Non-Supersymmetric Black Holes and Topological Strings

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Non-supersymmetric Black Holes and Topological Strings

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We study non-supersymmetric, extremal 4 dimensional black holes which arise upon compactification of type II superstrings on Calabi-Yau threefolds. We propose a generalization of the OSV conjecture for higher derivative corrections to the non-supersymmetric black hole entropy, in terms of the one parameter refinement of topological string introduced by Nekrasov. We also study the attractor mechanism for non-supersymmetric black holes and show how the inverse problem of fixing charges in terms of the attractor value of CY moduli can be explicitly solved.

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1. Introduction

String theory provides microscopic description of the entropy of certain types of black holes through the counting of D-brane bound states. The predictions of string theory include not only a confirmation of the leading semi-classical entropy formula of Bekenstein and Hawking which was first confirmed in [1] (see, e.g. [2,3] for a review and references) but also all the subleading quantum gravitational corrections which was proposed in [4] (building on the work of [5,6,7,8,9]). These higher derivative corrections have been confirmed by explicit microscopic enumeration in a number of examples [10,11,12,13,14,15].

An important feature of extremal black hole solutions in $\mathcal{N} = 2, 4, 8$ supergravity in four space-time dimensions is that some of the scalar fields (lowest components of the vector multiplets) acquire fixed values at the horizon. These values are determined by the magnetic and electric charges (p^I, q_I) of the black hole and does not depend on the asymptotic values of the fields at infinity. The so-called attractor mechanism, which is

responsible for such fixed point behavior of the solutions, was first studied in [16,17,18,19] in the context of the BPS black holes in the leading semiclassical approximation. Later, the attractor equations describing these fixed points for BPS black hole solutions were generalized to incorporate the higher derivative corrections to $\mathcal{N} = 2$ supergravity Lagrangian (see [20] for a review).

Using these supergravity results OSV [4] conjectured a simple relation of the form $Z_{\mathrm{BH}} = |Z_{\mathrm{top}}|^2$ between the (indexed) entropy of a four-dimensional BPS black hole in a Type II string Calabi-Yau compactification, and topological string partition function, evaluated at the attractor point on the moduli space. Viewed as an asymptotic expansion in the limit of large black hole charges, this relation predicts all order perturbative contributions to the black hole entropy due to the F-term corrections in the effective $\mathcal{N} = 2$ supergravity Lagrangian. Over the last few years, this led to a significant progress in understanding the spectrum of D-brane BPS states on compact and non-compact Calabi-Yau manifolds, and gave new insights on the topological strings and quantum cosmology. For a recent review and references on this subject, see [21].

Define a mixed partition function for a black hole with magnetic charge p^I and electric potential ϕ^I by

$$Z_{\mathrm{BH}}(p^I, \phi^I) = \sum_{q_I} \Omega(p^I, q_I)\, e^{-\phi^I q_I}, \tag{1.1}$$

where $\Omega(p^I, q_I)$ represent supersymmetric black hole degeneracies for a given set of charges (p^I, q_I). Then the OSV conjecture [4] reads

$$\Omega(p^I, q_I) = \int d\phi^I e^{q_I \phi^I} \left| Z_{\mathrm{top}}(p^I, \phi^I) \right|^2. \tag{1.2}$$

As was already mentioned in [4], expression (1.2) needs some additional refinement. In particular, rigorous definition of (1.2) requires taking care of the background dependence of the topological string partition function Z_{top}, governed by the holomorphic anomaly [22]. Also, the integration measure, as well as the choice of a suitable integration contour needs to be specified. Some of these issues were investigated in [12,14,15,23,24], see [25] for a recent discussion of these and other subtleties.

In this paper we will address an even more fundamental ambiguity in (1.2) that is present already at the semiclassical level (without considering higher genus topological string corrections). The problem is that although the right hand side of (1.2) can be defined for any set of charges (p^I, q_I), it is well known [26] that not for all such (p^I, q_I) a

supersymmetric spherically symmetric black hole solution exists. Typically, there is a real codimension one 'discriminant' hypersurface $\mathcal{D}(p^I, q_I) = 0$ in the space of charges, such that supersymmetric black hole solutions exist only when $\mathcal{D}(p^I, q_I) < 0$. Therefore, in this case $\Omega(p^I, q_I)$ on the left hand side of (1.2), representing a suitable index of BPS states of charge (p^I, q_I), is zero.

This phenomenon can be illustrated by several examples. Consider compactification of Type IIB string theory on the diagonal $T^6 = \Sigma_\tau \times \Sigma_\tau \times \Sigma_\tau$ [26,27], where Σ_τ is the elliptic curve with modular parameter τ, with $D3$-brane wrapping a real 3-cycle on T^6. This can be viewed as part of the Calabi-Yau moduli when we orbifold T^6. In this paper when we refer to the diagonal T^6 we have in mind the corresponding locus in the moduli of an associated Calabi-Yau 3-fold with $\mathcal{N} = 2$ supersymmetry where part of the homology of the CY 3-cycles is identified with the charges (p^I, q_I). Let the charge configuration be invariant under the permutation symmetry of the three elliptic curves Σ_τ. Note also that the diagonal T^6 model is a good approximation to the generic behavior of Type IIB compactification on a one-modulus Calabi-Yau threefold in the large radius limit. If we label homology of 3-cycles on T^6 according to the mirror IIA D-brane charges as $(u, q, p, v) = (D0, D2, D4, D6)$, the leading contribution to the corresponding black hole degeneracy takes the form

$$\Omega_{\text{susy}}(p, q, u, v) \approx \exp\left(\pi\sqrt{-\mathcal{D}(p, q, u, v)}\right), \tag{1.3}$$

where the discriminant is $\mathcal{D}(p, q, u, v) = -\left(3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2\right)$. It is clear that for some sets of charges this quartic polynomial can become positive (for example, it is always the case for $D0 - D6$ system, where $\mathcal{D}(0, 0, u, v) = u^2v^2$), and (1.3) breaks down. Similar situations occurs in $\mathcal{N} = 2$ truncation of the heterotic string on T^6, the so-called STU model, where $\mathcal{D}$ becomes Cayley's hyperdeterminant [28] that can also be either positive or negative. Another example of this phenomenon arises in Type IIB compactification on $K3 \times T^2$. This leads to $\mathcal{N} = 4$ supergravity in four dimensions, and corresponding expression for the degeneracy [19,29,30]

$$\Omega_{\text{susy}}(p^I, q_I) \approx \exp\left(\pi\sqrt{(P \cdot P)(Q \cdot Q) - (P \cdot Q)^2}\right). \tag{1.4}$$

breaks down when $(P \cdot Q)^2 > (P \cdot P)(Q \cdot Q)$.

Thus, the OSV formula (1.2) needs to be modified even at the semiclassical level. One remedy one may think is to sum in (1.1) only over the charges that support BPS

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states: $Z_{\mathrm{BH}}(p^I, \phi^I) = \sum\limits_{q_I : \mathcal{D}(p^I, q_I) \leq 0} \Omega_{\mathrm{susy}}(p^I, q_I) \, e^{-\phi^I q_I}$. This, however, will not work because the inverse transform of the topological string partition function would have to automatically give zero when (p^I, q_I) are non-supersymmetric. This however turns out not to be the case, and one gets the naive analytic continuation of the BPS case (leading to imaginary entropy!). Instead, we can use an observation that in many examples studied recently in the literature [31,32,33,34,35,36] there exists a non-supersymmetric extremal black hole solution for those sets of charges that do not support a BPS black hole: $\mathcal{D}(p^I, q_I) > 0$. The attractor behavior of a non-supersymmetric extremal black hole solutions [37,38,39,40,41,42,43,44,45] is similar to the BPS black hole case, since it is a consequence of extremality rather than supersymmetry [46]. Moreover, in the simplest examples, the macroscopic entropy of a non-supersymmteric extremal black holes is proportional to the square root of the discriminant: $S_{\mathrm{BH}}^{\mathrm{n-susy}} \approx \pi\sqrt{\mathcal{D}}$, so that a general expression for the extremal black holes degeneracy takes the form

$$\Omega_{\mathrm{extrm}}(p^I, q_I) \approx \exp\left(\pi\sqrt{\left|\mathcal{D}(p^I, q_I)\right|}\right), \tag{1.5}$$

which is valid both for supersymmetric $\mathcal{D} \leq 0$ and non-supersymmetric $\mathcal{D} > 0$ solutions.

Therefore, it is natural to look for an extension of the OSV formula (1.2) that can be applied *simultaneously* for both BPS and non-BPS extremal black holes and obtain corrections to their entropy due to higher derivative terms in the Lagrangian as a perturbative series in the inverse charge. Recently, several steps in this direction were taken from the supergravity side. A general method (the entropy function formalism) for computing the macroscopic entropy of extremal black holes based on $\mathcal{N} = 2$ supergravity action in the presence of higher-derivative interactions was developed in [47,48], and applied for studying corrected attractor equations and corresponding entropy formula for non-supersymmetric black holes in [49,50,51,52,53,54,55,56,57]. A five-dimensional viewpoint on higher derivative corrections to attractor equations and entropy, based on the c-function extremization, was developed in [58,59]. Black hole partition function for non-supersymmetric extremal black holes was discussed in [51,60].

In this paper we propose a generalization of (1.2) motivated by the topological string considerations as well as the work [50]: It was observed in [50] that the higher order corrections to the non-supersymmetric black hole entropy needs higher derivative corrections in the $\mathcal{N} = 2$ theory which are not purely antiself-dual in the 4d sense, because unlike the BPS case, the radii of AdS_2 and S^2 factors of the near horizon geometry are not the

same. Thus, more information than F-terms computed by topological strings, which only capture antiself-dual geometries, is needed. Indeed if one considers only the antiself-dual higher derivative corrections to the 4d action, there is already a contradiction with the microscopic count of the non-supersymmetric black hole at one loop [50]. Instead it is natural to look for an extension of topological string which incorporates non-antiself-dual corrections as well. Such a generalization of topological strings, in the context of geometrically engineered gauge theories have been proposed by Nekrasov [61], where the string coupling constant is replaced by a pair of parameters (ϵ_1, ϵ_2) which roughly speaking capture the strength of the graviphoton field strength in the 12 and 34 directions of the 4d non-compact spacetime respectively. In the limit when $\epsilon_1 = -\epsilon_2 = g_{\text{top}}$ one recovers back the ordinary topological string expansion. However when $\epsilon_1 \neq -\epsilon_2$ this refinement of the topological string partition function computes additional terms in the 4d effective theory, as appears to be needed for a correct accounting of the entropy for non-supersymmetric black holes. This includes a term proportional to $\mathcal{R}^2$ which as was found in [54] is needed to get the correct one loop correction which is captured by the refined topological string partition function, but not the standard one.

Motivated by this observation and identifying (ϵ_1, ϵ_2) with physical fluxes in the non-supersymmetric black hole geometry, and motivated by the computations in [50] we propose a conjecture for the partition function of an OSV-like ensemble which includes both BPS and non-supersymmetric extremal black holes. We conjecture

$$\Omega_{\text{extrm}}(p^I, q_I) = \int d\phi^I e^{q_I \phi^I} \sum_{\text{susy,n-susy}} \left| e^{\frac{i\pi}{2}\mathcal{G}(p^I, \phi^I)} \right|^2, \tag{1.6}$$

where $\mathcal{G}(p^I, \phi^I)$ is obtained from the $\mathcal{G}$-function

$$\mathcal{G} = \frac{1}{2}\left(P_\epsilon^I - X^I\right)\left(P_\epsilon^J - X^J\right)\overline{F}_{IJ}(\overline{X}, \overline{\epsilon}) + \left(P_\epsilon^I - X^I\right)F_I(X, \epsilon) + F(X, \epsilon) + $$

$$+ \frac{1}{2}(\epsilon_1 + \epsilon_2)\overline{X}^I F_I(X, \epsilon) - \frac{1}{2}(\epsilon_1 + \epsilon_2)\left(\epsilon_1 \partial_{\epsilon_1} - \epsilon_2 \partial_{\epsilon_2}\right)F(X, \epsilon) + \mathcal{O}(\epsilon_1 + \epsilon_2)^2, \tag{1.7}$$

$$P_\epsilon^I = -\epsilon_2 p^I + \frac{i}{\pi}\epsilon_1 \phi^I,$$

by extremizing $\text{Im}\mathcal{G}$ with respect to the parameters $\epsilon_{1,2}$ and (extended) Calabi-Yau moduli X^I, and then substituting corresponding solution $\epsilon_{1,2} = \epsilon_{1,2}(p, \phi)$, $X^I = X^I(p, \phi)$ back into $\mathcal{G}$ (1.7). The sum in (1.6) is over all such solutions to the extremum equations $\partial_{\epsilon_{1,2}}\text{Im}\mathcal{G} = \partial_I \text{Im}\mathcal{G} = 0$, one of which ends up being the supersymmetric one given by $X^I(p, \phi) = p^I + \frac{i}{\pi}\phi^I$, reproducing the OSV conjecture for this case. The function

$F(X, \epsilon) \equiv F(X^I, \epsilon_1, \epsilon_2)$ in (1.7) denotes Nekrasov's refinement of the topological string free energy[1]. Depending on the choice of the charges (p^I, q_I), integration over ϕ^I near the saddle point picks out supersymmetric or non-supersymmetric black hole solution. In the supersymmetric case it reduces to the OSV formula. In the non-supersymmetric case the corrections have the general structure suggested by [50] (however the exact match cannot be made because [50] only consider higher derivative terms captured by standard topological string corrections).

The above conjecture is the minimal extension of OSV needed to incorporate non-supersymmetric corrections. It is conceivable that there are further $O(\epsilon_1 + \epsilon_2)^2$ corrections to this conjecture. Such corrections will not ruin the fact that supersymmetric saddle point still reproduces the OSV conjecture.

The rest of the paper is organized as follows: In section 2 we review the attractor equations and entropy formula for supersymmetric and non-supersymmetric extremal black holes of $d = 4$, $\mathcal{N} = 2$ supergravity arising in the leading semiclassical approximation. In section 3 we discuss an alternative formulation of the attractor equations which helps us to treat supersymmetric and non-supersymmetric black holes in a unified way, suitable for using in an OSV-like formula. In section 4 we formulate the *inverse problem* that allows us to find magnetic and electric charges of the extremal black hole in terms of the values of the moduli in vector multiplets fixed at the horizon. We give a solution to this problem for a general one-modulus Calabi-Yau compactification. In section 5 we discuss semiclassical approximation to the generalized OSV formula for extremal black holes. In section 6 we review the results [50,51,57] for a corrected black hole entropy in $\mathcal{N} = 2$ supergravity with higher-derivative couplings, obtained using the entropy function formalism. In section 7 we observe that matching with the supergravity computations requires replacing the string coupling constant with two variables on the topological string side, and identify these variables as an equivariant parameters in Nekrasov's extension of the topological string. This allows us to formulate a generalization of the OSV entropy formula which is conjectured to be valid asymptotically in the limit of large charges both for the supersymmetric and non-supersymmetric extremal black holes. We conclude in section 8 with a discussion of our results and directions for future research. Appendix A contains explicit solutions of the inverse and direct problems relating the charges and corresponding attractor complex structures for the diagonal T^6 model.

