of maps $\rho:\mathbb{Z}^{1,n}\to\mathrm{Pic}(X)$.

Marked pairs. Let (S,ϕ) be a marked blowup. An *anticanonical curve* is a reduced curve $Y\subset S$ whose cohomology class in $H^2(S,\mathbb{Z})$ satisfies

$$[Y]=\left[3H-\sum E_i\right]=-K_S. \tag{6.1}$$

A *marked pair* (S,Y,ϕ) is a marked blowup with a distinguished anticanonical curve. An *isomorphism* between marked pairs (S,Y,ϕ) and (S',Y',ϕ') is a biholomorphic map $F:S\to S'$, compatible with markings, that sends Y to Y'.

Proposition 6.3 *If $n\geq 10$, then S carries at most one irreducible anticanonical curve Y.*

Proof. In this case we have $Y^2=9-n<0$. ∎

From surfaces to cubics. Let $\pi:S\to\mathbb{P}^2$ be a projection compatible with ϕ, presenting S are a blowup with basepoints $(p_i)_1^n$. Then (6.1) implies that

$$X=\pi(Y)\subset\mathbb{P}^2$$

is a cubic curve, passing through each basepoints p_i with multiplicity one. Moreover, the fact that $E_i\cdot Y=1$ implies $\pi:Y\to X$ is an isomorphism. Combining the identification $H^2(S,\mathbb{Z})=\mathrm{Pic}(S)$ with the restriction map $r:\mathrm{Pic}(S)\to\mathrm{Pic}(Y)$, we obtain a natural marking

$$\rho:\mathbb{Z}^{1,n}\xrightarrow{\phi}H^2(S,\mathbb{Z})=\mathrm{Pic}(S)\xrightarrow{r}\mathrm{Pic}(Y)\xrightarrow{\pi_*}\mathrm{Pic}(X).$$

Thus a marked pair (S, Y, ϕ) canonically determines a marked cubic curve (X, ρ).

From cubics to surfaces. Conversely, let (X, ρ) be a marked cubic curve. Then we have basepoints $p_i \in X$ determined by $\rho(e_i)_1^n$, and an embedding $X \subset \mathbb{P}^2$ determined by $\rho(e_0)$. Letting (S, ϕ) be the marked blowup with basepoints $p_i \in \mathbb{P}^2$, and $Y \subset S$ the strict transform of X, we obtain a marked pair

$$(S, Y, \phi) = \mathrm{Bl}(X, \rho)$$

which we call the *blowup* of (X, ρ). It is easy to see that this functorial construction inverts the preceding one; summing up, we have:

Theorem 6.4 *The functor $(X, \rho) \mapsto (S, Y, \phi) = \mathrm{Bl}(X, \rho)$ establishes an equivalence between the category of marked cubic curves and the category of marked pairs.*

Volume. Since Y is an anticanonical curve, there is a meromorphic $(2, 0)$ form η on S, unique up to scale, with a simple pole along Y and no other poles or zeros.

The form η determines a natural volume measure

$$\mathrm{vol}(U) = \int_U \eta \wedge \overline{\eta},$$

locally finite on $S - Y$ but of infinite total mass.

Determinant. Let $\mathrm{Aut}(S, Y)$ denote the group of automorphisms of S stabilizing Y. The forms η and $F^*\eta$ are proportional for any $F \in \mathrm{Aut}(S, Y)$, and thus we have a natural homomorphism

$$\delta : \mathrm{Aut}(S, Y) \to \mathbb{C}^*$$

characterized by

$$F^*\eta = \delta(F) \cdot \eta.$$

We call $\delta(F)$ the *determinant* of F, since $\delta(F) = \det DF_p$ for all $p \in S - Y$ fixed by F.

Note that every $F \in \mathrm{Aut}(S, Y)$ covers a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ stabilizing X.

Theorem 6.5 *For any $F \in \mathrm{Aut}(S, Y)$, we have $D(f|X) = \delta(F)$.*

Proof. The Poincaré residue map [GH, p.500] sends η to a nonzero 1-form $\omega \in \Omega(Y) \cong \Omega(X)$, satisfying $F^*\eta/\eta = f^*\omega/\omega$. ■

Realizations. An element $w \in W_n$ is realized by $F \in \mathrm{Aut}(S, Y)$ if $\phi \circ w = F_* \circ \phi$ on $\mathbb{Z}^{1,n}$. Let

$$\mathrm{Aut}(S, Y, \phi) \subset \mathrm{Aut}(S, \phi) \subset W_n$$

be the group of elements so realized. We can now show that $\mathrm{Aut}(S, Y, \phi) = \mathrm{Aut}(X, \rho) \cap W(S, \phi)$.

Proof of Theorem 6.1. Let $w \in W_n$ lie in the intersection of the groups on the right. Since $w \in W(S, \phi)$, there is a projection $\pi' : S \to \mathbb{P}^2$ corresponding to the marking $\phi' = \phi \circ \rho$. The corresponding basepoints p_i' lie on the cubic curve

$$X' = \pi' \circ \iota(X) \subset \mathbb{P}^2,$$

furnishing it with a marking ρ'. By construction, (X', ρ') is isomorphic to $(X, \rho \circ w)$. Since $w \in \mathrm{Aut}(X, \rho)$, there is an automorphism $g : \mathbb{P}^2 \to \mathbb{P}^2$ sending X to X' and p_i to p_i'. This map is covered by a unique $F \in \mathrm{Aut}(S)$ realizing w. Since $g \circ \pi = \pi' \circ F$, we also have $F(Y) = Y$, and thus $w \in \mathrm{Aut}(S, Y, \phi)$.

For the reverse inclusion, observe that any $F \in \mathrm{Aut}(S, Y)$ covers a birational map with $f|X \in \mathrm{Aut}(X)$. ■

A similar argument appears in [Ha1, Cor. 4.4].

The irreducible case. Let us specialize the preceding discussion to the case where X is irreducible. We first remark:

Theorem 6.6 *If $0 \notin \rho_0(\Phi_n)$, then (S, ϕ) has no nodal roots.*

Proof. By Corollary 5.5, if (S, ϕ) has a nodal root then it also has a root α represented by an effective sum D of smooth rational curves. By irreducibility, the strict transform Y of X is either a singular rational curve or a smooth elliptic curve (recall we only blowup smooth points of X), so it is not contained in D. Thus

$$\rho(\alpha) = [D \cap Y] \in \mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$$

is a line bundle of degree zero represented by an effective divisor, so it is trivial. ■

Proof of Theorem 6.2. Once there are no nodal roots, we have $W(S, \phi) = W_n$, and thus $\mathrm{Aut}(S, Y, \phi) = \mathrm{Aut}(X, \rho)$ by Theorem 6.1; and Proposition 6.3 implies $\mathrm{Aut}(S, Y, \phi) = \mathrm{Aut}(S, \phi)$ when $n \geq 10$. ■

Comparison to K3 surfaces. The surface $S-Y$ behaves in many ways like a K3 surface, with its canonical bundle trivialized by η. One can similarly regard Theorem 6.4 as an elementary Torelli theorem, stating that (S,Y) is determined by its 'periods' $H^2(S,\mathbb{Z}) \to \mathrm{Pic}(Y)$. Compare [Lo], [Ha2].

7 Cuspidal cubics

In this section we will establish:

Theorem 7.1 *Suppose λ is an eigenvalue of $w \in W_n$, λ is not a root of unity and w has no periodic roots. Then there is a unique marked pair (S,Y,ϕ) and $F \in \mathrm{Aut}(S,Y)$ such that Y is irreducible, $\delta(F)=\lambda$ and F realizes w.*

Moreover $\langle F\rangle$ has finite index in the full group $\mathrm{Aut}(S)$.

