On Non-Handlebody Instantons in 3D Gravity

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Abstract

In this note we describe the contribution from non-handlebody geometries to the partition function of three-dimensional pure gravity with negative cosmological constant on a Riemann surface of genus greater than one, extending previous considerations for handlebodies.

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1. Introduction

The three-dimensional pure quantum gravity with a negative cosmological constant has been conjectured to be dual to a holomorphically factorized extremal conformal field theory (ECFT), of central charge $c = 24k$ [1]. While it is not known whether ECFTs with $k > 1$ exist, one may compute its partition function on a Riemann surface from the gravity path integral, by doing a perturbative expansion around gravitational instantons and sum over all instantons. A step toward computing the gravity partition function was carried out in [2], and it was conjectured that the classical regularized Einstein-Hilbert action evaluated on a handlebody hyperbolic instanton agrees[1] with the leading term in the $1/k$ expansion of the "fake" CFT partition function, which captures the part of the full ECFT partition function that factorizes on Virasoro descendants of the identity operator. It is then further conjectured in [2] that the full contribution from the handlebody instanton is given by the fake CFT partition function, the latter determined entirely by sphere correlation functions of Virasoro descendants of the identity.

A priori, one should sum over all hyperbolic three-manifolds M whose conformal boundary is the given Riemann surface Σ, of genus g. When $g > 1$, such manifolds are not all handlebodies. The question remains how to calculate these non-handlebody contributions. It should be of the form

$$e^{kS_0+S_1+k^{-1}S_2+\cdots} \tag{1.1}$$

where S_l is the l-loop contribution around the instanton background, and depends holomorphically on the moduli of the Riemann surface. $S_{cl} = -k(S_0 + \overline{S_0})$ is the regularized

[1] A (three-dimensional) handlebody is a three manifold homeomorphic to the domain enclosed by a surface in $\mathbf{R}^3$. When the boundary Riemann surface has genus two, this conjecture was checked to nontrivial orders near the factorization limit.

classical instanton action. In this note we will describe how to compute S_{cl}, and hence S_0. It should have the following properties:

(1) S_{cl} is a harmonic function on the moduli space of Σ, and hence can be written as $-k(S_0 + \overline{S_0})$ for some holomorphic function S_0.

(2) Let $\Gamma' \subset Sp(2g, \mathbf{Z})$ be the subgroup of the mapping class group of Σ that extends to M (hence "preserving" M). e^{S_0} transforms under Γ' as a modular form of weight 12. This is needed to be consistent with the full partition function transforming as a modular form of weight $12k$.

(3) When a handle of Σ is pinched, and if M does not fill in the handle, then $e^{kS_0(M;\Sigma)}$ only contributes to the factorization on operators in the CFT of dimension $\Delta \geq k$. This is needed to be consistent with the fact that the handlebody contribution already captures the factorization on operators of $\Delta < k$.[2]

The regularized Einstein-Hilbert action on M has been computed by Krasnov [3] when M is a handlebody, and by Takhtajan and Teo [4] for more general hyperbolic manifolds,[3] and was shown to coincide with a suitably defined Liouville action evaluated at its critical point. The S_{cl} with the above desired properties, especially property (1), is related to the Liouville action of [4] by a shift, due to conformal anomaly.

In section 2 we will sketch a topological classification of the hyperbolic three-manifold instantons. Section 3 describes the general strategy in computing the instanton contribution. In section 4 we consider the factorization limits of the instanton action.

2. A classification of hyperbolic gravitational instantons

Consider a hyperbolic three-manifold $M = \mathbf{H}_3/G$, where $G \subset SL(2, \mathbf{C})$ is a torsion free Kleinian group. Suppose that M has a conformal boundary Σ, which is a connected Riemann surface of genus g. In other words, $\Sigma = U/G$, where $U = \mathbf{P}^1 - \Lambda$ is the domain of discontinuity for G on the boundary $\mathbf{P}^1$ of $\mathbf{H}_3$, and Λ is the set of limiting points of G.

[2] The first nontrivial primaries in the ECFT have dimension $k+1$, so the factorization on operators of dimension $\Delta \leq k$ involves Virasoro descendants of the identity only. In [2] it was found that the fake CFT partition function, when summed over its modular images, factorizes correctly on states of dimension $\Delta < k$, at least in the genus two case. On the other hand, the factorizations on $\Delta = k$ operators may not be correctly reproduced by the handlebody contributions alone.

[3] In [4] M is required to be of "Class A", and have in general multiple boundaries, as recalled below. We are interested in the case where the boundary of M is connected, and M can be lifted to a finite cover $\tilde{M}$ which is of Class A.

