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Citation
Street, Ethan J. 2012. Towards an Instanton Floer Homology for Tangles. Doctoral dissertation, Harvard University.

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Dissertation Advisor: Peter B. Kronheimer Author: Ethan J. Street

Towards an Instanton Floer Homology for Tangles

Abstract

In this thesis we investigate the problem of defining an extension of sutured instanton Floer homology to give an instanton invariant for a *tangle*. We do this in three separate steps. First, we investigate the representation variety of singular flat connections on a punctured Riemann surface Σ. Suppose Σ has genus g and that there are n punctures. We give formulae for the Betti numbers of the space $\mathcal{R}_{g,n}$ of flat SU(2)-connections on Σ with trace 0 holonomy around the punctures. By using a natural extension of the Atiyah-Bott generators for the cohomology ring $H^*(\mathcal{R}_{g,n})$, we are able to write down a presentation for this ring in the case $g = 0$ of a punctured sphere. This is accomplished by studying the intersections of Poincarè dual submanifolds for the new generators and reducing the calculation to a linear algebra problem involving the symplectic volumes of the representation variety.

We then study the related problem of computing the instanton Floer homology for a product link in a product 3-manifold

$$(Y_g, K_n) := (S^1 \times \Sigma, S^1 \times \{n \text{ pts}\}).$$

It is easy to see that the Floer homology of this pair, as a vector space, is essentially the same as the cohomology of $\mathcal{R}_{g,n}$, and so we set ourselves to determining a presentation for the natural algebra structure on it in the case $g = 0$. By leveraging a stable parabolic bundles calculation for $n = 3$ and an easier version of this Floer homology, $I_*(Y_0, K_n, u)$, we are able to write down a complete presentation for the Floer homology $I_*(Y_0, K_n)$ as a ring. We recapitulate somewhat the techniques in [**27**] in order to do this. Crucially, we deduce

that the eigenspace for the top eigenvalue for a natural operator $\mu^{\mathrm{orb}}(\Sigma)$ on $I_*(Y_0, K_n)$ is 1-dimensional.

Finally, we leverage this 1-dimensional eigenspace to define an instanton tangle invariant THI and several variants by mimicking the definition of sutured Floer homology SHI in [**22**]. We then prove this invariant enjoys nice properties with respect to concatenation, and prove a nontriviality result which shows that it detects the product tangle in certain cases.

Contents

Acknowledgements

The work contained in this thesis was carried out during the period beginning January 2011 and ending in February 2012 while the author was a graduate student in the Mathematics Department at Harvard University. It is necessary that he first and foremost acknowledge the completely indispensible role played by his advisor Prof. Peter B. Kronheimer in preparing these results. Prof. Kronheimer's careful guidance and keen insight into technical issues were essential at every step along the way. Indeed, the entire idea of the thesis - defining an instanton tangle invariant in the final chapter - is entirely due to him. For this and the countless hours and enormous patience the author would like to extend his sincerest thanks. In addition, he would like to thank his other thesis committee members, Profs. Clifford Taubes and Tomasz Mrowka, for their encouragement and guidance.

The author would also like to express his gratitude for the entire Harvard Mathematics Department, for the use of its resources, his own private office, and five years of free coffee, tea, and cookies. In particular, he would like to thank Susan Gilbert for her expert stewardship through the administrative details of being a graduate student. In addition, he would like to thank the department's support staff, especially Roberta Miller, without whom the goings-on at Harvard Math would grind to a screeching halt.

The author owes a great deal to the other graduate students, especially those in his entering class of 2007, for the positive influence they have had on his mathematics and overall sanity. He would like to thank Prof. Noam Elkies, for his endless mathematical enthusiasm and motivational influence which got the author to the gym on a regular basis. It would also be difficult to overstate the positive effect on the author's state of mind by the members of the Bender in Cambridge, MA. For years of laughter- and joy-filled Wednesday evenings, he would like to say thank you.

Last but not least, the author would like thank his parents for their support, both fiduciary and emotional, throughout the past five years, and for maintaining a healthy interest in his progress, but without asking too many questions.

CHAPTER 0

Introduction

Given a 3-manifold Y with link K and cohomology class ω in $H^2(Y, L; \mathbb{Z}/2)$ satisfying a non-integrality assumption, Kronheimer and Mrowka have defined a relatively $\mathbb{Z}/4$-graded instanton invariant $I_*(Y, K, \omega)$, which roughly speaking counts singular flat connections and instantons on Y and $\mathbb{R} \times R$, respectively [**23**]. This theory generalizes the original definition of instanton Floer homology for a homology 3-sphere or 3-manifold with odd U(2)-bundle (without link) due to Floer [**6**]. In this thesis we study the cohomology ring of a specific moduli space of flat connections on a punctured Riemann surface Σ, and subsequenctly the instanton Floer homology of a product 3-manifold $S^1 \times \Sigma$ with a product link $S^1 \times \{n \text{ pts}\}$ and class $\omega = 0$ or $\mathrm{PD}([S^1 \times \mathrm{pt}])$. We are able to completely compute the ring structures for both the cohomology ring and the Floer homology in the case that the genus of Σ is zero. We then use this computation in order to define a new invariant for a class of sutured 3-manifolds with embedded tangle and briefly study its properties.

We briefly review the sequence of events for the analogous story of a compact (not punctured) Rieman surface, product 3-manifold without link, and sutured manifold without tangle. Suppose that Σ is a compact, closed genus g Riemann surface. The moduli space $\mathcal{M}_g^0(2, 1)$ of flat connections on an odd U(2) bundle with fixed determinant modulo the determinant 1 gauge group is a compact $3g - 3$ complex dimensional Kähler manifold. Its cohomology ring was worked out completely in [**29**] and [**14**], the salient feature being a presentation in terms of canonical generators and a set of relations which are defined recursively in the genus g. Now, we consider the 3-manifold $S^1 \times \Sigma$ and cohomology class $w = \mathrm{PD}([S^1 \times \mathrm{pt}])$. As a vector space, the instanton Floer homology $I_*^w(S^1 \times \Sigma)$ is isomorphic to a direct sum of two copies of the cohomology $H^*(\mathcal{M}_g^0(2, 1))$. However, with respect to the natural ring structure on $I_*^w(S^1 \times \Sigma)$ obtained by applying the functoriality of I_* to a pair of pants cobordism from two copies of $S^1 \times \Sigma$ to itself, this isomorphism does not respect the multiplications on the two spaces. A presentation of $I_*^w(S^1 \times \Sigma)$ as a ring was worked out by Muñoz in [**27**], using results from the theory of Donaldson polynomials of 4-manifolds, and computations of those invariants for certain elliptic surfaces. A key result, as noted in [**22**], is that the eigenspace for the top eigenvalue $2g-2$ for multiplication by a natural generator in $I_*^w(S^1 \times \Sigma)$ is 1-dimensional. This is used in [**22**] to define an instanton invariant SHI for a class of sutured 3-manifolds. There it is proved, for example, that SHI detects whether a "taut" sutured 3-manifold is a product.

This thesis recapitulates this story in the singular or parabolic case. In Chapter 1, we study the moduli space $\mathcal{M}_{g,n}$ of flat connections on Σ which are singular at n punctures. For us, singular means the holonomy of the connection around one of the punctures gives a trace 0 element of SU(2). It is possible to study the more general case that the trace be some other number between ± 2, but our eventual goal is to apply our results to singular instanton Floer homology, where the holonomy is forced to be trace 0 for technical reasons (monotonicity requirements for the underlying infinite-dimensional Morse theory). We investigate the cohomology ring of $\mathcal{M}_{g,n}$ by finding a set of codimension 2 submanifolds Poincaré dual to a subcollection of natural generators and studying their intersections. In the case that $g = 0$, we are able to write down all the relations in the cohomology ring for these generators by solving a linear algebra problem involving the top pairings. We obtain a presentation for the ring $H^*(\mathcal{M}_{g,n})$, which consists of relations defined recursively in the number n (throughout, n must always be an odd number to avoid reducible connections).

In Chapter 2, we investigate the ring structure on the Floer homology

$$\mathbb{V}_{g,n} := I_*(S^1 \times \Sigma, S^1 \times \{n \text{ pts}\}, \emptyset),$$

which is isomorphic as a vector space to $H^*(\mathcal{M}_{g,n})$. We first review some of the salient features of the construction of the functor I_*, and define an extended version which generalizes relative Donaldson invariants to the singular case. Using the method in [**27**] of leveraging the inductive nature of $\mathbb{V}_{g,n}$ in the number of points n, we reduce the question to studying the eigenvalues of a natural operator $\mu^{\mathrm{orb}}([\Sigma])$, corresponding to multiplication by a particular generator. We effect a calculation in the case $g = 0$ and $n = 3$ by studying a 2-dimensional moduli space of stable parabolic holomorphic bundles on $S^2 \times S^2$. We then bootstrap by leveraging an easier version of the Floer homology for a product link in a product 3-manifold in order to obtain the eigenvalues for for $g = 0$ and all n. This is enough to write down a complete presentation of $\mathbb{V}_{0,n}$. It is then proved that the top eigenvalue for $\mu^{\mathrm{orb}}([\Sigma])$ has 1-dimensional (generalized) eigenspace, analogous to the original nonzingular case of $n = 0$ and $g > 1$.

In Chapter 3 we take advantage of this 1-dimensional subspace of $\mathbb{V}_{0,n}$ in order to define an invariant THI for a class of tangles in certain sutured 3-manifolds, and prove a non-triviality result which shows this invariant detects the product tangle. The crucial ingredient in the definition of THI is a generalization of the *excision* formula for I_* due to Floer [**6**] to the case of cutting and regluing along 2-spheres intersecting a link in an odd number of points. This generalization is analogous to the version of excision for cutting along genus g surfaces proved in [**22**]. We also define "unreduced" invariants $THI^\sharp$ and $THI^\sharp_{\mathrm{even}}$ which obey multiplicative laws for vertical concatenation and horizontal juxtaposition. Finally, we prove that $THI^\sharp$ detects the product tangle and product sutured manifold by comparing it to the sutured instanton Floer homology SHI, defined in [**22**], of a derived sutured manifold without tangle, and leveraging the analogous result for SHI.

CHAPTER 1

The Cohomology Ring of The Moduli Space of Flat Connections

1.1. Introduction

Given a compact Riemann surface Σ of genus g, it is well known that the moduli space of stable, rank 2 holomorphic bundles $\mathcal{E}$ with given fixed odd degree determinant line bundle L forms a smooth compact manifold $\mathcal{M}_g^0(2,1)$ of real dimension $6g-6$ [**1**]. The space $\mathcal{M}_g^0(2,1)$ is naturally a Kähler manifold, in fact a rational projective variety over $\mathbb{C}$, and has been studied extensively. This space can also be studied from the point of view of representations of the fundamental group of Σ into the group SU(2). Its cohomology ring has been studied in great detail and presentations for it have been given by several authors [**14, 29**].

In this chapter we are interested in the analgous problem for holomorphic vector bundles on an orbifold surface $\check{\Sigma}$. The relevant object of study is the moduli space of parabolic vector bundles over $\check{\Sigma}$. A parabolic vector bundle is a vector bundle on the (smoothed) surface Σ along with a filtration of the fibers over the orbifold points, and a weight value for each subspace in the filtration. There is a natural notion of a morphism and stability for such bundles and, under suitable genericity assumptions on the weights, a corresponding smooth, compact moduli space. It is the cohomology of this moduli space we study in the first part of this thesis.

For a surface of arbitrary genus g with n orbifold points and arbitary weights, this moduli space can be quite a bit more complicated than $\mathcal{M}_g^0(2,1)$. Our approach will be to use the classical theorem of Mehta and Seshadri [**25**] that it is isomorphic to a suitable moduli space of representations of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\check{\Sigma})$. We eventually wish to apply the results to a question in Floer homology, and the setup of that theory necessitates that we restrict attention here to the case that all the parabolic weights are $1/4$ and n is odd. In this case, the problem becomes more tractable. We denote the moduli space of parabolic bundles (with fixed determinant) in this special case $\mathcal{M}^0_{g,n}$. The theorem of Mehta and Seshadri shows that $\mathcal{M}^0_{g,n}$ is isomorphic to a certain representation variety $\mathcal{R}_{g,n}$ of $\pi_1^{\mathrm{orb}}(\check{\Sigma})$ into SU(2). We shall prove the following result for the genus 0 case:

THEOREM 1.1.1. *Let $n = 2m+1 \geq 3$. The cohomology ring $H^*(\mathcal{R}_{0,n};\mathbb{Q})$ is zero in odd dimensions, and is generated as a ring by the class α of twice the natural symplectic form on $\mathcal{R}_{0,n}$ and degree two classes $\delta_1,\ldots,\delta_n$ corresponding to the parabolic points, satisfying $\delta_j^2 = \delta_k^2$ for all j,k. Let $n = 2m+1$, define $\beta := \delta_k^2 \in H^4(\mathcal{M}_{0,n})$ and for each $m \geq 0$ define the polynomial $r_{0,2m+1}(\alpha,\beta)$ for $n \geq 3$ recursively via the relation:*

$$r_{0,2m+3} = \alpha \cdot r_{0,2m+1} - m^2\beta \cdot r_{0,2m-1}$$

with $r_{0,1} = 1$, $r_{0,3} = \alpha$. Then for each subset $J \subset \{1,\ldots,n\}$ with $|J| = s \leq m$, the polynomial

$$R^J_{0,2m+1} = r_{0,2m-2s+1} \cdot \prod_{k\in J} \delta_k$$

is a relation in $H^*(\mathcal{R}_{0,n};\mathbb{Q})$. This set of relations, along with $\delta_k^2 = \delta_j^2$ is complete.

We shall find it necessary to consider both points of view of the space $\mathcal{R}_{g,n}$, as a space of parabolic bundles, and a space of representations. As a go between, we will also give a description of this moduli space as a space of flat connections, where the natural symplectic form on it is easiest to describe. To get a handle on the topology of $\mathcal{R}_{g,n}$, we begin by determining its Betti numbers. Methods for computing the Poincaré polynomials of moduli spaces of parabolic bundles have been described by several authors. Hans Boden, in his thesis [**4**], provides a comprehensive account of an adaptation of Atiyah and Bott's equivariant Morse theory method from [**1**] to the parabolic case that achieves this. Using his exposition, we are able to read off a recursive formula in our case for the Betti numbers of $\mathcal{M}_{g,n}$.

In order to determine generators and relations for the cohomology ring, we adapt the method in [**32**], which can be summarized as follows. Lying over $\mathcal{R}_{0,n}$, there are natural U(1) bundles coming from each of the marked points. The Chern classes of these bundles, along with the class of a natural symplectic form on $\mathcal{R}_{0,n}$, fully generate the cohomology as a ring. These bundles are trivial outside certain real codimension two submanifolds given by representations where two generators are mapped to equal or opposite elements of SU(2). This allows us to identify representatives for the Poincaré duals of the first Chern classes of these circle bundles. Each of these submanifolds behaves like a 2-sphere bundle over a copy of the moduli space for two fewer parabolic points. We use this fact to analyze the intersection numbers of the corresponding homology classes recursively in the number n of parabolic points. We reduce the problem to linear algebra, and solve it by quoting a result from the theory of orthogonal polynomials and an identity involving the so called *Euler numbers*. This gives us relations in the cohomology ring. Once we write down the collection of relations in Theorem 1.1.1, a dimension counting argument similar to that appearing in [**29**] suffices to prove that they are a complete set.

1.2. Preliminaries

In this section we define the moduli spaces whose cohomology we are interested in. Specifically, we give the three points of view - flat connections, representations, and parabolic stable bundles - we shall need, and describe a canonical set of generators for the cohomology ring. The thesis of Hans Boden [**4**] gives an excellent introduction to the various points of view of the moduli spaces we are interested in. Our treatment of parabolic bundles comes from the discussion there, while our treatment of flat connections is modelled on the presentation in [**32**].

1.2.1. Moduli Spaces and the Symplectic Structure. We first describe the three moduli spaces and explain briefly the isomorphisms between them. We then define the symplectic structure on the moduli space of flat connections.

Stable Parabolic Bundles. The notion of a parabolic vector bundle on a Riemann surface really arises from considering holomorphic orbifold bundles on an compact orbifold Riemann surface $\check{\Sigma}$. By smoothing, such an orbifold can be viewed as simply a smooth Riemann surface Σ, with cone angles specified at each of several marked points $x_1, \ldots, x_n$. One can also "smooth" a holomorphic orbifold bundle and this process naturally yields a standard holomorphic bundle along with a filtration and corresponding sequence of numerical weights at each marked point. It turns out that the moduli space of *stable* parabolic bundles on Σ (suitably defined) with fixed specified weights is isomorphic to a representation variety of the orbifold fundamental group of $\check{\Sigma}$ into a Lie group. In this paper we will eventually only be concerned with the representation theoretic point of view, keeping in mind applications to the algebraic and analytic sides of the story as motivation. The parabolic bundle picture is, however, indispensible, as it is the origin of the canonical cohomology classes on moduli space and will pop up several times in later sections. We review both sides of the story pertaining to the moduli space we are interested.

Our starting point is the compact Riemann surface Σ of genus g with the n marked points $x_1, \ldots, x_n$.

Definition 1.2.1. A *parabolic* vector bundle $\mathcal{E}$ on Σ with respect to the marked points is holomorphic vector bundle along with the data of a descending filtration at each marked point x_k:

$$E_{x_k} = F_k^{(0)} \supset F_k^{(0)} \supset \cdots \supset F_k^{(s_k)}$$

and weights $t_k^{(0)} < \ldots < t_k^{(s_k)}$ in $[0, 1]$.

The notions of subbundles, quotients, morphisms, and exact sequences all have parabolic analogs, which are generalizations of the normal concepts with additional coherence conditions with respect to the weighted filtrations. There is also a notion of parabolic degree, which is simply the degree plus the sum of the weights at all the marked points counted with multiplicity according to the size of the quotients $F_k^{(j)}/F_k^{(j+1)}$. We define the parabolic slope $\mu^{\mathrm{par}}(\mathcal{E})$ to be the quotient $\deg^{\mathrm{par}}(\mathcal{E})/\operatorname{rk}(\mathcal{E})$. The bundle $\mathcal{E}$ is then called *stable* if for any proper parabolic subbundle $\mathcal{F}$, one has $\mu^{\mathrm{par}}(\mathcal{F}) < \mu^{\mathrm{par}}(\mathcal{E})$ (for definitions and facts regarding parabolic bundles and stability, we refer the reader to §3 of [**4**]). For a suitable (that is, generic) fixed collection of weights, there is a smooth, compact moduli space of stable parabolic bundles. In this paper, we are concerned only with the case when $\mathcal{E}$ is rank two, and it will be convenient to assume that $\deg(\mathcal{E}) = 1$. There are then only two weights $t_k^{(0)}$

and $t_k^{(1)}$ at each marked point, and the filtration consists of a choice of line $F_k \subset \mathcal{E}_{x_k}$. We also make the additional assumption that the sum $t_k^{(0)} + t_k^{(1)} = 1$, and set $t_k = t_k^{(0)}$. Denote the moduli space in this case by $\mathcal{M}_{g,n}(t_1, \ldots, t_n)$. Finally, for a given fixed degree one line bundle $\mathcal{L}$ (non-parabolic), there is a subset of the moduli space $\mathcal{M}_{g,n}^{\mathcal{L}}(t_1, \ldots, t_n)$ consisting of those bundles whose underlying determinant bundle is isomorphic to $\mathcal{L}$. When the weights are understood, we will simply write $\mathcal{M}_{g,n}$ for the full moduli space and since the choice of particular $\mathcal{L}$ will be inconsequential, we use $\mathcal{M}_{g,n}^0$ to denote the fixed determinant space.

Let us briefly describe a construction of the moduli space of parabolic bundles. This will be convenient when we discuss generators for its cohomology. We fix a C^∞ bundle E of rank two over Σ with $c_1(E) = 1$, and fixed parabolic data at each of the n marked points; for us this just means a 1-dimensional subspace $F_k \subset E_{x_k}$ at the fiber over each x_k (recall we have fixed the weights to be $1/4$, so these need not be specified). A holomorphic structure on E is uniquely specified by a partial connection operator $\overline{\partial}_E : \Omega^0(\Sigma) \otimes E \to \Omega^{0,1}(\Sigma) \otimes E$. Denote by $\mathcal{C}$ the space of holomorphic structures on Σ. It is an infinite dimensional affine space modeled on the vector space $\Omega^{0,1}(\Sigma) \otimes E$ of $(0,1)$-forms on Σ with values in E. Inside of $\mathcal{C}$ is the space $\mathcal{C}^{\mathrm{s}}$, the space of holomorphic structures for which, along with the parabolic data we have fixed, the corresponding parabolic holomorphic bundles are stable. Two natural spaces of automorphisms act on $\mathcal{C}$. We denote the first by $\mathcal{G}^{\mathrm{c}}$. It is the natural complex analog of our $\mathcal{G}$ for flat connections, and consists of all smooth complex automorphisms of the bundle E of determinant 1, ignoring the parabolic structure. The other is the subset of those automorphisms preserving the parabolic structure, the "parabolic gauge group":

$$\mathcal{G}^{\mathrm{c}}_{\mathrm{par}} = \{u \in \mathrm{SL}_{\mathbb{C}}(E) \colon u \text{ is smooth, and } u(F_k) = F_k \text{ for all } k\}.$$

Since elements of this automorphism group fix the parabolic filtration, $\mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$ preserves the subspace $\mathcal{C}^{\mathrm{s}} \subset \mathcal{C}$. The quotient $\mathcal{C}^{\mathrm{s}}/\mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$ is then the space of isomorphism classes of stable parabolic bundles, which we call $\mathcal{M}_{g,n}$. Within $\mathcal{G}^{\mathrm{c}}_{\mathrm{par}} \subset \mathcal{G}^{\mathrm{c}}$ there is the subgroup of constant automorphisms, which is a copy of $\mathbb{C}^\times$. These act trivially on holomorphic structures and so we denote the quotients of our two gauge groups by this $\mathbb{C}^\times$ by $\overline{\mathcal{G}}^{\mathrm{c}}$ and $\overline{\mathcal{G}}^{\mathrm{c}}_{\mathrm{par}}$ (the "reduced" gauge groups). It is consequence of the stability condition that $\overline{\mathcal{G}}^{\mathrm{c}}_{\mathrm{par}}$ acts freely on $\mathcal{C}^{\mathrm{s}}$.

The space $\mathcal{M}_{g,n}$ admits a map det to the space $\mathrm{Pic}^1(\Sigma)$ of isomorphism classes of degree one line bundles, which is a torsor for the Jacobian variety $J(\Sigma)$. Fixing a line bundle $\mathcal{L} \in \mathrm{Pic}^1(\Sigma)$, denote the fiber of det over $\mathcal{L}$ by $\mathcal{M}^0_{g,n}$ (the moduli space with fixed determinant). The Jacobian $J(\Sigma)$ also acts on $\mathcal{M}_{g,n}$ by tensor product and restricting this action gives a map

$$\mathcal{M}^0_{g,n} \times J(\Sigma) \to \mathcal{M}_{g,n}$$

which is a (connected, so nontrivial) degree 4^g covering.

Representation Varieties. Denote by Σ^* the noncompact surface obtained by removing the x_k's. Let $a_1, \ldots, a_{2g}$ be standard set of loops in Σ generating the fundamental group so that their homology classes are a symplectic basis for $H_1(\Sigma)$ (that is, a_j pairs with a_{j+1}) and for each removed point x_k let d_k denote a simple loop going once around the puncture. To get a relationship between representations of the fundamental group of Σ^* and parabolic bundles with an *odd* first Chern class, we look at the $\mathbb{Z}/2$ central extension $\widehat{\Gamma}$ of the fundamental group $\Gamma = \pi_1(\Sigma^*)$ which has an extra order two central generator ζ and is determined by the single relation

$$\prod_{j=1}^{g}[a_{2j-1}, a_2 j] \cdot \prod_{k=1}^{n} d_k = \zeta$$

We consider a space of representations of $\widehat{\Gamma}$ into $\mathrm{SU}(2)$. We bring the parabolic weights t_k into the picture by only allowing our representations to send the generator d_k to an element of trace $2\cos(2\pi t_k)$. Let $\underline{t}$ denote the weight vector $(t_1, \ldots, t_k)$

Definition 1.2.2. The *representation variety* $\mathcal{R}^{\mathrm{odd}}_{g,n}(\underline{t})$ for the weight vector $\underline{t}$ is the quotient of the space of representations of

$$\rho : \widehat{\Gamma} \to \mathrm{SU}(2)$$

such that $\mathrm{Tr}(\rho(d_k)) = 2\cos(2\pi t_k)$ and $\rho(\zeta) = -1$ under the action of conjugation by SU(2) on the target.

Denote by $\widetilde{\mathcal{R}}^{\mathrm{odd}}_{g,n}(\underline{t})$ space of representations before quotienting by conjugation. The action of SU(2) descends to one of $\mathrm{PU}(2) = \mathrm{SU}(2)/\{\pm 1\}$. This action is free and the quotient $\mathcal{R}^{\mathrm{odd}}_{g,n}(\underline{t})$ is a smooth, compact manifold only as long as the t_k's are generic: for any collection of signs $\epsilon_k = \pm 1$, no sum $\Sigma_{k=1}^{n} \epsilon_k t_k$ may be an integer. This condition is exactly the one that precludes *reducible* representations. Let $\underline{t}$ denote the weight vector $(t_1, \ldots, t_k)$. We have the following important correspondence:

THEOREM 1.2.3. *(Mehta, Seshadri, [**25**]) For a generic collection of weights $\underline{t}$, there is a diffeomorphism*

$$\mathcal{M}^0_{g,n}(\underline{t}) \cong \mathcal{R}^{odd}_{g,n}(\underline{t}).$$

This diffeomorphism, is not at all obvious. A map from $\mathcal{R}_{g,n}(\underline{t}) \to \mathcal{M}^0_{g,n}(\underline{t})$ can be constructed by going through orbifolds: a representation gives rise to a flat orbifold connection $\check{d}_A$ over $\check{\Sigma}$. Composing this operator with the projection to bundle-valued $(0,1)$ forms then gives a holomorphic orbifold $\overline{\partial}$ operator, whence via smoothing we obtain a parabolic bundle. See [**4**] for details.

The topology of these moduli spaces depends on the weights t_k. It is known (see [**5**], for example) that the genericity condition splits the space of weight vectors $\underline{t}$ into chambers separated by codimension one walls, and that as the weight vector passes through a wall, the moduli space undergoes a well-understood monoidal transformation consisting of a blow up and blow down along a submanifold. The geometric application (singular rank 2 instanton Floer homology) we have in mind requires us to fix $t_k = 1/4$ for all k. In this case we denote the representation space

$$\mathcal{R}_{g,n} := \mathcal{R}^{\mathrm{odd}}_{g,n}(1/4, \ldots, 1/4).$$

The space $\mathcal{R}_{g,n}$ is a quotient of the space $\widetilde{\mathcal{R}}_{g,n}$ consisting of $2g+n$-tuples $(S_1,\ldots,S_{2g},T_1,\ldots,T_n)$ of elements of SU(2) satisfying

$$\prod_{j=1}^{g}[S_{2j-1},S_2j]\cdot\prod_{k=1}^{n}T_k=-1.$$

and such that $\mathrm{Tr}(T_k)=0$, under the action of SU(2) by conjugation. The quotient is a smooth, compact manifold of real dimension $6g-6+2n$, as long as n is an odd number $2m+1$, which we henceforth assume. For example, $\mathcal{R}_{0,3}$ consists of a single point: any representation may be conjugated to one sending the three generators to $\mathbf{i},\mathbf{j},\mathbf{k}\in\mathrm{SU}(2)$.

NOTATION. It will be convenient later onto use the notation $\underline{S}$ and $\underline{T}$ for the $2g$ and n tuples of elements $(S_1,\ldots,S_{2g})$ and $(T_1,\ldots,T_n)$ of SU(2). Denote the equivalence class in $\mathcal{R}_{g,n}$ of a point $(S_1,\ldots,S_{2g},T_1,\ldots,T_n)$ in $\widetilde{\mathcal{R}}_{g,n}$ by $[S_1,\ldots,S_{2g},T_1,\ldots,T_n]$ or simply $[\underline{S},\underline{T}]$.

Flat Connections. The fact that isomorphism classes of flat connections and representations of the fundamental group are in bijective correspondence is an old idea in topology. In the presence of the parabolic points, the discussion of this correspondence becomes somewhat awkward, and necessitates a careful discussion of exactly what kinds of connections we allow, which we now undertake. We essentially copy the setup for this from [**12**]. In what follows, all maps and forms are C^∞. Recall that we have fixed a smooth vector bundle E on Σ with $c_1(E)=1$. This removes the possibility for reducible connections in the case that there are no marked points. We fix a hermitian inner product $\langle\cdot,\cdot\rangle$ on E, and it will be useful to introduce the principal U(2) bundle P corresponding to $(E,\langle\cdot,\cdot\rangle)$. Denote by $\mathrm{Ad}P$ is the associated principal PU(2) bundle and $\mathrm{Ad}E$ the associated rank three real vector bundle via the representation $\mathrm{PU}(2)\to\mathrm{SO}(3)$. The bundle $\mathrm{Ad}E$ sits naturally inside $\mathrm{End}\,E$ as the skew hermitian trace-free subspace. On the noncompact surface Σ^*, for each k we identify a neighborhood U_k of the kth puncture with the cylinder $(0,1)\times S^1$. This gives coordinates (s_k,θ_k) on U_k, with $\theta_k\in[0,2\pi]$. We want to study SU(2) connections on E, but $c_1(E)\neq 0$ and E cannot be made to have structure group SU(2). We therefore use the familiar technique of studying connections on $\mathrm{Ad}E$, but use a subgroup of the full gauge group of $\mathrm{Ad}E$ consisting of those automorphisms which are induced by determinant 1 gauge transformations on E. For each k, we fix a trivialization of E over U_k such that for $1 \leq k \leq n$ the preferred line F_k is spanned by the vector $(1,0) \in \mathbb{C}^2$. With respect to these trivializations, connections on $\mathrm{Ad}E$ over can be identified with 1-forms with values in $\mathfrak{so}(3)$. We denote by $\mathcal{A}$ the following space of connections on Σ^* which are "standard" near the punctures:

$$\mathcal{A} = \Big\{ A \text{ a } C^\infty,\ \mathrm{SO}(3) \text{ connection on } \mathrm{Ad}E \text{ such that:} \\ A(s_k,\theta_k) = \tfrac{1}{4}\mathrm{ad}\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right) d\theta_k = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} d\theta_k \text{ for } s_k \geq 1/2,\ 1 \leq k \leq n \Big\} \tag{1.2.1}$$

These connections are fixed on the ends of Σ in order to ensure that the corresponding connection has fixed holonomy around the punctures:

$$\mathrm{hol}_{\partial U_k}(A) = \exp\left(2\pi \cdot \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SO}(3). \tag{1.2.2}$$

At the puncture x_k, the line $F_k \subset E_{x_k}$ picks out a real 2-dimensional subspace H_k of $(\mathrm{Ad}E)_{x_k}$ corresponding to endomorphisms h for which $h(F_k) \subset F_k^\perp$, which is actually a complex line by precomposing with complex scalars. Our space of connections is rigged so that the holonomy around x_k is exactly the element which is the identity on the real line $\ell_k := H_k^\perp$ and -1 on the plane H_k.

Given a unitary automorphism g of E, there is an induced automorphism $\mathrm{Ad}g$ on $\mathrm{Ad}E$ which arises by conjugation by g. We define the gauge group $\mathcal{G}$ to be the space of smooth determinant 1 sections of $\mathrm{Aut}(E)$ of unitary automorphisms of E, and restrict to those automorphisms whose action preserves the end behavior of our 1-forms. Namely, we define

$$\mathcal{G} = \left\{ \begin{array}{c} g \in \mathrm{Aut}(E)\text{:}\ \ \det(g) = 1, \text{ and } \forall\, k, \text{ there is } z \in \mathbb{C},\ |w| = 1 \\ \text{such that } g(s_k,\theta_k) = \left(\begin{smallmatrix} w & 0 \\ 0 & -w \end{smallmatrix}\right), \text{ for } s_k \geq 1/2 \end{array} \right\} \tag{1.2.3}$$

where we use the trivialization near each each puncture to identify g with an $\mathrm{SU}(2)$-valued function. An element g acts by pulling back connections via the automorphism $\mathrm{Ad}g$ on $\mathrm{Ad}E$. In other words a section t of $\mathrm{Ad}E$ is $(g \cdot A)$-parallel if and only if $\mathrm{Ad}(g) \cdot t = g \circ t \circ g^{-1}$ is

A parallel. The gauge group is rigged so that $\mathrm{Ad}g$ preserves the orthogonal decomposition $H_k \oplus \ell_k$. We will think of $\mathcal{G}$ as acting on the bundle $\mathrm{Ad}E$ over the entire surface Σ, while $\mathcal{A}$ consists of connections only over Σ^*.

Let $\mathcal{A}_{\mathrm{flat}} \subset \mathcal{A}$ denote the subset of flat connections, that is, those connections satisfying $F_A = d_A^2 = 0$. It is not difficult to see that any smooth flat SO(3) connection on Σ^* with the proper holonomy around the punctures will be gauge equivalent to one in $\mathcal{A}_{\mathrm{flat}}$. Denote by $\mathcal{B}_{\mathrm{flat}}$ the quotient $\mathcal{A}_{\mathrm{flat}}/\mathcal{G}$. It is a standard fact that $\mathcal{B}_{\mathrm{flat}}$ is diffeomorphic to $\mathcal{R}_{g,n}$. It will be useful to understand this diffeomorphism later when we begin analyzing the symplectic form in more detail. Fix a basepoint $z \in \Sigma$ away from the punctures and trivialize E outside z_0. This gives a reduction of the structure group of $P|_{\Sigma \setminus \{z_0\}}$ to SU(2). With respect to the trivialization, a connection on $\mathrm{Ad}E$ over $\Sigma^* \setminus \{z_0\}$ becomes an $\mathfrak{so}(3)$ valued 1-form a. An SU(2) connection B on E becomes an $\mathfrak{su}(2)$ valued 1-form b, and the induced connection $\mathrm{Ad}B$ on $\mathrm{Ad}E$ has 1-form given by $\mathrm{ad}b$. The Lie algebra homomorphism $\mathrm{ad} : \mathfrak{su}(2) \to \mathfrak{so}(3)$ is an isomorphism so there is a unique b for which $\mathrm{ad}b = a$, and B is flat if and only if A is. Letting z be a new basepoint near z_0, the holonomy $\mathrm{hol}_z(A)$ gives a representation of $\pi_1(\Sigma^* \setminus \{z_0\})$ into SO(3) and $\mathrm{hol}_z(B)$ gives one to SU(2) which lifts the homomorphism $\mathrm{hol}_z(A)$. The holonomy of B around a small loop γ_0 based at z around z_0 must be -1 since the bundle doesn't extend across z_0 but this holonomy must lift the holonomy of A, which is the identity in SO(3). Hence, from a flat connection A we get a representation of $\widehat{\Gamma} \cong \pi_1(\Sigma^* \setminus \{z_0\})/\langle \gamma_{z_0}^2 \rangle$ into SU(2) given by the holonomy of the lift B. What remains is to check that the holonomy of B around the punctures has trace 0. To see this, we simply note that we are free to fix our trivialization of E away from z_0 in a way agreeing with the fixed trivializations already chosen near each punctures. With respect to this, the 1-form of A near the punctures has been fixed so that the holonomy around a small loop is the 3×3 matrix in (1.2.2). The two SU(2) lifts of this matrix are $\pm \mathbf{i}$, which have trace 0. Now, the action of the gauge group serves to conjugate the representation $\mathrm{hol}_z(B)$, hence we get the desired map $\mathcal{B}_{\mathrm{flat}} \to \mathcal{R}_{g,n}$. For the rest of the paper, we fix the point z_0 and trivialization of E on $\Sigma \setminus \{z_0\}$, writing $\mathfrak{su}(A)$ for the unique SU(2) connection with $\mathrm{Ad}(\mathfrak{su}(A)) = A$ away from z_0 (which we were calling B).

The Symplectic Structure. The tangent space to a point $A \in \mathcal{A}$ is naturally identified with the space of 1-forms a with values in the bundle $\mathfrak{so}(\mathrm{Ad}E) \subset \mathrm{End}(\mathrm{Ad}E)$ of skew-adjoint endomorphisms, which are zero near the punctures. The subspace corresponding to $T_A\mathcal{A}_{\text{flat}}$ is the subspace satisfying $d_A a = 0$, the linearization of the flatness condition. Let $\mathrm{Tr}(\cdot,\cdot) : \mathfrak{so}(3) \otimes \mathfrak{so}(3) \to \mathbb{R}$ denote the invariant positive definite form $\mathrm{Tr}(X, Y) = -\mathrm{Tr}(X \circ Y)$, which gives an inner product on $\mathfrak{so}(\mathrm{Ad}E) \subset \mathrm{End}(\mathrm{Ad}E)$. We then define a 2-form $\widetilde{\omega}$ on $\mathcal{A}_{\text{flat}}$ via:

$$\widetilde{\omega}(a \wedge b) = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}(a \wedge b) \tag{1.2.4}$$

Here, we use the natural composite of Tr with the wedge product $\wedge$ on 1-forms. The tangent space to an equivalence class $[A] \in \mathcal{B}_{\text{flat}}$ is the quotient $T_A\mathcal{A}_{\text{flat}}/T_A(\mathcal{G} \cdot A)$. The key point is that ω is annihilated by $T_A(\mathcal{G} \cdot A)$. A vector in $T_A(\mathcal{G} \cdot A)$ is given by $d_A v \in \Omega^1(\Sigma^*) \otimes \mathfrak{so}(\mathrm{Ad}E)$ for some $v \in \Omega^0(\Sigma^*) \otimes \mathfrak{so}(\mathrm{Ad}E)$. We have:

$$\widetilde{\omega}(d_A v \wedge b) = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}(d_A v, b) = -\frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}(v, d_A b) = 0$$

The second equality is due to Stoke's theorem (Σ^* is not closed, but all functions involved are constant or zero near the boundary), and the third is because b is infinitesimally flat ($d_A b = 0$). Hence, $\widetilde{\omega}$ descends to a 2-form ω on the quotient $\mathcal{B}_{\text{flat}}$. Nondegeneracy and closedness are standard results going back to [**1**], where it is shown that ω is the symplectic form on $\mathcal{R}_{g,n}$ arising from an infinite dimensional symplectic reduction of $(\mathcal{A}, \widetilde{\omega})$ with moment map the curvature $F_\bullet : \mathcal{A} \to \Omega^2(\Sigma, \mathfrak{so}(\mathrm{Ad}E))$.

1.2.2. Universal Bundles and Generators for Cohomology. Fundamental to the computation of the cohomology of $\mathcal{M}_g^0(2, 1)$ is the knowledge that a certain canonical collection of cohomology classes generate the full rational cohomology ring. These classes were first described in [**1**], where the methods of infinite dimensional equivariant Morse theory are used to show that they are generators. This technique was generalized in [**3**] (using [**28**]) to the case of parabolic bundles, the essential aspects of which we review here. Because we will be making explicit use of these specific cohomology classes in our computations of the ring structure on $H^*(\mathcal{M}^0_{g,n})$, we will go into a fair amount of detail here.

The cohomology classes we describe all arise naturally from one or more "universal bundles" over a product of Σ with a moduli space or classfiying space. This terminology is used for several distinct concepts in this circle of ideas, so we first review the ones we need here.

One kind of universal bundle we can consider is a universal holomorphic parabolic bundle $\mathbb{E} \to \mathcal{M}^0_{g,n} \times \Sigma$.

Definition 1.2.4. A *universal parabolic bundle* over $\mathcal{M}^0_{g,n} \times \Sigma$ is a pair consisting of a holonomorphic bundle $\mathbb{E}$ over the product and for each parbaolic point x_k a rank one subbundle$\mathbb{F}_k \subset \mathbb{E}|_{\mathcal{M}^0 \times \{x_k\}}$ over the parabolic point x_k, with the following property: restricting the pair $(\mathbb{E}, \{\mathbb{F}_k\})$ to the slice $\{\mathcal{E}\} \times \Sigma$ should give a stable parabolic bundle on Σ isomorphic to $\mathcal{E}$.

In the case that such an $\mathbb{E}$ exists, we can get cohomology classes in $\mathcal{M}^0$ using characteristic classes and the slant product operation

$$/ : H^i(\mathcal{M}^0 \times \Sigma) \otimes H_j(\Sigma) \to H^{i-j}(\mathcal{M}^0)$$

Theorem 1.2.5. *(Biswas, [**3**]) Suppose that a universal parabolic bundle $(\mathbb{E}, \{\mathbb{F}_k\})$ exists on $\mathcal{M}^0 \times \Sigma$. Then as $h \in H_j(\Sigma)$ ranges over a basis for the homology of Σ, the classes $c_2(\mathbb{E})/h \in H^{4-i}(\mathcal{M}^0; \mathbb{Q})$ for $j = 1, 2$ and $c_1(\mathrm{Hom}(\mathbb{F}_k, \mathbb{E})) \in H^2(\mathcal{M}^0; \mathbb{Q})$ generate $H^*(\mathcal{M}^0; \mathbb{Q})$ as a $\mathbb{Q}$-algebra.*

The existence of a universal parabolic bundle, however, is a delicate matter and is not always guaranteed. These difficulties mostly arise from an obstruction to the lifting of an easily constructed projective universal bundle to a vector bundle. For us, all we really need is this projective bundle, which has all the properties one would expect of the projectivization of a universal vector bundle were it to exist.

PROPOSITION 1.2.6. *There exists a $\mathbb{C}P^1$-bundle $\mathbb{P}$ over $\mathcal{M}^0 \times \Sigma$, along with a section $\mathbf{s}_k$ over the slice $\mathcal{M}^0 \times \{x_k\}$ for each k with the following properties:*

- *The bundle $\mathbb{P}$ is semi-holomorphic: it is smooth in $\mathcal{M}^0$ directions and holomorphic upon restriction to Σ slices (in a way varying smoothly over $\mathcal{M}^0$)*
- *Over $\{\mathcal{E}\} \times \Sigma$ the pair $(\mathbb{P}, \{\mathbf{s}_k\})$ restricts to a projective bundle pair holomorphically isomorphic to the projectivization of the parabolic pair $(\mathcal{E}, \{F_k\})$.*

PROOF. We begin on the infinite dimensional space $\mathcal{C}^{\mathrm{s}} \times \Sigma$, letting π denote the projection to Σ. The pullback π^*E is equipped with the following tautological semi-holomorphic structure: over $\{\overline{\partial}_A\} \times \Sigma$ the holomorphic structure is the one given by $\overline{\partial}_A$, and over $\mathcal{C}^{\mathrm{s}} \times \{x_k\}$ has line subbundle π^*F_k. Using the natural smooth structure on the affine space $\mathcal{C}^{\mathrm{s}}$, this assignment gives a smoothly varying holomorphic structure on Σ slices. This induces a semi-holomorphic structure on the projectivization $\mathbb{P}(\pi^*E)$, which naturally carries the section $\mathbb{P}(\pi^*F_k)$.

There is an action of the gauge group $\mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$ on the total space of $\mathbb{P}(\pi^*E)$ covering its action on $\mathcal{C}^{\mathrm{s}}$, as follows. Given a point $(\ell, \overline{\partial}_A, x)$ in the the total space of $\mathbb{P}(\pi^*E)$ with $\ell \subset E_x$, an element g of the gauge group acts via:

$$g \cdot (\ell, \overline{\partial}_A, x) = (g_x(\ell), g \cdot \overline{\partial}_A, x)$$

Because $\mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$ preserves the F_k's, this action preserves the subbundles $\pi^*(F_k)$. The action is not free, since it factors through the quotient $\mathcal{G}^{\mathrm{c}}_{\mathrm{par}} \to \overline{\mathcal{G}}^{\mathrm{c}}_{\mathrm{par}}$: constant scalar automorphisms certainly act trivially on connections and lines. However, the resulting action of $\overline{\mathcal{G}}^{\mathrm{c}}_{\mathrm{par}}$ is free (in fact, free on the base). Hence, we may form the quotient projective bundle $\mathbb{P} \to \mathcal{M} \times \Sigma$. The group $\overline{\mathcal{G}}^{\mathrm{c}}_{\mathrm{par}}$ preserves slices through points in Σ and also the section $\mathbb{P}(\pi^*F_k)$, so that we get $\mathbf{s}_k \to \mathcal{M} \times \{x_k\}$.

That there is a holomorphic structure in Σ directions is clear: the action by an element $g \in \mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$ on $\mathbb{P}(\pi^*E)$ is by a holomorphic isomorphism from the restriction to the slice through $\{\overline{\partial}_A\}$ to the restriction through $\{g \cdot \overline{\partial}_A\}$. This also make it clear that the pair $(\mathbb{P}, \{\mathbf{s}_k\})$ has exactly the desired tautological property on restriction to Σ slices. Continuity in $\mathcal{M}^0$ directions is obvious, so we obtain the desired semi-holomorphic structure. Restricting everything to the fixed determinant subset $\mathcal{M}^0$ gives a universal bundle on the correct space. $\square$

REMARK. The technique of passing to the projective bundle is technical device which allows us to get an action of the reduced gauge group, and is analogous to passing from a U(2)-bundle to the adjoint SO(3)-bundle in the flat connection story.

There is an analogous universal bundle over $\mathcal{R}_{g,n}\times\Sigma$, carrying a family of flat connections over $\mathcal{R}_{g,n}\times\Sigma^*$, seen as follows. Recall that $\mathcal{A}_{\text{flat}}$ is our space of flat connections on $\mathrm{Ad}E$ over Σ^*. Letting π now denote the projection to $\mathcal{A}_{\text{flat}}\times\Sigma\to\Sigma$, there is an obvious tautological family of connections on $\pi^*(\mathrm{Ad}E)$, which on the slice through A is just the connection d_A. This family is smooth in $\mathcal{A}_{\text{flat}}$ directions, and is preserved by the natural action of the smooth SU(2) gauge group $\mathcal{G}$. The center $\{\pm1\}$ of $\mathcal{G}$ of constant scalar automorphisms acts trivially on $\mathcal{A}_{\text{flat}}$. In the holomorphic case, the center of the gauge group acted trivially on the base but not on the bundle; here we avoid this problem by passing to the adjoint bundle. The gauge group $\mathcal{G}$ acts on $\pi^*(\mathrm{Ad}E)$ via:

$$g\cdot(t,A,x)=(g_x\circ t\circ g_x^{-1},g\cdot A,x)$$

where $t\in(\mathrm{Ad}E)_x\subset\mathrm{End}(E_x)$. The center acts trivially, so this action descends to a free one of the quotient $\overline{\mathcal{G}}$. The quotient of $\pi^*\mathrm{Ad}E$ by $\overline{\mathcal{G}}$ is an SO(3) vector bundle $\mathbf{E}^{\text{ad}}\to\mathcal{R}_{g,n}\times\Sigma$. Since an automorphism g preserves the complex line $H_k\subset\mathrm{Ad}E_{x_k}$ for each k, it preserves the complex line subbundle $\pi^*(H_k)\subset\mathrm{Ad}E|_{\mathcal{A}_{\text{flat}}\times\Sigma}$. Hence, inside of $\mathbf{E}^{\text{ad}}$, there is a subbundle $\mathbf{V}_k$ over $\mathcal{R}_{g,n}\times\{x_k\}$ coming from the quotient of $\pi^*(H_k)$ for each k. The pair $(\mathbf{E}^{\text{ad}},\{\mathbf{V}_k\})$ has a flat structure $\mathbf{A}$ in Σ directions which moves continuously in $\mathcal{R}_{g,n}$ directions, and on $\{[A]\}\times\Sigma$ the bundle restricts to one with a flat connection on Σ^* isomorphic to A, with the isomorphism carrying $(\mathbf{V}_k)_{[A]}$ to the subspace H_k.

In order to recover generating sets of characteristic classes, let us relate the classes coming from our adjoint bundles to a hypothetical bona fide universal bundle. For any complex vector bundle $\mathbf{E}$, we have:

$$p_1(\mathrm{Ad}\mathbf{E}) = -4c_2(\mathbf{E})$$

In addition, given $\mathbf{F}_k \subset \mathbf{E}$, the subbundle $\mathbf{W}_k \cong \mathrm{Hom}(\mathbf{F}_k^\perp, \mathbf{F}_k)$ of matrices with no diagonal terms with respect to the orthogonal decomposition has

$$c_1(\mathbf{W}_k) = -c_1(\mathrm{Hom}(\mathbf{F}_k, \mathbf{E}))$$

over the copy of Σ. We conclude:

THEOREM 1.2.7. *The rational cohomology ring $H^*(\mathcal{M}^0; \mathbb{Q})$ is generated as a $\mathbb{Q}$-algebra by elements $p_1(\mathbf{E}^{\mathrm{ad}})/h$, for $h \in H_*(\Sigma; \mathbb{Q})$, and the classes $c_1(\mathbf{V}_k)$.*

1.3. Betti Numbers of The Moduli Space

While §1.2.2 provides a generating set for the cohomology ring $H^*(\mathcal{R}_{g,n}; \mathbb{Q})$, and §1.5 will describe relations among them, proving that these relations are a complete set for $g = 0$ will require knowledge of the Betti numbers of $\mathcal{R}_{0,n}$. A convenient formula for these that is general enough to accommodate an arbitrary choice of parabolic weights appears in [**4**], and we can actually obtain formulas for the general case $g \neq 0$ without much additional work. To apply this to the specific case of equal weights treated here, we will need to interpret that formula a bit. To this end, we briefly review the methodology in [**4**].

As seen in [**28**], the space of holomorphic structures $\mathcal{C}$ on $E \to \Sigma$ comes with a stratification analogous to the stratification described in [**1**] in the non-parabolic case. The subset of $\mathcal{C}^{\mathrm{s}} \subset \mathcal{C}$ for which the fixed weights and 1 dimensional subspaces F_k give a stable parabolic structure on the rank two bundle E is an open subset. The point is that not all bundles in the complement are born equal: some unstable bundles are *more* unstable than others. Given an unstable rank two bundle $\mathcal{E}$, there is a unique destabilizing line subbundle $\mathcal{F}$ sitting in a slope-decreasing short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0$$

Still following [**4**], we define $\lambda = \deg(\mathcal{F})$. Conceptually, the larger λ is, the more unstable $\mathcal{E}$ is. We can get more data from $\mathcal{F}$ by recording whether it coincides with F_k at each parabolic point. Namely, let $e_k = \dim(\mathcal{F}_{x_k} \cap F_k)$, and denote by $\underline{e}$ the vector of the e_k's, which are either 0 or 1. The invariant $\underline{e}$ is a property of $\mathcal{E}$ and is exactly the *intersection matrix* considered in [**28**]. The pair $(\lambda, \underline{e})$ is called the *type* of $\mathcal{E}$, and we let $\mathcal{C}_{\lambda,\underline{e}}$ denote the locally closed submanifold of $\mathcal{C}$ consisting of holomorphic structures on E of type $(\lambda, \underline{e})$. The crucial point is that this strata for a fixed type is *connected* ([**28**], Prop. 3.5). The codimension $d_{\lambda,\underline{e}}$ of the strata $\mathcal{C}_{\lambda,\underline{e}}$ is

$$d_{\lambda,\underline{e}} = 2\lambda + n + g - 1 + \sum_k e_k \tag{1.3.1}$$

which essentially appears in [**4**] as equation (17).

REMARK 1.3.1. In [**4**], the assumption is made that $\deg^{\mathrm{par}}(E) = 0$, which is achieved by taking $\deg(E) = -n$. The moduli space in this case is isomorphic to ours as can be easily seen by comparing the representation varieties. For the rest of this section, we temporarily assume $\deg(E) = -n$ so as to use the formulas appearing there without attempting modification.

The essential content of [**4**] is that the stratification of $\mathcal{C}$ via the strata $\mathcal{C}_{\lambda,\underline{e}}$ is *equivariantly perfect* for the action of the complex gauge group $\mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$. As a result, one can record the equivariant Poincaré polynomial of the total space as a sum over terms coming from the individual strata. These contributions can be computed exactly for the unstable strata, thus yielding a formula for the Poincaré polynomial for the quotient. We may now record the result in [**4**] on the betti numbers of $\mathcal{M}^0_{g,n}$. For a topological space X, let $P_t(X)$ denote its Poincaré polynomial $\sum_d \mathrm{rk} H_*(X) \cdot t^d$.

THEOREM 1.3.2. *The Poincaré polynomial of the representation variety $\mathcal{R}_{g,n}$ for $n \geq 1$ is given by:*

$$P_t(\mathcal{R}_{g,n}) = \frac{(1+t)^{2g-2}}{(1-t)^2}\left((1-t+t^2)^{2g}(1+t^2)^{n-1} - (1-t^2)\sum_{\lambda,\underline{e}} t^{2d_{\lambda,\underline{e}}} \right) \tag{1.3.2}$$

where the sum ranges over all types $(\lambda, \underline{e})$ *which are destabilizing.*

Proof. We know that $\mathcal{R}_{g,n} \cong \mathcal{M}^0_{g,n}$. Formula (1.3.2) is just equation (21) of [**4**] divided by the Poincaré polynomial of the Jacobian $J(\Sigma)$, since the equation there is for the moduli space $\mathcal{M}_{g,n}$, which is a cohomologically trivial bundle over $\mathcal{M}^0_{g,n}$ with fiber homeomorphic to $J(\Sigma)$. □

Example 1.3.3. Let us illustrate how to use (1.3.2) to compute the Poincaré polynomial in a simple example: $\mathcal{R}_{0,3}$, which consists of the single equivalence class $[\mathbf{i}, \mathbf{k}, \mathbf{j}]$. We need to understand the domain of the sum in the formula. The parabolic degree of the destabilizing line bundle $\mathcal{F}$ of a parabolic bundle $\mathcal{E}$ of type $(\lambda, \underline{e})$ is:

$$\deg^{\mathrm{par}}(\mathcal{F}) = \lambda + \sum_{k=1}^{3} \left(\tfrac{1}{4}(1-e_k) + \tfrac{3}{4}e_k\right) = \lambda + \tfrac{3}{4} + \tfrac{1}{2}\sum_{k=1}^{3} e_k.$$

For such $\mathcal{F}$ to be destabilizing we need $\mu(\mathcal{F}) \geq \mu(\mathcal{E}) = 3/2$. Letting $e = \sum_{k=1}^{3} e_k$, we therefore need

$$4\lambda + 2e \geq -3 \tag{1.3.3}$$

It is convenient to visualize the pairs (λ, e) satisfying 1.3.3; see Figure 1.3.3.

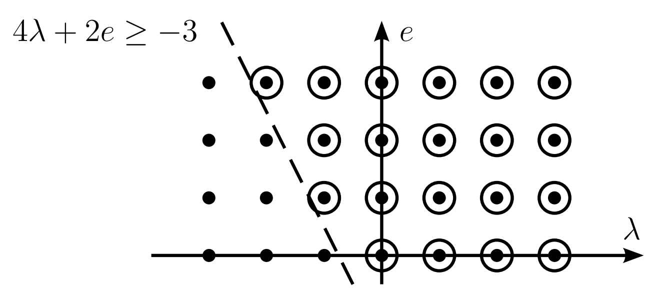

Figure 1.1. The locus of unstable pairs (λ, e) for $n = 3$.