The surface S is constructed explicitly by blowing up the points $(x_i, x_i^3)_1^n$ on the cuspidal cubic $y^3 = x$ in $\mathbb{C}^2$, where $x_i = -v_i - v_0/3$ and $w(v) = \lambda^{-1}v$.

More generally, let $\mathbb{C}^{1,n} = \mathbb{Z}^{1,n} \otimes \mathbb{C}$ with the complex bilinear Minkowski form, and let

$$(\mathbb{C}^{1,n})^* = \{v \in \mathbb{C}^{1,n} \ : \ 0 \notin v \cdot \Phi_n\}.$$

For each $v \in (\mathbb{C}^{1,n})^*$, we define

$$W_n^v = \{w \in W_n \ : \ [v] \in \mathbb{C}^{1,n}/\mathbb{C}k_n \text{ is an eigenvector for } w\}.$$

(This group includes all w for which v itself is an eigenvector; the formulation above allows for a uniform treatment of the case $n=9$.) We will construct a marked blowup along a cuspidal cubic for every $v \in (\mathbb{C}^{1,n})^*$, and establish:

Theorem 7.2 *For every $v \in (\mathbb{C}^{1,n})^*$, we have* $\mathrm{Aut}(S^v, Y^v, \phi^v) = W_n^v$.

Corollary 7.3 *If $n \geq 10$, we have* $\mathrm{Aut}(S^v) \cong W_n^v$.

Corollary 7.4 *A Coxeter element $w \in W_n$ is realized on (S^v, ϕ^v) whenever v is a leading eigenvector for w and $n \neq 9$.*

These Corollaries are immediate from Theorems 6.2 and 2.6.

Structure of the automorphism group. It is easy to see, as in the proof of Theorem 7.1, that W_n^v is virtually abelian whenever v is timelike or null ($v \cdot v \geq 0$). This is also true for $n = 10$ by [Bor, Thm. 3.9.1]. (I am grateful to D. Allcock for this reference.)

Question. Is the group W_n^v, $v \in (\mathbb{C}^{1,n})^*$, always virtually abelian?

Question. Is there a automorphism of a pair (S, Y) with positive entropy and $\delta(F) = 1$?

Blowups along a cuspidal cubic. Let $X^* \subset \mathbb{C}^2 \subset \mathbb{P}^2$ be the smooth locus of the cuspidal cubic defined by $y = x^3$, parameterized by $p(t) = (t, t^3)$. Given $v \in (\mathbb{C}^{1,n})^*$, define $(t_i)_0^n$ and $(p_i)_0^n$ by

$$\begin{aligned} 3t_0 &= v \cdot e_0, \\ t_i &= v \cdot e_i, \quad i > 0, \quad \text{and} \\ p_i &= p(t_i - t_0). \end{aligned}$$

Then $p_i = (x_i, x_i^3)$ with $x_i = -v_i - v_0/3$ as above. The condition $0 \notin v \cdot \Phi_n$ implies

$$v \cdot \alpha_{ij} = v \cdot (e_i - e_j) = t_i - t_j \neq 0,$$

so the points $(p_i)_1^n$ are distinct. Thus the given embedding $X \subset \mathbb{P}^2$, together with the basepoints $(p_i)_1^n$, determines a marking $\rho^v : \mathbb{Z}^{1,n} \to \mathrm{Pic}(X)$.

We let (S^v, ϕ^v) denote the marked blowup of $\mathbb{P}^2$ at the basepoints of (X, ρ^v), and $Y^v \subset S^v$ the strict transform of X. Note that (S^v, ϕ^v) only depends on the location of v in the quotient $\mathbb{C}^{1,n}/\mathbb{C}k_n$.

Theorem 7.5 *The marking homomorphism $\rho_0^v : L_n \to \mathrm{Pic}_0(X)$ is given by*

$$\rho_0^v(u) = (u \cdot v)D,$$

where $D = [p(1) - p(0)]$.

Proof. Identify $\mathrm{Pic}_0(X)$ with $\mathbb{C}$ by $\sum n_j p(s_j) \mapsto \sum n_j s_j$; then $D = 1$. Let $u = de_0 + \sum_1^n m_i e_i$ be an element of L_n; then $3d + \sum m_i = 0$. Since p_0 is the unique flex of X, we have $\rho^v(e_0) = [3p_0]$, and hence:

$$\begin{aligned} \rho^v(u) &= \left[3dp_0 + \sum m_i p_i\right] = \left[3dp(0) + \sum m_i p(t_i - t_0)\right] \\ &= \left(\sum m_i t_i\right) - t_0 \left(\sum m_i\right) = 3dt_0 + \sum m_i t_i = u \cdot v. \end{aligned}$$

■

Corollary 7.6 *We have $(X, \rho^v \circ w^{-1}) \cong (X, \rho^{w(v)})$.*

Proof. By Theorem 4.3, ρ_0 determines (X, ρ) up to isomorphism, and we have $\rho_0^v \circ w^{-1} = \rho_0^{w(v)}$ because $w^{-1}(u) \cdot v = u \cdot w(v)$. ■

Corollary 7.7 *We have* $(X,\rho^a) \cong (X,\rho^b)$ *iff* $a = \lambda b + \mu k_n$ *for some* $\lambda, \mu \in \mathbb{C}$.

Proof. An isomorphism is given by an $f \in \mathrm{Aut}(X)$ satisfying $f_* \circ \rho_0^a = \rho_0^b$. Since $\mathrm{Aut}(X)$ acts on $\mathrm{Pic}_0(X) \cong \mathbb{C}$ by scalar multiplication, the result follows. ∎

Proof of Theorem 7.2. The assumption $v \in (\mathbb{C}^{1,n})^*$ implies $0 \notin (v \cdot \Phi_n)D = \rho^v(\Phi_n)$, and thus $\mathrm{Aut}(S^v, Y^v, \phi^v) = \mathrm{Aut}(X, \rho^v)$ by Theorem 6.2. By the preceding observations, $w \in \mathrm{Aut}(X,\rho^v)$ if and only $[v]$ is an eigenvector for w on $\mathbb{C}^{1,n}/\mathbb{C}k_n$. ∎

Proof of Theorem 7.1. Since λ is an eigenvalue of w, so is λ^{-1}. Let $v \in \mathbb{C}^{1,n}$ be a corresponding eigenvector. By Theorem 2.7, we have $v \in (\mathbb{C}^{1,n})^*$ and thus w is realized by an automorphism $F \in \mathrm{Aut}(S^v, Y^v)$, covering a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. This realization satisfies

$$D(f)\rho_0^v = f_*\rho_0^v = \rho_0^v \circ w = \rho_0^{w^{-1}(v)} = \rho_0^{\lambda v} = \lambda \rho_0^v,$$

and thus $\delta(F) = D(f) = \lambda$ as required.

For uniqueness, observe that we have $\mathrm{Pic}_0(Y) \cong \mathbb{C}$ by Proposition 4.2, and thus $X = \pi(Y)$ is a cuspidal cubic, by irreducibility of Y. Hence (S, Y, ϕ) has the form (S^u, Y^u, ϕ^u) for some u satisfying $w(u) = \lambda^{-1}u$. Since w preserves a form of signature $(1, n)$, any eigenvalue which is not a root of unity has multiplicity one; thus u is proportional to v, and hence $(S, Y, \phi) \cong (S^v, Y^v, \phi^v)$.