If G is freely generated and purely loxodromic, it is a Schottky group. In this case M is a handlebody. In general, consider the map

$$i_* : \pi_1(\Sigma) \to \pi_1(M) \tag{2.1}$$

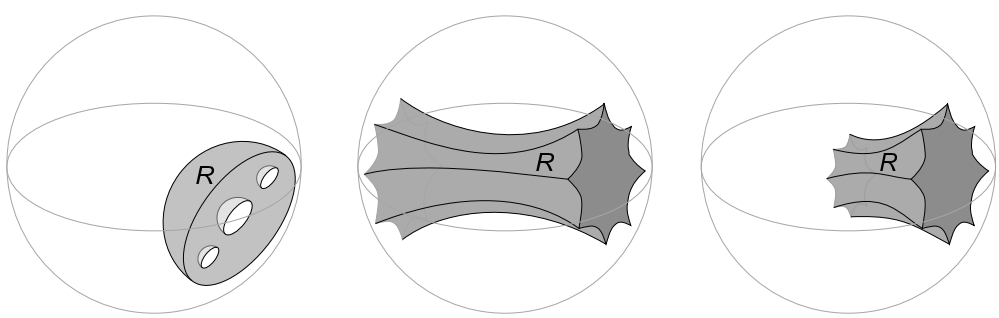
induced by $i : \Sigma = \partial M \hookrightarrow M$. If i_* is not injective, suppose γ is a loop in Σ such that $i(\gamma)$ is null-homotopic in M. By Dehn's lemma (see for example [5]) there is an embedded disc $D \subset M$ such that $\partial D = \gamma$. By cutting along D, we can reduce (M, Σ) to one of the following three geometries:

(i) (M', Σ'), where M' is connected, and Σ' has genus $g - 1$;

(ii) two disconnected three-manifolds (M_1', Σ_1') and (M_2', Σ_2'), such that $g_1' + g_2' = g$.

(iii) $(M', \Sigma_1' \sqcup \Sigma_2')$, where Σ_1' and Σ_2' are the two connected boundary components of M'.

Note that in the case (iii) we will be forced to consider manifolds with multiple boundaries. Such gravitational instantons are rather pathological, as will be discussed later. By repeating such surgeries, we can reduce M to one or several disconnected three-manifolds whose boundaries are π_1-injective. We will call the hyperbolic manifold M with a π_1-injective connected conformal boundary Σ a "tight" manifold. A simple class of tight manifolds are given topologically by twisted I-bundles over an unoriented surface S, namely $I \to M \to S$, such that Σ is a two-fold covering of S. These are in fact all tight manifolds with the property that $i_*\pi_1(\Sigma)$ is a finite index subgroup of $\pi_1(M)$ (and the index is 2). On the other hand, there are also tight manifolds with $[\pi_1(M) : i_*\pi_1(\Sigma)] = \infty$.[4]



(a) (b) (c)

Figure 1. A typical fundamental domain R of (a) a handlebody, (b) a class A manifold with two boundaries, and (c) a non-handlebody with one boundary, in the hyperbolic 3-space $\mathbf{H}_3$.

[4] I'm grateful to C. McMullen for explaining to me such examples.

A fundamental domain for $M = \mathbf{H}_3/G$ in $\mathbf{H}_3$ is of the form (R, F), where R is a fundamental domain of G in $\mathbf{H}_3$, and $F = G \cap U$ a fundamental domain for $\Sigma = U/G$. In general, R can be described as a cell complex, with $3, 2, 1, 0$-cells, corresponding to the bulk of R, its faces, edges, and corners. G is called a "Class A" Kleinian group if one can choose the fundamental domain R to have no 0-cells in the bulk of $\mathbf{H}_3$. General non-handlebody class A manifolds M will have multiple boundary components, $\Sigma_1, \cdots, \Sigma_n$. We shall consider the case when (M, Σ) can be lifted to a finite covering space $(\tilde{M}, \Sigma_1 \sqcup \cdots \sqcup \Sigma_n)$, such that $\tilde{M}$ is of class A.

3. The holomorphically factorized classical action of $M = \mathbf{H}_3/G$

The regularized Einstein-Hilbert action of [6,3] takes the form

$$S_{EH}(M; \Sigma) = \frac{4k}{\pi} \lim_{\epsilon \to 0} \left(V_\epsilon - \frac{1}{2} A_\epsilon + 2\pi(2g-2) \ln \epsilon \right) \tag{3.1}$$

Here V_ϵ and A_ϵ are the volume of the bulk hyperbolic three-manifold and the area of the boundary cutoff surface, respectively; ϵ is the cutoff parameter. The cutoff surface is chosen so that its induced metric has constant curvature $-1/\epsilon^2$. The $\ln \epsilon$ divergence in the Einstein-Hilbert action with boundary term is related to the conformal anomaly in the boundary CFT.