We see that we can take the sum to over all e and $\lambda \geq -1$, except that we must add a contribution for $(-2, 3)$ and subtract one for $(-1, 0)$. We have:

$$\begin{aligned}
\sum_{\lambda,\underline{e}} t^{2d_{\lambda,\underline{e}}} &= t^2 - 1 + \sum_{\lambda \geq -1, \underline{e}} t^{2(2\lambda+2+e)} \\
&= t^2 - 1 + t^4 \sum_{\lambda \geq -1} t^{4\lambda} \left(\sum_{e_1=0,1} t^{2e_1} \sum_{e_2=0,1} t^{2e_2} \sum_{e_3=0,1} t^{2e_3} \right) \\
&= t^2 - 1 + (1-t^4)^{-1}(1+t^2)^3
\end{aligned}$$

Since $g = 0$ here, formula (1.3.2) reduces to

$$\frac{1}{(1-t^2)^2} \left((1+t)^2 - (1-t^2) \sum_{\lambda,\underline{e}} t^{2d_{\lambda,\underline{e}}} \right).$$

Plugging in for the summation and applying simple algebra, we see that $P_t(\mathcal{R}_{0,3}) = 1$, as expected.

Focusing on the genus 0 case temporarily, we can use (1.3.2) to get a beautiful result which yields explicit formulae for the Poincaré polynomials.

Proposition 1.3.4. *Suppose $n = 2, +1 > 3$. Then up to its middle dimension $n - 3$, the Poincaré polynomial $P_t(\mathcal{R}_{0,n})$ equals the Poincaré polynomial of the graded algebra $\mathbb{C}[\alpha, \beta, \delta_1, \ldots, \delta_n]/(\delta_i^2)$, where α, β, δ_i have degrees 2, 4, and 2, respectively.*

Proof. The proof is an exercise in bookkeeping for the sum over strata in (1.3.2). For a type $(\lambda, \underline{e})$ to be destabilizing, we now require (againg letting e denote the sum of the e_k's):

$$4\lambda + 2e \geq -n \tag{1.3.4}$$

Let $Q_n \subset \mathbb{Z}^2$ be the subset of the lattice points in the (λ, e) plane satisfying (1.3.4) and bounded by $0 \leq e \leq n$. It can be approximated by the subset $\lambda \geq -m$, $0 \leq e \leq n$; this approximation leaves out a small triangle of lattice points to the left of $\lambda = -m$, but errantly includes a triangle of the same size but rotated 180°, to the right of $\lambda = -m$ (see Figure 1.3).

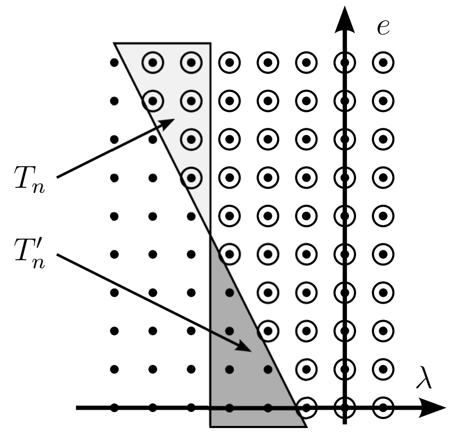

FIGURE 1.2. The triangles T_n and T'_n.

Let T_n denote the triangle of lattice points in Q_n with $\lambda < -m$. Each lattice point (λ, e) in Q_n contributes a sum of terms $t^{2d_{\lambda,\underline{e}}}$ consisting of vectors $\underline{e}$ with $\sum_k e_k = e$. There are $\binom{n}{e}$ such vectors, so the contribution of the point (λ, e) is $\binom{n}{e} t^{4\lambda+2n-2+2e}$. Each point (λ, e) in T_n pairs with the point $(\lambda', e') = (-n-\lambda, n-e)$ obtained by rotating $180°$ about the center $(-\frac{n}{2}, \frac{n}{2})$. The approximating half space $\lambda \geq -m$ misses $(\lambda, e) \in T_n$, but wrongfully includes (λ', e') in the rotated triangle T'_n. The correction for this in the sum over unstable strata is therefore

$$\binom{n}{e} \left(t^{n-2+(4\lambda+n+2e)} - t^{n-2-(4\lambda+n+2e)} \right). \tag{1.3.5}$$

We want to separate the sum $\sum_{Q_n} t^{2d_{\lambda,\underline{e}}}$ into the part $\sum_{\lambda \geq -m, \underline{e}} t^{2d_{\lambda,\underline{e}}}$, plus the correction which we momentarily denote by $\sum_{T_n}(\cdots)$, consisting of a sum of terms (1.3.5). We focus first on the the rectangular sum. We have:

$$\begin{aligned}
\sum_{\lambda \geq -m, \underline{e}} t^{2d_{\lambda,\underline{e}}} &= t^{2n-2} \sum_{\lambda \geq -m} t^{4\lambda} \left(\sum_{e_1=0,1} t^{2e_1} \cdots \sum_{e_n=0,1} t^{2e_n} \right) \\
&= (1-t^4)^{-1}(1+t^2)^n
\end{aligned}$$

Going back to the formula for the Poincaré polynomial plugging in $g = 0$ we get:

$$\begin{aligned} P_t(\mathcal{R}_{0,n}) &= \frac{1}{(1-t^2)^2}\left((1+t^2)^{n-1} - (1-t^2)\left[(1-t^4)^{-1}(1+t^2)^n + \sum_{T_n} t^{2d_{\lambda,\underline{e}}}\right]\right) \\ &= -\frac{1}{1-t^2}\sum_{T_n}(t^{2d_{\lambda,\underline{e}}}) \end{aligned}$$

We see that the entire cohomology is determined by the contributions from the triangle T_n. Let T_n' denote the rotation of T_n about the center point $(-\frac{n}{2}, \frac{n}{2})$. Pairs $(\lambda, e) \in T_n'$, that is with $4\lambda + 2e < -n$ and $\lambda \geq -m$, are in bijection with pairs (μ, e) with $4\mu + 2e \leq n-3$ and $\mu \geq 0$. The contribution to the above sum for such a pair is obtained by dividing equation (1.3.5) by $t^2 - 1$:

$$\binom{n}{e}\sum_{i=0}^{D} t^{4\mu+2e+2i} \tag{1.3.6}$$

where $D = n - 3 - (4\mu + 2e)$. Now, we consider the graded algebra

$$\mathbb{B}_{0,n} := \mathbb{C}[\alpha, \beta, \delta_1, \ldots, \delta_n]/(\delta_i^2),$$

where α, β, δ_k have degrees 2, 4, and 2, respectively. The dimenension in each grading of $\mathbb{B}_{0,n}$ can be found by counting monomials of the form $\alpha^a\beta^b\delta_1^{e_1}\cdots\delta_n^{e_n}$ for $e_k = 0, 1$. For fixed $e = \sum_k e_k$, the number of such monomials with fixed a and b is $\binom{n}{e}$. The contribution to $P_t(\mathcal{R}_{0,n})$ from such a monomial can be viewed as coming from the term $t^{2a+4b+2e}$ in the above sum (1.3.6) for $\mu = b$ and $i = a$. This shows that the Poincaré polynomials of $\mathcal{R}_{0,n}$ and $\mathbb{B}_{0,n}$ are equal up to the middle dimension $n - 3$. $\square$

We can use this theorem to get our first glimpses of the recursive nature of the $\mathcal{R}_{g,n}$ in the number of parabolic points n, and eventually the genus g.

Corollary 1.3.5. *Suppose $n = 2m + 1 > 3$. For the genus 0 representation varieties, we have:*

$$P_t(\mathcal{R}_{0,n+2}) = (1+t^2)^2 P_t(\mathcal{R}_{0,n}) + 2^{n-1}t^{n-1} \tag{1.3.7}$$

PROOF. This follows from the isomorphism

$$\mathbb{B}_{0,n+2} \cong \mathbb{B}_{0,n} \otimes \mathbb{C}[\delta_{n+1}, \delta_{n+2}]/(\delta_{n+1}^2, \delta_{n+2}^2).$$

Let $\mathcal{T}_{0,n}^d$ denote the collection of monomials $\alpha^a \beta^b \delta_1^{e_1} \cdot \ldots \cdot \delta_n^{e_n}$ in $\mathbb{B}_{0,n}$ of degree d. For example,

$$T_{0,n}^4 = \{\alpha^2, \beta, \alpha\delta_1, \ldots, \alpha\delta_n, \delta_1\delta_2, \ldots, \delta_{n-1}\delta_n\}$$

For d up to the middle dimension of $\mathcal{R}_{0,n}$, the coefficient of t^d in $P_t(\mathcal{R}_{0,n})$ is the cardinality of $T_{0,n}^d$. For d strictly less than the middle dimension $2m$ for $\mathcal{R}_{0,n+2}$, any monomial in $T_{0,n+2}^d$ can be obtained by multiplying a monomial in $T_{0,n}^{d'}$ by δ_{n+1}, δ_{n+2}, both, or neither, for $d' \leq 2m-2$, which is the middle dimension for $\mathcal{R}_{0,n}$. This argument shows that $P_t(\mathcal{R}_{0,n+2})$ and $(1+t^2)^2 P_t(\mathcal{R}_{0,n})$ agree up to degree $2m-2$.

In degree $2m$, we consider the collection of those monomials in $T_{0,n+2}^{2m}$ arising from multiplying a monomial in $T_{0,n}^{2m-2e_1-2e_2}$ by $\delta_{n+1}^{e_1}\delta_{n+2}^{e_2}$ for $(e_1, e_2) = (1,0), (0,1), (1,1)$, along with those arising from multiplying a monomial in $T_{0,n}^{2m-4}$ by β. These 4 disjoint subsets contribute $2\left|T_{0,n}^{2m-4}\right| + 2\left|T_{0,n}^{2m-2}\right|$ to $\left|T_{0,n+2}^{2m}\right|$, which is the same as the contribution to the coefficient of t^{2m} in $P_t(\mathcal{R}_{0,n+2})$ from $(1+t^2)^2 P_t(\mathcal{R}_{0,n})$ by Poincaré duality. It remains to count those monomials in $T_{0,n+2}^{2m}$ not arising this way. These are precisely those monomials in only the variables α and δ_i's for $i \leq n$. Of course, these are in bijection with vectors $\underline{e} = (e_1, \ldots, e_n)$ with $e = \sum_k e_k \leq m$, which is easy seen to equal $2^{n-1} = 2^{2m}$. The equation (1.3.7) now follows immediately. $\square$

With a bit more work, we get:

COROLLARY 1.3.6. *We have the following recursive formulas for the the Poincaré polynomials, where $n \geq 1$:*

$$P_t(\mathcal{R}_{g,n+2}) = (1+t^2)^2 P_t(\mathcal{R}_{g,n}) + 2^{n-1} t^{2g+n-1}(1+t)^{2g} \tag{1.3.8}$$

$$P_t(\mathcal{R}_{g+1,n}) = (1+t^3)^2 P_t(\mathcal{R}_{g,n}) + 2^{n-1} t^{2g+n-1}(1+t)^{2g}(1+t^2) \tag{1.3.9}$$

PROOF. Let $S_{g,n}$ denote the sum $\sum_{\lambda,\underline{e}} t^{2d_{\lambda,\underline{e}}}$ in (1.3.2). We can solve (1.3.7) for an equation giving $S_{0,n+2}$ in terms of $S_{0,n}$:

$$S_{0,n+2} = (1+t^2)^2 S_{0,n} - 2^{n-1}(1-t^2)t^{n-1} \tag{1.3.10}$$

Increasing the genus does not change the domain of the sum in (1.3.2), but multiplies it by the factor t^{2g} so that $S_{g,n} = t^{2g} S_{0,n}$. Hence:

$$S_{g,n+2} = (1+t^2)^2 S_{g,n} - 2^{n-1}(1-t^2)t^{n-1} \tag{1.3.11}$$

holds for all g. Plugging this equation into (1.3.2) for arbitrary g, the equation (1.3.8) follows by some straightforward algebra. To prove formula (1.3.9), we may argue by induction on n. Let $\Delta_{g,n}$ denote the difference $P_t(\mathcal{R}_{g+1,n}) - (1+t^3)^2 P_t(\mathcal{R}_{g,n})$. Suppose that $\Delta_{g,n} = 2^{2g+n-1}t^{n-1}(1+t)^{2g}(1+t^2)$. Now, we use (1.3.8) to compute $\Delta_{g,n+2}$. We have:

$$\begin{aligned}
\Delta_{g,n+2} &= P_t(\mathcal{R}_{g+1,n+2}) - (1+t^3)^2 P_t(\mathcal{R}_{g,n+2}) \\
&= (1+t^2)^2 \Delta_{g,n} + 2^{n-1}t^{2g+n+1}(1+t)^{2g+2} - 2^{n-1}t^{2g+n-1}(1+t)^{2g}(1+t^3)^2 \\
&= 2^{n-1}t^{2g+n-1}(1+t)^{2g}\left[(1+t^2)^3 + t^2(1+t)^2 - (1+t^3)^2\right] \\
&= 2^{n-1}t^{2g+n-1}(1+t)^{2g}\left(4t^2+4t^4\right) = 2^{n+1}t^{2g+n+1}(1+t)^{2g}(1+t^2)
\end{aligned}$$

which is exactly what need for (1.3.8) to hold in general. The base cases of $n=1$ are easy to check and left to the reader. □

REMARK 1.3.7. The formula (1.3.8) should be interpreted as saying that the homology of $\mathcal{R}_{g,n+2}$ is the same as that of $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathcal{R}_{g,n}$, with 2^{n-1} times the homology of the Jacobian $J(\Sigma)$ added in the middle dimensions. On the other hand, formula (1.3.9) says that the homology of $\mathcal{R}_{g+1,n}$ is obtained as the homology of $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathcal{R}_{g,n}$ with 2^{n-1} times the homology of the genus g Jacbobian crossed with $\mathbb{P}^1$ added in the middle dimensions. Corollary 1.3.5 is strong evidence that the cohomology *ring* is a free algebra on generators α, β, and δ_i, modulo $\delta_i^2 = x$ for some x, up to the middle dimension, with relations in degree 2 higher than the middle.

COROLLARY 1.3.8. $H^2(\mathcal{R}_{g,n}) = n+1$

EXAMPLE 1.3.9. If we set $g=0$ but let $n=5$, we expect a 4-dimensional moduli space. By Corollary 1.3.5:

$$P_t(\mathcal{R}_{0,5}) = (1+t^2)^2 P_t(\mathcal{R}_{0,3}) + 4t^2 = 1 + 6t^2 + t^4$$

(or more simply we could just use Corollary 1.3.8). As noticed by Boden in [**4**], the paper [**15**] proves that the only possibility for a four-dimensional representation variety (in genus 0) with $b_2 = 6$ is $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$. This follows by considering the structure of a toric variety on these moduli spaces.

Finally, let us relate the topology of $\mathcal{R}_{g,n}$ for positive n to the original non-parabolic case $\mathcal{R}_{g,0}$. The formula (1.3.2) in Theorem 1.3.2 only applies for positive n. However, the sum $S_{g,1}$ is very easy to evaluate: we have either $\underline{e} = (0)$ or (1), and in either case the sum is over all $\lambda \geq 0$. Thus the sum becomes $(1-t^4)^{-1}(1+t^2)$, and after some algebra we obtain:

$$P_t(\mathcal{R}_{g,1}) = \frac{(1+t^3)^2g - t^2g(1+t)^2g}{(1-t^2)^2}. \tag{1.3.12}$$

This is exactly $(1+t^2)$ times the now well known formula for the Poincaré polynomial of $\mathcal{R}_{g,0}$:

$$P_t(\mathcal{R}_{g,1}) = (1+t^2) P_t(\mathcal{R}_{g,0}) \tag{1.3.13}$$

This is to be expected since $\mathcal{R}_{g,1}$ is a $\mathbb{P}^1$-bundle over $\mathcal{R}_{g,0}$.

1.4. Canonical Line Bundles on $\mathcal{R}_{g,n}$

In this section we describe a natural collection of line bundles over $\mathcal{R}_{g,n}$ as well as explicit submanifolds of the moduli space whose Poincaré duals can be used to write down the first Chern classes of these line bundles. These line bundles and submanifolds are exactly of the type studied in [**32**], though the assumptions there on the genericity of parabolic weights and methods of symplectic reduction preclude the full application of his results to our case.

Specifically, setting all the weights to be 1/4 makes the calculation of the Poincaré duals of the first Chern classes of the line bundles somewhat more tedious, and the inductive technique in that paper of reducing the number of parabolic points by one fails, because we require n to be odd. However, we give an analogous scheme by which n can be reduced by two. The resulting recursive description of $\mathcal{R}_{g,n}$ will eventually allow us to completely write down the cohomology ring for $g = 0$, using these Chern classes as generators. We first define these line bundles, and show that their first Chern classes are naturally identified with a subset of the generating set of classes found in §1.2.2.

1.4.1. Line bundles V_k and submanifolds $D^{\pm}_{k,l}$. The line bundles we want to study are those associated to explicit U(1) bundles over $\mathcal{R}_{g,n}$. Recall our standard generating set $\{a_j, d_k, \zeta\}$ (for $j = 1, \ldots, 2g$ and $k = 1, \ldots, n$) with $\zeta^2 = 1$ for the central extension $\widehat{\Gamma}$ of the fundamental group of Σ^*. The unreduced representation space $\widetilde{\mathcal{R}}$ consists of all maps $\widehat{\Gamma} \to \mathrm{SU}(2)$, with a trace condition on the d_k and with ζ mapping to -1. Viewing $\widetilde{\mathcal{R}}_{g,n}$ as sitting in $\mathrm{SU}(2)^{2g+n}$, inside of $\widetilde{\mathcal{R}}_{g,n}$ we have, for each k, a subspace

$$V_k := \{(S_1, \ldots, S_{2g}, T_1, \ldots, T_n) \in \widetilde{\mathcal{R}}_{g,n} |\ T_k = \mathbf{i}\}$$

where the k parabolic coordinate is exactly $\mathbf{i}$. Any representation in $\widetilde{\mathcal{R}}_{g,n}$ can be conjugated to one in V_k, so the quotient map $V_k \to \mathcal{R}_{g,n}$ is surjective. A fiber can be identified with the stabilizer of $\mathbf{i}$: if $\rho \in V_k$ maps to $\overline{\rho}$ in $\mathcal{R}_{g,n}$, any other element of the preimage of $\overline{\rho}$ arises by conjugating ρ, but to stay in V_k the conjugating element $g \in \mathrm{SU}(2)$ must fix $\mathbf{i}$. This stabilizer is the one parameter subgroup $S^1_{\mathbf{i}}$ through $\mathbf{i}$. The action of $S^1_{\mathbf{i}}$ on the fiber decends to a free one of the quotient $S^1_{\mathbf{i}}/\{\pm 1\}$; both are isomorphic to the group U(1), so V_k becomes a principal U(1)-bundle over $\mathcal{R}_{g,n}$.

For each pair k, l, let we define the subspace $D^{\pm}_{k,l} \subset \mathcal{R}_{g,n}$:

$$D^{\pm}_{k,l} = \{[\underline{S}, \underline{T}] \mid T_k = \pm T_l\}.$$

The union $D^+_{k,l} \cup D^-_{k,l}$ is exactly the locus where T_k and T_l commute. It is not difficult to see that the spaces $D^\pm_{k,l}$ are smooth, orientable, connected, real codimension 2 submanifolds of $\mathcal{R}_{g,n}$. It is a careful study of these spaces and their intersections which gives us a great deal of information about the cohomology ring of $\mathcal{R}_{g,n}$.

The first step, as seen in [**32**], is to notice the line bundles V_k and the submanifolds $D^\pm_{k,l}$ are intimately related.

LEMMA 1.4.1. *The* U(1) *bundle V_k is trivial on the complement of $D^+_{k,l} \sqcup D^-_{k,l}$ in $\mathcal{R}_{g,n}$, for any $l \neq k$.*

PROOF. We shall describe an explicit section $\mathcal{R}_{g,n} \to V_k$ outside $D^+_{k,l} \sqcup D^-_{k,l}$. Let $[\underline{S}, \underline{T}] \in \mathcal{R}_{g,n}$ be such that T_k and T_l do not commute. It is not hard to see that there is a unique representative of $[\underline{S}, \underline{T}]$ for which $T_k = \mathbf{i}$, the $\mathbf{k}$-component of T_l is 0, and the $\mathbf{j}$-component of T_l is positive, assuming T_k and T_l are not equal or antipodal. Perhaps the best intuition for this comes from geography: there is a unique oriented orthogonal transformation of the earth which maps a given point to the north pole and any other point not the south pole to lie on the Prime Meridian. This recipe gives a unique representative $\underline{T}$ for any given conjugacy class, and this assignment is certainly continuous in $\mathcal{R}_{g,n}$, so gives a section. $\square$

COROLLARY 1.4.2. *Suppose $n \geq 3$, and let $[D^\pm_{k,l}]$ denote the homology classes associated to $D^\pm_{k,l}$ with some choice of orientations. Then there are integers r, s such that*

$$PD(c_1(V_k)) = r[D^+_{k,l}] + s[D^-_{k,l}] \tag{1.4.1}$$

This raises the question of what happens when $n = 1$. The author can find no obvious projection $\mathcal{R}_{g,1} \to \mathcal{R}_{g,0}$ by studying the representation varieties. Nonetheless, such a projection map is easily defined by studying the correpsonding moduli spaces of stable parabolic bundles. Suppose $\mathcal{E}$ is a stable parabolic bundle over Σ with one parabolic point x_1. Assuming $\deg \mathcal{E} = -1$ (which we are free to do by tensoring with a line bundle of degree -1), we have $\deg^{\mathrm{par}} \mathcal{E} = 0$. We claim that $\mathcal{E}$ is actually stable as a nonparabolic bundle. Indeed, if $\mathcal{L}$ were a (nonparabolic) destabilizing line bundle, then we must have $\deg \mathcal{L} > -\frac{1}{2}$ and so

$\deg \mathcal{L} \geq 0$. But then $\deg^{\mathrm{par}} \mathcal{L} \geq \frac{1}{4}$, so $\mathcal{L}$ would be a parabolic destabilizing line bundle for the parabolic bundle $\mathcal{E}$, which is a contradiction. Hence, we have a forgetful map

$$f : \mathcal{M}^0_{g,1} \to \mathcal{M}^0_{g,0}$$

It is a surjective morphism of projective algebraic varieties, whose fiber is the $\mathbb{P}^1$ coming from the choice of parabolic line $F_1 \subset \mathcal{E}_{x_1}$.

In fact what this shows is that when $n = 1$, $\mathcal{C}^{\mathrm{s}} \subset \mathcal{C}$ actually parametrizes stable non-parabolic holomorphic structure on $E \to \Sigma$ as well. To get $\mathcal{M}_{g,0}$, we simply take the quotient by the larger (nonparabolic) gauge group $\mathcal{G}^{\mathrm{c}}$. The parabolic gauge group $\mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$ is a subgroup of $\mathcal{G}^{\mathrm{c}}$ and the coset space $\mathcal{G}^{\mathrm{c}}/\mathcal{G}^{\mathrm{c}}_{\mathrm{par}}$ is a $\mathbb{P}^1$. After passing to the fixed determinant subspaces, this realizes the $\mathbb{P}^1$ bundle

$$\mathcal{M}^0_{g,1} = \mathcal{C}^{\mathrm{s}}/\mathcal{G}^{\mathrm{c}}_{\mathrm{par}} \to \mathcal{C}^{\mathrm{s}}/\mathcal{G}^{\mathrm{c}} = \mathcal{M}^0_{g,0}.$$

Let us study the corresponding universal bundles. Over $\mathcal{M}^0_{g,1} \times \Sigma$ we have $\mathbb{P}_{g,1}$ and section $\mathbf{s}_k$ over $x_1 \in \Sigma$, constructed as a quotient of $\mathbb{P}(\pi^* E)$ over $\mathcal{C}^{\mathrm{s}} \times \Sigma$. Over $\mathcal{M}^0_{g,0}$ we have $\mathbb{P}_{g,0}$. There is an obvious fiberwise map $\mathbb{P}_{g,1} \to \mathbb{P}_{g,0}$ covering the bundle map $\mathcal{M}^0_{g,1} \times \Sigma \to \mathcal{M}^0_{g,0} \times \Sigma$ which explicitly realizes the first as isomorphic to the pullback of the second. It is straightforward to check also that for the SO(3) universal bundle $\mathbf{E}^{\mathrm{ad}}(g, n)$ over $\mathcal{R}_{g,n} \times \Sigma$, we have

$$\mathbf{E}^{\mathrm{ad}}_{g,1} \cong f^*(\mathbf{E}^{\mathrm{ad}}_{g,0}).$$

Before tackling the computation of the constants r, s in Corollary 1.4.2, we first relate the line bundles V_k to the universal bundle constructions from §1.2.2 and make a few notes on the symmetry inherent in our setup.

1.4.2. The Universal Bundle and V_k's. We would like to identify V_k with natural bundles arising from the universal pair over $\mathcal{R}_{g,n} \times \Sigma$. We recall the universal bundle pair $(\mathbf{E}^{\mathrm{ad}}, \{\mathbf{V}_k\})$ over $\mathcal{R}_{g,n} \times \Sigma$, constructed as the quotient of a pullback pair $(\pi^* \mathrm{Ad} E, \{\pi^* H_k\})$ by the gauge group $\mathcal{G}$. Fix a basepoint $z \in \Sigma^*$ away from the punctures, and consider the subgroup

$$\mathcal{G}_z \subset \mathcal{G},$$

called the "based" gauge group, of gauge transformations which are the identity at z. The quotient $\mathcal{G}/\mathcal{G}_z$ is isomorphic to SU(2), and $\mathcal{A}_{\text{flat}}/\mathcal{G}_z$ is naturally an SO(3) bundle over $\mathcal{R}_{g,n} = \mathcal{A}_{\text{flat}}/\mathcal{G}$. Given a connection A on AdE, we write $[A]$ for the $\mathcal{G}$ equivalence class, and $\langle A \rangle_z$ for the equivalence class modulo $\mathcal{G}_z$.

LEMMA 1.4.3. *The residual* SO(3) *action on the quotient* $\mathcal{A}_{\mathit{flat}}/\mathcal{G}_z$ *is isomorphic as a principal bundle to* $\widetilde{\mathcal{R}}_{g,n} \to \mathcal{R}_{g,n}$.

PROOF. We define an explicit map

$$\mathcal{A}_{\text{flat}}/\mathcal{G}_z \to \widetilde{\mathcal{R}}_{g,n}$$

using the holonomy of flat connections. The point is that by modding out by the based gauge group, we can still get from a flat connection its true holonomy representation and not just its conjugacy class. To wit, recall we have fixed an auxillary point z_0 near z and trivialization of E outside z_0, so that a flat connection A on AdE gives rise to a flat SU(2) connection $\mathfrak{su}(A)$ away from z_0. This gives a homomorphism

$$\text{hol}_z(\mathfrak{su}(A)) : \widehat{\Gamma} \to \text{SU}(2).$$

The based gauge group preserves the holonomy up to conjugation by the local group $(\mathcal{G}_z)|_z$, which is trivial by definition, and so $\text{hol}_z(\mathfrak{su}(A))$ is independent of the based gauge representative. We therefore get a map $\langle A \rangle_z \mapsto \text{hol}_z(\mathfrak{su}(A))$ which is a fibered bijection between the bundles, and clearly SO(3)-equivariant. □

LEMMA 1.4.4. *Let W be the rank three $\mathbb{R}$ vector bundle associated to the principal* SO(3) *bundle* $\widetilde{\mathcal{R}}_{g,n} \to \mathcal{R}_{g,n}$. *Then for any $z \in \Sigma^*$ away from the punctures, W is isomorphic to* $\mathbf{E}^{\text{ad}}|_{\mathcal{R}_{g,n} \times \{z\}}$.

PROOF. By definition, W is the bundle $\widetilde{\mathcal{R}}_{g,n} \times_{\mathrm{SO}(3)} \mathfrak{so}(3)$, and a vector in the total space of the restriction of $\mathbf{E}^{\mathrm{ad}}$ to the $\mathcal{R}_{g,n}$ slice through z is the $\mathcal{G}$ orbit of a pair (t, A) where $t \in (\mathrm{Ad}E)_z$. A map $\mathbf{E}^{\mathrm{ad}} \to W$ over the $\mathcal{R}_{g,n}$ slice can be defined by

$$\mathcal{G} \cdot (t, A) \mapsto \overline{(\mathrm{hol}_z(\mathfrak{su}(A)), t)}$$

where t is an element of $\mathrm{Ad}E_z \cong \mathfrak{so}(3)$. Independence on the choice of representative (t, A) is seen by a straightforward unwinding of the definitions, which we carry out for completeness. For $g \in \mathcal{G}$, we must show that $(\mathrm{hol}_z(\mathfrak{su}(g \cdot A))), g_z \circ t \circ g_z^{-1})$ and $(\mathrm{hol}_z(\langle A \rangle_z), t)$ are SO(3)-equivalent. Indeed: the holonomy of $\mathfrak{su}(g \cdot A)$ is that of $\mathfrak{su}(A)$ conjugated by g_z, so these two pairs are identified by the action of the element $\overline{g}_z \in \mathrm{SO}(3)$. The map is certainly a linear isomorphism on fibers. $\square$

When the basepoint z is near an x_k (say $s_k(z) < 3/4$ for some k), we get a different kind of based gauge group which we shall denote by $\mathcal{G}_k$, consisting of gauge transformations which are the identity near the kth puncture. Since the original gauge group $\mathcal{G}$ was only allowed to take values $\mathrm{U}(1) \subset \mathrm{SU}(2)$ near the punctures, the quotient $\mathcal{G}/\mathcal{G}_k$ is isomorphic to U(1). Hence, the quotient $\mathcal{A}_{\mathrm{flat}}/\mathcal{G}_k$ is a principal U(1) bundle over $\mathcal{R}_{g,n}$. Let $\langle A \rangle_k$ denote the $\mathcal{G}_k$ equivalence class of A. We now finally bring the discussion back to our line bundles V_k defined at the beginning of the section.

LEMMA 1.4.5. *The* U(1) *bundle* $\mathcal{A}_{\mathit{flat}}/\mathcal{G}_k \to \mathcal{R}_{g,n}$ *is isomorphic to the bundle* $V_k \to \mathcal{R}_{g,n}$.

PROOF. The proof is analogous to that of Lemma 1.4.3. Since x is near the puncture, we have a chosen unitary identification $E_z \to \mathbb{C}^2$ under which the line F_k is spanned by $(1, 0)$. Given a connection A, the holonomy $\mathrm{hol}_z(\mathfrak{su}(A))$ gives an element of $\widetilde{\mathcal{R}}_{g,n}$. Since E_z is fixed by the based gauge group, this element is independent of the representative for the orbit $\mathcal{G}_k \cdot A$. Moreovoer, because we have fixed the 1-form of A (and thus of $\mathfrak{su}(A)$) near the punctures, the holonomy of $\mathfrak{su}(A)$ around a small loop around x_k staying inside the locus where $t \geq 1/2$ will be precisely $\mathbf{i}$. This implies that $\mathrm{hol}_z(\mathfrak{su}(A)) \in \widetilde{\mathcal{R}}_{g,n}$ is in V_k. The identification $\langle A \rangle_k \mapsto \mathrm{hol}_z(\mathfrak{su}(A))$ is manifestly a U(1)-equivariant bijection. $\square$

LEMMA 1.4.6. *There is an isomorphism*

$$\mathbf{V}_k \cong V_k$$

PROOF. We mimic the proof of Lemma 1.4.4. For clarity of notation, we let L_k specifically denote the complex line bundle $V_k \times_{\mathrm{U}(1)} \mathbb{C}$ associated to the principal U(1)-bundle $V_k \subset \widetilde{\mathcal{R}}_{g,n}$ over $\mathcal{R}_{g,n}$. Here, $w \in \mathrm{U}(1) \subset \mathbb{C}$ acts on $\mathbb{C}$ by simple multiplication, but the action on V_k (in order to make it a free one) is by conjugating a representation by a choice of square root $\mathrm{diag}(w^{1/2}, w^{-1/2})$. Now, a vector in the total space of $\mathbf{V}_k$ is a gauge orbit of a pair (t, A) where $t \in H_k \subset (\mathrm{Ad}E)_{x_k} \subset \mathrm{End}(E_{x_k})$. With our fixed isomorphism $E_{x_k} \cong \mathbb{C}^2$, t is just a matrix $\left(\begin{smallmatrix} 0 & w \\ -\overline{w} & 0 \end{smallmatrix}\right)$. Still letting z be a basepoint near x_k, we define a map $\mathbf{V}_k \to L_k$ via:

$$\mathcal{G} \cdot \left(\left(\begin{smallmatrix} 0 & w \\ -\overline{w} & 0 \end{smallmatrix}\right), A\right) \mapsto \overline{(\mathrm{hol}_z(\mathfrak{su}(A)), w)}$$

Let us prove the independence on gauge representative. Suppose $g \in \mathcal{G}$, and let $g_z = \left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)$ for a unit length $v \in \mathbb{C}$. Then we have:

$$\mathrm{hol}_z\left(\mathfrak{su}(g \cdot A)\right) = \left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right) \cdot \mathrm{hol}_z(\mathfrak{su}(A)) \cdot \left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)^{-1}$$
$$\text{and: } \left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & w \\ -\overline{w} & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & v^2 w \\ -v^{-2}\overline{w} & 0 \end{smallmatrix}\right).$$

The pair

$$\left(\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right) \cdot \mathrm{hol}_z(\mathfrak{su}(A)) \cdot \left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)^{-1}, v^2 w\right)$$

is equivalent to $(\mathrm{hol}_z(\mathfrak{su}(A)), b)$ under the action of $v^2 \in \mathrm{U}(1)$. This shows independence of the representative, and the resulting map is certainly a linear isomorphism. $\square$

COROLLARY 1.4.7. *We have $c_1(V_k) = c_1(\mathbf{V}_k)$ in $H^2(\mathcal{R}_{g,n}; \mathbb{Q})$.*

The upshot of this is that we may now use the $D^{\pm}_{k,l}$'s to study the cohomology ring of $\mathcal{R}_{g,n}$: we know that their first Chern classes of the V_k's are Poincaré dual to a pair of $D^{\pm}_{k,l}$'s, and we also know that these classes are part of a natural generating set for the cohomology. What remains is to determine the relations they satisfy.

1.4.3. Action of the Mapping Class Group and Flips. When analyzing the $\mathcal{D}^{\pm}_{k,l}$'s, it will be convenient to exploit the considerable symmetry in $\mathcal{R}_{g,n}$ arising from the symmetries of Σ^*.

Action of the Mapping Class Group. On the space of connections on $E \to \Sigma^*$ there is a (left) action of the basepoint preserving mapping class group $\mathrm{Mod}^*_{g,n}$ by the operation of pullback. Let us make this precise. Fix a basepoint $z \in \Sigma$ and for each $\phi \in \mathrm{Mod}_{g,n}$, fix representative $f : \Sigma^* \to \Sigma^*$ for ϕ with $f(z) = z$ and with the property that on the coordinate cylinders $U_k \cong (0,1) \times S^1$ near each puncture, f agrees with the coordinates. In other words, if f maps the kth puncture to the jth, then it gives a diffeomorphism U_k to U_j which is just $(s_k, \theta_k) \mapsto (s_j, \theta_j + \xi)$ for some constant ξ. In addition, fix a fiberwise isometry $F : E \to E$ covering f which is the identity at z, and let $\widehat{F} : \mathrm{Ad}E \to \mathrm{Ad}E$ denote the induced isometry. Given a connection $A \in \mathcal{A}$, we define a new connection $\widetilde{M}_\phi(A)$ by setting, for a local section t of $\mathrm{Ad}E$ and tangent vector $X \in T_x\Sigma^*$:

$$d_{\widetilde{M}_\phi(A)}(t) \cdot X = \widehat{F}(d_A(\widehat{F}^{-1} \circ t \circ f) \cdot df(X)) \tag{1.4.2}$$

This complicated looking formula is exactly what is required so that a section t is $M_\phi(A)$-flat if and only if $\widehat{F} \circ t \circ f^{-1}$ is A-flat. We see that $\widetilde{M}_\phi$ gives a map on $\mathcal{A}$ which preserves the set of flat connections. Now, if g is a gauge transformation, then t is $\widetilde{M}_\phi(g \cdot A)$-flat if and only if $\mathrm{Ad}g \circ \widehat{F} \circ t \circ f^{-1}$ is A-flat, if and only if $(\widehat{F}^{-1} \circ \mathrm{Ad}g \circ \widehat{F}) \circ t$ is $\widetilde{M}_\phi(A)$-flat. Moreover, it is clear that

$$(\widehat{F}^{-1} \circ \mathrm{Ad}g \circ \widehat{F}) = \mathrm{Ad}(F^{-1} \circ g \circ F)$$

and thus

$$\widetilde{M}_\phi(g \cdot A) = (F^{-1} \circ g \circ F) \cdot \widetilde{M}_\phi(A)$$

This shows that $\widetilde{M}_\phi$ preserves gauge equivalence classes. Hence, $\widetilde{M}_\phi$ descends to a diffeomorphism M_ϕ on $\mathcal{R}_{g,n}$. One can check that although the map $\widetilde{M}_\phi$ on $\mathcal{A}$ itself may depend very much on the choices made (for example, f and F), these choices do not affect the holonomy of a flat connection and so M_ϕ depends only the mapping class ϕ. Indeed, the holonomy of

$M_\phi(A)$ around a loop γ based at z is clearly just the holonomy of A around the loop $f(\gamma)$, which is certainly invariant under the choice of f for ϕ and completely independent of F. We therefore get an action of the mapping class group $\mathrm{Mod}^*_{g,n}$ on $\mathcal{R}_{g,n}$. Another way to see this action is simply by looking at the action of $\mathrm{Mod}^*_{g,n}$ on the $\mathbb{Z}/2$ extension $\widehat{\Gamma}$ of $\pi_1(\Sigma^*)$: precomposing representations by this action induces an one on $\widetilde{\mathcal{R}}_{g,n}$ (which lifts the action from $\mathcal{R}_{g,n}$).

Inside of $\mathrm{Mod}^*_{g,n}$, there is a copy of the braid group B_n on n strands arising from those diffeomorphisms which are supported on a small disk containing the punctures. There is a natural surjective homomorphism $\tau : B_n \to S_n$ where S_n is the symmetric group on n elements given by observing how a mapping class permutes the punctures.

LEMMA 1.4.8. *Let $\phi \in B_n \subset \mathrm{Mod}_{g,n}$, and set $\sigma = \tau(\phi)$. Then $M_\phi(D^{\pm}_{\sigma(k),\sigma(l)}) = D^{\pm}_{k,l}$, and $M^*_\phi(V_{\sigma(k)}) \cong V_k$.*

PROOF. The first statement is straightforward. If ϕ is a mapping class carrying neighborhoods of x_k and x_l to those of $x_{\sigma(k)}$ and $x_{\sigma(l)}$ respectively, then the holonomies of the connection $M_\phi(A)$ around x_k and x_l are just those of A around $x_{\sigma(k)}$ and $x_{\sigma(l)}$. Hence, M_ϕ carries connections with equal or antipodal holonomies around $x_{\sigma(k)}$ and $x_{\sigma(l)}$ to those with equal or antipodal connections around x_k to x_l, respectively.

For the second statement about V_k, let us make use of an explicit presentation for B_n. It is well known that B_n is generated by $n-1$ elementary braids $b_1, \dots, b_{n-1}$ subject to the relation $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for each i, and where b_i and b_j commute for $|i-j| \geq 2$. The action of B_n on $\pi_1(\Sigma^*)$ is given by

$$b_i(d_k) = \begin{cases} d_k, & \text{if } k \neq i, i+1 \\ d_{i+1}, & \text{if } k = i \\ d_{i+1}^{-1} d_i d_{i+1}, & \text{if } k = i+1 \end{cases},$$

and B_n fixes the a_j's. The lemma will be proved if we can show it is true when ϕ arises from an elementary braid b_i, since these generate the action. Since $\tau(b_i)$ is just the transposition

$(i\ \ i+1)$, we need to show that $M_\phi^*(V_{i+1})$ and V_i are isomorphic. Clearly, the action of M_ϕ on $\widetilde{\mathcal{R}}_{g,n}$ maps the subset V_{i+1} to V_i (since $\rho \circ b_i(d_i) = \rho(d_{i+1}) = \mathbf{i}$). In fact, it is easy to check that this map is equivariant with respect to the action of U(1) by conjugation. This proves directly that the pullback of V_{i+1} by the action of b_i on the base is isomorphic to V_i as a U(1) principal bundle, completing the proof. □

Flips. There is an additional set of symmetries in $\mathcal{R}_{g,n}$ which we call "flips". For any subset $J \subset \{1, \ldots, n\}$ of even cardinality, we get an involution on $\widetilde{\mathcal{R}}_{g,n}$, denoted by $\widetilde{M}_J$:

$$\widetilde{M}_J(\underline{S}, T_1, \ldots, T_n) = (\underline{S}, \epsilon_1 T_1, \ldots, \epsilon_n T_n)$$

where ϵ_k takes the value -1 if $k \in J$ and 1 if not. Since multiplication by -1 commutes with conjugation, this map descends to a map M_J on $\mathcal{R}_{g,n}$. The maps $\widetilde{M}_J$ and M_J are called *flips*.

Suppose $\phi \in \mathrm{Mod}_{g,n}^*$ is a mapping class which permutes the punctures according to a permutation σ of the indices. Then we have

$$M_\phi \circ M_{\sigma(J)} = M_J \circ M_\phi \tag{1.4.3}$$

Let us record the effect of the maps M_J on the line bundles V_k and submanifolds $D\pm_{k,l}$

LEMMA 1.4.9. *We have $M_J^*(V_k) \cong V_k^*$ if $k \in J$, otherwise $M_J^*(V_k) \cong V_k$. In addition, $M_J(D_{k,l}^\pm) = D_{k,l}^\pm$ if $|\{k,l\} \cap J|$ is even, and $M_J(D_{k,l}^\pm) = D_{k,l}^\mp$ if it is odd.*

PROOF. The second statement is completely obvious, so we focus on the line bundles V_k. In the case that $k \notin J$, the map $\widetilde{M}_J$ simply maps V_k to itself (U(1)-equivariantly). So, we suppose $k \in J$. As an S^1 fiber bundle, $M_J^*(V_k)$ is just $\widetilde{M}_J^{-1}(V_k)$ by definition of pullback, which is just the set of representations $[\underline{S}, \underline{T}]$ with $T_k = -\mathbf{i}$. The map on $\widetilde{\mathcal{R}}_{g,n}$ which conjugates a representation by $\mathbf{j}$ brings $\widetilde{M}_J^{-1}(V_k)$ back to V_k, but this map is not U(1)-equivariant. In fact, acting by $z \in \mathrm{U}(1)$ on $\widetilde{M}_J^{-1}(V_k)$ is the same as acting by $\overline{z}$ on V_k, so that these bundles are conjugates. □

Interaction with the Symplectic Structure. The maps M_J and M_ϕ have the convenient property that they are actually *symplectomorphisms* of the sympletic manifold $(\mathcal{R}_{g,n}, \omega)$.

PROPOSITION 1.4.10. *$M_J^*(\omega) = M_\phi^*(\omega) = \omega$.*

PROOF. Let us first prove that $M_\phi^*(\omega) = \omega$. Let $[\rho] \in \mathcal{R}_{g,n}$, choose a connection $A \in \mathcal{A}_{\text{flat}}$ with $\text{hol}_z(\mathfrak{su}(A)) = \rho$, and let $B = \widetilde{M_\phi}(A)$. The tangent space to the equivalence class $[\rho] \in \mathcal{R}_{g,n}$ consists of equivalence classes 1-forms with values in $\mathfrak{so}(\text{Ad}E)$ which vanish on the loci $s_k \geq 1/2$ near each puncture and are in the kernel of the connection operator d_A. For such a 1-form a, the pushforward 1-form $a' = (d\widetilde{M_\phi})_A(a)$ is given at $x \in \Sigma^*$ by (letting $X \in T_x\Sigma^*$ and $v \in E_x$):

$$a'_x(X) \cdot v = \widehat{F}_x^{\,-1}\left(a_{f(x)}(df(X)) \cdot \widehat{F}_x(v)\right)$$

Despite this complicated-looking formula, given two such 1-forms a, b, it is not hard to see that their pushforwards satisfy $\text{Tr}(a' \wedge b') = f^* \text{Tr}(a \wedge b)$, since the map $\widehat{F}$ preserves the bilinear form Tr on $\mathfrak{so}(\text{Ad}E)$. Hence:

$$\begin{aligned} M_\phi^*(\omega)\left([a] \wedge [b]\right) =& \frac{1}{4\pi^2}\int_\Sigma \text{Tr}\left(a' \wedge b'\right) \\ =& \frac{1}{4\pi^2}\int_\Sigma f^*\left(\text{Tr}\left(a \wedge b\right)\right) = \frac{1}{4\pi^2}\int_\Sigma \text{Tr}\left(a \wedge b\right) = \omega\left([a] \wedge [b]\right). \end{aligned}$$

This proves that elements of $\text{Mod}_{g,n}$ preserve ω.

As for the map M_J, we will need to lift this diffeomorphism to one on $\mathcal{A}_{\text{flat}}$ where the definition of ω originates; we need an operation on connections A which negate the holonomies of $\mathfrak{su}(A)$ around x_k for $k \in J$. The idea is that if A' is a connection with $\text{hol}_z(\mathfrak{su}(A')) = M_J(\text{hol}_z(\mathfrak{su}(A)))$, then A and A' will not be guage equivalent with respect to $\mathcal{G}$ but will be gauge equivalent with respect to the full gauge group $\widehat{\mathcal{G}}$ of SO(3) automorphisms of AdE over $\Sigma^* \setminus \{z_0\}$, not necessarily arising from an SU(2) automorphism of E. The group $\mathcal{G}$ is naturally a subgroup of $\widehat{\mathcal{G}}$ of index $|H^1(\Sigma; \mathbb{Z}/2)|$. Still using our trivialization of E outside the auxillary point z_0, let u_J be an SO(3) automorphism of AdE over $\Sigma^* \setminus \{z_0\}$ such that with respect to this trivialization, u_J gives a function $\Sigma^* \setminus \{z_0\} \to \mathrm{SO}(3)$ with $g(z) = \mathrm{Id}$ for whom the induced homomorphism $u_{J*} : \widehat{\Gamma} \to \pi_1(\mathrm{SO}(3), \mathrm{Id}) \cong \mathbb{Z}/2$ satisfies:

- $u_{J*}(a_j) = \overline{0}$, for $1 \le j \le 2g$.
- $u_{J*}(d_k) = \overline{1}$ for $k \in J$, and $\overline{0}$ for $k \notin J$.
- $u_{J*}(\zeta) = \overline{0}$, where ζ is the central order two generator.

Such a u_J can be shown to exist using a homotopy equivalence of $\Sigma^* \setminus \{z_0\}$ with a wedge of circles. Moreover, we can arrange u_J to be the identity near the puncture x_k for $k \notin J$, and for $k \in J$, the explicit function:

$$u_J(s_k, \theta_k) = \begin{pmatrix} \cos\theta_k & 0 & -\sin\theta_k \\ 0 & 1 & 0 \\ \sin\theta_k & 0 & \cos\theta_k \end{pmatrix} = \mathrm{Ad}\begin{pmatrix} 0 & e^{i\theta_k/2} \\ -e^{-i\theta_k/2} & 0 \end{pmatrix}. \tag{1.4.4}$$

Even though the matrix $\begin{pmatrix} 0 & e^{i\theta_k/2} \\ -e^{-i\theta_k/2} & 0 \end{pmatrix}$ is not well defined on $U_k \setminus \{x_k\}$, the expression does give an adjoint which is single-valued. Near x_k for $k \in J$, if a is the fixed connection 1-form of A, the 1-form of $u_J \cdot A$ is

$$\begin{aligned} \mathrm{Ad}_{u_J}(a) + u_J d(u_J^{-1}) &= \mathrm{ad}\Big[\tfrac{1}{4}\begin{pmatrix} 0 & e^{i\theta_k/2} \\ -e^{-i\theta_k/2} & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & -e^{i\theta_k/2} \\ e^{-i\theta_k/2} & 0 \end{pmatrix} \\ &\qquad + \tfrac{1}{2}\begin{pmatrix} 0 & e^{i\theta_k/2} \\ -e^{-i\theta_k/2} & 0 \end{pmatrix}\begin{pmatrix} 0 & ie^{i\theta_k/2} \\ ie^{-i\theta_k/2} & 0 \end{pmatrix}\Big] \\ &= \mathrm{ad}\Big[\tfrac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\Big] = \tfrac{1}{4}\mathrm{ad}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = a \end{aligned}$$

We see that acting by u_J preserves A near the punctures and so gives a map on $\mathcal{A}$, which certainly preserves the flat locus and descends to a function on $\mathcal{G}$ equivalence classes. The holonomy of $\mathfrak{su}(u_J \cdot A)$ around the loop d_k will clearly be that of $\mathfrak{su}(A)$ for $k \notin J$. Moreover, since u_J is just an SO(3) gauge transformation with $u_J(z) = \mathrm{Id}$, the SO(3) holonomy of $u_J \cdot A$ based at z is exactly that of A. Suppose if $k \in J$ and let $s(\theta_k)$ denote a section $[0, 2\pi] \to E$ over the loop d_k that is $\mathfrak{su}(A)$-parallel. Along $d_k : [0, 2\pi] \to \Sigma^*$, with respect to the trivialization of E, u_J is $\mathrm{Ad}v_J$ for some $v_J : [0, 2\pi] \to \mathrm{SU}(2)$. Since u_J does not extend across the kth puncture for $k \in J$, it must be the case that the lift v_J takes the value $v_J(2\pi) = -1$ instead of 1. Hence, the holonomy of $\mathfrak{su}(u_J \cdot A)$ around d_k must be -1 times that of $\mathfrak{su}(A)$. This shows that u_J lifts the map M_J to $\mathcal{A}_{\text{flat}}$. Because u_J is really just a gauge transformation, it certainly preserves the closed 2-form $\widetilde{\omega}$ on $\mathcal{A}_{\text{flat}}$. We conclude that M_J preserves ω as desired. □

Interaction with the Universal Bundle. There is one last result on the effect of flips and mapping classes we will need when we begin working with cohomology classes.

PROPOSITION 1.4.11. *For a mapping class $\phi \in \mathrm{Mod}_{g,n}$, let f be the chosen representative, and let σ be the corresponding permutation of the punctures. Then as bundles on $\mathcal{R}_{g,n} \times \Sigma$, we have*

(i) $(M_\phi \times f^{-1})^*(\mathbf{E}^{\text{ad}}) \cong \mathbf{E}^{\text{ad}}$.
(ii) $M_\phi^*(\mathbf{V}_k) \cong \mathbf{V}_{\sigma(k)}$.

For $J \subset \{1, \ldots, n\}$ with $|J|$ even, we also have

(iii) $(M_J \times \mathrm{Id})^*(\mathbf{E}^{\text{ad}}|_{\mathcal{R}_{g,n}\times\Sigma^*}) \cong \mathbf{E}^{\text{ad}}|_{\mathcal{R}_{g,n}\times\Sigma^*}$
(iv) $M_J^*(\mathbf{V}_k) \cong \begin{cases} \mathbf{V}_k, \text{ if } k \notin J \\ \mathbf{V}_k^*, \text{ if } k \in J \end{cases}$

PROOF. Recall that given the mapping class ϕ we have chosen a diffeomorphism representative f with controlled behavior near the punctures, and lifting isomorphism $\widehat{F} : \mathrm{Ad}E \to \mathrm{Ad}E$. To prove (i), we will construct a fiberwise isomorphism of $\mathbf{E}^{\text{ad}}$ to itself covering $M_\phi \times f^{-1}$. A point in the total space of $\mathbf{E}^{\text{ad}}$ is a $\mathcal{G}$-equivalence class $\overline{(t, A, x)}$ with $t \in \mathrm{Ad}E_x$. We define the map via:

$$\left(\overline{(t, A, x)}\right) \mapsto \overline{\left(\widehat{F}_x{}^{-1}(t), \widetilde{M}_\phi(A), f^{-1}(x)\right)}.$$

To show that this is independent of the choice of representative (t, A, x), let $g \in \mathcal{G}$. A different representative is $g \cdot (t, A, x) = (g_x \circ t \circ g_x^{-1}, g \cdot A, x)$, which gets sent to

$$\left(\widehat{F}_x{}^{-1}(g_x \circ t \circ g_x^{-1}), M_\phi(g \cdot A), f^{-1}(x)\right) \tag{1.4.5}$$

Since $M_\phi(g \cdot A) = (F^{-1} \circ g \circ F) \cdot M_\phi(A)$, it is clear that acting by the gauge transformation

$$F^{-1} \circ g \circ F \text{ on } \left(\widehat{F}_x^{\,-1}(t), \widetilde{M}_\phi(A), f^{-1}(x)\right)$$

gives exactly the output (1.4.5), proving independence of gauge representative. Hence, the recipe (1.4.5) is well defined on equivalence classes and gives the desired isomorphism of bundles. The second isomorphism (ii) of line bundles follows by an entirely analogous argument.

For (iii), we recall from the proof of Proposition 1.4.10 that there is an SO(3) gauge transformation u_J lifting M_J to the space of flat connections. We define a fiberwise isomorphism from $\mathbf{E}^{\mathrm{ad}}$ to itself covering $M_J \times \mathrm{Id}$ via:

$$\overline{(t, A, x)} \mapsto \overline{((u_J)_x \cdot t, u_J \cdot A, x)}. \tag{1.4.6}$$

It is exactly because u_J does not extend across the puncture that we cannot extend this map to one on $\mathbf{E}^{\mathrm{ad}}$ over all of Σ. To see independence on gauge representative, for $g \in \mathcal{G}$ the different representative $(g_x \circ t \circ g_x^{-1}, g \cdot A, x)$ is sent to:

$$\left((u_J)_x(g_x \circ t \circ g_x^{-1}), u_J \cdot (g \cdot A), x\right) = (u_J \circ \mathrm{Ad}_g \circ u_J^{-1}) \cdot ((u_J)_x(t), u_J \cdot A, x)\,.$$

It remains to show that $u_J \circ \mathrm{Ad}_g \circ u_J^{-1} = \mathrm{Ad}_{g'}$ for a gauge tranformation $g' \in \mathcal{G}$. A determinant 1 automorphism g' of E satisfying this equation will certainly exist and be unique up to sign but it remains to check that g' is a constant diagonal element of SU(2) near a puncture. This is obvious when $k \notin J$. Near the kth puncture for $k \in J$, we have:

$$\begin{aligned} u_J \circ \mathrm{Ad}_g \circ u_J^{-1} &= \mathrm{Ad}\begin{pmatrix} 0 & e^{i\theta_k/2} \\ -e^{-i\theta_k/2} & 0 \end{pmatrix} \circ \mathrm{Ad}\begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} \circ \mathrm{Ad}\begin{pmatrix} 0 & -e^{i\theta_k/2} \\ e^{-i\theta_k/2} & 0 \end{pmatrix} \\ &= \mathrm{Ad}\begin{pmatrix} \overline{w} & 0 \\ 0 & w \end{pmatrix} \end{aligned}$$

We prove the final isomorphism (iv) in the case $k \in J$. Let $h \in H_k \subset \mathrm{Ad}E_x$ so that $h = \mathrm{ad}\begin{pmatrix} 0 & -\overline{v} \\ v & 0 \end{pmatrix}$ for $v \in \mathbb{C}$. We define a fiberwise bijection $\mathbf{V}_k \to \mathbf{V}_k$ covering M_J on $\mathcal{R}_{g,n} \times \{x_k\}$ via:

$$\overline{\left(\mathrm{ad}\begin{pmatrix} 0 & -\overline{v} \\ v & 0 \end{pmatrix}, A\right)} \mapsto \overline{\left(\mathrm{ad}\begin{pmatrix} 0 & -v \\ \overline{v} & 0 \end{pmatrix}, u_J \cdot A\right)}. \tag{1.4.7}$$

Suppose $g_{x_k} = \mathrm{diag}(w, \overline{w})$. The different representative

$$g \cdot (h, A) = \left(\mathrm{ad}\left[\left(\begin{smallmatrix} w & 0 \\ 0 & \overline{w} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -\overline{v} \\ v & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{w} & 0 \\ 0 & w \end{smallmatrix}\right)\right], g \cdot A\right) = \left(\mathrm{ad}\left(\begin{smallmatrix} 0 & -w^2\overline{v} \\ \overline{w}^2 v & 0 \end{smallmatrix}\right), g \cdot A\right)$$

is sent to $\left(\mathrm{ad}\left(\begin{smallmatrix} 0 & -\overline{w}^2 v \\ w^2\overline{v} & 0 \end{smallmatrix}\right), u_J \cdot (g \cdot A)\right)$. It is not hard to check that this is the same as acting by $u_J \circ \mathrm{Ad}_g \circ u_J^{-1}$ on $\left(\mathrm{ad}\left(\begin{smallmatrix} 0 & -v \\ \overline{v} & 0 \end{smallmatrix}\right), u_J \cdot A\right)$. The proof is completed by noting that (1.4.7) is complex conjugate-linear, and so gives a fiberwise isomorphism $\mathbf{V}_k \to \mathbf{V}_k^*$ covering M_J. $\square$

PROPOSITION 1.4.12. *In Corollary 1.4.2 above, we have* $|r| = |s| = 1$.