To analyze the full group $\mathrm{Aut}(S) \cong W_n^v$, we note that $\sigma(w) > 1$ and thus w acts by translation along a geodesic γ in the hyperbolic space $\mathbb{H}^n \subset \mathbb{PR}^{1,n}$. The Galois conjugates of $[v]$ include one of the endpoints $[u]$ of $\gamma \subset \overline{\mathbb{H}^n}$. It then follows readily from discreteness of W_n that $W_n^v = W_n^u$ stabilizes γ, so W_n^v is a finite extension of $\langle w \rangle$. Consequently $\mathrm{Aut}(S)$ is a finite extension of $\langle F \rangle$. ∎

Modular perspective. The discussion above can be summarized on the level of moduli spaces as follows: there is a W_n-equivariant map

$$\mathbb{P}(\mathbb{C}^{1,n})^* \cong M_n(X)^* \to \mathcal{P}_n^* \subset \mathcal{P}_n,$$

given by $[v] \mapsto (S^v, \phi^v)$, and hence fixed points (eigenvectors) for w in $\mathbb{P}(\mathbb{C}^{1,n})^*$ furnish fixed points (realizations) for w in $\mathcal{P}_n^*$.

Nodal cubics. Parallel results can be formulated for other irreducible cubics. For example, let X be the nodal cubic $y^2 = 4x^2(x-1)$, parameterized by $p(t) = (q(t), q'(t))$ where $q(t) = 1/\sin^2(t)$. Let (S^v, ϕ^v) be the marked blowup with basepoints $p_i = p(v \cdot (e_i - e_0/3))$. This corresponds to the marking $\rho_0^v : \mathbb{Z}^{1,n} \to \mathrm{Pic}_0(X) \cong \mathbb{C}^*$ given by

$$\rho_0^v(u) = \exp(2iu \cdot v).$$

Using the fact that $\mathrm{Aut}(X)$ acts on $\mathrm{Pic}_0(X) \cong \mathbb{C}^*$ by $z \mapsto z^{\pm 1}$, Theorem 6.2 readily implies:

Theorem 7.8 *If $\rho_0^v \circ w(u) = (\rho_0^v(u))^{\pm 1}$, and $1 \notin \exp(2iv \cdot \Phi_n)$, then w is realized on (S^v, ϕ^v).*

The case of an elliptic curve is similar.

8 Reducible cubics

When $n \geq 8$, any n points on a reducible cubic curve satisfy a nodal relation. Nevertheless, some interesting automorphisms can be realized in this configuration. In this section we give two such constructions: one for three lines through a point, and the other for a conic with a tangent line. These are the reducible cubics with $\mathrm{Pic}_0(X) \cong \mathbb{C}$.

Standard Coxeter element. Let $\pi_n = s_1 s_2 \cdots s_{n-1} \in \Sigma_n \subset W_n$ be the cyclic permutation $(123\ldots n)$. We will study realizations of the *standard* Coxeter element, defined by

$$w = \pi_n \circ \kappa_{123} \in W_n.$$

This element satisfies

$$\begin{aligned} w(e_0) &= 2e_0 - e_2 - e_3 - e_4, \\ w(e_1) &= e_0 - e_3 - e_4, \\ w(e_2) &= e_0 - e_2 - e_4, \\ w(e_3) &= e_0 - e_2 - e_3, \end{aligned} \tag{8.1}$$

$w(e_i) = e_{i+1}$ for $4 \leq i < n$, and $w(e_n) = e_1$.

I. Three lines through a point. Let $X_j \subset \mathbb{P}^2$ be the line defined in affine coordinates (x, y) by $y = j$, and let $X = X_1 \cup X_2 \cup X_3$. Then X is a reducible cubic, consisting of three lines meeting in a single point at infinity. Their fundamental classes determine a natural basis $([X_j])_1^3$ for $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^3$.

Assume $n \equiv 0 \bmod 3$. We will show the standard Coxeter element can be realized by an automorphism that cyclically permutes the irreducible components of X.

Let $[i] \in \{1,2,3\}$ denote the residue class of $i \bmod 3$. Define $R : \mathbb{Z}^{1,n} \to H^2(X,\mathbb{Z})$ by

$$R(e_i) = \begin{cases} [X_1]+[X_2]+[X_3] & \text{if } i=0, \text{ and} \\ [X_{[i]}] & \text{if } i \geq 1. \end{cases}$$

Then R transports the action of the Coxeter element $w|\mathbb{Z}^{1,n}$ to the order 3 automorphism of $H^2(X,\mathbb{Z})$ that sends $[X_i]$ to $[X_{[i+1]}]$.

Given $v \in \mathbb{C}^{1,n}$, let $t_i = v \cdot e_i$ and define $(p_i)_1^n$ in (x,y)-coordinates by

$$p_i = \begin{cases} (t_i - t_0, 1) & \text{if } [i]=1, \\ (-t_i/2, 2) & \text{if } [i]=2 \text{ and} \\ (t_i, 3) & \text{if } [i]=3. \end{cases}$$

Assuming the points p_i are distinct, we then obtain a marked cubic (X, ρ^v).

Theorem 8.1 *The marked cubic (X,ρ^v) satisfies $\deg(\rho^v(u)) = R(u)$ and $\rho_0^v(u) = u \cdot v$, for a suitable choice of isomorphism $\mathrm{Pic}_0(X) \cong \mathbb{C}$.*

Proof. Observing that a transverse line meets X_j in a single point, we have $\deg(\rho^v(e_0)) = [X_1]+[X_2]+[X_3] = R(e_0)$; and similarly, $\deg(\rho^v(e_i)) = \deg([p_i]) = R(e_i)$ for $i \geq 1$. This shows $\deg(\rho^v(u)) = R(u)$.

For the second part, note that if $(x_1,1)$, $(x_2,2)$ and $(x_3,3)$ lie on a line, then $x_1 - 2x_2 + x_3 = 0$. Thus we have an isomorphism $\mathrm{Pic}_0(X) \cong \mathbb{C}$ given by

$$\sum n_k(x_k,y_k) \mapsto \left(\sum_{y_k=1} n_k x_k\right) - 2\left(\sum_{y_k=2} n_k x_k\right) + \left(\sum_{y_k=3} n_k x_k\right).$$

Note also that if $u = de_0 + \sum m_i e_i$ and $\deg \rho(u) = 0$, then $d + \sum_{[i]=1} m_i = 0$. Hence

$$\begin{aligned} \rho_0^v(u) &= \sum_{[i]=1} m_i(t_i - t_0) - 2\sum_{[i]=2} m_i(-t_i/2) + \sum_{[i]=3} m_i t_i \\ &= dt_0 + \sum_1^n m_i t_i = u \cdot v, \end{aligned}$$

as desired. ∎

Realizations. Let (S^v, ϕ^v) be the marked blowup determined by (X, ρ^v), and let Y^v be the strict transform of X.

Theorem 8.2 *Let v be a leading eigenvector for $w = \pi_n \circ \kappa_{123}$, where $n \equiv 0 \bmod 3$ and $n \neq 9$. Then w is realized by an automorphism on the marked pair (S^v, Y^v, ϕ^v).*

Proof. It suffices, by Theorem 6.1, to show that (i) (X, ρ^v) is a marked cubic, (ii) $w \in \mathrm{Aut}(X, \rho^v)$, and (iii) $w \in W(S^v, \phi^v)$.

Let v be a leading eigenvector for w; then $0 \notin v \cdot \Phi_n$, by Theorem 2.6. Hence $\rho_0^v(e_i - e_j) \neq 0$ for $i > j \geq 1$; thus the points $(p_i)_1^n$ are distinct, and we have (i).

Next, consider the automorphism of X given by $f(x, j) = (\lambda x, [j + 1])$, where $w(v) = \lambda^{-1} v$. Since f cyclically permutes the lines (X_1, X_2, X_3), we have

$$\deg \circ \rho^v \circ w = R \circ w = f_* \circ \rho^v;$$

and since f acts on $\mathrm{Pic}_0(X) \cong \mathbb{C}$ by multiplication by λ, we have

$$\rho_0^v \circ w = \rho_0^{w(v)} = \lambda \rho_0^v = f_* \circ \rho_0^v.$$

Hence Theorem 4.3 implies $(X, \rho^v \circ w) \cong (X, f_* \rho^v) \cong (X, \rho^v)$; thus $w \in \mathrm{Aut}(X, \rho^v)$, and we have (ii).