The main result of [4] is that, if $(\tilde{M}, \Sigma_1 \sqcup \cdots \sqcup \Sigma_n)$ is of class A, then the regularized Einstein-Hilbert action on $\tilde{M}$ is related to the classical Liouville action evaluated at its critical point, $S_L(\tilde{M}, \Sigma_1 \sqcup \cdots \sqcup \Sigma_n)$, by

$$S_{EH}(\tilde{M}, \Sigma_1 \sqcup \cdots \sqcup \Sigma_n) = -k \left[S_L(\tilde{M}, \Sigma_1 \sqcup \cdots \sqcup \Sigma_n) + \sum_{i=1}^{n} (2g_i - 2) \cdot const \right]. \tag{3.2}$$

where g_i is the genus of Σ_i. We refer to [3,4] for the precise definition of S_L (which, importantly, depends not only on Σ but on the Kleinian group G as well).[5] An important property is that S_L is a Kahler potential for the Weil-Petersson metric on the Teichmüller space of $\Sigma_1 \sqcup \cdots \sqcup \Sigma_n$. More generally, if M is not in class A but can be lifted to its n-fold covering space $\tilde{M}$ which is in class A, then the regularized Einstein-Hilbert action on M is given by

$$S_L(M; \Sigma) = \frac{1}{n} S_L(\tilde{M}; \Sigma, \cdots, \Sigma) \tag{3.3}$$

[5] Our convention for S_L differs from that of [4] by a factor of π.

Clearly, $S_L(M;\Sigma)$ will also be a Kahler potential for the Weil-Petersson metric on the Teichmüller space of Σ. Consequently, if M_1, M_2 have the same conformal boundary Σ, then $S_L(M_1;\Sigma) - S_L(M_2;\Sigma)$ is a harmonic function on the Teichmüller space of Σ, i.e. $\exp(S_L(M_1;\Sigma) - S_L(M_2;\Sigma))$ is holomorphically factorized.

We define the "holomorphically factorized" classical action $S_{cl} = -kS_0$ by

$$\begin{aligned} S_0(M;\Sigma) + \overline{S_0(M;\Sigma)} &= S_L(M;\Sigma) + 12\ln\frac{\det'\Delta}{\det \mathrm{Im}\Omega} \\ &= S_L(M;\Sigma) + 12\ln\frac{\zeta'_\Sigma(1)}{\det \mathrm{Im}\Omega} + (2g-2)c_0 \end{aligned} \tag{3.4}$$

where $\zeta_\Sigma(s)$ is the Selberg zeta function for the Riemann surface Σ [7], and c_0 is a constant. By Zograf's factorization formula for $\det'\Delta$ [8], the RHS of (3.4) is harmonic when M is a handlebody; by the above argument, this must also be the case for all M whose boundary is Σ. (3.4) still leaves the ambiguity of adding an imaginary constant to $S_0(M;\Sigma)$, which may depend on the topology of M; this corresponds to the overall phase of the contribution e^{kS_0} to the holomorphic partition function. A natural choice of the phase is such that e^{S_0} is real when $\mathrm{Re}\Omega = 0$. This is consistent with the factorization of the partition function. This still leaves an overall sign ambiguity for e^{S_0}. The sign may potentially be different for distinct topologies.

The full quantum holomorphic partition function on M, $Z_k(M;\Sigma)$, should be a weight $12k$ holomorphic modular form under $\Gamma_G \subset Sp(2g,\mathbf{Z})$, the subgroup of the mapping class group of Σ that leaves M invariant, defined on the Teichmüller space of Σ.[6] It takes the form

$$Z_k(M;\Sigma) = \exp\left[kS_0(M;\Sigma) + S_1(M;\Sigma) + \frac{1}{k}S_2(M;\Sigma) + \cdots\right] \tag{3.5}$$

where $S_1, S_2, \cdots$ are loop corrections, suppressed by powers of $1/k$. In (3.4), $\det'\Delta$ is modular invariant, and $S_L(M;\Sigma)$ is invariant under G. Due to the $\det \mathrm{Im}\Omega$ factor, $|e^{kS_0}|^2$ transforms under Γ_G with holomorphic and anti-holomorphic weight $12k$, as expected.