PROOF. Let us first treat the case $n \geq 5$. By Lemma 1.4.8, we can assume without loss of generality that $k = 3$ and $l = 4$. We will use an auxillary submanifold to cut down the moduli space so that the necessary computation may be performed on a copy of the two-sphere. Let

$$\mathcal{S} \subset \mathcal{R}_{g,n}$$

denote the subset of points $[\underline{S}, \underline{T}]$ where the S_j's are all 1. There is an obvious identification $\mathcal{S} \leftrightarrow \mathcal{R}_{0,n}$ which carries $D_{k,l}^{\pm} \cap \mathcal{S} \subset \mathcal{R}_{g,n}$ to $D_{k,l}^{\pm} \subset \mathcal{R}_{0,n}$ and for which the pullback of V_k is just V_k. Hence, we can assume without a loss of generality that $g = 0$. Define another submanifold $Z \subset \mathcal{R}_{0,n}$ by:

$$Z = D_{4,5}^{+} \cap D_{5,6}^{+} \cap \cdots \cap D_{n-1,n}^{+}$$

As a general rule, all intersections of the above type will turn out to be transverse. An outline of a proof of transversality in this case would go as follows. Observe first that issues of transversality can be dealt with upstairs in $\widetilde{\mathcal{R}}_{0,n}$ with the preimages of the $D_{k,l}^{+}$'s. Next, argue that show that any 2-way intersection is transverse by reducing to the case of $n = 5$. Then, note that everytime another pair is added to the intersection, we are really studying an intersection in representation variety for 2 fewer points. The general case then follows by induction.

A point in Z is an equivalence class $[T_1, T_2, T_3, T_4, \ldots, T_4]$, where T_4 appears $n-3 = 2m-2$ (and so an even number of) times. Hence, we must have $T_1T_2T_3 = (-1)^m$. Up to conjugation, the representation is $[(-1)^{m+1}\mathbf{j}, \mathbf{k}, \mathbf{i}, T_4, \ldots, T_4]$, which defines T_4 uniquely. This gives both a map $f : Z \mapsto C_{\mathbf{i}}$, as well as a trivializing section of $V_3|_Z$. To compute r and s, we will study this restriction $V_3|_Z$ near the intersections $D^{\pm}_{3,4} \cap Z$. These intersections are transverse, and each is easily seen to be isomorphic to a representation variety for three parabolic points: a single point. Denote these intersection points by $\rho^{\pm}$.

In order to compute the values of r and s, we need to compute the winding number of the trivialization $\tau_{3,4}$ of $V_3|_{Z\setminus\{\rho^{\pm}\}}$ coming from Lemma 1.4.1 and $D^{+}_{3,4}$, with respect to the full trivialization τ' over all of Z. We will study the trivializations along the loop $\gamma : S^1 \to C_{\mathbf{i}}$ defined by

$$\gamma(e^{i\theta}) = [(-1)^{m+1}\mathbf{j}, \mathbf{k}, \mathbf{i}, e^{i\theta}\mathbf{j}, \dots, e^{i\theta}\mathbf{j}]$$

where the multiplication occurs in the unit quaternions. For a point $\gamma(e^{i\theta})$ in the loop, the trivialization $\tau_{3,4}$ requires us to find a number in the complex circle $S^1_{\mathbf{i}}$ by which we may conjugate $e^{\mathbf{i}\theta} \cdot \mathbf{j}$ so that it just becomes $\mathbf{j}$. This number is just $e^{i\theta/2}$ (or its opposite). Since the trivialization τ' is just given by

$$\tau'\left([(-1)^{m+1}\mathbf{j}, \mathbf{k}, \mathbf{i}, e^{i\theta}\mathbf{j}]\right) = \left((-1)^{m+1}\mathbf{j}, \mathbf{k}, \mathbf{i}, e^{i\theta}\mathbf{j}\right),$$

we see that the difference between the two trivialization is conjugating by $e^{\mathbf{i}\theta/2}$. Recall that the U(1) structure of V_3 is by the quotient of $S^1_{\mathbf{i}}$ by $\{\pm 1\}$, so the "clutching" function is really just $e^{i\theta} \mapsto e^{i\theta}$ and so is of degree ± 1. There is an ambiguity in signs arising from the lack of orientation chosen for the $D^{+}_{l,k}$'s and general disregard for sign conventions regarding the definition and computation of the first Chern class. This completes the proof in the case $n \geq 5$.

If $n = 3$, consider the embedding $\iota : \mathcal{R}_{g,3} \hookrightarrow \mathcal{R}_{g,5}$ given by

$$[\underline{S}, T_1, T_2, T_3] \mapsto [\underline{S}, T_1, T_2, T_3, -T_3, T_3]$$

We may assume without loss of generality that $k, l = 1, 2$. It is straightforward to see that $V_1(g,3) \cong \iota^* V_1(g,5)$. Since $D^{\pm}_{k,l}(g,3) = \iota^{-1}(D^{\pm}_{k,l}(g,5))$, the result follows from the $n = 5$ case. $\square$

1.5. Relations in the Cohomology Ring

We know from Theorem 1.2.7 that the cohomology ring of $\mathcal{R}_{g,n}$ comes with a generating set of classes of degrees 2, 3, and 4, arising from a universal bundle pair $(\mathbf{E}^{\mathrm{ad}}, \{\mathbf{V}_k\})$ on $\mathcal{R}_{g,n} \times \Sigma$. In order to understand the relations between these classes, it is necessary to describe them more concretely. For example, by the definition of slant product / the degree four class $p_1(\mathbf{E}^{\mathrm{ad}})/[\mathrm{pt}]$ is just $p_1(\mathbf{E}^{\mathrm{ad}}|_{\mathcal{R}_{g,n}\times\{\mathrm{pt}\}})$. This, we know by Lemma 1.4.4, is just $p_1(W)$, where again W is the rank three $\mathbb{R}$ vector bundle associated to the SO(3) bundle $\widetilde{\mathcal{R}}_{g,n} \to \mathcal{R}_{g,n}$.

In §1.4.2 we identified the classes $c_1(\mathbf{V}_k) \in H^2(\mathcal{R}_{g,n})$ with the classes $c_1(V_k)$, and we found explicit submanifolds representating their Poincaré duals. These submanifolds have the property that intersections between two of them (with one common index) behave like representation variety for two fewer marked points. The key to exploiting this property is identifying the restrictions of the other cohomology classes to these intersections with the corresponding classes in the lower representation varieties. This will give information about how the pairings of these other classes with the $c_1(V_k)$'s behave. In what follows, for a homology class $h \in H_*(\Sigma)$, we will use the simple shorthand $[h]$ to denote $p_1(\mathbf{E}^{\mathrm{ad}})/[h]$, so that, for example, $[\mathrm{pt}] = p_1(W)$.

1.5.1. The Class of the Symplectic Structure. There is another natural cohomology class on $\mathcal{R}_{g,n}$: the degree two class $[\omega]$ of the symplectic form. By Corollary 1.3.8 and Theorem 1.2.7, we know that $H^2(\mathcal{R}_{g,n};\mathbb{Q})$ is spanned by the classes $[\Sigma]$ and $c_1(V_1), \ldots, c_1(V_n)$. Hence, $[\omega]$ must be a linear combination of these:

$$[\omega] = A[\Sigma] + \sum_{k=1}^{n} D_k c_1(V_k) \tag{1.5.1}$$

By symmetry the D_k's must all be equal. We claim that $A \neq 0$. To see this, suppose on the contrary that $[\omega] = D\sum_{k=1}^{n} c_1(V_k)$. We average M_J^* applied to this equation over all even J. The number of even J containing an index is the same as the number not containing it and $M_J^*(c_1(V_k)) = \pm c_1(V_k)$ depending on whether $k \in J$, so the right hand side of the averaged equation vanishes. However $M_J^*([\omega]) = [\omega]$, so we arrive at $[\omega] = 0$. But $[\omega]$ cannot be zero as it is the class of a symplectic form on a compact manifold. We conclude that $A \neq 0$, and we have proved:

PROPOSITION 1.5.1. *The $n+1$ classes $[\omega], c_1(V_1), \ldots, c_1(V_n)$ are a basis for $H^2(\mathcal{R}_{g,n})$.*

The class $[\omega]$ is more convenient than $[\Sigma]$ for us because it behaves well under the symmetries, including flips. What we lack is a geometric description for the Poincaré dual of $[\omega]$. Instead, for us the important data concerning this class will be the pairing of its top power with the fundamental class of the moduli space, called the symplectic volume. The paper [**12**] gives a formula for this top pairing in the case of arbitrary rational weights t_k. We note that in that paper, their weights in $(0,1)$ correspond to twice the value of our weights in $(0,1/2)$. We have rewritten their formula to agree with our conventions.

THEOREM 1.5.2. *([**12**], Prop. 4.12, and eq. (5.3)) The pairing of the top power of $[\omega]$ with the moduli space is given by:*

$$\langle [\omega]^{3g-3+n}, \mathcal{R}_{g,n}(\underline{t})\rangle = \frac{(3g+n-3)!}{2^{g-2}\pi^{2g-2+n}} \sum_{N=1}^{\infty} \frac{1}{N^{2g-2+n}} \prod_{k=1}^{n} \sin(2\pi N t_k) \tag{1.5.2}$$

COROLLARY 1.5.3. *For the case $n \geq 1$ and odd and $\underline{t} = (1/4, \ldots, 1/4)$, we have*

$$\langle [\omega]^{3g+n-3}, \mathcal{R}_{g,n}(\underline{t})\rangle = \frac{(3g+n-3)!}{2^{3g+n-3} g!} |E_{2g+n-3}| \tag{1.5.3}$$

where E_j is the jth Euler number defined to be the coefficient of $x^j/j!$ in the Taylor series of $\operatorname{sech}(x) = 1/\cosh(x)$.

PROOF. We plug in the weight $t_k = 1/4$ in (1.5.2) and simplify. For these values of t_k, the even N terms in the sum in (1.5.2) vanish, are we are left with:

$$\langle [\omega]^{3g-3+n}, \mathcal{R}_{g,n}(\underline{t})\rangle = \frac{(3g+n-3)!}{2^{g-2}\pi^{2g-2+n}} \sum_{M=1}^{\infty} \frac{(-1)^M}{(2M+1)^{2g-2+n}} \tag{1.5.4}$$

The sum now appearing consists of alternating negative powers of the odd integers, which is a well documented function of the exponent $2g-2+n$ known as a the Dirichlet β function.

It can be computed exactly, and one has:

$$\beta(2l+1) = \frac{(-1)^l \pi^{2l+1} E_{2l}}{2^{2l+2}(2l)!}$$

Plugging $\beta(2g-2+n)$ in for the sum in (1.5.4) gives the desired formula. □

1.5.2. Inductive Properties of the Moduli Space. In order to get information about pairings with the classes $c_1(V_k)$, let us further analyze the $D^{\pm}_{k,l}$'s and their intersections. Given a pair (k, l) of indices, we can construct a map to a smaller representation variety $D^{\pm}_{k,l} \to \mathcal{R}_{g,n-2}$ in the following way. Let $b \in B_n$ be such that the correpsonding permutation carries $(n-1, n)$ to (k, l). Then b gives a diffeomorphism from $D^{\pm}_{k,l}$ to $D^{\pm}_{n-1,n}$. There is then a map $D^{\pm}_{n-1,n} \to \mathcal{R}_{g,n-2}$ given by:

$$[\underline{S}, T_1, \ldots, T_{n-2}, T_{n-1}, \pm T_{n-1}] \mapsto [\underline{S}, T_1, \ldots, T_{n-3}, \mp T_{n-2}] \tag{1.5.5}$$

Indeed, $T_n \cdot T_n = -1$. This map is clearly surjective, and its fiber is just the freedom is choosing T_n: it is a copy of the sphere $C_{\mathbf{i}}$. Hence, by composing it with b, we see that $D^{\pm}_{k,l}$ is a $C_{\mathbf{i}}$ fiber bundle over $\mathcal{R}_{g,n-2}$.

Remark 1.5.4. This situation is to be contrasted with that in [**32**], where all of the parabolic weights t_k are distinct. There, the analogue of $D^{\pm}_{k,l}$ is also a (connected component of a) subspace where T_k and T_l commute but admits an isomorphism to an actual representation variety for one fewer parabolic point, where the parabolic weights t_k and t_l are replaced by the single weight $t_k \pm t_l$. Such a representation variety for us does not exist (smoothly), of course.

Consider now an intersection $D^{+}_{j,k} \cap D^{+}_{k,l}$ with j, k, l distinct. Rechoose b so that its permutation carries $(n-2, n-1, n)$ to (j, k, l), so b gives a diffeomorphism $D^{+}_{j,k} \cap D^{+}_{k,l}$ to $D^{+}_{n-2,n-1} \cap D^{+}_{n-1,n}$. There is then a map from $D^{+}_{n-2,n-1} \cap D^{+}_{n-1,n}$ to $\mathcal{R}_{g,n-2}$ given by restricting the map (1.5.5):

$$[\underline{S}, T_1, \ldots, T_{n-2}, T_{n-2}, T_{n-2}] \mapsto [\underline{S}, T_1, \ldots, T_{n-3}, -T_{n-2}] \tag{1.5.6}$$

which is a diffeomorphism. Composed with the map induced by b, we get a diffeomorphism of $D^+_{j,k} \cap D^+_{k,l}$ with $\mathcal{R}_{g,n-2}$. There are many such maps as a result of, for example, the choice of the braid b, but they all differ by postcomposing with the maps from flips and mapping classes on $\mathcal{R}_{g,n-2}$. There are similar isomorphisms for intersections $D^\pm_{j,k} \cap D^\pm_{k,l}$. Moreover, the inverse of (1.5.6) gives a section of the 2-sphere fiber bundle (1.5.5). It is also easy to see that 1.5.6 carries $D^\pm_{a,b} \cap D^+_{n-2,n-1} \cap D^+_{n-1,n}$ to the corresponding $D^\pm_{a,b}$ inside $\widetilde{\mathcal{R}}_{g,n-2}$, for $a, b \leq n-3$, and to $D^\mp_{a,b}$ if one of a, b is $n-2$, $n-1$, or n. The moral is that successive intersections of the $D^\pm_{k,l}$'s behave like representation varieties for fewer parabolic points. This recursive property is the key to understanding the cohomology of $\mathcal{R}_{g,n}$.

Proposition 1.5.5. *Let $\iota : D^\pm_{k,l} \to \mathcal{R}_{g,n}$ be the inclusion, and $\pi : D^\pm_{k,l} \to \mathcal{R}_{g,n-2}$ be a map arising from from formula (1.5.5) and the discussion preceding it, and let $\tau : \{1, \ldots, n-2\} \hookrightarrow \{1, \ldots, n\}$ denote the corresponding inclusion of index sets. If $\mathbf{E}^{\mathrm{ad}}(g, n)$ denotes the universal bundle on $\mathcal{R}_{g,n} \times \Sigma$, then we have:*

- $\mathbf{E}^{\mathrm{ad}}(g, n)|_{D^\pm_{k,l} \times \Sigma} \cong (\pi \times \mathrm{id})^* \mathbf{E}^{\mathrm{ad}}(g, n-2)$
- $\mathbf{V}_{\tau(k)}(g, n)|_{D^\pm_{k,l} \times \Sigma} \cong (\pi \times \mathrm{id})^* \mathbf{V}_k(g, n-2)$ *or* $(\pi \times \mathrm{id})^* \mathbf{V}_k(g, n-2)^*$

Proof. As in the proof of Proposition 1.5.6, we again assume we are working with $D^-_{n-1,n}$, let π be the map (1.5.7) and we use all the same notation from there. Let $\widehat{\mathcal{A}}_\nu$ denote the space of flat connections on $\widehat{\Sigma}^*$ arising from the operation of extending connections in $\mathcal{V}$ by the product connection; it is the space of flat connections on a surface with two fewer punctures whose 1-forms with respect to the induced trivialization on $\widehat{\Sigma}^* \setminus \{z_0\}$ are zero on a disk U. Denote the composition $\mathcal{A}_\nu \to \mathcal{V} \to \widehat{\mathcal{A}}_\nu$ by using a hat. Now, since $\mathcal{A}_\nu / \mathcal{G}_\nu = \mathcal{A}_{\mathrm{flat}} / \mathcal{G}$, the bundles $\mathbf{E}^{\mathrm{ad}}(g, n)$ and $\mathbf{V}_k(g, n)$ can also be constructed as the quotient of the pullback of $\mathrm{Ad}E$ and H_k to $\mathcal{A}_\nu \times \Sigma$ and $\mathcal{A}_\nu \times \{x_k\}$. We claim that a fiberwise isometry $\mathbf{E}^{\mathrm{ad}}(g, n) \to \mathbf{E}^{\mathrm{ad}}(g, n-2)$ covering $\pi \times \mathrm{id}$ may be defined by the formula:

$$\overline{(t, A, x)} \mapsto \overline{(t, \widehat{A}, x)}$$

for $A \in \mathcal{A}_\nu$ and $t \in \mathrm{Ad}E_x$. Note that the underlying compact surface Σ and bundle E for any number of punctures is the same, so we are free to repeat "x" and "t" on the right hand side. It is obvious that this is well-defined: if $g \in \mathcal{G}_\nu$ and $\widehat{g}$ is the tranformation g replaced by 1 on U, then $g \cdot (t, A, x)$ is sent to $\widehat{g} \cdot (t, \widehat{A}, x)$. This proves the first bundle isomorphism. The second is proved via an almost identical argument. The ambiguity between the line bundle and its dual arises from the effect of flips on the $\mathbf{V}_k$'s. $\square$

As a corollary, we see that the classes [pt] and $[a_j]$ restricted to $D^{\pm}_{k,l}$ are all pulled back from $\mathcal{R}_{g,n-2}$ in the expected way. Finally, we need to study how the projection $D^{\pm}_{k,l} \to \mathcal{R}_{g,n-2}$ behaves with respect to the class $[\omega]$. We have:

Proposition 1.5.6. *Denote by $\omega_{g,n}$ the symplectic form on $\mathcal{R}_{g,n}$. Let $\pi : D^{\pm}_{k,l} \to \mathcal{R}_{g,n-2}$ be any of the natural 2-sphere fiber bundles arising from the map (1.5.5), and let $\iota : D^{\pm}_{k,l} \hookrightarrow \mathcal{R}_{g,n}$ be the inclusion. Then we have $\iota^*(\omega_{g,n}) = \pi^*(\omega_{g,n-2})$.*

Proof. We illustrate the Proposition in the case $D^{-}_{k,l}$, as the other case follows by invoking Lemma 1.4.9. Without loss of generality, we may take $k, l = n-1, n$, and π is the map:

$$[\underline{S}, T_1, \ldots, T_{n-2}, T_{n-1}, -T_{n-1}] \mapsto [\underline{S}, T_1, \ldots, T_{n-2}]. \tag{1.5.7}$$

Any of the other natural choices for π differ by postcomposing with the known symplectomorphisms of $\mathcal{R}_{g,n-2}$. We need to realize this map as an operation on flat connections in $\mathcal{A}$ in order to understand its interaction with the symplectic form. It will be convenient to work with SU(2)-connections, so fix a basepoint $z \in \Sigma^*$ away from the punctures and recall our trivialization of E away from the auxillary point z_0. Connections are now the same as $\mathfrak{su}(2)$-valued 1-forms on $\Sigma^{**} = \Sigma^* \setminus \{z_0\}$, and these 1-forms are fixed near the punctures. Let $U \in \Sigma$ be a disk containing the neighborhoods of only the punctures x_{n-1}, x_n and such that $z \in \partial U$ and $z_0 \notin U$, let $\widehat{\Sigma}^*$ denote the surface obtained by filling in these two punctures, and set $V = \Sigma \setminus U$. Let $\mathcal{A}^{-}_{n-1,n}$ denote the space of connections A for whom $[A] \in D^{-}_{n-1,n}$. Every connection in $\mathcal{A}^{-}_{n-1,n}$ has SU(2) holonomy around ∂U equal to the product of two antipodal elements of SU(2), which is just the identity. Since $\pi_1(\mathrm{SU}(2)) = 1$, for any $A \in \mathcal{A}^-_{n-1,n}$ we can find a gauge equivalent A' such that the 1-form of $\mathfrak{su}(A')$ vanishes on a fixed annular neighborhood ν of ∂U. Let $\mathcal{A}_\nu$ denote the subset of such (flat) connections; this subset is acted on by the subgroup $\mathcal{G}_\nu$ of gauge transformations which are constant on ν and the quotient is all of $D^-_{n-1,n}$.

Let $\mathcal{U}, \mathcal{V}$ denote the spaces of flat connections on U, V respectively, with the desired behavior near the punctures and whose 1-forms vanish near the boundary. Restriction of connections gives a homeomorphism $\widetilde{\eta} : \mathcal{A}_\nu \to \mathcal{U} \times \mathcal{V}$. If $\widetilde{\omega}_{\mathcal{U}}$ and $\widetilde{\omega}_{\mathcal{V}}$ denote the 2-forms on $\mathcal{U}$ and $\mathcal{V}$ coming from restricting the domain of integration in the definition (1.2.4) of $\widetilde{\omega}$ on $\mathcal{A}_\nu \subset \mathcal{A}$, then splitting the domain of integration implies that

$$\widetilde{\eta}^*(\widetilde{\omega}_{\mathcal{U}} \oplus \widetilde{\omega}_{\mathcal{V}}) = \widetilde{\omega} \tag{1.5.8}$$

Letting $\mathcal{G}^1_\nu \subset \mathcal{G}_\nu$ denote the subgroup of g with $g|_\nu = 1$, the map $\widetilde{\eta}$, the 2-forms in equation (1.5.8), and the relationship 1.5.8 descend to

$$\widetilde{\eta}' : \mathcal{A}_\nu/\mathcal{G}^1_\nu \to \mathcal{U}/\mathcal{G}^1_\nu \times \mathcal{V}/\mathcal{G}^1_\nu,$$

2-forms $\widetilde{\omega}'_{\mathcal{V}}$, $\widetilde{\omega}'_{\mathcal{U}}$, and $\widetilde{\omega}'$ and the equation $\widetilde{\eta}'^*(\widetilde{\omega}'_{\mathcal{U}} \oplus \widetilde{\omega}'_{\mathcal{V}}) = \widetilde{\omega}'$. Now, given $A' \in \mathcal{V}$ and $A'' \in \mathcal{U}$, there are corresponding connections $\widehat{A}'$ and $\widehat{A}''$ on $\widehat{\Sigma}^*$ and $\widehat{U}$, the sphere gotten by capping off U with another disk, obtained by extending via the trivial connection. This identifies (using the ideas of Lemma 1.4.3) the factor $\mathcal{U}/\mathcal{G}^1_\nu$ with the space

$$\widetilde{\mathcal{R}}_U = \{T_{n-1}, T_n \in C_{\mathbf{i}} : \; T_{n-1}T_n = 1\} \cong C_{\mathbf{i}}$$

and the factor $\mathcal{V}/\mathcal{G}^1_\nu$ with $\widetilde{\mathcal{R}}_{g,n-2}$. The space $\mathcal{A}_\nu/\mathcal{G}^1_\nu$ is clearly just the preimage $\widetilde{D}^-_{n-1,n}$ in $\widetilde{\mathcal{R}}_{g,n}$. Implicitly using all these identifications, it is straightforward to check that $\widetilde{\omega}'_{\mathcal{V}}$ corresponds to the closed 2-form $\widetilde{\omega}'_{g,n-2}$ on $\widetilde{\mathcal{R}}_{g,n-2}$ which is the pullback of $\omega_{g,n-2}$, and $\widetilde{\omega}'$ is the pullback of $\omega_{g,n}$ on $\mathcal{R}_{g,n}$. We have achieved now the isomorphism of manifolds with 2-forms:

$$\widetilde{\eta}' : (\widetilde{D}^-_{n-1,n}, \widetilde{\omega}'_{g,n}) \cong (C_{\mathbf{i}}, \widetilde{\omega}'_{\mathcal{U}}) \times (\widetilde{\mathcal{R}}_{g,n-2}, \widetilde{\omega}'_{g,n-2})$$

The quotient $\mathcal{G}_\nu/\mathcal{G}_\nu^1$ is isomorphic to SU(2) and the residual action of this group on $\widetilde{\mathcal{R}}_{g,n-2}$ is projectively free. The quotient on both sides (where the action on the right hand side is the simultaneous one) gives the $C_{\mathbf{i}}$-fiber bundle π. We are done then if we can show that $\widetilde{\omega}'_{\mathcal{U}}$ is actually 0. The key point is that all the connections in $\mathcal{U}$ are actually gauge equivalent; the quotient of $C_{\mathbf{i}}$ by SU(2) is a single point. Any two tangent vectors in $T_{A''}\mathcal{U}$ then differ by $(dg)g^{-1}$ for some gauge transformation g, which is in the annihilator of the linear form $\widetilde{\omega}'_{\mathcal{U}}$ (see the discussion on the symplectic structure in §1.2.1). Hence, $\widetilde{\omega}'_{\mathcal{U}}$ must vanish. □

Corollary 1.5.7. *The restriction of the map π in 1.5.6 to the intersection $D^{\epsilon_1}_{j,k} \cap D^{\epsilon_2}_{k,l}$ is a symplectomorphism.*

Proof. This restriction is a diffeomorphism by the discussion preceding Proposition 1.4.12, and its inverse gives a section of the 2-sphere bundle π. □

Up until now, we have been entirely focused on the line bundles V_k and degree two classes $c_1(V_k)$. We now consider the degree three classes $[a_j]$ for $1 \leq j \leq 2g$. Their role in the cohomology ring has been well understood for over twenty years, as we now review. As shown in [**31**], it is convenient to introduce the class $\gamma_j = \frac{1}{16}[a_{2j-1}][a_{2j}]$ (our normalization here will be justified later on) for $1 \leq j \leq g$. For each j, there is a natural embedding $\iota_j : \mathcal{R}_{g-1,n} \hookrightarrow \mathcal{R}_{g,n}$ given by

$$[\underline{S}, \underline{T}] \mapsto [S_1, \ldots, S_{2j-2}, 1, 1, S_{2j-1}, \ldots, S_{2g-2}, \underline{T}]$$

whose image is exactly the collection of representations $[\underline{S}, \underline{T}]$ for which $S_{2j-1} = S_{2j} = 1$. In [**31**], it is proved that the submanifold $\iota_j(\mathcal{R}_{g-1,n})$ is Poincaré dual to γ_j (at least in the case $n = 0$, and the proof adapts readily to our situation). It is straightforward to check using the methods of the current paper that ι_j respects the symplectic forms, and the universal bundle pair over $\mathcal{R}_{g-1,n}$ is pulled back via ι_j. This fact shows that the cohomology ring also has an inductive structure in the genus g. Putting everything together, we have:

Proposition 1.5.8. *The product class $\gamma_{j_1} \cdots \gamma_{j_r} c_1(V_{k_1}) \cdots c_1(V_{k_s})$ is a constant multiple of the Poincaré dual to a collection of 2^s submanifolds of $\mathcal{R}_{g,n}$ each symplectomorphic to a* *copy of* $\mathcal{R}_{g-r,n-2s}$ *by a map under which the classes* $[\mathrm{pt}]$, $[a_j]$, *and* $c_1(V_k)$ *(where* $j \neq 2j_i, 2j_i-1$ *and* $k \neq k_i$ *for any* i*) are all pulled back accordingly.*

PROOF. This is all a straightforward synthesis of facts proved up to this point. The result follows by induction after proving it in the case $r, s = 0, 1$ or $1, 0$. The case when the class is $c_1(V_k)$ for some k follows from Propositions 1.5.5 and 1.5.6. The case of the class γ_j follows from [**31**] and the discussion preceding the proposition. $\square$

COROLLARY 1.5.9. *Suppose* $f(a,b)$ *is a polynomial such that* $f([\omega],[\mathrm{pt}]) \in H^*(\mathcal{R}_{g,n};\mathbb{Q})$ *equals the zero class. Then the polynomial*

$$\gamma_{j_1}\cdots\gamma_{j_r}c_1(V_{k_1})\cdots c_1(V_{k_s})\cdot f([\omega],[\mathrm{pt}]) = 0$$

equals the zero class in the ring $H^*(\mathcal{R}_{g+r,n+2s};\mathbb{Q})$.

1.5.3. The Four Dimensional Class of a Point. We would like to record some properties of the degree four class [pt] and the degree three classes $[a_j]$. We first prove an easy equation relating [pt] to the degree two classes $c_1(V_k)$.

LEMMA 1.5.10. *For any* k*, we have* $[\mathrm{pt}] = -c_1(V_k)^2$

PROOF. We will prove this by studying a corresponding isomorphism of universal bundles on $\mathcal{R}_{g,n}$. By definition $[\mathrm{pt}] = p_1(\mathbf{E}^{\mathrm{ad}}|_{\mathcal{R}_{g,n}\times\{\mathrm{pt}\}})$ where $\mathrm{pt} \in \Sigma$. Since the isomorphism type of $\mathbf{E}^{\mathrm{ad}}|_{\mathcal{R}_{g,n}\times\{\mathrm{pt}\}}$ is independent of the choice of pt as $\mathcal{R}_{g,n}$ is connected, we are free to choose $\mathrm{pt} = x_k$ for any k. Write $\mathbf{E}^{\mathrm{ad}}_k \to \mathcal{R}_{g,n}$ for the restriction of $\mathbf{E}^{\mathrm{ad}}$ in this case. By definition of $\mathbf{V}_k$, it is clear that $\mathbf{E}^{\mathrm{ad}}_k$ is isomorphic as a real vector bundle to $\underline{\mathbb{R}} \oplus \mathbf{V}_k$. Hence, $p_1(\mathbf{E}^{\mathrm{ad}}_k) = -c_1(\mathbf{V}_k)^2$. $\square$

REMARK 1.5.11. One can prove Lemma 1.5.10 directly by studying the associated vector bundle constructions of W and V_k from PU(2) and U(1) bundles. Let L_k again denote the line bundle $V_k \times_{\mathrm{U}(1)} \mathbb{C}$ and W the associated bundle $\widetilde{\mathcal{R}}_{g,n} \times_{\mathrm{SO}(3)} \mathbb{R}^3$. Then for $\overline{(\rho,w)}$ in V_k, a map $L_k \oplus \underline{\mathbb{R}} \to W$ is given by

$$\left(\overline{(\rho, w)}, r\right) \mapsto \left(\rho, \begin{pmatrix} r & w \\ -\overline{w} & -r \end{pmatrix}\right)$$

LEMMA 1.5.12. *The class* [pt] *is Poincaré dual to a union of four, disjoint, codimension four submanifolds* D_1, D_2, D_3, *and* D_4 *each with a symplectomorphism* τ_κ *to* $\mathcal{R}_{g,n-2}$ *for* $\kappa = 1, 2, 3, 4$. *Letting* $\iota_\kappa : D_\kappa \to \mathcal{R}_{g,n}$ *denote the inclusions, these symplectomorphisms also satisfy* $\tau_\kappa^*[\mathrm{pt}] = \iota_\kappa^*[\mathrm{pt}]$ *and* $\tau_\kappa^*[a_j] = \iota_\kappa^*[a_j]$ *for* $1 \leq j \leq 2g$.

PROOF. By Lemma 1.5.10, we have $[\mathrm{pt}] = -c_1(V_k)^2$ for all k. By Proposition 1.4.12, we see that

$$\mathrm{PD}([\mathrm{pt}]) = ([D_{12}^+] + [D_{12}^-]) \cap ([D_{23}^+] + [D_{23}^-])$$

for some choice of orientations for these submanifolds. Each of the four terms $D_{12}^\pm \cap D_{23}^\pm$ in the expansion is symplectomorphic to $\mathcal{R}_{g,n-2}$ by 1.5.7, through a map under which the universal bundle $\mathbf{E}^{\mathrm{ad}}$ pulls back to the restriction. The classes [pt] and $[a_k]$ therefore also pull back, being defined through the universal bundle. □

COROLLARY 1.5.13. *Suppose* f *is a polynomial in the* $[a_j]$*'s,* $c_1(V_k)$*'s,* $[\omega]$*, and* [pt] *which is a relation in* $H^*(\mathcal{R}_{g,n}; \mathbb{Q})$*. Then the polynomial* $f \times [\mathrm{pt}]^s$ *is a relation in* $H^*(\mathcal{R}_{g,n+2s}; \mathbb{Q})$

1.5.4. The Classes [Σ] and [ω]. We now make a brief digression on the relationship between the class $[\Sigma]$ and $[\omega]$. It will be convenient later to nail down precisely the linear combination (1.5.1). What we need is to describe the action of the flips M_J on the class $[\Sigma]$. The issue is that $[\Sigma]$ is not preserved; by Proposition 1.4.11, the pullback of the universal bundle $\mathbf{E}^{\mathrm{ad}}$ by M_J is only an isomorphism away from the x_k's. Fix an even J, and suppose $k \in J$. Recall that we have small disk neighborhoods U_k with polar coordinate (s_k, θ_k) around each puncture x_k and a trivialization of the U(2) bundle E. Let Σ° denote $\Sigma \setminus \cup_{l=1}^n U_l$. The map M_J lifts to the space of flat connections via an SO(3) gauge transformation u_J, and the map (1.4.6) is an isomorphism of $\mathbf{E}^{\mathrm{ad}}$ to $(M_J \times \mathrm{Id})^*\mathbf{E}^{\mathrm{ad}}$ over $\mathcal{R}_{g,n} \times \Sigma^*$. The bundles are also isomorphic when restricted to $\mathcal{R}_{g,n} \times U_k$. To see this, we note that the bundles are isomorphic when restricted to $\mathcal{R}_{g,n} \times \{x\}$ for any point $x \in \Sigma^\circ$, and so this is also true for $x \in U_k$. Since $\mathcal{R}_{g,n} \times U_k$ contracts to $\mathcal{R}_{g,n}$, the isomorphism is automatic. In fact, the proof of Lemma 1.5.10 shows that the restriction of $\mathbf{E}^{\mathrm{ad}}$ to $\mathcal{R}_{g,n} \times \{x_k\}$ is isomorphic to $\underline{\mathbb{R}} \oplus \mathbf{V}_k$, and so the restriction of $(M_J \times \mathrm{Id})^*\mathbf{E}^{\mathrm{ad}}$ to $\mathcal{R}_{g,n} \times \{x_k\}$ is isomorphic to $\underline{\mathbb{R}} \oplus \mathbf{V}_k^*$, which is isomorphic to $\underline{\mathbb{R}} \oplus \mathbf{V}_k$ as a real bundle. Hence, we can describe the new bundle $(M_J \times \mathrm{Id})^*\mathbf{E}^{\mathrm{ad}}$ as being obtained by cutting $\mathbf{E}^{\mathrm{ad}}$ along $\mathcal{R}_{g,n} \times \partial U_k$ for each k and regluing with a "clutching function". We can compute this function as follows: it is the composition of the following circle of maps:

$$\begin{tikzcd}
\mathbf{E}^{\mathrm{ad}} \arrow[d] & & (M_J \times \mathrm{Id})^*\mathbf{E}^{\mathrm{ad}} \arrow[ll] \\
\underline{\mathbb{R}} \oplus \mathbf{V}_k \arrow[r, "-1\oplus\mathrm{Id}"] & \underline{\mathbb{R}} \oplus \mathbf{V}_k^* \arrow[r] & \underline{\mathbb{R}} \oplus (M_J \times \mathrm{Id})^*\mathbf{V}_k \arrow[u]
\end{tikzcd}$$

The maps, beginning with the vector $\overline{(t, A, \theta)}$ in $\mathbf{E}^{\mathrm{ad}}|_x$ for $x = (1, \theta_k) \in \partial U_k$, $t = \mathrm{ad}\left(\begin{smallmatrix} s & -\bar{z} \\ z & -s \end{smallmatrix}\right) \in \mathrm{Ad}E|_x$ and A a flat connection, compose to:

$$\begin{aligned}
\overline{\left(\mathrm{ad}\left(\begin{smallmatrix} s & -\bar{z} \\ z & -s \end{smallmatrix}\right), A\right)}_{\theta_k} \mapsto\ & \left(s, \overline{\left(\mathrm{ad}\left(\begin{smallmatrix} 0 & -\bar{z} \\ z & 0 \end{smallmatrix}\right), A\right)}\right)_{\theta_k} \mapsto \\
& \left(-s, \overline{\left(\mathrm{ad}\left(\begin{smallmatrix} 0 & -z \\ \bar{z} & 0 \end{smallmatrix}\right), u_J \cdot A\right)}\right)_{\theta_k} \mapsto \qquad \text{(by equation (1.4.7))} \\
& \overline{\left(\mathrm{ad}\left(\begin{smallmatrix} -s & -z \\ \bar{z} & s \end{smallmatrix}\right), u_J \cdot A\right)}_{\theta_k} \mapsto \\
& \overline{\left(\mathrm{ad}\begin{pmatrix} 0 & -e^{i\theta_k/2} \\ e^{-i\theta_k/2} & 0 \end{pmatrix}\left(\begin{smallmatrix} -s & -z \\ \bar{z} & s \end{smallmatrix}\right)\begin{pmatrix} 0 & e^{i\theta_k/2} \\ -e^{-i\theta_k/2} & 0 \end{pmatrix}, A\right)}_{\theta_k} \\
=\ & \overline{\left(\mathrm{ad}\begin{pmatrix} s & -e^{i\theta_k}\bar{z} \\ e^{-i\theta_k} z & -s \end{pmatrix}, A\right)}_{\theta_k}
\end{aligned} \tag{1.5.9}$$

We see that near x_k, $\mathbf{E}^{\mathrm{ad}}$ is isomorphic to the pullback of $\mathbb{R} \oplus \mathbf{V}_k$ to $\mathcal{R}_{g,n} \times \widetilde{U}_k$ for a slightly larger disk neighborhood $\widetilde{U}_k \supset U_k$, and $(M_J \times \mathrm{Id})^*\mathbf{E}^{\mathrm{ad}}$ is obtained by cutting this bundle along $\mathcal{R}_{g,n} \times \partial U_k$ and regluing $\underline{\mathbb{R}} \oplus \mathbf{V}_k$ to itself via $(s, v) \mapsto (s, e^{-i\theta_k} v)$, for each $k \in J$. It is not difficult to see that the characteristic classes of these two bundles are therefore related by:

LEMMA 1.5.14. *Upon slant product with* $[\Sigma]$, *the first Pontryagin classes of* $\mathbf{E}^{\mathrm{ad}}$ *and* $(M_J \times \mathrm{Id})^*\mathbf{E}^{\mathrm{ad}}$ *are related by:*

$$p_1\left((M_J \times \mathrm{Id})^*\mathbf{E}^{\mathrm{ad}}\right)/[\Sigma] = p_1\left(\mathbf{E}^{\mathrm{ad}}\right)/[\Sigma] - 2\sum_{k\in J} c_1(\mathbf{V}_k)$$

PROOF. This is standard bundle theory and unwinding the definition of the slant product. □

We can use the lemma to determine the relationship between the cohomology classes $[\Sigma]$ and $[\omega]$. As before, by symmetry we know $[\omega] = s[\Sigma] + t\sum_{k=1}^n c_1(V_k)$ for some constants s, t. The class ω is invariant under flips, and the lemma implies that the only linear combinations of $[\Sigma]$ and $c_1(V_k)$ which are flip-invariant are scalar multiples of $[\Sigma] - \sum_{k=1}^n c_1(V_k)$. We conclude that

$$[\omega] = A\left([\Sigma] + \sum_{k=1}^n c_1(V_k)\right) \tag{1.5.10}$$

for some nonzero constant A. In fact, from [**12**] we see that in our notation $A = -1/4$.

Define the graded commutative polynomial algebra

$$\mathbb{A}_{g,n} := \mathbb{C}[\alpha, \beta, \delta_1, \ldots, \delta_n] \otimes \wedge^*[\psi_1, \ldots, \psi_{2g}],$$

where we assign α and δ_k degree 2, ψ_j degree 3, and β degree 4 (i.e. α, β, and the d_k's are commutative and the ψ_j's anti-commute with each other). We denote by $\mathbb{H}_{g,n}$ the $\mathbb{C}$-algebra $H^*(\mathcal{R}_{g,n};\mathbb{C})$. We can define a map $\Psi : \mathbb{A}_{g,n} \to \mathbb{H}_{g,n}$ via:

$$\alpha \mapsto 2[\omega]$$

$$\beta \mapsto -\tfrac{1}{4}[\mathrm{pt}]$$

$$\psi_j \mapsto -\tfrac{1}{4}[a_j]$$

$$\delta_k \mapsto \tfrac{1}{2}c_1(V_k)$$

The fractional factors in front of each generator are used to make the presentation of the cohomology ring simpler and stem from the fact that we use the Pontryagin class of an adjoint universal bundle, rather than the second Chern class of a standard universal bundle. We remark that this notation gives

$$\Psi(\alpha) = -\tfrac{1}{2}[\Sigma] + \sum_{k=1}^{n} \Psi(\delta_k). \tag{1.5.11}$$

What we have proved so far is that $\mathbb{H}_{g,n}$ is isomorphic to the ring $\mathbb{A}_{g,n}/\mathcal{I}_{g,n}$ for some ideal of relations $\mathcal{I}_{g,n} = \psi^{-1}(0)$, which includes the relations $\beta - \delta_k^2$ for each k. For each g, n, $n \geq 1$, there is a natural inclusion

$$\iota_{g,n}^{0,1} : \mathcal{R}_{g,n} \hookrightarrow \mathcal{R}_{g,n+2}$$

arising from the isomorphism $D_{n,n+1}^{-} \cap D_{n+1,n+2}^{+} \mapsto \mathcal{R}_{g,n}$, which by virtue of the results of this section has the property that pulling back the images under Ψ of the generators α, β, ψ_j, and δ_k (for $k = 1, \ldots, n$) gives the corresponding generators for the smaller cohomology ring. It can also be checked that δ_{n+1} pulls back to $-\delta_n$ and δ_{n+2} pulls back to δ_n. Let $\pi_{g,n}^{0,1} : \mathbb{A}_{g,n+2} \to \mathbb{A}_{g,n}$ denote the corresponding ring map. Then Corollary 1.5.9 implies that $\pi(\mathcal{I}_{g,n}) \subset \mathcal{I}_{g,n}$. More generally, there are inclusions

$$\iota_{g,n}^{r,s} : \mathcal{R}_{g,n} \hookrightarrow \mathcal{R}_{g+r,n+2s}$$

with corresponding ring maps $\pi_{g,n}^{r,s} : \mathbb{A}_{g+r,n+2s} \to \mathbb{A}_{g,n}$, under which

$$\delta_k \mapsto \begin{cases} (-1)^{k-n}\delta_n, & k > n \\ \delta_k, & k \leq n \end{cases}, \qquad \alpha_j \mapsto \begin{cases} \pm 0, & j > 2g \\ \alpha_j, & j \leq 2g \end{cases}$$

and we have the inclusion $\pi_{g,n}^{r,s}(\mathcal{I}_{g+r,n+2s}) \subset \mathcal{I}_{g,n}$. This is a direct consequence of Corollary 1.5.9 and [**28**]. This encapsulates the inductive structure of the moduli spaces, and in the case of no marked points is well known.

NOTATION. From now on, we denote by a hat the image under Ψ in $\mathbb{H}_{g,n}$ of a generator in $\mathbb{A}_{g,n}$ by a hat. For example, we have $\widehat{\beta} = \widehat{\delta}_k^2$ for all k.

There is an immediate relation in the cohomology ring resulting from comparing different versions of the volume class on $\mathcal{R}_{g,n}$. Namely, if we let $2D = \dim_{\mathbb{R}}(\mathcal{R}_{g,n}) = 6g + 2n - 6$ and set $e = D \mod 2 = 0, 1$, then the images of $\widehat{\alpha}^D$ and $\widehat{\beta}^{\lfloor D/2 \rfloor}\widehat{\alpha}^e$ are both multiples of eachother. We are thus interested in pairings of the form $\left\langle \widehat{\alpha}^r \widehat{\beta}^s, \mathcal{R}_{g,n} \right\rangle$ for $r + 2s = D$. Lemma 1.5.12 allows us to compute these, once we know the symplectic volume formula (1.5.3), at least for $s \leq m$:

$$\langle \widehat{\alpha}^r \widehat{\beta}^s, \mathcal{R}_{g,n} \rangle = \frac{(3g + n - 2s - 3)!}{(2g + n - 2s - 3)!} |E_{2g+n-2s-3}| = \frac{r!}{(r - g)!} |E_{r-g}| . \tag{1.5.12}$$

However, there will be relations of smaller degree involving only $\widehat{\alpha}$ and $\widehat{\beta}$, which arises from this formula and Poincaré duality. The role of Poincaré duality is in the following statement: if $f \in \mathbb{A}_{g,n}$ with $\deg f = r$ and $\langle \Psi(f)\Psi(f'), \mathcal{R}_{g,n}, =\rangle \, 0$ for all f' of degree $2D - f'$, then f is a relation in $\mathcal{I}_{g,n}$. It also implies that there must be at least one relation of degree $D + 2$, or just over half the dimension, since the Betti numbers must be symmetric about the middle dimension and the dimension of the degree d part of $\mathbb{A}_{g,n}$ strictly increases with d.

1.6. The Case $g = 0$

Recall that we have set $n = 2m + 1 > 1$, and we now set $g = 0$. In this case $\mathbb{A}_{0,n}$ is generated by α, β, and the δ_k's, and $D = n - 3 = 2m - 2$ (in fact, β is redundant).

PROPOSITION 1.6.1. *There is a unique polynomial $r_{0,n}(\alpha, \beta)$ in $\mathcal{I}_{0,n}$ of degree $2m$ monic with respect to α. It is obtained via the recursion*

$$\begin{gathered} r_{0,2m+3}(\alpha, \beta) = \alpha \cdot r_{0,2m+1}(\alpha, \beta) - m^2 \beta \cdot r_{0,2m-1}(\alpha, \beta) \\ r_{0,1}(\alpha, \beta) = 1, \qquad r_{0,3}(\alpha, \beta) = \alpha \end{gathered} \tag{1.6.1}$$

PROOF. We will argue by induction. We know a relation of degree $2m$ must exist due to Poincaré duality. The content of the lemma is that there is a relation involving only α and β, that it is unique, and that there is a recursive formula in n which it satisfies. To see that there is a relation in α and β and that it is unique, let W_m denote the vector space of possible polynomials $r_{0,n}(\alpha,\beta)$ of degree $2m$. Letting e denote the remainder upon dividing m by 2, such a polynomial looks like

$$r_{0,n}(\alpha,\beta) = A_m\alpha^m + A_{m-2}\alpha^{m-2}\beta + \ldots + A_e\alpha^e\beta^{\lfloor m/2\rfloor}$$

so that $\dim W_m = \lfloor m/2\rfloor + 1$. For each of the $\lfloor m/2\rfloor$ choices of $s = \lfloor m/2\rfloor - 1, \lfloor m/2\rfloor - 2, \ldots, 0$, there is a monomial $\alpha^{m-2-2s}\beta^s$ of complementary degree $2m-4$. We get $\lfloor m/2\rfloor$ linear functionals

$$\ell_s : r_{0,n} \mapsto \left\langle r_{0,n}\widehat{\alpha}^{m-2-2s}\widehat{\beta}^s, \mathcal{R}_{0,n}\right\rangle$$

and the relation we seek will lie in the kernel of each of them. Since $\dim W_m$ has one greater dimension than this collection of functionals, a (nonzero) $r_{0,n}$ will certainly exist.

We can rephrase this as saying the vector $(A_e,\ldots,A_{m-2},A_m) \in \mathbb{R}^{\lfloor m/2\rfloor+1}$ of coefficients is in the kernel of the $(\lfloor m/2\rfloor - 1)\times\lfloor m/2\rfloor$ matrix (E_{ij}) where $E_{ij} = \left\langle \widehat{\alpha}^{2e+2j+2i}\widehat{\beta}^{\lfloor m/2\rfloor-1-i}, \mathcal{R}_{0,n}\right\rangle$ with $i = 0,\ldots,\lfloor m/2\rfloor - 1$ and $j = 0,\ldots,\lfloor m/2\rfloor$. By the formula (1.5.12), we have in the case $g = 0$ the very simple expression $E_{ij} = |E_{2e+2i+2j}|$. This is an example of a “Hankel” matrix, a type of matrix which arises when studying the so-called “moment problem” in connection with the theory of orthogonal polynomials and continued fractions (here, with the sequence of moments $|E_0|, |E_2|, |E_4|, \ldots$). This theory, along with the known continued fraction expansion for the formal generating function $\sum_{i=0}^{\infty} |E_i|\, x^i$, implies that the polynomial $r_{0,n}$ is unique up to scale and satisfies the beautiful recurrence relation (1.6.1). We relegate the proof of this formula to the appendix.

The polynomial $r_{0,n}$ is rigged to pair to 0 with each of the complementary degree monomials involving just $\widehat{\alpha}$ and $\widehat{\beta}$. It remains to check that this $r_{0,n}$ pairs to 0 with complementary polynomials in not just $\widehat{\alpha}$ and $\widehat{\beta}$ but also the δ_k's. Because of the relation $\widehat{\delta}_k^2 = 4\widehat{\beta}$, we simply need to check that $r_{0,n}$ pairs to 0 with terms of the form $\widehat{\alpha}^r\widehat{\beta}^s\widehat{\delta}_{k_1}\cdots\widehat{\delta}_{k_t}$ with $r+2s+t = m-2$ and the k_i's distinct. By inductive hypothesis, we have the relation $r_{0,n-2k}$ in the ideal $\mathcal{I}_{0,n-2s}$. The recurrence relation implies that $r_{0,n}$ is also a relation in $\mathcal{I}_{0,n-2s}$. By Corollary 1.5.9 the product $\delta_{k_1} \cdots \delta_{k_t} \cdot r_{0,n}$ is a relation in $\mathcal{I}_{0,n}$, which is enough to ensure the vanishing of all pairings in $\mathbb{A}_{0,n}$ with $r_{0,n}$. □

From this, we can write down a large collection of relations which must hold in the ring $\mathbb{H}_{0,n}$.

COROLLARY 1.6.2. *For each $J \subset \{1, \ldots, n\}$ with $|J| = s \leq m$, the polynomial*

$$R^J_{0,n} = r_{0,n-2s}(\alpha, \beta) \cdot \prod_{k \in J} \delta_k \tag{1.6.2}$$

is in the ideal of relations $\mathcal{I}_{0,n}$

PROOF. Simply combine Proposition 1.6.1 and Corollary 1.5.9. □

All that remains to show is that this is a complete set of relations. For this we mimic the approach in [**29**] and describe an explicit basis for $H^*(\mathcal{R}_{0,n})$, and show that any other monomial can be expressed a linear combination of monomials in the basis and the relations $R^J_{0,n}$. In what follows, we denote $\underline{\delta}^J = \prod_{k \in J} \delta_k$.

LEMMA 1.6.3. *Let $\mathcal{S}_{0,n}$ denote the collection of monomials $\alpha^a \beta^b \underline{\delta}^J$ with $a + b + |J| < m$. Then any other monomial $\alpha^{a'} \beta^{b'} \underline{\delta}^{J'}$ with $a' + b' + |J'| \geq m$ can be reduced to a linear combination of monomials in $\mathcal{S}_{0,n}$ and the relations $R^J_{0,n}$ and $\delta_i^2 - \beta$.*

PROOF. We can certainly assume $|J| < m$, since $\underline{\delta}^J$ is a relation if $|J| \geq m$. We first treat the case $J = \emptyset$. Because the leading term (with respect to α) of $R^{\emptyset}_{0,n} = r_{0,n}$ is α^m and all other terms are monomials in α and β with lower exponent sum, we can certainly reduce the monomial $\alpha^a \beta^b$ for $a + b \geq m$ to a linear combination of $\alpha^r \beta^s$ with $r + s < m$. Hence, we suppose $|J| \geq 1$. Now, suppose that $\phi \in \mathrm{Mod}_{0,n}$. If we can reduce $M^*_\phi(z)$ for a monomial z, then we can certainly reduce z, because the collection of relations $R^J_{0,n}$ is preserved by the mapping class group action. Since the mapping class group action serves to permute the δ_i's, without a loss of generality we may prove the lemma for monomials with $J = J' \cup \{\delta_n\}$ where $J' \subset \{1, \ldots, n-2\}$.

We argue by induction on n. Suppose the lemma is true for n and for $J \subset \{1, \ldots, n+2\}$ assume that $J = J' \cup \{\delta_{n+2}\}$ with $J' \subset \{1, \ldots, n\}$. Suppose that $a + b + |J| \geq m + 1$. Then $a + b + |J'| \geq m$ and so by inductive hypothesis, the monomial $\alpha^a \beta^b \underline{\delta}^{J'}$ may be reduced to a linear combination of relations $R^K_{0,n}$, $\delta_i^2 - \beta$, and monomials in $\mathcal{S}_{0,n}$. Multiplying $R^K_{0,n}$ by δ_{n+2} gives the relation $R^{K \cup \{\delta_{n+2}\}}_{0,n+2}$, and so multiplying this linear combination by δ_{n+2} gives a reduction for our monomial $\alpha^a \beta^b \underline{\delta}^J$, as desired. This completes the proof. □

PROPOSITION 1.6.4. *Along with the relations $\delta_i^2 - \beta$, the set of relations $R^J_{0,n}$ is a complete set.*

PROOF. We saw in §1.3 that the Poincaré polynomial of $\mathcal{R}_{0,n}$ agreed with that of the graded algebra $\mathbb{C}[\alpha, \beta, \delta_1, \ldots, \delta_n]/(\delta_i^2)$ up to the middle dimension. The algebra

$$\mathbb{C}[\alpha, \beta, \delta_1, \ldots, \delta_n]/(\delta_i^2 - \beta)$$

has the same Poincaré polynomial. Hence, there can be no relations other than $\delta_i^2 - \beta$ in $H^*(\mathcal{R}_{0,n})$ below degree $2m$, which is the degree of $R^J_{0,n}$. Now, by Lemma 1.6.3 the relations imply that $\mathcal{S}_{0,n}$ as above contains a basis. But it is a simple matter to check that the if $\mathcal{S}_{0,n}(d)$ denotes the number of monomials of degree d, then $\mathcal{S}_{0,n}(d)$ is the same as the the dimension of the degree d part of $\mathbb{C}[\alpha, \beta, \delta_1, \ldots, \delta_n]/(\delta_i^2 - \beta)$ up to the middle dimension. Moreover, it is easy to check that $\mathcal{S}_{0,n}(2n - 6 - d) = \mathcal{S}_{0,n}(d)$, and so by Poincaré duality, we must have that $\mathcal{S}_{0,n}$ actually is a basis, and so we have a complete set of relations. □

We have therefore proved Theorem 1.1.1.