For (iii), suppose $p_k \in \overline{p_i p_j}$ for three distinct indices (i, j, k), with $i, j \in \{1, 2, 3\}$. Then there is exactly one point on each of the three lines X_1, X_2, X_3. For these three points to be collinear, the divisor $[p_i + p_j + p_k]$ must be linearly equivalent to the hyperplane section, which implies $\rho_0^v(\kappa_{ijk}) = 0$. But we have seen that $0 \notin v \cdot \Phi_n$, so no such triple (i, j, k) exists; consequently $\kappa_{123} \in W(S^v, \phi^v)$ by Theorem 5.2. The permutation π_n simply reorders the basepoints (cf. equation (5.3)), so we also have $w = \pi_n \circ \kappa_{123} \in W(S^v, \phi^v)$, as desired. ∎

Theorem 8.3 *The standard Coxeter element $w = \pi_9 \circ \kappa_{123}$ is realized on (S^v, ϕ^v) whenever $w(v) = \lambda v$, $v \neq 0$, and λ is a primitive 5th root of unity.*

Proof. One can check that the eigenvector has the form $v = (v_0, v_1, \ldots, v_9)$ with $v_1 = v_5$ and $v_4 = v_9$, but no other coincident entries. Since the $1 \neq 5 \bmod 3$ and $4 \neq 9 \bmod 3$, the corresponding points $(p_i)_1^9$ on X are distinct. Similarly, a direct computation shows the only lines through three basepoints are X_1, X_2, X_3 and $\overline{p_5 p_7 p_9}$. Thus Theorem 5.2 applies again, to show $\kappa_{123} \in W(S^v, \phi^v)$ and hence $w \in \mathrm{Aut}(S^v, \phi^v)$. ∎

II. Conic and a tangent line. Now let $X = X_1 \cup X_2 \subset \mathbb{P}^2$, where X_2 is the conic $xy = 1$, and X_1 is its tangent line $y = 0$. Their fundamental classes $[X_1]$, $[X_2]$ determine a basis for $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^2$.

Assume n is odd. By a similar argument, we will show the standard Coxeter element can be realized by an automorphism that exchanges X_1 and X_2.

Let $[i] = 2$ if $i \in \{1, 2, 3\}$ or i is even, and let $[i] = 1$ otherwise. Define $R : \mathbb{Z}^{1,n} \to H^2(X, \mathbb{Z})$ by

$$R(e_i) = \begin{cases} [X_1] + 2[X_2] & \text{if } i = 0, \text{ and} \\ [X_{[i]}] & \text{if } i \geq 1. \end{cases}$$

Then R transports the action of the Coxeter element $w|\mathbb{Z}^{1,n}$ to the involution of $H^2(X, \mathbb{Z})$ that exchanges $[X_1]$ and $[X_2]$.

Given $v \in \mathbb{C}^{1,n}$, let $t_i = v \cdot e_i$ and define $(p_i)_1^n$ by

$$p_i = \begin{cases} (t_0 - t_i, 0) & \text{if } [i] = 1, \text{ and} \\ (t_i, 1/t_i) & \text{if } [i] = 2. \end{cases}$$

Assuming the points p_i are distinct, we again obtain a marked cubic (X, ρ^v).

Theorem 8.4 *The marked cubic (X, ρ^v) satisfies $\deg(\rho^v(u)) = R(u)$ and $\rho_0^v(u) = u \cdot v$, for a suitable choice of isomorphism $\mathrm{Pic}_0(X) \cong \mathbb{C}$.*

Proof. Observing that a transverse line meets X_j in j points, we have $\deg(\rho^v(e_0)) = [X_1] + 2[X_2] = R(e_0)$; and similarly, $\deg(\rho^v(e_i)) = \deg([p_i]) = R(e_i)$ for $i \geq 1$. This shows $\deg(\rho^v(u)) = R(u)$.

For the second part, note that if $(a, 0)$, $(b, 1/b)$ and $(c, 1/c)$ lie on a line, then $a = b + c$. Thus we have an isomorphism $\mathrm{Pic}_0(X) \cong \mathbb{C}$ given by

$$\sum n_k(x_k, y_k) \mapsto \left(\sum_{y_k \neq 0} n_k x_k \right) - \left(\sum_{y_k = 0} n_k x_k \right).$$

Note also that if $u = de_0 + \sum m_i e_i$ and $\deg \rho(u) = 0$, then $d + \sum_{[i]=1} m_i = 0$. Hence

$$\rho_0^v(u) = \sum_{[i]=2} m_i t_i - \sum_{[i]=1} m_i(t_0 - t_i) = dt_0 + \sum_1^n m_i t_i = u \cdot v,$$

as desired. ∎

Realizations. Letting (S^v, Y^v, ϕ^v) be the marked blowup determined by (X, ρ^v) as before, we can now state:

Theorem 8.5 *Assume n is odd and $n \neq 9$. Then the standard Coxeter element $w = \pi_n \circ \kappa_{123}$ belongs to $\mathrm{Aut}(S^v, Y^v, \phi^v)$ whenever v is a leading eigenvector for w.*

Theorem 8.6 *The standard Coxeter element $w = \pi_9 \circ \kappa_{123}$ is realized on (S^v, ϕ^v) whenever $w(v) = \lambda v$, $v \neq 0$, and λ is a primitive 5th root of unity.*

The proofs follow the same lines as the proofs of Theorems 8.2 and 8.3.

9 Expanding dynamics

In this section we describe the global dynamics of $F \in \mathrm{Aut}(S, Y)$ when the natural volume form $\eta \wedge \overline{\eta}$ is expanded by F.

Fixed points. Let $F : S \to S$ be an automorphism of a blowup of $\mathbb{P}^2$, preserving an anticanonical curve Y. Let $X = \pi(Y) \subset \mathbb{P}^2$ as usual, and assume $\mathrm{Pic}_0(X) \cong \mathbb{C}$; then X is either a cuspidal cubic, three lines through a point or a conic with a tangent line.

The unique singular point $p \in Y \cong X$ is necessarily fixed by F, and it is straightforward to compute the eigenvalues (λ_1, λ_2) of DF_p in terms of $\delta = \delta(F)$. Indeed, since Y has multiplicity two at p, $F|Y$ determines DF_p, and we find:

$$(\lambda_1, \lambda_2) = \begin{cases} (\delta^{-2}, \delta^{-3}) & \text{for a cuspidal cubic;} \\ (\epsilon\delta^{-1}, \epsilon^{-1}\delta^{-1}) & \text{for three lines through a point; and} \\ (\epsilon\delta^{-2}, \epsilon^{-1}\delta^{-1}) & \text{for a conic with a tangent line.} \end{cases} \tag{9.1}$$

Here ϵ is a primitive kth root of unity, with $k = 1, 2$ or 3 depending on the period of $F^*|H^2(X, \mathbb{Z})$.

Expanding maps. We say $F \in \mathrm{Aut}(S, Y)$ is *expanding* if $|\delta(F)| > 1$. This guarantees $\mathrm{Pic}_0(X) \cong \mathbb{C}$ as assumed above, by Proposition 4.2. By equation (9.1), F has an attracting fixed point at p.

The *Julia set* $J^+(F) \subset S$ is the smallest closed set such that the forward iterates $\langle F^n, n > 0\rangle$ form a normal family when restricted to $S - J^+(F)$.