Suppose (M,Σ) can be reduced to (M',Σ') by cutting along an embedded disc $(D,\partial D)$. For a general holomorphic CFT of central charge $c = 24k$, the partition function on Σ and Σ' are related by

$$Z(\check{\Sigma}) = G(\Sigma', z_1, z_2; q)^k \sum_i q^{\Delta_i} \langle \mathcal{A}_i(z_1)\mathcal{A}_i(z_2)\rangle_{\check{\Sigma}'} \tag{3.6}$$

[6] When M is a handlebody, $Z_k(M;\Sigma)$ is simply invariant under Γ_∞. We are working in the convention that the partition function of a chiral boson on Σ is normalized to 1; in other words, Z is the partition function of the holomorphic CFT divided by that of $24k$ chiral bosons.

where Σ is obtained from Σ' by gluing a handle of modulus q to z_1, z_2. The notation $\check{\Sigma}, \check{\Sigma}'$ indicates a compatible choice of basis 1-cycles on the Riemann surfaces, as the partition functions are modular forms of nonzero weight. G is a universal holomorphic correction factor that depends only on the gluing procedure, with the property $G(\Sigma', z_1, z_2; q = 0) = 1$. The conjecture of [2] is that we can compute the gravity partition function of Σ from that of Σ' by

$$Z_k(M;\Sigma) = G(\Sigma', z_1, z_2; q)^k \sum_{\mathcal{A}_i \in Vir(k)} q^{\Delta_i} \langle \mathcal{A}_i(z_1)\mathcal{A}_i(z_2)\rangle_{fake;M',\Sigma'} \tag{3.7}$$

where the sum is only over Virasoro descendants of the identity (denoted by $Vir(k)$). On the RHS, the "fake" two-point function of $\mathcal{A}_i \in Vir(k)$ on Σ' is completely determined by $Z_k(M';\Sigma')$, since all correlators of the stress tensor on Σ' can be obtained by taking derivatives of $Z_k(M';\Sigma')$ with respect to the complex moduli.

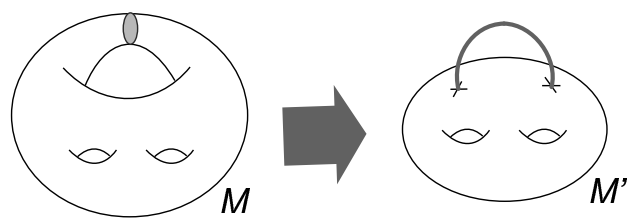


Figure 2. Reducing M to M' along a filled handle.

When M' has two connected components M_1' and M_2', with conformal boundary Σ_1' and Σ_2', the gravity partition functions can be similarly related as

$$Z_k(M,\Sigma) = G(\Sigma_1', z_1; \Sigma_2', z_2; \epsilon)^k \sum_{\mathcal{A}_i \in Vir(k)} \epsilon^{\Delta_i} \langle \mathcal{A}_i(z_1)\rangle_{fake;M_1',\Sigma_1'} \langle \mathcal{A}_i(z_2)\rangle_{fake;M_2',\Sigma_2'} \tag{3.8}$$

where Σ is obtained by sewing Σ_1' and Σ_2' together along a tube of modulus ϵ, attached to the points $z_1 \in \Sigma_1'$ and $z_2 \in \Sigma_2'$. $G(\Sigma_1', z_1; \Sigma_2', z_2; \epsilon)$ is the appropriate holomorphic correction factor in factorizing a ($c = 24$) CFT partition function on Σ into the one-point functions on Σ_1' and Σ_2', with the property $G(\Sigma_1', z_1; \Sigma_2', z_2; \epsilon = 0) = 1$.[7]

[7] When Σ_1' and Σ_2' are of genus one, $G(\Sigma_1', z_1; \Sigma_2', z_2; \epsilon)$ is related to the holomorphic correction factor of [9,2] by a normalization factor $\chi_{10}(\Omega)/(\epsilon^2\Delta(\tau_1)\Delta(\tau_2))$, due to our different convention of the genus g partition function.

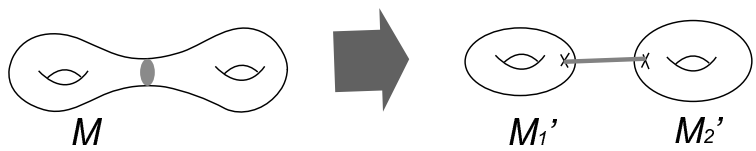


Figure 3. Reducing M to $M_1' \sqcup M_2'$ along a filled tube.

Finally, when M' is connected but have two boundary components Σ_1' and Σ_2', the contribution from M and M' should be related by

$$Z_k(M;\Sigma) = G(\Sigma_1', z_1; \Sigma_2', z_2; \epsilon)^k \sum_{\mathcal{A}_i \in Vir(k)} \epsilon^{\Delta_i} \mathcal{D}_{\Sigma_1'}^{\mathcal{A}_i(z_1)} \mathcal{D}_{\Sigma_2'}^{\mathcal{A}_i(z_2)} Z_k(M'; \Sigma_1' \sqcup \Sigma_2') \tag{3.9}$$

Here $\mathcal{D}_\Sigma^{\mathcal{A}(z)}$ is a differential operator in the moduli of Σ, defined by the property $\mathcal{D}_\Sigma^{\mathcal{A}(z)} Z(\Sigma) = \langle \mathcal{A}(z) \rangle_\Sigma$, where $Z(\Sigma)$ and $\langle \mathcal{A}(z) \rangle_\Sigma$ are the partition function and one-point function of a general $c = 24k$ CFT on Σ, $\mathcal{A} \in Vir(k)$.