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[1] Supersymmetric solution corresponds to $\epsilon_1 = -\epsilon_2 = 1$, in this case we use the same conventions as in [4], and find $\mathcal{G}_{\text{susy}}(p^I, \phi^I) \equiv F(p^I + \frac{i}{\pi}\phi^I, 256)$. Nekrasov's extension of the topological string is discussed in section 7.1 below.

2. The Black Hole Potential and Attractors

Let us review the attractor equations for extremal black holes in $d = 4$, $\mathcal{N} = 2$ super-gravity, arising in the context of type IIB compactification on a Calabi-Yau manifold M. We start by choosing a symplectic basis of 3-cycles $(A^I, B_I)_{I=0,\dots h^{2,1}}$ on M, such that

$$X^I = \int_{A^I} \Omega, \qquad F_I = \partial_I F = \int_{B_I} \Omega, \tag{2.1}$$

where Ω is a holomorphic 3-form and F is the prepotential of the Calabi-Yau manifold. We also choose a basis of 3-forms $(\alpha_I, \beta^I) \in H^3(M, \mathbb{Z})$ dual to (A^I, B_I). The Kähler potential is given by[2]

$$K(X, \overline{X}) = -\log\left(-i\int_M \Omega \wedge \overline{\Omega}\right) = -\log i\left(\overline{X}^I F_I - X^I \overline{F}_I\right). \tag{2.2}$$

It defines the Kähler metric $g_{i\bar{j}} = \partial_i \overline{\partial}_{\bar{j}} K$. Let us introduce the superpotential

$$\mathcal{W} = \int_M \Omega \wedge H, \tag{2.3}$$

where

$$H = p^I \alpha_I + q_I \beta^I \tag{2.4}$$

is the RR 3-form, parameterized by a set of (integral) magnetic and electric charges (p^I, q_I). The central charge is defined by

$$Z = e^{\frac{K}{2}} \mathcal{W}. \tag{2.5}$$

Attractor points are the solutions minimizing the so-called black hole potential [18,19,46,62]

$$V_{\text{BH}} = |Z|^2 + |DZ|^2. \tag{2.6}$$

Here D is a fully covariant derivative[3], and $|DZ|^2 = g^{i\bar{j}} D_i Z \overline{D}_{\bar{j}} \overline{Z}$. Notice that for a fixed complex structure on Calabi-Yau the central charge (2.5) is linear in the charges (p^I, q_I), and therefore the black hole potential (2.6) is quadratic in the charges.

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[2] We use the Einstein convention and always sum over repeated indices in the paper.

[3] On the objects of Kähler weight w it acts as $D = \partial + w\partial K + \Gamma$, where Γ is the Levi-Civita connection of the Kähler metric. For example, $DZ = \partial Z + \frac{1}{2} Z \partial K$.

We are interested in describing the extremum points of the potential (2.6). These points correspond to the solutions of the following equations [46]

$$
\begin{aligned}
\partial_i V_{\mathrm{BH}} &= 2\overline{Z} D_i Z + g^{k\overline{j}} \big(D_i D_k Z \big) \overline{D}_{\overline{j}} \overline{Z} = 0, \\
\overline{\partial}_{\overline{i}} V_{\mathrm{BH}} &= 2 Z \overline{D}_{\overline{i}} \overline{Z} + g^{j\overline{k}} \big(\overline{D}_{\overline{i}} \overline{D}_{\overline{k}} \overline{Z} \big) D_j Z = 0.
\end{aligned}
\tag{2.7}
$$

There are two types of the solutions, which can be identified as follows. From the second equation in (2.7) we find, assuming $Z \neq 0$

$$
\overline{D}_{\overline{j}} \overline{Z} = -\frac{g^{l\overline{k}} \big(\overline{D}_{\overline{j}} \overline{D}_{\overline{k}} \overline{Z} \big)}{2Z} D_l Z.
\tag{2.8}
$$

By substituting this into the first equation in (2.7), we obtain[4]

$$
\mathrm{M}_i{}^j D_j Z = 0,
\tag{2.9}
$$

where

$$
\mathrm{M}_i{}^j = 4|Z|^2 \delta_i{}^j - \big(D_i D_k Z \big) g^{k\overline{m}} \big(\overline{D}_{\overline{m}} \overline{D}_{\overline{n}} \overline{Z} \big) g^{\overline{n}j}
\tag{2.10}
$$

Now it is clear that there are two types of solutions to (2.9):

$$
\begin{aligned}
\text{susy}: \quad & \det\mathrm{M} \neq 0, \quad D_i Z = 0 \\
\text{non}-\text{susy}: \quad & \det\mathrm{M} = 0, \quad D_i Z = v_i,
\end{aligned}
\tag{2.11}
$$

where v_i are the null-vectors: $\mathrm{M}_i{}^j v_j = 0$ of the matrix (2.10).

Solutions to the extremum equations (2.7) minimize the black hole potential (2.6), if the Hessian

$$
Hess(V_{\mathrm{BH}}) = \begin{pmatrix} \partial_i \partial_j V_{\mathrm{BH}} & \partial_i \overline{\partial}_{\overline{j}} V_{\mathrm{BH}} \\ \overline{\partial}_{\overline{i}} \partial_j V_{\mathrm{BH}} & \overline{\partial}_{\overline{i}} \overline{\partial}_{\overline{j}} V_{\mathrm{BH}} \end{pmatrix},
\tag{2.12}
$$

computed at the extremal point, is positive definite: $Hess(V_{\mathrm{BH}})\big|_{\partial V_{\mathrm{BH}}=0} > 0$. We will refer to such solutions as attractor points. According to the classification (2.11), these attractors can be supersymmetric or non-supersymmetric. It is easy to show that all supersymmetric

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[4] Similar expression was derived in [40], see eq. (3.5). In fact, it is straightforward to see that up to a term which annihilates $D_j Z$ due to (2.7), the matrix $\mathrm{M}_i{}^j$ is the square $\mathrm{M} \sim M\overline{M}$ of the matrix M_{ij} used in [40]. Also, note that the matrix $\mathrm{M}_i{}^j$ can be used to classify the attractor solutions without assuming $Z \neq 0$ (see, e.g. [35] for explicit examples of the non-supersymmetric attractor solutions with $Z = 0$). We thank S. Ferrara for this clarification.

solutions (2.11) minimize the black hole potential. This is, however, not true in general for the non-supersymmetric solutions, see e.g. [31,36,40] for some examples.

The black hole potential (2.6) is related to the Bekenstein-Hawking entropy of the corresponding black hole in a simple way. In the classical geometry approximation (at the string tree level) the entropy is just π times the value of the potential (2.6) at the attractor point

$$S_{\text{BH}} = \pi V_{\text{BH}}\big|_{\partial V_{\text{BH}}=0}. \tag{2.13}$$

After appropriate modification of the black hole potential this formula gives corrections to the classical Bekenstein-Hawking entropy in the presence of higher derivative terms. This can be effectively realized using the entropy function formalism [47,48].

3. An Alternative Form of the Attractor Equations

In this section we discuss an alternative form of the attractor equations describing extremal black holes in $d = 4, \mathcal{N} = 2$ supergravity coupled to n_V vector multiplets in the absence of higher derivative terms. We describe two versions of attractor equations, one involving inhomogeneous and another involving homogeneous coordinates on Calabi-Yau moduli space. A natural generalization of these equations in the presence of higher derivative corrections will be introduced later in section 7.

It is convenient to start with the following representation of the black hole potential [46]

$$V_{\text{BH}} = -\frac{1}{2}\big(q_I - \mathcal{N}_{IJ}p^J\big)\Big(\frac{1}{\text{Im}\mathcal{N}}\Big)^{IJ}\big(q^J - \overline{\mathcal{N}}^{JK}p^K\big), \tag{3.1}$$

where

$$\mathcal{N}_{IJ} = \overline{F}_{IJ} + 2i\frac{\text{Im}\big(F_{IK}\big)X^K\text{Im}\big(F_{JL}\big)X^L}{\text{Im}\big(F_{MN}\big)X^M X^N}, \qquad F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J}. \tag{3.2}$$

Notice that $\mathcal{N}_{IJ}$ is $(n_V+1)\times(n_V+1)$ symmetric complex matrix, and $\text{Im}\mathcal{N}_{IJ}$ is a negative definite matrix, as opposed to $\text{Im}F_{IJ}$, which is of signature $(1, n_V)$. This is clear from the following identity [62]

$$-\frac{1}{2}\Big(\frac{1}{\text{Im}\mathcal{N}}\Big)^{IJ} = e^K\big(X^I\overline{X}^J + g^{i\bar{j}}D_iX^J\overline{D}_{\bar{j}}\overline{X}^J\big). \tag{3.3}$$

One can use (3.3) and the defining relation [62]

$$F_I = \mathcal{N}_{IJ}X^J \tag{3.4}$$

to bring (3.1) into the form (2.6). Indeed, since

$$(q_I - \mathcal{N}_{JI}p^J)X^I = q_I X^I - p^I F_I = \mathcal{W}, \tag{3.5}$$

the black hole potential (3.1) takes the form

$$V_{\text{BH}} = e^K \big(\mathcal{W}\overline{\mathcal{W}} + g^{i\bar{j}} D_i \mathcal{W} \overline{D}_{\bar{j}} \overline{\mathcal{W}} \big), \tag{3.6}$$

which is equivalent to (2.6).

3.1. Attractor equations and inhomogeneous variables

Let us introduce an auxiliary field P^I that later will be identified with the complexified magnetic charge p^I, and consider a modified black hole potential

$$V_{\text{BH}} = \frac{1}{2}P^I \text{Im}(\mathcal{N}_{IJ})\overline{P}^J - \frac{i}{2}P^I(q_I - \mathcal{N}_{IJ}p^J) + \frac{i}{2}\overline{P}^I(q_I - \overline{\mathcal{N}}_{JK}p^K), \tag{3.7}$$

where P^I serves as a Lagrange multiplier. We want to describe the extrema of V_{BH}. Variation of (3.7) with respect to $\overline{P}^I$ gives

$$P^I = -\frac{i}{\text{Im}\mathcal{N}_{IJ}}\big(q_J - \overline{\mathcal{N}}_{JK}p^K\big). \tag{3.8}$$

By plugging this expression form P^I back to (3.7) we obtain the original black hole potential (3.1). It is straightforward to solve equations (3.8) in terms of the charges:

$$\boxed{\begin{aligned} p^I &= \text{Re}\big(P^I\big) \\ q_I &= \text{Re}\big(\mathcal{N}_{IJ}P^J\big) \end{aligned}} \tag{3.9}$$

Variation of (3.7) with respect to the Calabi-Yau moduli $\partial_i V_{\text{BH}} = 0$ gives

$$P^I \overline{P}^J \partial_i \text{Im}\mathcal{N}_{IJ} + i\big(P^I \partial_i \mathcal{N}_{IJ} - \overline{P}^J \partial_i \overline{\mathcal{N}}_{IJ}\big)p^J = 0. \tag{3.10}$$

After using the solution (3.9), we obtain

$$P^I \partial_i \mathcal{N}_{IJ} P^J - \overline{P}^I \partial_i \overline{\mathcal{N}}_{IJ} \overline{P}^J = 0. \tag{3.11}$$

This set of the extremum equations can also be written in a compact form as follows

$$\boxed{\partial_i \text{Im}(P^I \mathcal{N}_{IJ} P^J) = 0.} \tag{3.12}$$

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For a fixed set of charges (p^I, q_I), solutions to the combined system of equations (3.9) and (3.12) which minimize the modified potential (3.7) correspond to the extremal black holes.

Among these, there is always a special solution of the form

$$P^I = CX^I, \tag{3.13}$$

where C is the complex constant. Indeed, in this case extremum equations (3.12) read

$$C^2 X^I X^J \partial_i \mathcal{N}_{IJ} - \overline{C}^2 \overline{X}^I \overline{X}^J \partial_i \overline{\mathcal{N}}_{IJ} = 0. \tag{3.14}$$

The second term in (3.14) vanishes since $\overline{X}^I \partial_i \overline{\mathcal{N}}_{IJ} = \partial_i(\overline{\mathcal{N}}_{IJ}\overline{X}^J) = \partial_i \overline{F}_I = 0$ according to (3.4). The first term in (3.14) vanishes because of the special geometry relation

$$0 = \int_M \Omega \wedge \partial_i \Omega = X^I \partial_i F_I - F_I \partial_i X^I = X^I X^J \partial_i \mathcal{N}_{IJ}. \tag{3.15}$$

The solution (3.13) describes supersymmetric attractors [16,17,18], since (3.9) gives in this case the well-known equations

$$\begin{cases} p^I = \mathrm{Re}(CX^I) \\ q_I = \mathrm{Re}(CF_I). \end{cases} \tag{3.16}$$

3.2. Attractor equations and homogeneous variables

Consider the following potential:

$$V_{\mathrm{BH}} = q_I \mathrm{Im}P^I + \mathrm{Im}(F_{IJ})\mathrm{Re}\big((P^I - X^I)(P^J - X^J)\big) - \frac{1}{2}\mathrm{Im}(F_{IJ}P^I P^J). \tag{3.17}$$

We will keep P^I fixed (in particular, $\mathrm{Re}P^I = p^I$) and vary X^I. In order to get rid of the scaling of X^I let us introduce a new variable T by

$$X^I = \widehat{X}^I T, \tag{3.18}$$

and integrate out T as follows:

$$e^{\widehat{V}_{\mathrm{BH}}} \approx \int dT e^{V_{\mathrm{BH}}}. \tag{3.19}$$

The potential (3.17) is quadratic in T

$$V_{\mathrm{BH}} = q_I \mathrm{Im}P^I + \mathrm{Im}(F_{IJ})\mathrm{Re}\big(P^I P^J + \widehat{X}^I \widehat{X}^J T^2 - 2\widehat{X}^I P^J T\big) - \frac{1}{2}\mathrm{Im}(F_{IJ}P^I P^J), \tag{3.20}$$

since F_{IJ} has zero weight under the rescaling (3.18). Variation with respect to T gives:

$$T = \frac{\widehat{X}^I \text{Im}(F_{IJ}) P^J}{\widehat{X}^I \text{Im}(F_{IJ}) \widehat{X}^J} \tag{3.21}$$

Therefore, the semiclassical approximation to (3.19) gives

$$\widehat{V}_{\text{BH}} = q_I \text{Im} P^I + \frac{i}{4} P^I \mathcal{N}_{IJ} P^J - \frac{i}{4} \overline{P}^I \overline{\mathcal{N}}_{IJ} \overline{P}^J, \tag{3.22}$$

where

$$\mathcal{N}_{IJ} = \overline{F}_{IJ} + 2i \frac{\text{Im}(F_{IK}) \widehat{X}^K \text{Im}(F_{JL}) \widehat{X}^L}{\widehat{X}^K \text{Im}(F_{KL}) \widehat{X}^L}. \tag{3.23}$$

The expression (3.22) should be compared to the modified black hole potential (3.7), which reduces to (3.22) if we use $\text{Re} P^I = p^I$.

