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THE TOPOLOGY OF NORMAL SINGULARITIES OF AN ALGEBRAIC SURFACE AND A CRITERION FOR SIMPLICITY

By DAVID MUMFORD

Let a variety V^n be embedded in complex projective space of dimension m. Let $P \in V$. About P, choose a ball U of small radius ε, in some affine metric $ds^2 = \Sigma dx_j^2 + \Sigma dy_j^2$, $z_j = x_j + iy_j$ affine coordinates. Let B be its boundary and $M = B \cap V$. Then M is a real complex of dimension $2n-1$, and a manifold if P is an isolated singularity. The topology of M together with its embedding in B ($=$ a $2m-1$-sphere) reflects the nature of the point P in V. The simplest case and the only one to be studied so far, to the author's knowledge, is where $n=1, m=2$, i.e. a plane curve (see [3], [14]). Then M is a disjoint union of a finite number of circles, knotted and linked in a 3-sphere. There is one circle for each branch of V at P, the intersection number of each pair of branches is the linking number of the corresponding circles, and the knots formed by each circle are compound toroidal, their canonical decomposition reflecting exactly the decomposition of each branch via infinitely near points.

The next interesting case is $n=2, m=3$. One would hope to find knots of a 3-sphere in a 5-sphere in this case; this would come about if P were an isolated singularity whose normalization was non-singular. Unfortunately, isolated non-normal points do not occur on hyper-surfaces in any Cohen-MacCaulay varieties. What happens, however, if the normalization of P is non-singular, is that M is the image of a 3-sphere mapped into a 5-sphere by a map which (i) identifies several circles, and (ii) annihilates a ray of tangent vectors at every point of another set of circles. In many cases the second does not occur, and we have an immersion of the 3-sphere in the 5-sphere. It would be quite interesting to know Smale's invariant in $\pi_3(V_{3,5})$ in this case (see [10]).

From the standpoint of the theory of algebraic surfaces, the really interesting case is that of a singular point on a *normal* algebraic surface, and m arbitrary. M is then by no means generally S^3 and consequently its own topology reflects the singularity P! In this paper, we shall consider this case, first giving a partial construction of $\pi_1(M)$ in terms of a resolution of the singular point P; secondly we shall sketch the connexion between $H_1(M)$ and the algebraic nature of P. Finally and principally, we shall demonstrate the following theorem, conjectured by Abhyankar:

Theorem. — $\pi_1(M) = (e)$ if and only if P is a simple point of F (a locally normal surface); and F topologically a manifold at P implies $\pi_1(M) = (e)$.

I. — ANALYSIS OF M AND PARTIAL CALCULATION OF $\pi_1(M)$

A normal point P in F is given. A finite sequence of quadratic transformations plus normalizations leads to a non-singular surface F′ dominating F [15]. The inverse image of P on F′ is the union of a finite set of curves $E_1, E_2, \dots, E_n$. By further quadratic transformations if necessary we may assume that all E_i are non-singular, and, if $i \neq j$, and $E_i \cap E_j \neq \emptyset$, then that E_i and E_j intersect normally in exactly one point, which does not lie on any other E_k. This will be a great technical convenience.

We note at this point the following fundamental fact about E_i: the intersection matrix $S = ((E_i . E_j))$ is negative definite. (This could also be proven by Hodge's Index Theorem.)

Proof. — Let H_1 and H_2 be two hyperplane sections of F, H_1 through P, and H_2 not (and also not through any other singular points of F). Let $(f) = H_1 - H_2$. Let H_1' be the proper transform of H_1 on F′, and H_2' the total transform of H_2. Then $H_2' \equiv H_1' + \Sigma m_i E_i$, where $m_i > 0$, all i (here m_i is positive since $m_i = \mathrm{ord}_{E_i}(f)$, f a function that is regular and zero at P on F, and moreover P is the center of the valuation of E_i on F).

Let $S' = ((m_i E_i . m_j E_j)) = M.S.M$, where M is the diagonal matrix with $M_{ii} = m_i$. To prove S′ is negative definite is equivalent with the desired assertion. Now note (*a*) $S_{ij} \geq 0$, if $i \neq j$, (*b*) $\sum_i S'_{ij} = \sum_i (m_i E_i . m_j E_j) = -(H_1' . m_j E_j) \leq 0$, all j. For any symmetric matrix S′, these two facts imply negative indefiniteness. To get definiteness, look closer: we know also (*c*) $\sum_i S'_{ij} < 0$, for some j (since H_1' passes through some E_j), and (*d*) we cannot split $(1, 2, \dots, n) = (i_1, i_2, \dots, i_k) \cup (j_1, j_2, \dots, j_{n-k})$ disjointly so that $S'_{i_a j_b} = 0$, any a, b (since $\cup E_i$ is connected by Zariski's main theorem [16]). Now these together give definiteness: Say

$$0 = \sum_{ij} \alpha_i \alpha_j S'_{ij} = \Sigma \alpha_i^2 S'_{ii} + 2 \sum_{i<j} \alpha_i \alpha_j S'_{ij}$$
$$= \sum_j \Big(\sum_i S'_{ij} \Big) \alpha_j^2 - \sum_{i<j} S'_{ij} (\alpha_i - \alpha_j)^2$$

where α_i are real. Then by (*c*), some $\alpha_j = 0$, and by (*d*), $\alpha_i = \alpha_j$, all i, j.

Our first step is a close analysis of the structure of M. We have defined it informally in the introduction in terms of an affine metric (depending apparently on the choice of this metric). Here we shall give a more general definition, and show that all these manifolds coincide, by virtue of having identical constructions by patching maps.

In the introduction, M is a level manifold of the positive C^∞ fcn.

$$p^2 = |Z_1|^2 + \dots + |Z_n|^2,$$

(Z_i affine coordinates near $P \in F$). Now notice that M may also be defined as the level manifolds of p^2 on the non-singular F′ (p^2 being canonically identified to a fcn. on F′). It is as a "tubular neighborhood" of $\cup E_i \subset F'$ that we wish to discuss M. Now the general problem, given a complex $K \subset E^n$, Euclidean n-space, to define a tubular neighborhood, has been attacked by topologists in several ways although it does not appear to have been treated definitively as yet. J. H. C. Whitehead [13], when K is a subcomplex in a triangulation of E^n, has defined it as the boundary of the star of K in the second barycentric subdivision of the given triangulation. I am informed that Thom [11] has considered it more from our point of view: for a suitably restricted class of positive C^∞ fcns. f such that $f(P)=0$ if and only if $P\in K$, define the tubular neighborhood of K to be the level manifolds $f=\varepsilon$, small ε. The catch is how to suitably restrict f; here the archtype for f^{-1} may be thought of as the potential distribution due to a uniform charge on K. In our case, as we have no wish to find the topological ultimate, we shall merely formulate a convenient, and convincingly broad class of such f, which includes the p^2 of the introduction.

Let us say that a positive C^∞ real fcn. f on F' such that $f(P)=0$ iff $P\in E_i$, is *admissible* if

1) $\forall P\in E_i-\bigcup_{j\neq i} E_j$, if $Z=0$ is a local equation for E_i near P, $f=|Z|^{2n_i}.g$, where g is C^∞ and neither 0 nor ∞ near P.

2) If $P_{ij}=E_i\cap E_j$, and $Z=0$, $W=0$ are local equations for E_i, E_j respectively then $f=|Z|^{2n_i}.|W|^{2n_j}.g$, where g is C^∞ and neither 0 nor ∞ near P_{ij}.

The following proposition is left to the reader.

Proposition: (i) If F'' dominates F', and f is admissible for $\bigcup E_i$ on F', and $g: F''\to F'$ is the canonical map, then $f\circ g$ is admissible for $g^{-1}(\bigcup E_i)$ on F'.

(ii) For a suitable F'' dominating F', p^2 is an admissible map for $g^{-1}(\bigcup E_i)$.

Let me say, however, that in (ii), the point is to take F'' high enough so that the linear system of zeroes of the functions $(\Sigma\alpha_i Z_i)$ less its fixed components, has no base points.