Theorem 9.1 *If F is expanding, then $J^+(F)$ has measure zero, and $F^n(z) \to p$ for all $z \in S - J^+(F)$.*

Proof. For concreteness we treat the case where $X \cong Y$ is a cuspidal cubic. The other two cases are similar.

Let U be the open set of $z \in S$ such that $F^n(z) \to p$. We will first show that

$$\mathrm{vol}(S - U) = \int_{S-U} \eta \wedge \overline{\eta} < \infty.$$

Let $x : Y \to \mathbb{P}^1$ be the inverse of the normalization of the singular curve Y, sending Y^* to $\mathbb{C}$ and p to infinity. Composing with a translation, we can assume $F|Y$ has the form $x \mapsto \delta x$; then $x = 0$ gives the unique fixed point $q \in Y^*$. Since $|\delta| > 1$, all other points of Y converge to p under iteration, and thus $Y - \{q\} \subset U$.

We extend x to a local coordinate system (x, y) on S, in which $q = (0, 0)$, Y is the x-axis, and $\eta = dx\,dy/y$. We assume (x, y) maps a neighborhood of q to the polydisk $\Delta^2 = \{(x, y) : |x|, |y| < 1\}$. Since $F(Y) = Y$, in this coordinate system we have

$$(x', y') = F(x, y) = (\delta x, 0) + O(|y|).$$

Equivalently, there is a $C > 0$ such that $|y'| \le C|y|$ and $|x' - \delta x| \le C|y|$.

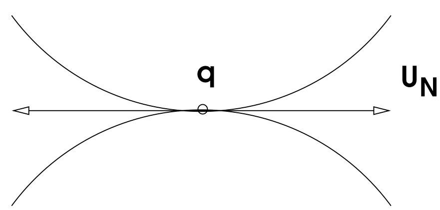

Figure 3. The region U_N lies in the basin of p.

Consider the region $U_N = \{(x, y) \in \Delta^2 : |y| < |x|^N/N\}$ shown in Figure 3. If $(x, y) \in U_N$ and $(x', y') = F(x, y) \in \Delta^2$, then for N sufficiently large we have

$$|x'| \ge |\delta x| - C|y| \ge |\delta x|(1 - |x|^{N-1}/N) \ge |\delta|^{1/2}|x|. \tag{9.2}$$

Choose N larger still, we can assume $|\delta|^{N/2} > C$; then we have

$$|y'| \le C|y| \le C|x|^N/N \le |x'|^N/N. \tag{9.3}$$

In other words, (x', y') is also in U_N.

Because of (9.2), every point $z \in U_N$ eventually escapes from Δ^2 under iteration, but by (9.3) it remains in U_N until it does so. For N large, the escaping points are very close to Y yet a definite distance from q; thus they lie in U, and consequently $U_N \subset U$.

Since the measure V is locally finite on $S - Y$, and U contains all of Y except q, to show $\mathrm{vol}(S - U)$ is finite it suffices to show the volume V_N of $\Delta^2 - U_N$ with respect to the form $\eta \wedge \overline{\eta}$ is finite. But this is straightforward, since $\int_{\Delta^2} \eta \wedge \overline{\eta}$ is only borderline divergent. In detail, we have:

$$\begin{aligned} V_N &= 4 \int_{\Delta^2 - U_N} |dx|^2\, |dy|^2/|y|^2 = 4\pi \int_{\Delta} (N|y|)^{2/N} |dy|^2/|y|^2 \\ &= 8\pi^2 \int_0^1 (Nr)^{2/N} dr/r < \infty. \end{aligned}$$

Thus $\mathrm{vol}(S - U)$ is finite. The closed set $S - U$ is also invariant under F, so we have

$$\mathrm{vol}(S - U) = \mathrm{vol}(F(S - U)) = |\delta|^2 \,\mathrm{vol}(S - U).$$

Since $|\delta| > 1$, this implies $S - U$ has measure zero. Clearly $J^+(F) \subset S - U$, so the Julia set also has measure zero.

Finally suppose $z \notin J^+(F)$; then $\langle F^n | B \rangle$ forms a normal family on some ball B containing z. Since U has full measure, it must meet B, and hence $F^n | B$ converges to the constant function p; in particular $F^n(z) \to p$, as desired. ∎

Corollary 9.2 *Any F-invariant probability measure is supported on $J^+(F)$, and hence it is singular with respect to Lebesgue measure.*

In particular the unique measure of maximal entropy for F is singular. For more on the maximal measure, see [BD], [Duj] and references therein.

10 Siegel disks

In this section we show:

Theorem 10.1 *For all n sufficiently large with $n \neq 2, 4 \bmod 6$, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a Siegel disk.*

The proof rests on a more general statement that also yields concrete examples.

Irrational rotations. A pair of numbers $\alpha, \beta \in \mathbb{C}^*$ are *multiplicatively independent* if they satisfy no relation of the form $\alpha^i \beta^j = 1$ with $(i,j) \neq (0,0)$. (In particular, neither α nor β is a root of unity.)

A linear map $R : \mathbb{C}^2 \to \mathbb{C}^2$ is an *irrational rotation* if its eigenvalues α, β lie on the unit circle and are multiplicatively independent. (We exclude rotations that preserve a pencil of curves of the form $x^i = ty^j$.)

Let $F : S \to S$ be a surface automorphism with a fixed point p. A domain $U \subset S$ is a *Siegel disk* for F if $F(U) = U$ and $F|U$ is analytically conjugate to an irrational rotation $R|\Delta^2$. Using results from transcendence theory, one can show [Mc2, Thm 5.1]:

Theorem 10.2 *If $F(q) = q$ and DF_q is an irrational rotation with algebraic eigenvalues, then F has a Siegel disk centered at q.*

Eigenvalues. Now let $F : S \to S$ be an automorphism of a blowup $\pi : S \to \mathbb{P}^2$, preserving an anticanonical curve Y with $\pi(Y) = X$. We will assume F has the following properties.

1. The determinant $\delta = \delta(F)$ lies on the unit circle, but is not a root of unity.
2. The Lefschetz number $L(F) = 2$; equivalently, $\operatorname{Tr} F^*|H^2(X, \mathbb{C}) = 0$.
3. F has only isolated fixed points.
4. No irreducible component of Y is invariant under F.

These assumptions imply that X is either three lines through a point or a conic with a tangent line; that F fixes the unique singular point $p \in Y$; that F has exactly one other fixed point q; and that $q \notin Y$. We aim to produce a Siegel disk centered at q.

Let (α, β) be the eigenvalues of DF_q. Since $q \notin Y$, we have $\alpha\beta = \det DF_q = \delta$. We also have

$$2 + (\alpha/\beta) + (\beta/\alpha) = T_m(\delta),$$

where $T_m(\delta)$ is a rational function whose form depends on the number of irreducible components m of Y.