The choice of the potential (3.17) can be motivated by looking at the $\mathcal{N} = 2$ supergravity action [63]. At tree level, the coupling of the vector fields can be described as

$$8\pi S_{\text{vec}}^{\text{tree}} = \int d^4 x \Big(\frac{i}{4} F_{IJ} \mathcal{F}_{\mu\nu}^{-I} \mathcal{F}^{-J\mu\nu} + \frac{1}{4} \text{Im}(F_{IJ}) \overline{X}^J \mathcal{F}_{\mu\nu}^{-I} T^{-\mu\nu} - \\ - \frac{1}{32} \text{Im}(F_{IJ}) \overline{X}^I \overline{X}^J T_{\mu\nu}^- T^{-\mu\nu} + h.c. \Big). \tag{3.24}$$

Then $V_{\text{BH}} - q_I \text{Im} P^I$ in (3.20) can be interpreted as a zero-mode reduction of (3.24), with the following identification:

$$\mathcal{F}_{\mu\nu}^{-I} \to i \overline{P}^I$$
$$X^I \to \widehat{X}^I$$
$$T_{\mu\nu}^- \to 4i \overline{T} \tag{3.25}$$
$$\int d^4 x \to 1.$$

Let us now discuss the attractor equations that describe the minima of the modified black hole potential (3.17). We can derive them in two equivalent ways. First, we can vary (3.22) with respect to the Calabi-Yau moduli, which gives (3.12). Or, second, we can vary the potential (3.17) with respect to the homogeneous coordinates X^I *before* we integrate out the overall scale T. This gives $\partial_I V_{\text{BH}} = 0$ and we obtain the following attractor equations:

$$-\frac{i}{2} C_{IJK} \text{Re}\big((P^J - X^J)(P^K - X^K)\big) - \text{Im}(F_{IK})\big(P^K - X^K\big) + \frac{i}{4} C_{IJK} P^J P^K = 0, \tag{3.26}$$

where

$$C_{IJK} = \partial_I F_{JK} = \partial_I \partial_J \partial_K F. \tag{3.27}$$

Using the identity

$$C_{IJK} X^K = 0, \tag{3.28}$$

which follows from the homogeneity relation $X^I F_I = 2F$, we can write (3.26) as

$$\boxed{C_{IJK}\left(\overline{P}^J - \overline{X}^J\right)\left(\overline{P}^K - \overline{X}^K\right) = 4i\mathrm{Im}(F_{IJ})\left(P^J - X^J\right)}$$
$$\tag{3.29}$$

It is clear that $X^I = P^I$ is the solution of (3.26). If we identify $T \to C$, $X^I \to \widehat{X}^I$, we obtain $P^I = C\widehat{X}^I$, which is the supersymmetric solution (3.13),(3.16). Moreover, if we contract (3.29) with X^I and use (3.28), we get

$$\mathrm{Im}(F_{IJ})X^I\left(P^J - X^J\right) = 0. \tag{3.30}$$

In the next section will use this relation to find all other solutions $P^I(X)$ of the attractor equations (3.29) in the one-modulus Calabi-Yau case.

4. The Inverse Problem

For a given set of charges (p^I, q_I) solutions to the system (2.7) define the complex structure on M. However, since these equations are highly non-linear, it is hard to write down solutions explicitly for a general Calabi-Yau manifold. On the other hand, since the black hole potential (2.6) is quadratic in charges[5] (p^I, q_I), we can try to solve the inverse problem: For a given point t^i on the Calabi-Yau moduli space, find corresponding set of the charges (p^I, q_I) that satisfy (2.7). Similar techniques were used in [64] to solve the inverse problem for metastable non-supersymmetric backgrounds in the context of flux compactifications.

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[5] This is clear from looking at the alternative representation (3.1) of the black hole potential.

4.1. Inverse problem and inhomogeneous variables

Strictly speaking, the physical charges (p^I, q_I) are quantized, but in semiclassical approximation in the limit of large charges we can ignore this integrality problem and treat the charges as continuous coordinates. Another ambiguity in defining the inverse problem is related to the fact that all sets of charges (p^I, q_I) connected by an $Sp(2n_V + 2, \mathbb{Z})$ transformations give the same point on the moduli space, since the black hole potential (2.6) and hence the extremum equations (2.7) are symplectically invariant. Therefore, we need to choose some canonical symplectic basis in $H^3(M, \mathbb{Z})$ and keep it fixed. However, even including that, the inverse problem is not well-defined, since the extremization of (2.6) gives only $2n_V$ real equations (2.7) for $2n_V + 2$ real variables (p^I, q_I). In order to fix this ambiguity, we suggest to look only at the critical points where the superpotential (2.3) takes some particular value:

$$\mathcal{W} = \omega, \tag{4.1}$$

where ω is a new complex parameter. This can be viewed as a convenient gauge fixing. Therefore, we are interested in solving the system of equations

$$\partial_i V_{\mathrm{BH}} = \overline{\partial}_{\bar{i}} V_{\mathrm{BH}} = 0, \qquad \mathcal{W} = \omega. \tag{4.2}$$

at some particular point t^i on the Calabi-Yau moduli space. Then solution of this inverse problem gives a (multivalued) map: $(t^i, \omega) \rightarrow (p^I, q_I)$.

Since $\int_M \Omega \wedge H = q_I X^I - p^I F_I$, the equation (4.1) can be written as

$$X^I \big(q_I - \mathcal{N}_{IJ} p^J \big) = \omega. \tag{4.3}$$

If we then use (3.9), this gives $X^I \mathrm{Im}(\mathcal{N}_{IJ}) \overline{P}^J = i\omega$. Therefore, the solution of the inverse problem is given by the following system of equations:

$$
\begin{aligned}
p^I &= \mathrm{Re}\big(P^I\big) & \partial_i \mathrm{Im}(P^I \mathcal{N}_{IJ} P^J) &= 0 \\
q_I &= \mathrm{Re}\big(\mathcal{N}_{IJ} P^J\big) & X^I \mathrm{Im}\big(\mathcal{N}_{IJ}\big) \overline{P}^J &= i\omega
\end{aligned}
\tag{4.4}
$$

In other words, fixing Calabi-Yau moduli and the gauge (4.1) allows one to solve for P^I from the two equations on the right of (4.4). Then the charges are given by the two equations on the left of (4.4).

Among the solutions to (4.4), there always is a supersymmetric solution (3.13), that can be written as

$$P^I = 2ie^K \overline{\omega} X^I, \tag{4.5}$$

where we used $K = -\log\left(-2X \cdot \mathrm{Im}\mathcal{N} \cdot \overline{X}\right)$ to fix the constant C as

$$C = 2i\overline{\omega}e^K = 2i\left(q_I\overline{X}^I - p^I\overline{F}_I\right)e^K = 2i\overline{Z}e^{\frac{K}{2}}. \tag{4.6}$$

An example of the explicit solution of the inverse problem in the diagonal T^6 model is presented in Appendix A.1

4.2. Inverse problem and homogeneous variables: one-modulus Calabi-Yau case

We can think of the homogeneous variables X^I as parameterizing extended space $\widetilde{\mathcal{M}}$ of the complex structures on a Calabi-Yau threefold M. This space can also be viewed as a total space $\widetilde{\mathcal{M}}$ of the line bundle $\mathcal{L} \to \mathcal{M}$ of the holomorphic 3-forms $H^{3,0}(M,\mathbb{C})$ over the Calabi-Yau moduli space (to be precise, the Teichmüller space) $\mathcal{M}$. Let us comment on the dimension of the space of solutions to the system (3.29). For a fixed extended Calabi-Yau moduli, this is a set of $n_V + 1$ complex quadratic equations for $n_V + 1$ complex variables P^I. Therefore, this system can have at most 2^{n_V+1} solutions. One of them describes supersymmetric black hole and thus there are at most $2^{n_V+1} - 1$ non-supersymmetric solutions.

Let us discuss the inverse problem for a one-modulus Calabi-Yau case, when

$$F = (X^0)^2 f(\tau), \qquad \tau = \frac{X^1}{X^0}. \tag{4.7}$$

The homogeneity relation gives $F_0 = 2X^0 f - X^1 f'$, where $f' \equiv \partial_\tau f$, and we obtain the following matrix of second derivatives

$$F_{IJ} = \begin{pmatrix} 2f - 2\tau f' + \tau^2 f'' & f' - \tau f'' \\ f' - \tau f'' & f'' \end{pmatrix}. \tag{4.8}$$

an the matrix of third derivatives

$$C_{0IJ} = -\tau C_{1IJ} = \frac{1}{X^0}\begin{pmatrix} -\tau^3 f''' & \tau^2 f''' \\ \tau^2 f''' & -\tau f''' \end{pmatrix} \tag{4.9}$$

To simplify expressions below, let us introduce the notation

$$y^I = P^I - X^I. \tag{4.10}$$

Then the attractor equations (3.26) read

$$\begin{cases} C_{0JK}\overline{y}^J\overline{y}^K = 4i\mathrm{Im}(F_{0J})y^J \\ C_{1JK}\overline{y}^J\overline{y}^K = 4i\mathrm{Im}(F_{1J})y^J. \end{cases} \tag{4.11}$$

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Using the relation (4.9), we find from (4.11)

$$\text{Im}(F_{0I})y^I = -\tau\text{Im}(F_{1I})y^I, \tag{4.12}$$

which is equivalent to (3.30). To shorten the notations, let us define

$$X_I \equiv X^J\text{Im}F_{JI}. \tag{4.13}$$

For example, $X_0 \equiv X^0\text{Im}F_{00} + X^1\text{Im}F_{10}$. Then we find from (4.12)

$$y^1 = -\frac{X_0}{X_1}y^0. \tag{4.14}$$

If we plug this back into (4.11), we obtain

$$(\overline{y}^0)^2 = \mathcal{Y}y^0, \tag{4.15}$$

where

$$\mathcal{Y} = -4iX_1\frac{\left(X^0\right)^4\det\|\text{Im}F_{IJ}\|}{f'''\left(X^IX_I\right)^2} \tag{4.16}$$

For future reference, let us write down an explicit expression for the ingredients entering (4.16), in terms of the holomorphic function f defining the prepotential (4.7):

$$\begin{aligned}
X_1 &= X^0(\text{Im}f' - \text{Im}(\tau)\overline{f''}) \\
X^IX_I &= 2(X^0)^2\left(\text{Im}f - \text{Im}(\tau)\overline{f'} - i(\text{Im}\tau)^2\overline{f''}\right) \\
\det\|\text{Im}F_{IJ}\| &= 2\text{Im}(f)\text{Im}(f'') - (\text{Im}f')^2 + 2\text{Im}(\tau)\text{Im}(f'\overline{f''}) - (\text{Im}\tau)^2|f''|^2.
\end{aligned} \tag{4.17}$$

In order to solve (4.15), we take the square of the complex conjugate equation and then use (4.15). This gives

$$(y^0)^4 = \overline{\mathcal{Y}}^2\mathcal{Y}y^0. \tag{4.18}$$

Therefore, in terms of the original variables (4.10) we find the following four solutions:

$$\begin{cases} P^0_{(0)} = X^0 \\ P^1_{(0)} = X^1, \end{cases} \tag{4.19}$$

and

$$\begin{cases} P^0_{(k)} = X^0 + \left(\overline{\mathcal{Y}}^2\mathcal{Y}\right)^{1/3}e^{2\pi ik/3} \\ P^1_{(k)} = X^1 - \frac{X_0}{X_1}\left(\overline{\mathcal{Y}}^2\mathcal{Y}\right)^{1/3}e^{2\pi ik/3}, \qquad k = 1, 2, 3. \end{cases} \tag{4.20}$$

where the first solution corresponds to a supersymmetric black hole and the other three are non-supersymmetric. Corresponding black hole charges are given by

$$\begin{cases} p^I = \mathrm{Re}P^I \\ q_I = \mathrm{Re}(\mathcal{N}_{IJ}P^J). \end{cases} \tag{4.21}$$

5. Semiclassical Entropy in the OSV Ensamble

In this section we develop a semiclassical version of OSV formalism which applies to both supersymmetric and non-supersymmetric black holes. We then illustrate it using $D0-D4$ system in the diagonal T^6 model as an example. This will serve as a preparation for the discussions in section 6 and the conjecture in section 7 taking into account perturbative corrections to the extremal black hole entropy.

We begin by recalling some basic ingredients of the OSV formalism. The formula [4]

$$Z_{\mathrm{BH}}(p^I, \phi^I) = \left| e^{F_{\mathrm{top}}(p^I + \frac{i}{\pi}\phi^I)} \right|^2. \tag{5.1}$$

describes a relation between the mixed partition function of the supersymmetric (BPS) black hole and topological string free energy. Here F_{top} denotes the topological string free energy. It is well known [22] that the higher genus contributions to F_{top} depend non-holomorphically on the background complex structure. This dependence, originally described in [22] as the holomorphic anomaly in the topological string amplitudes coming from the boundary of the moduli space, was interpreted in [65] as a dependence of the wave-function $\Psi_{\mathrm{top}} = e^{F_{\mathrm{top}}}$ on the choice of the polarization. This viewpoint on the topological string partition function as a wave-function was further studied in [23,66,67].