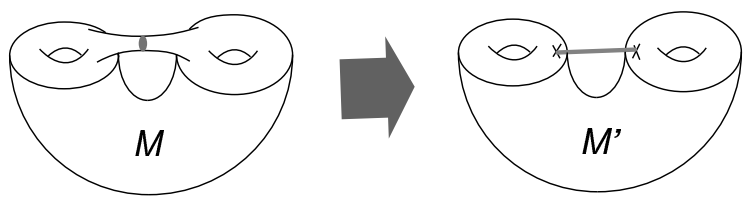


Figure 4. Reducing M to M' with two boundary components.

The last case is however puzzling from the dual CFT perspective, as it appears to spoil the factorization of the partition function on Σ into the product of the partition functions on Σ_1' and Σ_2' in the pinching limit, barring miraculous cancelations. There are two possible interpretations: (1) the dual CFT does not exist, due to the failure of the factorization of the gravity partition function; or (2) gravitational instantons that lead to connected M''s with multiple boundary components under the cutting surgery (Figure 4) should be excluded from the gravity path integral. Note that since M is a hyperbolic manifold, it is atoroidal, which implies that Σ_1' and Σ_2' must have genus $g_1', g_2' > 1$.[8] So the potential failure of the factorization of the partition function on Σ can only show up

[8] An explicit example of such M is obtained topologically by attaching a solid handle to the two sides of $\Sigma' \times I$, where Σ' is a genus $g > 1$ surface. Now M has a genus $2g$ boundary, and admits a hyperbolic metric. The corresponding Kleinian group G is a free product of a (quasi-)Fuchsian group with $\mathbf{Z}$, the latter generated by a loxodromic element of $SL(2, \mathbf{C})$ of sufficiently large multiplier.

at genus $g \geq 4$. Also note that $[\pi_1(M) : i_*\pi_1(\Sigma)] = \infty$ in this case. From now on we will adopt the second interpretation above, and exclude these pathological gravitational instantons. This may seem rather ad hoc from the perspective of the gravity path integral; on the other hand, it leads to dual CFT partition functions with consistent factorization property, and one may be able to extract CFT correlation functions from them.

In general, the above conjectured relations between $Z_k(M, \Sigma)$ for different pairs (M, Σ) related by surgeries encode nontrivial relations between fake Virasoro correlators and the Liouville action $S_L(M; \Sigma)$, generalizing the conjectures of [2] for handlebodies.

We know S_0 explicitly in two special classes of examples. When M is a handlebody, as explained in [2],

$$S_0(M; \Sigma) = 12 \sum_{\gamma \ prim.} \sum_{m=1}^{\infty} \ln(1 - q_\gamma^m) \tag{3.10}$$

where the sum runs through all primitive conjugacy classes of the Schottky group G, and q_γ is the multiplier of $\gamma \in SL(2, \mathbf{C})$, with $|q_\gamma| < 1$.

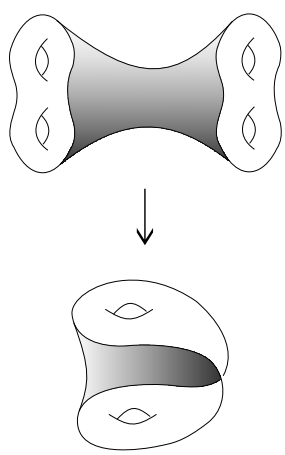


Figure 5. The twisted I-bundle as a $\mathbf{Z}_2$ quotient.

When M is *topologically* a twisted I-bundle $I \to M \to S$, we can lift $(M; \Sigma)$ to its double cover $(\tilde{M}; \Sigma \sqcup \Sigma)$, such that $M = \tilde{M}/\iota$ for an involution ι on $\tilde{M}$. When $\Sigma = \overline{\Sigma}$, $\tilde{M}$ is the quotient of $\mathbf{H}_3$ by a Fuchsian group $\tilde{G}$. In this case we can explicitly write the hyperbolic metric on $\tilde{M}$ as

$$ds^2 = dr^2 + \cosh^2 r ds_\Sigma^2, \tag{3.11}$$

where ds_Σ^2 is a hyperbolic metric on Σ. ι acts as $r \to -r$ together with an orientation reversing, fixed-point free involution on Σ. For example, suppose Σ has genus two, with

period matrix $\Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix}$. Then Σ admits such an involution at the real locus of its moduli space, $\rho = -\bar{\sigma}$, $\nu = i\nu_2$, $\nu_2 \in \mathbf{R}$. At a generic point on the moduli space, the metric on $\tilde{M}$ does not take the form (3.11), and $\tilde{G}$ will be a quasi-Fuchsian group instead of a Fuchsian group (it fixes a Jordan curve on $\partial \mathbf{H}_3 = \mathbf{P}^1$, rather than the equator of the $\mathbf{P}^1$).