Theorem 10.3 *The function $T_m(\delta)$ is given by*

$$T_m(\delta) = \begin{cases} \delta(1+\delta)^2/(1+\delta+\delta^2)^2 & \text{when } m = 2, \text{ and} \\ \delta/(1+\delta)^2 & \text{when } m = 3. \end{cases}$$

Proof. Let $(d_1, d_2) = (\det(DF_p), \det(DF_q))$, and let $(t_1, t_2) = (\operatorname{Tr} DF_p, \operatorname{Tr} DF_q)$. By assumption (4) above, the m irreducible components of Y are cyclically permuted by F. Thus ϵ is a primitive mth root of unity in equation (9.1), and hence

$$(d_1, t_1) = \begin{cases} (\delta^{-3}, -\delta^{-1} - \delta^{-2}) & \text{when } m = 2, \text{ and} \\ (\delta^{-2}, -\delta^{-1}) & \text{when } m = 3. \end{cases}$$

We also have $d_2 = \delta$ as already noted. By the Atiyah-Bott formula [AB, (4.9-4.10)], we also have:

$$L^r(F) = \sum_{s=0}^{4} (-1)^s \operatorname{Tr} F^* | H^{r,s}(S) = \sum_{F(z)=z} \frac{\operatorname{Tr} \wedge^r DF_z}{\det(I - DF_z)};$$

applying this formula with $r = 0$ and $r = 1$, and using the fact that $\det(I - A) = 1 - \operatorname{tr}(A) + \det(A)$, we obtain

$$w_1 + w_2 = 1 \quad \text{and} \quad t_1 w_1 + t_2 w_2 = 0$$

where $1/w_i = 1 - t_i + d_i$. These relations imply

$$T_m(\delta) = \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{t_2^2}{d_2} = \frac{(t_1 w_1)^2}{d_2(1 - w_1)^2},$$

and the formulas stated above then follow. ∎

Note that $T_m(\delta) \in \mathbb{R}$ when $\delta \in S^1$.

Theorem 10.4 *Suppose $T_m(\delta) \in [0, 4]$ but $T_m(\delta') \notin [0, 4]$, where $\delta' \in S^1$ is a Galois conjugate of δ. Then F has a Siegel disk centered at q.*

Proof. The eigenvalues of DF_q satisfy $\alpha\beta = \delta$ and $(\alpha/\beta) + (\beta/\alpha) + 2 = T_m(\delta)$; since δ is algebraic, so are α and β. The condition $T_m(\delta) \in [0, 4]$ implies $|\alpha/\beta| = 1$; since $|\alpha\beta| = |\delta| = 1$ as well, the eigenvalues lie on the unit circle.

To check multiplicative independence, let (α', β') be Galois conjugates of (α, β) corresponding to δ'; then $|\alpha'\beta'| = |\delta'| = 1$ but $|\alpha'/\beta'| \neq 1$, because $F(\delta') \notin [0, 4]$. Thus if $\alpha^i \beta^j = 1$, we have $(\alpha')^i (\beta')^j = 1$ and hence $i = j$; but then $(\alpha\beta)^i = \delta^i$ and hence $i = 0$, since δ is not a root of unity.

By Theorem 10.2, F has a Siegel disk centered by q. ∎

Theorem 10.5 *For all n sufficiently large, any Coxeter element $w \in W_n$ has a pair of conjugate leading eigenvalues (δ, δ') with $T_m(\delta) \in [0,4]$ and $T_m(\delta') \notin [0,4]$.*

Proof. Note that $T_2(S^1) = [4/9, \infty]$ and $T_3(S^1) = [1/4, \infty]$. In either case we can find open intervals $I, I' \subset S^1$ such that $T_m(I) \subset [0,4]$ and $T_m(I') \cap [0,4] = \emptyset$. Then by the equidistribution Theorem 2.5, for all n sufficiently large the Salem factor $Q_n(t)$ of $\det(tI - w)$ has at least one root $\delta \in I$ and another $\delta' \in I'$. ∎

Proof of Theorem 10.1. For any δ as above, the constructions of §8 apply (when n is odd or divisible by three) to yield a realization of the standard Coxeter element with $\delta(F) = \delta$. We have $\mathrm{Tr}(F^*|H^2(S, \mathbb{Z})) = \mathrm{Tr}(w|\mathbb{Z}^{1,n}) = 0$, F cyclically permutes the irreducible components of Y, and F has no curve of fixed points (e.g. by Theorem 11.1 below); thus F has a Siegel disk by Theorem 10.4. ∎

Remark. A similar discussion in the setting of K3 surfaces appears in [Mc2].

11 Examples

This section presents some specific examples of automorphisms of rational surfaces, including expanding maps and maps with Siegel disks.

The birational model. Consider the birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ given in affine coordinates by

$$f(x, y) = (a, b) + (y, y/x) \tag{11.1}$$

for some $(a, b) \in \mathbb{C}^2$. This map blows up the vertices of the triangle $\Delta(p_1, p_2, p_3)$, and blows down its sides to yield the triangle $\Delta(p_2, p_3, p_4)$, where $p_1 = (0, 0)$, $p_2 = (\infty, 0)$, $p_3 = (0, \infty)$ and $p_4 = (a, b)$. (The points p_2 and p_3 are on the line at infinity.) See Figure 4.

When the parameters (a, b) are chosen so that $p_4 = p_1$, the triangle $\Delta(p_1, p_2, p_3)$ is invariant under f. Upon blowing up these three points we obtain a realization of the standard Coxeter element in W_3, which has order six. More generally, defining $p_{i+4} = f^i(p_4)$, we have:

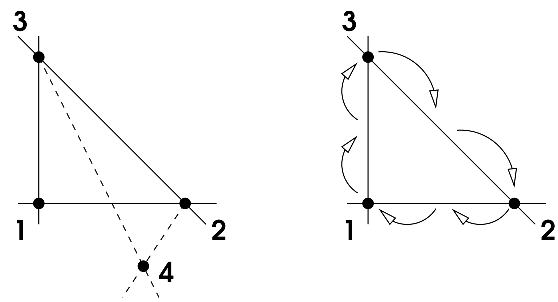

Figure 4. Birational dynamics $f : \mathbb{P}^2 \to \mathbb{P}^2$, with p_1, p_2, p_3 and p_4 labeled. The case $p_4 = p_1$, shown at the right, has order 6.

Theorem 11.1 *Realizations of the standard Coxeter element in W_n correspond to values of $(a,b) \in \mathbb{C}^2$ such that*

$$p_i \notin \overline{p_1p_2} \cup \overline{p_2p_3} \cup \overline{p_3p_1}, \quad 4 \le i \le n,$$

and $p_{n+1} = p_1$.

Proof. Assume the orbit of p_4 cycles as above, and let $\pi : S \to \mathbb{P}^2$ be the marked blowup with basepoints $(p_i)_1^n$. Then certainly f lifts to a morphism $F_0 : S \to \mathbb{P}^2$, since the points of indeterminacy $\{p_1, p_2, p_3\}$ have been replaced by lines. But now every $p_i \in \mathbb{P}^2$ is the image $F_0(L_i)$ of a line in S, so F_0 lifts to an automorphism $F : S \to S$ covering f.

To compute the element $w \in W_n$ realized by F, note that f sends a generic line to a conic through (p_2, p_3, p_4), and thus $w(e_0) = 2e_0 - e_2 - e_3 - e_4$. The point p_1 blows up to the line $\overline{p_3p_4}$, so we have $w(e_1) = e_0 - e_3 - e_4$. By similar reasoning we can compute $w(e_i)$ for the remaining basis elements, and observe that the result agrees with the answer for standard Coxeter transformation (given by equation (8.1)).

Conversely, if an automorphism of a marked blowup $F : S \to S$ realizes the standard Coxeter transformation $w = \pi_n \circ \kappa_{123}$, we can normalize the basepoints so that $(p_1, p_2, p_3) = ((0,0), (\infty, 0), (0, \infty))$; then the birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ covered by F is a composition of the standard Cremona involution (5.4) with an automorphism sending (p_1, p_2) to (p_2, p_3). Such a map has the form $f(x,y) = (a', b') + (Ay, By/x)$; conjugating by $(x,y) \mapsto (Bx, By/A)$ puts it into the form $f(x,y) = (a,b) + (y, y/x)$. ■

n	(a_n, b_n)	$\delta(F) = \lambda_n$	Salem polynomial $Q_n(t)$
10	$(0.49949650, -0.08373582)$	1.17628081	$t^{10}+t^9-t^7-t^6-t^5-t^4-t^3+t+1$
11	$(0.58778739, -0.04883156)$	1.23039143	$t^{10}-t^7-t^5-t^3+1$
12	$(0.64057213, -0.03135002)$	1.26123096	$t^{10}-t^8-t^5-t^2+1$
13	$(0.67470764, -0.02114197)$	1.28063815	$t^8-t^5-t^4-t^3+1$
14	$(0.69769622, -0.01469740)$	1.29348595	$t^{10}-t^8-t^7+t^5-t^3-t^2+1$
15	$(0.71359046, -0.01042890)$	1.30226880	$t^{12}-t^{11}-t^7+t^6-t^5-t-1$
16	$(0.72478872, -0.00750881)$	1.30840900	$t^{16}+t^{15}-t^{13}-t^{12}-\cdots-t^3+t+1$
50	$(0.75487582, -0.00000044)$	1.32471696	$t^{50}+t^{49}-t^{47}-t^{46}-\cdots-t^3+t+1$
∞	$(0.75487766, 0)$	1.32471795	t^3-t-1

Table 5. The expanding series.