What we must now show is that there is a unique manifold M such that, if f is any admissible fcn., M is homeomorphic to $\{P\,|\,f(P)=\varepsilon\}$ for all sufficiently small ε. Fix a fcn. f to be considered. Notice that at each of the points P_{ij}, there exist real C^∞ coordinates X_{ij}, Y_{ij}, U_{ij}, V_{ij}, such that

$$f=(X_{ij}^2+Y_{ij}^2)^{n_i}\,(U_{ij}^2+V_{ij}^2)^{n_j}\alpha_{ij},$$

α_{ij} *a constant*, valid in some neighborhood U given by

$$\begin{gathered} X_{ij}^2+Y_{ij}^2<1\\ U_{ij}^2+V_{ij}^2<1. \end{gathered}$$

Assume E_i is $X_{ij}=Y_{ij}=0$, and E_j is $U_{ij}=V_{ij}=0$.

Our first trick consists of choosing a C^∞ metric $(ds)^2$ (depending on f), such that within

$$U'=\begin{Bmatrix} X_{ij}^2+Y_{ij}^2<1/2\\ U_{ij}^2+V_{ij}^2<1/2 \end{Bmatrix},$$

$$ds^2=dX_{ij}^2+dY_{ij}^2+dU_{ij}^2+dV_{ij}^2.$$

Such a metric exists, e.g. by averaging a Hodge metric with these Euclidean metrics by some partition of unity. Now let

$$\begin{array}{ccc} N_i & & S_i \\ \downarrow \pi_i & \text{and} & \downarrow \psi_i \\ E_i & & E_i \end{array}$$

be the normal 2-plane bundle to E_i and normal S^1-bundle to E_i in F' respectively. Consider the map $(\exp)_i$: $N_i \to F'$ obtained by mapping N_i into F along geodesics perpendicular to E_i. Let $f_i = f\circ(\exp)_i$. Now for every point $Q \in E_i - \bigcup_{j \neq i} E_j$, there is a neighborhood W of $Q \in E_i$, and an ε_0 such that if $\varepsilon < \varepsilon_0$, the locus $f_i(P) = \varepsilon$, $\pi_i(P) \in W$ cuts once each ray in $\pi_i^{-1}(W)$ (because f_i^{1/n_i} is a well-defined pos. C^∞ fcn. vanishing on the zero cross-section, with non-degenerate Hessian in normal directions; this is the standard situation of Morse theory, see [9]). Consequently, for any $W \subset E_i$ open, such that $E_j \cap W = \emptyset$, $j \neq i$, there is an ε_0 such that if $\varepsilon < \varepsilon_0$, the locus $f(P) = \varepsilon$ canonically contains a homeomorphic image of $\psi_i^{-1}(W)$ (recall $(\exp)_i$ is a local homeomorphism near the zero-section of N_i). Therefore, we see that the manifold M for which we are seeking a definition independent of f, is to be put together out of pieces of S_i; we need only seek its structure near P_{ij}. Let us therefore look in U'. Let us fix neighborhoods U_{ij} of $P_{ij} \in E_i$ and U_{ji} of $P_{ij} \in E_j$ by $(U_{ij}^2 + V_{ij}^2) < 1/4$ and $(X_{ij}^2 + Y_{ij}^2) < 1/4$ respectively. Let $E_k^* = E_k - \bigcup_{j \neq k} U_{kj}$ for all k. Now choose $\varepsilon_0 < \alpha_{i,j}/8^{n_i+n_j}$ and so that if $\varepsilon < \varepsilon_0$, $f(P) = \varepsilon$ contains $\psi_i^{-1}(E_i^*)$ and $\psi_j^{-1}(E_j^*)$ canonically. Then in the local coordinates in U' about P_{ij}, $\psi_i^{-1}(\partial E_i^*) \subset \{P \mid f(P) = \varepsilon\}$ equals

$$\left\{(X_{ij}, Y_{ij}, U_{ij}, V_{ij}) \mid U_{ij}^2 + V_{ij}^2 = 1/4,\ X_{ij}^2 + Y_{ij}^2 = \left(\frac{4^{n_j}\varepsilon}{\alpha_{ij}}\right)^{1/n_i}\right\}$$

and $\psi_j^{-1}(\partial E_j^*) \subset \{P \mid f(P) = \varepsilon\}$ equals

$$\left\{(X_{ij}, Y_{ij}, U_{ij}, V_{ij}) \mid X_{ij}^2 + Y_{ij}^2 = 1/4,\ U_{ij}^2 + V_{ij}^2 = \left(\frac{4^{n_i}\varepsilon}{\alpha_{ij}}\right)^{1/n_j}\right\}$$

(because of the Euclidean character of the metric ds^2 near P_{ij}, $\exp_i$ takes the simplest possible form!). Note $\left(\frac{4^{n_j}\varepsilon}{\alpha_{ij}}\right)^{1/n_i} < 1/8$. Therefore, we see that $\psi_i^{-1}(E_i^*)$ and $\psi_j^{-1}(E_j^*)$ are patched by a standard "plumbing fixture":

$$\{(x, y, u, v) \mid (x^2 + y^2) \leq 1/4,\ (u^2 + v^2) \leq 1/4,\ (x^2 + y^2)^n \cdot (u^2 + v^2)^m = \varepsilon < 1/8^{n+m}\}$$

where n and m are integers.

One sees immediately that this is simply $S^1 \times S^1 \times [0, 1]$, and if we set $M_i^* = \psi_i^{-1}(E_i^*)$, then it simply attaches ∂M_i^* to ∂M_j^*. Moreover, what is this attaching? There is a coordinate system on both ∂M_i^* and ∂M_j^* via

$$\left(\frac{X_{ij}}{\sqrt{X_{ij}^2+Y_{ij}^2}}, \frac{Y_{ij}}{\sqrt{X_{ij}^2+Y_{ij}^2}}\right) = \xi \in S^1 \text{ (in the usual embedding in } E^2)$$

$$\left(\frac{U_{ij}}{\sqrt{U_{ij}^2+V_{ij}^2}}, \frac{V_{ij}}{\sqrt{U_{ij}^2+V_{ij}^2}}\right) = \eta \in S^1 \text{ (in the usual embedding in } E^2)$$

and relative to these coordinates, the attaching is readily seen to be the identity. To complete the invariant topological description of M, we need only to show that the cycles $\{(\xi, \eta_0) \mid \xi \in S^1, \eta_0 \text{ fixed}\}$ and $\{(\xi_0, \eta) \mid \xi_0 \text{ fixed}, \eta \in S^1\}$ are invariantly determined (since an identification of 2 tori is determined up to isotopy by an identification of a basis of 1-cycles). But on M_i^* for instance, the 1st one is just the fibre of S_i over a point of E_i, and the 2nd is the loop ∂E_i^* lifted to S_i so that it is contractible in $\psi_i^{-1}(U_{ij})$; similarly on M_j^*, but *vice versa*.

This determines M uniquely. We have essentially found, moreover, not only M but also for any fixed f, maps

$$\varphi : M \to \cup E_i$$
$$\psi : \{P \mid 0 < f(P) \leqslant \varepsilon\} \to M$$

where ψ induces a homeomorphism of any $\{P \mid f(P) = \varepsilon' \leqslant \varepsilon\}$ onto M. Namely, define φ on M_i^* by ψ_i: projection into E_i, and in U′ near P_{ij}, define it as follows (fig. 1):

$$\begin{aligned}\varphi((X_{ij}, Y_{ij}, U_{ij}, V_{ij})) &= (0, 0, U_{ij}, V_{ij}) \in E_i \quad \text{if } U_{ij}^2+V_{ij}^2 \geq 1/4\\ &= (0, 0, pU_{ij}, pV_{ij}) \in E_i \quad \text{if } X_{ij}^2+Y_{ij}^2 \leqslant U_{ij}^2+V_{ij}^2 \leqslant 1/4\\ &= (p'X_{ij}, p'Y_{ij}, 0, 0) \in E_j \quad \text{if } U_{ij}^2+V_{ij}^2 \leqslant X_{ij}^2+Y_{ij}^2 \leqslant 1/4\\ &= (X_{ij}, Y_{ij}, 0, 0) \in E_j \quad \text{if } X_{ij}^2+Y_{ij}^2 \geqslant 1/4,\end{aligned}$$

where
$$p = \tau(X_{ij}^2+Y_{ij}^2, U_{ij}^2+V_{ij}^2)$$
$$p' = \tau(U_{ij}^2+V_{ij}^2, X_{ij}^2+Y_{ij}^2)$$

and where
$$\tau(\alpha, \beta) = \frac{\beta - \alpha}{1 - 4\alpha}.$$

As for ψ, away from P_{ij}, define ψ by first $(\exp)_i^{-1}$, then the projection of N_i — (0-section) to S_i, and then the identification of S_i into M; near P_{ij}, define it by identifying those points whose ξ and η coordinates are equal, and that have the same image in $E_i \cup E_j$ under the map φ.