As noted in [4], the formula (1.1) can be inverted, and resulting expression

$$\Omega(p^I, q_I) = \int d\chi^I e^{-i\pi\chi^I q_I} \Psi^*_{\mathrm{top}}(p^I - \chi^I)\Psi_{\mathrm{top}}(p^I + \chi^I). \tag{5.2}$$

can be interpreted as the Wigner function[6] associated to the topological string wave function. Here $\Psi_{\mathrm{top}}(p^I) = \langle p^I | \Psi_{\mathrm{top}} \rangle$ represents the topological string wave function in real polarization (see [69] for a comprehensive review and references), and the chemical potentials are restored after deforming the integration contour as $\phi^I = -i\chi^I$.

---

[6] Let us recall that in quantum mechanics the Wigner function defines the quasi-probablity measure $f(\mathrm{x}, \mathrm{p}) = \frac{1}{2\pi} \int dy e^{-iy\mathrm{p}} \psi^*(\mathrm{x} - \frac{\hbar}{2}y)\psi(\mathrm{x} + \frac{\hbar}{2}y)$ on the *phase* space, see e.g. [68]. Here the canonical commutation relation is $[\widehat{\mathrm{p}}, \widehat{\mathrm{x}}] = -i\hbar$. In the topological string setup $\hbar = \frac{2}{\pi}$.

5.1. Black hole potential and OSV transformation

Let us rewrite modified black hole potential (3.17) in the form

$$V_{\text{BH}}^{(0)} = q_I \text{Im} P^I + \left(\frac{i}{4}(P^I - X^I)(P^J - X^J)\overline{F}_{IJ}^{(0)} + \frac{i}{2}(P^I - X^I)F_I^{(0)} + \frac{i}{2}F^{(0)} + c.c. \right). \tag{5.3}$$

We put the superscript (0) to stress that the prepotential $F^{(0)}$ corresponds to a genus zero part of the topological string free energy. As in the OSV setup [4], we can parameterize the Lagrange multiplier P^I (which can also be viewed as a complexified magnetic charge) as

$$P^I = p^I + \frac{i}{\pi}\phi^I, \tag{5.4}$$

so that the first of the attractor equations (3.9) is automatically satisfied. At the next step, we rewrite the semiclassical entropy $S_{\text{BH}}^{(0)} = \pi V_{\text{BH}}^{(0)}$ as

$$S_{\text{BH}}^{(0)} = q_I \phi^I - \pi \text{Im} \mathcal{G}^{(0)}, \tag{5.5}$$

where we introduced a function $\mathcal{G}^{(0)}$ defined by

$$\mathcal{G}^{(0)} = \frac{1}{2}(P^I - X^I)(P^J - X^J)\overline{F}_{IJ}^{(0)} + (P^I - X^I)F_I^{(0)} + F^{(0)}. \tag{5.6}$$

In order to compute the entropy in (5.5) we should find the values of ϕ^I and X^I that extremize the black hole potential (5.3). Extremization with respect to the (extended) Calabi-Yau moduli $\partial_I V_{\text{BH}}^{(0)} = 0$ gives the equations (3.29). Let us use the index s to label all solutions to these equations, $X_s^I = X_s^I(P)$. There are two types of these solutions, supersymmetric ($s = \text{susy}$) and non-supersymmetric ($s = \text{n} - \text{susy}$) ones. In particular, the supersymmetric solution is given by $X_{\text{susy}}^I(P) = P^I$. By substituting these solutions in (5.6) we obtain the functions $\mathcal{G}_s^{(0)}(P^I) = \mathcal{G}_s^{(0)}(p^I, \phi^I)$. In the supersymmetric case $\mathcal{G}_{\text{susy}}^{(0)}(P^I) = F^{(0)}(p^I + \frac{i}{\pi}\phi^I)$. Let us define a mixed partition functions corresponding to each of the solutions $X_s^I = X_s^I(P)$ by

$$Z_s^{(0)}(p^I, \phi^I) = e^{i\frac{\pi}{2}\mathcal{G}_s^{(0)}(p^I, \phi^I)}. \tag{5.7}$$

For example, the supersymmetric mixed partition function

$$Z_{\text{susy}}^{(0)}(p^I, \phi^I) = e^{i\frac{\pi}{2}F^{(0)}(p^I + \frac{i}{\pi}\phi^I)} \tag{5.8}$$

describes the leading contribution to (5.1).

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For a fixed charge vector (p^I, q_I) the extremal black hole degeneracy can be written symbolically as $\Omega_{\text{extrm}} = \Omega_{\text{susy}} + \Omega_{\text{n-susy}}$. Therefore, the leading semiclassical contribution to Ω_{extrm} is given by an OSV type integral

$$\Omega_{\text{extrm}}^{(0)}(p^I, q^I) = \int d\phi^I e^{q_I \phi^I} \sum_s \left| Z_s^{(0)}(p^I, \phi^I) \right|^2, \qquad (5.9)$$

where the sum is over all solutions to the extremum equations (3.29). We will discuss perturbative corrections to this formula later in section 7, but before that let us comment on the possible wave function interpretation of this expression.

Define

$$\Psi(X, P) = \exp \frac{i\pi}{2} \left(\frac{1}{2}(P^I - X^I)(P^J - X^J)\overline{F}_{IJ}^{(0)} + (P^I - X^I)F_I^{(0)} + F^{(0)} \right). \qquad (5.10)$$

This is essentially the off-shell version of the partition function (5.7), since we have not substituted the extremum solution $X_s^I = X_s^I(P)$ into (5.10) yet. This can be achieved by integrating out the fields X^I in the semiclassical approximation

$$\sum_s \left| Z_s^{(0)}(p^I, \phi^I) \right|^2 \approx \int dX^I d\overline{X}^I \sqrt{\det \|\text{Im}F_{IJ}\|} \Psi(X, P)\Psi^*(X, P). \qquad (5.11)$$

The function $\Psi(X, P)$ given in (5.10) is holomorphic in P^I and non-holomorphic in X^I. It turns out that (up to some numerical factors due to a difference in conventions) it coincides exactly with the DVV 'conformal block' [66] appearing in study of the five-brane partition function! In particular, as was shown in [66], it satisfies the holomorphic anomaly equation [22]. Using results of [69], it can be identified as the intertwining function $\Psi(X, P) =_{(X,\overline{X})}\langle X^I | P^I \rangle$ between the coherent state $|P^I\rangle$ in the real polarization and the coherent state $|X^I\rangle_{(X,\overline{X})}$ in the holomorphic polarization appearing in quantization of $H^3(M, \mathbb{C})$. The integral in (5.11) then can naturally be interpreted as averaging over the wave function polarizations, thus effectively removing the background dependence. We should stress, however, that only semiclassical approximation to this integral is needed for (5.9). This would be interesting to develop further, especially in connection with the topological M-theory [70,71] interpretation of the black hole entropy counting.

We now turn to a simple example of the diagonal T^6 model, where semiclassical formula (5.9) for extremal black hole entropy can be illustrated.

5.2. Semiclassical entropy in the diagonal T^6 compactification

Consider Type IIB compactification on the diagonal T^6 threefold [26] (see Appendix A for more details about this model). The prepotential is

$$F = \frac{(X^1)^3}{X^0}, \qquad f(\tau) = \tau^3, \tag{5.12}$$

where the complex structure parameter $\tau = \frac{X^1}{X^0}$. We compute:

$$F_{IJ} = \begin{pmatrix} 2\tau^3 & -3\tau^2 \\ -3\tau^2 & 6\tau \end{pmatrix}, \quad C_{IJ0} = -\frac{6\tau}{X^0}\begin{pmatrix} \tau^2 & -\tau \\ -\tau & 1 \end{pmatrix}, \quad C_{IJ1} = \frac{6}{X^0}\begin{pmatrix} \tau^2 & -\tau \\ -\tau & 1 \end{pmatrix}. \tag{5.13}$$

Let us denote $y^I = P^I - X^I$. The attractor equations (3.29) read

$$\begin{cases} C_{0IJ}\overline{y}^I\overline{y}^J = 4i\mathrm{Im}(F_{0I})y^I \\ C_{1IJ}\overline{y}^I\overline{y}^J = 4i\mathrm{Im}(F_{1I})y^I. \end{cases} \tag{5.14}$$

In order to compute the function $\mathcal{G}^{(0)}(p^I, \phi^I)$, we need to find from these equations a solution $X^I = X^I(P)$ of the direct problem. This can be done by inverting the solutions of the inverse problem (4.19)-(4.20). However, it turns out that it is easier to find $X^I = X^I(P)$ directly from (5.14).

According to (4.9) and (5.13), the third derivatives of the prepotential are related as $C_{0IJ} = -\tau C_{1IJ}$, and therefore (5.14) reduces to

$$2y^0\mathrm{Im}(\tau^3) - 3y^1\mathrm{Im}(\tau^2) = 3\tau y^0\mathrm{Im}(\tau^2) - 6\tau y^1\mathrm{Im}(\tau). \tag{5.15}$$

Apart from the supersymmetric solution $y^0 = y^1 = 0$, this gives

$$\frac{y^1}{y^0} = \mathrm{Re}\tau - \frac{i}{3}\mathrm{Im}\tau, \tag{5.16}$$

If we recall that $y^I = P^I - X^I$, we can solve (5.16) for X^1:

$$X^1 = X^0\frac{4\mathrm{Re}(X^0\overline{P}^1) - 2\overline{P}^1P^0 + P^1\overline{P}^0}{4\mathrm{Re}(X^0\overline{P}^0) - |P^0|^2}. \tag{5.17}$$

Then we plug this into the second equation of (5.14) and find[7]

$$\left(\overline{X}^0 - \overline{P}^0\right)^2 = 3X^0\left(X^0 - P^0\right). \tag{5.18}$$

---

[7] assuming $\mathrm{Im}\left(P^0\overline{P}^1\right) \neq 0$.

This should be compared to (4.15). To solve the equation (5.18), is convenient to work with the real and imaginary parts of X^0 and P^0. Then (5.18) can be reduced to a quartic equation for $\mathrm{Re}X^0$. For a generic choice of $\mathrm{Re}P^0$ and $\mathrm{Im}P^0$, two of the roots of this quartic equation are complex, and two are real. These real roots lead to the two solutions of the attractor equations (5.14), supersymmetric

$$
\begin{aligned}
X^0 &= P^0, \\
X^1 &= P^1,
\end{aligned}
\tag{5.19}
$$

and non-supersymmetric one. Explicit expression for the non-supersymmetric solution depends on the signs of $\mathrm{Re}P^0$ and $\mathrm{Im}P^0$. For example, when $\mathrm{Im}P^0 > |\mathrm{Re}P^0|$, it is given by[8]

$$
\begin{aligned}
\mathrm{Re}X^0 &= \frac{1}{4}\mathrm{Re}P^0 + \frac{3}{8}\left(\mathrm{Re}P^0 + \mathrm{Im}P^0\right)^{\frac{2}{3}}\left(\mathrm{Im}P^0 - \mathrm{Re}P^0\right)^{\frac{1}{3}} - \\
&\quad - \frac{3}{8}\left(\mathrm{Re}P^0 + \mathrm{Im}P^0\right)^{\frac{1}{3}}\left(\mathrm{Im}P^0 - \mathrm{Re}P^0\right)^{\frac{2}{3}}, \\
\mathrm{Im}X^0 &= \frac{1}{4}\mathrm{Im}P^0 - \frac{1}{4}\sqrt{9\left(\mathrm{Im}P^0\right)^2 - 8\left(\mathrm{Re}P^0\right)^2 - 8\mathrm{Re}\left(X^0\right)\mathrm{Re}\left(P^0\right) + 16\left(\mathrm{Re}X^0\right)^2}.
\end{aligned}
\tag{5.20}
$$

We can use these solutions and study a system of $kD0$ and $ND4$ branes on the diagonal T^6. This corresponds to the charge vector of the form $(k, 0, N, 0)$. In this case the discriminant $\mathcal{D} = -(3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2)$ reduces to $\mathcal{D} = -4kN^3$, so that the system is supersymmetric when $kN > 0$ and non-supersymmetric when $kN < 0$. Complexified magnetic charges are given by

$$
P^0 = \frac{i}{\pi}\varphi, \qquad P^1 = N + \frac{i}{\pi}\phi,
\tag{5.21}
$$

and the black hole degeneracy (5.9) in this case reads

$$
\Omega^{(0)}_{\mathrm{extrm}}(k, N) = \int d\phi d\varphi\, e^{k\varphi}\left(e^{-\pi\mathrm{Im}\mathcal{G}^{(0)}_{\mathrm{susy}}\left(\frac{i}{\pi}\varphi, N + \frac{i}{\pi}\phi\right)} + e^{-\pi\mathrm{Im}\mathcal{G}^{(0)}_{\mathrm{n-susy}}\left(\frac{i}{\pi}\varphi, N + \frac{i}{\pi}\phi\right)}\right).
\tag{5.22}
$$

Let us now compute expressions for $\mathcal{G}^{(0)}$-functions entering into (5.22). Using (5.19), we find from (5.6)

$$
-\pi\mathrm{Im}\mathcal{G}^{(0)}_{\mathrm{susy}}\left(\frac{i}{\pi}\varphi, N + \frac{i}{\pi}\phi\right) = \frac{N^3\pi^2 - 3N\phi^2}{\varphi}.
\tag{5.23}
$$

---

[8] Corresponding solution for X^1 is obtained by plugging this expression into (5.17).

The non-supersymmetric solution (5.20) in the case (5.21) reads

$$X^0 = -\frac{i}{2\pi}\varphi$$
$$X^1 = \frac{1}{2}\left(N - \frac{i}{2\pi}\phi\right).$$

(5.24)

Therefore, from (5.6) we obtain the following expression

$$-\pi\mathrm{Im}\mathcal{G}_{\mathrm{n-susy}}\left(\frac{i}{\pi}\varphi, N + \frac{i}{\pi}\phi\right) = -\frac{N^3\pi^2 - 3N\phi^2}{\varphi}.$$

(5.25)

The integral over ϕ in (5.22) is quadratic, and (ignoring the convergence issue) in the semiclassical approximation $\phi = 0$. The critical points in the φ direction are given by

$$\partial_\varphi(k\varphi - \pi\mathrm{Im}\mathcal{G}_{\mathrm{susy}}) = 0 \quad \Rightarrow \quad \varphi_{\mathrm{susy}} = \pi\sqrt{\frac{N^3}{k}}$$

(5.26)

for supersymmetric term, and

$$\partial_\varphi(k\varphi - \pi\mathrm{Im}\mathcal{G}_{\mathrm{n-susy}}) = 0 \quad \Rightarrow \quad \varphi_{\mathrm{n-susy}} = \pi\sqrt{-\frac{N^3}{k}}$$

(5.27)

for the non-supersymmetric term. Since we are integrating over the real axis, the leading contribution to (5.22) comes only from one of the two terms, depending on the sign of the ratio $\frac{N}{k}$. This gives:

$$\Omega^{(0)}_{\mathrm{extrm}}(k, N) \approx \exp\left(2\pi\sqrt{|N^3 k|}\right),$$

(5.28)

which is a correct expression for extremal black hole degeneracy, valid both in the supersymmetric and non-supersymmetric cases. Using the same method, it is also easy to obtain an expression $\Omega^{(0)}_{\mathrm{extrm}}(N_0, N_6) \approx \exp\left(\pi|N_0 N_6|\right)$ for the degeneracy of $D0 - D6$ system on diagonal T^6, which agrees with [72].