Corollary 11.2 *If $3 \leq n \leq 8$ then f is periodic, and its period agrees with the Coxeter number $h_n = 6, 5, 8, 12, 18$ or 30.*

The expanding series; $n \geq 10$. Recall that for each $n \geq 10$, a Coxeter element $w \in W_n$ has a unique eigenvalue $\lambda_n > 1$ outside the unit circle; that λ_n is a Salem number; and that the other eigenvalues of w are conjugates of λ_n or roots of unity (§2).

By §7, for each $n \geq 10$ there is also a unique realization of $w = \pi_n \circ \kappa_{123}$ by an automorphism $F_n : (S_n, Y_n) \to (S_n, Y_n)$ such that Y_n is irreducible and $\delta(F_n) = \lambda_n$. The entropy of this map satisfies

$$h(F_n) = \log \lambda_n,$$

since λ_n is the largest eigenvalue of $F_n^*|H^*(S_n)$ (see the Appendix). The basepoints $(p_i)_1^n$ for S_n lie on a cuspidal cubic and are immediately computable from the corresponding eigenvector for w. Indeed, if $w(v) = \lambda_n^{-1} v$, then one can take $p_i = (x_i, x_i^3) \in \mathbb{C}^2$ where $x_i = v_i + v_0/3$, as in §7.

Changing coordinates, one easily obtains the coefficients for the birational map $f_n(x, y) = (a_n, b_n) + (y, y/x)$ covered by F_n. The results for several values of n are summarized in Table 5; the last column gives the Salem polynomial satisfied by λ_n.

As noted in §2 we have $\lambda_n \to \lambda_{\text{Pisot}}$ as $n \to \infty$, and one can similarly check that $(a_n, b_n) \to (a_\infty, b_\infty) = (1/\lambda_{\text{Pisot}}, 0)$. We have included these values in Table 5 under $n = \infty$; they can be regarded as the parameters for an automorphism of a blowup with infinitely many basepoints.

Lehmer's automorphism. As previously remarked, we have $h(F_{10}) = \log(\lambda_{\text{Lehmer}})$, and thus F_{10} is a surface automorphism with the smallest possible positive entropy. Its behavior over the real projective plane is depicted in Figure 1 of the Introduction.

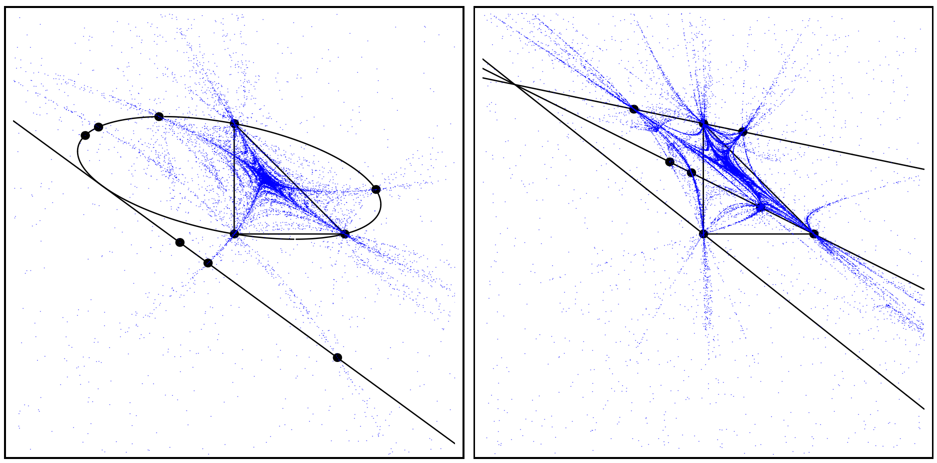

Figure 6. Automorphisms preserving a conic with a tangent line ($n = 11$) and three lines through a point ($n = 12$).

The reducible series. The standard Coxeter element can also be realized by an expanding map on a blowup over a reducible cubic when n is odd (using a conic with a tangent line) or $n \equiv 0 \bmod 3$ (using three lines through a single point). These realizations, constructed in §8, also satisfy $\delta(F) = \lambda_n$. Examples for $n = 11$ and $n = 12$ are shown in Figure 6; the corresponding parameter values are $(a, b) = (-0.92607569, 0.61173015)$ and $(-2.26123096, 1.79287619)$ respectively.

As in Figure 1, the basepoints $(p_i)_1^n$ are shown as round dots lying on the cubic X (some are outside of the frame of the Figure). The points (p_1, p_2, p_3) form the vertices of the central right triangle. The scatter plot gives an approximation to the Julia set $J^+(F)$, obtained by backward iteration of random points in $\mathbb{RP}^2$. By Theorem 9.1 all other points converge, under forward iteration, to the unique singular point $p \in Y$ (the point of tangency when $n = 11$, and the triple point when $n = 12$).

***n* = 9 : Pencils of cubics.** By Theorems 8.3 and 8.6, the standard Coxeter element in W_9 can also be realized by blowing up points along reducible cubics. We obtain four such realizations, one for each choice of a 5th root of unity. The corresponding parameter values are $(a, b) = (a, -\overline{a})$, where a is a root of the polynomial $t^4 + 3t^3 + 4t^2 + 2t + 1$.

It can be verified that the realizations obtained using a conic with a tangent line are the same as those obtained using three lines through a point. Thus there is a pencil of conics through the basepoints $(p_i)_1^9$, generated by these two reducible curves.

We remark that the maps of the form $f(x, y) = (a, -a) + (y, y/x)$ also leave invariant a pencil of conics, and were studied as early as 1945 by Lyness (see e.g. [BR], [PR]). These maps (for $a \neq 0, 1$) realize the standard Coxeter element in W_9 in a generalized sense (it is necessary to blow up infinitely near points on $\mathbb{P}^2$).

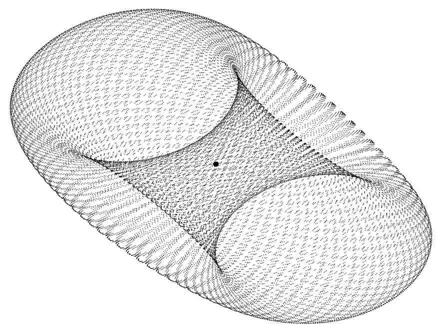

Figure 7. An orbit in a Siegel disk.

Siegel disks. We conclude with two explicit examples of surface automorphisms with Siegel disks.

The first is obtained using the two roots

$$\begin{aligned} \delta &\approx -0.09996672 + 0.99499078i, \\ \delta' &\approx -0.63841984 + 0.76968831i \end{aligned}$$

of the Salem polynomial $Q_{11}(t)$. Applying the conic and tangent line construction of §8, we obtain a realization of the standard Coxeter in W_{11} by an automorphism with $\delta(F) = \delta$. Since $T_2(\delta) \approx 2.81$ lies in $[0, 4]$ while $T_2(\delta') \approx 9.43$ does not, Theorem 10.4 implies that F has a Siegel disk centered at the unique fixed point $q \notin Y$.