In the case $\Sigma = \bar{\Sigma}$ and $\tilde{G}$ is a Fuchsian group, the Liouville action simply evaluates to $S_L(\tilde{M}; \Sigma, \Sigma) = 2S_L(M; \Sigma) = 2c(2g-2)$ for some constant c [10,4]. In other words,

$$2\mathrm{Re}S_0(M; \Sigma = \overline{\Sigma}) = c(2g-2) + 12 \ln \frac{\zeta'_\Sigma(1)}{\det \mathrm{Im}\Omega_\Sigma} \tag{3.12}$$

$\zeta_\Sigma(s)$ can be defined as

$$\zeta_\Sigma(s) = \prod_{\Upsilon \ prim.} \prod_{m=0}^{\infty} (1 - q_\Upsilon^{m+s}), \tag{3.13}$$

where the first product is over all primitive conjugacy classes of the Fuchsian group of Σ, and q_Υ is the multiplier of $\Upsilon \in SL(2, \mathbf{R})$, $q_\Upsilon < 1$. Υ also corresponds to a primitive geodesic on the surface Σ equipped with hyperbolic metric, and $q_\Upsilon = e^{-l(\Upsilon)}$ where $l(\Upsilon)$ is the length of the geodesic. (3.12) could be used to determine the harmonic function $\mathrm{Re}S_0(M; \Sigma)$ on the entire Teichmüller space of Σ.

Now that we know how to compute S_0, at least in principle, a remaining question is how to compute the $1/k$ corrections $S_1, S_2, \cdots$ for tight manifolds (M, Σ). Once these are known, $Z_k(M, \Sigma)$ for the tight manifolds will be determined, and it can be used to determine the partition functions of all (M, Σ) by the sewing procedure described earlier. It would also be nice to have a formula analogous to (3.10) for all M (non-handlebodies).

4. Factorization

4.1. Degenerating limits of Selberg zeta function

Starting with a Riemann surface Σ of genus g, let us consider the limit where a handle is pinched, and Σ is reduced to a Riemann surface Σ' of genus $g-1$. In order to examine the behavior of Selberg zeta function on Σ in this limit, we will assume that Σ is equipped with a hyperbolic metric, and let the length of the short geodesic around the pinched handle be $2\pi l$. Along the pinched handle, the metric can be approximated by the hyperbolic metric on an infinite tube,

$$d\chi^2 + l^2 \cosh^2 \chi d\phi^2 = \frac{|dw|^2}{\sin^2(\mathrm{Re}\, w)} \tag{4.1}$$

where

$$w = -i \ln \frac{\sinh \chi + i}{\cosh \chi} + il\phi, \qquad \mathrm{Re}\, w \in (0, \pi) \tag{4.2}$$

The modulus of the tube, τ, is related to the length of the short geodesic by $\tau_2 = \pi/(2\pi l) = 1/(2l)$. dw approximates a holomorphic 1-form on Σ, restricted to the tube. The period matrix of Σ takes the form

$$\Omega_\Sigma \to \begin{pmatrix} \tau & * \\ * & \Omega'_\Sigma \end{pmatrix} \tag{4.3}$$

in the pinching limit, where $*$ stands for finite entries.

We can write the Selberg zeta function $\zeta_\Sigma(s)$ as

$$\zeta_\Sigma(s) = \tilde{f}_\Sigma(s) \prod_{m=0}^{\infty} (1 - e^{-2\pi l(m+s)})^2, \tag{4.4}$$

where we singled out the contributions from the short geodesics (counted with both orientations). $\tilde{f}_\Sigma(s)$ denotes the contribution from all other closed geodesics on Σ. $\tilde{f}_\Sigma(s)$, just like $\zeta_\Sigma(s)$, has a simple zero at $s = 1$. This can be understood from the fact that the density of closed geodesics of length L grows as $\rho(L) \sim e^{-L}/L$ [11]. Therefore we have

$$\zeta'_\Sigma(1) = \tilde{f}'_\Sigma(1) \prod_{m=1}^{\infty} (1 - e^{-2\pi ml})^2. \tag{4.5}$$

Now $\tilde{f}'_\Sigma(1)$ is finite in the $l \to 0$ limit, since it does not involve the contribution from the short geodesic of length $2\pi l$.