It is instructive to compare this prediction of (5.9) with the original OSV formula [4]

$$\Omega(p^I, q_I) = \int d\phi^I e^{q_I \phi^I + \mathcal{F}(p^I, \phi^I)}.$$

(5.29)

Because of our choice of the non-canonical $D3$-brane intersection matrix (see Appendix A) on T^6, we have $q_I\phi^I = -u\phi^0 - 3q\phi$. Also,

$$\mathcal{F}(p^I, \phi^I) = -\pi\mathrm{Im}\left(\frac{(p + \frac{i}{\pi}\phi)^3}{v + \frac{i}{\pi}\phi^0}\right).$$

(5.30)

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In the semiclassical approximation, the leading contribution to $\ln \Omega(u, q, p, v)$ can be computed by extremizing the exponent in (5.29). This gives

$$2q = -\frac{(p + \frac{i}{\pi}\phi)^2}{v + \frac{i}{\pi}\phi^0} - \frac{(p - \frac{i}{\pi}\phi)^2}{v - \frac{i}{\pi}\phi^0},$$

$$2u = \frac{(p + \frac{i}{\pi}\phi)^3}{(v + \frac{i}{\pi}\phi^0)^2} - \frac{(p - \frac{i}{\pi}\phi)^3}{(v - \frac{i}{\pi}\phi^0)^2}.$$

(5.31)

which essentially are the supersymmetric attractor equations (3.16). The general solution to (5.31) is easy to write:

$$\phi^0 = \pm \pi \frac{2p^3 + 2pqv - uv^2}{\sqrt{-\mathcal{D}}},$$

$$\phi = \mp \pi \frac{2p^2q + 2q^2v + puv}{\sqrt{-\mathcal{D}}},$$

(5.32)

where the discriminant $\mathcal{D} = -(3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2)$. The sign ambiguity in (5.32) can be fixed by imposing physically natural condition

$$\mathrm{Im}\,\tau = \mathrm{Im}\,\frac{p + \frac{i}{\pi}\phi}{v + \frac{i}{\pi}\phi^0} > 0.$$

(5.33)

Notice that the potentials (5.32) become pure imaginary when $\mathcal{D} > 0$. Therefore, if one is allowed to do the analytical continuation when computing the integral (5.29), the answer for the microcanonical entropy reads

$$\ln \Omega(u, q, p, v) \approx \pi \sqrt{3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2}.$$

(5.34)

This expression, of course, becomes pure imaginary on the non-supersymmetric side $\mathcal{D} > 0$ of the discriminant hypersurface $\mathcal{D} = 0$, which is meaningless. This thus illustrates the shortcoming of OSV formalism in the context of non-BPS black holes.

6. Including Higher Derivative Corrections: The Entropy Function Approach

The Wald's formula provides a convenient tool for computing the macroscopic black hole entropy in the presence of higher derivative terms. It can be written as

$$S_{\mathrm{BH}} = 2\pi \int_H d^2x \sqrt{h}\, \epsilon_{\mu\nu} \epsilon_{\lambda\rho} \frac{\delta \mathcal{L}}{\delta \mathcal{R}_{\mu\nu\lambda\rho}},$$

(6.1)

where $\mathcal{L}$ is the Lagrangian density and the integral is computed over the black hole horizon. Sen [47,48] showed that in the case of a spherically symmetric extremal black holes with

$AdS^2 \times S_2$ near horizon geometry Wald's formula simplifies drastically. This gives an effective method for computing a macroscopic entropy of a spherically symmetric extremal black holes in a theory of gravity coupled to gauge and scalar fields, called the entropy function formalism.

In this section we briefly describe, following [20], a formulation of $\mathcal{N} = 2$ supergravity coupled to n_V abelian gauge fields, in the presence of higher-derivative corrections. Then we review recent computations of the extremal black hole entropy in this setup [50,51,57], performed in the framework of the entropy function formalism.

6.1. $d = 4$, $\mathcal{N} = 2$ Supergravity with F-term $\mathcal{R}^2$ corrections

The Lagrangian density of $\mathcal{N} = 2$ Poincare supergravity coupled to n_V vector multiplets can be conveniently formulated using the off-shell description [63]. The idea is to start with an $\mathcal{N} = 2$ conformal supergravity and then reduce it to Poincare supergravity by gauge fixing and adding appropriate compensating fields. The advantage of working with $\mathcal{N} = 2$ superconformal approach is that it provides many powerful tools, such as superconformal tensor calculus and a general *density formula* for the Lagrangian.

One introduces the Weil and matter chiral superfields

$$
\begin{aligned}
W_{\mu\nu}(x,\theta) &= T^-_{\mu\nu} - \frac{1}{2}\mathcal{R}^-_{\mu\nu\lambda\rho}\epsilon_{\alpha\beta}\theta^\alpha\sigma^{\lambda\rho}\theta^\beta + \dots \\
\Phi^I(x,\theta) &= X^I + \frac{1}{2}\mathcal{F}^{-I}_{\mu\nu}\epsilon_{\alpha\beta}\theta^\alpha\sigma^{\mu\nu}\theta^\beta + \dots
\end{aligned}
\tag{6.2}
$$

where $T^-_{\mu\nu}$ is an auxiliary antiself-dual tensor field[9], and $\mathcal{F}^{-I}_{\mu\nu}$ and $\mathcal{R}^-_{\mu\nu\lambda\rho}$ denote the antiselfdual parts the field-strength and curvature tensors correspondingly. The conventions are $*T_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}$ and $T^\pm_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} \pm i * T_{\mu\nu})$, so that $T^-_{\mu\nu} = \overline{T}^+_{\mu\nu}$ for Minkovski signature. The superconformally covariant field strength

$$
\mathbf{F}^I_{\mu\nu} = \mathcal{F}^I_{\mu\nu} - \left(\frac{1}{4}\overline{X}^I T^-_{\mu\nu} + \epsilon_{ij}\overline{\psi}^i{}_{[\mu}\gamma_{\nu]}\Omega^{jI} + \epsilon_{ij}\overline{X}^I\overline{\psi}^i{}_\mu\psi^j_\nu + h.c.\right)
\tag{6.3}
$$

enters into the bosonic part of the Lagrangian through the combination $\mathcal{F}^{+I}_{\mu\nu} - \frac{1}{4}X^I T^+_{\mu\nu}$. The F-terms can be reproduced from the generalized prepotential

$$
F(X^I, W) = \sum_g F^{(g)}(X^I)W^{2g},
\tag{6.4}
$$

---

[9] At tree-level this field is identified with the graviphoton by the equations of motion.

where $F^{(g)}$ can be computed from the topological string amplitudes [22,73]. In particular, the topological string free energy is given by

$$F_{\text{top}}(X^I, g_{\text{top}}) = \sum_g (g_{\text{top}})^{2g-2} F^{(g)}(X^I). \tag{6.5}$$

The function $F^{(g)}$ is homogeneous of degree $2 - 2g$, so that

$$F(\lambda X^I, \lambda W) = \lambda^2 F(X^I, W). \tag{6.6}$$

This homogeneity relation for the generalized prepotential (6.4) can also be written as

$$X^I \partial_I F + W \partial_W F = 2F. \tag{6.7}$$

Notice that another notation

$$\widehat{A} \equiv W^2, \qquad F(X^I, \widehat{A}) \equiv F(X^I, W) \tag{6.8}$$

is sometimes used in the supergravity literature.

The coupling of the vector fields to the gravity is governed by the generalized prepotential (6.4) as follows

$$
\begin{aligned}
8\pi S_{\text{vect}} &= 8\pi S_{\text{vect}}^{\text{tree}} + \int d^4x d^4\theta \sum_{g=1}^{\infty} F_g(\Phi^I) \big(W_{\mu\nu} W^{\mu\nu}\big)^g + h.c. = \\
&= 8\pi S_{\text{vect}}^{\text{tree}} + \int d^4x \sum_{g=1}^{\infty} F_g(X^I) \big(\mathcal{R}_-^2 T_-^{2g-2} + \dots\big) + h.c.
\end{aligned}
\tag{6.9}
$$

The terms in the Lagrangian density, relevant for the computation of the entropy are [20]

$$
8\pi\mathcal{L} = -\frac{i}{2}\Big[\frac{1}{2}\big(\mathcal{F}_{\mu\nu}^{+I} - \frac{1}{4}X^I T_{\mu\nu}^+\big)\big(\mathcal{F}^{+J\mu\nu} - \frac{1}{4}X^J T^{+\mu\nu}\big)\overline{F}_{IJ} + \frac{T^{+\mu\nu}}{4}\big(\mathcal{F}_{\mu\nu}^{+I} - \frac{1}{4}X^I T_{\mu\nu}^+\big)F_I + \frac{\widehat{\overline{A}}}{16}F - \\
- \overline{X}^I F_I \mathcal{R} - F_{\widehat{A}}\widehat{C} - h.c.\Big] + \dots
\tag{6.10}
$$

Here

$$
\begin{aligned}
\widehat{C} &= 64\mathcal{R}_{\nu\mu\rho\sigma}^- \mathcal{R}^{-\nu\mu\rho\sigma} + 16 T^{-\mu\nu} f_\mu^\rho T_{\rho\nu}^+ + \dots \\
f_\mu^\nu &= -\frac{1}{2}\mathcal{R}_\mu^\nu + \frac{1}{32}T_{\mu\rho}^- T^{+\nu\rho} + \dots \\
F &= F(X^I, \widehat{A}), \qquad F_{\widehat{A}} \equiv \partial_{\widehat{A}} F,
\end{aligned}
\tag{6.11}
$$

and ... in (6.10)-(6.11) denotes the terms (auxiliary fields, fermions, etc.) that will vanish or cancel out on the black hole ansatz.

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6.2. Review of the entropy function computation

We are interested in a spherically symmetric extremal black hole solutions arising in the supergravity theory defined by the Lagrangian (6.10). Consider the most general $SO(2,1) \times SO(3)$ ansatz [50] for a field configurations consistent with the $AdS_2 \times S^2$ near horizon geometry of the black hole

$$
ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + \sin^2 \theta d\varphi^2),
$$
$$
X^I = x^I, \quad \mathcal{F}^I_{rt} = -\frac{\phi^I}{\pi}, \quad \mathcal{F}^I_{\theta\varphi} = p^I \sin\theta, \quad T^-_{rt} = v_1 w, \tag{6.12}
$$

and all other fields presents in (6.10) are set to zero[10]. The entropy function [47] is defined as

$$
\mathcal{E} = q_I \phi^I - 2\pi \int_H d\theta d\varphi \sqrt{-\det g} \mathcal{L}. \tag{6.13}
$$

This function depends on free parameters $(x^I, v_1, v_2, w, \phi^I)$ of the $SO(2,1) \times SO(3)$ ansatz (6.12). The entropy of an extremal black hole is obtained as an entropy of a non-extremal black hole in the extremal limit, when the function (6.13) is extremized with respect to a free parameters

$$
\frac{\partial \mathcal{E}}{\partial x^I} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial w} = 0, \quad \frac{\partial \mathcal{E}}{\partial \phi^I} = 0. \tag{6.14}
$$

The black hole entropy (6.1) is given by the value of $\mathcal{E}$ at the extremum

$$
S_{\mathrm{BH}} = \mathcal{E}|_{\partial \mathcal{E}=0}. \tag{6.15}
$$

The result of computation [50] reads

$$
\mathcal{E} = q_I \phi^I - i\pi v_1 v_2 \Bigg[\frac{1}{4} \Big(-\frac{\phi^I}{\pi v_1} + i\frac{p^I}{v_2} - \frac{1}{2} x^I \overline{w} \Big) \Big(-\frac{\phi^J}{\pi v_1} + i\frac{p^J}{v_2} - \frac{1}{2} x^J \overline{w} \Big) \overline{F}_{IJ} +
$$
$$
+ \frac{\overline{w}}{4} \Big(-\frac{\phi^I}{\pi v_1} + i\frac{p^I}{v_2} - \frac{1}{2} x^J \overline{w} \Big) F_I + \frac{\overline{w}^2}{8} F -
$$
$$
- \Big(\frac{1}{v_1} - \frac{1}{v_2} \Big) \overline{x}^I F_I + \Big(|w|^4 - 8|w|^2 \Big(\frac{1}{v_1} + \frac{1}{v_2} \Big) + 64 \Big(\frac{1}{v_1} - \frac{1}{v_2} \Big)^2 \Big) F_{\widehat{A}} - c.c. \Bigg], \tag{6.16}
$$

where

$$
\widehat{A} = -4w^2. \tag{6.17}
$$

---

[10] The dilaton is set to $1/3\mathcal{R}$, so that the combination $D - 1/3\mathcal{R}$ vanishes.

Note that the entropy function (6.16) is invariant under the following rescaling

$$x^I \to \lambda x^I, \quad w \to \lambda w, \quad v_{1,2} \to \frac{1}{\lambda\bar{\lambda}} v_{1,2}, \quad \phi^I \to \phi^I, \quad q_I \to q_I, \quad p^I \to p^I, \qquad (6.18)$$

since the Lagrangian (6.10) was derived from a superconformally invariant expression. This means that there is a linear relation between the extremum equations (6.14). One can switch to inhomogeneous variables to fix this symmetry.

The above form of the entropy function does not take into account all the relevant higher derivative corrections needed for the non-supersymmetric black hole, as has been observed in [50]. For example at least an $\mathcal{R}^2$ term is needed in certain cases. We will come back to this point in the next section when we propose our conjecture.