Note that φ induces a map $\varphi_*: \pi_1(M) \to \pi_1(\cup E_i)$, which is onto as all the "fibres" are connected (1). In order not to be lost in a morass of confusion, we shall now restrict ourselves to computing only H_1 in general, and π_1 only if $\pi_1(\cup E_i) = (e)$. Note thats this last is equivalent to (*a*) E_i connected together as a tree (i.e. it never happens $E_1 \cap E_2 \neq \emptyset$, $E_2 \cap E_3 \neq \emptyset$, ..., $E_{k-1} \cap E_k \neq \emptyset$, $E_k \cap E_1 \neq \emptyset$ and $k > 2$ for some ordering of the E_i's), (*b*) all E_i are rational curves.

First, to compute $H_1(M)$, start with $H_i(\cup E_i)$. Let $\cup E_i$, as a graph, be p-connected,

(1) M is, of course, not a fibre space in the usual sense. However, the map φ_* in question is onto for any simplicial map such that the inverse image of every point is connected.

i.e. there exist some $P_1, \ldots, P_p$ such that if these points are deleted from $\cup E_i$, then $\cup E_i$ becomes a tree, but this does not happen for fewer P_i. Choose such P_i, and to $\cup E_i - \cup P_i$, for each P_i, add two points P_i' and P_i'', one to each E_j to which P_i belonged. The result, T, is, up to homotopy type, simply the wedge of the (closed) surfaces E_i (1). $\cup E_i$ is itself obtained from T by identifying the p pairs of points P_i', P_i''; therefore up to homotopy

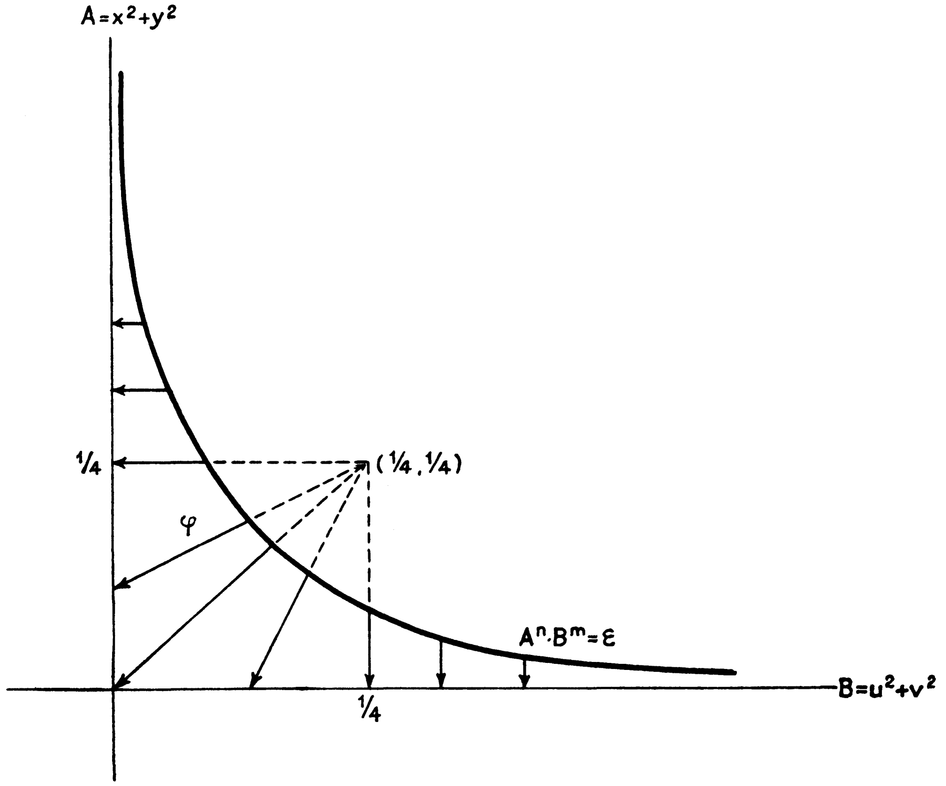

Fig. 1

type, it is the wedge of E_i and p loops. Therefore $H_1(\cup E_i) = \mathbf{Z}^{p+2\Sigma g_i}$, where g_i is the genus of E_i.

Now φ_* induces an onto map $H_1(M) \to H_1(\cup E_i)$, by passing modulo the commutators. Let K be its kernel. Let α_i be the loop or cycle of M consisting of the fibre of M over some point in $E_i - \cup_{j \neq i} E_j$ with the following sense: if $f_i = 0$ is a local equation for E_i,

$$\int_{\alpha_i} \frac{df_i}{f_i} = +2\pi i$$

or equivalently α_i as a loop about the origin of a fibre of the normal bundle N_i to E_i should have positive sense in its canonical orientation. I claim α_i generate K, and their relations are exactly $\Sigma(E_i \cdot E_j)\alpha_j = 0$, $i = 1, \ldots, n$.

(1) For example, proceeding surface by surface in any order, we may deform the complex $\cup E_i$ so that all the E_j which meet some one E_i meet it at the same point.

Proof. — First introduce the auxiliary cycles β_{ij} on $\varphi^{-1}(E_i)=M_i$, whenever $E_i \cap E_j = (P_{ij}) \neq \emptyset$. Namely, move the cycle α_j along the fibres until it lies on $\varphi^{-1}(P_{ij}) \subset M_i$, and there call it β_{ij}. By my construction of the patching of M_i and M_j, we know that β_{ij} is what I called η, while α_i is ξ. Now compute the subgroup K_i of $H_1(M_i)$ defined by

$$\begin{array}{ccccccccc} 0 & \to & K_i & \longrightarrow & H_1(M_i) & \to & H_1(E_i) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K & \longrightarrow & H_1(M) & \to & H_1(\cup E_i) & \to & 0. \end{array}$$

As above, let U_{ij} be a small disc on E_i about P_{ij}, and $E_i^* = E_i - \cup U_{ij}$, and $M_i^* = \varphi^{-1}(E_i^*)$. Then M_i^* is a deformation retract of M_i, and is, on the one hand canonically the restriction of the bundle S_i to E_i^*, and on the other hand uncanonically homeomorphic to $S^1 \times E_i^*$. In this last description, α_i is canonically identified to $S^1 \times$ (point), while β_{ij} are identified to (point) $\times \partial(U_{ij})$ only up to adding a multiple of α_i. Therefore we see that K_i is generated by α_i, β_{ij}, with one relation ([1])

$$\sum_j \beta_{ij} + N\alpha_i = 0, \text{ some } N.$$

To evaluate N, note that β_{ij} considered as cycles in S_i are locally contractible (i.e. in the neighborhood of $\varphi^{-1}(P_{ij})$ described by my plumbing fixture). It is well known that when the oriented fundamental 2-cycle of E_i is lifted to S_i, its boundary is $(E_i^2)\alpha_i$. Therefore, this same lifting in M_i^* will have boundary $\sum_j \beta_{ij} + (E_i^2)\alpha_i$. Now by the Mayer-Vietoris sequence, $H_1(M)$ is generated by $H_1(M_i)$, hence K is by K_i, and has extra relations imposed by the identification of cycles on $M_i \cap M_j$. Since $H_1(M_i \cap M_j)$ is generated by β_{ij} and β_{ji}, these relations are implicit in our choice of generators.

As a consequence of our result, since $\det(E_i . E_j) = \mu \neq 0$, K is a finite group of order μ, and is the torsion subgroup of $H_1(M)$.