Using the modular transformation of the Dedekind eta function

$$\prod_{m=1}^{\infty} (1 - e^{-2\pi ml}) = e^{\frac{\pi l}{12}} \eta(il) = \frac{e^{\frac{\pi l}{12}}}{\sqrt{l}} \eta(i/l) \sim \frac{e^{-\frac{\pi}{12l}}}{\sqrt{l}}, \qquad (l \to 0) \tag{4.6}$$

and $\det \mathrm{Im}\Omega_\Sigma \sim \tau_2 \sim \frac{1}{2l}$, we find

$$\ln \frac{\zeta'_\Sigma(1)}{\det \mathrm{Im}\Omega_\Sigma} \sim -\frac{\pi}{3}\tau_2 + finite \tag{4.7}$$

in the pinching limit. This result is also known from [12].

Let us now consider the limit in which Σ is pinched at a tube connecting two components Σ'_1 and Σ'_2, of genus g'_1 and g'_2, $g'_1 + g'_2 = g$. Along the tube, the metric is again approximated by (4.1); the modulus of the tube is related to l in the same way as before. A difference is that, in the separating degeneration limit,

$$\Omega_\Sigma \to \begin{pmatrix} \Omega_{\Sigma'_1} & 0 \\ 0 & \Omega_{\Sigma'_2}, \end{pmatrix} \tag{4.8}$$

and in particular $\det \mathrm{Im}\Omega_\Sigma$ is non-degenerate. The Selberg zeta function takes the form

$$\zeta_\Sigma(s) \to f_1(s) f_2(s) \prod_{m=0}^{\infty} (1 - e^{-2\pi l(m+s)})^2 \tag{4.9}$$

where $f_1(s)$ and $f_2(s)$ involve closed geodesics on Σ_1' and Σ_2', respectively. At the separating degeneration, each $f_i(s)$ has simple zero at $s = 1$, since the number of geodesics of length $\sim L$ on each punctured Riemann surface Σ_i' grows like $\rho(L) \sim e^L/L$. In particular, $\partial_s|_{s=1}(f_1(s)f_2(s)) \to 0$ in the $l \to 0$ limit. In order to extract the l dependence, we make use of the estimates of [12]

$$\zeta_\Sigma'(1) \sim \lambda_1 \prod_{m=1}^{\infty} (1 - e^{-2\pi m l})^2 \times finite, \qquad l \to 0 \tag{4.10}$$

where λ_1 is the smallest nonzero eigenvalue of the Laplacian Δ on Σ (all other nonzero eigenvalues of Δ are of order 1 in the degeneration limit). It is easy to see that $\lambda_1 \sim l$, and hence (4.7) still holds in the separating degeneration limit.

4.2. S_L in the factorization limits

Now let us consider the Liouville action S_L in the factorization limits. A special case is when M is a twisted I-bundle, and the complex structure of Σ is such that it admits an orientation reversing $\mathbf{Z}_2$ involution. As discussed in section 3, S_L takes constant value along this real locus of the moduli space of Σ. It then immediately follows from (3.4) that when a handle or tube of Σ is pinched (along the real locus), the contribution from M to the partition function behaves like

$$e^{kS_0(M;\Sigma)} \sim q^k f(M', \Sigma') + \mathcal{O}(q^{k+1}) \tag{4.11}$$

where $q = e^{2\pi i\tau}$ is the pinching modulus parameter, and f is a generic function that depends on the pinched geometry. This means that M can only contribute to the factorization on states of dimension $\Delta \geq k$ along any tube of the Riemann surface.

More generally, the Liouville action $S_L(M;\Sigma)$ in (3.4) is bounded when a cycle corresponding to an element γ of G is pinched, or equivalently, when the pinched loop is not contractible in M. In fact, the thin tube that is being pinched is formed by gluing a thin strip of the fundamental domain F of Σ on the $\mathbf{P}^1$ by identifying the two sides by γ, and $\gamma'(z)$ approaches 1 in the pinching limit along the strip. One may worry about the potential divergence in S_L due to the singular behavior of the Liouville field ϕ near

the pinching point. To see this, let us represent the thin strip as the domain between two circles in the complex z-plane, both centered on the real axis and touching say at $z = 0$ (Figure 6). Near the pinching point, the Liouville field ϕ is approximately given by

$$\phi \simeq -\ln(\mathrm{Im} z)^2, \tag{4.12}$$

and so that $|\partial_z \phi|^2 \simeq e^{\phi}$. It follows that in the Liouville action S_L, both the integral of the Liouville Lagrangian over the bulk of the fundamental domain F, as well as the boundary integrals, are finite. The singular behavior of S_0 (3.4) then entirely comes from the term $12 \ln(\zeta'_\Sigma(1)/\det \mathrm{Im}\Omega)$, as analyzed earlier.