1.7. The Case $n = 1$

The case when $g \neq 0$ and $n = 1$ turns out to be readily accessible for us as it is closely related to the original story for no marked points. Recall that $\mathcal{R}_{g,1}$ is a $\mathbb{C}P^1$-bundle over $\mathcal{R}_{g,0}$, with the projection given by a forgetful map on parabolic stable bundles.

PROPOSITION 1.7.1. *The cohomology ring* $\mathbb{H}_{g,1} = H^*(\mathcal{R}_{g,1})$ *is naturally an algebra over* $\mathbb{H}_{g,0} = H^*(\mathcal{R}_{g,0})$*, and has a presentation*

$$\mathbb{H}_{g,1} \cong \mathbb{H}_{g,0}[\delta_1]/(\delta_1^2 - \beta). \tag{1.7.1}$$

PROOF. Let $f : \mathcal{R}_{g,1} \to \mathcal{R}_{g,0}$ denote the bundle projection. This establishes $\mathbb{H}_{g,1}$ as an algebra over $\mathbb{H}_{g,0}$. By the discussion following Corollary 1.4.2, the universal bundle on $\mathcal{R}_{g,1}$ is pulled back from $\mathcal{R}_{g,0}$, and so if a relation in $[\Sigma]$, [pt], and the $[a_j]$'s holds in $\mathcal{R}_{g,0}$, then it holds in $\mathcal{R}_{g,1}$. It is well-known that the cohomology ring of a $\mathbb{C}P^1$-bundle is generated as an algebra over the cohomology of the base by an additional generator τ satisfying a single additional relation $\tau^2 + c\tau + d = 0$, where c and d are characteristic classes. For us, we know that the relation $c_1[V_1]^2 + [\mathrm{pt}]^2 = 0$ (and thus $\delta_1^2 - \beta = 0$) holds, by Lemma 1.5.10. The Poincaré polynomial of $\mathbb{H}_{g,0}[\delta_1]/(\delta_1^2 - \beta)$ is clearly exactly $(1 + t^2)$ times the Poincaré polynomial of $\mathbb{H}_{g,0}$, which is exactly what we know to be the polynomial of $\mathcal{R}_{g,1}$, by (1.3.13). We conclude that $\tau^2 + c\tau + d$ and $\delta_1^2 - \beta$ must be proportional and the theorem follows. □

It is now well-known that the cohomology ring of the moduli space $\mathcal{R}_{g,0}$ has a presentation based on a triple of relations which are recursive in the genus g. Let $(r^{(1)}_{g,0}, r^{(2)}_{g,0}, r^{(3)}_{g,0})$ be the relations given in [**29**] as Proposition 3.2 (noting that we switch sub- and superscripts from that notation): $(r^{(1)}_{1,0}, r^{(2)}_{1,0}, r^{(3)}_{1,0}) = (\alpha, \beta, \gamma)$ and:

$$\begin{aligned} r^{(1)}_{g+1,0} &= \alpha r^{(1)}_{g,0} + g^2 r^{(2)}_{g,0} \\ r^{(2)}_{g+1,0} &= \beta r^{(1)}_{g,0} + \frac{2g}{g+1} r^{(3)}_{g,0} \\ r^{(3)}_{g+1,0} &= \gamma r^{(1)}_{g,0} \end{aligned} \tag{1.7.2}$$

Then it is a result of previous work on $\mathcal{R}_{g,0} \cong \mathcal{M}^0_g(2,1)$ that the ideal $\mathcal{I}_{g,0} \subset \mathbb{A}_{g,0}$ of relations is generated by relations $R^{(i),J}_{g,0}$, where $J \subset \{1, \ldots, g\}$ and

$$R^{(i),J}_{g,0} = R^{(i)}_{g-s,0} \prod_{j \in J} \gamma_j \tag{1.7.3}$$

where $s = |J|$.

Let $\widehat{\alpha}$ denote the image of the class α in $\mathbb{A}_{g,0}$ in $\mathbb{H}_{g,1}$ under the isomorphism (1.7.1. The class $\widehat{\alpha}$ in $\mathbb{H}_{g,0}[\delta_1]/(\delta_1^2 - \beta)$ is no longer proportional to the class of the symplectic form and thus differs from the image of $\alpha \in \mathbb{A}_{g,1}$ under the map $\Psi : \mathbb{A}_{g,1} \to \mathbb{H}_{g,1}$: it is pulled back from the symplectic form on $\mathcal{R}_{g,0}$. In fact, since the universal bundle on $\mathcal{R}_{g,1}$ is pulled back (again, see the discussion following Corollary 1.4.2), we have $\widehat{\alpha} = -\frac{1}{2}p_1(\mathbf{E}^{\mathrm{ad}})/[\Sigma]$. By (1.5.11), we have $\Psi(\alpha) = \widehat{\alpha} + \delta_1$. Under the change of variable $\alpha \mapsto \alpha - \delta_1$, we have the relations:

$$
\begin{aligned}
r^{(1)}_{g+1,1} &= (\alpha - \delta_1) r^{(1)}_{g,1} + g^2 r^{(2)}_{g,1} \\
r^{(2)}_{g+1,1} &= \beta r^{(1)}_{g,1} + \frac{2g}{g+1} r^{(3)}_{g,1} \\
r^{(3)}_{g+1,1} &= \gamma r^{(1)}_{g,1}
\end{aligned}
\tag{1.7.4}
$$

and $(r^{(1)}_{1,1}, r^{(2)}_{1,1}, r^{(3)}_{1,1}) = (\alpha - \delta_1, \beta, \gamma)$. We set

$$
R^{(i),J}_{g,1} = R^{(i)}_{g-s,1} \prod_{j \in J} \gamma_j
\tag{1.7.5}
$$

Corollary 1.7.2. *The ideal of relations $\mathcal{I}_{g,1} \subset \mathbb{A}_{g,1}$ is generated by the relations $R^{(i),J}_{g,1}$ above and $\delta_1^2 - \beta$.*

CHAPTER 2

Instanton Floer Homology of a Product Link

2.1. Introduction

Given a closed, oriented 3-manifold Y with SU(2)-bundle E and $w_2(E) = w$, instanton Floer homology gives a vector space $I^w_*(Y)$ obtained as the homology of a chain complex whose generators are isomorphism classes flat SU(2) connections on Y. The differential of this complex is defined by studying instantons on the cylinder $\mathbb{R} \times Y$. This construction is informed by viewing it as an infinite dimensional, S^1-valued Morse theory with Morse function the Chern-Simons functional, whose critical points are simply the flat connections.

A natural problem is understanding the vector space $\mathbb{V}_{g,0} := I^w_*(Y_g)$ for Y_g the product $S^1 \times \Sigma$ and w dual to the S^1 factor. It has the additional structure of an algebra by applying the functoriality of I_* to the 4-manifold with boundary $F \times \Sigma$, where F is a pair of pants, in order to get a multiplication map $\mathbb{V}_{g,0} \otimes \mathbb{V}_{g,0} \to \mathbb{V}_{g,0}$. This space was studied in [**27**], where a presentation for this ring was given in terms of natural generators α, β and ψ_j, $j = 1, \ldots, 2g$ and relations. An important piece information extracted from this ring presentation is the spectrum of the multiplication by α map, which is shown to be the set of even integers between $\pm(2g-2)$. This information, along with the fact that the eigenspaces for the top values $\pm(2g-2)$ are 1-dimensional, was used in [**22**] to defined a version of instanton Floer homology for sutured 3-manifold.

A version of instanton Floer homology for three-manifolds Y with link K, denoted $I_*(Y, K)$ was described by Kronheimer and Mrowka in [**17**]. There, the critical points are flat connections which have a singularity along K of a presecribed type, causing the holonomy around a small meridian to give a trace 0 element of SU(2). A natural generalization of the product 3-manifold story above is to allow Σ to be a surface of genus g along with marked points $x_1, \ldots, x_n$ and to study the Floer homology of the product $Y_g = S^1 \times \Sigma$ with the product link $K_n = S^1 \times \{x_1, \ldots, x_n\}$ (here n must be odd). There is an analogous ring structure on the vector space

$$\mathbb{V}_{g,n} := I_*(Y_g, K_n) \otimes \mathbb{C}.$$

In this chapter we shall study the case $g = 0$ by combining our results from Chapter 1 on the ring structure of $H^*(\mathcal{R}_{0,n}; \mathbb{C})$ with techniques in Floer homology to determine a presentation for the ring $\mathbb{V}_{0,n}$.

We model our approach on Muñoz's, but it does not seem that there are suitable tools readily available to perform a strictly analogous computation, appealing to Donaldson invariants of algebraic surfaces; the relevant surface would include a sphere of square 0. Instead, we find an eigenvector by leveraging a somewhat simpler version $\mathbb{U}_{0,n}$ of the Floer homology of the pair (Y_0, K_n) obtained by using a bundle on $Y_0 \setminus K_n$ which does not extend to Y_0. Machinery for studying the instanton Floer homology for such a bundle is provided by the recent paper [**23**] of Kronheimer and Mrowka. The vector space $\mathbb{U}_{0,n}$ is not a ring but there is a multiplication map $\mathbb{U}_{0,n}^{\otimes 2} \to \mathbb{V}_{g,n}$. An eigenvector in $\mathbb{U}_{0,n}$ is easy to find, and this gives one in $\mathbb{V}_{0,n}$. As a consequence, we are able to write down a complete presentation of the ring $\mathbb{V}_{0,n}$ (see Theorem 2.9.10). From this presentation it is easy to read off the spectrum for multiplication by a natural generator α, which is the set of odd integers between $\pm(n-2)$, and the eigenspaces for the values $\pm(n-2)$ are 1-dimensional. We shall use this fact in the next chapter to define a version of instanton Floer homology for sutured manifolds with embedded tangle.

2.2. Preliminaries on Floer Homology

There are by now several different constructions and versions of instanton Floer homology in the literature. Our work here requires the full power of the singular instanton theory for a three-manifold Y with link K. In fact, we will need a slightly more general construction, laid out in detail by Kronheimer and Mrowka in [**23**], which allows for a bundle on the complement $Y \setminus K$ of the link which does not necessarily extend to all of Y. We summarize here the definitions and constructions required to build this version of I_*. All of this material and more can be found in [**23**] and the references therein.

2.2.1. Singular Connections. As with most constructions in gauge theory, the heart of the technical content of [**23**] required to define the Floer homology groups is describing the particular Sobolev spaces of connections and gauge groups acting on them. Here, we will study connections on a bundle which are singular along a codimension 2 locus. The issues involved in setting this up fall into two categories: global topological issues coming from the kinds of vector bundles our connections live in, and analytical issues arising from defining suitable Banach spaces. Once the topological issues have been settled, the analytical ones follow mostly standard lines, once the language and techniques of orbifold bundles is introduced.

We begin with a closed Riemannian 4-manifold X with embedded surface Σ, and P a $\mathrm{PU}(2)$-bundle on $X \setminus \Sigma$. Rather than choosing a smooth metric, we pick an orbifold metric $\check{g}$ on $X \setminus \Sigma$ which has cone angle π around Σ, so that it is locally the quotient of a smooth metric on $D^2 \times D^2$ by the rotation $(z_1, z_2) \mapsto (z_1, -z_2)$. We view X as an orbifold singular along Σ which local group $\mathbb{Z}/2$ at points on Σ, writing $\check{X}$ to denote X viewed in this way. See [**18**] for details on orbifold metrics.

We are interested in connections A on P away from Σ, with the following behavior near Σ: for each $x \in \Sigma$, there is a B^4 neighborhood U of x over which there is a trivialization of P and local cylindrical coordinates x_1, x_2, r, θ (for which Σ is the locus $r = 0$) such that the connection 1-form of A is given by

$$a_0 + \frac{1}{4}\mathrm{ad}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} d\theta. \tag{2.2.1}$$

Here, a_0 is a regular $\mathfrak{so}(3)$-valued 1-form over U, and $\mathrm{ad} : \mathfrak{su}(2) \to \mathfrak{so}(3)$ is the canonical isomorphism of Lie algebras. This connection is singular along Σ, and has holonomy around a small circle linking Σ equal to an order 2 element of $\mathrm{PU}(2) \cong \mathrm{SO}(3)$. Any connection with this local behavior determines a double cover $\Sigma_\Delta \to \Sigma$: locally there are two possible ways of modifying A to extend across Σ by adding a singular 1-form, giving a $\mathbb{Z}/2$ local system Δ. The approach in [**23**] is to start with this double cover Σ_Δ. We let X_Δ be a non-Hausdorff space in which Σ has been replaced with Σ_Δ, which admits a map

$$\pi : X_\Delta \to X_\Delta,$$

where π is 2-to-1 over Σ and is a homoemorphism away elsewhere. One then studies a bundle

$$P_\Delta \to X_\Delta$$

along with additional information called "singular bundle data", subject to some additional conditions. The topological issue, and the reason for introducing the double cover Σ_Δ, is exactly the fact that Δ may not be a trivial local system. In that case, the bundle $P = P_\Delta|_{X\setminus\Delta}$ will not extend across Σ, complicating somewhat the topological classification of possible bundles. We recall the definition of singular bundle data (see [**23**], Definition 2.1). In what follows $\widetilde{\nu}_\Delta$ is the pull back of the normal disk bundle ν_Σ over Σ to Σ_Δ, and ν_Δ is its non-Hausdorff quotient under π as a subset of X_Δ.

Definition 2.2.1. For a 4-manifold X with embedded surface Σ, *singular bundle data* for the pair (X, Σ) will mean the following data:

- a double cover $\Sigma_\Delta \to \Sigma$ and non-Hausdorff space X_Δ with a map π to X realizing the cover Σ_Δ;
- a principal PU(2)-bundle P_Δ on X_Δ;
- a 2-plane bundle $\widetilde{Q} \to \Sigma_\Delta$ with an identification of the orientation bundles $o(\widetilde{Q}) \to o(\Sigma_\Delta)$;
- an orientation-preserving bundle isometry

$$\rho : N_{\Sigma_\Delta} \to \mathrm{Hom}^-(\tau^*\widetilde{Q}, \widetilde{Q})$$

where $\tau : X_\Sigma \to X_\Sigma$ is the involution inducing the quotient map π, N_{Σ_Σ} is the normal 2-plane bundle to Σ_Δ inside X_Σ, and Hom^- denotes the 2-plane of orientation reversing conformal maps.

- an identification on ν_Δ of the resulting quotient bundle Q_Δ with an O(2) reduction of $P_\Delta|_{\nu_\Delta}$.

As noted in [**23**], when this data exists, the double cover Σ_Δ is determined by the PU(2)-bundle $P = P_\Delta|_{X_\Delta \setminus \Sigma_\Delta}$ on the complement of Σ in X.

Orbifolds and the Space of Connections. Given the singular bundle data and bundle P_Δ, [**23**] provides a recipe for writing down a model connection A_1 on the bundle P over $X \setminus \Sigma$ having the desired local behavior near Σ. The connection A_1 does not extend across Σ, but it does extend as an *orbifold* connection on an orbifold bundle PU(2)-bundle $\check{P}$ over $\check{X}$. For us, an orbifold bundle will be an orbifold total space admitting a map of orbifolds to $\check{X}$ and having fibers affine copies of PU(2) or quotients of PU(2) by the local group, which in our case is just $\mathbb{Z}/2$. There is a notion of a section for such a bundle, and then of an orbifold connection, which is a differential operator on the space of sections. The connection A_1 can be seen to extend to a smooth orbifold connection $\check{A}_1$ on $\check{P}$, at which point the story becomes fairly standard. Write $\mathfrak{g}_{\check{P}}$ for the adjoint orbifold bundle of Lie algebras (which has fiber $\mathfrak{so}(3)$ over $X \setminus \Sigma$. The orbifold connection $\check{A}_1$ provides us with kth derivative operators

$$\nabla^k_{\check{A}_1} : \Gamma\left(\wedge^p T^*\check{X} \otimes \mathfrak{g}_{\check{P}}\right) \to \Gamma\left(\mathrm{Sym}^k(T^*\check{X}) \otimes \wedge^p T^*\check{X} \otimes \mathfrak{g}_{\check{P}}\right).$$

The metric induces norms on the spaces of sections of all these bundles, allowing us to define Sobolev spaces $\check{L}^2_{k,\check{A}_1}$ of 1-forms. We define a space of connections

$$\mathcal{C}_k(X, \Sigma, \check{P}) := \{A_1 + a \mid a \in \check{L}^2_{k,\check{A}_1}(\wedge^1 TX|_{X\setminus\Sigma} \otimes \mathfrak{g}_{\check{P}})\}$$

The Banach manifold $\mathcal{C}_k(X, \Sigma, \check{P})$ can be seen to be independent of the choice smooth orbifold connection $\check{A}_1$. In fact, $\mathcal{C}_k(X, \Sigma, \check{P})$ only depends on the choice of orbifold bundle $\check{P}$, which in turn depends on the topological data of P_Δ and the auxilliary singular bundle data.

The Gauge Group. Given the orbifold principal PU(2)-bundle $\check{P}$, we can form an orbifold bundle of groups $\mathrm{SU}(\check{P})$ with fiber SU(2) by taking the bundle associated to $\check{P}$ by the action of PU(2) on SU(2) by conjugation. As usual, the bundle $\mathrm{SU}(\check{P})$ acts on $\check{P}$ by automorphisms in the standard way, but not all automorphisms of $\check{P}$ arise this way. We define the gauge group of $\check{P}$ to be those bundle automorphisms coming from sections of $\mathrm{SU}(\check{P})$: the determinant 1 gauge group. We consider the Banach Lie group

$$\mathcal{G}_{k+1}(X, \Sigma, \check{P})$$

of such automorphisms coming from the $\check{L}^2_{k+1}$ Sobolev completion of smooth sections of $\mathrm{SU}(\check{P})$. There is the usual action of the gauge group on the space of connections by pullback, and we use $k+1$ when acting on $\mathcal{C}_k$ to preserve regularity. In the end we wish to study connections A on $X \setminus \Sigma$ satisfying an approximate anti-self-duality condition $F_A^+ = V(A)$, where V is a "small" gauge-invariant smooth function $\mathcal{C}_k \to \Gamma(\wedge^+(\check{X}) \otimes \mathfrak{g}_{\check{P}})$. However, we only want to consider such connections up to gauge. Thus, we define the space of gauge equivalence classes:

$$\mathcal{B}_k(X, \Sigma, \check{P}) := \mathcal{C}_k(X, \Sigma, \check{P})/\mathcal{G}_{k+1}(X, \Sigma, \check{P})$$

This space is not a Banach manifold because of the presence of reducible connections. If $A \in \mathcal{C}$ is reducible then its stabilizer in $\mathcal{G}$ will be larger than the identity and $\mathcal{B}$ will be singular at $[A]$. We denote the set of irreducible connections by $\mathcal{C}^*$ and the quotient by the gauge group $\mathcal{B}^*$, which is an open subset and an infinite-dimensional Banach manifold.

The 3-dimensional Case. We also need to study the corresponding situation on a 3-manifold Y. In this case, our connections should be singular along a link $K \subset Y$. The story for 4-manifolds is readily modified for this case: we view Y as an orbifold $\check{Y}$ with point groups $\mathbb{Z}/2$ along K and study orbifold connections on an orbifold bundle $\check{P}$ over $\check{Y}$. We obtain a Banach manifold $\mathcal{C}_k(Y, K, \check{P})$ and gauge group $\mathcal{G}_{k+1}(Y, K, \check{P})$. The space of mod gauge $\mathcal{B}_k(Y, K, \check{P})$ is just the quotient $\mathcal{C}_k(Y, K, \check{P})/\mathcal{G}_{k+1}(Y, K, \check{P})$. In general, there will be reducible connections in $\mathcal{C}$ so we copy the above and use a star to denote the irreducible locus.

2.2.2. The Instanton Floer Functor. Since it is awkward to work directly with the orbifold bundles $\check{P}$ on a 3-manifold Y, we follow [**23**] by introducing a category whose objects naturally determine such bundles up to isomorphism. The data we need to specify just consists of a geometric representative for the Stiefel-Whitney class of P. We let $\widetilde{\text{WINK}}$ be the collection of triples (Y, K, ω) consisting of:

- A compact, closed, oriented, (possibly disconnected) 3-manifold Y
- An link $K \subset Y$
- An embedded 1-manifold with boundary $\omega \subset Y$ meeting K normally at its endpoints.

Given such a triple, there is a unique isomorphism class of orbifold bundles $\check{P}$ for which $w_2(\check{P}|_{Y\setminus K}) = \mathrm{PD}(\omega)$ in $H^2(Y, K; \mathbb{Z}/2)$. To define the Floer homology of a 3-manifold Y with link K and orbifold bundle $\check{P}$, one constructs a chain complex whose generators are a finite collection of points in $\mathcal{B}_k(Y, K, \check{P})$ which are critical points for a perturbed Chern-Simons functional. In order to avoid reducible connections, we restrict the class $\widetilde{\text{WINK}}$ to include only those objects (Y, K, ω) satisfying the following non-integrality condition ([**23**], Definition 3.1):

Definition 2.2.2. We say that the triple (Y, K, ω) satisfies the non-integrality condition if, in each connected component Y_i of Y, there is an embedded surface $\Sigma \subset Y_i$ such that either

- Σ is disjoint from K and ω intersects Σ transversely an odd number of times;
- Σ intersects K transversely an odd number of times.

Let WINK be the collection of triples (Y, K, ω) satisfying the non-integrality condition. For each such triple, the constructions of instanton Floer homology give an abelian group $I_*(Y, K, \omega)$. We review briefly the contents of this process, which is described in detail in [**23**].

The Chain Complex. Our chain complex will be generated by almost flat connections in $\mathcal{C}(Y, K, \check{P})$. In order to define these, we will need a perturbation of the flatness condition. As shown in [**17**], there is a Banach space $\mathcal{P}$ of perturbation functions $f : \mathcal{B}_k \to \mathbb{R}$ and a residual subset $\mathcal{U}$ of such f for which the perturbed Chern-Simons functional $CS + f : \mathcal{B}_k \to \mathbb{R}/\mathbb{Z}$ has non-degenerate critical points. There exists an $\epsilon > 0$ for which $|f| < \epsilon$ implies that the critical point set does not contain any reducibles. This is a consequence of the non-integrality assumption. The critical point set $\mathfrak{C}_f$ will be finite, and we construct a chain complex $C_*(Y, K, \check{P})$ with generators the elements of $\mathfrak{C}_f$. To define the differential ∂, suppose we are given any two generators β_0, β_1. We choose gauge representatives B_0, B_1 and consider the orbifold bundle $\check{P} = \mathbb{R} \times \check{P}$ over the 4-orbifold $\check{X} = \mathbb{R} \times \check{Y}$. Choosing an orbifold metric $\check{g}$ on $\check{Y}$, $\check{X}$ acquires the product metric, and we study a perturbation of the ASD equation for connections on $\check{P}$. Fix a smooth orbifold connection $\check{A}_0$ on $\check{P}$ which agrees with the pullbacks of B_0 and B_1 for large positive and large negative $t \in \mathbb{R}$, respectively. We define the following Banach space of connections on the cylinder:

$$\mathcal{C}_k^{\text{cyl}}(Y, K, \check{P}, B_1, B_0) = \{\check{A}_0 + a \,|; |a \in L^2_{k,\check{A}_0}(\wedge^1 \check{X} \otimes \mathfrak{g}_{\check{P}})\}$$

The fact that connections in the space must differ from $\check{A}_0$ by an $\check{L}^2_k$ 1-form implies they are all asymptotic to B_0 and B_1 on the ends, but this Banach space does not depend on the particular choice of $\check{A}_0$. In addition, because the B_i are irreducible, so will the connections in $\mathcal{C}^{\text{cyl}}$. Define the gauge group

$$\mathcal{G}_{k+1}^{\text{cyl}}(Y, K, \check{P})$$

consisting of $\check{L}^2_{k+1}$ determinant 1 gauge transformations which are asymptotic to the identity on the ends. We then define the quotient:

$$\mathcal{B}_k^{\text{cyl}}(Y, K, \check{P}, B_0, B_1) = \mathcal{C}_k^{\text{cyl}}(Y, K, \check{P}, B_1, B_0)/\mathcal{G}_{k+1}^{\text{cyl}}(Y, K, \check{P})$$

Given the perturbation f, there is a corresponding function

$$V_f : \mathcal{B}_k^{\text{cyl}} \to \Omega^+(X) \otimes \mathfrak{g}_{\check{P}}$$

and approximate anti-self duality equation $F_A^+ + V_f(A) = 0$. The function V_f is rigged so that the pull backs along the cylinder of critical points in $\mathcal{B}_k$ are solutions to this equation in the case $B_0 = B_1$. We obtain a moduli space of solutions

$$\mathcal{M}_f(Y, K, B_1, B_0)$$

which is independent of k for k large enough, and is a smooth finite dimensional manifold. The definition of this moduli space depended on the representatives B_0, B_1. In reality it only depends on the homotopy class z of the path in $\mathcal{B}_k$ from β_1 to β_0 given by a connection in $\mathcal{C}_k$. The moduli space is the space of trajectories to the downward gradient flow of $CS + f$ in the homotopy class z, so we write it as

$$\mathcal{M}_{z,f}(Y, K, \check{P}, \beta_1, \beta_0),$$

or simply $\mathcal{M}_{z,f}(\beta_1, \beta_0)$. Its expected dimension is given by a spectral flow index $gr_z(\beta_1, \beta_0)$, and the moduli spaces of trajectories from β_1 to β_0 of a certain dimension d is a union

$$\mathcal{M}_f(\beta_1, \beta_0)_d = \bigcup_{z | gr_z = d} \mathcal{M}_{f,z}(\beta_1, \beta_0).$$

By monotonicity, a result of choosing the singularity of the connections to have holonomy an order 2 element of SO(3) around circles linking the singular locus, and the resulting finiteness theorem (Corollary 3.25) of [**17**], this union is finite. On each of these spaces there is a translation action by $\mathbb{R}$ and in low dimensions they are smooth manifolds with boundary. The quotient of the 1-dimensional moduli spaces by this action is a finite collection of points, which when counted up to sign, gives the matrix coefficient from β_1 to β_0 of the differential ∂:

$$\partial \cdot \beta_1 = \sum_{\beta_0} \left(\sum_{z | gr_z(\beta_1, \beta_0) = 1} \# \frac{\mathcal{M}_z(\beta_1, \beta_0)}{\mathbb{R}} \right) \cdot \beta_0 \tag{2.2.2}$$

Monotonicity also gives a compactness result for the trajectory spaces, and so this sum is finite as well. Keeping track of the correct signs is a delicate issue involving assigning orientations to the moduli spaces of trajectories. One then checks that $\partial^2 = 0$ by studying the 2-dimensional moduli space between critical points. Taking a quotient of this space by the $\mathbb{R}$-action yields a compact 1-manifold with boundary, and the compactness theorem implies that these boundary points give exactly the terms in ∂^2 between the two critical points. They thus pair up, and the orientation scheme implies their signs cancel, giving zero.

In the end, given the triple (Y, K, ω) we get a chain complex $C_*(Y, K, \check{P})$ and define

$$I_*(Y, K, \omega) := H_*(C_*(Y, K, \check{P}))$$

to be its homology. Independence of all the choices (metric and perturbations) is proved by constructing product cobordisms interpolating two choices. There is a natural relative grading modulo 4 on $I_*(Y, K, \omega)$ coming from the possible values of the index $gr(\cdot, \cdot)$. In the case that Y is a disjoint union of triples (Y_i, K_i, ω_i), it is a trivial consequence of the definition of the chain complex that there is the isomorphism of relatively $\mathbb{Z}/4$-graded vector spaces:

$$I_*(Y, K, \omega) = \bigotimes_i I_*(Y_i, K_i, \omega_i).$$

Functoriality. What remains is to discuss the maps between Floer homologies of triples arising from cobordisms. Our approach to this is to turn WINK into a category whose morphisms are certain isomorphism classes of cobordism triples (W, S, Ω). This approach is laid out in detail in [**23**], and we recall the specifics here. To set this up, let (Y_0, K_0, ω_0), (Y_1, K_1, ω_1) be triples in WINK, and suppose that (W, S, Ω) is a triple consisting of oriented 4-manifold W with boundary, properly embedded 2-manifold with boundary S, and 2-manifold Ω with corners such that:

- There is an orientation preserving identification $\phi : \partial W \to \overline{Y}_1 \sqcup Y_0$ where $\overline{Y}_1$ is Y_1 with the opposite orientation;
- The map ϕ restricts to an identification of ∂S to $K_1 \sqcup K_0$;
- The boundary of Ω consists of two pieces, one meeting S normally in a compact embedded 1-manifold, and the other corresponding to $\omega_0 \sqcup \omega_1$ in ∂W under ϕ.

Furthermore, Ω only intersects S in its boundary plus a union of transverse intersections.

DEFINITION 2.2.3. An *oriented cobordism triple* from (Y_1, K_1, ω_1) to (Y_0, K_0, ω_0) consists of the triple (W, S, Ω) and data above. Two triples are considered isomorphic if there is a diffeomorphism between them respecting the boundary identifications.

The class WINK becomes a category by specifying that the morphisms are exactly the isomorphism classes of such cobordism triples. What we desire is a functor from WINK to the category AB of abelian groups, which means we must assign to each cobordism triple (W, S, Ω) as above a map

$$I(W, S, \Omega) : I_*(Y_1, K_1, \omega_1) \to I_*(Y_0, K_0, \omega_0).$$

We describe this procedure briefly, as all the details are covered in depth in [**23**].

To construct $I(W, S, \Omega)$, we study moduli spaces of approximately ASD connections on an orbifold bundle $\check{P}$. Unlike in 3 dimensions, the data (W, S, Ω) does not specify a unique isomorhpism class of orbifold bundle on W restricting to the unique induced orbifold bundle $\check{P}$ on Y, but rather a collection of such bundles which are pairwise isomorphic outside a finite collection of points. This is due to the presence of 4-dimensional characteristic classes. As a result, we will consider a disjoint union of moduli spaces on W arising from the all the possible choices giving the right dimension.

Fix one such bundle $\check{P}$. We choose an orbifold metric $\check{g}$ on the interior of (W, S) which has two cylindrical ends $\mathbb{R}^- \times Y_1$ and $\mathbb{R}^+ \times Y_0$. Let f_i be a perturbation function for Y_i such that there is again a finite non-degenerate critical point set $\mathfrak{C}_{f_i}$ for $CS + f_i$ on $\mathcal{B}_k(Y_i, K_i, \check{P}_i)$, where $\check{P}_i$ is the unique orbifold bundle on Y_i for the given triple. Fix two critical points $\beta_i \in \mathfrak{C}_{f_i}$ and let B_i be gauge representatives. Let $\check{P}$ be an orbifold bundle on (W, S, Ω) with the correct Stiefel-Whitney class, with isomorphisms of $\check{P}$ on the cylindrical ends to the pullbacks of $\check{P}_i$. We choose a smooth orbifold connection $\check{A}_0$ on $\check{P}$ which agrees with the pullbacks of the B_i's on the ends for large positive and negative $t \in \mathbb{R}$, and define a Banach space

$\mathcal{C}_k(W, S, \check{P}, B_1, B_0)$, gauge group $\mathcal{G}_{k+1}(W, S, \check{P})$, and connection space $\mathcal{B}_k(W, S, \check{P}, B_1, B_0)$ just as in the cylinder case. The functions f_i give a perturbation function V_{f_1,f_2} into the self-dual forms on the connection space by using a cutoff function on the ends of W (see [**23**] eq. (17)), and we have a moduli space

$$\mathcal{M}(W, S, \check{P}, B_1, B_0)$$

of solutions to $F_A^+ + V_{f_1,f_2}(A) = 0$, depending on the representatives B_i, and not depending on k (for k large enough).

Following [**23**], we can separate the possible triples $(\check{P}, B_1, B_0)$ of bundles and gauge representatives into equivalence classes. We fix one such triple $(\check{P}, B_0, B_1)$.

DEFINITION 2.2.4. A $(\check{P}, B_0, B_1)$-*marked bundle* is an equivalence class of data

$$(\check{P}', B_0', B_1'; T')$$

where T' is an identification of bundles $\check{P}' \to \check{P}_0$ ouside a finite set in W for which B_i pulls back to B_i' for large positive or negative $t \in \mathbb{R}$. Two sets of data $(\check{P}', B_0', B_1'; T')$ and $(\check{P}'', B_0', B_1'; T'')$ are equivalent if there is a determinant 1 isomorphism (over all of W) between them (where we use the identifications T' and T'' to define determinant).

We will often denote simply by the letter z such an equivalence class, in order to highlight its resemblence to a homotopy class of paths in the space of connections on a 3-manifold. We get a countable collection of moduli spaces $\mathcal{M}_z(W, S, \Omega, \beta_1, \beta_0)$. The map $\Psi(W, S, \Omega)$ is defined by looking at those which are 0-dimensional. For generic perturbations the moduli spaces will be cut out transversely and can be given orientations after making some additional choices (for example, basepoints in $\mathcal{B}_k(Y, K, \check{P})$). The dimension of $\mathcal{M}_z(W, S, \Omega, \beta_0, \beta_1)$ depends on the "homotopy class" z between β_1 and β_0 (even though there is not naturally a path in a space of connections mod gauge to which z corresponds); we denote it by $gr_z(\beta_1, \beta_0)$.

We set:

$$I(W,S,\Omega)([\beta_1]) = \sum_{\beta_0} \sum_{z|gr_z(\beta_1,\beta_0)=0} \#\mathcal{M}_z(W,S,\Omega,\beta_1,\beta_0)\cdot[\beta_0]. \tag{2.2.3}$$

Again, this sum is finite due to a compactness theorem, in turn relying on a monotonicity result due to the particular choice of conjugacy class for the holonomy of our connections around the singularities.

There is quite a bit more that needs to be checked at this point. Firstly, one must check that $I(W,S,\Omega)$ is actually a chain map - it is defined in terms of generators of the chain complex and we must check that the given matrix commutes with the boundary map ∂. The proof of this fact is related to the proof that $\partial^2 = 0$: looking at the 1-dimensional moduli spaces on (W,S,Ω) gives compact 1-manifolds with boundary. Let M denote the matrix given by (2.2.3). Each 1-dimensional component has 2 boundary components, one contributing to the matrix coefficient for M between $\partial(\beta_1)$ and β_0 and the other between β_1 and $\partial(\beta_0)$. This provides an equality for the matrices of $\partial\circ M$ and $M\circ\partial$. One must also check that $I(W,S,\Omega)$ does not depend on the perturbation functions or the metric. This can be shown by studying parametrized moduli spaces over a path in the space of perturbations and metrics, as is covered in some detail in [**23**]. However, this will fail if W has a closed 4-dimensional component W_0 - the invariant obtained by counting instantons on this component may actually depend on the metric in the same as the dependence of Donaldson invariants for manifolds with $b^+ = 1$. The space of metrics C_{W_0} is broken into a collection of chambers by walls along which reducible connections may arise. We can fix the problem if whenever W contains a closed component W_0, along with the triple (W,S,Ω) there is specified a chamber $\boldsymbol{\sigma}$ of C_{W_0}.

We also have an issue relating to signs and orientations. The term $\#\mathcal{M}_z(W,S,\Omega,\beta_1,\beta_0)$ refers to a signed count of points - these signs can all be pinned down by orienting all the moduli spaces, except that in the end the overall sign of (2.2.3) depends on some arbitrary choice separate from the triple (W,S,Ω). Thus, we will understand that $I(W,S,\Omega)$ denotes the pair of homomorphisms defined by (2.2.3) and its negative. As a result, for now I_* gives a functor from WINK to P-AB, the "projective" category of abelian groups whose morphisms are oppositely signed pairs of morphisms in AB. Later, we will describe a way to nail down the sign of $I(W, S, \Omega)$ for most of the specific cobordisms we work with.

For the considerations of this paper, it will be convenient to take I_* to be a vector space rather than an abelian group; we will want to consider dual vector spaces and eigenspaces of various operators. From now on, we let I_* denote the vector space obtained by taking the homology of the complex chain complex $(C_* \otimes \mathbb{C}, \partial \otimes \mathbb{C})$, and will speak of a projective functor into the category $\mathbb{C}$-VEC of $\mathbb{C}$ vector spaces. It will be convenient to have a definition of I_* when the 3-manifold is empty. In this case, we assign it the vector space $\mathbb{C}$. A cobordism from $\emptyset$ to (Y, K, ω) will just be a 4-manifold with this as its oriented boundary triple. Functoriality in this case gives a pair of maps $\mathbb{C} \to I_*(Y, K, \omega)$, which is equivalent to giving a pair $\{v, -v\}$ of elements in the Floer homology. We call the vector v the "relative" invariant of the 4-manifold triple. In the case that both incoming and outgoing ends of a cobordism triple (W, S, Ω) are empty, we just have a closed 4-manifold and the invariant becomes a single number in $\mathbb{C}$. This number will correspond to the simple integer Donaldson invariant for this 4-manifold.

Composition. In order to complete the description of our functor WINK $\to \mathbb{C}$-VEC, we must check that composition of cobordisms agrees with composition of the corresponding maps. In other words, suppose we have two cobordism triples $(W_{10}, S_{10}, \Omega_{10})$ and $(W_{21}, S_{21}, \Omega_{21})$ between (Y_1, K_1, Ω_1), (Y_0, K_0, ω_0) and (Y_1, K_1, Ω_1), (Y_2, K_2, Ω_2), and let $(W_{20}, S_{20}, \Omega_{20})$ denote the cobordism obtained by gluing the incoming boundary piece of the first to the outgoing piece of the second. The equation we desire is:

$$I(W_{20}, S_{20}, \Omega_{20}) = I(W_{10}, S_{10}, \Omega_{10}) \circ I(W_{21}, S_{21}, \Omega_{21}) \tag{2.2.4}$$

with the understanding that we choose sign representatives for each of these and are content with equality up to sign. We can prove such an equality by standard means, but *only* in the case that Y_1 is *connected*. The reason for this is that the necessary gluing argument relies on the assumption that the stabilizer of a flat connection B_1 on Y_1 in the gauge group is, under the hypothesis of non-integrality (Definition 2.2.2), the constant subgroup $\{\pm 1\}$. This will be the case when Y_1 is connected, but in general the stabilizer is $\{\pm 1\}^n$ if there are n connected components. As discussed in [**23**], in the disconnected case most of the stabilizing gauge transformations will not extend to ones on the cobordism. As a result, there will be 2^{n-1} different ways to glue connections on the different pieces, leading to a larger count on the glued up cobordism than the product over the two halves. There are a few ways around this difficulty. The approach of [**23**] is to modify the definition of $I_*(Y, K, \omega)$ to include the specification of a subgroup ϕ of the cohomology group $H^1(Y; \mathbb{Z}/2)$, which acts on the space of connections mod gauge (and thus critical point set $\mathfrak{C}_f$ and moduli spaces). This is equivalent to enlarging the gauge group to include more automorphisms of $\check{P}$ than those arising as determinant 1 gauge transformations. A cobordism triple (W, S, Ω) will also come with a subgroup $\Phi \subset H^1(W, S, \Omega)$ and one demands that when gluing two cobordism maps with subgroups Φ_{10} and Φ_{21}, the group Φ_{20} to be used on the glued-up triple is the largest one restricting to Φ_{10} and Φ_{21} on the smaller pieces.

Another approach is to simply ignore the issue, and assert that in each situation, (2.2.4) holds up to a nonzero constant. We lose strict functoriality for I_*, but recover.

Proposition 2.2.5. *For triples $(W_{10}, S_{10}, \Omega_{10})$, $(W_{21}, S_{21}, \Omega_{21})$, and $(W_{20}, S_{20}, \Omega_{20})$ as above, suppose that there are no closed components of the 4-manifolds. Then there is a constant C such that*

$$I(W_{20}, S_{20}, \Omega_{20}) = C \cdot I(W_{10}, S_{10}, \Omega_{10}) \circ I(W_{21}, S_{21}, \Omega_{21}) \tag{2.2.5}$$

In the case that W_{20} carries an almost complex structure, the constant C is a nonzero power of 2.

Proof. The standard approach to proving such a composition formula is to consider a family of metrics on W_{20} for which there is a product "neck" between the pieces having been glued together of length, say, T. This family $\check{g}_t$ is parametrized by $\mathbb{R}^+$, and over $W_{20} \times \mathbb{R}^+$

there is a parametrized moduli space $\mathcal{M}_t$ built from the 0-dimensional moduli spaces for each metric. It can be shown that for suitable perturbations and family $\check{g}_t$ the total space of $\mathcal{M}_t$ is a smooth 1-manifold in $\mathcal{B}_k \times \mathbb{R}^+$. There is a natural compactification of this family obtained by adding ∞ to $\mathbb{R}^+$ to get the segment $[0, \infty]$. We define the metric corresponding to ∞ to be a "broken" one, given by a metric $\check{g}_\infty$ on $W_{20} \setminus Y_1$ and having cylindrical ends on each of the pieces W_{21} and W_{10} on either side of Y_1. For fixed incoming critical point β_2 and outgoing β_0, we need a gluing theorem that says that the fiber of the compactification of $\mathcal{M}_t$ in $\mathcal{B}_k \times [0, \infty]$ over ∞ is related to the product of 0-dimensional moduli spaces $\mathcal{M}_{10}$ and $\mathcal{M}_{21}$ over W_{21} and W_{10}. These moduli spaces should be defined to consist of approximately ASD connections asymptotically equal to β_2 and β_0 on the far left and far right ends, respectively, and the product should be a fiber product over the common maps of each to the critical point set $\mathfrak{C}_1$ for Y_1 obtained by looking at the limiting connections on the Y_1 ends. Such a theorem is true, by appealing to an implicit function theorem argument; for each "broken" trajectory in the fiber product of $\mathcal{M}_{10}$ and $\mathcal{M}_{21}$, there should be a nearby glued up trajectory in $\mathcal{M}_{20}$.

As discussed above, a different stablizer in the gauge group for critical points on Y_1 and those on the glued up pieces of the cobordism leads to multiple possible glued up trajectories. Hence, we obtain (2.2.5) for the constant C. □

Unfortunately, we cannot rule out cancellations between the various (non-determinant 1) gauge equivalent trajectories, and thus that C is non-zero, without further assumptions. However, we have:

Lemma 2.2.6. *Suppose that W_{20} in Proposition 2.2.5 carries an almost complex structure. Then the constant C is a power of 2, and hence is nonzero.*

Proof. In order to conclude that C is nonzero, we need a way to compare the signs associated to the multiple flowlines arising from different gluings. The different gluings are the same up the SO(3) automorphisms on Y_1, so flowlines are actually gauge equivalent modulo the full SO(3) gauge group on W_{20}. These different flowlines might cancel if some of these gauge transformations are orientation reversing. Let $\mathrm{Aut}(\check{P}_{20})$ denote the full group of

SO(3) gauge transformations which are L^2_k regular over W_{20}, and let $\widehat{\mathcal{B}}_{W_{20}}$ denote the quotient of the space of connections by $\mathrm{Aut}(\check{P}_{20})$, so that there is a quotient map $\mathcal{B}_{W_{20}} \to \widehat{\mathcal{B}}_{W_{20}}$. Defining the chain maps associated to W_{20} requires orienting the space $\mathcal{B}_{W_{20}}$; one must first prove it is orientable and then make a choice. As shown in [**23**], if W_{20} has an almost complex structure, then $\widehat{\mathcal{B}}_{W_{20}}$ is orientable as well. This implies that each of the non-determinant 1 gauge transformations in $\mathrm{Aut}(\check{P})$ is orientation preserving. Hence, C will just be equal to the number of different flowlines obtained by gluing two fixed flowlines by different gluings. This is exactly the cardinality of the image of $H_1(W_{20};\mathbb{Z}/2)$ in $H_0(Y_1;\mathbb{Z}/2)$ under the image of the Mayer-Vietoris boundary map, which must be a power of 2. In particular, C is nonzero. $\square$

The condition on closed components is necessary for the instanton Floer functor to be defined. In general, if any of W_{10}, W_{21}, or W_{20} have closed components, we must specify chambers in the space of metrics. Let $\boldsymbol{\sigma}_{10}$, $\boldsymbol{\sigma}_{20}$ be chambers in the space of metrics on the collection of closed components of W_{10}, W_{21}, respectively. Gluing W_{10} and W_{21} may produce new closed components in W_{20}; let $\boldsymbol{\sigma}_{20}$ be the chamber in the space of metrics on the collection of closed components of W_{20} obtained by using metrics specified by $\boldsymbol{\sigma}_{10}$ and $\boldsymbol{\sigma}_{21}$ on old closed components, and, on any new closed components, the chamber given by metrics having a long neck along the gluing locus. In this case, we have:

$$I(W_{20}, S_{20}, \Omega_{20}; \boldsymbol{\sigma}_{20}) = C \cdot I(W_{10}, S_{10}, \Omega_{10}; \boldsymbol{\sigma}_{10}) \circ I(W_{21}, S_{21}, \Omega_{21}; \boldsymbol{\sigma}_{21}) \tag{2.2.6}$$

This will be particularly useful when comparing cohomological pairings on the space of flat connections with pairings in Donaldson theory in §2.4.1.

Flips. Given a 4-manifold X with embedded surface Σ or 3-manifold Y with link K, there are a natural set of symmetries between the moduli spaces of connections on the pair (X, Σ) and the Floer homologies of triples (Y, K, ω) arising from the connected components of the singular loci, known as "flips". Begin with the pair (X, Σ) and suppose we have singular bundle data $\mathbf{P}$ specified on our triple, consisting of the data in Definition 2.2.1. Suppose that Σ is connected, and let τ be the involution on X_Σ inducing the quotient map π to $X_\Sigma \to X$.

We will construct a new collection of singular bundle data by pulling back P_Δ via τ. We do not change the double cover Σ_Δ, but immediately obtain a new bundle $\tau^* P_\Delta$. We can pull back $\widetilde{Q}$ to obtain $\tau^*\widetilde{Q}$, and since τ identifies $T_x\Sigma_\Delta$ with $T_{\tau(x)}\Sigma_\Delta$, we obtain the necessary identifications of orientation bundles. Now, pulling back the map ρ via τ gives

$$\rho : \tau^* N_{\Sigma_\Delta} \to \tau^* \operatorname{Hom}^-(\tau^*\widetilde{Q}, \widetilde{Q}).$$

There is an obvious isomorphism $\tau^* N_{\Sigma_\Delta} \to N_{\Sigma_\Delta}$ (N_{Σ_Δ} is pulled back from a bundle $N_\Sigma \to \Sigma$) and we have

$$\tau^* \operatorname{Hom}^-(\tau^*\widetilde{Q}, \widetilde{Q}) = \operatorname{Hom}^-(\widetilde{Q}, \tau^*\widetilde{Q}),$$

so we obtain another orientation-preserving map

$$\rho : N_{\Sigma_\Delta} \to \tau^* \operatorname{Hom}^-(\widetilde{Q}, \tau^*\widetilde{Q}).$$

Lastly, it is easy to see that the resulting quotient bundle $\widetilde{Q}'_\Delta$ on ν_Δ is just the pull back of the original $\widetilde{Q}_\Delta$ via τ, and so the last data of the identification with an $\mathrm{O}(2)$ reduction of P_Δ comes for free: it is the pullback of the original one. We see that given singular bundle data $\mathbf{P}$, pulling back via τ gives a new set $\mathbf{P}^\tau$. The construction may also be carried out without modification for the 3-dimensional pair (Y, K).

Definition 2.2.7. The singular bundle data $\mathbf{P}^\tau$ obtained as above will be said to arise from the *flip* on Σ or K.

In general, the flip changes the singular bundle data to something non-isomorphic. Suppose that we are in the situation where the bundle $P_\Delta|_{X\setminus\Sigma}$ extends to a true bundle on all of X, and suppose momentarily that Σ is connected. Then, as in [**18**,**20**], the singular bundle is classified by the topological invariants k and l, called the instanton and monopole numbers, respectively, and the second Stiefel-Whitney class Ω. The effect of the flip is to change these invariants to k', l', and Ω' via:

$$
\begin{aligned}
k' &= k + l - \tfrac{1}{4}\Sigma\cdot\Sigma \\
l' &= \tfrac{1}{2}\Sigma\cdot\Sigma - l \\
\Omega' &= \Omega + [\Sigma]
\end{aligned}
\tag{2.2.7}
$$

See §A1 of [**20**] for details.

Now, suppose that Σ has a decomposition $\Sigma_1 \sqcup \ldots \sqcup \Sigma_n$ into connected components and fix an index i. We can form an involution τ_i on X_Δ which is the identity over Σ_j for $j \neq i$ and which is the order two involution over Σ_i. The constructions above may be repeated for τ_i, turning singular bundle data $\mathbf{P}$ into new data $\mathbf{P}^{\tau_i}$. In this way we get a collection of commuting "partial" flips. In the case that there is a global extension of $P_\Delta|_{X\setminus\Sigma}$ over all of X, we have defined the instanton number k, and monopole numbers $l_1, \ldots, l_n$, and these numbers change in the intuitive way under the partial flip via:

$$
\begin{aligned}
k' &= k + l_i - \tfrac{1}{4}\Sigma_i\cdot\Sigma_i \\
l_i' &= \tfrac{1}{2}\Sigma_i\cdot\Sigma_i - l_i \\
\Omega' &= \Omega + [\Sigma_i]
\end{aligned}
\tag{2.2.8}
$$

and $l_j' = l_j$ for $j \neq i$. Since the partial flips all commute, we can more generally define a partial flipping involution τ_J for any $J \subset \{1, \ldots, n\}$.

The value of the flip is that not only can it be used to pass between different sets of singular bundle data, it can be used to transform the connections on them. Fixing (X, Σ) and bundle data $\mathbf{P}$, we have a Sobolev space of singular connections $\mathcal{C}_k(\mathbf{P})$, determinant 1 gauge group $\mathcal{G}_{k+1}(\mathbf{P})$, and connections mod gauge $\mathcal{B}_k(\mathbf{P})$. Likewise, we have similar spaces for the flipped data $\mathbf{P}^\tau$. Given $[A] \in \mathcal{B}_k(\mathbf{P})$, we can pull it back to a connection $[\tau^* A] \in \mathcal{B}_k(\mathbf{P}^\tau)$. It is easy to verify that the pulled-back connection (which we view as a standard connection on an SO(3)-bundle away from Σ) lives in the desired Sobolev space, and that the pullback operation preserves gauge orbits. Now, choose a perturbation functional V such that it and

$V \circ \tau^*$ are suitably generic. The corresponding approximately ASD equation

$$F^+(A) + V(A) = 0$$

cuts out a moduli space $\mathcal{M}_V(\mathbf{P})$. We also have a moduli space $\mathcal{M}_{V\circ\tau^*}(\mathbf{P}^\tau)$, and by construction, pullback via τ gives a diffeomorphism between them.

2.3. Extended Instanton Floer Homology

There is an extension of our projective functor $I_* : \textsc{wink} \to \mathbb{C}\text{-}\textsc{vec}$ to include the specification of, along with a cobordism triple (W, S, Ω), a homology class $h \in H_*(W)$. More generally, we will allow an element of a graded commutative tensor algebra over the homology and additional generators coming from the connected components of S. The model for this extension is Donaldson's polynomial invariants and their extension by Kronheimer and Mrowka to the singular instanton case. The idea is that a homology class h in W gives rise to a cohomology class $\mu(h)$ in the space of connections modulo gauge through a characteristic class of the universal bundle over $\mathcal{B} \times W$. The Poincaré duals of these cohomology classes can be represented by compact embedded subspaces in $\mathcal{B}$. If we have a collection of homology class $h_1, \ldots, h_s$ we get subspaces V_i representing the $\mu(h_i)$'s, and if d is the sum of the codimensions, the d-dimensional moduli space $\mathcal{M}_d$ intersect $V_1 \cap \cdots \cap V_s$ in a collection of points, at least generically. Given the triple (W, S, ω), we will then define a map whose matrix coefficient between β_1 and β_0 is gotten not by looking at the 0-dimensional moduli spaces, but by looking at the intersection of the d-dimensional moduli spaces with $V_1 \cap \cdots \cap V_s$. The material here is not strictly speaking new, as it uses ideas going back to the original attempts to compute Donaldson invariants by splitting along a 3-manifold [**6**]. However, we extend the techniques to our situation where the connections are singular, as well as introduce new operators coming from the singular loci. These new operators are modelled on the classes σ_i from [**16**].

We suppose that (W, S, Ω) is a cobordism triple between (Y_i, K_i, ω_i) and we denote by $\mathbb{A}(W)$ the graded commutative tensor algebra on the vector space $H_{\leq 2}(W; \mathbb{C})$, whose elements consist of polynomials in generators of $H_2(W)$, $H_1(W)$, and $H_0(W)$. In other words, we set:

$$\mathbb{A}(W) = \mathrm{Sym}^*(H_0(W) \oplus H_2(W)) \otimes \wedge^* H_1(W).$$

A polynomial $z \in \mathbb{A}(W)$ naturally gives a cohomology class in the space of connections mod gauge, analogous to the way we constructed generators for cohomology ring of the moduli space of flat connections on a punctured surface in Chapter 1.

2.3.1. Geometric Representatives in $\mathcal{B}$. We fix as before an orbifold bundle $\check{P}$, perturbations, metric, critical points β_i, gauge representatives B_i, and then corresponding space of connections, gauge group and space of connections mod gauge, calling them $\mathcal{C}_W$, $\mathcal{G}_W$, $\mathcal{B}_W$ (each depending on a Sobolev regularity k, which is suppressed in the notation). There is a universal (orbifold) bundle

$$\check{\mathcal{P}} \to \mathcal{B}_W \times W$$

carrying a connection $\check{\mathcal{A}}$ in W directions which varies smoothly in $\mathcal{B}_W$ directions. Were we in the situation that S is empty, we could take the first Pontryagin class of the universal bundle and use slant product against homology classes in W to get cohomology classes in $\mathcal{B}_W$. Even in this case, there are issues with pairing these classes against the moduli spaces because in the general the higher dimensional moduli spaces are not compact. There is an added difficulty involving orbifolds and singular connections. We first outline here solutions to these technical issues, beginning with descriptions of the classes in $\mathcal{B}_W$ that we will use, and then proceed to define the extended Floer homology maps.

The Class of a Surface. Let $\Sigma \subset W$ be an embedded surface, and suppose that it intersects S transversely in a collection of point $x_1, \ldots, x_n$. Let $\mathcal{B}_W$ denote the space $\mathcal{B}_k(W, S, \check{P}, \beta_1, \beta_0)$. The topology of this space is largely determined by the topology of the gauge group, via the usual arguments of homotopy theory. There are natural classes associated to Σ arising from considering just what happens to connections upon restriction to Σ. Let $\mathcal{B}^*_{\nu(\Sigma)}$ denote the $\check{L}^2_k$ space of *irreducible* connections $\mathcal{C}^*_{\nu(\Sigma)}$ on $\nu(\Sigma)$ modulo the gauge group $\mathcal{G}_{\nu(\Sigma)}$ of local gauge transformations. Define the dense open subset of connections

$$\mathcal{B}^{**}_W = \{[A] \in \mathcal{B}_W : A|_{\nu(\Sigma)} \text{ is irreducible}\}$$

and write

$$r_{\nu(\Sigma)} : \mathcal{B}^{**}_W \to \mathcal{B}^*_{\nu(\Sigma)}$$

for the restriction map.