Now consider the case E_i rational, and $\cup E_i$ tree-like. We shall compute $\pi_1(M)$, using $\pi_1(M_i)$ as building blocks. In order to keep these various groups, with their respective base points, under control, it is necessary to define a skeleton of basic paths leading throughout E_i. Let $Q_i \in E_i - \bigcup_{j \neq i} E_j$ be chosen as base point in E_i. On E_i, choose a path l_i as illustrated in Diagram II touching on each $P_{ij} \in E_i$. Lift all the l_i together into M by a map s, so that $\varphi(s(l_i)) = l_i$, and so that at $\varphi^{-1}(P_{ij})$, $s(l_i) \cap s(l_j) \neq \emptyset$. Choose, e.g. $s(Q_1)$ as base point for all of M. Let $G = \cup l_i$. Now the lifting s enables us to give the following recipe for paths α_i:

1. Go along $s(G)$ from $s(Q_1)$ to a point P in M_i.
2. Go once around the fibre of M_i through P in the canonical direction explained above.
3. Go back to $s(Q_1)$ along $s(G)$.

([1]) In the map $H_1(E_i^*) \to H_1(E_i)$, the kernel is generated by $\{\partial(U_{ij})\}$ with the single relation $\Sigma_{j \neq i} \partial(U_{ij}) = \partial$(fundamental 2-cycle of E_i^*) ~ 0.

This is clearly independent of the choice of P.

Our result can now be stated: firstly, the α_i generate π_1; secondly, their only relations are (*a*) α_i and α_j commute if $E_i \cap E_j \neq \emptyset$, (*b*) if $k_i = (E_i^2)$, and $E_{j_1}, E_{j_2}, \ldots, E_{j_m}$ are those E_j intersecting E_i, written in the order in which they intersect l_i, then

$$e = \alpha_{j_1} \alpha_{j_2}, \ldots, \alpha_{j_m} \alpha_i^{k_i}.$$

To prove this, we use the following theorem of Van Kampen (see [8], p. 30): if X and Y are subcomplexes of a complex Z, and $Z = X \cup Y$, while $X \cap Y$ is connected, then $\pi_1(Z)$ is the free product of $\pi_1(X)$ and $\pi_1(Y)$ modulo amalgamation of the sub-

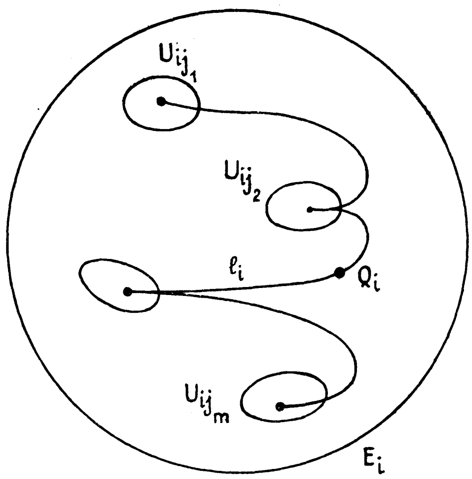

Fig. 2

groups $\pi_1(X \cap Y)$. Now since E_i is tree-like, M can be gotten from the M_i by successively joining on a new M_i with *connected* intersection with the part so far built up. Let $\pi_1(M_i)$ be mapped into $\pi_1(M)$ by mapping a loop in M_i with base point $s(Q_i)$ to one in M with base point $s(Q_1)$ by simply tagging on to both ends of it the section of $s(G)$ joining these two points. Then $\pi_1(M)$ is simply the free product of the $\pi_1(M_i)$ with amalgamation of the loops in $M_i \cap M_j$. Now recalling the structure of M_i^*, we have an exact sequence that splits:

$$0 \to \pi_1(S^1) \to \pi_1(M_i^*) \xrightarrow{\pi} \pi_1(E_i^*) \to 0$$

(S^1 the fibre of M_i, a 1-sphere). The path α_i is clearly a generator of $\pi_1(S^1)$ here, and hence in the center of $\pi_1(M_i^*)$.

Now the important thing to notice is that if E_i meets E_j, then α_j in $\pi_1(M_j)$ can be moved by modifying the point P on $s(G)$ where α_j detours around the fibre S^1; in particular, it may do this at $s(l_i) \cap s(l_j)$. In that position the loop α_j may be regarded canonically as in $\pi_1(M_i)$. Under the identification of $\pi_1(M_i)$ to $\pi_1(M_i^*)$ and the projection π of this group onto $\pi_1(E_i^*)$, what happens to the loop α_j? Recalling the patching map on the boundaries of M_i^* and M_j^* which was examined above, we see that this path proceeds along G from Q_i to near P_{ij}, then circles around the boundary of U_{ij}

in a positively oriented direction, then returns along G to Q_i. Referring again to our diagram, we see the relation $e=\pi(\alpha_{j_1}).\pi(\alpha_{j_2}).\dots.\pi(\alpha_{j_m})$. Now it is well-known that these loops $\pi(\alpha_{j_k})$ generate the fundamental group of the m-times punctured sphere, and that this is the unique relation. Consequently, looking at the above exact sequence, it is clear that $\alpha_i, \alpha_{j_1}, \dots, \alpha_{j_m}$ (when distorted into M_i as indicated above) generate $\pi_1(M_i)$. Moreover, the only relations among these generators are, therefore, that α_i and α_{j_k} commute, and $\alpha_{j_1}.\dots.\alpha_{j_m}\in\pi_1(S^1)$, i.e. $=\alpha_i^N$. But, using our results on $H_1(M)$, $N=-(E_i^2)$.

It follows that α_i generate $\pi_1(M)$ with relations (a) and (b), and that the only additional relations are those coming from the amalgamation of $\pi_1(M_i\cap M_j)=\mathbf{Z}+\mathbf{Z}$. But α_i and α_j are generators here, and as loops in M_i and M_j, these have already been identified. Hence we are through, Q.E.D.

II. — ALGEBRO-GEOMETRIC SIGNIFICANCE OF $H_1(M)$

(a) Local Analytic Picard Varieties and Unique Factorization.

We shall study in this section two questions of algebro-geometric interest in the solution of which the topological structure of M, in particular its homological structure, is reflected. The first of these is the problem of the local Picard Variety at $P\in F$. Generally speaking, this, as a group, should be the group of *local* divisors at P modulo local linear equivalence to zero. (We shall be more precise below.) However, if by divisor one refers to an algebraic divisor and by local one means in the sense of the Zariski topology, one sees by example that the resulting group has little significance: it is not local enough. Ideally, one should mean by an irreducible local divisor a minimal prime ideal in the formal completion of the local ring of the point in question. However, I have been unable to establish the structure of the resulting Picard group. A compromise between these two groups is possible over the complex numbers. Take as divisors *analytic* divisors, and the usual complex topology to interpret local. There results a local *analytic* Picard variety that is quite accessible. In this section, we shall first analyze the group of local *analytic* divisors near $\cup E_i$ modulo local linear equivalence and then consider the singular point P. Here by local analytic divisors we mean formal sums of irreducible analytic divisors defined in a neighborhood of $\cup E_i$ (including the divisors E_i themselves). Such a sum, $\Sigma n_i D_i$, is said to be locally linearly equivalent to zero if there exists a neighborhood U of $\cup E_i$ where all D_i are defined and a meromorphic function f on U such that $(f)=\Sigma n_i(D_i\cap U)$. This quotient we shall call the local analytic Picard Variety at $\cup E_i$, or Pic $(\cup E_i)$.

Denote by Ω the sheaf of germs of holomorphic functions on F'; by $\Omega^*\subset\Omega$ the sheaf of germs of non-zero holomorphic functions. One has the usual exact sequence:

$$0\to\mathbf{Z}\to\Omega\xrightarrow{\exp(2\pi ix)}\Omega^*\to 0$$

where $\mathbf{Z}$ is the constant sheaf of integers. Let $\pi\colon F'\to F$ be the regular projection from the non-singular surface F' to the singular F.

Proposition. — $\text{Pic}(\cup E_i) \simeq (R^1\pi)(\Omega^*)_P$.