To further motivate our conjecture, let us analyze the structure of the entropy function (6.16). First of all, compared to the topological string partition function, it depends on one more parameter. Indeed, using the scaling invariance of the entropy function (inherited from the formulation in terms of the superconformal action) we can gauge away w, and identify $(X^I, W^2) \sim (x^I, \hat{A})$. However, after that the entropy function still depends on the relative magnitude of the variables v_1 and v_2, describing correspondingly the radii squared of AdS^2 and S_2 factors in the black hole near horizon geometry, and there is no such parameters in (5.3). Therefore, in order to match with the macroscopic computations on the supergravity side we need a modification of the topological string depending on an additional parameter. Moreover because of the observations of [50,54] this extension of topological string should be computing additional higher derivative corrections, including extra $\mathcal{R}^2$ terms. These observations naturally lead to our conjecture in the next section.

7. A Conjecture

In the last section we saw that we need a one parameter extension of topological string which captures non-antiself-dual $4d$ geometries, for higher derivative corrections for non-supersymmetric black holes. In fact on the topological string side there is a natural candidate that can be used for this purpose: a one parameter extension of the topological string that appeared in the works of Nekrasov [61,74,75,76,77,78] on instanton counting in Seiberg-Witten theory. There, a function $F(X^I, \epsilon_1, \epsilon_2)$ was introduced. In the special limit $-\epsilon_2 = \epsilon_1 = g_{\text{top}}$ this function reduces to the ordinary topological string free energy (6.5) according to

$$F(X^I, \epsilon_1, \epsilon_2)\big|_{\epsilon_1 + \epsilon_2 = 0} = F_{\text{top}}(X^I, g_{\text{top}}), \qquad g_{\text{top}}^2 = -\epsilon_1 \epsilon_2, \qquad (7.1)$$

In order to make a connection with the supergravity ansatz (6.12) we will need to identify the parameters as

$$\epsilon_1 = \frac{16}{|w|^2 v_1}, \quad \epsilon_2 = -\frac{16}{|w|^2 v_2}. \tag{7.2}$$

This is consistent with the fact that the field theory limit $\epsilon_{1,2} \to 0$ in the Nekrasov's approach corresponds to the flat space approximation in the ansatz (6.12).

Since the Nekrasov's extension of the topological string may not be familiar, we will first review the necessary background from [61,77,79]. Then we will be able to make a proposal about the corresponding generalization of the OSV formula.

7.1. Review of the Nekrasov's extension of the topological string

The instanton corrections to the prepotential of $\mathcal{N} = 2$ gauge theory can be computed by a powerful application of localization technique introduced by Nekrasov [61]. This localization, in the physical context gets interpreted as turning on non-antiself-dual graviphoton background,

$$T = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4. \tag{7.3}$$

This reproduces the $\mathcal{N} = 2$ prepotential by considering the most singular term as $\epsilon_i \to 0$, which scales as $F^{(0)}/\epsilon_1 \epsilon_2$. However there is more information in the localization computation of Nekrasov: One can also look at the subleading terms and identify their physical significance. For the case of $\epsilon_1 = -\epsilon_2$ there is a natural answer, as this gets mapped to the $\mathcal{N} = 2$ F-terms which capture (anti)-selfdual graviphoton corrections, of the type studied in [22,73]. In fact the two can get identified using geometric engineering of $\mathcal{N} = 2$ gauge theories [80,81] by considering, in the type IIA setup, a local Calabi-Yau given by ALE fibrations over some base space (e.g. $\mathbb{P}^1$). Thus Nekrasov's gauge theory computation leads, *indirectly*, to a computation of topological string amplitudes, upon the specialization $\epsilon_1 = -\epsilon_2 = g_{\text{top}}$:

$$\lim_{\epsilon_2 \to -\epsilon_1} F(X^I, \epsilon_1, \epsilon_2) = \sum_{g=0}^{\infty} (g_{\text{top}})^{2g-2} F^{(g)}(X^I), \quad g_{\text{top}} = \epsilon_1. \tag{7.4}$$

It has been checked [82,83,84,85] using the topological vertex formalism [86,87] that this indeed agrees with the direct computation of topological string amplitudes in such backgrounds, see also [88,89,90].

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However, it is clear that there is still more to the story: Nekrasov's computation has more information than the topological string in such backgrounds as it depends on an extra parameter, which is visible when $\epsilon_1 + \epsilon_2 \neq 0$. In fact Nekrasov's extension $F(X^I, \epsilon_1, \epsilon_2)$ satisfies the homogeneity condition

$$\left[\epsilon_1 \frac{\partial}{\partial \epsilon_1} + \epsilon_2 \frac{\partial}{\partial \epsilon_2} + X^I \frac{\partial}{\partial X^I} \right] F(X^I, \epsilon_1, \epsilon_2) = 0. \tag{7.5}$$

which means that it does depend on one extra parameter compared to the topological strings. Below we will use a shorthand notation

$$F(X, \epsilon) \equiv F(X^I, \epsilon_1, \epsilon_2). \tag{7.6}$$

Even though the exact effective field theory terms that $F(X, \epsilon)$ computes has not been worked out, it is clear from the derivation that it has to do with constant, non-antiself-dual configurations of graviphoton and Riemann curvature. The origin of first such correction has been identified in [79] which we will now review. In general one can expand $F(X, \epsilon)$ as follows [77,78,79]

$$F = \frac{1}{\epsilon_1 \epsilon_2} F^{(0)} + \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} H_{\frac{1}{2}} + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} G_1 + F^{(1)} + \mathcal{O}(\epsilon_1, \epsilon_2). \tag{7.7}$$

Let us discuss a geometrical meaning of the genus one terms in (7.7). Recall a general relations

$$\frac{1}{32\pi^2} \int_{\mathcal{X}} \text{Tr} \mathcal{R} \wedge *\mathcal{R} = \chi, \qquad \frac{i}{32\pi^2} \int_{\mathcal{X}} \text{Tr} \mathcal{R} \wedge \mathcal{R} = \frac{3}{2}\sigma, \tag{7.8}$$

where χ is the Euler characteristic of a Euclidean 4-manifold $\mathcal{X}$ and σ is the signature. The curvature tensor $\mathcal{R}$ in (7.8) is viewed as a 2-form $\mathcal{R}^a{}_b = \mathcal{R}^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$ with values in Lie algebra of $SO(4)$. As is clear from (6.9), the ordinary topological strings compute contributions to the effective action of the form[11]

$$\frac{1}{16\pi^2} \int_{\mathcal{X}} F^{(1)}(X)\mathcal{R}_- \wedge \mathcal{R}_- + \text{higher genus} = \frac{1}{2} F^{(1)}(X)\left(\chi - \frac{3}{2}\sigma\right) + \text{higher genus}. \tag{7.9}$$

On the other hand, more general couplings to χ and σ can be seen in the Donaldson theory. As was explained by Witten [91], the low energy effective action of twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on an arbitrary four-manifold $\mathcal{X}$ contains terms proportional to χ and σ. The Donaldson invariant D_ξ in general has three contributions

$$D_\xi = Z_u + Z_+ + Z_-, \tag{7.10}$$

---

[11] there is of course a similar antiholomorphic contribution starting with $\overline{F}^{(1)}(\chi - \frac{3}{2}\sigma)$.

where $Z_\pm$ are Seiberg-Witten invariants defined via the moduli space of monopoles, and Z_u is non-zero when $b_+(\mathcal{X}) = 1$ and is given by the u-plane integral [92]

$$Z_u = \int_{u-\text{plane}} dad\bar{a} A(u)^\chi B(u)^\sigma e^{pu+S^2 T} \Psi. \tag{7.11}$$

As shown in [79], the functions A and B are related to genus one terms in (7.7) as

$$F^{(1)} = \ln A - \frac{2}{3} \ln B, \qquad G_1 = \frac{1}{3} \ln B \tag{7.12}$$

Note that the equivariant integral of the superfield $\Phi = \Phi^{(0)} + \Phi^{(1)}\theta + \ldots + \Phi^{(4)}\theta^4$ in the case $\mathcal{X} = \mathbb{C}^2$ is given by

$$\int_{\mathcal{X}} d^4x \int d^4\theta \Phi = \frac{\Phi^{(0)}(0)}{\epsilon_1 \epsilon_2}. \tag{7.13}$$

It is also instructive to write down [79] the equivariant Euler number and signature for $\mathbb{C}^2$:

$$\chi(\mathbb{C}^2) = \epsilon_1 \epsilon_2, \qquad \sigma(\mathbb{C}^2) = \frac{\epsilon_1^2 + \epsilon_2^2}{3}. \tag{7.14}$$

Let us introduce another notation:

$$\widetilde{F}^{(1)} = 4G_1 + F^{(1)}, \qquad G_1 = \frac{1}{4}(\widetilde{F}^{(1)} - F^{(1)}). \tag{7.15}$$

Then (7.7) can be rewritten as

$$\epsilon_1 \epsilon_2 F = F^{(0)} + (\epsilon_1 + \epsilon_2) H_{\frac{1}{2}} + \frac{1}{2}\left(\chi - \frac{3}{2}\sigma\right) F^{(1)} + \frac{1}{2}\left(\chi + \frac{3}{2}\sigma\right) \widetilde{F}^{(1)} + \epsilon_1 \epsilon_2 \mathcal{O}(\epsilon_1, \epsilon_2). \tag{7.16}$$

The term $\widetilde{F}^{(1)} = 4G_1 + F^{(1)}$ is not captured by the ordinary topological string!

Extra terms are needed to obtain a correct macroscopic entropy for non-supersymmetric black holes in addition to the standard terms computed by the topological strings [50,54]. In fact the particular term needed, which is discussed in [54] reduces, upon compactification to $4d$, to the term of the form $t \cdot \text{Tr}\mathcal{R} \wedge \mathcal{R}$ for large t, where t is the overall Kähler moduli of the CY. Such a correction is indeed captured by the leading behavior of $G_1(t)$ for large t, as follows from (7.12). This gives us further confidence about the relevance of Nekrasov's extension of topological strings for a correct accounting of the non-supersymmetric black hole entropy.

In general, as pointed out in [84] one would expect that implementation of Nekrasov's partition function for general Calabi-Yau will mix hypermultiplet and vector multiplets. The case studied in [61] involved the case where there were no hypermultiplets so the question of mixing does not arise. In the context of the conjecture in the next section, this would suggest that higher derivative corrections may also fix the vevs for the hypermultiplet moduli in the context of non-supersymmetric black holes.

We now turn to a minimal conjecture for extremal black hole entropy which uses Nekrasov's extension of topological strings.

7.2. Minimal ϵ-deformation

Let us start with a semiclassical expression (5.6) for the $\mathcal{G}^{(0)}$-function

$$\mathcal{G}^{(0)} = \frac{1}{2}(P^I - X^I)(P^J - X^J)\overline{F}_{IJ}^{(0)} + (P^I - X^I)F_I^{(0)} + F^{(0)}, \qquad (7.17)$$

where $F^{(0)} = F^{(0)}(X)$ is the Calabi-Yau prepotential, identified with genus zero topological string free energy, and $P^I = p^I + \frac{i}{\pi}\phi^I$. Our goal is to find an ϵ-deformation $\mathcal{G}^{(0)} \to \mathcal{G}$ of (7.17), such that corresponding extremum equations

$$\frac{\partial \mathrm{Im}\mathcal{G}}{\partial \epsilon_1} = \frac{\partial \mathrm{Im}\mathcal{G}}{\partial \epsilon_2} = \frac{\partial \mathrm{Im}\mathcal{G}}{\partial X^I} = 0 \qquad (7.18)$$

still admit a supersymmetric attractor solution

$$\epsilon_1 = 1, \quad \epsilon_1 + \epsilon_2 = 0, \quad X^I = P_\epsilon^I = p^I + \frac{i}{\pi}\phi^I, \qquad (7.19)$$

and the extremum value of $\mathrm{Im}\mathcal{G}$ computed using this solution is such that it describes correctly corresponding contribution [4] to the supersymmetric black hole entropy

$$-\mathrm{Im}\mathcal{G}_{\mathrm{susy}}(p^I, \phi^I) = -\mathrm{Im}F\big(p^I + \frac{i}{\pi}\phi^I, 256\big) = 2\mathrm{Re}F_{\mathrm{top}}\big(p^I + \frac{i}{\pi}\phi^I\big). \qquad (7.20)$$

We will obtain this deformation of $\mathcal{G}$-function in two steps. First, we will use Nekrasov's refinement of the topological string to deform the prepotential as

$$F^{(0)}(X) \; \to \; F(X^I, \epsilon_1, \epsilon_2), \qquad (7.21)$$

and at the same time, motivated from [50], deform the complexified magnetic charge as[12]

$$P^I \; \to \; P_\epsilon^I = -\epsilon_2 p^I + \frac{i}{\pi}\epsilon_1 \phi^I. \qquad (7.22)$$

Second, in order to satisfy conditions (7.18)-(7.20) after the deformation (7.21)-(7.22), we will need to add some compensating terms to $\mathcal{G}$. As we will see, there is some freedom in choosing these terms, but there is a minimal choice that does the job.

At the first step, after substituting (7.21)-(7.22) directly into (7.17), we obtain

$$\widetilde{\mathcal{G}} = \frac{1}{2}\big(P_\epsilon^I - X^I\big)\big(P_\epsilon^J - X^J\big)\overline{F}_{IJ}(\overline{X}, \overline{\epsilon}) + \big(P_\epsilon^I - X^I\big)F_I(X, \epsilon) + F(X, \epsilon). \qquad (7.23)$$

---

[12] When $\epsilon_2 = -\epsilon_1$, this is just a rescaling of P^I, while general deformation with $\epsilon_2 \neq -\epsilon_1$ involves a change of the complex structure in $H^3(M, \mathbb{C})$.