We will study cohomology classes on the space $\mathcal{B}^*_{\nu(\Sigma)}$. Any orbifold bundle on Σ extends to a non-orbifold bundle across the singular locus. Likewise, the bundle $\check{P}|_{\nu(\Sigma)}$ extends to a non-orbifold bundle P on all of the 2-disk bundle $\nu(\Sigma)$. Let D_k denote the fiber of $\nu(\Sigma)$ over x_k as a disk bundle over Σ, i.e., D_k is a connected component of $S \cap \nu(\Sigma)$. By construction of the space of connections, for any singular connection A on $\nu(\Sigma)$, there is a globally defined connection A_0 on P such that near each disk D_k there is a trivialization of P and polar coordinates for which, as 1-forms:

$$A = A_0 + \frac{1}{4}\mathrm{ad}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} d\theta.$$

As in Chapter 1, if we fix a bump function η, there is a well-defined procedure by which we may pass from A to A_0 by adding 1-forms near the punctures. Without first producing the extension P, this procedure takes a singular connection and produces a new connection with holonomy around small circles linking S limiting to the identity map. This can be done in families of connections as well. By standard arguments, over $\mathcal{B}^*_{\nu(\Sigma)} \times (\nu(\Sigma) \setminus S)$ there is a universal bundle $\mathcal{P}$ and family of connections $\mathcal{A}_0$ with a tautological property upon restriction to $\nu(\Sigma)$ slices. The procedure then gives a new family $\mathcal{A}$ with trivial limiting holonomy around small circles linking S. This new family defines an extension of the universal bundle $\mathcal{P}$ over $\mathcal{B}^*_{\nu(\Sigma)} \times S$, using the holonomy.

We may now take the characteristic class $p_1(\mathcal{P})$ in $H^4(\mathcal{B}^*_{\nu(\Sigma)} \times \nu(\Sigma))$. Slant product with $[\Sigma]$ produces an integral degree 2 class in $\mathcal{B}^*_{\nu(\Sigma)}$. However, this class is not invariant under the flip operation on connections. For us, this issue manifests itself in the dependence of the resulting class $p_1(\mathcal{P})/[\Sigma]$ on the particular process of passing from the connection A to A_0. If Σ is disjoint from S, there is no issue. Otherwise, near each singular locus, there are actually two natural 1-forms to add and these give different bundle extensions. As a result, the class $p_1(\mathcal{P})/[\Sigma]$ is somewhat awkward to use in the context of the gluing theorems and independence on the choice the representative for the homology class $[\Sigma]$ needed later on. We shall find it convenient to instead use a different degree 2 class, and for this we must introduce classes associated to points on S.

The Class of a Point on S. We can define additional degree 2 cohomology classes in $\mathcal{B}_W$, one for each component S_i of S. Choose a point $y \in S$, and a small ball neighborhood $\nu(y)$ intersecting S in a disk. As in the discussion for the class associated to a surface $[\Sigma]$, there is an extension of $\check{P}$ to a non-orbifold bundle P over $\nu(y)$, and connections can be viewed as being defined on P away from S. The holonomy of a connection around small loops around S near y picks out a preferred complex line K_A in the fiber of the associated rank 3 vector bundle to P. Now, let $\mathcal{B}_{\nu(y)}$ denote the Banach space of singular connections on the ball mod gauge. Over $\mathcal{B}_{\nu(y)} \times \nu(y)$, there is a universal bundle $\mathcal{P}$ and universal tautological singular connection $\mathcal{A}$ in $\nu(y)$-directions. The complex lines K_A fit together to form a line subbundle $\mathcal{K}_y$ over $\mathcal{B}_{\nu(y)} \times \{y\}$. Its first Chern class gives a degree 2 cohomology class on $\mathcal{B}_{\nu(y)}$, and this pulls back to a class which we still denote by $c_1(\mathcal{K}_y)$ in $\mathcal{B}^{**}_W$, which we now take to be the space of connections mod gauge which are irreducible on $\nu(y)$. We now remark that, as in the case of a surface Σ, the line bundle $\mathcal{K}_y$ depends on a choice of extension P across the $\mu(y) \cap S$. It is not difficult to see that choosing the other extension yields the class $-c_1(\mathcal{K}_y)$. For a fixed point $y \in S$, let $\{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\}$ be the two possible choices of bundle extension. In order to define the class $c_1(\mathcal{K}_y)$, we have to specify a choice of $\boldsymbol{\kappa}_i$. Denote the corresponding class by $\mu([y])_{\boldsymbol{\kappa}_i}$.

The Orbifold Class of a Surface. Returning to the degree 2 class associated to a surface Σ, our discussion in Chapter 1 can be adapted here to show that if we choose an intersection point $y \in \Sigma \cap S$ and change our extension P of $\check{P}$ by adding the *other* 1-form to connections near y (i.e. passing from $\boldsymbol{\kappa}_1$ to $\boldsymbol{\kappa}_2$), then the class $p_1(\mathcal{P})/[\Sigma]$ changes by adding $c_1(\mathcal{K}_y)$ (defined using $\boldsymbol{\kappa}_1$). If we define the line bundle $\mathcal{K}_y$ by using the restriction to $\nu(y)$ of the extension P on $\nu(\Sigma)$, we call this choice $\boldsymbol{\kappa}_\Sigma$. We see that the class

$$p_1(\mathcal{P})/[\Sigma] - \sum_{k=1}^{n} c_1(\mathcal{K}_{y_k})$$

where $n = |\Sigma \cap S|$ and y_k is the kth intersection point, is a well-defined class, not depending on the particular choice of extension P over Σ. We denote this class by $\mu^{\mathrm{orb}}([\Sigma])$.

In order to evaluate this class on potentially non-compact moduli spaces, we will find a closed codimension 2 submanifold of $\mathcal{B}^{**}_W$ representing it. For this we follow §5.2.2 of [**8**]: we let $\mathcal{L}_\Sigma$ be a smooth line bundle over $\mathcal{B}_{\nu(\Sigma)}$ with $\mu^{\mathrm{orb}}([\Sigma])$ as its first Chern class. There will be a smooth, transverse (to zero) section s of $\mathcal{L}_\Sigma$ whose pullback to the bundle $r^*_{\nu(\Sigma)}(\mathcal{L})$ is transverse. The zero set $(r^*_{\nu(\Sigma)}s)^{-1}(0)$ is an (oriented) codimension 2 smooth submanifold of $\mathcal{B}^{**}_W$, which we shall denote by V_Σ, representing the Poincaré dual of $\mu^{\mathrm{orb}}([\Sigma])$. For the class $c_1(\mathcal{K}_y)$, we perform the same maneuver and get a codimension 2 submanifold V_y for y in S, with the understanding that the orientation of V_y depends on a choice of extension of $\check{P}$ across $\nu(y)$.

The Submanifold $V_{y,\boldsymbol{\kappa}_i}$ for $y \in S$. In order to evaluate the class $c_1(\mathcal{K}_y)$ on moduli spaces, we use the technique of [**16**], where the corresponding cohomology class is denoted σ. We choose a small ball neighborhood $\nu(y)$ of y instersection S in a disk. There is a space of connections mod local gauge $\mathcal{B}^*_{\nu(y)}$, a subspace $\mathcal{B}^{**}_W$ of connection irreducible on $\nu(y)$, and a correpsonding restriction map. On $\mathcal{B}^*_{\nu(y)}$ there is defined a line bundle $\mathcal{K}_y$, depending on the choice of $\boldsymbol{\kappa}_i$. As in the case of a surface, we choose a transverse section of $\mathcal{K}_y$ such that the zero section pulls back to a smooth submanifold intersecting all moduli spaces transversely. Denote this oriented submanifold by $V_{y,\boldsymbol{\kappa}_i}$.

The Class of a Point in $W \setminus S$. Let x be a point in W away from S. We can obtain a cohomology class $\mu(\mathrm{pt})$ of degree 4 in exactly the way described in §9.2.3 of [**8**]. That is, we take the first Pontryagin class $p_1(\mathcal{P})$ of a universal bundle (which only needs to exist over $\mathcal{B}_W \times (W \setminus S)$). Like the case of a surface, we pick a neighborhood of x in $W \setminus S$ and look at the pullback of a natural degree 4 class on the space $\mathcal{B}_{\nu(x)}$ of connections on this small ball by the restriction map from $\mathcal{B}_W^{**}$, now the set of connections irreducible on $\nu(x)$. While we cannot always find a codimension 4 submanifold representing this cohomology class, we can find a codimension 4 stratified subspace $V_{\nu(x)}$, with a real codimension 4 strata and a closed codimension 12 strata [**19**], and this is enough to define the pairings we need. Let V_x be the inverse image of $V_{\nu(x)}$ under the restriction map. Membership for this subspace for a connection A depends only on the behavior of A on a small ball around x.

The Class of a Loop. Let γ be a smoothly embedded loop in $W \setminus \Sigma$ representing an element of $H_1(W)$. The class $[\gamma]$ gives a cohomology class of degree 3 in $\mathcal{B}_W$, and like the case of both a point and surface, we can find a codimension 3 submanifold V_γ representing it which is the (transverse) inverse image of a submanifold in the space of connections on a tubular neighborhood of γ (see [**19**]).

Details for the last two of these constructions can be found in §2(ii) of [**19**].

2.3.2. Cobordisms and Polynomials. The algebra $\mathbb{A}(W)$ comes with a natural grading which assigns $[\Sigma] \in H_2$ degree 2, $[\gamma] \in H_1$ degree 3, and $[x] \in H_0$ degree 4. We also define a graded commutative algebra $\mathbb{A}(W, S) = \mathbb{A}(W) \otimes \mathrm{Sym}^*(H_0(S))$, assigning the generators $[y] \in H_0(S)$ degree 2. Suppose first that we have a monomial $\chi \in \mathbb{A}(W)$ which is the product of the degree 2 classes $[\Sigma_1], \ldots, [\Sigma_r]$, degree 3 classes $[\gamma_1], \ldots, [\gamma_s]$, degree 4 classes $[x_1], \ldots, [x_t]$, and lastly degree 2 classes $[y_1], \ldots, [y_n]$. At the outset, we fix representatives y_l for these latter classes, and the data of $\underline{\boldsymbol{\kappa}} = \{\boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_n\}$ of choices of bundle extension near each of them. We find smoothly embedded submanifolds (or more generally stratified subspaces) of W representing them, and arrange that there are no triple intersections, single intersections are transverse, and the intersections with S are all transverse. Using each of the recipes above, we can find closed subspaces V_{Σ_i}, V_{γ_j}, V_{x_k}, and $V_{y_l,\boldsymbol{\kappa}_l}$ of $\mathcal{B}_W$ corresponding to each of them, using small enough tubular neighborhoods of the Σ_i, γ_j, x_k, and y_l with no triple intersections. We denote by $\mathcal{V}_\chi$ the intersection of all of these, which has codimension $d = 2r + 3s + 4t + 2n$. Each $V_\bullet$ in reality is a class in an open subset $\mathcal{B}_W^{**}$ consisting of connections irreducible on a tubular neighborhood. Let $\mathcal{B}_W^\circ$ denote the subspace of connections mod gauge which are irreducible on *all* the tubular neighborhoods (individually). Then each $V_\bullet$ and V_χ can be taken to be inside $\mathcal{B}_W^\circ$. For generic perturbations, the moduli spaces will all be contained in this subspace as well.

Given any $(\check{P}, B_0, B_1)$-marked bundle data z with corresponding moduli space of approximately ASD connections $\mathcal{M}_z(W, S, \Omega, \beta_1, \beta_0)$ having dimension d, the intersection

$$\mathcal{M}_z(W, S, \Omega, \beta_1, \beta_0) \cap \mathcal{V}_\chi \tag{2.3.1}$$

has expected dimension 0. As in [**6**], one can arrange the perturbations involved to be generic enough that this intersection is completely transverse and orient everything so that the points of intersection have signs. We define a map

$$I(W, S, \Omega; \chi) : I_*(Y_1, K_1, \omega_1) \to I_*(Y_0, K_0, \omega_0)$$

using these intersection numbers as the matrix entry between β_1 and β_0 at the chain level. So far, we have only defined the $V_\bullet$'s in a way depending on the critical points β_i. However, these subspaces are pulled back from spaces of connections on open subsets of W, which are independent of the β_i. The proof that the intersection numbers (2.3.1) define a chain map (commuting with ∂) then goes in much the same way as for without polynomials. Again, while we can choose coherent orientations for everything, the overall sign of $I(W, S, \Omega; \chi)$ depends on an arbitrary choice, so we interpret this to mean a pair of oppositely signed maps between the Floer homologies of the Y_i.

PROPOSITION 2.3.1. *The intersection (2.3.1) is a compact zero-manifold and the map* $I(W, S, \Omega; \chi)$ *described above depends only on the homology classes of the submanifolds* Σ_i,

γ_j and x_k of W and connected components of S in which the y_l lie. It is graded symmetric multilinear in these classes.

Proof. The compactness argument in the proof of Proposition 2.2 of [**16**] carries over here with some modifications, the first being to treat the degree 3 classes arising from the γ_i. The dimension counting argument for compactness carries through without modification, as the γ_i are away from S, Σ_i's, and x_k's. This argument also handles the case of the classes $\mu([y])_{\boldsymbol{\kappa}}$ and $\mu([\Sigma])$ separately, and thus the introduction of the class $\mu^{\mathrm{orb}}([\Sigma])$ presents no difficulties. The key difference lies with the argument required to show that if a sequence of connections A_i converges weakly to an A_∞ but not strongly, then the lost action (integral of norm-squared curvature) implies that A_∞ lives in a moduli space of lower dimension. In [**16**] and [**18**], this is arranged by assuming the holonomy parameter (which for us is fixed to be 1/4) lies in an "interval of compactness", determined by the cone angle of the orbifold metric. Letting the angle tend to 0 increases the size of the interval. For us, we get that A_∞ lives in a moduli space of lower dimension for free since for our choice of singular connection spaces, the action of the connection is monotonic in the dimension of the moduli space. This fact was already used to define the maps from cobordisms in §2.2.2, but it applies in this more general situation.

The proof that the map $I(W, S, \Omega; \chi)$ only depends on the *homology* classes of the embedded submanifolds of W needs no major modification in the case that no Σ_i intersects S (see, for example, Theorem 9.2.12 of [**8**]), as is the case with multilinearity. The key point is that we may reduce to the case of a 2, 3, or 4 dimensional moduli space, and by compactness we may view the intersection numbers as being an honest cohomology pairing, where the statements are obvious. In the case that some Σ_i intersects S the relevant cohomology class is the modified class $\mu^{\mathrm{orb}}([\Sigma_i])$. We must check that if Σ and Σ' are homologous then $\mu^{\mathrm{orb}}([\Sigma])$ and $\mu^{\mathrm{orb}}([\Sigma'])$ are equal cohomology classes in $\mathcal{B}_W$. Suppose that there exists an extension of $\check{P}$ to a non-orbifold bundle P on W. Then the classes $\mu([\Sigma])$, $\mu([\Sigma'])$ can be defined, and it is clear that they are equal. We also get, for $y \in \Sigma \cap S$, the class $\mu_S([y])$.

The key is that $\Sigma \cap S$ and $\Sigma' \cap S$ are homologous in S, and so we get an equality of sums

$$\sum_{y \in \Sigma \cap S} \mu_S([y]) = \sum_{y' \in \Sigma' \cap S} \mu_S([y']).$$

We conclude that $\mu^{\text{orb}}([\Sigma])$ and $\mu^{\text{orb}}([\Sigma'])$ are equal. Now, in general $\check{P}$ does not extend to a non-orbifold bundle on W, but it will do so outside a finite collection of points Q in S. The surfaces Σ and Σ' are still homologous in $W \setminus Q$ and $\Sigma \cap S$ and $\Sigma' \cap S$ are homologous in $S \setminus Q$, so the previous argument is still valid. □

The result of Proposition 2.3.1 implies that we can extend the definition of $I(W, S, \Omega; \chi)$ from monomials to elements of $\mathbb{A}(W, S)$ by linearity. Once a coherent system of orientations is chosen (there will be two choices), we get a map

$$I(W, S, \Omega, -) : I_*(Y_1, K_1, \omega_1) \otimes \mathbb{A}(W, S) \to I_*(Y_0, K_0, \omega_0).$$

Without choosing an orientation, we get a pair of such maps out of the tensor product, differing in sign.

Gluing. So far we have extended the definition of I_* to take a cobordism triple (W, S, Ω) and polynomial $\chi \in \mathbb{A}(W, S)$ and produce a map between the boundary Floer homologies. In the case that χ is just the polynomial 1, we recover our original cobordism maps, and in this case the assignment is (projectively) functorial; composition of cobordisms corresponds to composition. We would like to have an analogue of this composition for our extended theory for nontrivial polynomials χ. To set this up, we suppose again that we have two triples $(W_{10}, S_{10}, \Omega_{10})$ and $(W_{21}, S_{21}, \Omega_{21})$ which can be glued up to give the triple $(W_{20}, S_{20}, \Omega_{20})$. Let the boundary 3-manifolds be (Y_i, K_i, ω_i), for $i = 0, 1, 2$. The inclusions of (W_{10}, S_{10}) and (W_{21}, S_{10}) into (W_{20}, S_{20}) induce maps of $\mathbb{C}$-algebras:

$$\mathbb{A}(W_{10}, S_{10}), \mathbb{A}(W_{21}, S_{21}) \longrightarrow \mathbb{A}(W_{20}, S_{20}) \tag{2.3.2}$$

Proposition 2.3.2. *The extended Floer homology maps enjoy the following gluing property. Let $\chi \in \mathbb{A}(W_{10}, S_{10})$ and $\chi' \in \mathbb{A}(W_{21}, S_{21})$, and denote by $\hat{\chi}$, $\hat{\chi}'$ their images in $\mathbb{A}(W_{20})$*

under the maps (2.3.2). Then for some non-negative integer C we have:

$$I(W_{20}, S_{20}, \Omega_{20}; \hat{\chi}\hat{\chi}') = C \cdot I(W_{10}, S_{10}, \Omega_{10}; \chi) \circ I(W_{21}, S_{21}, \Omega_{21}; \chi')$$

In the above, $C = \pm 1$ if Y_1 is connected.

Proof. The essential details of this go back to the original results of Fukaya on gluing formulae for Donaldson invariants [**11**]. The proof of this is modelled on the proof of the corresponding result (2.2.5), via "stretching the neck". The only new details to consider are related to compactness and these were already handled in the proof of Proposition 2.3.1. □

Operators on $I_*(Y, K, \omega)$. We can use the results of this section to endow the Floer homology of a 3-manifold triple (Y, K, ω) with an action of an algebra on the homology of Y and K. Let $\mathbb{A}(Y)$ denote the graded commutative tensor $\mathbb{C}$-algebra over $H_{\leq}2(Y; \mathbb{C})$, i.e.:

$$\mathbb{A}(Y) = \mathrm{Sym}^*(H_0(Y) \oplus H_2(Y)) \otimes \wedge^* H_1(Y).$$

In the case that K is nonempty, will use the algebra $\mathbb{A}(Y, K) = \mathbb{A}(Y) \otimes \mathrm{Sym}^* H_0(K)$. There is a natural action of these algebras on the Floer homology $I_*(Y, K, \omega)$, giving a map $\mu(\chi)$ for a polynomial χ in $\mathbb{A}(Y)$, or $\mathbb{A}(Y, K)$:

$$\mu(\chi) : I_*(Y, K, \omega) \to I_*(Y, K, \omega)$$

$$\mu(\chi) = I([0, 1] \times Y, [0, 1] \times K, [0, 1] \times \omega; \hat{\chi})$$

where $\hat{\chi}$ is the image of χ in $\mathbb{A}(W)$ or $\mathbb{A}(W, S)$ under map induced by the inclusion of (Y, S) as the slice at $t \in [0, 1]$ in the interior of the interval. These algebras come with the same grading as in the 4-dimensional case, and the operator $\mu(\chi)$ is a graded map on the Floer homology which shifts degree by $\deg(\chi)$ (recall that $I_*(Y, K, \omega)$ has a relative grading mod 4). The operators $\mu(\chi)$ can be understood in terms of the Morse theoretic picture of $I_*(Y, K, \omega)$. For example, if $\chi = [\Sigma] \in H_2(Y)$, the recipe of the previous section can be used to construct a codimension 2 submanifold V_Σ of the space $\mathcal{B}(Y)$ of connections mod gauge on $\check{P} \to Y$. The moduli spaces over the cylinder $[0,1] \times Y$, with the right metric, are just those over $\mathbb{R} \times Y$ and correspond to spaces of flowlines in $\mathcal{B}$ for the downward gradient flow of the perturbed Chern-Simons functional. The map $\mu([\Sigma])$ is given at the chain level by counting intersection points of the 2-dimensional moduli spaces with V_Σ.

Of course, these operators are so far a priori only defined up to a choice of sign. As discussed before, the sign can be pinned down by choosing an almost complex structure on the cylinder $\mathbb{R} \times Y$, and this is equivalent to choosing a nonvanishing vector field on Y tangent to K along K [**23**]. Eventually, we will be using these operators in the context of a product link in a product 3-manifold, for which there is a canonical such vector field, namely the tangent vectors to the S^1 factor. The corresponding almost complex structure on $\mathbb{R} \times S^1 \times \Sigma$ will simply be the obvious product one. When this is the case, we will be able to speak of the operators as having a fixed sign without further specification, understanding that we are using the orientations on connection spaces arising from this canonical almost complex structure. Lastly, in order to pin down the sign of $\mu([y])$ for $[y] \in H_0(K)$ and K a disjoint union of components $K_1, \ldots, K_n$, we must make a collection of choices

$$\underline{\boldsymbol{\kappa}} = (\boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_n).$$

Donaldson Invariants. As before, if we take $(W, S, \Omega, \boldsymbol{\sigma})$ to be a closed 4-manifold triple with a specified chamber $\boldsymbol{\sigma}$ of the space of metrics C_W on W, the constructions above give a map (up to sign)

$$I(W, S, \Omega, \boldsymbol{\sigma}; -) : \mathbb{A}(W, S) \to \mathbb{C}.$$

These correspond to Donaldson polynomial invariants considered in [**20**] and [**16**]. Our arguments here for gluing show that these invariants enjoy the usual gluing properties for splittings along 3-manifolds, at least for homology classes living entirely in one half of the decomposition.

The Effect of Flips. Suppose we are again in the case that $(W, S, \Omega; \boldsymbol{\sigma})$ is a closed 4-manifold triple with metric chamber. We investigate the map $\mathbb{A}(W, S) \to \mathbb{C}$ in the context of the flipping operation discussed in §2.2. Let $S = S_1 \sqcup \ldots \sqcup S_n$ be the decomposition of S into connected components, and fix a monomial $\chi \in \mathbb{A}(W, S)$ of degree d. The value of $I(W, S, \Omega, \boldsymbol{\sigma}; \chi)$ is determined by looking at the d-dimensional moduli space $\mathcal{M}_d$ on (W, S, Ω) and evaluating cohomology classes. Let $J \subset \{1, \ldots, n\}$ be such that

$$\sum_{j \in J} [S_j]$$

is a mutliple of 2 in $H_2(W)$. Then the flip τ_J preserves the homology class of Ω modulo 2 and so (after choosing a flip-invariant perturbation function) gives an involution on the space $\mathcal{B}_W$ of connections mod gauge and the moduli space $\mathcal{M}_d$.

LEMMA 2.3.3. *The classes $\mu(\mathrm{pt})$ and $\mu(\gamma)$ for $\gamma \in H_1(W)$ are preserved under τ_J^*. For $y \in S_j$ and set of extension choices $\{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2\}$, we have:*

$$\tau_J^*(\mu([y])_{\boldsymbol{\kappa}_i}) = \mu([y])_{\boldsymbol{\kappa}_{2-i}} = -\mu([y])_{\boldsymbol{\kappa}_i}.$$

Finally, for $[\Sigma] \in H_2(W)$, we have

$$\tau_J^*(\mu^{\mathrm{orb}}(\Sigma)) = \mu^{\mathrm{orb}}(\Sigma).$$

PROOF. The proof is entirely analogous to the work carried out in Chapter 1 for the flips M_J. One studies how the universal bundle over $\mathcal{M}_d \times W$ is pulled back under τ_J^*, showing that the bundle is preserved, while the line subbundles lying over $\mathcal{M}_d \times S$ corresponding to the point classes $\mu([y])$ are transformed by conjugation or are preserved according to whether y is in a flipped component of S. While the naïve class $\mu(\Sigma)$ is not preserved under the flip, the modified class $\mu^{\mathrm{orb}}(\Sigma)$ is (see Lemma 1.5.14 and equation (1.5.10). □

2.4. The Floer Homology of a Product $S^1 \times \Sigma$: A Review

The major problems associated with instanton Floer homology are related to its computation, even for simple 3-manifolds. For 3-manifolds Y without a link, the theory laid out above goes back to Floer's original work (see [**6**]). The first basic cases to consider are fibered 3-manifolds, or simply product 3-manifolds $S^1 \times \Sigma$ for Σ a compact Riemann surface.

2.4.1. Relative Invariants and Relations. In this section we review the calculation of the Floer homology in this basic case. To satify the condition of non-integrality, there are two obvious choices for the Stiefel-Whitney class ω. One obvious choice for ω for the 3-manifold $S^1 \times \Sigma$ is the product curve $w := S^1 \times \mathrm{pt}$. This case was studied by Dostoglou and Salamon (who considered the more general case of a fibered 3-manifold) [**9**] and later Muñoz [**27**], who gave a presentation for the natural *ring* structure on this Floer homology. Let Y_g be the 3-manifold $S^1 \times \Sigma$ where Σ is a compact Riemann surface of genus $g \geq 2$. The surface $\mathrm{pt} \times \Sigma$ is a non-integral surface for the triple $(Y_g, \emptyset, w)$, and so the Floer homology $I_*(Y_g, w)$ is defined. Let F be a pair of pants surface, and view the 4-manifold triple $(F \times \Sigma, \emptyset, F \times \mathrm{pt})$ as a cobordism from two copies of Y_g to one copy of Y_g. Functoriality gives a map

$$I_*(Y_g, w) \otimes I_*(Y_g, w) \to I_*(Y_g, w)$$

which is graded commutative and associative in the incoming factors.

DEFINITION 2.4.1. Let $\mathbb{V}_{g,0}$ be the $\mathbb{C}$-algebra given by $I_*(Y_g, w)$ with the ring structure induced by the above pair of pants cobordism.

By definition, as a vector space $\mathbb{V}_{g,0}$ is the homology of a chain complex whose generators are approximately flat connections on Y_g for some perturbation of the Chern-Simons functional. The bundle on Y_g is the unique $\mathrm{PU}(2) = \mathrm{SO}(3)$ bundle on $S^1 \times \Sigma$ with w_2 dual to the curve w, which is the pullback of the unique nontrivial $\mathrm{SO}(3)$ bundle on Σ. This bundle is associated to the $\mathrm{U}(2)$ bundle with $c_1 = 1$ on Σ via the homomorphism $\mathrm{U}(2) \to \mathrm{PU}(2)$, and we can view the space of connections as a space of $\mathrm{U}(2)$ connections with fixed determinant connection θ. It is easy to see that the set of equivalence classes of flat connections is in bijection with the space of representations of $\pi_1(Y_g \setminus w) \to \mathrm{SU}(2)$ modulo conjugation which send a meridian of a small loop around w to the element -1. This is the same as two copies of the moduli space of flat connections $\mathcal{M}_g(2,1)$ of rank 2 and degree 1 on Σ, with the two copies arising from a choice of the holonomy along the S^1 factor, which must be ± 1. Upon perturbing the critical point set becomes discrete, but the differential is a deformation of the regular Morse differential for (two copies of) the finite dimensional compact manifold $\mathcal{M}_g(2,1)$. The homology will then be a sub-quotient of $H_*(\mathcal{M}_g(2,1))^{\oplus 2}$. However, it turns out that $\mathcal{V}_g$ is actually the full homology of the space of flat connections, as first argued by Dostoglou and Solomon [**9**]. There is a quicker proof of this fact, however, which uses the extended Floer homology construction above, and relies on a description of cohomology of the moduli space of flat connections $\mathcal{M}_g(2,1)$ due to Siebert and Tian among others [**29**]. We briefly review this argument, which can also be found in [**26**], as we will revisit it to prove an analogous result for a product 3-manifold with product link.

The strategy for obtaining a description of this Floer homology is to use the extended invariants for the bounding 4-manifold $D^2 \times \Sigma$. Let $a_1, \ldots, a_{2g}$ be a standard generating set of loops in $\pi_1(\Sigma)$ so that their homology classes are a symplectic basis for H_1. We obtain elements in $\mathbb{V}_{g,0}$ by taking the relative invariant $I(D^2 \times \Sigma, \emptyset, D^2 \times \mathrm{pt}; \chi)$ for a polynomial $\chi \in \mathbb{A}(D^2 \times \Sigma)$. The algebra $\mathbb{A}_{g,0} := \mathbb{A}(D^2 \times \Sigma)$ is simply the graded commutative tensor algebra on $H_*(\Sigma)$, and so is generated by the $[a_i]$'s, $[\Sigma]$, and a class $[x]$ for $x \in Y_g$. Let us introduce notation for these generators analogous to the notation used in Chapter 1. We set $\alpha = -\frac{1}{4}[\Sigma]$, $\beta = -\frac{1}{4}[\mathrm{pt}]$, and $\psi_j = -\frac{1}{4}[a_j]$, where we use the normalizing factor $-\frac{1}{4}$ to account for the fact that we have used the characteristic class p_1 of the universal bundle in the extended invariants, rather than the second Chern class c_2. We can get a map $\Phi : \mathbb{A}_{g,0} \mapsto \mathbb{V}_{g,0}$ via

$$\Phi(\chi) = I(D^2 \times \Sigma, D^2 \times \mathrm{pt}; \chi) \tag{2.4.1}$$

From now on we will denote this multiplication by juxtaposition, or a dot $\cdot$. The ring structure on $\mathbb{V}_{g,0}$ is natural because it interacts well with Φ.

Lemma 2.4.2. *The map (2.4.1 is a homomorphism of algebras.*

PROOF. Let $\chi_1, \chi_2 \in \mathbb{A}_{g,0}(D^2 \times \Sigma) = \mathbb{A}_{g,0}$ be two polynomials. Then the statement of the lemma is exactly the equation

$$I(D^2 \times \Sigma, D^2 \times \mathrm{pt}; \chi_1) \cdot I(D^2 \times \Sigma, D^2 \times \mathrm{pt}; \chi_2) = I(D^2 \times \Sigma, D^2 \times \mathrm{pt}; \chi_1\chi_2).$$

This is a straightforward application of our gluing results. The product of the relative invariants $I(D^2 \times \Sigma, D^2 \times \mathrm{pt}; \chi_i)$ is the same as the relative invariant associated to the 4-manifold obtained by plugging the incoming ends of F with disks, taking a product with Σ, and using the polynomial $\chi_1\chi_2$. This is the same as $I(D^2 \times \Sigma, D^2 \times \mathrm{pt}; \chi_1\chi_2)$. $\square$

The most important fact concerning the Floer homology $\mathbb{V}_{g,0}$ is that Φ is actually surjective onto a half dimensional subspace. Now, the approach in [**26**] is to redefine the Floer homology of the product by enlarging the gauge group so that the critical points coming from the different copies of $\mathcal{M}_g(2,1)$ are identified by a gauge transformation, giving a new vector space $\widehat{\mathbb{V}}_g$. There is then a surjective map $\mathbb{A}_{g,0} \to \widehat{\mathbb{V}}_g$. To get fully surjective map, we instead enlarge the algebra $\mathbb{A}_{g,0}$ to include a variable ϵ with $\epsilon^2 = 1$, that is, we define $\widetilde{\mathbb{A}}_g = \mathbb{A}_{g,0}[\epsilon]/(\epsilon^2 - 1)$, and set

$$\Phi(\epsilon) = E(\Phi(1)).$$

Define the map $E : \widetilde{\mathbb{A}}_g \to \mathbb{V}_{g,0}$ via:

$$E = I([0,1] \times Y_g, \emptyset, [0,1] \times S^1 \times \mathrm{pt} + \mathrm{pt} \times \Sigma)$$

LEMMA 2.4.3. *We have $E^2 = 1$, E has degree 4, and commutes with the multiplication structure on $\mathbb{V}_{g,0}$: we have*

$$E(v) = \Phi(\epsilon) \cdot v.$$

PROOF. That E has degree 4 mod 8 follows from the Donaldson dimension formula for moduli spaces. By gluing, we have:

$$E = I([0,1] \times Y_g, \emptyset, [0,1] \times S^1 \times \mathrm{pt} + 2\mathrm{pt} \times \Sigma)$$

Now, the Stiefel-Whitney 2-cycle $[0,1]\times S^1\times \mathrm{pt}+2\mathrm{pt}\times\Sigma$, which is the same as $[0,1]\times S^1\times\mathrm{pt}$ with $\mathbb{Z}/2$ coefficients. This implies that we have $E^2=\pm 1$. The formula §A1.4(ii) of [**20**], however, implies that the sign works out to 1. That E commutes with the multiplication is a straightforward application of our gluing theory. □

Using the map E, we can extend the homomorphism Φ to $\widetilde{\mathbb{A}}_g$, which is surjective. Before we prove this, however, we must recall some facts about the cohomology of the representation variety $\mathcal{M}_g(2,1)$. Recall that over $\mathcal{M}_g(2,1)\times\Sigma$ there is a universal rank 2 vector bundle $\mathcal{U}$ and that the cohomology ring of $\mathcal{M}_g(2,1)$ is generated as a $\mathbb{C}$-algebra by the elements $c_2(\mathcal{U})/h$ for $h\in H_*(\Sigma)$. Following [**14**], let $\mathcal{S}$ denote the set of monomials in $\mathbb{A}_{g,0}$ in α, β, and the ψ_j's of the form

$$\alpha^i\beta^j\psi_{j_1}\cdot\ldots\cdot\psi_{j_s}$$

where the j_i's are distinct and in increasing order, and for which $i+s<g$, $j+s<g$. Sending α to $c_2(\mathcal{U})/[\Sigma]$, β to $c_2(\mathcal{U})/[\mathrm{pt}]$, and ψ_j to $c_2(\mathcal{U})/[a_j]$ gives an algebra map from $\Psi:\mathbb{A}_{g,0}\to H^*(\mathcal{M}_g(2,1))$, and the image of $\mathcal{S}$ is a basis for $H^*(\mathcal{M}_g(2,1);\mathbb{C})$ (Remark 5.3, [**14**]).

Proposition 2.4.4. *The map $\Phi:\widetilde{\mathbb{A}}_g\to\mathbb{V}_{g,0}$ is surjective, and the image of $\mathcal{S}\sqcup\epsilon\mathcal{S}$ is a basis.*

Proof. To prove the proposition, we use the natural pairing $\langle,\rangle$ on the vector space $\mathbb{V}_{g,0}$ obtained by taking the product cobordism $[0,1]\times Y_g$ and identifying the outgoing end with Y_g in an orientation reversing way. This gives an elbow macaroni cobordism from two copies of Y_g to the empty set. Since the underlying 4-manifold here gave the identity map, this pairing is nondegenerate. The basis $\mathcal{S}$ can be decomposed into subsets by cohomological degree. We can construct an $|\mathcal{S}|$-by-$|\mathcal{S}|$ matrix M whose entries are the pairings between the elements $\Phi(\chi)$. If χ,χ' are two basis elements, the pairing $\langle\Phi(\chi),\Phi(\chi')\rangle$, by gluing, is the same as the integer Donaldson invariant associated to the closed manifold $S^2\times\Sigma$ with $\Omega=S^2\times\mathrm{pt}$, polynomial $\chi\chi'$, and metric for which the S^2 is much larger than Σ. On this 4-manifold, the ASD moduli spaces have dimension $8k+6g-6$, for k a nonnegative integer. Since $g\geq 2$, the least dimension is $6g-6>0$, and so the pairing $\langle \Phi(\chi), \Phi(\chi')\rangle$ is zero unless $\deg(\chi\chi') \geq 6g-6$, which is the dimension of the space $\mathcal{M}_g(2,1)$. With respect to the block decomposition of the pairing matrix M by degree, we get a matrix which is block anti-lower-triangular (that is, a matrix for whom the i,j [indexed by degree] block is zero unless $i+j \geq 6g-6$, and invertible if $i+j=6g-6$ as a result. The key is that the anti-diagonal blocks where $\deg(\chi)+\deg(\chi') = 6g-6$ are the *same* as the blocks of the corresponding pairing matrix for top cohomology pairings in $\mathcal{M}_g(2,1)$. To see this, we simply note that intersection numbers coming from the extended Floer homology arise from a Morse homology picture which can be arranged to be isomorphic to a Morse homology picture for $\mathcal{M}_g(2,1)$. The cohomology classes $\Psi(\alpha)$, $\Psi(\beta)$, and $\Psi(\psi_i)$'s are exactly the cohomology classes corresponding to the $V_\bullet$'s in $\mathcal{B}^*_{Y_g}$, restricted to the flat locus $\mathcal{M}_g(2,1)$. The approximate ASD equation reduces to approximate flatness, since the $6g-6$ dimensional moduli space corresponds to zero action connections. By Poincaré duality, the pairing matrix in $H^*(\mathcal{M}_g(2,1))$ is nondegenerate because $\mathcal{S}$ is a basis. As a result, M is also nondegenerate, so the image of $\mathcal{S}$ in $\mathbb{V}_{g,0}$ is a linearly independent set. Now, we can construct a $2\,|\mathcal{S}|$-by-$2\,|\mathcal{S}|$ matrix N corresponding to the pairings between monomials in $\Phi(\mathcal{S}) \sqcup \Phi(\epsilon\mathcal{S})$, which has 4 blocks. The upper left block corresponds to M, and the lower right block gives M as well because $\epsilon^2 = 1$. For the off diagonal blocks, the pairing corresponds to a Donaldson invariant for the bundle on $S^2 \times \Sigma$ with $w_2 = \mathrm{PD}(S^2 \times \mathrm{pt} + \mathrm{pt} \times \Sigma)$. Thus, in the ASD dimension formula $d = 8k+6g-6$, k is a half integer, so the lowest dimension is $6g-2$. Hence, these off diagonal blocks are *strictly* block anti-lower-triangular (the diagonal blocks are zero). This implies that we have:

$$N = \begin{pmatrix} M & P \\ P & M \end{pmatrix}$$

where M is block anti-lower-triangular, and P is strictly so. Hence, $\det(N) = \det(M)^2 \neq 0$, so $\Phi(\mathcal{S}) \sqcup \Phi(\epsilon\mathcal{S})$ is linearly independent. Since we know the rank of $\mathbb{V}_{g,0}$ is no more than twice the rank of $H^*(\mathcal{M}_g(2,1))$, we see that in fact these ranks are equal and $\Phi(\mathcal{S}) \sqcup \Phi(\epsilon\mathcal{S})$ is a basis. $\square$

DEFINITION 2.4.5. The *ideal of relations* in $\widetilde{\mathbb{A}}_g$ is the ideal $\widetilde{\mathcal{J}}_g = \Phi^{-1}(0)$. The map Φ expresses the algebra $\mathbb{V}_{g,0}$ as a quotient

$$\mathbb{V}_{g,0} \cong \widetilde{\mathbb{A}}_g / \widetilde{\mathcal{J}}_g.$$

2.5. Excision

Our computations of $\mathbb{V}_{0,n}$ will utilize a formula for Floer homology for surgery along genus 1 surfaces known as *excision*, originally due to Floer. The following result is a recast and slightly generalized version of that theorem and is the prototype for a genus 0 analogue we will prove in the next chapter. Let (Y, K, ω) to be a 3-manifold triple and suppose that T_1, T_2 are 2-tori disjoint from K and for which $\omega \cdot T_i$ is odd for each i. We can construct a new triple (Y', K', ω') by cutting Y along T_1 and T_2 and regluing in the opposite (orientation preserving) way, choosing some identification of T_1 and T_2 and perhaps replacing ω with something homologous.

NOTATION. For Y connected, we denote by $I_*(Y, K, \omega)_{\mathrm{pt},\pm 8}$ the ± 8 generalized eigenspace for the operator $\mu(\mathrm{pt})$. In the case that Y is a disjoint union of connected components $Y = Y_1 \sqcup \ldots \sqcup Y_s$, we understand that $I_*(Y, K, \omega)_{\mathrm{pt},\pm 8}$ refers to the tensor product of the ± 8 generalized eigenspaces of $\mu(y_i)$ for $y_i \in Y_i$.

THEOREM 2.5.1. *(Floer, [**6**]) Suppose that* (Y', K', ω') *is obtained from* (Y, K, ω) *as above. Then we have:*

$$I_*(Y', K', \omega')_{\mathrm{pt},\pm 8} \cong I_*(Y, K, \omega)_{\mathrm{pt},\pm 8}.$$

Before proving the Theorem, we recall some basic facts about the Floer homology of the manifold $Y_1 = T^3$.

PROPOSITION 2.5.2. *The Floer homology* $I_*(T^3, \emptyset, S^1)$ *is rank 2, with generators in the same* $\mathbb{Z}/4$ *grading and for which the matrix of the operator* $\mu(\mathrm{pt})$ *is* $\left(\begin{smallmatrix} 0 & -8 \\ -8 & 0 \end{smallmatrix}\right)$.

COROLLARY 2.5.3. *Suppose there is an embedded genus 1 surface* Σ *in the triple* (Y, K, ω) *disjoint from* K *and intersecting* ω *in an odd number of points. Then the spectrum of* $\mu(\mathrm{pt})$

on $I_*(Y,K,\omega)$ *is* ± 8. *In other words,*

$$I_*(Y,K,\omega) \cong I_*(Y,K,\omega)_{\mathrm{pt},-8} \oplus I_*(Y,K,\omega)_{\mathrm{pt},+8}$$

Proof. Let $(\mathbf{Y}_\Sigma, \mathbf{K}, \Omega)$ be the 4-manifold triple obtained by taking $[0,1]\times(Y,K,\omega)$ and deleting a tubular neighborhood of an internal copy of Σ, thought of as a cobordism with incoming ends (Y,K,ω) and $(T^3,\emptyset,S^1)$. Then by gluing, the operator $(\mu(\mathrm{pt})-8)(\mu(\mathrm{pt})+8)$ acting on $I_*(Y,K,\omega)$ is the same as the composite of capping the T^3 end, acting via $(\mu(\mathrm{pt})-8)(\mu(\mathrm{pt})+8)$ on that end, and applying $I(\mathbf{Y}_\Sigma,\mathbf{K},\Omega)$

$$I_*(Y,K,\omega)\otimes\mathbb{C} \xrightarrow{1\otimes I(D^2\times T^2,\emptyset,D^2\times\mathrm{pt})} I_*(Y,K,\omega)\otimes I_*(T^3,\emptyset,S^1)$$
$$\xrightarrow{1\otimes(\mu(\mathrm{pt})-8)(\mu(\mathrm{pt})+8)} I_*(Y,K,\omega)\otimes I_*(T^3,\emptyset,S^1) \xrightarrow{I(\mathbf{Y}_\Sigma,\mathbf{K},\Omega)} I_*(Y,K,\omega)$$

The operator $(\mu(\mathrm{pt})-8)(\mu(\mathrm{pt})+8)$ certainly vanishes on $I_*(T^3,\emptyset,S^1)$ by the Proposition, so this composite is zero. $\square$

We now prove Theorem 2.5.1.

Proof. We can argue almost exactly as in [**6**]. There is a standard cobordism triple $(W,\overline{K},\overline{\omega})$ from (Y,K,ω) to (Y',K',ω') which may be reflected to give a triple $(W^*,\overline{K}^*,\overline{\omega}^*)$ in the opposite direction. We will consider the two ways of composing W with W^* and compare the results with the identity map. The key is that the $\mu(\mathrm{pt})$ operators commute with all operators arising from cobordisms. We shall prove the theorem for the $+8$ eigenspace, as the negative case is exactly the same.

We begin with the glued-up cobordism $Z = W^*\circ W$ from (Y,K,ω) to itself. Inside of Z there is a copy γ of T^3 along which cutting produces a 4-manifold with four boundary pieces consisting of two copies of Y and two copies of T^3. The 2-cycle $\overline{\omega}^*\circ\overline{\omega}$ in Z intersects γ in an odd number of circles by assumption on $\omega\cdot T_i$, so the 3-manifold triple correpsonding to γ satisfies non-integrality. Since we cut along γ, we recover Z by gluing $U=[0,1]\times T^3$ into the two T^3 boundary components. The triple corresponding to the cylinder U has Stiefel-Whitney class homologous to $[0,1]\times S^1$, and the corresponding map on the Floer homology of (T^3, S^1) is the identity. This Floer homology is well known to be rank 2 with $\mu(\mathrm{pt})$ exchanging the two basis vectors and scaling by 8.

Restriction to the $+8$ generalized eigenspace of $\mu(\mathrm{pt})$ in the Floer homologies is equivalent to acting by the operator corresponding to the polynomial

$$\pi_{\mathrm{pt},+8} = \prod_i \prod_{\lambda \neq 8} ([x_i] - \lambda)^N$$

where λ ranges all the other eigenvalues of $\mu(\mathrm{pt})$ appearing in any of the 3-manifolds, N is larger than any of the ranks, and x_i is a point in the ith connected component of Z. The operator $\mu(\pi_{\mathrm{pt},+8})$ annihilates the other eigenspaces, and the polynomial $\pi_{\mathrm{pt},+2}$ can "slide" between all the cobordisms, by our general gluing result. This operator projects to a one-dimensional subspace of $I_*(T^3, S^1)$. Viewing U as a cobordism from the empty set to two copies of T^3, the relative invariant associated to U and the polynomial $\pi_{\mathrm{pt},+8}$ simply gives $v \otimes v$ where $v \in I_*(T^3, S^1)$ is a $+2$-eigenvector for $\mu(\mathrm{pt})$. Up to a nonzero constant multiple, this relative invariant also arises from the 4-manifold with boundary U' given by two disjoint copies of $D^2 \times T^2$ with Stiefel-Whitney class $D^2 \times \mathrm{pt}$ and polynomial $\pi_{\mathrm{pt},+8}$. Gluing U' into Z cut along M simply gives the identity cobordism, and so we see that the map $I_*(Z; \pi_{\mathrm{pt},+8})$ is just a constant multiple of the map $I_*([0,1] \times Y; \pi_{\mathrm{pt},+8})$. This itself is a constant multiple of the identity map upon restriction to the $+8$ generalized eigenspace of $\mu(\mathrm{pt})$.

Composing the cobordisms in the opposite direction produces $Z^* = W \circ W^*$ and the same argument shows that the induced map on $I_*(Y', K', \omega')$ is a constant multiple of the identity on restriction to $I_*(Y', K', \omega')_{\mathrm{pt},+8}$. Hence, the triple $(W, \overline{K}, \overline{\omega})$ induces the desired isomorphism. The exact same argument works for the -8 eigenspace. $\square$

The Multiplicative Case. There is a much simpler version of the Floer homology for a product 3-manifold Y_g (with empty link), obtained by letting the 2-cycle ω be the loop $u := \mathrm{pt} \times c$, where c is a simple closed curve in Σ shown in PICTURE. The vector space $I_*(Y_g, u)$ can be computed by using a decomposition of Σ into lower genus surfaces and the above excision result, and behaves multiplicatively in $g - 1$. Suppose that c_1 and c_2 are two circles in Σ intersecting c in 1 point each and such that cutting along them produces a genus $g-2$ surface and a genus 1 surface each with 2 boundary components, then $T_i = S^1 \times c_i$ for $i = 1, 2$ is a pair of 2-tori in Y_g satisfying the hypotheses of Theorem 2.5.1. Cutting and regluing produces a copy of Y_2 and Y_{g-1}. We know the Floer homology of a disjoint union is a tensor product and the same is true of the $+8$ generalized eigenspace. We see that

$$I_*(Y_g, u)_{\mathrm{pt},+8} \cong I_*(Y_{g-1}, u)_{\mathrm{pt},+8} \otimes I_*(Y_2, u)_{\mathrm{pt},+8}$$

The basic case to discuss then is $g = 2$. As mentioned in [**21**] this Floer homology was computed by Braam and Donaldson ([**6**], Proposition 1.15). Since the singular set is empty here, the Floer homology has a relative $\mathbb{Z}/8$ grading. As in the case $\omega = S^1 \times \mathrm{pt}$, the Floer homology for $\omega = u$ is equal to the homology of the representation variety, which is now just two copies of a 2-torus. The discussion in [**21**] implies that the sequence of vector spaces in each grading is:

$$I_*(Y_2, u) \cong 0 \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$$

and the operator $\mu(\mathrm{pt})$, which has degree 4, is an isomorphism in each (mod 4) grading. The operator $\mu(\mathrm{pt})$ has 4-dimensional eigenspace for the each of ± 8, which nails down its matrix with respect to a graded basis. Hence, $I_*(Y_2, u)_{\mathrm{pt},+8}$ has rank 4, and

$$I_*(Y_g, u)_{\mathrm{pt},+8} \cong (I_*(Y_2, u)_{\mathrm{pt},+8})^{\otimes(g-1)} \cong \mathbb{C}^{4^{g-1}}. \tag{2.5.1}$$

The key result for the more complicated Floer homology $\mathbb{V}_{g,0}F = I_*(Y_g, w)$ considered in §2.4.1 is that the simultaneous eigenspace for the top eigenvalue of $\mu(\Sigma)$ and $+8$ for $\mu(\mathrm{pt})$ is one-dimensional. The same is true for $I_*(Y_2, u)$, except that now the proof is much quicker. The operator $\mu(\Sigma_2)$ on $I_*(Y_2, u)$ has 2-dimensional generalized eigenspace for each of the values $8i^r$, $i = 0, 1, 2, 3$ and the simultaneous eigenspace for $(\mu(\mathrm{pt}), \mu(\Sigma_2))$ is rank 1 as shown in [**21**]. By stringing together cobordisms from Y_g to $Y_{g-1} \sqcup Y_2$, one gets a cobordism W_g^u from Y_g to $g-1$ copies of Y_2, realizing the isomorphism (2.5.1). The surface $\Sigma \subset Y_g$ is homologous in W_g^u to the sum of the $g-1$ copies $\Sigma_2^{(i)}$, where $\Sigma_2^{(i)}$ is the surface in the ith copy $Y_2^{(i)}$ of Y_2, so the operators $\mu(\Sigma)$ on $I_*(Y_g, u)_{\mathrm{pt},+8}$ and $\sum_{i=1}^{g-1} \mu(\Sigma_2^{(i)})$ on $I_*(Y_2, u)_{\mathrm{pt},+8}^{\otimes(g-1)}$ are intertwined by the isomorphism (2.5.1). It follows that the eigenspace of $\mu(\Sigma)$ for the value $2g-2$ is one-dimensional, spanned by the preimage of $\otimes_{i=1}^{g-1} v_2$ under (2.5.1) where v_2 is an eigenvector with value 2. This argument is not, however, independent of the calculations in [**27**], as our knowledge of the spectrum of $\mu(\Sigma_2)$ on $I_*(Y_2, u)$ in fact comes from an understanding on how $\mu(\Sigma_2)$ acts on $\mathbb{V}_{2,0}$. The above discussion could have been carried out for either ± 8, and so we have proved:

PROPOSITION 2.5.4. *The isomorphism (2.5.1) intertwines $\mu(\Sigma)$ on $I_*(Y_g, u)_{\mathrm{pt},\pm 8}$ and $\sum_{i=1}^{g-1} \mu(\Sigma_2^{(i)})$ on $I_*(Y_2, u)_{\mathrm{pt},\pm 8}^{\otimes(g-1)}$. The spectrum of $\mu(\Sigma)$ is the set*

$$\{-8(g-1), -8(g-2), \ldots, -8, 0, 8, \ldots, 8(g-2), 8(g-1)\}$$

with the simultaneous eigenspace of the pair $(\mu(\mathrm{pt}), \mu(\Sigma))$ for the values $(\pm 8(g-1), \pm 8)$ being 1-dimensional.

2.6. The Floer Homology of (Y_g, K_n)

We now set ourselves to the task of solving the main problem of this thesis, which is a computation of the Floer homology of a product link K_n in $Y_0 = S^1 \times S^2$. We first make some remarks applicable to the general case of (Y_g, K_n). To get a 3-manifold triple satisfying non-integrality, we must impose the condition that n be odd (or zero). In this case a Σ fiber of Y_g will be a non-integral surface. There are several possibilities for the Stiefel-Whitney class: we could choose $S^1 \times \mathrm{pt}$, or a union of arcs $u_i \subset S^2$ joining several pairs of marked points, the the empty set, or some linear combination of these. It is not difficult to see that adding $S^1 \times \mathrm{pt}$ to the Stiefel-Whitney class does not change the Floer homology. We can recover the classical case $n = 0$ as well as long as we do use $\omega = S^1 \times \mathrm{pt}$. The most difficult and relevant case for us is the case $\omega = \emptyset$, and this is the analogue of the case $\omega = S^1 \times \mathrm{pt}$ in the nonsingular story. However, our calculations for this space will use the case of a nonzero Stiefel-Whitney class. In the case $\omega = \emptyset$, the Floer homology of (Y_g, K_n) inherits a product structure analogous to that of $\mathbb{V}_{g,0}$.

DEFINITION 2.6.1. Let $\mathbb{V}_{g,n} = I_*(Y_g, K_n, \emptyset)$ denote the $\mathbb{C}$-algebra obtained by using the product structure induced by applying functoriality of I_* to the cobordism $F \times (\Sigma_g, \{n \text{ pts}\}, \emptyset)$.

Our first result is that $\mathbb{V}_{g,n}$ is isomorphic as a vector space to the homology of two copies of the corresponding representation variety of flat connections $\mathcal{R}_{g,n}$ on the punctured surface $\Sigma_g \setminus \{x_1, \ldots, x_n\}$.

To prove this, we need an analogue of the Φ-map 2.4.1. We use the free graded-commutative algebra

$$\mathbb{A}_{g,n} = \mathbb{A}_{g,0} \otimes \mathbb{C}[\delta_1, \ldots, \delta_n]$$

which is exactly the algebra appearing in Chapter 1. We proceed as before and construct elements in the Floer homology now via the bounding 4-manifold pair $(A_g, S_n) = D^2 \times (\Sigma_g, \{x_1, \ldots, x_n\})$. Let $\Phi : \mathbb{A}_{g,n} \to \mathbb{V}_{g,n}$ be defined on generators via:

$$\begin{aligned}
\alpha &\mapsto -\tfrac{1}{4}\mu^{\mathrm{orb}}([\Sigma]) \cdot I_*(A_g, S_n, \emptyset) \\
\beta &\mapsto -\tfrac{1}{4}\mu(\mathrm{pt}) \cdot I_*(A_g, S_n, \emptyset) \\
\psi_j &\mapsto -\tfrac{1}{4}\mu([a_j]) \cdot I_*(A_g, S_n, \emptyset) \\
\delta_k &\mapsto \tfrac{1}{2}\mu([x_k])_{\boldsymbol{\kappa}_k} I_*(A_g, S_n, \emptyset).
\end{aligned}$$

where x_k refers to a point on the singular locus component $D^2 \times \{x_k\}$, and we again use the normalizing factors $-\frac{1}{4}$ and $\frac{1}{2}$ to account for the fact that we have used the characteristic class p_1 of the universal bundle, rather than the second Chern class c_2. Strictly speaking in order to pin down the sign of δ_k, we need to specify a choice of $\boldsymbol{\kappa}_k$. Later on, however, we will see that changing the signs of any of the δ_k's induces an automorphism of $\mathbb{V}_{g,n}$ so that the choice of sign is immaterial, and we simply make an arbitrary choice. The map Φ respects the relative $\mathbb{Z}/4$ grading and is a homomorphism of $\mathbb{C}$ algebras.