Proof. — Define $\text{Pic}(\cup E_i) \to (R^1\pi)(\Omega^*)_P$, by associating to $\Sigma n_i D_i$, defined in $U \supset \cup E_i$, the following 1-cocycle: assume $P \in V$, $\pi^{-1}(V) \subset U$, assume f_j is a local equation for $\Sigma n_i D_i$ in V_j, $\{V_j\}$ a covering of V, then $\{f_{j_1}/f_{j_2}\} \in H^1(\{V_j\}, \Omega^*)$ induces an $\alpha \in H^1(\pi^{-1}(V), \Omega^*)$, hence an $\alpha' \in (R^1\pi)(\Omega^*)_V$, hence an $\alpha'' \in (R^1\pi)(\Omega^*)_P$. It is well known that $\alpha \in H^1(\pi^{-1}(V), \Omega^*)$ is uniquely determined by $\Sigma n_i D_i$, hence so is α''.

To see that $\Sigma n_i D_i \to \alpha''$ is $1-1$, say $\alpha''=0$. Therefore $\exists V' \subset V$ say, $\text{Res}_{V'}\alpha'=0$, i.e. $\text{Res}_{\pi^{-1}(V')}(\alpha)=0$. Therefore the covering $\{V_j \cap \pi^{-1}(V')\} = \{V'_j\}$ has a refinement $\{V''_k\}$ such that there exist non-zero functions g_k on V''_k such that $g_{k_1}/g_{k_2} = f_{\tau k_1}/f_{\tau k_2}$ (for some map τ from the indices of $\{V''_k\}$ to those of $\{V_j\}$ such that $V''_k \subset V'_{\tau k}$). Therefore $f = \dfrac{f_{\tau k}}{g_k}$ defines a function throughout $\pi^{-1}(V')$ such that $(f) = \Sigma n_i D_i$.

To see that $\Sigma n_i D_i \to \alpha''$ is onto $(R^1\pi)(\Omega^*)_P$, let $\beta'' \in (R^1\pi)(\Omega^*)_P$ be represented by $\beta \in H^1(\pi^{-1}(V), \Omega^*)$ and let this define the line bundle L over $\pi^{-1}(V)$ in the usual way. Let $\mathscr{I}$ be the sheaf of germs of cross-sections of L: a coherent sheaf. Now by a result of Grauert and Remmert (cf. Borel-Serre [2], p. 104), $(R^0\pi)(\mathscr{I})$ is coherent on F. But $(R^0\pi)(\mathscr{I})$ is not the zero sheaf on F (at all points $Q \neq P$, $\mathscr{I}_Q \simeq (R^0\pi)(\mathscr{I})_Q$), hence there exists some element $S \in (R^0\pi)(\mathscr{I})_P$, $S \neq 0$. S corresponds to a section in $\mathscr{I}_{\pi^{-1}(V')}$, for some open $V' \ni P$, $V' \subset V$. Therefore, the line bundle $L|\pi^{-1}(V)$ has a section S. But if β is represented by a cocycle f_{ij} with respect to a covering $\{V_i\}$ of V, then S is given by a set of holomorphic functions f_i on V_i such that $f_j = f_i(f_{ij})$. It follows that $f_i = 0$ define a divisor which is represented by β.

A. Grothendieck has posed the problem, for any proper map $f: V_1 \to V_2$ (onto), to define a relative Picard Variety of the map f. It seems clear, in the classical case, that if Ω^* is the sheaf of holomorphic units on V_1, $(R^1 f)(\Omega^*)$ is the logical choice although no nice properties have been established in general so far as the writer knows. In our case, $(R^1 f)(\Omega^*)_Q$, for $Q \neq P$, is simply (1), but at P, we have seen it to be $\text{Pic}(\cup E_i)$. We now wish to show that in our case, $(R^1 f)(\Omega^*)_P$ is an analytic group variety. This is seen by the exact sequence for derived functors:

$$\begin{aligned} 0 &\to (R^0\pi)(\mathbf{Z}) \to (R^0\pi)(\Omega) \xrightarrow{\varphi} (R^0\pi)(\Omega^*) \to \\ &\to (R^1\pi)(\mathbf{Z}) \xrightarrow{\chi} (R^1\pi)(\Omega) \to (R^1\pi)(\Omega^*) \xrightarrow{\psi} \\ &\to (R^2\pi)(\mathbf{Z}) \to \dots \end{aligned}$$

(i) Note first that if $x \in (R^0\pi)(\Omega^*)_P$, then x is a non-zero function on $\pi^{-1}(V)$, $P \in V$, and necessarily constant on $\cup E_i$ which is connected and compact, therefore, at least on some $\pi^{-1}(V')$, $P \in V' \subset V$, $x = \exp(2\pi i y)$, y a holomorphic function on $\pi^{-1}(V')$, hence $x = \varphi(y)$, $y \in (R^0\pi)(\Omega)_P$.

(ii) Note secondly that $(R^i\pi)(\mathbf{Z})_p \simeq H^i(\cup E_i, \mathbf{Z})$, since for $P \in V$, V small, $\pi^{-1}(V)$ is contractible to $\cup E_i$.

(iii) Note thirdly that if $i>0$, $(R^i\pi)(\Omega)_Q=(0)$ for $Q\neq P$, and being a coherent sheaf, for $Q=P$ must be a finite dimensional vector space over $\mathbf{C}$.

(iv) Note fourthly that if $\gamma\in H^2(\bigcup E_i, \mathbf{Z})\simeq(R^2\pi)(\mathbf{Z})_P$, there exists $\alpha\in(R^1\pi)(\Omega^*)_P$ such that $\psi\alpha=\gamma$. To show this, note that $H^2(\bigcup E_i, \mathbf{Z})\simeq\mathbf{Z}^n$, ($n$ = number of irreducible curves in $\bigcup E_i$) with generators γ_i whose value on the 2-cycle E_j is δ_{ij}; it is enough to verify it for the generators γ_i. But let D_i be an irreducible analytic curve through $Q\in E_i-\bigcup_{j\neq i} E_i$, with a simple point at Q, and tangent transversal to that of E_i at Q. If $D_i\to\alpha_i\in(R^1\pi)(\Omega^*)$, I claim $\psi\alpha_i=\gamma_i$. This is left to the reader. Therefore, we obtain

$$0\to H^1(\bigcup E_i, \mathbf{Z})\xrightarrow{\chi}(R^1\pi)(\Omega)_P\to \mathrm{Pic}\,(\bigcup E_i)\to H^2(\bigcup E_i, \mathbf{Z})\to 0$$
$$\wr\|$$
$$\mathbf{C}^N, \text{ some } N.$$

(v) Note lastly that χ maps $H^1(\bigcup E_i, \mathbf{Z})$ into a *closed* subgroup of $(R^1\pi)(\Omega)_P$, hence the connected component of $\mathrm{Pic}(\bigcup E_i)$ is an analytic group. If this were false, there would be a *real* sum of elements of $H^1(\bigcup E_i, \mathbf{Z})$ that was zero without having to be, i.e. $\{\alpha_{ij}\}\in H^1(\pi^{-1}(V), \mathbf{R})$ (with respect to some covering $\{U_i\}$) such that $\{\alpha_{ij}\}\sim 0$ in the sheaf Ω (in some $\pi^{-1}(V')$, $V'\subset V$). In other words, $\alpha_{ij}=f_i-f_j$, f_i holomorphic in U_i. But let p_i be a real, C^∞ function on U_i such that $\alpha_{ij}=p_i-p_j$ (Poincaré's lemma). Then $f_i-p_i=F$, $df_i=\omega$ and $dp_i=\eta$, are defined all over $\pi^{-1}(V')$, $\omega-\eta=dF$. I claim actually all the periods of η are zero (which implies $\eta=df$, and $\{\alpha_{ij}\}\sim 0$ in $H^1(\bigcup E_i, \mathbf{R})$ and we are through). First of all, the periods of η equal those of ω. Look at its periods on the 1-cycles of any E_i: since η is real, all the periods of the holomorphic differential ω are also real. But it is wellknown that then all the periods of ω must be identically zero, and therefore ω reduces to *zero* on paths in E_i. Since this is true for all i, ω has no periods along *any* path in $\bigcup E_i$, and since $\pi^{-1}(V')$ is contractible to $\bigcup E_i$, ω has no periods at all. Therefore neither does η and we are through.