This, however, is not the full answer, since the derivatives of $\text{Im}\widetilde{\mathcal{G}}$ with respect to ϵ-parameters are not zero on the supersymmetric solution (7.19). This can be corrected at the second step, by adding to $\widetilde{\mathcal{G}}$ two terms, proportional to $\epsilon_1 + \epsilon_2$, so that the value (7.20) of the potential is not affected when $\epsilon_1 + \epsilon_2 = 0$. This leads to the following minimal ϵ-deformation

$$
\boxed{
\begin{aligned}
\mathcal{G} = {} & \frac{1}{2}\left(P_\epsilon^I - X^I\right)\left(P_\epsilon^J - X^J\right)\overline{F}_{IJ}(\overline{X},\overline{\epsilon}) + \left(P_\epsilon^I - X^I\right)F_I + F(X,\epsilon) + \\
& + \frac{1}{2}(\epsilon_1 + \epsilon_2)\overline{X}^I F_I - \frac{1}{2}(\epsilon_1 + \epsilon_2)\left(\epsilon_1 \partial_{\epsilon_1} - \epsilon_2 \partial_{\epsilon_2}\right)F(X,\epsilon)
\end{aligned}
}
$$
(7.24)

We call (7.24) a minimal ϵ-deformation because we can also add to (7.24) any terms proportional to $(\epsilon_1 + \epsilon_2)^2$ without affecting conditions (7.18)-(7.20):

$$
\mathcal{G} \to \mathcal{G} + \mathcal{O}(\epsilon_1 + \epsilon_2)^2.
$$
(7.25)

It is straightforward to check, using the homogeneity condition (7.5) and the relations

$$
p^I = -\frac{1}{2\epsilon_2}(P_\epsilon^I + \overline{P}_\epsilon^I), \qquad \phi^I = -\frac{i\pi}{2\epsilon_1}(P_\epsilon^I - \overline{P}_\epsilon^I),
$$
(7.26)

which follow from the definition

$$
P_\epsilon^I = -\epsilon_2 p^I + \frac{i}{\pi}\epsilon_1 \phi^I,
$$
(7.27)

that the extremum equations (7.18) for (7.24) indeed admit a solution (7.19), which corresponds to a supersymmetric BPS black hole. Moreover, in this case (7.20) is also satisfied.

Expression $q_I \phi^I - \pi\text{Im}\mathcal{G}$ should be compared to the entropy function (6.16). Then our notations are related to those of [50] as follows. We identify

$$
\epsilon_1 = \frac{16}{|w|^2 v_1}, \qquad \epsilon_2 = -\frac{16}{|w|^2 v_2}.
$$
(7.28)

The supersymmetric attractor equations of [50] read $p^I = -\frac{i}{4}v_2(\overline{w}x^I - w\overline{x}^I)$, while in our conventions the supersummetric case is $p^I = \text{Re}X^I$. Therefore,

$$
X^I = -\frac{i}{2}\overline{w}x^I, \qquad x^I = \frac{2i}{\overline{w}}X^I.
$$
(7.29)

We also set in this case

$$
w\overline{w} = 16, \qquad v_1 = v_2 = 1.
$$
(7.30)

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7.3. Putting it all together

Now we are ready to make a proposal about the extremal black holes entropy. We want to write down a generalization of the semiclassical expression for the extremal black hole degeneracy from section 5, that would reduce to the OSV formula (1.2) for the supersymmetric charge vector (p^I, q_I). The expression (7.24) for the deformed black hole potential provides a natural way to do this, and allows to treat supersymmetric and non-supersymmetric cases simultaneously.

We introduce a function $\mathcal{G} = \mathcal{G}(p, \phi; X, \epsilon)$ defined by

$$
\begin{aligned}
\mathcal{G} = &\frac{1}{2}\left(P_\epsilon^I - X^I\right)\left(P_\epsilon^J - X^J\right)\overline{F}_{IJ}(\overline{X}, \overline{\epsilon}) + \left(P_\epsilon^I - X^I\right)F_I(X, \epsilon) + F(X, \epsilon) + \\
&+ \frac{1}{2}(\epsilon_1 + \epsilon_2)\overline{X}^I F_I(X, \epsilon) - \frac{1}{2}(\epsilon_1 + \epsilon_2)\left(\epsilon_1 \partial_{\epsilon_1} - \epsilon_2 \partial_{\epsilon_2}\right)F(X, \epsilon) + \mathcal{O}(\epsilon_1 + \epsilon_2)^2,
\end{aligned}
\tag{7.31}
$$

where $\mathcal{O}(\epsilon_1 + \epsilon_2)^2$ denotes an ambiguity that cannot be fixed just by requiring that $\mathrm{Im}\mathcal{G}$ gives correct description of the supersymmetric black holes. In the minimal deformation case we set $\mathcal{O}(\epsilon_1 + \epsilon_2)^2 = 0$. In general, there are two types of solutions to the extremum equations

$$
\frac{\partial}{\partial X^I}\mathrm{Im}\mathcal{G} = \frac{\partial}{\partial \epsilon_i}\mathrm{Im}\mathcal{G} = 0,
\tag{7.32}
$$

the supersymmetric one (7.19) $X^I = p^I + \frac{i}{\pi}\phi^I$, and non-supersymmetric ones (all other). Let us denote the functions obtained by substituting these non-supersymmetric solutions $X^I = X^I(p, \phi)$, $\epsilon_{1,2} = \epsilon_{1,2}(p, \phi)$ into (7.31) as $\mathcal{G}(p^I, \phi^I)$. For supersymmetric solution $\mathcal{G}_{\mathrm{susy}}(p^I, \phi^I) = F(p^I + \frac{i}{\pi}\phi^I)$. We conjecture the following relation for the extremal black hole degeneracy

$$
\boxed{\Omega_{\mathrm{extrm}}(p^I, q_I) = \int d\phi^I e^{q_I \phi^I}\left(\left|e^{\frac{i\pi}{2}F(p^I + \frac{i}{\pi}\phi^I)}\right|^2 + \sum_{n-\mathrm{susy}}\left|e^{\frac{i\pi}{2}\mathcal{G}(p^I, \phi^I)}\right|^2\right),}
\tag{7.33}
$$

which is expected to be valid asymptotically in the limit of large charges. The sum in (7.33) runs over all non-supersymmetric solutions to the extremum equations (7.32). However, it is expected that for a given set of charges (p^I, q_I) only one solution (supersymmetric or non-supersymmetric, depending on the value of the discriminant) dominates, and contributions from all other solutions, including the ones with non-positive Hessian, are exponentially suppressed.

As noted before, it is expected that for general non-toric Calabi-Yau compactifications, which lead to hypermultiplets, the analog of Nekrasov's partition function would mix hypermultiplets with vector multiplets and therefore will fix their values at the horizon. This would be interesting to develop further.

8. Conclusions and Further Issues

We studied the black hole potential describing extremal black hole solutions in $\mathcal{N} = 2$ supergravity and found a new formulation of the semi-classical attractor equations, utilizing homogeneous coordinates on the Calabi-Yau moduli space. This allowed us to solve the inverse problem (that is, express the black hole charges in terms of the attractor Calabi-Yau moduli) completely in the one-modulus Calabi-Yau case. We found three non-supersymmetric solutions in addition to the supersymmetric one. In the higher dimensional case we found a bound $\#_{\mathrm{n-susy}} \leq 2^{n_V+1} - 1$ on the possible number of non-supersymmetric solutions to the inverse problem.

We then investigated a generalization of the attractor equations and OSV formula in the case when other corrections are turned on. We conjectured that corresponding corrected extremal black hole entropy needs an additional ingredient: the Nekrasov's extension of the topological string free energy $F(X^I, \epsilon_1, \epsilon_2)$. We related this to the black hole entropy using a minimal deformation conjecture given in (7.24),(7.33), that reduces to $F_{\mathrm{top}}(X^I + \frac{i}{\pi}\phi^I)$ for the choice of the black hole charges that support a supersymmetric solution. We were unable to fix the $\mathcal{O}(\epsilon_1 + \epsilon_2)^2$ ambiguity in (7.31), though it could be that the minimal conjecture is correct.

One important open question is how to test our conjecture. One possible test may be using the local Calabi-Yau geometry for which Nekrasov's partition function is known. Another important question is to find out what is exactly computed by Nekrasov's partition function[13] and how to extend it to the case where there are both hypermultiplets and vector multiplets. Clearly there is a long road ahead. We hope to have provided strong evidence that Nekrasov's extension of topological string is a key ingredient in a deeper understanding of non-supersymmetric black holes.

---

[13] for example, in the $AdS_2 \times S^2$ setup of [50], the ϵ-parameters corresponding to the radii of AdS_2 and S^2 factors were real, but from the topological string viewpoints it is natural to consider a complexification of $\epsilon_{1,2}$. This suggests that there should exist corresponding deformation of the $AdS_2 \times S^2$ near horizon geometry.

Appendix A. The Diagonal Torus Example

Consider the case [26] when $M = T^6$ is the so-called diagonal torus:

$$M = \Sigma_\tau \times \Sigma_\tau \times \Sigma_\tau, \tag{A.1}$$

where $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is the elliptic curve with modular parameter τ. Let us introduce complex coordinates $dz^i = dx^i + \tau dy^i, i = 1, 2, 3$ on each Σ_τ. As in [27] can label the relevant 3-cycles of M according to their mirror branes in IIA picture:

$$
\begin{aligned}
D0 \rightarrow & - dy^1 dy^2 dy^3 \\
D2 \rightarrow & dy^1 dy^2 dx^3 + dy^1 dx^2 dy^3 + dx^1 dy^2 dy^3 \\
D4 \rightarrow & dx^1 dx^2 dy^3 + dx^1 dy^2 dx^3 + dy^1 dx^2 dx^3 \\
D6 \rightarrow & - dx^1 dx^2 dx^3
\end{aligned} \tag{A.2}
$$

The intersection matrix of these 3-cycles is

$$
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 3 & 0 \\
0 & -3 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}. \tag{A.3}
$$

We denote the brane charge vector as $(D0, D2, D4, D6) = (u, q, p, v)$. Then

$$\mathcal{W} = u + 3q\tau - 3p\tau^2 - v\tau^3. \tag{A.4}$$

The black hole potential is

$$V_{\mathrm{BH}} = e^K \left(|\mathcal{W}|^2 + g^{\tau\bar{\tau}} |\partial\mathcal{W} + \mathcal{W}\partial K|^2 \right) \tag{A.5}$$

where

$$K \sim \log(\mathrm{Im}\tau)^3, \qquad g^{\tau\bar{\tau}} = \frac{3}{4(\mathrm{Im}\tau)^2}. \tag{A.6}$$

Therefore, we have

$$V_{\mathrm{BH}} = \frac{8}{(\mathrm{Im}\tau)^3}\left(\left|u+3q\tau-3p\tau^2-v\tau^3\right|^2 + 3\left|2i\mathrm{Im}\tau(q-2p\tau-v\tau^2)-u-3q\tau+3p\tau^2+v\tau^3\right|^2\right). \tag{A.7}$$

A.1. Solution of the inverse problem

Let us decompose τ into the real and imaginary parts

$$\tau = \tau_1 + i\tau_2, \tag{A.8}$$

and introduce new variables α, β, γ that are real linear combination of the charges

$$\begin{aligned}
\alpha &= \mathcal{W}\big|_{\tau_2=0} = u + 3q\tau_1 - 3p\tau_1^2 - v\tau_1^3, \\
\beta &= \frac{1}{3}\frac{\partial\mathcal{W}}{\partial\tau}\bigg|_{\tau_2=0} = q - 2p\tau_1 - v\tau_1^2, \\
\gamma &= -\frac{1}{6}\frac{\partial^2\mathcal{W}}{\partial\tau^2}\bigg|_{\tau_2=0} = p + v\tau_1.
\end{aligned} \tag{A.9}$$

Using (A.9), we can rewrite the superpotential (A.4) as

$$\mathcal{W} = \alpha + 3i\beta\tau_2 + 3\gamma\tau_2^2 + iv\tau_2^3. \tag{A.10}$$

Then (4.1) gives

$$\begin{aligned}
\alpha + 3\gamma\tau_2^2 &= \omega_1, \\
3\beta\tau_2 + v\tau_2^3 &= \omega_2,
\end{aligned} \tag{A.11}$$

where $\omega = \omega_1 + i\omega_2$. The black hole potential (2.6) in new variables is given by

$$V_{\mathrm{BH}} = \frac{32}{\tau_2^3}\left(\alpha^2 + 3\beta^2\tau_2^2 + 3\gamma^2\tau_2^4 + v^2\tau_2^6\right) \tag{A.12}$$

The extremum equations $\frac{\partial V_{\mathrm{BH}}}{\partial\tau_1} = \frac{\partial V_{\mathrm{BH}}}{\partial\tau_2} = 0$ take the form:

$$\begin{aligned}
\alpha\beta - 2\beta\gamma\tau_2^2 + v\gamma\tau_2^4 &= 0, \\
-\alpha^2 - \beta^2\tau_2^2 + \gamma^2\tau_2^4 + v^2\tau_2^6 &= 0,
\end{aligned} \tag{A.13}$$

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If we express α and v in terms of β and γ using (A.11)

$$\alpha = \omega_1 - 3\gamma\tau_2^2, \qquad v = \frac{\omega_2 - 3\beta\tau_2}{\tau_2^3}, \tag{A.14}$$

assuming $\tau_2 \neq 0$, the first equation in (A.13) gives

$$\beta = \frac{\gamma\omega_2\tau_2}{8\tau_2^2\gamma - \omega_1}. \tag{A.15}$$

Here we also assumed that $8\tau_2^2\gamma \neq \omega_1$. We will discuss this special case later. The second equation in (A.13) then takes the form

$$\left(4\tau_2^2\gamma - \omega_1\right)\left(128\tau_2^6\gamma^3 - 96\tau_2^4\gamma^2\omega_1 + 18\tau_2^2\gamma\omega_1^2 - 6\tau_2^2\gamma\omega_2^2 + \omega_1\omega_2^2 - \omega_1^3\right) = 0. \tag{A.16}$$

We immediately see that $\gamma = \frac{\omega_1}{4\tau_2^2}$, and therefore

$$\alpha = \frac{\omega_1}{4}, \qquad \beta = \frac{\omega_2}{4\tau_2}, \qquad \gamma = \frac{\omega_1}{4\tau_2^2}, \qquad v = \frac{\omega_2}{4\tau_2^3}, \tag{A.17}$$

gives a solution to (A.13). In fact, it describes a supersymmetric branch of the extremum equations (2.7). The cubic equation for γ in (A.16) has three non-susy solutions that can be described by the formula:

$$\gamma = \frac{2\mathrm{Re}(\omega) + |\omega|\left(|\omega|/\omega\right)^{1/3} + |\omega|\left(|\omega|/\omega\right)^{-1/3}}{8\tau_2^2}, \tag{A.18}$$

where one can choose any of three cubic root branches. It is obvious that all solutions (A.18) are real. Correspondingly, in this case

$$\begin{aligned}
\alpha =&\frac{1}{4}\mathrm{Re}(\omega) - \frac{3}{8}|\omega|\left(|\omega|/\omega\right)^{1/3} - \frac{3}{8}|\omega|\left(|\omega|/\omega\right)^{-1/3}, \\
\beta =&\frac{\mathrm{Im}(\omega)}{8\tau_2} \cdot \frac{2\mathrm{Re}(\omega) + |\omega|\left(|\omega|/\omega\right)^{1/3} + |\omega|\left(|\omega|/\omega\right)^{-1/3}}{\mathrm{Re}(\omega) + |\omega|\left(|\omega|/\omega\right)^{1/3} + |\omega|\left(|\omega|/\omega\right)^{-1/3}}, \\
\gamma =&\frac{2\mathrm{Re}(\omega) + |\omega|\left(|\omega|/\omega\right)^{1/3} + |\omega|\left(|\omega|/\omega\right)^{-1/3}}{8\tau_2^2}, \\
v =&\frac{\mathrm{Im}(\omega)}{8\tau_2^3} \cdot \frac{2\mathrm{Re}(\omega) + 5|\omega|\left(|\omega|/\omega\right)^{1/3} + 5|\omega|\left(|\omega|/\omega\right)^{-1/3}}{\mathrm{Re}(\omega) + |\omega|\left(|\omega|/\omega\right)^{1/3} + |\omega|\left(|\omega|/\omega\right)^{-1/3}}.
\end{aligned} \tag{A.19}$$