Likewise, for each polynomial $\chi \in \mathbb{A}_{g,n}$, we obtain an element $\Psi(\chi)$ of $H^*(\mathcal{R}_{g,n})$ via characteristic classes of the universal bundle over $\mathcal{R}_{g,n} \times \Sigma$, by Chapter 1. We would like to show that Φ is surjective, but as before we need to enlarge the algebra $\mathbb{A}_{g,n}$. We define a map $E : \mathbb{V}_{g,n} \circlearrowleft$ via

$$E = I([0,1] \times Y_g, [0,1] \times K_n, \mathrm{pt} \times \Sigma),$$

noting that E^2 is either 1 or -1. We no longer can be confident about the sign, so we write

$$E^2 = \zeta_{g,n}$$

and define the extended algebra

$$\widetilde{\mathbb{A}}_{g,n} := \mathbb{A}_{g,n}[\epsilon]/(\epsilon^2 - \zeta_{g,n}).$$

The map Φ extends to one from $\widetilde{\mathbb{A}}_{g,n}$.

THEOREM 2.6.2. *Let $\mathcal{R}_{g,n}$ denote the moduli space of representations from Chapter 1 for n odd and $3g + n - 3 \geq 0$. The vector space $\mathbb{V}_{g,n}$ is isomorphic to $H^*(\mathcal{R}_{g,n}, \mathbb{C})^{\oplus 2}$.*

PROOF. Let $\mathcal{S} \subset \mathbb{A}_{g,n}$ be a finite set which maps to a basis of $H^*(\mathcal{R}_{g,n})$ under Ψ. As before there is a pairing $\langle , \rangle$ on $\mathbb{V}_{g,n}$ which can be checked to agree with the cohomological pairing on $H^*(\mathcal{R}_{g,n})$ for pairs $z, z' \in \mathcal{S}$ of complimentary degree. We get a $2\,|\mathcal{S}| \times 2\,|\mathcal{S}|$ matrix N of pairings in the Floer homology with 4 blocks. The diagonal blocks are invertible by Poincaré duality and the off diagonal blocks are stricly block anti-lower-triangular, and thus N is invertible. We see that $\Phi(\mathcal{S}) \sqcup \Phi(\epsilon\mathcal{S})$ is a basis for $\mathbb{V}_{g,n}$, completing the proof. $\square$

2.6.1. Comparing the Ring Structures. We wish to compute the ring structure of $\mathbb{V}_{g,n}$, using as our model the techniques in [**27**]. The main idea is that the relations in the ring $\mathbb{V}_{g,n}$ are graded deformations of those in $H^*(\mathcal{R}_{g,n})$. The task then becomes computing the terms in the deformation. Again, our approach here differs from [**27**] since we do not quotient by a larger gauge group and so the Floer homology is twice as large as the cohomology ring of $\mathcal{R}_{g,n}$. Instead, let us define a map $\Phi^+ : \mathbb{A}_{g,n} \to \mathbb{V}_{g,n}$ via

$$\begin{aligned}\Phi^+(z) = \tfrac{1}{2}\left[\Phi(z) + \sqrt{\zeta_{g,n}}^{-1}\Phi(\epsilon z)\right] \\ \tfrac{1}{2}\left[\Phi(z) + \sqrt{\zeta_{g,n}}^{-1}E(\Phi(z))\right]\end{aligned}$$

where $\sqrt{\zeta_{g,n}}$ is a choice of square root.

LEMMA 2.6.3. *The map Φ^+ is a homomorphism of algebras and is a surjection onto the $\sqrt{\zeta_{g,n}}$-eigenspace of E, which is half-dimensional.*

PROOF. Let $z, z' \in \mathbb{A}_{g,n}$. We have:

$$\begin{aligned}\Phi^+(z)\Phi^+(z') &= \tfrac{1}{4}\left[\Phi(z) + \sqrt{\zeta_{g,n}}^{-1}\Phi(\epsilon)\Phi(z)\right]\left[\Phi(z') + \sqrt{\zeta_{g,n}}^{-1}\Phi(\epsilon)\Phi(z')\right] \\ &= \tfrac{1}{4}\left[2\Phi(z)\Phi(z') + 2\sqrt{\zeta_{g,n}}^{-1}\Phi(\epsilon)\Phi(z)\Phi(z')\right] = \Phi^+(zz')\end{aligned}$$

so Φ^+ is a homomorphism. By basic linear algebra the image of $\mathcal{S}$ is a linearly independent set. If we choose the other square root $-\sqrt{\zeta_{g,n}}$, we obtain a map Φ^- and the images of Φ^+ and Φ^- are orthogonal under the pairing. This can be seen by considering the dimensions of moduli spaces on $S^2 \times (\Sigma_g, \{x_1, \ldots, x_n\})$ and pairings of homogeneous elements. The span of the images of Φ^+ and Φ^- must then be all of $\mathbb{V}_{g,n}$, and so the image of Φ^+ is half-dimensional. That this is exactly the $\sqrt{\zeta_{g,n}}$-eigenspace of E is now clear. □

As a corollary, we see that the $\sqrt{\zeta_{g,n}}$-eigenspace of E is a half-dimensional subalgebra $\mathbb{V}^+_{g,n}$, and the map $\Phi^+ : \mathbb{A}_{g,n} \to \mathbb{V}^+_{g,n}$ induces an isomorphism of vector spaces

$$\mathfrak{F} : H^*(\mathcal{R}_{g,n}) \to \mathbb{V}^+_{g,n}$$

by extending the set map from $\Psi(\mathcal{S})$ to $\Phi^+(\mathcal{S})$ by linearity. The analogue of the statement on deformations for us is:

PROPOSITION 2.6.4. *Under the above vector space isomorphism, the ring structure of $\mathbb{V}^+_{g,n}$ is a mod 2 graded deformation of the cup product on cohomology. More precisely if $f, f' \in H^*(\mathcal{R}_{g,n})$ are homogeneous of degree i, j, then we have:*

$$((\mathfrak{F}(f) \cdot \mathfrak{F}(f')) = \mathfrak{F}\Big(f \smile f' + \sum_{\substack{d<i+j \\ d\equiv(i+j) \bmod 2}} g_d\Big)$$

for a sequence of $g_d \in H^*(\mathcal{R}_{g,n})$ of degree $d \leq i + j - 2$.

PROOF. The proof is essentially the same as for Theorem 5 of [**27**], with the only difference that the dimensions of moduli spaces on $S^2 \times \Sigma$ are now $\dim(\mathcal{R}_{g,n}) + M$ for M a multiple of 2 rather than 4. $\square$

The ring structure of $\mathbb{V}^+_{g,n}$ is determined by the kernel $\mathcal{J}^+_{g,n}$ of Φ^+. The surjection $\mathbb{A}_{g,n} \to H^*(\mathcal{R}_{g,n})$ induces an isomorphism of the cohomology ring with a quotient $\mathbb{A}_{g,n}/\mathcal{I}_{g,n}$ for an ideal $\mathcal{I}_{g,n}$. In the genus 0 case, Proposition 2.6.4 and our results from Chapter 1 then imply:

COROLLARY 2.6.5. *The ideal $\mathcal{J}^+_{0,n}$ contains a collection of relations $Q^J_{0,n}$ for each $J \subset \{1, \ldots, n\}$ of cardinality at most $m = (n-1)/2$. The polynomial $Q^J_{0,n}$ has leading term $R^J_{0,n}$ from (1.6.2) and is a mod 2 graded deformation of this term. That is:*

$$Q^J_{0,n} = R^J_{0,n} + \text{lower order terms.}$$

The remaining relations in $\mathcal{J}^+_{0,n}$ are deformations of the relations $\delta_k^2 - \beta$ in $\mathcal{I}_{0,n}$. We shall study these deformations later.

2.6.2. Induction on n. As in our computations for the cohomology rings $\mathbb{H}_{g,n}$, we will leverage an inductive structure for the rings $\mathbb{V}_{g,n}$, which arise as natural surjections $\mathbb{V}_{g,n+2} \to \mathbb{V}_{g,n}$. Let $(A_g, S_n, \emptyset)$ denote the 4-manifold triple, as in Theorem 2.6.2, consisting of the cap $A_g = D^2 \times \Sigma$ (for Σ genus g) and singular locus $S_n = D^2 \times \{x_1, \ldots, x_n\}$. The boundary of this triple is the 3-manifold triple

$$(Y_g, K_n, \emptyset) = (S^1 \times \Sigma, S^1 \times \{x_1, \ldots, x_n\})$$

We will find homologous copy of Σ intersecting the singular locus in $n+2$ points rather than n. Let C denote an interior copy of Σ, say $\{\underline{0}\} \times \Sigma$ in A_g. Let γ be the image of an embedding of the interval $(-1, 1)$ into D^2 sending 0 to $(0, 0)$. Let y denote the image of $1/2$, and let B be a small 3-ball around (y, x_n) inside the 3-manifold $\gamma \times \Sigma$. Then ∂B is a small 2-sphere intersecting S_n in just two points, as it hits $D^2 \times \{x_n\}$ twice and misses the other singular disks. We can then tube together the 2-spheres ∂B and C in a way missing the singular locus, giving a new 2-sphere Σ' intersecting the singular locus $n+2$ times,

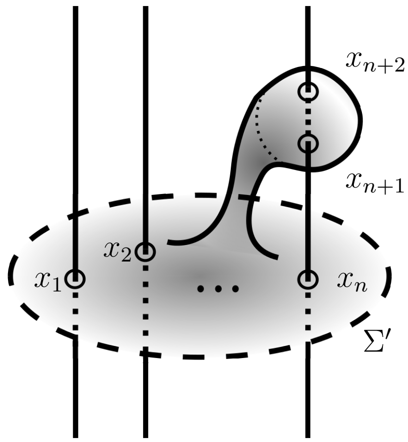

FIGURE 2.1. The surface Σ' intersecting S_n in $n+2$ points.

hitting $D^2 \times \{x_n\}$ 3 times (see figure 2.6.2). Let $\nu\Sigma'$ denote a narrow tubular neighborhood of Σ', and let $(W_{g,n}^{g,n+2}, S_{g,n}^{g,n+2}, \emptyset)$ be the cobordism triple obtained by subtracting $\nu\Sigma'$ from $(A_g, S_n, \emptyset)$. This cobordism triple goes from $(Y_g, K_{n+2}, \emptyset)$ to $(Y_g, K_n, \emptyset)$, and induces a map

$$\pi_{g,n}^{g,n+2} : \mathbb{V}_{g,n+2} \to \mathbb{V}_{g,n}.$$

To be precise, we must specify an identification of the new boundary component of $W_{g,n}^{g,n+2}$ with $(Y_g, K_{n+2}, \emptyset)$. We take a diffeomorphism which respects the natural S^1-bundle structure of $\partial\nu\Sigma'$, and which is the identity away from the tube and B, and which maps the portion of Σ' due to the tube and B to a small neighborhood of x_{n+1} and x_{n+2}. Orienting the disk $D^2 \times \{x_n\}$ such that its intersection with Σ is positive, one of the new intersections with Σ' will be positive and one will be negative. We choose our identification so that the negative one is mapped to x_{n+1} and the positive one to x_{n+2}. In addition, we pin down the sign for this cobordism by choosing the restriction of the product complex structure on $D^2 \times \Sigma$, which agrees with the product complex structure on the cylinder over $(Y_0, K_n, \emptyset)$.

REMARK 2.6.6. These maps are the exact analogue of the genus-increasing maps used by Muñoz in [**27**] (see Lemma 9). Indeed, there are also natural maps

$$\pi_{g,n}^{g,n+2} : \mathbb{V}_{g,n+2} \to \mathbb{V}_{g,n}$$

$$\pi_{g,n}^{g+1,n} : \mathbb{V}_{g+1,n} \to \mathbb{V}_{g,n}$$

arising from cobordism triples $(W_{g,n}^{g+1,n}, S_{g,n}^{g+1,n}, \emptyset)$.

The major fact that we need about these maps is that they behave well with respect to the maps $\Phi : \mathbb{A}_{g,n} \to \mathbb{V}_{g,n}$.

LEMMA 2.6.7. *Let $\tau_{g,n}^{g,n+2} : \mathbb{A}_{g,n+2} \to \mathbb{A}_{g,n}$ denote the map sending α to α, β to β, ψ_j to ψ_j, and δ_k to δ_k for $k \neq n-1$, and δ_{n+1} to $-\delta_n$. The following diagram commutes:*

$$\begin{tikzcd}
\mathbb{A}_{g,n+2} \arrow[r, "\tau_{g,n}^{g,n+2}"] \arrow[d, "\Phi"] & \mathbb{A}_{g,n} \arrow[d, "\Phi"] \\
\mathbb{V}_{g,n+2} \arrow[r, "\pi_{g,n}^{g,n+2}"] & \mathbb{V}_{g,n}
\end{tikzcd}$$

PROOF. The point is that if we compose the capping cobordism $(A_g, S_{n+2}, \emptyset)$ with the cobordism $(W_{g,n}^{g,n+2}, S_{g,n}^{g,n+2}, \emptyset)$ from Remark 2.6.6 then the result is isomorphic to $(A_g, S_n, \emptyset)$. Moreover, the surface Σ in the copy of $(A_g, S_{n+2}, \emptyset)$ is homologous - isotopic, in fact - to the copy of Σ in $(A_g, S_n, \emptyset)$ under this identification. This show that the diagram is commutative for the generators α and β, and in fact also for the ψ_j's. As for the δ_k's, under this identification the piece $D^2 \times \{x_k\}$ of singular locus in $(A_g, S_{n+2}, \emptyset)$ is a subset of the locus $D^2 \times \{x_k\}$ in $(A_g, S_n, \emptyset)$ for $k \leq n$. The locus $D^2 \times \{x_{n+i}\}$ for $i = 1, 2$ is a (small) subset of $D^2 \times \{x_n\}$ in $(A_g, S_n, \emptyset)$ as well, but the orientation on the D^2 factor is flipped for $i = 1$. This accounts for the sign on δ_{n+1}. $\square$

Now, since we do not have an isomorphism $\mathbb{V}_{g,n} \cong H^*(\mathcal{R}_{g,n}; \mathbb{C})$ as vector spaces but rather an isomorphism $\mathbb{V}_{g,n}^+ \cong H^*(\mathcal{R}_{g,n}; \mathbb{C})$, where the plus sign indicates the $\sqrt{\zeta_{g,n}}$-eigenspace for the degree 2 map E, we need to amplify the above result for this subspace of $\mathbb{V}_{g,n}$. The result will clearly hold with $\mathbb{V}_{g,n}^+$ replacing $\mathbb{V}_{g,n}$ as long as E commutes with the map $\pi_{g,n}^{g,n+2}$.

That this is the case is clear, since the 2-cycle representing the Stiefel-Whitney class in the cobordism (which is just the cylinder) defining E is a copy of the surface Σ, and the "incoming" Σ is isotopic to the "outgoing" Σ, with all the almost complex structures agreeing up to isotopy in the gluings. Recall that $\mathbb{V}^+_{g,n} \cong \mathbb{A}_{g,n}/\mathcal{J}^+_{g,n}$ for an ideal $\mathcal{J}^+_{g,n} = (\Phi^+)^{-1}(0)$.

Corollary 2.6.8. *Under the map $\tau^{g,n+2}_{g,n} : \mathbb{A}_{g,n+2} \to \mathbb{A}_{g,n}$, we have the following inclusion of ideals:*

$$\mathcal{J}^+_{g,n+2} \subset \left(\tau^{g,n+2}_{g,n}\right)^{-1} (\mathcal{J}^+_{g,n}) \tag{2.6.1}$$

Proof. Suppose that $f \in \mathbb{A}_{g,n+2}$ is a polynomial which is a relation in $\mathbb{V}^+_{g,n+2}$. Then by Lemma 2.6.7, the polynomial $\tau^{g,n+2}_{g,n}(f)$ is a relation in $\mathbb{V}^+_{g,n}$. □

Lastly, there are also natural injections $\iota^{g,n}_{g,n+2}$ and $\iota^{g,n}_{g+1,n}$ obtained by reversing the direction of the cobordisms defining the maps $\pi^{\bullet}_{\bullet}$. We will want to leverage these later, but note now that these injections do not fit into obvious commutative diagrams with injections $\mathbb{A}_{g,n} \hookrightarrow \mathbb{A}_{g,n}, \mathbb{A}_{g,n+2}$.

The Flips on $\mathbb{V}_{g,n}$. While the flip construction has been described and used for moduli spaces on 4-manifolds, we can use it to generate automorphisms of the 3-manifold Floer homology $\mathbb{V}_{g,n}$.

Lemma 2.6.9. *Let $J \subset \{1, \ldots, n\}$ be an even cardinality subset. The automorphism $\widetilde{m}_J$ of $\mathbb{A}_{g,n}$ sending δ_k to δ_k for $k \notin J$ and $-\delta_k$ for $k \in J$ preserves the ideal $\mathcal{J}^+_{g,n}$. This induces an automorphism m_J of $\mathbb{V}^+_{g,n}$.*

Proof. The lemma is equivalent to the following statement: Iforf $\chi \in \mathbb{A}_{g,n}$, we have an equality of Donaldson invariants

$$I(S^2 \times \Sigma, S^2 \times \{x_1, \ldots, x_n\}, \emptyset; \chi) = I(S^2 \times \Sigma, S^2 \times \{x_1, \ldots, x_n\}, \emptyset; \widetilde{m}_J(\chi)),$$

where the metric chamber we use on each is the one for which the area of S^2 is much larger than that of Σ. This second statement is an immediate consequence of Lemma 2.3.3. Indeed the Donaldson invariants are simple pairings of cohomology classes with moduli spaces of connections, and the flip carries the moduli space on the left to that on the right, and by construction the cohomology class $\mu(\chi)$ to $\mu(\widetilde{m}_j(\chi))$. $\square$

From now on, the maps $\widetilde{m}_J$ and m_J will also be called the "flips" on J.

LEMMA 2.6.10. *Suppose that $f \in \mathcal{J}^+_{g,n}$ is a polynomial relation. Break f into different δ_k terms:*

$$f = \sum_{I \subset \{1,\dots,n\}} \underline{\delta}^I f_I(\alpha, \beta, \psi_1, \dots, \psi_{2g})$$

for polynomials f_I not involving any δ_k's. Then for any individual I, the polynomial $\underline{\delta}^I f_I$ is itself a relation.

PROOF. We prove this by induction on the number of nonzero terms in the sum. Suppose that $I' \neq I$ and f_I and $f_{I'}$ are nonzero. Then we can find some pair k, l such that the cardinalities of $\{k,l\} \cap I$ and $\{k,l\} \cap I'$ are of different parities. By Lemma 2.6.9, the flipping automorphism $\widetilde{m}_{\{k,l\}}$ of $\mathbb{A}_{g,n}$ preserves $\mathcal{J}^+_{g,n}$. Hence, if $\{k,l\} \cap I$ is even, then we have:

$$\tfrac{1}{2}(1 + \widetilde{m}_{\{k,l\}}) \cdot f = \sum_{J : |J \cap \{k,l\}| = \text{even}} \underline{\delta}^J f_J$$

and this polynomial is also a relation. This sum contains the same term $\underline{\delta}^I f_I$, but the term for I' goes away. If $\{k,l\} \cap I$ were odd, we could look at $\frac{1}{2}(1 - \widetilde{m}_{\{k,l\}}) \cdot f$ and again get a new relation with fewer terms and in which the I term remains. By induction, we find that $\underline{\delta}^I f_I$ is itself a relation. $\square$

2.7. The Case $g = 0$ and $n = 3$

Our calculations for the ring structure on $I_*(Y_g, K_n, \emptyset)$ will leverage calculations performed for the simplest nontrivial case $g = 0$ and $n = 3$. There are essentially two different flavors of the Floer homology of (Y_0, K_3) to consider. In general, we have already defined $\mathbb{V}_{g,n}$, the Floer homology $I_*(Y_g, K_n, \emptyset)$. For any g and $n \geq 1$ (not necessarily odd) we define:

DEFINITION 2.7.1. The *twisted* Floer homology of a product (Y_g, K_n) is the vector space

$$\mathbb{U}_{g,n} := I_*(Y_g, K_n, u)$$

where u is an arc in Σ_g between x_1 and x_2.

Here, n need not be odd since for a loop d_1 around x_1, the 2-torus $S^1 \times d_1$ is a non-integral surface. While $\mathbb{U}_{g,n}$ is not naturally a ring, it is a module over $\mathbb{V}_{g,n}$. We now will study the case $n = 3$ and $g = 0$ in depth.

2.7.1. The Ring $\mathbb{V}_{0,3}$. When $n = 3$, both $\mathbb{V}_{0,3}$ and $\mathbb{U}_{0,3}$ are 2 dimensional. This is clear for $\mathbb{V}_{0,3}$, since the representation variety $\mathcal{R}_{0,3}$ is point. We can also see this directly as follows. Let d_i be a loop going around x_i and t the loop $S^1 \times \mathrm{pt}$. A flat connection gives a representation into SU(2) modulo conjugation, where x_i maps to the trace zero conjugacy class $C_{\mathbf{i}}$. The only triple (T_1, T_2, T_3) satisfying this is, up to ocnjugation, $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, and t must map then to ± 1. We get exactly 2 flat connections ρ_i for $i = 0, 1$ with S^1 holonomy $(-1)^i$ and it is not hard to see that the Chern-Simons functional is of Morse type, so that we need not perturb. It remains to check that the differential vanishes, which follows from the following.

LEMMA 2.7.2. *The two flat connections on (Y_0, K_3) differ in grading by 2 mod 4.*

PROOF. Consider the 4-manifold with boundary $A_0 = D^2 \times S^2$, with singular locus $S_3 = D^2 \times \{x_1, x_2, x_3\}$. The relative invariant $I_*(A_0, S_3, \emptyset)$ in $\mathbb{V}_{0,3}$ is found by looking at instantons on A_0. We are searching for 0-dimensional moduli spaces. The connection ρ_0 extends to one on the trivial bundle on $(A_0, S_3, \emptyset)$ and by inspection this is the unique flat connection. The nondegenerateness of CS on flat connections implies this single flat connection is the full transverse 0-dimensional moduli space; all others are higher dimensional. We see that up to sign $I_*(A_0, S_3, \emptyset)$ is the vector $[\rho_0]$. Similarly, if we change the Stiefel-Whitney class, there is a flat connection for the triple $(A_0, S_3, D^2 \times \mathrm{pt})$ extending ρ_1 and up to sign $I_*(A_0, S_3, D^2 \times \mathrm{pt}) = [\rho_1]$.

To see that the ρ_i lie in degrees differing by 2, consider the elbow macaroni cobordism diffeomorphic to $[0, 1] \times Y_0$ going from two copies of Y_0 to the empty set. Plugging in two copies of $(A_0, S_3, \emptyset)$, we get a 4-manifold diffeomorphic to $S^2 \times S^2$, with 0 Steifel-Whitney class for the bundle. The dimension formula for the ASD moduli spaces from [**18**] gives $8k + 4l$ where k is the instanton number and l is the monopole number. Since the class w_2 is 0, l and k are integers and so the moduli space has dimension a multiple of 4. Using what we know about the relative invariant for $(A_0, S_3, \emptyset)$, by stretching the metric near the ends of $[0, 1] \times Y_0$ we see that the moduli spaces on the cylinder going between ρ_0 to ρ_0 have dimension a multiple of 4. On the other hand if we replace one of the $(A_0, S_3, \emptyset)$ with $(A_0, S_3, D^2 \times \mathrm{pt})$, we see that the in the dimension formula $8k+4l$, because the singular locus intersects w_2 in an odd number (three) of points, l will now be a half-inetger, and $8k + 4l$ will be 2 mod 4. Thus, the moduli spaces going from ρ_0 to ρ_1 will have dimension 2 mod 4, allowing us to conclude ρ_0 and ρ_1 are off by 2 in grading. □

REMARK 2.7.3. So far, the signs of the generator $[\rho_i]$ has been determined by letting it be the image of the flat connection ρ_i under the quotient map upon passing to homology (of course, the differential in this case is 0). It will be convenient to instead choose $[\rho_0]$ to be generator obtained as the relative invariant associated to $(A_0, S_3, \emptyset)$, and $[\rho_1]$ to be $I_*(A_0, S_3, D^2 \times \mathrm{pt})$. The signs are fixed by using the product almost complex structure on the underlying 4-manifold $A_0 = D^2 \times S^2$. From now this is what we will mean by $[\rho_i]$.

While this essentially determines the ring $\mathbb{V}_{0,3}$, the important information we need from $\mathbb{V}_{0,3}$ are the matrices for the operators $\mu^{\mathrm{orb}}(\Sigma)$ and $\mu(\mathrm{pt})$. The operator $\mu(\mathrm{pt})$ can be determined by comparing with the corresponding operator for the nonsingular case of $Y_1 = T^3$. However, we first want to understand the operator $\mu(\Sigma)$, namely its eigenvalues. Despite a full answer for the corresponding question for the nonsingular case $\mathbb{V}_{g,0}$, the author is unable to devise an argument leveraging that computation in this situation. Instead, we proceed by performing the computation by directly understanding a 2-dimensional moduli space. We shall prove:

THEOREM 2.7.4. *The operator $\mu(\Sigma)$ on $\mathbb{V}_{0,3}$ sends the generator $[\rho_i]$ to $\pm 4[\rho_{1-i}]$, where the sign is the same in each case.*

In order to effect this computation, we will need to delve into the world of parabolic stable bundles.

2.7.2. Stable Parabolic Bundles. We will need a result relating singular anti-self dual connections on a Kähler surface to stable parabolic holomorphic bundles. The analogous result in 2 real dimensions was used in Chapter 1 to understand the moduli space of flat connections on a punctured Riemann surface. The 4 dimensional case was treated in [**20**], and it is this treatment we model our approach after. We first review the notion of a rank 2 parabolic bundle and the stability condition in 4 dimensions, and refer the reader to more detailed accounts [**30**] and [**24**] for more general notions and more complete discussion.

Suppose that (X, κ) is a closed Kähler 4-manifold, and S is a smooth holomorphic curve with connected components $S^{(1)}, \dots, S^{(n)}$. A rank 2 holomorphic bundle on X with parabolic structure S is holomorphic vector bundle $\mathcal{E}$ along with a line subbundle $\mathcal{F}$ of $\mathcal{E}|_S$ and pair of weights $\lambda_0 > \lambda_1 \in [0, 1)$. For us, we will have a fixed smooth rank 2 bundle on X in mind with structure group U(2), and since our holomorphic bundles will be required to have a fixed determinant bundle we will have $\lambda_0 + \lambda_1 = 1$ in our correspondence, so we set $\lambda = \lambda_1$ and force $\lambda_0 = 1 - \lambda$ (soon, λ will just be $1/4$). The parabolic degree of $\mathcal{E}$ is defined to the pairing with the Kähler class $[\kappa]$ of the parabolic first Chern class ([**30**], Definition 3.6):

$$c_1^{\mathrm{par}}(\mathcal{E}) = c_1(\mathcal{E}) + \sum_{i=1}^{n} \mathrm{PD}[S^{(i)}].$$

In order to define the stability condition in rank 2, we only need to consider the possible line subbundles of $\mathcal{E}$. If $\mathcal{L} \to \mathcal{E}$ is a map of a line bundle into $\mathcal{E}$, it acquires weights $\lambda^{(i)}$ at each $S^{(i)}$ depending on whether the image of $\mathcal{L}|_{S^{(i)}}$ is contained in $\mathcal{F}_i$ and is nonzero. We set $\lambda^{(i)} = 1 - \lambda$ if it is, and $\lambda^{(i)} = \lambda$ otherwise, and define the parabolic first Chern class of $\mathcal{L}$ via

$$c_1^{\mathrm{par}}(\mathcal{L}) = c_1(\mathcal{L}) + \sum_{i=1}^{n} \lambda^{(i)} \mathrm{PD}[S^{(i)}].$$

We then define the parabolic *slope* μ^{par} of $\mathcal{E}$ or $\mathcal{L}$ to be the parabolic degree divided by the rank:

$$\mu^{\text{par}}(\mathcal{E}) = \tfrac{1}{2}\left\langle c_1^{\text{par}}(\mathcal{E}) \smile [\kappa], X\right\rangle \tag{2.7.1}$$

$$\mu^{\text{par}}(\mathcal{L}) = \left\langle c_1^{\text{par}}(\mathcal{L}) \smile [\kappa], X\right\rangle \tag{2.7.2}$$

with the understanding that the notion for a line bundle $\mathcal{L}$ depends on a particular map into $\mathcal{E}$.

DEFINITION 2.7.5. The parabolic rank 2 bundle $(\mathcal{E}, \mathcal{F})$ is said to be *stable* if for any nonzero map $\mathcal{L} \to \mathcal{E}$ for a line bundle $\mathcal{L}$ we have $\mu^{\text{par}}(\mathcal{L}) < \mu^{\text{par}}(\mathcal{E})$.

DEFINITION 2.7.6. The *monopole* number of a parabolic rank 2 bundle $(\mathcal{E}, \mathcal{F})$ on the component $S^{(i)}$ of S is the number

$$l_i(\mathcal{E}, \mathcal{F}) := \tfrac{1}{2}(c_1(\mathcal{F}^{\perp}) - c_1(\mathcal{F})) = \tfrac{1}{2}c_1(\mathcal{E}|_{S^{(i)}}) - c_1(\mathcal{F})$$

Two parabolic bundles $(\mathcal{E}, \mathcal{F}, \lambda)$ and $(\mathcal{E}', \mathcal{F}', \lambda')$ are declared to be isomorphic if $\lambda = \lambda'$ and there is a holomorphic isomorphism $\mathcal{E} \to \mathcal{E}'$ under which $\mathcal{F}$ is carried to $\mathcal{F}'$. General theory of parabolic bundles implies that the set of isomorphism classes of stable parabolic bundles with fixed generic weight λ and characteristic classes is naturally an algebraic variety and in nice cases is smooth and compact.

Now, we suppose that (X, κ) is an orbifold Kähler surface singular with cone angle π along a holomorphic curve $S \subset X$, and suppose Ω is a Stiefel-Whitney 2-cycle with no boundary components, so that any singular orbifold bundle $\check{P}$ for the triple $(X, S\Omega)$ extends to a non-orbifold SO(3)-bundle P on all of X. Let k and $\underline{l}$ be the instanton and monopole numbers for $\check{p}$. Fix a smooth $\mathbb{C}^2$ bundle E on X with $c_1(E) \equiv \text{PD}(\Omega) \mod 2$ and $c_2(E) = k$. Then we have:

THEOREM 2.7.7. *There is a one-to-one correspondence between elements of the moduli space $\mathcal{M}_{\kappa}(\check{P})$ of singular anti-self dual connections (with respect to the metric κ) with monopole invariants $(k, \underline{l})$ and the moduli space $\mathcal{M}^{\text{par}}(E)$ of isomorphism class of stable (with* *respect to κ) parabolic bundles structures on E with the same monopole invariants and the weight $\lambda = 1/4$. In the case that the moduli space $\mathcal{M}_\kappa(\check{P})$ is a smooth compact manifold, this bijection gives a smooth diffeomorphism between the moduli spaces. Over $\mathcal{M}_\kappa(\check{P})$ there is a universal family of bundles $\underline{P}$ with family of singular connections $\underline{A}$, and suppose that over $\mathcal{M}(E)$ there is a universal family of parabolic stable bundles $\underline{\mathcal{E}}$. Then there is an isomorphism between the associated* $\mathrm{SO}(3)$*-bundle* $\mathrm{ad}\underline{\mathcal{E}}$ *to $\underline{\mathcal{E}}$ and the bundle $\underline{P}$.*

PROOF. This is essentially a restatement of the results of [**20**], specialized to the weight 1/4. The first statement follows from the bijection in Theorem 8.21 and the statement about universal bundles follows from the discussion on deformations in Proposiiton 8.23. □

We will prove Theorem 2.7.4 by computing a singular Donaldson invariant, in turn done transferring the question to a problem of computing a moduli space of parabolic stable bundles on an algebraic curve. From the proof of Lemma 2.7.2, using the notation there, the 4-manifold triple $(A_0, S_3, \emptyset)$ gives relative invariant $[\rho_0]$, and the triple $(A_0, S_3, \mathrm{pt} \times \Sigma)$ with opposite orientation gives the linear functional sending $[\rho_1]$ to ± 1. Our gluing theory then implies that the coefficient of $[\rho_1]$ of $\mu(\Sigma) \cdot [\rho_0]$ is the same as the integer invariant associated to the 4-manifold triple

$$(X_0, \overline{S}_3, \mathrm{pt} \times \Sigma)$$

where $X_0 = S^2 \times \Sigma \cong \mathbb{C}P^1 \times \mathbb{C}P^1$ and $\overline{S}_3 = S^2 \times \{x_1, x_2, x_3\}$ with the degree 2 class $[\Sigma] \in \mathbb{A}_{0,3}$. Since this is a closed 4-manifold with $b_2^+ = 1$, we use a metric coming from a very long neck in the gluing of A_0 to $\overline{A}_0$. We will want to use an orbifold Kähler metric in order to apply the previous discussion, and so define κ to be such a 2-form in the cohomology class

$$[\kappa] = \mathrm{PD}[S^2 \times \mathrm{pt}] + M\mathrm{PD}[\mathrm{pt} \times \Sigma],$$

where $M \gg 1$ so as to be in the same chamber as metrics with long neck. Admissible orbifold bundles $\check{P}$ on this space are classified by the instanton number k and monopole numbers l_i, where l_i is the monopole number on the component $S_3^{(i)} = S^2 \times \{x_i\}$. Writing $l = l_1 + l_2 + l_3$

and $\underline{l} = (l_1, l_2, l_3)$, the expected dimension of the moduli space is

$$d = 8k + 4l.$$

All such bundles have preferred extensions to non-orbifold bundles on X, and arise as the adjoint SO(3) bundle of a smooth U(2)-bundle E with $c_1(E) \equiv \mathrm{PD}(\mathrm{pt} \times \Sigma) \mod 2$. Each connection in our moduli space determines a line subbundle $L_i \subset E|_{S_3^{(i)}}$ via its asymptotic holonomy around linking circles, and the monopole number is given by

$$l_i = -\tfrac{1}{2}\left[c_1(L_i) - c_1(L_i^{\perp})\right].$$

Wwrite $\mathcal{M}_{k,\underline{l}}$ for the moduli space $\mathcal{M}_\kappa(\check{P})$ with $\check{P}$ the orbifold bundle with invariants k and $\underline{l}$ and Stiefel-Whitney class $\mathrm{PD}(\mathrm{pt} \times \Sigma)$. The number we wish to compute is the sum of the evaluations of $\mu^{\mathrm{orb}}(\Sigma)$ on all the 2-dimensional moduli spaces $\mathcal{M}_{k,\underline{l}}$, i.e. for which $8k + 4l = 2$. By Theorem 2.7.7, this number is the same as the evaluation of the analogue of $\mu^{\mathrm{orb}}(\Sigma)$ on the collection of 2-dimensional moduli spaces of stable parabolic bundles on $\mathbb{C}P^1 \times \mathbb{C}P^1$ for the metric κ and singular along the 3-component curve S_3.

We will need the following classification of rank 2 holomorphic bundles on a ruled surface. Let $\pi : X \to C$ be an algebraic $\mathbb{P}^1$ bundle over a smooth curve C, and fix a section C_0 and fiber F_0.

THEOREM 2.7.8. *(Brînzănescu, [**7**]) Let $\mathcal{E} \to X$ be a rank 2 holomorphic bundle. Then there are integers a, b, r, s, a codimension 2 subscheme $Y \subset X$, and line bundles $\mathcal{L}_1, \mathcal{L}_2 \in$* $\mathrm{Pic}^0(C)$ *such there is an exact sequence*

$$0 \to \mathcal{O}_X(aC_0 + rF_0) \otimes \mathcal{L}_1 \to \mathcal{E} \to \mathcal{O}_X(bC_0 + sF_0) \otimes \mathcal{L}_2 \otimes I_Y \to 0$$

where I_Y is the ideal sheaf for the subscheme Y, and $a \geq b$.

We now proceed to prove Theorem 2.7.4

PROOF. We think of $X_0 = S^2 \times \Sigma$ as fibering over the S^2 factor with Σ as fiber, and write $C = S^2$ and $X_0 = C \times \Sigma$. The theorem implies that for any holomorphic bundle $\mathcal{E}$, there is an exact sequence

$$0 \to \mathcal{O}_C(a) \boxtimes \mathcal{O}_\Sigma(r) \to \mathcal{E} \to \mathcal{O}_C(b) \boxtimes \mathcal{O}_\Sigma(s) \otimes I_Y \to 0$$

for some locally complete intersection Y and integers a, b, r, s with $a \geq b$. This implies that

$$c_1(\mathcal{E}) = (a+b)\mathrm{PD}[\mathrm{pt} \times \Sigma] + (r+s)\mathrm{PD}[C \times \mathrm{pt}] \tag{2.7.3}$$

$$c_2(\mathcal{E}) = \tfrac{1}{2}(b-a)(s-r) + |Y| \tag{2.7.4}$$

where $|Y|$ counts points in the subscheme with multiplicity. Thus, we must have $r + s = 0$, $a + b = 1$, and the instanton number k is given by $\frac{1}{2}(b-a)(s-r) + |Y|$. We therefore write $\mathcal{E}$ as an extension

$$0 \to \mathcal{O}_{X_0}(1+a, -r) \to \mathcal{E} \to \mathcal{O}_{X_0}(-a, r) \otimes I_Y \to 0 \tag{2.7.5}$$

where $\mathcal{O}_{X_0}(x, y)$ is the line bundle $\mathcal{O}_C(x) \boxtimes \mathcal{O}_\Sigma(y)$, and $a \geq 0$. The value of a determines the generic splitting type of the bundle $\mathcal{E}$ upon restriction to sections $C \times \mathrm{pt}$: we have $\mathcal{E}|_{\pi^{-1}(x)} \cong \mathcal{O}_C(1+a+t) \oplus \mathcal{O}_C(-a-t)$ and $t \geq 0$ is generically 0 and jumps only if $C \times \{x_i\} \cap Y$ is non-empty.

We now consider the parabolic structure. This amounts to choosing, for each $i = 1, 2, 3$, a line bundle $\mathcal{F}_i \subset \mathcal{E}_{C \times \{x_i\}}$. For a fixed vector $\underline{l} = (l_1, l_2, l_3)$, the degree of $\mathcal{F}_i$ is determined; we have $c_1(\mathcal{F}_i^\perp) = c_1(\mathcal{E}|_{c \times \{x_i\}}) - c_1(\mathcal{F}_i)$ and so $l_i = \frac{1}{2} - c_1(\mathcal{F}_i)$. Because we know how $\mathcal{E}$ splits upon restriction to C slices, we see that $c_1(\mathcal{F}_i)$ must be $1 + a + t$ or $-a - t$.

Finally, we introduce the stability condition. By (2.7.2) and our choice of κ, we have:

$$\mu^{\mathrm{par}}(\mathcal{E}) = \tfrac{1}{2} + \tfrac{3}{2}M.$$

Any line bundle $\mathcal{L}$ on X_0 is of the form $\mathcal{O}_{X_0}(x, y)$. Suppose that $\mathcal{L} \to \mathcal{E}$ is a nonzero map, and define e_i to be 0 if the image of $\mathcal{L} \in \mathcal{E}|_{C \times \{x_i\}}$ is 0, and 1 otherwise. Letting $e = e_1 + e_2 + e_3$, by (2.7.2) the slope is given by:

$$\mu^{\mathrm{par}}(\mathcal{L}) = x + yM + (\tfrac{3}{4} + \tfrac{e}{2})M.$$

Stability of $\mathcal{E}$ then requires that for any map $\mathcal{L} \to \mathcal{E}$, we must have

$$x + yM + (\tfrac{3}{4} + \tfrac{e}{2})M < \tfrac{1}{2} + \tfrac{3}{2}M$$

For $M \gg 1$, this inequality is equivalent to:

$$c_1(\mathcal{L}|_{\mathrm{pt}\times\Sigma}) + \tfrac{e}{2} < \tfrac{3}{4}. \tag{2.7.6}$$

Claim. *We must have $r = 0$ or $r = 1$ in (2.7.5).*

Suppose first that $r \leq -1$. Then the map $\mathcal{O}_{X_0}(1 + a, -r) \to \mathcal{E}$ in the splitting of $\mathcal{E}$ is destabilizing, since even if $e = 0$, the line bundle $\mathcal{O}_{X_0}(1 + a, -r)$ does not satisfy (2.7.6). Hence, r must be greater than or equal to 0. To show that $r < 2$, we will argue that there are no bundles having $r \geq 2$ lying in 2-dimensional moduli spaces, that is with $8k + 4l = 2$. If $r \geq 2$, then by (2.7.4), $8k = 8(2a + 1)r + 8|Y| \geq 32a + 16 + 8|Y|$. Now, the minimal value for $4l$ is achieved when $c_1(\mathcal{F}_i)$ is maximal for each i. If the splitting type of $\mathcal{E}$ on $C \times \{x_i\}$ is

$$\mathcal{O}_C(1 + a + t_i) \oplus \mathcal{O}_C(-a - t_i) \tag{2.7.7}$$

(where $t_i \geq 0$), then the lowest l_i can be is $-\frac{1}{2} - a - t_i$. The value of t_i is zero if $C \times \{x_i\} \cap Y$ is empty, and the maximum value of $t_1 + t_2 + t_3$ is $|Y|$, by elementary algebraic geometry. Hence:

$$4l \geq -6 - 12a - 4|Y|. \tag{2.7.8}$$

Thus, $8k + 4l \geq 10 + 20a + 4|Y|$, which is always at least 10. We see that no moduli space of dimension 2 contains a bundle with $r \geq 0$, proving the claim. Note that this argument obviously fails for $r = 1$.

We therefore have two cases: $r = 0$, and $r = 1$. The inequality (2.7.8) gives a general lower bound on $4l$. For convenience, we establish an analogous upper bound. The largest possible value for l is realized when $c_1(\mathcal{F}_i)$ is minimal for each i, which the value $-a - t_i$ and this gives $l_1 = \frac{1}{2} + a + t_i$. We therefore have the inequality:

$$4l \le 6 + 12a + 4\,|Y|\,. \tag{2.7.9}$$

Case $r = 0$: Suppose we have an extension $\mathcal{O}_{X_0}(1+a,0) \to \mathcal{E} \to \mathcal{O}_{X_0}(-a,0) \otimes I_Y$, so that $8k = 8\,|Y|$. In this case, in order for the map of the line bundle $\mathcal{O}_{X_0}(1+a,0)$ to not be destabilizing, (2.7.6) implies that we need $e \le 1$. The restriction of $\mathcal{E}$ to $C \times \{x_i\}$ splits as (2.7.7), and the map $\mathcal{O}_{X_0}(1+a,0) \to \mathcal{E}$ restricts to

$$\iota_C : \mathcal{O}_C(1+a) \to \mathcal{O}_C(1+a+t_i) \oplus \mathcal{O}_C(-a-t_i).$$

By degree considerations, we have $\iota_C = (f,0)$ where f is a nonzero map $\mathcal{O}_C(1+a) \to \mathcal{O}_C(1+a+t_i)$ (which must have t_i zeroes counted with multiplicity). Thus, to ensure $e \le 1$, we must have that $\mathcal{F}_i = \mathcal{O}_C(-a-t_i)$ for at least two values of i, which we assume without loss of generality are $i = 1,2$ and possibly $i = 3$. This means that

$$\begin{aligned} 4l = 6 - 4\sum_i c_1(\mathcal{F}_i) &\ge 6 + 8a + 4t_1 + 4t_2 - 4 - 4a - 4t_3 \\ &= 2 + 4a - 4\,|Y| \end{aligned}$$

and so $8k + 4l \ge 2 + 4a + 4\,|Y|$. We conclude that $a = 0$ and $|Y| = 0$.

The set of isomorphism classes of non-split extensions $\mathcal{O}_{X_0}(1,0) \to \mathcal{E} \to \mathcal{O}_{X_0}(0,0)$ is, by well know techniques in bundle theory, in bijection with the projectivization of the first cohomology $H^1(\mathrm{Hom}(\mathcal{O}_{X_0}(0,0), \mathcal{O}_{X_0}(1,0)))$. This is just $H^1(\mathcal{O}_{X_0}(1,0))$, which vanishes by the Künneth theorem. We see that $\mathcal{E} \cong \mathcal{O}_{X_0}(1,0) \oplus \mathcal{O}_{X_0}(0,0)$. In order to have $8k + 4l = 2$, we must have $l = \frac{1}{2}$ and so $\mathcal{F}_i$ must be a degree 0 for two values of i and the third $\mathcal{F}_i$ should be degree 1. Let us assume $\mathcal{F}_i$ are degree 0 for $i = 1,2$. Then $\mathcal{F}_3$ is rigid; there is a unique degree 1 line bundle in $\mathcal{E}|_{C \times \{x_3\}} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(0)$. The bundles $\mathcal{F}_i$ for $i = 1,2$ are each determined by a section (f_i, c_i) of $\mathcal{O}_C(1) \oplus \mathcal{O}_C(0)$ where the constant c_i is not zero and not both of f_1 and f_2 are zero. We must now consider automorphisms τ of $\mathcal{E}$. It is easy to see that any automorphism is constant in Σ directions, and upon restriction to Σ must take the form $\tau = \left(\begin{smallmatrix} c & f \\ 0 & d \end{smallmatrix}\right)$ for nonzero constants c, d and section f of $\mathcal{O}_C(1)$. Hence, we can always use an automorphism to bring (c_1, f_1) to $(0, 1)$, and then (f_2, c_2) into $(f_2, 1)$. There is $H^0(\mathcal{O}_C(1)) \cong \mathbb{C}^2$ worth of f_2, and since the line bundle $\mathcal{F}_i$ is insensitive to scaling these sections, we see that we get a $\mathbb{C}P^1$'s worth of isomorphism types of parabolic bundles. It remains to check that these are all stable, but a quick glance at (2.7.6) shows that the only case that needs to be consider is the case of map from $\mathcal{O}_{X_0}(1, 0)$ into $\mathcal{E}$, and the discussion here works to eliminate the possibility of a destabilizing bundle in this case. Since we made the choice $l_3 = -\frac{1}{2}$, we actually get 3 distinct copies of $\mathbb{C}P^1$.

Case $r = 1$: If $r = 1$, then $8k = 16a + 8 + 8|Y|$ and (2.7.8) implies $8k + 4l \geq 2 + 4a + 4|Y|$, and so $a = 0$ and $|Y| = 0$. Hence, $\mathcal{E}$ sits in an extension

$$0 \to \mathcal{O}_{X_0}(1, -1) \to \mathcal{E} \to \mathcal{O}_{X_0}(0, 1) \to 0 \tag{2.7.10}$$

and $8k = 8$, so $l = -\frac{3}{2}$ and $l_i = -\frac{1}{2}$ for each i. Again we have $\mathcal{E}|_{C\times\{x_3\}} \cong \mathcal{O}_\Sigma(1) \oplus \mathcal{O}_\Sigma(0)$ and we have no choice but to let $\mathcal{F}_i$ be the $\mathcal{O}_\Sigma(1)$ factor for each i. Now, suppose that the extension is split. Then $\mathcal{O}_{X_0}(0, 1)$ maps in and this violates (2.7.6). Hence, we can classify the possible extensions via a cohomology group again, which in this case becomes $H^1(\mathcal{O}_\Sigma(1, -2))$. By the Künneth theorem and Riemann-Roch, this is isomorphic to $H^0(\mathcal{O}_C(1)) \otimes H^1(\mathcal{O}_\Sigma(-2)) \cong \mathbb{C}^2 \otimes \mathbb{C}$, whose projectivation is $\mathbb{C}P^1$. For a non-split extension arising this way, it is clear that there will be no non-scalar automorphisms, and so by uniqueness of the $\mathcal{F}_i$'s we get a $\mathbb{C}P^1$'s worth of isomorphism classes of parabolic bundles. The map of $\mathcal{O}_{X_0}(1, -1)$ into $\mathcal{E}$ certainly obeys (2.7.6) even though $e = 3$. There will be no other maps of $\mathcal{O}_{X_0}(a, b)$ into $\mathcal{E}$ with $b > 1$, so these bundles are all stable.

We must now calculate how the operator $\mu(\Sigma)$ acts, and for this we must study universal bundles over $\mathcal{N} \times X_0$ for our 2-dimensional moduli space $\mathcal{N}$, which consists of 4 copies of $\mathbb{C}P^1$. We will do this for the copy arising from the case $r = 1$, and appeal to a "flip" symmetry to handle the other 3. Let $\pi : (\mathbb{C}^2 \setminus \underline{0}) \times X_0 \to X_0$ denote the projection. Topologically, all the bundles $\mathcal{E}$ from the extension (2.7.10) are isomorphic to the direct sum of $\mathcal{O}_{X_0}(1, -1)$

and $\mathcal{O}_{X_0}(0,1)$. The extension is specified by an element in a cohomology group, which we may take to be a Hodge cohomology group $\mathcal{H} = H^{0,1}(\mathcal{O}_{X_0}(1,-2)) \cong \mathbb{C}^2$. We can choose a splitting $\mathcal{H} \hookrightarrow \Omega^{0,1}$ of the de Rham quotient map, and this allows us to write down a "tautological" extension

$$0 \to \pi^*\mathcal{O}_{X_0}(1,-1) \to \widetilde{\mathbb{E}} \to \pi^*\mathcal{O}_{X_0}(0,1) \to 0$$

which when restricted to $\{\xi\} \times X_0$ for $\xi \in \mathcal{H} \setminus \underline{0}$ is the non-split extension on X_0 corresponding to ξ in the isomorphism $\mathcal{H} \cong H^1(\mathcal{O}_\Sigma(1,-2))$. Modding out by scalar isomorphisms corresponds to the simultaneous action of $\mathbb{C}^\times$ on $(\mathbb{C}^2 \setminus \underline{0})$ and scalar multiplication on one of the two factors in the bundle direct sum, which we may as well take to be the first. The quotient of the analogous action on the trivial bundle $\underline{\mathbb{C}} \to \mathbb{C}^2$ by $\mathbb{C}^\times$ is the tautological bundle $\tau_{\mathbb{C}P^1} \to \mathbb{C}P^1$, and so the quotient of $\widetilde{\mathbb{E}}$ is a bundle $\mathbb{E} \to \mathbb{C}P^1 \times X_0$ given by:

$$\mathbb{E} \cong [\tau_{\mathbb{C}P^1} \boxtimes \mathcal{O}_{X_0}(1,-1)] \oplus \mathcal{O}_{X_0}(0,1)$$

Now, there are also line bundles $\mathbb{F}_i \subset \mathbb{E}|_{\mathbb{C}P^1 \times C \times \{x_i\}}$ corresponding to the parabolic structure. For the case $r = 1$, these are clearly each just given by

$$\mathbb{F}_i = \tau_{\mathbb{C}P^1} \boxtimes \mathcal{O}_C(1)$$

In order to determine $\mu^{\mathrm{orb}}([\Sigma])$, we must use the adjoint bundle $\mathrm{Ad}\mathbb{E}$, which has first Pontryagin class equal to $-4c_2(\mathbb{E})$. The point class for the component $C \times \{x_i\}$ of the singular loci is given by $c_1(\mathbb{E}|_{\mathbb{C}P^1 \times \mathrm{pt}}) - 2c_1(\mathbb{F}_i|_{\mathbb{C}P^1 \times \mathrm{pt}})$, and so $\mu^{\mathrm{orb}}([\Sigma])$ is equal to

$$-4c_2(\mathbb{E})/[\Sigma] - 3c_1(\mathbb{E}|_{\mathbb{C}P^1 \times \mathrm{pt}}) + 2\sum_i c_1(\mathbb{F}_i|_{\mathbb{C}P^1 \times \mathrm{pt}})$$

Since $c_1(\tau_{\mathbb{C}P^1}) = -h_{\mathbb{C}P^1}$, which is minus the hyperplane class, we have:

$$c_2(\mathbb{E}) = (-h_{\mathbb{C}P^1} + h_C - h_\Sigma) \cdot h_\Sigma = -h_{\mathbb{C}P^1} h_\Sigma + h_C h_\Sigma$$

and so $\langle c_2(\mathbb{E})/[\Sigma], \mathbb{C}P^1 \rangle = -1$. We also have

$$c_1(\mathbb{F}_i|_{\mathbb{C}P^1 \times \mathrm{pt}}) = h_{CP^1} + h_C,$$

which evaluates to -1 on $\mathbb{C}P^1$. Lastly $c_1(\mathbb{E}|_{\mathbb{C}P^1 \times \mathrm{pt}}) = h_{\mathbb{C}P^1}$, which again gives -1 on $\mathbb{C}P^1$. Hence:

$$\left\langle \mu^{\mathrm{orb}}([\Sigma]), \mathbb{C}P^1 \right\rangle = 4 + 3 - 6 = 1.$$

This is the contribution coming from the component of the moduli space corresponding to $\underline{l} = (-1/2, -1/2, -1/2)$ and $k = 1$. We now utilize the flip construction. Since we are in the case that the Stiefel-Whitney 2-cycle intersects the singular locus $\overline{S}_3$ transversely and has no boundary components, we can define monopole and instanton numbers $\underline{l}$ and k for each component of this moduli space. For $J \subset \{1, 2, 3\}$, since the components $S_3^{(j)}$ of the singular locus have zero self-intersection, the partial flip τ_J sends the component corresponding to $\underline{l}$ and k the component with numbers

$$k' = k + \sum_{j \in J} l_j$$

and $l'_j = -l_j$ for $j \in J$. In our case, flipping on a single component $C \times \{x_i\}$ changes the Stiefel-Whitney class, but flipping twice preserves it. Thus, for each pair of indices, say $\{1, 2\} \in\subset \{1, 2, 3\}$, the flip $\tau_{\{1,2\}}$ sends the component with $\underline{l} = (-1/2, -1/2, -1/2)$ and $k = 1$ to the one with $l_1 = l_2 = 1/2$, $l_3 = -1/2$, and $k = 0$. By Lemma 2.3.3, the class $\mu^{\mathrm{orb}}(\Sigma)$ is preserved under the flip, and so we need to determine whether it preserves orientation. For this, we use the formula (A1.5) and the preceding discussion from [**20**], which in our zero self-intersection case implies that whether the flip is orientation preserving or not depends only on the genus of the singular set. Everything is oriented by the presence of the obvious complex structure on $X_0 = C \times \Sigma$. The formula implies that a single flip on $C \times \{x_1\}$ is orientation reversing, though this moves us into a moduli space for the triple with 2-cycle $\mathrm{pt} \times \Sigma + C \times \{x_1\}$. If we flip again on a new index, the result is orientation preserving, but now our 2-cycle is really $\mathrm{pt} \times \Sigma + 2C \times \{x_1\}$. Even though this is equivlant mod 2 to the original 2-cycle, the equivalence requires us to compare orientations between isomorphic moduli spaces, which are oriented by choices of U(2) lifts. From [**20**] we see that this identification is actually orientation preserving, because the difference has square 0. Another way to see that orientation is preserved is to simply note that we can take two different paths from the $k = 1$ component and a given $k = 0$ component: we can flip on a single pair of indices, or the other two pairs in sequence. Whether the flip preserves orientation should be the same for all the flips by symmetry, and so either they are all reversing or all preserving. We immediately see by the parity that they are all preserving. As a result, the contributions from the other 3 components of the moduli space are also 1, so we obtain a total of 4. □

This result is the core computation of the entire paper. It will eventually allow us to compute the deformation of the cup product on $H^*(\mathcal{R}_{0,n})$ giving ring structure on the Floer homology. For convenience, we redefine the generator $[\rho_1]$ of $\mathbb{V}_{0,3}$ to be the image of $[\rho_0]$ under $-\frac{1}{4}\mu^{\text{orb}}(\Sigma)$. Thus, the matrix for $\mu^{\text{orb}}(\Sigma)$ in the basis $\{[\rho_0], [\rho_1]\}$ is $\left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right)$, and $[\rho_1] = \frac{1}{4}I^*(A_0, S_3, \emptyset; [\Sigma])$.