The corresponding birational map is given by $f(x, y) = (a, -\overline{a}) + (y, y/x)$ with $a \approx 0.04443170 - 0.44223856i$. The orbit closure of a typical point near q is a totally real torus; the projection of such an orbit to the real plane is shown in Figure 7.

The second example is constructed using the two roots

$$\begin{aligned}\delta &\approx -0.77526329 + 0.63163820i,\\ \delta' &\approx -0.96079798 + 0.27724941i\end{aligned}$$

of $Q_{12}(t)$ to obtain an automorphism F of a marked blowup along three lines through a single point, satisfying $\delta(F) = \delta$. Since $T_3(\delta) \approx 2.22$ while $T_3(\delta') \approx 12.75$, we again obtain a Siegel disk. The corresponding birational map is given by $f(x, y) = (a, -\overline{a}) + (y, y/x)$ with $a \approx -0.22473670 - 0.63163820i$.

Notes and references. Bedford and Kim determine the possible values of the entropy (or degree growth) for birational maps in the family $f(x, y) = (a, b) + (y, y/x)$ in [BK1]; the values $\log \lambda_n = h(F_n)$ appear as special cases, as do the parameter values for $n = 9$ and $n = 10$ obtained above. Theorem 11.1 and its Corollary explain the occurrence of Coxeter numbers in [BK1, Theorem 2].

12 Minimality

A surface automorphism $F : S \to S$ is *minimal* if for any birational morphism $\pi : S \to S'$ and $F' \in \mathrm{Aut}(S')$ that makes the diagram

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[d, "\pi"] & S \arrow[d, "\pi"] \\ S' \arrow[r, "F'"] & S' \end{tikzcd}$$

commute, the map π is an isomorphism. (This agrees with Manin's notion of a G-minimal surface [Man1], where the action of $G \cong \mathbb{Z}$ is generated by F.) In this section we will show:

Theorem 12.1 *Suppose $w \in W_n$ has infinite order, no periodic roots, and there is no $v \in W_n(e_1)$ fixed by w. Then any realization of w is minimal.*

Corollary 12.2 *Any realization of a Coxeter element $w \in W_n$, $n \geq 10$ is minimal.*

Proof. We have already seen in §2 that for $n \geq 10$, w has infinite order and no periodic roots. It also follows from equation (2.4) for $\det(tI - w)$ that the kernel of $(I - w)|\mathbb{Z}^{1,n}$ is rank one, generated by k_n. Since $e_1 \notin \mathbb{Z}k_n$ and W_n fixes k_n, w has no fixed vector in $W_n(e_1)$. ∎

Iterated blowups. An *iterated blowup* is a surface S equipped with a sequence of birational morphisms

$$S = S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots S_1 \xrightarrow{\pi_1} S_0 = \mathbb{P}^2, \tag{12.1}$$

such that S_i is the blowup S_{i-1} at a single point $p_i \in S_{i-1}$. Any birational morphism $\pi : S \to \mathbb{P}^2$ can be factored as above.

Let E_i denote the effective divisor $(\pi_i \circ \cdots \circ \pi_n)^{-1}(p_i) \subset S$, and let $H \subset S$ be the preimage of a generic line in $\mathbb{P}^2$. Then the data above determines a natural marking

$$\phi : \mathbb{Z}^{1,n} \to H^2(S, \mathbb{Z})$$

sending e_0 to $[H]$ and e_i to $[E_i]$.

The following two results from [Nag1] and [Nag2] are stated explicitly in [Ha3, Cor 1.2] and [Ha1, Thm 0.1]; the second is a more general form of Theorem 5.1.

Theorem 12.3 *If a rational surface S admits an automorphism F such that $F^*|H^2(S,\mathbb{Z})$ has infinite order, then S is an iterated blowup of $\mathbb{P}^2$.*

Theorem 12.4 *If both (S, ϕ) and $(S, \phi \circ w)$ are marked iterated blowups, then $w \in W_n$.*

Proof of Theorem 12.1. Let F be a realization of w on a marked surface (S, ϕ), and let $\pi : S \to S'$ present $F' \in \mathrm{Aut}(S')$ as a quotient of F. Then by Theorem 12.3 there is a birational morphism $\pi' : S' \to \mathbb{P}^2$. By the structure of birational morphisms [Ha, V.5], each term in the composition $\pi' \circ \pi : S \to \mathbb{P}^2$ can be factored to yield a presentation of S as an iterated blowup (12.1), with $S' = S_k$ for some k. By Theorem 12.4, the corresponding marking of S satisfies $\psi = \phi \circ g$ for some $g \in W_n$.

If $k = n$ then $\pi : S \to S'$ is an isomorphism as desired. If $k = n - 1$, then $\pi : S \to S'$ has a unique exceptional fiber E_n, which must be preserved by F (and mapped by π to a fixed point of F'). Since $[E_n] = \phi \circ g(e_n)$, the vector $g(e_n) \in W_n(e_n) = W_n(e_1)$ is fixed by w, contrary to assumption.

Similarly, if $k < n - 1$, then F stabilizes the curve

$$C = E_{k+1} \cup E_{k+2} \cup \cdots \cup E_n$$

consisting of the critical points of π. An iterate F^j, $j > 0$ then stabilizes each irreducible component of C, and hence fixes the class $[E_n] - [E_{n-1}]$. Consequently w^j fixes the root $\alpha = g(e_n - e_{n-1})$, contrary to our assumption that w has no periodic roots. ∎

We remark that the blowups along cubics considered in this paper, and the blowups along sextics considered by Coble, both produce surfaces with effective plurianticanonical divisors. It is natural to ask the following:

Question. Let $F : S \to S$ be a minimal realization of an element of infinite order in W_n. Then does some negative power of K_S admit a holomorphic section?

A similar question is formulated in [Ha3], which shows the answer is *no* if one drops the hypothesis of minimality.[1]

A Appendix: Entropy of surface automorphisms

This appendix presents a general lower bound for the entropy of a surface automorphism.

Entropy. Let $F : S \to S$ be an automorphism of a compact complex manifold, not necessarily projective. When S is a Kähler, results of Gromov and Yomdin show that the topological entropy of F is given simply by the spectral radius of its action on the cohomology [Gr, p. 233]; that is:

$$h(F) = \log \sigma(F^* | H^*(S, \mathbb{C})). \tag{A.1}$$

In the case of dimension two we find:

Theorem A.1 *Let $F : S \to S$ be an automorphism of a compact complex surface. Then either $h(F) = 0$ or $h(F) \geq \log \lambda_{Lehmer} \approx 0.16235761$.*

Proof. Suppose $h(F) > 0$. By [Ca2], a minimal model for S is either a K3 surface, an Enriques surface, an Abelian surface, or a rational surface. In particular, S is Kähler. In the first three cases, F descends to the minimal model and $h(F) = \log \sigma(F^* | H^2(S, \mathbb{C}))$ is the logarithm of a Salem number of degree at most 22 over $\mathbb{Q}$. It is known that $\lambda_{\mathrm{Lehmer}}$ is the smallest such Salem number [FGR], so the desired bound holds.

Now assume S is rational. Then by the results of Nagata from §12, S is an iterated blowup of $\mathbb{P}^2$ and $h(F) = \log \sigma(w)$ for some $w \in W_n$. Since W_n is a Coxeter group, we have $\sigma(w) \geq \lambda_{\mathrm{Lehmer}}$ by [Mc1], completing the proof. ∎

[1] A negative answer in the minimal case is announced in [BK2].

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