It is instructive to compute the values of the black hole potential (A.12) at the three non-supersymmetric extremal points (A.19). Using the second equation in (A.13), we obtain

$$V_{\mathrm{BH}} = \frac{64}{\tau_2}\left(\beta^2 + 2\gamma^2\tau_2^2 + v^2\tau_2^4\right). \tag{A.20}$$

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If we apply (A.14), after some algebra we find

$$\beta^2 + 2\gamma^2\tau_2^2 + v^2\tau_2^4 = \frac{128\tau_2^8\gamma^4 - 32\tau_2^6\gamma^3\omega_1 + 2\tau_2^4\gamma^2\omega_1^2 + 26\tau_2^4\gamma^2\omega_2^2 - 10\tau_2^2\gamma\omega_1\omega_2^2 + \omega_1^2\omega_2^2}{\tau_2^2(8\tau_2^2\gamma - \omega_1)^2} =$$

$$= \frac{\omega_1^2 + \omega_2^2}{2\tau_2^2} + \frac{\tau_2^2\gamma + \omega_1/2}{\tau_2^2(8\tau_2^2\gamma - \omega_1)^2}\left(128\tau_2^6\gamma^3 - 96\tau_2^4\gamma^2\omega_1 + 18\tau_2^2\gamma\omega_1^2 - 6\tau_2^2\gamma\omega_2^2 + \omega_1\omega_2^2 - \omega_1^3\right).$$
(A.21)

The last term in the second line vanishes at the non-supersymmetric extremum point due to (A.16), and we get a simple formula for the potential

$$V_{\text{BH}}^{\text{n-susy}} = 32\frac{|\omega|^2}{\tau_2^3}.$$
(A.22)

Notice that the value of the potential is the same for all three points (A.19). At the supersymmetric extremum point (A.17) we have

$$V_{\text{BH}}^{\text{susy}} = 8\frac{|\omega|^2}{\tau_2^3},$$
(A.23)

so that, as in [40]

$$V_{\text{BH}}^{\text{n-susy}} = 4V_{\text{BH}}^{\text{susy}}.$$
(A.24)

Note that this relation is written in terms of Calabi-Yau moduli rather then in terms of the black hole charges.

As we will see in a moment, all three non-supersymmetric extremum points provide a minimum of the black hole potential. In order to show this, let us look at the Hessian

$$Hess(V_{\text{BH}}) = \begin{pmatrix} \frac{\partial^2 V_{\text{BH}}}{\partial\tau_1^2} & \frac{\partial^2 V_{\text{BH}}}{\partial\tau_1\partial\tau_2} \\ \frac{\partial^2 V_{\text{BH}}}{\partial\tau_2\partial\tau_1} & \frac{\partial^2 V_{\text{BH}}}{\partial\tau_2^2} \end{pmatrix}.$$
(A.25)

Straightforward computation gives

$$Hess(V_{\text{BH}}) = \frac{192}{\tau_2^3}\begin{pmatrix} 3\beta^2 - 2\alpha\gamma + (4\gamma^2 - 2\beta v)\tau_2^2 + v^2\tau_2^4 & 4\gamma\tau_2(-\beta + v\tau_2^2) \\ 4\gamma\tau_2(-\beta + v\tau_2^2) & -\beta^2 + 2\gamma^2\tau_2^2 + 3v^2\tau_2^4 \end{pmatrix}.$$
(A.26)

At the non-supersymmetic extremal point (A.19), using (A.14) and (A.16), we obtain the following expression

$$M = \frac{\tau_2^3}{96}Hess(V_{\text{BH}}) =$$

$$= \begin{pmatrix} \frac{96\tau_2^4\gamma^2(2\omega_1^2+\omega_2^2) - 8\tau_2^2\gamma\omega_1(6\omega_1^2+\omega_2^2) + 3\omega_1^4 - \omega_1^2\omega_2^2}{\tau_2^2(8\tau_2^2\gamma - \omega_1)^2} & \frac{8\gamma(4\tau_2^2\gamma - \omega_1)\omega_2}{8\tau_2^2\gamma - \omega_1} \\ \frac{8\gamma(4\tau_2^2\gamma - \omega_1)\omega_2}{8\tau_2^2\gamma - \omega_1} & \frac{32\tau_2^4\gamma^2(2\omega_1^2+5\omega_2^2) - 8\tau_2^2\gamma\omega_1(2\omega_1^2+7\omega_2^2) + \omega_1^4 + 5\omega_1^2\omega_2^2}{\tau_2^2(8\tau_2^2\gamma - \omega_1)^2} \end{pmatrix}$$
(A.27)

The eigenvalues $h_{1,2}$ of the matrix (A.27) are solutions to the equation

$$0 = \det \left\| M - \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right\| = h^2 - 4\frac{|\omega|^2}{\tau_2^2} h + 3\frac{|\omega|^4}{\tau_2^4} -$$

$$-\frac{8\omega_2^2(4\tau_2^2\gamma - \omega_1)(16\tau_2^4\gamma^2 + 4\tau_2^2\gamma\omega_1 - \omega_1^2)}{\tau_2^4(8\tau_2^2\gamma - \omega_1)^4}\left(128\tau_2^6\gamma^3 - 96\tau_2^4\gamma^2\omega_1 + 6\tau_2^2\gamma(3\omega_1^2 - \omega_2^2) + \omega_1\omega_2^2 - \omega_1^3\right)$$

(A.28)

The last line vanishes because of the extremum equation (A.16), and we get

$$h^2 - 4\frac{|\omega|^2}{\tau_2^2} h + 3\frac{|\omega|^4}{\tau_2^4} = 0. \tag{A.29}$$

Therefore, the eigenvalues of the matrix (A.27)

$$h_1 = \frac{|\omega|^2}{\tau_2^2} \geq 0$$

$$h_2 = 3\frac{|\omega|^2}{\tau_2^2} \geq 0$$

(A.30)

are always non-negative. Since $\tau_2 > 0$, this means that the eigenvalues of the Hessian (A.26) are also positive if $\omega \neq 0$, and thus the non-supersymmetric extremum points minimize the potential.

A.2. Solution of the direct problem

The black hole potential (A.7) is given by

$$V_{\mathrm{BH}} = \frac{4}{\tau_2^3}\big(u^2 + 6qu\tau_1 + 9q^2\tau_1^2 - 6pu\tau_1^2 - 18pq\tau_1^3 - 2uv\tau_1^3 + 9p^2\tau_1^4 -$$
$$- 6qv\tau_1^4 + 6pv\tau_1^5 + v^2\tau_1^6 + 3q^2\tau_2^2 - 12pq\tau_1\tau_2^2 + 12p^2\tau_1^2\tau_2^2 - \tag{A.31}$$
$$- 6qv\tau_1^2\tau_2^2 + 12pv\tau_1^3\tau_2^2 + 3v^2\tau_1^4\tau_2^2 + 3p^2\tau_2^4 + 6pv\tau_1\tau_2^4 + 3v^2\tau_1^2\tau_2^4 + v^2\tau_2^6\big).$$

Straightforward calculation gives

$$\frac{\partial V_{\mathrm{BH}}}{\partial \tau_1} = \frac{24}{\tau_2^3}\Big((q - 2p\tau_1 - v\tau_1^2)(u + 3q\tau_1 - 3p\tau_1^2 - v\tau_1^3) - 2(p + v\tau_1)(q - 2p\tau_1 - v\tau_1^2)\tau_2^2 + (p + v\tau_1)v\tau_2^4\Big) \tag{A.32}$$

and

$$\frac{\partial V_{\mathrm{BH}}}{\partial \tau_2} = \frac{12}{\tau_2^4}\Big(-(u + 3q\tau_1 - 3p\tau_1^2 - v\tau_1^3)^2 - (q - 2p\tau_1 - v\tau_1^2)^2\tau_2^2 + (p + v\tau_1)^2\tau_2^4 + v^2\tau_2^6\Big). \tag{A.33}$$

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The extremal points are solutions to the equations $\frac{\partial V_{\mathrm{BH}}}{\partial \tau_1} = \frac{\partial V_{\mathrm{BH}}}{\partial \tau_2} = 0$. From (A.32) we find that for a generic set of charges (assuming $v\gamma \neq 0$)

$$\tau_2^2 = \frac{\beta\gamma \pm \sqrt{\beta\gamma(\beta\gamma - v\alpha)}}{v\gamma}, \tag{A.34}$$

where

$$\begin{aligned}
\alpha &= u + 3q\tau_1 - 3p\tau_1^2 - v\tau_1^3, \\
\beta &= q - 2p\tau_1 - v\tau_1^2, \\
\gamma &= p + v\tau_1.
\end{aligned} \tag{A.35}$$

If we plug (A.34) into (A.33), we obtain

$$\gamma\sqrt{\beta\gamma - v\alpha}\left(\beta\sqrt{\beta\gamma}(v^2\alpha - 3v\beta\gamma - 2\gamma^3) \mp \gamma\sqrt{\beta\gamma - v\alpha}(3v\beta^2 + v\alpha\gamma + 2\beta\gamma^2)\right) = 0. \tag{A.36}$$

Let us look at the solution $\beta\gamma - v\alpha = 0$ first. Due to (A.35) this is equivalent to

$$\tau_1 = \frac{pq - uv}{2(p^2 + qv)} \tag{A.37}$$

Then (A.34) gives, assuming $\tau_2 > 0$

$$\tau_2 = \frac{\sqrt{-\mathcal{D}}}{2(p^2 + qv)} \tag{A.38}$$

where

$$\mathcal{D} = -\left(3p^2q^2 + 4p^3u + 4q^3v + 6pquv - u^2v^2\right). \tag{A.39}$$

This is the supersymmetric solution obtained in [26]. Note that there is no such solution if the discriminant (A.39) is positive: $\mathcal{D} > 0$.

The non-supersymmetric solution will emerge from the second branch:

$$\beta\sqrt{\beta\gamma}(v^2\alpha - 3v\beta\gamma - 2\gamma^3) = \pm\gamma\sqrt{\beta\gamma - v\alpha}(3v\beta^2 + v\alpha\gamma + 2\beta\gamma^2) \tag{A.40}$$

Without loss of generality we can take the square of this equation. Then, after plugging in (A.35) we find massive cancellations, and obtain the following cubic equation

$$\begin{aligned}
\left(2p^6 + 6p^4qv + 3p^2q^2v^2 - 4p^3uv^2 - 2q^3v^3 - 6pquv^3 + u^2v^4\right)\tau_1^3 - \\
-3(p^5q + 5p^3q^2v + 3p^4uv + 5pq^3v^2 + 4p^2quv^2 - q^2uv^3 - pu^2v^3)\tau_1^2 - \\
-3\left(p^4q^2 + 2p^5u + 2p^3quv - 2q^4v^2 - 2pq^2uv^2 - p^2u^2v^2\right)\tau_1 + \\
+\left(2p^3q^3 + 3p^4qu + 3pq^4v + 6p^2q^2uv + p^3u^2v + q^3uv^2\right) = 0.
\end{aligned} \tag{A.41}$$

40

The discriminant of this equation is equal to

$$\Delta = 729\mathcal{D}^3\left(p^2+qv\right)^6\left(2p^6+6p^4qv+3p^2q^2v^2-4p^3uv^2-2q^3v^3-6pquv^3+u^2v^4\right)^2. \quad \text{(A.42)}$$

Only one solution of this equation can be real, if $\mathcal{D} > 0$, which implies $\Delta > 0$, but this is exactly what we are looking for. It is given by

$$
\begin{aligned}
\tau_1 = {} & \frac{1}{(2(p^2+qv)^3+v^2\mathcal{D})}\Big((p^2+qv)^2(pq-uv)-vp\mathcal{D}- \\
& -\frac{2^{1/3}(p^2+qv)^3\mathcal{D}}{\left(v(2p^3+3pqv-uv^2)\mathcal{D}^2+(2(p^2+qv)^3+v^2\mathcal{D})\mathcal{D}\sqrt{\mathcal{D}}\right)^{1/3}}+ \\
& +\frac{p^2+qv}{2^{1/3}}\left(v(2p^3+3pqv-uv^2)\mathcal{D}^2+(2(p^2+qv)^3+v^2\mathcal{D})\mathcal{D}\sqrt{\mathcal{D}}\right)^{1/3}\Big).
\end{aligned}
\quad \text{(A.43)}
$$

Corresponding expression for τ_2 is obtained by substituting (A.43) into (A.34).

Appendix B. Cubic equation

Consider a general cubic equation of the form

$$ax^3+3bx^2-3cx-d=0. \quad \text{(B.1)}$$

The discriminant of this equation is

$$\Delta = -(3b^2c^2+4c^3a+4b^3d+6abcd-a^2d^2). \quad \text{(B.2)}$$

The solutions are given by

$$x_1 = -\frac{b}{a}+\frac{2^{1/3}(b^2+ac)}{a\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}}+\frac{\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}}{2^{1/3}a}, \quad \text{(B.3)}$$

$$x_2 = -\frac{b}{a}-\frac{2^{1/3}(1+i\sqrt{3})(b^2+ac)}{2a\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}}-\frac{(1-i\sqrt{3})}{2^{1/3}2a}\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3},$$
$$\text{(B.4)}$$

$$x_3 = -\frac{b}{a}-\frac{2^{1/3}(1-i\sqrt{3})(b^2-ac)}{2a\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}}-\frac{(1+i\sqrt{3})}{2^{1/3}2a}\left(a^2d-3abc-2b^3+a\sqrt{\Delta}\right)^{1/3}$$
$$\text{(B.5)}$$

We are interested in the case $\Delta > 0$, when there is one real root and a pair of complex conjugate roots.

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