There is another way of looking at $\mathrm{Pic}(\bigcup E_i)$. Namely, let $\mathfrak{o}$ be the local ring of (convergent) holomorphic functions at P, i.e. $(R^0\pi)(\Omega)_P$ (by the theorem of Riemann, cf. the report of Behnke and Grauert ([1], p. 18)). Now every divisor D' in $\pi^{-1}(V')$, except for the E_i's, defines a divisor D in V', hence a minimal prime ideal $\mathfrak{p}$ in $\mathfrak{o}$. Let us set Pic(P) equal to the group of ideal classes in $\mathfrak{o}$: i.e. to the semi-group of pure rank 1 ideals $\mathfrak{a}$ of $\mathfrak{o}$, modulo the principal ideals (1). Then the association of D to $\mathfrak{p}$ defines a map from $\mathrm{Pic}(\bigcup E_i)\to\mathrm{Pic}(P)$, (if we define the image of each E_i to be (1), the identity). This is quite clear once one sees that every meromorphic function f in $\pi^{-1}(V)$ is a quotient

(1) The composition law is the "Kronecker" product treated so elegantly by Hermann Weyl [12], cf. chapter 2, namely:

$$(\mathfrak{a}, \mathfrak{b})\to \text{rank 1 component of } \mathfrak{a}.\mathfrak{b}$$
$$=\bigcup_{n=1}^{\infty}(\mathfrak{a}.\mathfrak{b}) : \mathfrak{m}^n$$

where $\mathfrak{m}$ = maximal ideal of $\mathfrak{o}$

(:) = residual quotient operation.

of two holomorphic functions in some $\pi^{-1}(V')$, $V' \subset V$: but given f, consider the coherent sheaf $\mathscr{J}$ given by $\{g \mid (fg) \text{ is a positive divisor}\}$. $(R^0\pi)(\mathscr{J})$ is coherent, hence there exists $g_1 \in (R^0\pi)(\mathscr{J})_P$, and if $fg_1 = g_2$, then $f = g_2/g_1$ is the desired decomposition. Now the map $\mathrm{Pic}(\cup E_i) \to \mathrm{Pic}(P)$ is onto as every minimal prime ideal $\mathfrak{p} \subset \mathfrak{o}$ defines some divisor through P. Its kernel is immediately seen to be generated by the E_i themselves. Hence we see

Proposition: $$\frac{\mathrm{Pic}(\cup E_i)}{\{\Sigma n_i E_i\}} \simeq \mathrm{Pic}(P)$$

Corollary. — We have

$$0 \to H^1(\cup E_i, \mathbf{Z}) \to (R^1\pi)(\Omega)_P \xrightarrow{\varphi} \mathrm{Pic}(P) \xrightarrow{\psi} H_1(M)_0 \to 0$$

where $H_1(M)_0$ = torsion subgroup of $H_1(M)$ and ψ associates to the divisor D through P, the 1-cycle $D \cap M$.

Proof of Corollary: Note that $\Sigma n_i E_i$ is never in the image of $(R^1\pi)(\Omega)_P$ since that would require $(\Sigma n_i E_i, E_j) = 0$ for all j. To see the exactness at ψ, note that the co-kernel of φ is obtained by associating to a divisor $\Sigma n_i D_i$ (where we may assume $E_i \cap E_j \cap (\bigcup_l \mathrm{Supp}\, D_l) = \emptyset$, all $i \neq j$) the formal sum

$$\sum_k \Big(\sum_i n_i D_i \cdot E_k\Big)\gamma_k \text{ modulo } \Big\{\sum_k (E_i \cdot E_k)\gamma_k\Big\},$$

the γ_k as in (iv) above. But ψ is given by associating to $\Sigma n_i D_i$, the element

$$\sum_k (\Sigma n_i D_i \cdot E_k)\alpha_k,$$

in terms of our basis for $H_1(M)_0$ in (I); but by our enumeration of the relations on the α_k we see γ_k can be interchanged with α_k.

Do these results have purely algebraic counterparts? First, note that it is hopeless to expect that the ideal structure of $\mathfrak{o}_0$ (= algebraic local ring of P on F) will reflect the homology of the singularity so well. This is seen in the following example: Take a non-singular cubic curve E in the projective plane, and let $P_1, \ldots, P_{15}$ be points on E in general position except that on E the divisor $\Sigma_1^{15} P_i \equiv 5 \times$ (plane section). Blow up every point P_i to a divisor E_i, and call F′ the resulting surface. On F′, the proper transform E′ of E is exceptional: it is shrunk by the linear system of quintics through the P_i. Then $E_i - E_j$ as a divisor in Pic(E′) is in the component of the identity, but as an algebraic divisor is not algebraically locally equivalent to zero: in fact F′ is regular, hence algebraic and linear equivalence are the same, but since $\mathrm{Tr}_{E'}(E_i - E_j) \not\equiv 0$, $E_i - E_j$ is not locally linearly equivalent to zero.

However, I conjecture that the ideal class group of $\mathfrak{o}^*$ (= completion of $\mathfrak{o}_0$ and $\mathfrak{o}$) is *identical* to that of $\mathfrak{o}$, and that sums of formal branches through $\cup E_i$ modulo holomorphic linear equivalence (in the sense of Zariski [17]) gives $\mathrm{Pic}(\cup E_i)$. If this is so, it should give $\mathrm{Pic}(\cup E_i)$ an *algebraic* structure, which would be a decided improvement on our results. At present, I am unable to prove these statements.

(b) **Intersection Theory on Normal Surfaces.**

We consider here the problem of defining, for divisors A, B through P on F, (*a*) total transforms A′, B′ on F′, and (*b*) intersection multiplicities $i(A.B; P)$. This problem has been posed by Samuel (see [7]) and considered by J. E. Reeve [19]. In this case, I suggest the following as a canonical solution:

a) To define $A' = A'_0 + \Sigma r_i E_i$, where A'_0 is the proper transform of A, require

$$(A'.E_i) = 0, \quad i = 1, 2, \ldots, n,$$

or

$$(A'_0.E_i) + \sum_j r_j (E_j.E_i) = 0, \quad i = 1, 2, \ldots, n.$$

Since $\det(E_i.E_j) = \mu \neq 0$, this has a unique solution.

b) To define $i(A.B; P)$, set it equal to

$$\begin{aligned} &(A'.B') \text{ over } P \\ &= \sum_{P' \text{ over } P} [i(A'_0.B'_0; P') + \Sigma r_i i(E_i.B'_0; P')] \\ &= \sum_{P' \text{ over } P} [i(A'_0.B'_0; P') + \Sigma s_i i(A'_0.E_i; P')] \end{aligned}$$

where

$$A' = A'_0 + \Sigma r_i E_i; \quad B' = B'_0 + \Sigma s_i E_i.$$

We note the following properties:

(i) $A = (f)_F$, then $A' = (f)_{F'}$; hence $A \equiv B$ implies $A' \equiv B'$.

Proof. — For $((f)_{F'}.E_i) = 0$.

(ii) A effective, then all r_i are positive.

Proof. — Say some $r_i \leq 0$. Say also $r_i/m_i \leq r_j/m_j$, all j, where the m_j are the same as in the proof of negative definiteness. Then we see:

$$\begin{aligned} 0 \geq \sum_j r_j (E_j.E_i) &= \sum_j r_j/m_j \, (m_j E_j.E_i), \\ &\geq r_i/m_i \sum_j (m_j E_j.E_i) \geq 0. \end{aligned}$$

Therefore, if $E_i \cap E_j \neq \emptyset$, $r_i/m_i = r_j/m_j$ and $r_j \leq 0$. As $\cup E_i$ is connected, this gives ultimately $r_i/m_i = R$, independent of i. But then also $(\Sigma m_j E_j.E_i) = 0$, all i, which contradicts property (*c*) in the proof just referred to.

(iii) $i(A.B; P)$ is symmetric and distributive.

(iv) A and B effective, then $i(A.B; P)$ is greater than 0.

(v) $i(A.B; P)$ independent of the choice of F′.