2.7.3. The Vector Space $\mathbb{U}_{0,2}$. If we allow the 1-cycle ω to have ends on the link K_n in Y_g, it is no longer necessary for n to be odd. In the case $n = 2$, we have the vector space $\mathbb{U}_{0,2} = I_*(Y_0, K_2, u)$, where u is an arc between x_1 and x_2. It is straightforward to check that the space of flat connections is a single point, and that $\mathbb{U}_{0,2} \cong \mathbb{C}$. Denoting the generating flat connection by $\rho_{0,2}$, the operator $\mu^{\text{orb}}(\Sigma)$, which has degree -2, must vanish by degree considerationgs. However, the operator $\mu(\text{pt})$ does not, as we shall see. This is a general phenomenon which occurs for any 3-manifold triple with a non-integral embedded 2-torus missing the singular locus. To show this, we will leverage the known computation of $\mu(\text{pt})$ on the Floer homology of T_3, specifically for the triple $(Y_1, \emptyset, \beta)$, where β is a curve in $\Sigma_1 = T^2$. Recall that $\mathbb{V}_{1,0} = I_*(Y_1, \emptyset, \beta)$ is two dimensional, generated by two flat connections τ_i, $i = 0, 1$, with SU(2) holonomy equal to $(-1)^i$ around the S^1 factor, off by grading 4 mod 8 from each other (the absense of K means the Floer homology has a relative $\mathbb{Z}/8$ grading).

The operator $\mu(\mathrm{pt})$ interchanges these generators and multiplies by 2. In particular, $\mu(\mathrm{pt})$ has eigenvalues ± 8.

LEMMA 2.7.9. *Suppose that $(Y, K, \omega) \in$ WINK is connected and that there is a 2-torus $T \subset Y \setminus K$ on which ω is odd. Then $\mu(\mathrm{pt})$ has only eigenvalues ± 8 on $I_*(Y, K, \omega)$.*

PROOF. Let N be the rank of $I_*(Y, K, \omega)$, so that the span of the ± 8 generalized eigenspace of $\mu(\mathrm{pt})$ in $V = I_*(Y, K, \omega)$ is given by the kernel of $\mu(\pi_{\pm 8})$ where $\pi_{\pm 8} = (\mathrm{pt} - 8)^N (\mathrm{pt} + 8)^N$ for a point $Y \setminus K$. We claim that $\mu(\pi_{\pm 8})$ is zero on V. In the presence of T, there is a natural cobordism (W_1, S, Ω) from (Y, K, ω) to a disjoint union of (Y, K, ω) and $(Y_1, \emptyset, \beta)$. This can be viewed as performing an excision operation along two paralell copies of T. Another viewpoint is to take the product cobordism $[0, 1] \times (Y, K\omega)$, and to remove a small tubular neighborhood $(A_1, \emptyset, D)$ $\{1/2\} \times T$ from this 4-manifold, where D is the disk intersection of the neighborhood of T with $[0, 1] \times \omega$. By our gluing theorem, the operator $\mu(\pi_{\pm 8})$ is equal to the composite of the map induced by (W_1, S, Ω) followed by the linear functional induced by $(A_1, \emptyset, D)$ on $\mathbb{V}_{1,0}$ along with the polynomial $\pi_{\pm 8}$, tensored with the identity on V. By what we know about $\mu(\mathrm{pt})$ on $\mathbb{V}_{1,0}$, this linear functional vanishes. We conclude that $\mu(\pi_{\pm 8}) = 0$ on V. □

COROLLARY 2.7.10. *The operator $\mu(\mathrm{pt})$ on $\mathbb{U}_{0,2}$ is multiplication by either 2 or -2.*

PROOF. If d_1 denotes a circle separating x_1 and x_2, then $S^1 \times d_1$ is odd for u in (Y_0, K_2, u). The conclusion follows because $\mathbb{U}_{0,2}$ is one-dimensional. □

We define the number Π in what follows by $-4\Pi = \pm 8$, using whatever sign on the right as appears in the Corollary.

2.7.4. The Vector Space $\mathbb{U}_{0,3}$. Essentially the same argument as for the vector space $\mathbb{V}_{0,3}$ shows that there are two generators for $\mathbb{U}_{0,3}$ and each are off in grading by 2. Unlike for $\mathbb{V}_{0,3}$, these generators $[\sigma_0]$ and $[\sigma_1]$ are interchanged by the orientation reversing diffeomorphism of Y_0 coming from reflecting the S^1 factor. There is a map on $\mathbb{U}_{0,3}$ analogous to E on $\mathbb{V}_{g,n}$, which has degree 2 and square ± 1, and which we shall also call E. There is

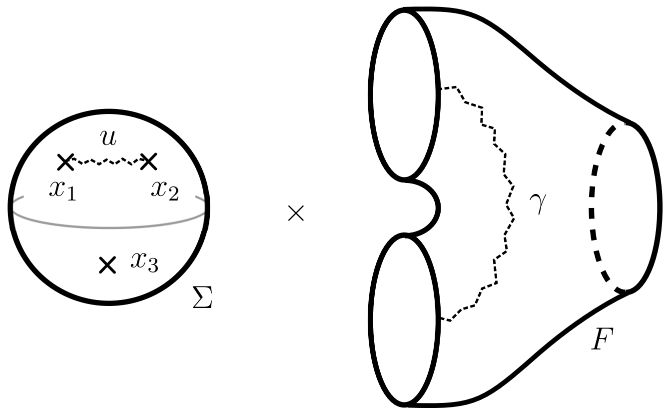

FIGURE 2.2. The cobordism defining Δ_3^u.

also a natural map from $(\mathbb{U}_{0,3})^{\otimes 2}$ to $\mathbb{V}_{0,3}$, obtained from the cobordism which is a product of $(\Sigma, 3 \text{ pts}, u)$ with (F, F, γ), where F is a pair of pants viewed as having two incoming ends and one outgoing end, and γ is an arc going between the two incoming ends. Denote this product cobordism by (W_3^u, F_3, Γ). There is a natural product almost complex structure on W_3^u and thus we get a well-defined sign for the map it induces, Δ_3^u.

LEMMA 2.7.11. *The map* $\Delta_3^u : (\mathbb{U}_{0,3})^{\otimes 2} \to \mathbb{V}_{0,3}$ *sends* $[\sigma_i] \otimes [\sigma_j]$ *to* $\pm[\rho_0]$ *if* $i \neq j$ *for a fixed sign, and* $\pm[\rho_1]$ *if* $i = j$ *for a possibly different fixed sign.*

PROOF. We can view the triple (W_3^u, F_3, Γ) as arising from an elbow macaroni cobordism $[0, 1] \times (Y_0, K_3, u)$ by deleting a tubular neighborhood of a copy of Σ_0 in the interior. The restriction of a product flat connection on $[0, 1] \times Y_0$ gives a flat connection on W_3^u, whose restriction to the outgoing end has trivial holonomy around the S^1 factor and thus is ρ_0. The two incoming flat connections are interchanged by the orientation-reversing diffeomorphism on Y_0 coming from flipping the S^1 factor, and so are σ_0 and σ_1 in some order. With a fixed incoming pair, it is not hard to see that this flat connection on W_3^u is unique and gives the actual ASD moduli space. The automorphism of F which interchanges the incoming ends via rotation by $180°$ gives an orientation-preserving automorphism of the space of connections on W_3^u and thus $[\sigma_i] \otimes [\sigma_{1-i}]$ maps to $\pm[\rho_0]$, each for the same sign.

On both $\mathbb{V}_{0,3}$ and $\mathbb{U}_{0,3}$ spaces, it is easy to see that the E maps interchange the two generators, possibly with a sign defect. In addition, the two versions of E are clearly intertwined (perhaps with sign) by Δ_3^u. This shows that $[\sigma_i] \otimes \epsilon[\sigma_i]$ maps to $\pm[\rho_1]$ for each i, and the sign is the same for each on account of the 180° rotation as before. □

LEMMA 2.7.12. *Just as for* $\mathbb{V}_{0,3}$*, the operator* $\mu^{\text{orb}}(\Sigma)$ *on* $\mathbb{U}_{0,3}$ *exchanges the generators (and multiplies by* -4*). The operator* $\mu(\text{pt})$ *is equal to* -4Π *(the eigenvalue* ± 8 *on* $\mathbb{U}_{0,2}$*) times the identity on* $\mathbb{U}_{0,3}$ *and* $\mathbb{V}_{0,3}$*.*

PROOF. To prove the first assertion, we use utilize the cobordism of Lemma 2.7.11 and corresponding map Δ_3^u. Let $\mu^{\text{orb}}(\Sigma) \cdot [\sigma_0] = C[\sigma_1]$. Since $\mu^{\text{orb}}(\Sigma) \times 1$ is intertwined with $\mu(\text{pt})$ by Δ_3^u, we have:

$$\begin{aligned} -4[\rho_1] = \mu^{\text{orb}}(\Sigma) \cdot [\rho_0] = \mu^{\text{orb}}(\Sigma) \cdot \pm\Delta_3^u([\sigma_0] \otimes [\sigma_1]) &= \pm\, \Delta_3^u(\mu^{\text{orb}}(\Sigma) \cdot [\sigma_0]) \otimes [\sigma_1]) \\ &= \pm\, \Delta_3(C[\sigma_1]) \otimes [\sigma_1]) \\ &= C[\rho_1]. \end{aligned}$$

Hence, $C = -4$ (the plus or minus sign throughout is the same and matches the sign appearing in Lemma 2.7.11). The same argument shows that $\mu^{\text{orb}}(\Sigma) \cdot [\sigma_1] = -4[\sigma_0]$. Now, since $\mu^{\text{orb}}([\Sigma])$, which is nonzero on both vector spaces, commutes with $\mu(\text{pt})$, it is clear in each case that the matrix of $\mu(\text{pt})$, which is diagonal by degree considerations, has equal diagonal entries. As in the case of $n = 2$, if d_1 is a small loop around x_1, then the torus $S^1 \times d_1$ is odd for u in (Y_0, K_3, u). Thus, by Lemma 2.7.9 $\mu(\text{pt})$ must be either $-4\Pi\text{Id}$ or $4\Pi\text{Id}$ on $\mathbb{U}_{0,3}$. As for $\mathbb{V}_{0,3}$, we again utilize the cobordism of Lemma 2.7.11. Since $\mu(\text{pt}) \times 1$ is intertwined with $\mu(\text{pt})$ by Δ_3^u, we have:

$$\mu(\text{pt}) \cdot \rho_0 = \Delta_3^u(\mu(\text{pt}) \cdot [\sigma_0] \otimes [\sigma_1]) = \pm 4\Pi\Delta_3^u([\sigma_0] \otimes [\sigma_1]) = \pm 4\Pi\rho_0.$$

Hence, the diagonal entries are both either $\pm 4\Pi$, with same value occuring for $\mathbb{U}_{0,3}$ and $\mathbb{V}_{0,3}$.

To nail down the sign, we study a cobordism from (Y_0, K_3, u) to a disjoint union of (Y_0, K_3, u) and (Y_0, K_2, u). This cobordism is given by excision along the tori $S^1 \times d_1$ and $S^1 \times d_2$ in Y_0. Let U be the 3-manifold with boundary given by the complement of small 2-balls B_1, B_2 inside a third 3-ball B. Let γ_1, γ_2 be 2 simple line segments going from ∂B

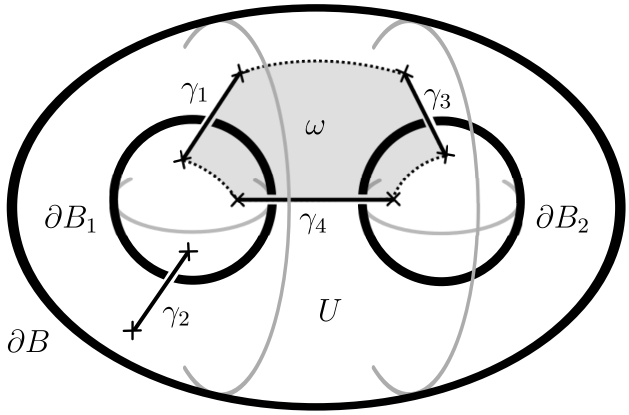

FIGURE 2.3. The cobordism from (Y_0, K_3, u) to $(Y_0, K_3, u) \sqcup (Y_0, K_2, u)$.

to ∂B_1, let γ_3 be a line segment from ∂B_1 to ∂B_2, and γ_4 be an arc going from ∂B_2 to ∂B. Finally, let ω be a disk whose boundary is the union of an arc on the boundary of each 3-ball, and the segments γ_1, γ_3, and γ_4 (see Figure 2.7.4). Then the excision cobordism is diffeomorphic to the product triple $(S^1, S^1, \mathrm{pt}) \times (U, \cup_i \gamma_i, \omega)$. By studying flat connections on this 4-manifold triple, it is easy to see that the map is nonzero. Every incoming vector is an eigenvector for $\mu(\mathrm{pt})$ with value $\pm 4\Pi$ and every outgoing vector is an eigenvector for $1 \otimes \mu(\mathrm{pt})$ on $\mathbb{U}_{0,3} \otimes \mathbb{U}_{0,2}$ with value -4Π. This is enough to conclude that the eigenvalue on $\mathbb{U}_{0,3}$ is equal to -4Π. □

It remains to discuss the action of the degree 2 operators $\mu([x_i])$ (arising from the ith singular S^1 in K_3). It turns out that they each vanish, not only on $\mathbb{U}_{0,3}$ but also on $\mathbb{V}_{0,3}$. Of course, this operator is not defined unless we choose a local extension

LEMMA 2.7.13. *The operator $\mu([x_i])_{\boldsymbol{\kappa}}$ (for either choice of $\boldsymbol{\kappa}$) is identically zero on $\mathbb{U}_{0,3}$ and $\mathbb{V}_{0,3}$.*

PROOF. The key idea is that the cohomology class $\mu([x_i])_{\boldsymbol{\kappa}}$ in the space of connections for (Y_0, K_3, u) is zero modulo torsion for $i = 1, 2$. To see this, we imagine moving the point $\{1\} \times \{x_i\}$ around the loop $S^1 \times \{x_1\}$. The presence of the Stiefel-Whitney 2-cycle u, which terminates on the circle $S^1 \times \{x_i\}$ for $i = 1, 2$, means that there is no global trivialization of $\check{P}$ on a the complement of $S^1 \times \{x_i\}$ in a small neighborhood of this circle. Following the trivialization around the loop, the choice $\boldsymbol{\kappa}$ is brought into the oppositive trivialization. Thus, we conclude that $\mu([x_i])_{\boldsymbol{\kappa}} = -\mu([x_i])_{\boldsymbol{\kappa}}$, and so the corresponding operator vanishes for $i = 1, 2$.

Now, there is also an operator $\mu([x_i])_{\boldsymbol{\kappa}'}$ on $\mathbb{V}_{0,3} = I_*(Y_0, K_3, \emptyset)$, and for one of the possible choices of $\boldsymbol{\kappa}'$, the operators $\mu([x_i])_{\boldsymbol{\kappa}} \otimes 1$ and $\mu([x_i])_{\boldsymbol{\kappa}'}$ are intertwined by the map Δ_3^u, by the gluing theorem. Since Δ_3^u is surjective, we conclude that $\mu([x_i])_{\boldsymbol{\kappa}'}$ vanishes on $\mathbb{V}_{0,3}$ for $i = 1, 2$. However, by symmetry, it must also be the case that it vanishes for $i = 3$, from which we conclude that $\mu([x_3])_{\boldsymbol{\kappa}}$ vanishes as well on $\mathbb{U}_{0,3}$. □

In fact, it is now possible for us to use known computations for Donaldson invariants in order to compute the exact sign of Π. We use a result from the paper [**16**].

LEMMA 2.7.14. *The value of* Π *is 2.*

PROOF. □

On $\mathbb{V}_{0,3}$, we now know that $\mu(\mathrm{pt})$ is the map $-8\mathrm{Id}$ and $\mu^{\mathrm{orb}}(\Sigma)$ exchanges the two generators and multiplies by -4. The map E exchanges the two generators $[\rho_i]$, $i = 0, 1$, possibly with signs. However, we do know that E commutes with $\mu^{\mathrm{orb}}(\Sigma)$ exactly, and this implies that $E^2 = 1$ on $\mathbb{V}_{0,3}$, and then by the discussion preceding Corollary 2.6.8 we must have $E^2 = 1$ on $\mathbb{V}_{0,n}$ and can choose $\sqrt{\zeta_{0,n}} = 1$. We then have $\Phi^+(1) = [\rho_0] + [\rho_1]$, and this vector generates $\mathbb{V}_{0,3}$. We have:

$$\Phi^+(\beta) = 2\Phi^+(1)$$

$$\Phi^+(\alpha) = \Phi^+(1)$$

And we have already shown that $\mu([x_k])$ vanishes on $\mathbb{V}_{0,3}$ so that $\Phi^+(\delta_k) = 0$. We conclude:

$$\mathcal{J}_{0,3}^+ = (\alpha - 1, \beta - 2, \delta_1, \delta_2, \delta_3). \tag{2.7.11}$$

We can now prove:

LEMMA 2.7.15. *The relation* $\delta_k^2 = \beta - 2$ *always holds for* $k = 1, \ldots, n$ *in* $\mathbb{V}_{0,n}$.

PROOF. Induction. The claim hold for $n = 3$, so suppose that the Lemma holds for n and there are relations $\delta_k^2 - \beta + 2$ inside $\mathcal{J}_{0,n}^+$. By Proposition 2.6.4 and the discussion following it, since the relation $\delta_k^2 = \beta$ holds in $H^*(\mathcal{R}_{0,n+2}; \mathbb{C})$, we know that there is a mod 2 deformed relation

$$\delta_k^2 + A\alpha + \sum_i D_i \delta_i - \beta + B$$

in $\mathcal{J}_{0,n+2}^+$. By Lemma 2.6.9, it is easy to see that

$$\delta_k^2 + A\alpha - \beta + B$$

will also be a relation. The image of this relation in $\mathbb{A}_{0,n}$ for $k \leq n$ will be itself, and for $k = n+1, n+2$ will be $\delta_n^2 + A\alpha - \beta + B$. By hypothesis $\delta_k^2 - \beta + 2$ is also a relation and so if $A \neq 0$, subtracting will yield a relation $\alpha - C$ for some C. Such a relation cannot hold in $\mathcal{J}_{0,n}^+$ for $n > 3$ because it is not a graded deformation of a relation in $H^*(\mathcal{R}_{0,n}; \mathbb{C})$. From this we can conclude that $A = 0$, and further that $B = 2$, proving the claim. □

2.7.5. The Case $g \neq 0$, $n = 1$. What is remarkable about our computations for $n = 3$ is that it allows us to gain information about the higher genus case, as we shall now explain. There is a way to obtain a higher genus surface Σ' from a given surface Σ by taking 2 nullhomotopic circles c_1 and c_2 bounding disjoint disks, cutting along them and regluing in the opposite way. In a product 3-manifold $S^1 \times \Sigma$, we can cut along $S^1 \times c_i$ and relguing along these 2-tori. This is exactly the kind of operation treated by the excision result Theorem 2.5.1. However, these two 2-tori are not suitable as yet because the Stiefel-Whitney 2-cycle has zero intersection with them. However, if our circles c_i enclose singular points x_i in K_n and there is an arc $u \subset \omega$ connected the circles $S^1 \times \{x_i\}$, then these 2-tori become non-integral surfaces. The result of excision produces an additional copy of $S^1 \times S^2$ with 2 marked points and an arc between them. In the case of a sphere with 3 marked points, this excision operation gives a cobordism map from

$$\mathbb{U}_{0,3} \to \mathbb{U}_{0,2} \otimes \mathbb{U}_{1,1}$$

where $\mathbb{U}_{g,1} := I_*(Y_g, K_1, \mathrm{pt} \times d)$ with d a simple loop in Σ. By Theorem 2.5.1, this map induces an isomorphism $\mathbb{U}_{0,3} \to (\mathbb{U}_{1,1})_{\mathrm{pt},-8}$, since $\mu(\mathrm{pt})$ is $-8\mathrm{Id}$ on $\mathbb{U}_{0,3}$ and $\mathbb{U}_{0,2}$, and the later is one-dimensional. Now, it is straightforward to verify that the representation variety for the triple $(Y_1, K_1, \mathrm{pt} \times d)$ consists of two isolated points, and thus the Floer homology is at most 2-dimensional. By everything we have said up to now, we can conclude that $\mathbb{U}_{1,1}$ is rank 2 with generators off in grading by 2, with $\mu(\mathrm{pt}) = -8\mathrm{Id}$, and $\mu^{\mathrm{orb}}[\Sigma]$ has matrix $\left(\begin{smallmatrix} 0 & -4 \\ -4 & 0 \end{smallmatrix}\right)$. Lastly, this implies that $\mathbb{U}_{0,3}$ and $\mathbb{U}_{1,1}$ are isomorphic via a map intertwining α and β. Now, we recall the vector space $I_*(Y_g, u) \cong \mathbb{C}4(g-1)$. By performing an excision along S^1 times a circle in $\Sigma_g \subset Y_g$ intersecting u in a point and a similar 2-torus in Y_1, we get an excision isomorphism

$$(\mathbb{U}_{g,1})_{\mathrm{pt},-8} \cong I_*(Y_g, u)_{\mathrm{pt},-8} \otimes \mathbb{U}_{1,1}.$$

Since the $+8$-eigenspace for $\mu(\mathrm{pt})$ on $\mathbb{U}_{1,1}$ is zero, we obtain:

$$\mathbb{U}_{g,1} \cong I_*(Y_g, u)_{\mathrm{pt},-8} \otimes \mathbb{U}_{1,1}. \tag{2.7.12}$$

Moreover, this isomorphism intertwines $\mu^{\mathrm{orb}}(\Sigma)$ on $\mathbb{U}_{g,1}$ and the sum $\mu(\Sigma_g) \otimes 1 + 1 \otimes \mu^{\mathrm{orb}}(\Sigma)$ on $I_*(Y_g, u)_{\mathrm{pt},-8} \otimes \mathbb{U}_{1,1}$. From Proposition 2.5.4, we conclude:

Proposition 2.7.16. *The spectrum of the operator $\mu^{\mathrm{orb}}(\Sigma)$ on $\mathbb{U}_{g,1}$ is the set*

$$\{-4(2g-1), -4(2g-3), \ldots, -4, 4, \ldots, 4(2g-3), 4(2g-1)\} \tag{2.7.13}$$

and the generalized eigenspaces for the values $\pm 4(2g-1)$ are 1-dimensional.

2.8. The Vector Space $\mathbb{U}_{0,n}$ and Eigenvectors

We can use our calculations for $\mathbb{U}_{0,3}$ to build up a description of $\mathbb{U}_{0,n}$ for arbitrary $n \geq 2$. We utilize the excision theorem to do this. Let $Y_0^{(1)}$ and $Y_0^{(2)}$ be two copies of $S^1 \times \Sigma_0$. There are 2-tori T_1 and T_2 coming from $S^1 \times d_1$ for a loop d_1 around $x_1 \in \Sigma_0$ in each. Performing an excision along these gives a cobordism $U^{3,3}_{4,2}$ from $Y_0^{(1)} \sqcup Y_0^{(2)}$ to a disjoint union of (Y_0, K_4, u) and (Y_0, K_2, u). We call this kind of excision cobordism a *point transfer* cobordism, as it "transfers" one point on Σ_0 to the other copy of Σ_0 (see Figure 2.8). By excision, this

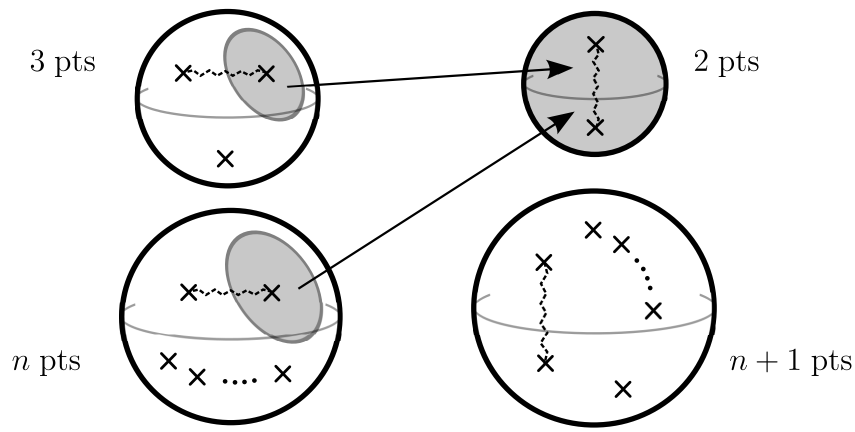

FIGURE 2.4. A schematic for the point transfer cobordism.

produces an isomorphism between the tensor products of the -8-generalized eigenspaces of $\mu(\mathrm{pt})$ on each vector space appearing. We need a quick result on the representation variety for (Y_g, K_n, u).

PROPOSITION 2.8.1. *The representation variety for the triple (Y_g, K_n, u) is a disjoint union of 2^{n-2} copies of the torus T^{2g}. The rank of the Floer homology is less than or equal to 2^{2g+n-2}.*

PROOF. Recall our standard generators $a_1, \ldots, a_{2g}$ for the fundamental group of Σ, and for each i let d_i be a small loop in Σ passing only around the point x_i. Letting S_i denote the holonomy around a_i, T_i around d_i, and Q around the S^1 factor, the representation variety is isomorphic to the quotient

$$\mathcal{R} \cong \left\{ S_1, \ldots, S_{2g}, T_1, \ldots, T_n, Q \in \mathrm{SU}(2) \;\middle|\; \prod_{j=1}^{g} [S_{2j-1}, S_{2j}] \prod_{k=1}^{n} T_k = 1,\ \forall k\ \mathrm{Tr}(T_k) = 0 \right.$$
$$\left. [S_j, Q] = 1,\ [T_k, Q] = 1 \text{ for } k \geq 3,\ [T_1, Q] = [T_2, Q] = -1 \right\} \Big/ \mathrm{SU}(2)$$

The condition $[T_1, Q] = -1$ implies that up to unique conjugation, $T_1 = i$ and $Q = j$. The value of T_2 must then be in the coset $\mathbf{i}S^1_{\mathbf{j}}$, where $S^1_{\mathbf{j}}$ is the circle subgroup through $\mathbf{j}$, and S_j must lie in the subgroup $S^1_{\mathbf{j}}$. In addition, T_k must be $\pm\mathbf{j}$ for $k \geq 3$ in order to have trace 0. The product of $T_3 T_4 \cdots T_n$ will be $\pm\mathbf{j}^{n-2}$, and the commutators $[S_{2j-1}, S_{2j}]$ are all 1. Thus,

T_2 is fixed to be either $\pm\mathbf{j}$ or $\pm\mathbf{k}$ depending on the other T_k's, which offer 2^{n-2} choices. The space of possible S_j's will give a torus $(S^1)^{2g} = T^{2g}$, which verifies the statement about the representation variety. By general considerations this representation is smoothly cut out and the Floer homology will be a subquotient of its homology, which has rank 2^{2g+n-2}. $\square$

We see that the rank of $\mathbb{U}_{0,4}$ is at most $2^2 = 4$. Since every vector in $\mathbb{U}_{0,3}$ and $\mathbb{U}_{0,2}$ is a -8-eigenvector for $\mu(\mathrm{pt})$, we know $\mathbb{U}_{0,4}$ contains a generalized -8-eigenspace for $\mu(\mathrm{pt})$ isomorphic to the tensor product of 2 copies of $\mathbb{U}_{0,3}$. The cobordism $U^{3,3}_{4,2}$ thus induces an isomorphism from $(\mathbb{U}_{0,3})^{\otimes 2}$ to $\mathbb{U}_{0,4}$, obtained by post-composing with the tensor product of Id on $\mathbb{U}_{0,4}$ with the linear functional $\ell_2 : \mathbb{U}_{0,3}$ sending $[\rho_{0,2}]$ to 1. Since it is surjective, every vector in $\mathbb{U}_{0,4}$ is actually an eigenvector with value -8 for $\mu(\mathrm{pt})$.

We can perform this operation more generally. There is an analogous point transfer cobordism $U^{n,3}_{n+1,2}$ inducing a map $\Delta^{n,3}_{n+1} : \mathbb{U}_{0,n} \otimes \mathbb{U}_{0,3} \to \mathbb{U}_{0,n+1}$, again after post-composing with $1 \otimes \ell_2$. By counting dimensions and using the bound from Proposition 2.8.1, we see that $\Delta^{n,3}_{n+1}$ is an isomorphism. Stringing together maps of this kind, we get a cobordism U^3_n from $n-2$ copies of (Y_0, K_3, u) to (Y_0, K_n, u) and $n-3$ copies of (Y_0, K_2, u). Post-composing with $1 \otimes \ell_2^{\otimes(n-3)}$, we get an isomorphism

$$\Delta^3_n : \underbrace{\mathbb{U}_{0,3} \otimes \cdots \otimes \mathbb{U}_{0,3}}_{n-2} \xrightarrow{\cong} \mathbb{U}_{0,n} \tag{2.8.1}$$

In particular, $\dim_{\mathbb{C}}(\mathbb{U}_{0,n}) = 2^{n-2}$, and $\mu(\mathrm{pt})$ is equal to -8 times the identity on $\mathbb{U}_{0,n}$. Conveniently, we can also pin down the operator $\mu^{\mathrm{orb}}(\Sigma)$ on $\mathbb{U}_{0,n}$. The key insight is that the copy of the 2-sphere Σ in the outgoing end (Y_0, K_n, u) is homologous in the cobordism U^3_n to the sum of the 2-spheres in all the other ends (including copies of (Y_0, K_2, u)). Since $\mu^{\mathrm{orb}}(\Sigma)$ vanishes on $\mathbb{U}_{0,2}$, we see that $\mu(\Sigma)$ on $\mathbb{U}_{0,n}$ and the map

$$\sum_{i=1}^{n-2} \underbrace{1 \otimes \cdots \otimes 1 \otimes \mu^{\mathrm{orb}}(\Sigma)}_{i} \otimes 1 \otimes \cdots \otimes 1$$

on $(\mathbb{U}_{0,3})^{\otimes(n-2)}$ are intertwined by Δ^3_n. Denote by $\sigma_\pm$ the ∓ 4 eigenvector of $\mathbb{U}_{0,3}$ obtained by taking a sum or difference of the two generators. We see that $\mu^{\mathrm{orb}}(\Sigma)$ has a basis of

2^{n-2} eigenvectors $\{\Delta_n^3(\sigma_{e_1} \otimes \cdots \otimes \sigma_{e_{n-2}})\}$, where e_i is choice of sign $\pm$. The eigenvalue of $\Delta_n^3(\sigma_{e_1} \otimes \cdots \otimes \sigma_{e_{n-2}})$ is equal to $-4\left(\sum_{i=1}^{n-2} e_i \cdot 1\right)$, and so the spectrum of $\mu(\Sigma)$ is the set

$$\{-4(n-2), -4(n-4), \ldots, 4(n-4), 4(n-2)\}.$$

We will see that for n odd, this information allows us to say the same thing about the operator $\mu^{\mathrm{orb}}(\Sigma)$ on $\mathbb{V}_{0,n}$. As it stands, we can at least show that the spectrum of $\mu(\Sigma)$ *contains* this set.

LEMMA 2.8.2. *For each* $\lambda \in \{-4(n-2), -4(n-4), \ldots, 4(n-4), 4(n-2)\}$*, there is a nonzero simultaneous eigenvector of* $(\mu^{\mathrm{orb}}(\Sigma), \mu(\mathrm{pt}))$ *in the kernel of the operators* $\mu([x_k])_{\boldsymbol{\kappa}_k}$ *with value* $(\lambda, -8)$ *in* $\mathbb{V}_{0,n}$.

PROOF. The pair of pants cobordism map (W_3^u, F_3, Γ) from the discussion preceding Lemma 2.7.11 has an obvious analogue (W_n^u, F_n, Γ) for general $n \geq 2$. It induces a map $\Delta_n^u : \mathbb{U}_{0,n} \otimes \mathbb{U}_{0,n} \to \mathbb{V}_{0,n}$. Post-composing with the product cap $(A_0, S_n, \emptyset)$ on $(Y_0, K_n, \emptyset)$ gives back the elbow macaroni cobordism inducing a bilinear form B_n^u on $\mathbb{U}_{0,n}$. Since the elbow macaroni cobordism is diffeomorphic to the identity cobordism, we conclude that this form is nondegenerate. Moreover, since the identity map preserves the various $\mu^{\mathrm{orb}}(\Sigma)$ eigenspaces, the bilinear form is non-degenerate when restricted to the λ eigenspace. Thus, there are a pair of vectors $v_\lambda, w_\lambda \in \mathbb{U}_{0,n}$ for which $B_n^u(v_\lambda, w_\lambda) \neq 0$. By our gluing theorem, this means that $\Delta_n^u(v_\lambda, w_\lambda)$ is a nonzero vector in $\mathbb{V}_{0,n}$ which is a λ-eigenvector. Since $\mu(\mathrm{pt}) = -8\mathrm{Id}$ and $\mu([x_k])_{\boldsymbol{\kappa}_k}$ vanishes on $\mathbb{U}_{0,n}$, this vector is a -8-eigenvector for $\mu(\mathrm{pt})$ and is in the kernel of the operator $\mu([x_k])_{\boldsymbol{\kappa}_k}$ for each k as well. $\square$

Because we actually wish to have a description of the subalgebra $\mathbb{V}_{0,n}^+ \subset \mathbb{V}_{0,n}$, we need to understand when the eigenvectors from Lemma 2.8.2 are in the image of Φ^+. The map E on $\mathbb{U}_{0,n}$ is given by the product cobordism with additional Stiefel-Whitney 2-cycle given by a copy of Σ. This copy of Σ is homologous to the *sum* of the different copies of Σ in $n-2$

the incoming ends of the cobordism U_n^3. We conclude that E on $\mathbb{U}_{0,n}$ and

$$\sum_{i=1}^{n} 1 \otimes \cdots \otimes 1 \otimes E \otimes 1 \otimes \cdots \otimes 1,$$

are intertwined by Δ_n^3, at least up to sign. It is this sign which will cause some issue for us, and so we denote it by e_n^3. On $\mathbb{U}_{0,3}$, we have ± 4 eigenvectors $\sigma_{\mp} = [\rho_0] \mp [\rho_1]$, and we have:

$$E(\sigma_+) = \sigma_+, \quad E(\sigma_-) = -\sigma_-$$

Thus, an eigenvector $\Delta_n^3(\sigma_{e_1} \otimes \cdots \otimes \sigma_{e_{n-2}})$ in $\mathbb{U}_{0,n}$ is E-invariant only if $\prod_i (e_i 1) = e_n^3$. Now, the operator E on $\mathbb{V}_{0,n}$ is intertwined with $E \otimes 1$ by Δ_n^u, also perhaps up to a sign e_n^u. Now, by the discussion following Lemma 2.7.14, $\mathbb{V}_{0,n}^+$ is the E-invariant subspace. In the proof of Lemma 2.8.2, we may take one of the eigenvectors, say v_λ, to be an elementary one

$$\Delta_n^3(\sigma_+ \otimes \cdots \otimes \sigma_+ \otimes \cdots \otimes \sigma_- \otimes \cdots \otimes \sigma_-)$$

with -4 times the difference in the number P of positives and number N of negatives equal to λ. Then the resulting eigenvector in $\mathbb{V}_{0,n}$ will be E invariant only if $(-1)^N = e_n^u e_n^3$. Let E_n denote this overall sign. Rather than carefully keeping track of orientation conventions and nailing this sign down precisely, we will eventually proceed by noticing that the inductive structure of the rings $\mathbb{V}_{g,n}$ actually force this sign to be alternating in n. What our discussion does show is the following result.

PROPOSITION 2.8.3. *For each λ in {-(n-2),-(n-4),...,n-4,n-2}, there is a simultanous eigenvector either with value $(\lambda, 2)$ or $(-\lambda, 2)$ for multiplication by the pair $\Phi^+(\alpha)$ and $\Phi^+(\beta)$ and in the kernel of multiplication by $\Phi^+(\delta_k)$.*

PROOF. If the vector

$$\Delta_n^3(\sigma_+ \otimes \cdots \otimes \sigma_+ \otimes \cdots \otimes \sigma_- \otimes \cdots \otimes \sigma_-)$$

has N minus signs appearing, then reversing $+$ and $-$ gives a vector with opposite $\Phi^+(\alpha)$ eigenvalue. Exactly one out of this pair is E-invariant, since n is odd. $\square$

2.9. The Case $g = 0$ and $n > 3$

We now move on to the general case. We have already shown that the relation $\delta_k^2 - \beta + 2$ is in $\mathcal{J}_{0,n}^+$ for each k and n. It will be convenient to use an intermediate ring $\mathbb{B}_{g,n}$ obtained by taking the quotient by the relations $\delta_k^2 - \beta + 2$.

Notation. Let $\mathbb{B}_{g,n}$ denote the quotient algebra $\mathbb{A}_{g,n}/(\delta_1^2 - \beta + 2, \ldots, \delta_n^2 - \beta + 2)$. We denote the passage of an element from $\mathbb{A}_{g,n}$ to $\mathbb{B}_{g,n}$ by a bar, so that we write, for example, $\overline{\delta}_k^2 = \overline{\beta} - 2$ in $\mathbb{B}_{g,n}$. We denote the projection map from $\mathbb{B}_{g,n}$ to $\mathbb{V}_{g,n}^+$ by $\overline{\Phi}^+$.

The ring $\mathbb{B}_{g,n}$ is no longer graded, but there is still a notion of degree defined by taking the lowest degree representative in $\mathbb{A}_{g,n}$ for an element in $\mathbb{B}_{g,n}$. There also remains a mod 2 grading. Let $\overline{\mathcal{J}}_{g,n}^+$ denote the image of the relation ideal $\mathcal{J}_{g,n}^+$ in $\mathbb{B}_{g,n}$. Then it is an easy corollary of Proposition 2.6.4 that $\overline{\mathcal{J}}_{0,n}^+$ is generated by mod 2 graded deformations $\overline{Q}_{0,n}^J$ of $\overline{R}_{0,n}^J$. Moreover, there are induced maps $\overline{\tau}_{g,n}^{g,n+2} : \mathbb{B}_{g,n} \to \mathbb{B}_{g,n}$ and it is clear that there is an inclusion of ideals:

$$\overline{\mathcal{J}}_{g,n+2}^+ \subset \left(\overline{\tau}_{g,n}^{g,n+2}\right)^{-1}\left(\overline{\mathcal{J}}_{g,n}^+\right) \tag{2.9.1}$$

Moreover, the flips m_J are well defined on $\mathbb{B}_{g,n}$ and are intertwined by the possible projections.

It is easy to use the flip symmetry to determine $\overline{Q}_{0,n}^J$ for $|J| = m$.

Lemma 2.9.1. *For all $|J| = m$, we can assume $\overline{Q}_{0,n}^J = \overline{R}_{0,n}^J = \underline{\overline{\delta}}^J$.*

Proof. We invoke Lemma 2.6.10 to conclude that, for any J, without a loss of generality we may always take $\overline{Q}_{0,n}^J = \underline{\overline{\delta}}^J \cdot p(\overline{\alpha}, \overline{\beta})$ for some polynomial p. When $|J| = m$, we must have $\deg(p) = 0$, and so are free to set $\overline{Q}_{0,n}^J = \underline{\overline{\delta}}^J$. □

As a corollary, we see that $(\overline{\beta} - 2)^m \in \overline{\mathcal{J}}_{g,n}^+$, and with a bit of thought it is easy to see that m is the smallest such power.

The surjections $\pi^{g,n+2}_{g,n} : \mathbb{V}_{g,n+2} \to \mathbb{V}_{g,n}$ will be fundamental to our calculations of these rings, as they establish inclusions of the ideals of relations. These maps are easy to understand, as they fit into commutative diagrams with the natural surjections $\mathbb{A}_{g,n+2} \to \mathbb{A}_{g,n}$. We shall also need the injective maps $\iota^{g,n}_{g,n+2} : \mathbb{V}_{g,n} \to \mathbb{V}_{g,n+2}$ arising from oppositely directed cobordisms. These maps do not fit into the obvious commutative diagrams, and so we shall need to work out exactly what these maps are. At the level of flat connections, it is easy to see that the cobordisms defining the $\iota^{\bullet}_{\bullet}$ maps induce the natural injections $\mathcal{R}_{g,n} \hookrightarrow \mathcal{R}_{g,n+2}$ and $\mathcal{R}_{g,n} \hookrightarrow \mathcal{R}_{g+1,n}$. We shall focus on the former type, as the latter was essentially studied by Muñoz [**27**]. Unfortunately, this injection of representation varieties is not "flip invariant", in that the particular injection is not preserved under flips on the target space. To remedy this, we shall take a sum over the 4 possible natural injections. We achieve this by composing $\iota^{g,n}_{g,n+2}$ with the four possible flips on 2 or 0 indices in the set $\{n, n+1, n+2\}$ on the space $\mathbb{V}_{g,n+2}$. We set:

$$\widetilde{\iota}^{g,n}_{g,n+2} = \frac{1}{4}\left(\iota^{g,n}_{g,n+2} + \iota^{g,n}_{g,n+2} \circ F_{\{n,n+1\}} + \iota^{g,n}_{g,n+2} \circ F_{\{n,n+2\}} + \iota^{g,n}_{g,n+2} \circ F_{\{n+1,n+2\}}\right) \tag{2.9.2}$$

The map $\widetilde{\iota}^{g,n}_{g,n+2}$, at the level of flat connections, is then clearly given by the map $1 \mapsto \Psi(\beta)$.

PROPOSITION 2.9.2. *The map $\widetilde{\iota}^{g,n}_{g,n+2} : \mathbb{V}_{g,n} \to \mathbb{V}_{g,n+2}$ is injective and given by*

$$\widetilde{\iota}^{g,n}_{g,n+2}(\Phi_{g,n}(z)) = \Phi_{g,n+2}((\beta - C)z)$$

for some constant C, $z \in \mathbb{A}_{g,n}$, and where we have implicitly used the natural injection $\mathbb{A}_{g,n} \hookrightarrow \mathbb{A}_{g,n+2}$. When $g = 0$, we must have $C = 2$.

PROOF. We shall use the notation of $\widetilde{\iota}$ and Φ undecorated, with the superscripts and subscripts to be understood from context. We first show that $\widetilde{\iota}$ is injective. Let $z \in \widetilde{\mathbb{A}}_{g,n}$, and suppose that

$$\langle \widetilde{\iota}(\Phi(z)), \Phi(z') \rangle_{g,n+2} = 0$$

for all $z' \in \widetilde{\mathbb{A}}_{g,n+2}$. Then since capping the cobordism on the right and left produces the 4-manifold triple $(S^2 \times S^2, \{n \text{ pts}\} \times S^2, \emptyset)$, we see that

$$\left\langle \Phi(z), \Phi(\pi_{g,n}^{g,n+2}(z')) \right\rangle_{g,n} = 0$$

Hence, since $\pi_{g,n}^{g,n+2}$ is surjective, we must have $\Phi(z) = 0$.

It is an easy application of our gluing theory and the fact that E commutes with $\widetilde{\iota}$ that

$$\widetilde{\iota}(\Phi(z)) = \Phi(z)\widetilde{\iota}(\Phi(1)),$$

so it is enough to prove that $\widetilde{\iota}(\Phi(1)) = \Phi(\beta - 2)$. Recall that we have a set $\mathcal{S}_{g,n} \subset \mathbb{A}_{g,n}$ consisting of monomials of degree at most the dimension $6g + 2n - 6$ of the moduli space $\mathcal{R}_{g,n+2}$ such that $\Phi(\mathcal{S}_{g,n}) \sqcup \Phi(\epsilon\mathcal{S}_{g,n})$ is a basis of $\mathbb{V}_{g,n}$. Let $x \in \mathbb{A}_{g,n}$ be in the span of $\mathcal{S}_{g,n+2}) \sqcup \epsilon\mathcal{S}_{g,n+2}$ such that $\widetilde{\iota}(\Phi(1)) = \Phi(x)$. We require that

$$\left\langle 1, \Phi(\pi_{g,n}^{g,n+2}(z)) \right\rangle_{g,n} = \langle \Phi(x), \Phi(z) \rangle$$

for all $z \in \widetilde{\mathbb{A}}_{g,n+2}$. Note that the left hand side is zero unless $\deg(z)$ is at least $6g + 2n - 6$ and congruent to it mod 4 (counting ϵ as degree 2), as this is the lowest dimension of moduli space on $(S^2 \times S^2, \{n \text{ pts}\} \times S^2, \emptyset)$. Since the lowest dimension of moduli space on $(S^2 \times S^2, \{n+2 \text{ pts}\} \times S^2, \emptyset)$ is 4 higher than this, it is easy to see by nondegenerateness of the pairings matrix for the basis vectors that we must have $\deg(x) \leq 4$ and that x has only terms of degree 4 and 0. Let $\mathfrak{m}$ denote the unique top degree element of $\mathcal{S}_{g,n+2}$, which we may assume pairs to 1 with the element 1. It is unique in $\mathcal{S}_{g,n+2}$ in having degree strictly greater than $6g + 2n - 6$ and congruent to it mod 4. Now, we compare the pairings

$$\langle \Phi(x), \Phi(z) \rangle \quad \text{and} \quad \langle \Phi(\beta), \Phi(z) \rangle$$

By construction of the $\widetilde{\iota}$ maps, these pairings are equal when $\deg(z) = 6g + 2n - 6$, as the pairings correspond to cohomology pairings on moduli spaces of *flat* connections. There is only one other potentially nonzero pairing with $\Phi(x)$, which is with $\mathfrak{m}$. Thus, for some constant C, we have

$$x = \beta - C.$$

It remains to determine C in the case $g = 0$. But we know that $(\Phi(\beta) - 2)^m = 0$ in $\mathbb{V}_{0,n}$ and no lower power of $(\Phi(\beta) - 2)$ vanishes. By induction, we see that $C = 2$ for all n. □

An immediate consequence is the following, which should be compared to Corollary 15 of [**27**].

Corollary 2.9.3. *There is an inclusion of ideals*

$$(\beta - 2)(\pi_{g,n}^{g,n+2})^{-1}(\mathcal{J}_{g,n}^+) \subset \mathcal{J}_{g,n+2}^+ \subset (\pi_{g,n}^{g,n+2})^{-1}(\mathcal{J}_{g,n}^+) \tag{2.9.3}$$

We can now finish the puzzle. We shall prove:

Proposition 2.9.4. *Let $n = 2m + 1$. Then the image $\overline{\mathcal{J}}_{0,n}$ of $\mathcal{J}_{0,n}$ under the projection to $\mathbb{B}_{0,n}$ is generated by mod 2 graded deformations*

$$Q_{0,n}^J(\overline{\alpha}, \overline{\beta}, \overline{\delta}_1, \ldots, \overline{\delta}_n) = q_{0,n-2s}(\overline{\alpha}, \overline{\beta}) \cdot \underline{\overline{\delta}}^J \tag{2.9.4}$$

of the polynomials $R_{0,n}^J$ in $\mathbb{B}_{0,n}$ for $|J| = s \leq m$, and where $q_{0,n}(\alpha, \beta)$ a polynomial defined recursively by

$$q_{0,n+2} = (\alpha - (-1)^m n)\, q_{0,n} - m^2(\beta - 2) q_{0,n-2}. \tag{2.9.5}$$

and $q_{0,1} = 1$, $q_{0,-1} = 0$.

Before doing this, we shall first need to study a natural subring of $\mathbb{V}_{0,n}^+$, denoted $\mathbb{V}_{0,n}^{\mathrm{inv}}$, which is invariant under the group of flips. That it is a subring is clear. Less obvious is its connection to the correpsonding subring $\mathbb{H}_{0,n}^{\mathrm{inv}}$ of $\mathbb{H}_{0,n} = H^*(\mathcal{R}_{0,n})$. It is easy to see that $\mathbb{H}_{0,n}^{\mathrm{inv}}$ is given as the quotient of $\mathbb{C}[\alpha, \beta]/(r_{0,n}, \beta r_{0,n+2})$. The map Φ^+ establishes $\mathbb{V}_{0,n}^{\mathrm{inv}}$ as a quotient of $\mathbb{C}[\alpha, \beta]$ by some ideal $\mathcal{J}_{0,n}^{\mathrm{inv}}$. It is an easy adaptation of our previous arguments to prove:

LEMMA 2.9.5. *The ideal $\mathcal{J}^{\mathrm{inv}}_{0,n}$ is generated by polynomials $q^1_{0,n}(\alpha,\beta)$ and $q^2_{0,n}(\alpha,\beta)$ which are mod 2 graded deformations of $r_{0,n}$ and $\beta r_{0,n+2}$. These polynomials are of degree $2m$ and $2m+2$ respectively.*

The culmination of discussion with Floer theory and eigenvectors is the following result.

PROPOSITION 2.9.6. *Let $q_{0,n}(\alpha,\beta)$ be the polynomial defined by the recursion (2.9.5). Then for the polynomials $q^i_{0,n}$ from Lemma 2.9.5 we may take $q^1_{0,n} = q_{0,n}$ and $q^2_{0,n} = (\beta - 2)q_{0,n-2}$.*

PROOF. We argue by induction on n, assuming the Proposition has been proved for $n' \leq n$. We will leverage the inclusion of ideals (2.9.3), which certainly holds for the invariant ideals $\mathcal{J}^{\mathrm{inv}}_{0,n}$. Under the inductive hypothesis this implies that $q^1_{0,n+2}$ and $q^2_{0,n+2}$ are linear combinations of $q_{0,n}$ and $(\beta-2)q_{0,n-2}$. By degree considerations, we conclude that there are polynomials

$$q^1_{0,n+2} = a(\alpha,\beta)q_{0,n} + b(\alpha,\beta)(\beta-2)q_{0,n-2}$$

$$q^2_{0,n+2} = c(\alpha,\beta)q_{0,n} + d(\alpha,\beta)(\beta-2)q_{0,n-2}$$

where $\deg(a) = \deg(d) = 2$, $\deg(b) = 0$, and $\deg(c) = 4$. By comparing leading terms with the corresponding recursions for the $r_{0,n}$'s, we conclude that there are constants λ, μ, τ and C such that:

$$q^1_{0,n+2} = (\alpha-\lambda)q_{0,n} - m^2(\beta-2)q_{0,n-2}$$

$$q^2_{0,n+2} = (\beta+\mu\alpha-\tau)q_{0,n} + C(\beta-2)q_{0,n-2}.$$

The constant λ should be interpreted as a new eigenvalue for multiplication by $\Phi^+(\alpha)$. Now, from Proposition 2.8.3, we know that there is a nonzero simultaneous eigenvector v_{n+2} in $\mathbb{V}^+_{0,n+2}$ for multipication by (α,β) with value either $(n,2)$ or $(-n,2)$. This eigenvector is the image of (the tensor square of) a basis eigenvector $b_+ \otimes \cdots \otimes b_+$ or $b_- \otimes \cdots \otimes b_-$ in $\mathbb{U}_{0,n+2}$. Let us impose the inductive hypothesis that the spectrum of $\Phi^+(\alpha)$ on $\mathbb{V}^{\mathrm{inv}}_{0,n}$ is

$\{1, -3, 5, \ldots, (-1)^m(n-2)\}$. The question of whether the new value is n or $-n$ is answered by determining which of the images of $b_{\epsilon_1} \otimes \cdots \otimes b_{\epsilon_n}$ are E-invariant, which depends only the parity of the number of minus signs. We conclude that, since the inclusion of ideals forces $(-1)^m(n-2)$ to be an eigenvalue, the correct signed new eigenvalue is $(-1)^{m+1}n$.

Now applying the two relations to v_{n+2} gives $D(\alpha - \lambda)v_{n+2} = 0$ for some nonzero D. We conclude that $\lambda = (-1)^{m+1}n$. Now, we are free to subtract μ times the first equation from the second, and we obtain:

$$q_{0,n+2}^2 = (\beta - \tau)q_{0,n} + C'(\beta - 2)q_{0,n-2}.$$

Now, by the inductive hypothesis and (2.9.3), we know that $(\beta - 2)q_{0,n} \in \mathcal{J}_{0,n+2}^{\mathrm{inv}}$. Subtracting this from the above relation gives a relation of strictly lower degree, which upon comparing leading terms is not a graded deformation of $r_{0,n+2}$ or $\beta r_{0,n}$. This is a contradiction unless this difference is the zero polynomial. We conclude that

$$q_{0,n+2}^2 = (\beta - 2)q_{0,n}$$

This completes the inductive step. The base case of $n = 3$ is just equation (2.7.11). $\square$

We can now finish the proof of Proposition 2.9.4

Proof. We know that $\overline{\mathcal{J}}_{0,n}$ is generated by certain relations $Q_{0,n}^J(\overline{\alpha}, \overline{\beta}, \overline{\delta}_1, \ldots, \overline{\delta}_n)$ which are mod 2 graded deformations of $R_{0,n}^J$. Applying flips, we may assume that $Q_{0,n}^J$ takes the form $\underline{\overline{\delta}}^J \cdot q_{0,n}^J(\overline{\alpha}, \overline{\beta})$. It remains to show that $q_{0,n}^J$ is just $q_{0,n-2|J|}$, where $q_{0,n'}$ is the sequence defined by (2.9.5). We proceed by induction, both on n, assuming the proposition has been proved for all $n' < n$, and on $|J|$, assuming $q_{0,n}^{J'} = q_{0,n-2|J'|}$ for all $|J'| < |J|$. Proposition 2.9.6 essentially verifies this in the base case $J = \emptyset$, so we assume $|j| = s \geq 1$ and without loss of generality $1 \in J$, and set $J' = J \setminus \{1\}$. We may assume that $q_{0,n}^J$ is monic in the top $\overline{\alpha}$ power (because this is true of $q_{0,n-2s}$ and these polynomials are mod 2 deformations of each other), and suppose that it is not equal to $q_{0,n-2s}$. By 2.9.3 and the inductive hypothesis, we know that

$$(\overline{\beta} - 2)\underline{\overline{\delta}}\Phi^{J'} \cdot q_{0,n-2s}(\overline{\alpha}, \overline{\beta})$$

is in $\overline{\mathcal{J}}_{0,n}$. By multiplying $Q^J_{0,n}$ by $\overline{\delta}_1$, we also have the relation

$$(\overline{\beta} - 2)\underline{\overline{\delta}}\Phi^{J'} \cdot q^J_{0,n}(\overline{\alpha}, \overline{\beta}).$$

Now, the difference $d^J_{0,n} = q^J_{0,n} - q_{0,n-2s}$ has degree at most $2m - 2s - 2$, since the top degree terms must agree, and thus we obtain a relation

$$(\overline{\beta} - 2)\underline{\overline{\delta}}\Phi^{J'} \cdot d^J_{0,n}(\overline{\alpha}, \overline{\beta})$$

of degree 2m, and thus it must be a $\mathbb{C}$-linear combination of the $Q^J_{0,n}$'s. This can only be the case if $(\overline{\beta} - 2)d^J_{0,n}$ is a scale multiple of $q^{J'}0, n$, which by induction is equal to $q_{0,n-2s-2}$. On the other hand, it is easy to see from (2.9.5) that the top degree part of $q_{0,n-2s-2}$ is equal to $\overline{\alpha}^{m-s-1} + \overline{\beta} \cdot p(\overline{\alpha}, \overline{\beta})$ for some degree $2m - 2s - 6$ polynomial p, which is not true of $(\overline{\beta} - 2)d^J_{0,n}$. This is a contradiction, and thus we have established that $q^J_{0,n} = q_{0,n-2s}$. □

We now wish the study the spectrum of multiplication by $\Phi^+(\alpha)$ on $\mathbb{V}^+_{0,n}$. In what follows, we set $\lambda_n = (-1)^{m+1}(n - 2)$.