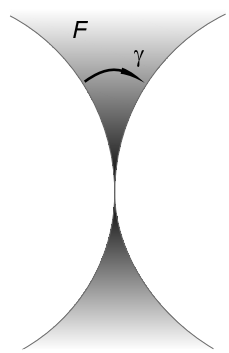


Figure 6. The strip in the fundamental domain of Σ corresponding to the pinched tube.

We conclude that when a loop γ of Σ is pinched, if γ is incontractible in M, then $e^{kS_0(M;\Sigma)}$ can only contribute to the factorization of the partition function on states of dimension $\geq k$.

Note that if γ were contractible in M, the pinching limit would correspond to shrinking a pair of circles whose interiors are excluded from F, rather than having two circles touching one another. In this case the Liouville action S_L will generically diverge. For example, suppose a circle $C : |z| = r_0$ is identified with $C' : |z - z_0| = r_0$ via the action

$$\gamma(z) = \frac{-e^{2i\theta_0} r_0^2}{z} + z_0 \tag{4.13}$$

We have $|\gamma'(z)| = 1$ along C. The Liouville action receives the contribution [8,4]

$$\frac{1}{\pi} \int_F d^2 z (|\partial_z \phi|^2 + e^{\phi}) - \frac{i}{\pi} \oint_C \phi d \ln \gamma'(z) + 4 \ln |c(\gamma)|^2 \tag{4.14}$$

from the domain near C and C'. Here $c(\gamma) = c = e^{-i\theta_0}/r_0$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C})$. Near C, in the $r_0 \to 0$ limit, the Liouville field behaves as

$$e^{\phi} \simeq \left(\frac{\pi}{2\ln r_0}\right)^2 \frac{1}{|z|^2 \sin^2(\frac{\pi}{2}\frac{\ln|z|}{\ln r_0})} \tag{4.15}$$

for $|z| \ll 1$. This is determined by rewriting the metric (4.1) in the coordinate $z = e^{(w-\pi)/l}$. From the integral of $|\partial_z \phi|^2$ near C and C', as well as the boundary term, the Liouville action behaves as

$$\begin{aligned} S_L &\sim \int_{r_0} r dr (\partial_r \phi(r))^2 - 4\phi(r_0) - 8\ln r_0 \\ &\sim -4\ln r_0 + \mathcal{O}(1). \end{aligned} \tag{4.16}$$

Note that the length of the short geodesic is

$$2\pi l = 2\pi r_0 e^{\phi(r_0)/2} = \frac{\pi^2}{-\ln r_0} \tag{4.17}$$

Therefore we have $S_L \sim 2\pi/l + \mathcal{O}(1)$ in the $l \to 0$ limit. This precisely cancels the singular term from (4.7), and hence contribution the holomorphically factorized partition function (3.4) remains finite in the $l \to 0$ limit, consistent with the expected factorization (3.6).

Let us write the full gravity partition function as $Z = Z_\gamma + \check{Z}_\gamma$, where Z_γ is the contribution from all hyperbolic three-manifolds M that fill in γ, and $\check{Z}_\gamma$ is the contribution from the remaining gravitational instantons, namely the ones such that γ is not contractible in M. By our conjectured relations (3.9), (3.8), as γ is pinched, Z_γ factorizes on the Virasoro descendants of the identity; if the dual CFT is extremal, this means that Z_γ already factorizes "correctly" on states of dimension $\Delta \leq k$, since all such states are Virasoro descendants of the identity. To avoid spoiling this factorization, one expects $\check{Z}_\gamma$ to contribute only to the factorization on states of dimension $\Delta \geq k+1$. The above discussion indicates that $\check{Z}_\gamma$ can only factorize on states with $\Delta \geq k$, which is consistent with the dual CFT having no nontrivial primaries up to dimension $k-1$. It is intriguing whether the contribution to the factorization on dimension $\Delta = k$ states in $\check{Z}_\gamma$ exactly cancel. This would require cancellation say between certain handlebody (subleading in the $Sp(2g, \mathbf{Z})$ Poincaré series of [2]) and non-handlebody contributions in the pinching limit. This issue is currently under investigation.

5. Summary

We have given a prescription for computing the classical contribution from all hyperbolic instantons, handlebody or not, to the holomorphically factorized partition function on a general Riemann surface. The $1/k$ quantum corrections to the contribution from (non-handlebody) "tight" manifolds remain to be understood. Once these are known, the gravity partition function is in principle determined completely. In the end, we would like to check the non-handlebody contributions against the dual ECFT, say by examining the factorization on states and extracting correlation functions in the CFT. It is also important to understand whether the gravitational instantons with multiple boundary components can be consistently excluded. These are left to future works.

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