Proof. — To show this, it suffices, since any two non-singular models are dominated by a third, see Zariski [15], to compare F′ with F″ gotten by blowing up some point P′ over P. But let A′, B′ be the total transforms of A, B on F′, and A″, B″ those on F″, and let T be the map from F″ to F′. Then with respect to T, A″ is the total transform of A′ on F″, and B″ that of B′. In that case it is well-known that, for any point set S in F′ (including all the points of any common components of A′, B′), $(A'.B')_S = (A''.B'')_{T^{-1}(S)}$.

(vi) A′ is integral if and only if $\Sigma(A'_0.E_i)\alpha_i = 0$ in $H_1(M)$.

Proof. — $\Sigma(A'_0.E_i)\alpha_i = 0$ if and only if there are integers k_j such that

$$(A'_0.E_i) = \Sigma k_j(E_j.E_i),$$

i.e. if the relation $\Sigma(A'_0.E_i)\alpha_i = 0$ is an integral sum of the relations defining $H_1(M)$. But this is equivalent to $(A'_0 + \Sigma k_j E_j.E_i) = 0$ for all i, i.e. $A' = A'_0 + \Sigma k_j E_j$, k_j integral. Q.E.D.

The element $\Sigma(A'_0.E_i)\alpha_i$ has this simple interpretation: if M is chosen near enough to P, it represents the 1-cycle $A \cap M$. We see that this is again the fundamental map: (Group of Local Divisors at P) $\to H_1(M)$ considered in the final corollary of part (*a*). By the results of part (*a*), moreover, we can interpret (vi) as saying: A′ is integral if and only if A is locally analytically equivalent to zero (i.e. A is in the connected component of Pic(P)). Essentially, our definition of intersection multiplicity on a normal surface is the unique linear theory that has the correct limiting properties for divisors that can be analytically deformed off the singular points.

III. — THE CASE $\pi_1(M) = (e)$

We shall prove the following theorem, stronger than that announced above:

Theorem. — Let F be a non-singular surface, and E_i, $i = 1, 2, \ldots, n$, a connected collection of non-singular curves on F, such that $E_i \cap E_j$ is empty, or consists of one point on a transversal intersection, and $E_i \cap E_j \cap E_k$ is always empty. Let M be a tubular neighborhood of $\cup E_i$, as defined in section I. If (*a*) $\pi_1(M) = (e)$, and (*b*) $((E_i.E_j))$ is negative definite, then $\cup E_i$ is exceptional of first kind, i.e. is the total transform of some simple point on a surface dominated by F and birational to it.

Proof. — As above, $\pi_1(M) = (e)$ implies that all E_i are rational, and connected together as a tree. Now suppose that $\cup E_i$ is not exceptional of first kind. Assume that among all collections of E_i with all the properties of the theorem, there is no collection not exceptional with *fewer* curves E_i. As a consequence, no E_i of our collection has the two properties (*a*) $(E_i^2) = -1$, (*b*) E_i intersects at most two other E_j. For if it did, one could shrink E_i by Castelnuovo's criterion, preserving all the properties required (that the negative definiteness is preserved is clear as follows: the self-intersection of a cycle of the E_j's on the blown down surface equals the self-intersection of its total transform on F which must be negative). We allow the case where there is only one E_i. Now the central fact on which this proof is based is the following group-theoretic proposition:

Proposition. — Let G_i, $i = 1, 2, 3$, be non-trivial groups, and a_i an element of G_i. Then denoting the free product of A and B by $A * B$, it follows $G_1 * G_2 * G_3$/modulo $(a_1 a_2 a_3 = e)$ is non-trivial.

Proof. — First of all, if $\infty \geq n_1, n_2, n_3 > 1$, then $Z_{n_1} * Z_{n_2} * Z_{n_3}/(a_1 a_2 a_3 = e)$ is non-trivial, where Z_k denotes the integers modulo k, and each a_i is a generator. For, as a matter of fact, these are well-known groups easily constructed as follows: choose a triangle with angles π/n_1, π/n_2, and π/n_3 (modular if some $n_i = \infty$), in one of the three standard planes. Reflections in the three sides of the triangle generate a group of motions of the plane, and the group we seek is the subgroup, of index 2, of the orientation preserving motions in this group. Secondly, reduce the general statement to this case by means of:

($\#$) If $n =$ order of a_1 in G_1, and a_1 is identified to a generator of $Z_n \subseteq G_1$, then $G_1 * G_2 * G_3/(a_1 a_2 a_3 = e)$ trivial $\Rightarrow Z_n * G_2 * G_3/(a_1 a_2 a_3 = e)$ trivial.

To show this, let $H = G_2 * G_3/((a_2 a_3)^n = e)$, and note that H is isomorphic to $Z_n * G_2 * G_3/(a_1 a_2 a_3 = e)$. Let n' be the order of a_1 in H. Then $G_1 * G_2 * G_3/(a_1 a_2 a_3 = e)$ is the free product of $G_1/(a_1^{n'} = e)$ and H with amalgamation of the subgroups generated by $a_2 a_3$ and a_1^{-1}. But by O. Schreier's construction of amalgamated free products (see [5], p. 29) this is trivial only if H is, hence ($\#$). Now the proposition is trivial if any $a_i = e$; hence let $n_i =$ order $(a_i) > 1$. By ($\#$) iterated, $G_1 * G_2 * G_3/(a_1 a_2 a_3 = e)$ trivial implies $Z_{n_1} * Z_{n_2} * Z_{n_3}/(a_1 a_2 a_3 = e)$ trivial, which is absurd. Q.E.D.

Returning to the theorem, we wish to show the absurdity of $\pi_1(M) = (e)$, while no E_i is such that (*a*) $(E_i^2) = -1$, and (*b*) E_i meets at most two other E_j. There are two cases to consider: either *some* E_i meets three or more other E_j; or every E_i meets at most two other E_j (this includes the case of only one E_i).

Case 1. — Let E_1 meet $E_2, \ldots, E_m$, where m is at least 4. For $i = 2, 3, \ldots, m$, let T_i be the set of E_j's (besides E_1) such that E_j is connected to E_i by a series of E_k other than E_1. The T_i's are disjoint. Let M_i be the manifold bounding a neighborhood of T_i as above. Let $G_i = \pi_1(M_i)$, and $G = \pi_1(M)$/modulo $\alpha_1 = e$, where α_1 represents, as in (I), the loop about E_1. Then by the results of (I),

$$G = G_2 * G_3, \ldots, * G_m/(\alpha_2 \alpha_3 \ldots \ldots \alpha_m = e),$$

if the G_i are ordered suitably, and α_i in G_i represents a loop about E_i. Now $m \geq 4$, and $\pi_1(M) = (e)$, hence $G = (e)$, hence by the above theorem, there exists an i (say $i = 2$) such that $G_2 = \pi_1(M_2) = (e)$. By the induction assumption, the tree of curves T_2 is exceptional of first kind. Therefore, by Zariski's theorem on the factorization of anti-regular transformations on non-singular surfaces (see [18]), some E_j in T_2 enjoys the properties (*a*) and (*b*) with respect to T_2. Then E_j would also enjoy them in $\cup E_i$ (which is impossible) *unless* $E_j = E_2$, in which case E_j could meet only two other E_k (say E_{m+1}, E_{m+2}) in T_2, but would meet *three* other E_k in $\cup E_i$. Pursuing this further, apply the same reasoning to the curve E_2 which meets exactly three other E_k. Again, either some curve shrinks, or else either E_1, E_{m+1}, or E_{m+2} has in any case property (*a*), i.e. self-intersection -1. But then compute $((E_2 + E_i)^2)$ ($i = 1, m+1$, or $m+2$ according as which E_i has property (*a*)), and we get 0, contradicting negative definiteness of the intersection matrix.

Case 2. — It remains to consider the case where no E_i intersects more than two others. Then the E_i are arranged as follows:

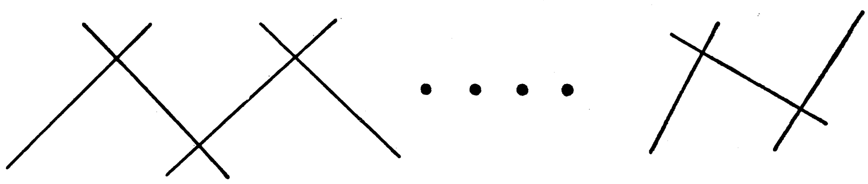

Fig. 3

In this case, it is immediate that π_1 is commutative, hence $= H_1$. It is given (in additive notation) by the equations:

$$\begin{aligned} k_1\alpha_1 - \alpha_2 \dots\dots\dots\dots &= 0 \\ -\alpha_1 + k_2\alpha_2 - \alpha_3 \dots\dots &= 0 \\ - \alpha_2 + k_3\alpha_3 \dots\dots &= 0 \\ \dots\dots\dots\dots\dots & \\ -\alpha_{n-1} + k_n\alpha_n &= 0, \end{aligned}$$

where $k_i = -(E_i^2)$. Assume all $k_i \geq 2$, and prove

$$\mu = \det \begin{pmatrix} k_1 & -1 & 0 & 0 \dots\dots & 0 \\ -1 & k_2 & -1 & 0 \dots\dots & 0 \\ 0 & -1 & k_3 & -1 \dots\dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 \dots & \dots & 0 & -1 & k_n \end{pmatrix} > 1,$$

hence the equations have a solution mod μ. To show this, use induction on n, using the stronger induction hypothesis $k_1 > 1$, $k_2, \dots, k_n \geq 2$, allowing k_i to be rational. Then note the identity:

$$\det \begin{pmatrix} k_1 & -1 & 0 \dots 0 \\ -1 & k_2 & -1 \dots 0 \\ \dots & \dots & \dots \\ 0 \dots\dots & -1 & k_n \end{pmatrix} = k_1 \det \begin{pmatrix} (k_2 - 1/k_1) & -1 \dots 0 \\ -1 & k_3 \dots 0 \\ \dots & \dots \\ 0 \dots\dots -1 & k_n \end{pmatrix}$$

This completes the proof of our theorem.

Corollary. — P a normal point of an algebraic surface F. If F has a neighborhood U homeomorphic to a 4-cell, P is a simple point of F.

Proof. — Let W be the intersection of an affine ball about P with F, as considered in the introduction, and so small that its boundary M lifted to a non-singular model F′ dominating F qualifies as a tubular neighborhood of the total transform of P. It suffices to show that $\pi_1(M) = (e)$, in view of the theorem just proven. Let U′ be a 4-cell-neighborhood of P contained in W, and let W′ be an affine ball about P contained in U′. We have constructed in section I a continuous map ψ from U′ — (P) to M that induces the canonical identification of M as the boundary of W′ to M (as the boundary of W). Therefore if γ is any path in M, regard γ as a path in the boundary of W′; as a path in U′ — (P) (which is homotopic to a 3-sphere) it can be contracted to a point; but then ψ maps this homotopy to contraction of γ as a path in M. Q.E.D.

IV. — AN EXAMPLE

It is instructive to note that there exist singular points P, for which $H_1(M) = (0)$, while, of course, $\pi_1(M) \neq (e)$. Take P to be the origin of the equation $0 = x^p + y^q + z^n$, where p, q, and n are pairwise relatively prime. Look at the equation as $-(z)^n = x^p + y^q$; this shows that M is an n-fold cyclic covering of the 3-sphere $|x|^2 + |y|^2 = 1$, x, y complex, branched along the points $x^p + y^q = 0$, i.e. along a torus knot, K, in S^3. Therefore M is a manifold of the type considered by M. Seifert [20], p. 222; he shows $H_1(M) = (0)$.

The singular point $0 = x^2 + y^3 + z^5$ is of particular interest as illustrating the possibility of a singular point on a surface whose local analytic Picard Variety is trivial contrary to a conjecture of Auslander. To show Pic(P) (P = (0, 0, 0)), is trivial amounts to showing $(R^1\pi)(\Omega)_P = (0)$, where $\pi\colon F' \to F$ is the map from a non-singular model to $0 = x^2 + y^3 + z^5$ (since we know $H_1(M) = (0)$ already). Let us choose a slightly better global surface F (our statement being local, we are free to choose a different model of $k(F)$ outside a neighborhood of P): namely take F_0 to be the double plane with sextic branch locus $B : u(u^2y^3 + z^5)$, where u, y, z are homogeneous coordinates. F_0 has two singularities: one is over $y = z = 0$ and this is P; the other is over $u = z = 0$ — call it Q. Let F_1 be the result of resolving Q alone, and F_2 be the non-singular surface obtained by resolving P and Q. Let $\pi : F_2 \to F_1$. We must show $(R^1\pi)(\Omega_{F_2})_P \simeq (0)$. But since $(R^1\pi)(\Omega_{F_2})$ is (0) outside of P, it is equivalent to show $H^0(F_1, (R^1\pi)(\Omega_{F_2})) = (0)$. First of all, note that F_2 is birational to P^2: indeed $0 = x^2 + y^3 + z^5$ is uniformized by the substitution:

$$x = 1/u^3v^5(u+v)^7,\ y = -1/u^2v^3(u+v)^5,\ z = -1/uv^2(u+v)^3.$$

Therefore $0 = H^1(F_2, \Omega_{F_2}) = H^2(F_2, \Omega_{F_2})$. Now consider the Spectral Sequence of Composite Functors:

$$H^i(F_1, (R^j\pi)(\Omega_{F_2})) \underset{i}{\Rightarrow} H^k(F_2, \Omega_{F_2}).$$

Noting that $(R^0\pi)(\Omega_{F_2}) = \Omega_{F_1}$, it follows:

a) $H^1(F_1, \Omega_{F_1}) = (0)$

b) $d_2^{0,1}\colon H^0(F_1, (R^1\pi)(\Omega_{F_2})) \to H^2(F_1, \Omega_{F_1})$
is 1 — 1, onto.

Therefore, it suffices to show $H^2(F_1, \Omega_{F_1}) = (0)$, or $0 \geqslant p_a(F_1)$ (= dim H^2 — dim H^1). Now unfortunately $p_a(F_0) = 1$, since, in general, if G is a double plane with branch locus of order $2m$, $p_a(G) = (m-1)(m-2)/2$ (none of the singularities of G being resolved, of course) (¹). To compute $p_a(F_1)$, embed F_0 in a family of double planes $F_{0,\alpha}$, where the branch locus B_α for $F_{0,\alpha}$ is

$$u(u^2y^3+z^5+\alpha u^4 z).$$

Now $F_{0,\alpha}$ have singularities over $u=z=0$ of identical type for all α, hence one may resolve these, and obtain a family of surfaces $F_{1,\alpha}$ containing F_1. But since B_α, for general α, has no singularity except $u=z=0$, the general $F_{1,\alpha}$ is non-singular. Now by the invariance of p_a [21], $p_a(F_1)=p_a(F_{1,\alpha})\leqslant \dim H^2(F_{1,\alpha}, \Omega)=\dim H^0(F_{1,\alpha}, \Omega(K))$, K the canonical class on $F_{1,\alpha}$. But if ω is the double *quadratic* differential (i.e. of type $A(dx\wedge dy)^2$ locally) on P^2 with poles exactly at B_α, one can readily compute $(f_\alpha^*\omega)$, where $f_\alpha\colon F_{1,\alpha}\rightarrow P^2$; it turns out strictly negative, and as it represents 2 K, it follows

$$p_g(F_{1,\alpha})=\dim H^0(F_{1,\alpha}, \Omega(K))=0.$$

For details on the behaviour of p_a of double planes, which include our result as a particular case, see the works of Enriques and Campedelli cited in [4], p. 203-4, and the doctoral thesis of M. Artin [Harvard, 1960].

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(¹) This may be seen by means of a suitable resolution of $(R^0 f)(\Omega_G)$, $f: G\rightarrow P^2$ being its double covering. It is, however, classical: cf. [4], p. 180-2 using the formula:

$$4p_a=n+P-3\pi-k/3-2 \text{ where } n=2,\ k=0,$$
$$\pi=m-1, \text{ and}$$
$$P=(2m-1)(2m-2)/2=p_a(\text{Branch Locus}).$$