Lemma 2.9.7. *The polynomial*

$$P_{0,n}(\alpha) = (\alpha - \lambda_n)(\alpha - \lambda_{n-2})^2 \cdots (\alpha - \lambda_3)^m \tag{2.9.6}$$

is a linear combination of $q_{0,n}$ *and* $(\beta - 2)q_{0,n-2}$.

Proof. From the recursion (2.9.5), it is easy to see that the polynomials $(\beta - 2)^s q_{0,n-2s}$ always lie in $\mathcal{J}^{\text{inv}}_{0,n}$ for all n (that is, they are linear combinations of $q_{0,n}$ and $(\beta - 2)q_{0,n-2}$), and so we are done if we can show that there are polynomials a^s_n in α and β such that

$$P_{0,n} = a^0_n q_{0,n} + a^1_n(\beta - 2)q_{0,n-2} + \ldots + a^{m-1}_n(\beta - 2)^{m-1}q_{0,3} + a^m_n(\beta - 2)^m \tag{2.9.7}$$

We argue by induction on n, and suppose that the polynomials a_n^s exist. We then desire a_{n+2}^s for which

$$P_{0,n+2} = a_{n+2}^0 q_{0,n+2} + \ldots + a_{n+2}^m (\beta - 2)^m q_{0,3} + a_{n+2}^{m+1} (\beta - 2)^{m+1}. \tag{2.9.8}$$

The left hand side of (2.9.8) is $\prod_{i=1}^{m+1}(\alpha - \lambda_{2i+1})$ times that of (2.9.8). We will solve for the a_{n+2}^s by comparing terms on the two right hand sides. Expand each $q_{0,n-2s}$ appearing in (2.9.8) via the recursion relation. We then equate coefficient polynomials for the $q_{0,n-2s}$'s, to obtain:

$$\begin{aligned}
a_{n+2}^0 &= (\alpha - \lambda_n) \cdot \ldots \cdot (\alpha - \lambda_3) a_n^0 \\
(\alpha - \lambda_n) a_{n+2}^1 - m^2 a_{n+2}^0 &= (\alpha - \lambda_n) \cdot \ldots \cdot (\alpha - \lambda_3) a_n^1 \\
(\alpha - \lambda_{n-2}) a_{n+2}^2 + (m-1)^2 a_{n+2}^1 &= (\alpha - \lambda_n) \cdot \ldots \cdot (\alpha - \lambda_3) a_n^2 \\
&\vdots \\
(\alpha - \lambda_3) a_{n+2}^m + 1^2 a_{n+2}^{m-1} &= (\alpha - \lambda_n) \cdot \ldots \cdot (\alpha - \lambda_3) a_n^2 \\
a_{n+2}^{m+1} &= 0
\end{aligned}$$

It is not difficult to see that these equations for the a_{n+2}^s's can be solved in sequence in terms of the a_n^s's. We omit the full details of this calculation. □

COROLLARY 2.9.8. *The spectrum of multiplication by* $\Phi^+(\alpha)$ *on* $\mathbb{V}_{0,n}^+$ *is*

$$\{\lambda_3, \lambda_5, \ldots, \lambda_n\}$$

where $\lambda_k = (-1)^{(k+1)/2}(k-2)$ *and the only eigenvalue for* $\Phi^+(\beta)$ *is 2. The generalized eigenspace for* $\Phi^+(\alpha)$ *for the top value* λ_n *is 1-dimensional and a spanning vector is an eigenvector for* $\Phi^+(\beta)$ *with value 2.*

PROOF. The previous lemma establishes the claim about the spectrum of $\Phi^+(\alpha)$. We see that the minimal polynomial for α divides $P_{0,n}(\alpha)$ and since $(\alpha - \lambda_n)$ appears with exponent 1, the generalized eigenspace for this value is the same as the true eigenspace. The eigenspace is exactly the the kernel of $\Phi^+(\alpha) - \lambda_n$. By rank-nullity, we have

$$\dim\left(\ker(\Phi^+(\alpha) - \lambda_n)\right) = \dim\left(\mathbb{V}^+_{0,n}/\operatorname{Im}(\Phi^+(\alpha) - \lambda_n)\right).$$

The vector space $\mathbb{V}^+_{0,n}/\operatorname{Im}(\Phi^+(\alpha) - \lambda_n)$ is isomorphic to $\mathbb{A}_{0,n}$ modulo the relations in $\mathcal{J}^+_{0,n}$ into which λ_n has been plugged in for α. From Lemma 2.9.7 and (2.9.3), we know that $(\beta - 2)P_{0,n-2}(\alpha)$ is in $\mathcal{J}^+_{0,n}$. Plugging in $\alpha = \lambda_n$, we get a nonzero scale multiple of $(\beta - 2)$, which shows that this vector space is at most 1-dimensional. It is at least this big because we know there is a nonzero eigenvector for $\Phi^+(\alpha)$, and so we conclude that it is indeed 1-dimensional. Any spanning vector is an eigenvector and must then be a scale multiple of the one guaranteed by Proposition 2.8.3, which is also an eigenvector for $\Phi^+(\beta)$ with value 2. □

The final loose end to tie up in this story is the fact that $\mathbb{V}^+_{0,n}$ is only half of the total Floer homology $\mathbb{V}_{0,n}$. The algebra $\mathbb{V}_{0,n}$ projects onto $\mathbb{V}^+_{0,n}$ via the map $\frac{1}{2}(1 + E)$, and so if $f \in \mathcal{J}^+_{0,n}$ is a relation in $\mathbb{A}_{0,n}$, then $(1 + \epsilon)f$ is a relation in $\widetilde{\mathcal{J}}_{0,n} \subset \widetilde{\mathbb{A}}_{0,n}$, the full ideal of relations holding in $\mathbb{V}_{0,n}$. On the other hand, we could have repeated the entire discussion, focusing on the -1-eigenspace of E, calling it $\mathbb{V}^-_{0,n}$. We would have obtained the exact same relations as in Proposition 2.9.4, except that each eigenvalue λ_n should be negated. Call this ideal $\mathcal{J}^-_{0,n}$. We see that

$$(1 + \epsilon)\mathcal{J}^+_{0,n} + (1 - \epsilon)\mathcal{J}^-_{0,n} \subset \widetilde{\mathcal{J}}_{0,n}.$$

In the case $n = 3$, for example, we get $\widetilde{\mathcal{J}}_{0,3} = (\alpha - \epsilon, \beta - 2, \delta_k^2 - \beta + 2)$. Let $q^+_{0,n}$ be the same as the polynomial $q_{0,n}$ of (2.9.5), and define $q^-_{0,n}$ by

$$q^-_{0,n+2} = (\alpha + \lambda_{n+2})q^-_{0,n} - m^2(\beta - 2)q^-_{0,n-2}. \tag{2.9.9}$$

and $q^-_{0,1} = 1$, $q^-_{0,-1} = 0$, so that $q^-_{0,n} \in \mathcal{J}^-_{0,n}$. Finally, define $\widetilde{q}_{0,n}$ via

$$\widetilde{q}_{0,n+2} = (\alpha - \lambda_{n+2}\epsilon)\widetilde{q}_{0,n} - m^2(\beta - 2)\widetilde{q}_{0,n-2}. \tag{2.9.10}$$

and $\widetilde{q}_{0,1} = 1$, $\widetilde{q}_{0,-1} = 0$.

LEMMA 2.9.9. *For all n, we have $(1+\epsilon)q^{+}_{0,n} + (1-\epsilon)q^{-}_{0,n} = 2\widetilde{q}_{0,n}$.*

PROOF. By induction. The cases $n = 3$ and $n = 1$ are trivial, so suppose the lemma holds for n and $n-2$. Then expanding $(1+\epsilon)q^{+}_{0,n} + (1-\epsilon)q^{-}_{0,n}$ via (2.9.5) and (2.9.9) immediately yields twice (2.9.10). $\square$

We can now prove:

THEOREM 2.9.10. *The ring $\mathbb{V}_{0,n}$ has a presentation as a quotient of $\widetilde{\mathbb{A}}_{0,n} = \mathbb{A}_{0,n}[\epsilon]/(\epsilon^2 - 1)$ by an ideal of relations $wt\mathcal{J}_{0,n}$ generated by the degree 4 relations $\delta_k^2 - \beta + 2$, and relations $\widetilde{Q}^J_{0,n}$ where $J \subset \{1, \ldots, n\}$ has cardinality $s \leq m$, defined by*

$$\widetilde{Q}^J_{0,n} = \widetilde{q}_{0,n-2s}\underline{\delta}^J \tag{2.9.11}$$

PROOF. We already know, via Lemma 2.9.9 and Proposition 2.9.4 that the relations $\widetilde{Q}^J_{0,n}$ are in $\widetilde{\mathcal{J}}_{0,n}$, so we must show this is a complete set (along with the degree 4 relations). They are graded deformations of the relations $R^J_{0,n}$ in $\mathcal{I}_{0,n}$, and, like them, all have different leading terms (in the lexicographic ordering of monomials, say). Hence, the quotient of $\widetilde{\mathbb{A}}_{0,n}$ by these relations has a basis given by the images of the monomials $\mathcal{S}_{0,n}$ (from Lemma 1.6.3) and $\epsilon\mathcal{S}_{0,n}$. The dimension of $\mathbb{V}_{0,n}$ is twice that of the cardinality of $\mathcal{S}_{0,n}$, so this establishes that these are all the relations. $\square$

Essentially the same argument as before shows that:

COROLLARY 2.9.11. *The spectrum of multiplication by $\Phi(\alpha)$ on $\mathbb{V}_{0,n}$ is*

$$\{-(n-2), -(n-4), \ldots, -1, 1, \ldots, n-4, n-2\}$$

and the only eigenvalue for $\Phi(\beta)$ is 2. The generalized eigenspace for $\Phi(\alpha)$ for the top values $\pm(n-2)$ are 1-dimensional and a spanning vector is an eigenvector for $\Phi(\beta)$ with value 2.

PROOF. One shows that the polynomial

$$\widetilde{P}_{0,n}(\alpha) = (\alpha^2 - (n-2)^2)(\alpha^2 - (n-4)^2)^2 \cdots (\alpha^2 - 1)^m \tag{2.9.12}$$

is in $\widetilde{\mathcal{J}}_{0,n}$, and argues as before. $\square$

CHAPTER 3

Excision and Tangles

3.1. Introduction

A sutured 3-manifold (M, γ) is called *balanced* if the two surfaces $R^{\pm}$ into which γ cuts ∂M have equal Euler characteristics: $\chi(R^{+}) = \chi(R^{-})$ (see [**13**]). Given a blanced (M, γ), Kronheimer and Mrowka have defined an instanton invariant $\mathrm{SHI}(M, \gamma)$. In this section we will define an invariant for *tangles* in sutured manifolds:

Definition 3.1.1. Let (M, γ) be a balanced sutured 3-manifold with positive and negative boundary pieces R^{+} and R^{-}. An *odd (even) balanced tangle* T in (M, γ) is a properly embedded 1-manifold such that $|T \cap R^{+}|$ and $|T \cap R^{-}|$ are equal odd (even) numbers, and T has no endpoints on the sutures.

We briefly recall the construction of SHI. This invariant is defined by first closing up (M, γ) in a particular way to give a closed manifold Y along with an embedded surface $\overline{R}$, and applying standard instanton Floer homology to Y. The key to the definition is in dealing with the complication that the closure Y is not uniquely defined, as it depeneds on a choice of diffeomorphism of the positive and negative boundary components as well as a choice of auxillary surface. In order to give a theory which is independent of these choices, the vector space $\mathrm{SHI}(M, \gamma)$ is defined to be a natural subspace of the instanton Floer homology of Y. More specifically, the closure Y comes with a surface $\overline{R}$ (coming from the glued up boundary), as well as a line bundle w, and we take the simultaneous $(-(8g(\overline{R}) - 8), -8)$-eigenspace of $I^{w}_{*}(Y)$ for the pair of operators $(\mu(\overline{R}), \mu(\mathrm{pt}))$ where $g(\overline{R})$ is the genus of $\overline{R}$ - the so-called "top"-eigenspace. One then shows that this subspace has isomorphism type invariant under the choices made in constructing the closure Y.

There are two operations on the set of closures one must study in order to prove this invariance. The first is changing the identification of top and bottom boundary components. This amounts to taking a particular closure Y, cutting along $\overline{R}$, and regluing via a different diffeomorphism. In the case that $\overline{R}$ is a torus, invariance comes for free because we may simply apply Floer's classical excision formula, which is described in [**6**]. One approaches the case $g(\overline{R}) > 1$ by generalizing that excision theorem for splittings of 3-manifolds along surfaces of higher genus. In genus 1, the theorem goes through because the Floer homology of the three torus $S^1 \times T^2$ is one dimensional (after passing to the -8-eigenspace for $\mu(\text{pt})$). In higher genus, the Floer homology of $S^1 \times \Sigma$ is much more complicated, but the simultaneous $(-(8g(\Sigma) - 8), -8)$-eigenspace for the pair $(\mu(\Sigma), \mu(\text{pt}))$ is 1-dimensional. Essentially the same proof as Floer's then gives the desired result. The second operation on the set of closures comes from increasing or descreasing the genus of R. This is handled with the genus 1 excision theorem.

This story serves as a model for our case of interest, that when there is a balanced tangle T in (M, γ) and thus a link in the closed up manifold Y intersecting the surface $\overline{R}$. Our knowledge gained in Chapter 2 of the Floer homology of a product link in $S^1 \times \Sigma$ for Σ a sphere means that we can prove an analogous of the excision formula which applies to the case when $\overline{R}$ is genus 0. This allows us to define an invariant $\text{THI}(M, T, \gamma)$ for a specific class of links in a certain class of sutured 3-manifolds. We then prove several properties enjoyed by $\text{THI}(M, T, \gamma)$, or rather a variant $\text{THI}^\sharp(M, T, \gamma)$, including a nontriviality result which shows that it characterizes the product tangle in a product sutured manifold.

3.2. Excision in Genus 0

In this section we prove an analogue of Theorem 2.5.1 for genus 0 surgeries in a 3-manifold Y with link L, where the embedded spheres intersect L in an equal odd number of points. The genus 1 theorem relied on the fact that the Floer homology of $S^1 \times T^2$ is only rank 2, and rank 1 upon restricting to the -8-eigenspace of $\mu(\text{pt})$. This is very far from the case of the Floer homology of (Y_0, K_n). To proceed, we must restrict to a subspace of the Floer homology. Recall that the top eigenvalues of the operator $\mu^{\mathrm{orb}}(\Sigma)$ on $I_*(Y_0, K_n, \emptyset)$ are

$$\pm 4(n-2) = \pm 4\chi(\Sigma \setminus \{n \text{ pts}\}),$$

where χ denotes Euler characteristic. Likewise, we know that the top eigenvalues for $\mu(\Sigma)$ on $I_*(Y_g, \emptyset, S^1)$ (for $g \geq 1$) are

$$\pm 4(2g-2) = \pm 4\chi(\Sigma_g),$$

and that in all cases the top generalized eigenspaces are 1-dimensional. We are motivated then to make the following definition.

Definition 3.2.1. Let (Y, K, ω) be a triple in WINK, and let R be an embedded surface intersecting K transversely. Define the *restricted* Floer homology of the triple with respect to R, denoted

$$I_*(Y, K, \omega | R)$$

to be the subspace of $I_*(Y, K, \omega)$ given by the simultaneous generalized eigenspace of the operator $\mu^{\mathrm{orb}}(R)$ for the value $-4\chi(\Sigma \setminus K)$ and the operator $\mu(\mathrm{pt})$ for the value -8.

Remark 3.2.2. For example, the content of Corollary 2.9.11 and the culmination of everything we have done in Chapters 1 and 2 is that for n odd:

$$I_*(Y_0, K_n, \emptyset | \mathrm{pt} \times \Sigma_0) \cong \mathbb{C}.$$

Now, for an arbitrary triple (Y, K, ω) and a *sphere* R intersecting K in an *odd* number of points, our first result is that the eigenvalues of $\mu^{\mathrm{orb}}(R)$ are a subset of (4 times) those in Corollary 2.9.11 (Compare to Corollary 7.2 of [**22**]).

Lemma 3.2.3. *For a sphere R as above, if λ is an eigenvalue for $\mu^{\mathrm{orb}}(R)$ then λ is 4 times an odd number between $\pm\chi(R \setminus K)$, where χ is the Euler characteristic.*

Proof. Let (W_R, S, Ω) denote the cobordism triple obtained by removing a tubular neighborhood of an interior copy of R from the tube $[0,1] \times (Y, K, \omega)$, viewed as having incoming ends (Y, K, ω) and (Y_0, K_n, ω') (where ω' is either empty or $S^1 \times \mathrm{pt}$). Let $\widetilde{P}_{0,n}$ be the polynomial (2.9.12). Then $\widetilde{P}_{0,n}(-1/4\mu^{\mathrm{orb}}(R))$ vanishes on $I_*(Y, K, \omega)$. Indeed, this operator can be viewed, using our gluing theory, as a composite of $\Phi(\widetilde{P}_{0,n}(\alpha)) \otimes \mathrm{Id}$ from $I_*(Y, K, \omega)$ with $I(W_R, S, \Omega)$, and $\Phi(\widetilde{P}_{0,n}(\alpha))$ is just the zero element. The statement of the lemma follows immediately. □

We have the following analogue of the excision result (Theorem 7.7) of [**22**].

THEOREM 3.2.4. *Let (Y, K, ω) be a connected 3-manifold triple in* WINK*, and let R_1 and R_2 be homologous embedded spheres each intersecting K transversely in the same odd number n of points. Suppose also that $\omega \cdot R_1$ and $\omega \cdot R_2$ are both zero or one. Let (Y_i', K_i', ω_i') be the two triples obtained by cutting Y along R_1 and R_2 and regluing according to some diffeomorphism carrying $(K, \omega) \cap R_1$ onto $(K, \omega) \cap R_2$. Within Y_i' there is a surface R_i' coming from R_1 and R_2. Then there is a natural cobordism from (Y, K, ω) to $\cup_i (Y_i', K_i', \omega_i')$ inducing an isomorphism*

$$I_*(Y, K, \omega | R_1) = I_*(Y, K, \omega | R_2) \longrightarrow I_*(Y_1', K_1', \omega_1' | R_1') \otimes I_*(Y_2', K_2', \omega_2' | R_2') \tag{3.2.1}$$

PROOF. The proof is essentially identical to that of Theorem 2.5.1, so content ourselves with a sketch, advising the reader to compare with that proof. We can form a cobordism in the following way, starting with the cylinder $[0, 1] \times (Y, K, \omega)$. Without loss of generality, assume $\omega \cdot R_i = 0$. Let νR_i be a small tubular neighborhood of R_i. To $\{1\} \times (\nu R_1 \sqcup \nu R_2)$ glue the pair $[1, 2] \times [-\epsilon, \epsilon] \times (S^2, n \text{ pts})$ such that $\{i\} \times [-\epsilon, \epsilon] \times S^2$ is identified with νR_i in a way carrying the n segments $[-\epsilon, \epsilon] \times \{n \text{ pts}\}$ to $\nu R_i \cap L$. We ensure that the induced diffeomorphism from R_1 to R_2 agrees with the surgery identification. The resulting 4-manifold triple with corners can be canonically smoothed to a cobordism triple (W, S, Ω) and has boundary components naturally identified with (Y, K, ω) and $(Y_1' \sqcup Y_2', K_1' \sqcup K_2', \omega_1' \sqcup \omega_2')$. One may repeat the construction "in reverse", gluing instead onto $\{0\} \times Y$, to get a cobordism (W^*, S^*, Ω^*) in the opposite direction. We can see that this cobordism induces the desired isomorphism by composing W and W^* in the two different orders. The only difference with the proof of Theorem 2.5.1 is that we further compose with the operator associated not to the polynomial $([\mathrm{pt}] \pm 8)^N$, but to the polynomial

$$\widetilde{P}_{0,n}(-1/4[R_1])/(-1/4[R_1] - (n-2)).$$

This annihilates the generalized eigenspaces for all eigenvalues of $\mu^{\mathrm{orb}}(R_1)$ not equal to $-4(n-2)$ and projects onto the remaining desired generalized eigenspace. □

Later on we shall need a modification of Theorem 2.5.1 for restricted Floer homology. Specifically, we need to analyze the effect of a genus 1 surgery along tori T_1, T_2 on restricted floer homology with respect to surfaces R_1 and R_2, where T_i and R_i intersect transversally.

THEOREM 3.2.5. *Suppose that (Y_i, K_i, ω_i) is a 3-manifold triple with embedded surface R_i intersecting K_i transversally, for $i = 1, 2$. Further, let T_i be an embedded torus in Y_i intersecting R_i transversely and which is odd for ω_i. Let $(\widetilde{Y}, \widetilde{K}, \widetilde{\omega})$ denote the triple obtained by a genus 1 excision surgery along the T_i performed in such a way that the R_i are cut and reglued to form a closed embedded surface $\widetilde{R}$. Then there is an isomorphism*

$$I_*(\widetilde{Y}, \widetilde{K}, \widetilde{\omega} | \widetilde{R}) \cong \bigotimes_{i=1,2} I_*(Y_i, K_i, \omega_i | R_i). \tag{3.2.2}$$

PROOF. Let us assume that $\widetilde{Y}$ is also connected, and that the curves $T_i \cap R_i$ are non-separating in R_i so that $\widetilde{R}$ is connected as well. The general case is a simple modification of the following argument. In the standard cobordism from the union of the (Y_i, K_i, ω_i) to $(\widetilde{Y}, \widetilde{K}, \widetilde{\omega})$, the surface $\widetilde{R}$ is homologous to the union of the R_i. Thus, the cobordism map intertwines the operators $\mu^{\mathrm{orb}}(R_1) \otimes 1 + 1 \otimes \mu^{\mathrm{orb}}(R_2)$ and $\mu^{\mathrm{orb}}(\widetilde{R})$. Moreover, we have

$$\chi(\widetilde{R}) = \chi(R_1) + \chi(R_2).$$

Thus, the cobordism map, which is an isomorphism on the generalized eigenspaces for $\mu(\mathrm{pt})$ for the value -8, carries $\otimes_{i=1,2} I_*(Y_i, K_i, \omega_i | R_i)$ to $I_*(\widetilde{Y}, \widetilde{K}, \widetilde{\omega} | \widetilde{R})$. This establishes (3.2.2). □

3.3. Tangles

Given a balanced sutured 3-manifold (M,γ), the regular sutured instanton Floer homology $\mathrm{SHI}(M,\gamma)$ is defined by taking a closure Y of M and using the top eigenspace for $\mu(\overline{R})$ on $I_*(Y)$, where $\overline{R}$ is the closed up image of its boundary pieces $R^{\pm}$ (see Definition 7.10 of [**22**]). We can make an analogue this construction for the the case that there is an odd balanced tangle T in M and the closure of M involves gluing along *spheres*. This will require us to include an additional hypothesis on our sutured manifold (M,γ):

DEFINITION 3.3.1. The balanced sutured manifold (M,γ) is said to have *type 0* if the surfaces $R^{\pm}$ which the sutures of γ break ∂M into are connected and homeomorphic to punctured spheres.

From now on, we will fix a balanced sutured manifold (M,γ) of type zero, and suppose that there is a balanced tangle T in it. The balanced condition on (M,γ) ensures that if it has type 0 then the surfaces $R^{\pm}$ are homeomorphic. Suppose they each have s boundary components and let F be a disjoint union of s disks. We shall call F the *auxilliary surface* for the closure, noting that we differ from [**22**] in that our auxilliary surface is disconnected. We attach the cylinder $[-\epsilon,\epsilon]\times F$ to M by gluing $[-\epsilon,\epsilon]\times\partial F$ to the union of the thickened sutures $A(\gamma)$ in an orientation preserving way. The resulting 3-manifold $\widetilde{M}$ with boundary will have two 2-sphere components $\widetilde{R}^{+}$ and $\widetilde{R}^{-}$, each containing the same odd number n of points of ∂T. Now, choose an identification

$$h:\widetilde{R}^{+}\to\widetilde{R}^{-}$$

which carries $\widetilde{R}^{+}\cap\partial T$ to $\widetilde{R}^{-}\cap\partial T$, bijectively (these sets are of equal size due to the balanced condition on T) and which restricts to the identity on F. Finally, close $\widetilde{M}$ up by gluing the boundary components together according to h. Let $\widehat{M}$ be the resulting 3-manifold, $\widehat{T}$ the resulting link in it, and $\overline{R}$ the embedded 2-sphere arising from gluing $\widetilde{R}^{\pm}$.

Definition 3.3.2. The *odd sutured tangle instanton Floer homology* of the odd tangle T in (M,γ) is the instanton Floer homology space

$$\mathrm{THI}_{\mathrm{odd}}(M,T,\gamma) := I_*(\widehat{M},\widehat{T},\emptyset|\overline{R}). \tag{3.3.1}$$

This Floer homology is defined because of the assumption that T is an odd tangle; the surface $\overline{R}$ is non-integral. As in the original sutured instanton Floer homology, the fact that this is an invariant of (M,T,γ) not depending on the identification h is a straightforward application of an excision theorem.

Proposition 3.3.3. *The isomorphism type of the vector space* $\mathrm{THI}_{\mathrm{odd}}(M,T,\gamma)$ *is independent of the identification* h.

Proof. Any other choice of h leads to a a closure triple $(\widehat{M'},\widehat{T}',\emptyset)$ which can be obtained from $(\widehat{M},\widehat{T},\emptyset)$ by cutting along $\overline{R}$ and regluing in a different way. We can view this as an excision operation between $(\widehat{M},\widehat{T},\emptyset)$ and $(\widehat{M'},\widehat{T}',\emptyset)\sqcup(Y_0,K_n,\emptyset)$. Since the Floer homology $I_*(Y_0,K_n,\emptyset|\mathrm{pt}\times\Sigma_0)$ is rank 1, the result follows immediately from Theorem 3.2.4. $\square$

While the definition above of $\mathrm{THI}_{\mathrm{odd}}(M,T,\gamma)$ is completely natural in light of the definition of $\mathrm{SHI}(M,\gamma)$ by Kronheimer and Mrowka [**23**], it will be convenient to use slightly different notion to obtain an invariant with useful properties, namely with respect to concatenation and juxtaposition of tangles. Let T be an n-stranded balanced tangle, not necessarily odd, in (M,γ).

Definition 3.3.4. The *stabilized* tangle, denoted $T^\sharp$, of T in (M,γ) is given by appending to T two "product" strands for each suture, which are push-offs into the interior of M of arcs traversing the annular components of γ (see Figure 3.3).

Form the closure $\widehat{M}$ as usual, and the closure $\widehat{T}^\sharp$ in such a way by ensuring the added strands form pairs of product loops. This is equivalent to closing up T to $\widehat{T}$ and appending two product loops which are glued up circles arising from arcs $[-\epsilon,\epsilon]\times\{2\text{ pts}\}$ in each of the s components of $[-\epsilon,\epsilon]\times F$. Let u_i be an arc in the glued-up surface $\overline{R}$ lying in the piece

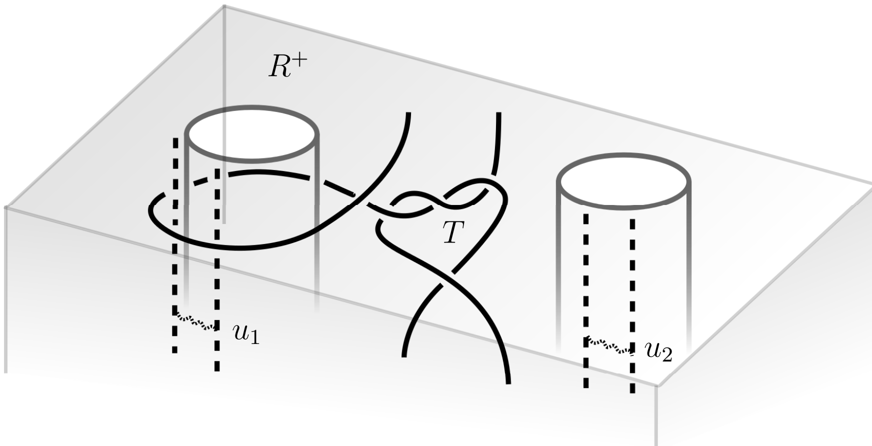

FIGURE 3.1. A schematic for the stablized tangle $T^\sharp$. The dashed components are the added product strands.

coming from F passing between the ith pair of added product components, and define:

$$u^\sharp := u_1 + \ldots + u_s.$$

DEFINITION 3.3.5. The "unreduced" instanton Floer homology of (M, T, γ) is the Floer homology

$$\mathrm{THI}^\sharp(M, T, \gamma) := I_*(\widehat{M}, \widehat{T}^\sharp, u^\sharp | \overline{R}) \tag{3.3.2}$$

Essentially the same proof as for $\mathrm{THI}_{\mathrm{odd}}$ shows the isomorphism type of this vectors space is independent of the identification of the top and bottom ends. The only hiccup is the case that T has an even number of strands, in which case we cannot use genus 0 excision to cut along S^2. To remedy this, we need to utilize our version of the genus 1 excision theorem for restricted Floer homology, Theorem 3.2.5.

PROPOSITION 3.3.6. *The isomorphism type of* $\mathrm{THI}^\sharp(M, T, \gamma)$ *is independent of the choices made in forming the closures* $\widehat{T}^\sharp$ *and* $\widehat{M}$.

PROOF. Let n be the number of strands of T. If n is odd, we are done by using genus 0 excision as before (since $T^\sharp$ has the same parity as T). Hence, we assume n is even, so that the 2-sphere $\overline{R}$ no longer intersects $\widehat{T}^\sharp$ in an odd number of points. Let S_1 be the torus boundary of a small tubular neighborhood of one of the added product components in $\widehat{T}^\sharp$. Let x_1, x_2, x_3 be three points in S^2, S_2 a torus containing $S^1 \times \{x_1\}$ in $Y_0 = S^1 \times S^2$, and u' an arc between x_1 and x_2 in S^2. Then we may perform a genus 1 excision surgery along S_1 and S_2 in the triples $(\widehat{M}, \widehat{T}^\sharp, u)$ and $(Y_0, S^1 \times \{x_1, x_2, x_3\}, u')$ to obtain the disjoint union

$$(\widehat{M}, \widehat{T}^\sharp \cup S^1 \times \mathrm{pt}, u^\sharp) \sqcup (Y_0, S^1 \times \{2 \text{ pts}\}, u'')$$

where u'' here denotes an arc between the two points. This surgery is entirely analogous to the point transfer construction used in §2.8; its effect is to add an additional product component T_0 to $\widehat{T}^\sharp$. Recall that we previously studied the Floer homogy of the triples $(Y_0, S^1 \times \{x_1, x_2, x_3\}, u')$ and $(Y_0, S^1 \times \{2 \text{ pts}\}, u'')$, which we called $\mathbb{U}_{0,3}$ and $\mathbb{U}_{0,2}$ and have restricted Floer homologies (with respect to the natural 2-spheres within them) are each of rank 1. Theorem 3.2.5 then guarantees an isomorphism

$$I_*(\widehat{M}, \widehat{T}^\sharp \cup T_0, u^\sharp | \overline{R}) \otimes \mathbb{C} \cong I_*(\widehat{M}, \widehat{T}^\sharp, u^\sharp | \overline{R}) \otimes \mathbb{C}. \tag{3.3.3}$$

Now, since $\widehat{T}^\sharp \cup T_0$ intersects $\overline{R}$ in an odd number of points, Theorem 3.2.4 applies as before. □

We can make one another special definition for the case of an even tangle whose properties we analyze in the next section. Let T be an even tangle in (M, γ) with $n = 2m$ strands, and let $\widehat{T}^\sharp$ be as before, obtained by adding two product loops for each suture. Group the strands of $\widehat{T}$ into pairs, and write

$$\underline{u} = v_1 + v_2 + \ldots + v_m + u^\sharp$$

where v_i is a strand connecting the ith pair of strands of $\widehat{T}$.

Definition 3.3.7. The *unreduced even instanton Floer homology* of (M, T, γ) is the vector space

$$\mathrm{THI}^\sharp_{\mathrm{even}}(M, T, \gamma) := I_*(\widehat{M}, \widehat{T}^\sharp, \underline{u} | \overline{R})$$

One may carry out an argument very similar to the proof of Proposition 3.3.6 to show that this invariant is independent of the gluing identification to form $\widehat{T}^\sharp$, using a point transfer cobordism to add a strand in order to have an odd-stranded link. It is also easy to see that it does not matter how we join the strands of T by the arcs v_i, as the resulting class in $H_1(\overline{R}, \partial(\overline{R} \setminus \nu T); \mathbb{Z}/2)$ (for νT a tubular neighborhood of T) is independent of that choice. In fact, it turns out that in the even case we have really just redefined the unreduced Floer homology of the tangle T.

Proposition 3.3.8. *There is an isomorphism*

$$\mathrm{THI}^\sharp_{\mathrm{even}}(M, T, \gamma) \cong \mathrm{THI}^\sharp(M, T, \gamma). \tag{3.3.4}$$

Proof. Let $n = 2m$ be the number of strands of T and s the number of boundary components of R^+, so that $T^\sharp$ has $n + 2s$ strands. We will utilize both genus 1 and genus 0 excision theorems. Using a point transfer cobordism (via genus 1 excision) as in the proof of Proposition 3.3.6, we know that

$$I_*(\widehat{M}, \widehat{T}^\sharp \cup T_0, \underline{u}|\overline{R}) \cong I_*(\widehat{M}, \widehat{T}^\sharp, \underline{u}|\overline{R})$$

where T_0 is an extra product component. Now, we consider the Floer homology

$$\mathbb{W}_{0,n+2s+1} := I_*(S^1 \times S^2, S^1 \times \{n + 2s + 1 \text{ pts}\}, \underline{u}^{(n)})$$

where $\underline{u}^{(n)}$ denotes a sum of arcs connecting n of the components of the product link of $n + 2s + 1$ components in $S^1 \times S^2$ pairwise. It is easy to see, via genus 0 excision, that $\mathbb{W}_{0,n+2s+1}$ is isomorphic to the vector space $\mathbb{U}_{0,n+2s+1}$ defined in Chapter 2:

$$\mathbb{W}_{0,n+2s+1} \cong \mathbb{U}_{0,n+2s+1}$$

via an isomorphism intertwining the operators $\mu^{\mathrm{orb}}(\mathrm{pt} \times S^2)$ on each side. Critically, the top eigenvalues of $\mu^{\mathrm{orb}}(\mathrm{pt} \times S^2)$ on $\mathbb{W}_{0,n+2s+1}$ are $\pm 4(n + 2s - 1)$, each with one-dimensional generalized eigenspace. Hence, we may perform a genus 0 excision on the disjoint union of

$(S^1 \times S^2, S^1 \times \{n+2s+1 \text{ pts}\}, \underline{u}^{(n2m)})$ and $(\widehat{M}, \widehat{T}^\sharp \cup T_0, \underline{u})$, to recover the pair $(\widehat{M}, \widehat{T}^\sharp \cup T_0)$. The new Stiefel-Whitney 1-cycle can be arranged, by choosing the regluing maps properly, to consist of now *two* arcs connecting each pair of original link components of $\widehat{T}$ previously connected by $\underline{u}$ before, and a single arc connecting the added product components. The doubled-up arcs "cancel" in the sense that the corresponding bundle has zero Stiefel-Whitney class. Hence, genus 0 excision provides an isomorphism

$$I_*(\widehat{M}, \widehat{T}^\sharp \cup T_0, \underline{u}|\overline{R}) \cong I_*(\widehat{M}, \widehat{T}^\sharp \cup T_0, u^\sharp|\overline{R})$$

We may now do a point transfer to "subtract" the component T_0 to obtain:

$$I_*(\widehat{M}, \widehat{T}^\sharp, \underline{u}|\overline{R}) \cong I_*(\widehat{M}, \widehat{T}^\sharp, u^\sharp|\overline{R}).$$

This is exactly the desired isomorphism (3.3.4). □

We will reuse this "stacking" argument again later on. The utility of defining the even version $\mathrm{THI}^\sharp_{\mathrm{even}}$ will become apparent in the next section. Before moving on, we make one last note on the choices of the Stiefel-Whitney class used to define the various versions of THI. In the definition of $\mathrm{THI}_{\mathrm{odd}}$, we were able to use an empty Stiefel-Whitney 1-cycle to compute the restricted Floer homology of the closed-up tangle. However, we could have chosen to use a cycle w which is the closure of a product arc $[-\epsilon, \epsilon] \times \mathrm{pt}$ in $[-\epsilon, \epsilon] \times F$. We could have done the same for $\mathrm{THI}^\sharp$, defining $\mathrm{THI}^{\sharp,w}$.

Lemma 3.3.9. *There is an isomorphism*

$$\mathrm{THI}^{\sharp,w}(M, T, \gamma) \cong \mathrm{THI}^\sharp(M, T, \gamma). \tag{3.3.5}$$

Proof. We use the stacking technique of the proof of Proposition 3.3.8 to introduce the 1-cycle arc u between two strands by doing genus 0 excision and by leveraging our knowledge of the vector space $\mathbb{U}_{0,n}$. We then perform a genus 1 excision, with one torus S_1 the boundary of a small tubular neighborhood of one of the added product components which intersects u and the other torus S_2 a boundary of a small tubular neighborhood of the component

$S^1 \times \{x_1\}$ in the triple

$$(S^1 \times S^2, S^1 \times \{x_1, x_2\}, u + w)$$

where u is an arc between x_1 and x_2 and w is $S^1 \times \mathrm{pt}$. If we arrange this w to lie close to $S^1 \times \mathrm{pt}$ so that it is within S_2, then doing an excision which glues the "inside" of S_2 to the inside of S_1 introduces a second copy of w in $(\widehat{M}, \widehat{T}^\sharp)$. Now, the triple $(S^1 \times S^2, S^1 \times \{x_1, x_2\}, u + w)$ is easily seen to give a vector space isomorphic to $\mathbb{U}_{0,2}$ and so $\mu^{\mathrm{orb}}(\mathrm{pt} \times S^2)$ vanishes on it. This implies the desired isomorphism. □

3.4. Properties and Applications

We now record some properties the various versions of instanton Floer homology for tangles enjoy. Again we assume our sutured manifolds are balanced and of type 0.

Concatentation. The first obvious consequence of the definition is multiplicativity with respect to vertical concatenation.

Proposition 3.4.1. *For $i = 1, 2$, let T_i be an n-stranded balanced tangle in the sutured manifold (M_i, γ_i), and suppose that for the top and bottom boundary pieces $R_i^\pm$ of ∂M_i for $i = 1$ have the same number of boundary components as for $i = 2$. Denote by $(M_1 \circ M_2, T_1 \circ T_2, \gamma_1 \circ \gamma_2)$ the sutured manifold with n-stranded tangle obtained by vertically stacking (M_i, T_i), using any identification of the bottom part of (M_1, T_1) and the top part of (M_2, T_2). Then we have*

$$\mathrm{THI}^\sharp(M_1 \circ M_2, T_1 \circ T_2, \gamma_1 \circ \gamma_2) \cong \mathrm{THI}^\sharp(M_1, T_1, \gamma_1) \otimes \mathrm{THI}^\sharp(M_2, T_2, \gamma_2) \tag{3.4.1}$$

Proof. It is clear that if $\widehat{T}_i^\sharp$ is the closed link in $\widehat{M}_i$ obtained by adding product strands to and closing up T_i, and likewise for $\widehat{T_1 \circ T_2}^\sharp$ from $T_1 \circ T_2$ in $\widehat{M_1 \circ M_2}$, then the triple

$$(\widehat{M_1 \circ M_2}, \widehat{T_1 \circ T_2}^\sharp, \emptyset). \tag{3.4.2}$$

can be obtained by an excision surgery along copies of $\Sigma = \mathrm{pt} \times S^2$ in the disjoint union

$$(\widehat{M}_1, \widehat{T}_1{}^\sharp, u^\sharp) \sqcup (\widehat{M}_2, \widehat{T}_1{}^\sharp, u^\sharp).$$

The Stiefel-Whitney class vanishes because the two versions of $u^\sharp$ cancel. In the case that n is odd, this establishes

$$I_*(Y_0, \widehat{T_1 \circ T_2}{}^\sharp, \emptyset|\Sigma) \cong I_*(Y_0, \widehat{T}_1{}^\sharp, u|\Sigma) \otimes I_*(Y_0, \widehat{T}_2{}^\sharp, u|\Sigma).$$

Now, the vector space $\mathrm{THI}^\sharp(M_1 \circ M_2, T_1 \circ T_2, \gamma_1 \circ \gamma_2)$ arises from the slightly different triple

$$(\widehat{M_1 \circ M_2}, \widehat{T_1 \circ T_2}{}^\sharp, u). \tag{3.4.3}$$

However, the triples (3.4.2) and (3.4.3) give identical restricted Floer homologies by the stacking argument from the proof of Proposition 3.3.8. Now, in the case that n is even, we may argue as in the proof of Proposition 3.3.6 to add a product strand. From here the argument goes exactly as in the odd case. □

The purpose of the added product strands in $T\sharp$ is two-fold. First, they allow us to carry out the operation of adding a strand, and so make the proof of invariance and the above result go through. Secondly, however, they allow us to prove a horizontal concatenation result for tangle in the cylinder $[-1,1] \times D^2$. In what follows, whenever T is a balanced tangle in $[-1,1] \times D^2$ thought of as a sutured manifold with a single suture $[-1,1] \times S^1$, we will simply write:

$$\mathrm{THI}^\sharp(T) := \mathrm{THI}^\sharp([-1,1] \times D^2, T, [-1,1] \times S^1).$$

Proposition 3.4.2. *Suppose T_i is an n_i-stranded balanced tangle in $[-1,1] \times D^2$ for $i = 1, 2$, and denote by $T_1|T_2$ the tangle obtained by horizontal juxtaposition of T_1 and T_2, which is a balanced tangle in $[-1,1] \times D^2$ with $N = n_1 + n_2$ strands. Then there is an isomorphism*

$$\mathrm{THI}^\sharp(T_1|T_2) \cong \mathrm{THI}^\sharp(T_1) \otimes \mathrm{THI}^\sharp(T_2) \tag{3.4.4}$$

PROOF. Let $\widehat{T_i}^\sharp$, $\widehat{T_1|T_2}^\sharp$ denote the closed up and appended versions of the tangles T_i and $T_1|T_2$. Let the added product components in $\widehat{T_i}^\sharp$ be s_i and t_i, u_i an arc between them, and S_i the boundary torus of a small tubular neighborhood of s_i. Then S_i is a non-integral surface and we may apply Theorem 2.5.1 to the excision surgery along these tori. The result is:

$$I_*(Y_0, \widehat{T_1}^\sharp, u_1)_{\mathrm{pt},-8} \otimes I_*(Y_0, \widehat{T_2}^\sharp, u_2)_{\mathrm{pt},-8} \cong I_*(Y_0, \widehat{T_1|T_2}^\sharp, u_{12})_{\mathrm{pt},-8} \otimes \mathbb{U}_{0,2} \tag{3.4.5}$$

where u_{12} is an arc between the product strands in $\widehat{T_1|T_2}^\sharp$ PICTURE. Denote by Σ the 2-sphere $\mathrm{pt} \times S^2$ in Y_0. As in the point transfer argument in the proof of Proposition 3.3.6, this isomorphism intertwines the operators

$$\mu^{\mathrm{orb}}(\Sigma) \otimes 1 + 1 \otimes \mu^{\mathrm{orb}}(\Sigma)$$

on each side, and since the top eigenvalue of interest on the right hand side is the sum of those on the left, we obtain the desired isomorphism (3.4.4) of restricted Floer homologies. □

A Non-triviality Result. So far, we have no evidence to suggest that either of our versions of THI give anything other than a rank 1 vector space as it does in the product case. We desire a result similar to the non-triviality results in [**22**], specifically Theorem 7.18. In other words, we would like to know whether THI detects the product tangle. However, as an analogue to the *taut* condition used in [**22**], we will need our tangles to satisfy an additional, easy checked hypothesis.

DEFINITION 3.4.3. The balanced tangle T in (M, γ) is said to be *vertical* if

(i) T has no closed components, and
(ii) each endpoint of T in R^+ is connected by a component of T to an endpoint of T in R^-.

For a balanced vertical tangle T, there are natural bijections between endpoint sets

$$f_T : R^+ \cap T \to R^- \cap T$$

$$f_T^\sharp : R^+ \cap T^\sharp \to R^- \cap T^\sharp$$

defined by following the strands of the tangle.

Remark. Perhaps a better definition than our notion of *vertical* is that of an *oriented* tangle. Such a tangle should be an oriented, properly embedded 1-manifold with boundary in (M, γ) such that the positive endpoints (using the orientation) lie on R^+ and the negative lie on R^-. This ensures the bijection f_T above exists. However, this does not preclude closed components. Hence, a vertical tangle is just an oriented tangle without closed components. The author suspects that neither hypothesis is necessary in the following result, but is unable to find an argument to this effect.

Theorem 3.4.4. *Let T be a balanced, vertical tangle with n strands in the connected sutured manifold (M, γ) of type 0. Suppose that (M, γ) is taut (as defined in [**22**]. Then $\mathrm{THI}^\sharp(M, T, \gamma) \cong \mathbb{C}$ if and only if (M, T, γ) is the product tangle in the product sutured manifold $[-1, 1] \times D^2$.*

Certainly if T is the product tangle and (M, γ) is the standard product sutured manifold $[-1, 1] \times D^2$ then $\mathrm{THI}^\sharp(M, T, \gamma) \cong \mathbb{C}$. The difficult direction is the converse. We will attack this by leveraging the analogous result in [**22**] and by using the even version $\mathrm{THI}^\sharp_{\text{even}}$. The idea is that $\mathrm{THI}^\sharp_{\text{even}}$ can be compared to the sutured instanton Floer homology of a derived 3-manifold with no singular locus, as we now explain. Suppose T is vertical with $n = 2m$ strands and γ has s loops. Number the strands of $T^\sharp$, denoting the ith original strand by T_i and letting T_{n+2i-1} and T_{n+2i} be the pair of added strands coming from the ith suture of γ. Let S_i be the boundary annulus of a small tubular neighborhood of T_i. If we pair up the S_i's in some way and cut and reglue M along them, we obtain a 3-manifold $\widetilde{M_T}$ with boundary with the natural structure of a sutured manifold with positive and negative pieces $\widetilde{R}_T^+$ and $\widetilde{R}_T^-$ surfaces of genus $m + s + 1$ with the same number s as before of boundary components (see Figure 3.4. Closing this manifold up by using a disjoint collection of disks F as auxilliary surface as before is equivalent to removing $[-1, 1] \times D^2$-neighborhoods of the tangle strands from $T^\sharp$ and placing a single, properly oriented suture on each new annular piece of ∂M, to arrive at a derived sutured manifold (M_T, γ_T), and constructing a closure

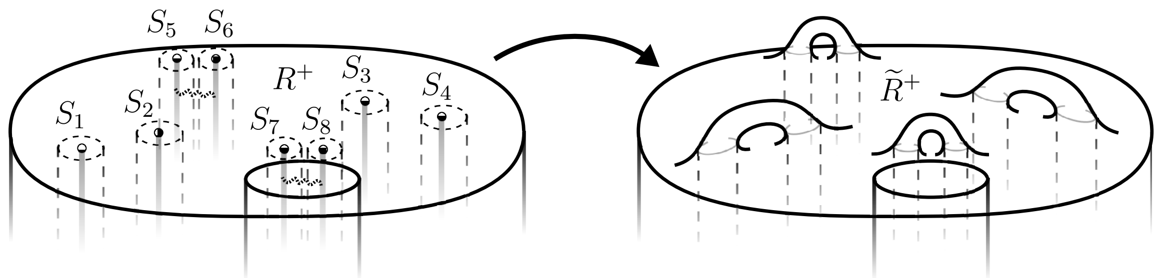

FIGURE 3.2. Cutting and regluing M to obtain $\widetilde{M_T}$. The example shown here begins with a tangle T wit 4 strands to which 4 strands are added to obtain $T^\sharp$, and the surface R^+ becomes a twice punctured genus 4 surface.

of this sutured manifold by using a different auxilliary surface which is a union of $m+s$ annuli and s disks. This process plays well with genus 1 excision, as we see in the proof of the following result.

PROPOSITION 3.4.5. *For an even, balanced, vertical tangle T with $n = 2m$ strands in (M,γ), we have an isomorphism:*

$$\mathrm{THI}^\sharp_{\text{even}}(M,T,\gamma) \cong \mathrm{SHI}(M_T,\gamma_T) \tag{3.4.6}$$

PROOF. Form the closures $\widehat{M}$ and $\widehat{T}^\sharp$ in such a way that the tangle $T^\sharp$ is closed up via the bijection $f^\sharp_T$. We assume the annuli S_i are glued up to tori $\overline{S}_i$, each of which are odd for the Stiefel-Whitney class $\underline{u}$. Let w denote a product loop in both of $\widehat{M}$ and $\widehat{M}_T$ as in Lemma 3.3.9, which tells us that we can compute $\mathrm{THI}^{\text{even}}(M,T,\gamma)$ by using the Stiefel-Whitney class $\underline{u}+w$. We can perform $m+s$ excision operations on $(\widehat{M},\widehat{T}^\sharp,\underline{u}+w)$ by taking the $\overline{S}_i$'s in pairs, making sure to pair up $\overline{S}_{n+2i-1}$ with $\overline{S}_{n+2i}$. The result is a sequence of 3-manifold triples

$$(\widehat{M},\widehat{T}^\sharp,\underline{u}+w) = (\widehat{M}_0,\widehat{T}^\sharp_0,\underline{u}_0+w),\ (\widehat{M}_1,\widehat{T}^\sharp_1,\underline{u}_1+w),\ \ldots,\ (\widehat{M}_{m+1},\widehat{T}^\sharp_{m+s},\underline{u}_{m+s}+w)$$

as well as a sequence of versions $\overline{R}_i$ of the surface $\overline{R}$. Theorem 3.2.5 then guarantees isomorphisms

$$I_*(\widehat{M}_i,\widehat{T}_i,\underline{u}_i+w|\overline{R}_i) \cong I_*(\widehat{M}_{i+1},\widehat{T}_{i+1},\underline{u}_{i+1}+w\overline{R}_{i+1}). \tag{3.4.7}$$

In reality, each time we perform a genus 1 excision, we introduce a new component of the 3-manifold isomorphic to the triple $(S^1 \times S^2, S^1 \times \{2 \text{ pts}\}, u)$ for u an arc between the two points. However, the operator $\mu(\text{pt} \times S^2)$ vanishes on its instanton Floer homology so this does not affect the argument. We conclude that there is an isomorphism

$$I_*(\widehat{M}, \widehat{T}, \underline{u} + w|\overline{R}) \cong I_*(\widehat{M}_{m+s}, \emptyset, \underline{u}_{m+s} + w|\overline{R}_{m+s}) \tag{3.4.8}$$

Now, as it stands the final manifold $\widehat{M}_{m+s}$ is isomorphic to a sutured manifold closure of (M_T, γ_T) obtained using an auxilliary surface G with m annulus components and s singly-punctured torus components PICTURE. The definition of SHI requires that we use a connected surface, so we need to go a bit further. We want to "join" up the components of G via genus 1 excision, but first we need to increase the genus of each annular component. Along each of the m new handles (corresponding to annular components of G) of the final surface $\overline{R}_{m+s}$ runs a loop component of u_{m+s}. Fixing one such, there is a perpendicular loop c and a torus $S^1 \times c$ which is odd for u_{m+s}. We will use the genus-increasing construction from [**22**], which leverages genus 1 excision. We take the triple $(S^1 \times \Sigma_2, \emptyset, u)$ with u a small loop in the genus 2 surface Σ_2, and let d be a complementary loop to u. Performing excision between $(\widehat{M}_{m+s}, \emptyset, \underline{u}_{m+s} + w)$ and this triple along $S^1 \times c$ and $S^1 \times d$ effectively increases the genus of the annular component of G running between the two joined pieces of boundary to a punctured torus. We repeat this for each of the s annular handles, obtaining a sequence of triples

$$(\widehat{M}_{m+s}, \emptyset, \underline{u}_{m+s} + w), (\widehat{M}^1_{m+s}, \emptyset, \underline{u}^1_{m+s} + w), \ldots, (\widehat{M}^s_{m+s}, \emptyset, \underline{u}^s_{m+s} + w). \tag{3.4.9}$$

The final triple is equivalent to closing up the sutured manifold (M_T, γ_T) using an auxilliary surface G' with s singly-punctured tori and m doubly-punctured tori. Each of these $m + s$ pieces have natural non-separating complementary loops d_i which yield tori $S^1 \times d_i$ odd for $\underline{u}^s_{m+s} + w$. By performing excision surgeries along these in sequence it is possible to join up the components of G' to form an auxilliary surface G'' which is an $2m + s$-punctured torus PICTURE. Each time we perform one of these "vertical" excision operations, it is easy to check as for (3.4.8) that the isomorphism type of the restricted Floer homology with respect to the relevant version of the surface $\overline{R}$ is unchanged. The final 3-manifold triple is isomorphic to a closure of (M_T, γ_T) using a connected, punctured genus 1 auxilliary surface G'', with a Stiefel-Whitney 1-cycle given by w plus a collection of loops $\underline{u}''$ in the closed up surface $\overline{R}''$, the result of $\overline{R}$ after performing all the excisions surgeries. To complete the proof, we need to demonstrate an isomorphism for the restricted Floer homologies with different Stiefel-Whitney 1-cycles:

$$I_*(\widehat{M_T}, \emptyset, \underline{u}'' + w|\overline{R}'') \cong I_*(\widehat{M_T}, \emptyset, w|\overline{R}'').$$

this isomorphism essentially appears in the discussion immediately preceding §7.5 of [**22**] and follows from a stacking argument similar to the proof of Proposition 3.3.8. Upon invoking Lemma 3.3.9 in order to remove the class w, we obtain the isomorphism desired in the Proposition. □

It is now a simple matter to prove Theorem 3.4.4.

Proof. One direction is clear so we assume that $\mathrm{THI}^\sharp(M, T, \gamma)$ is rank 1. Then by Proposition 3.4.5

$$\mathrm{THI}^\sharp(M_T, \gamma_T) \cong \mathrm{THI}^\sharp_{\mathrm{even}}(M_T, \gamma_T) \cong \mathbb{C},$$

(here we use equality of the even and and regular unreduced versions of THI). Now, it is clear that if $M, \gamma)$ is taut then so is (M_T, γ_T) and thus by Theorem 7.18 of [**22**] the sutured manifold (M_T, γ_T) is a product. But this clearly implies that (M, T, γ) is a product. □

APPENDIX A

Euler Numbers, Orthogonal Polynomials, and Continued Fractions

We owe the reader a discussion of how to arrive at the recursive relation (1.6.1) for the relations in the cohomology ring, given that the top pairings of the generators α and β are the Euler numbers E_n. This requires a brief digression on orthogonal polynomials, and an analysis of the ordinary generating function for the numbers E_n.

Orthogonal Polynomials. We begin by supposing we have a measure μ on the interval $[a,b] \subset \mathbb{R}$, which we suppose for simplicity is given by integrating against a continuous, nonnegative weighting function $w(x)$:

$$\int_a^b f(x)d\mu = \int_a^b f(x)w(x)dx.$$

This measure provides a linear functional $\mathcal{L}_\mu$, as well as an inner product and norm on the set of μ-integrable functions on $[a,b]$:

$$\begin{aligned}\mathcal{L}_\mu(f) &:= \int_a^b f(x)d\mu \\ \langle f,g\rangle_\mu &:= \int_a^b f(x)g(x)d\mu \\ \|f\|_\mu &:= \Big(\langle f,f\rangle_\mu\Big)^{\frac{1}{2}}\end{aligned}$$

Definition A.1. The sequence of numbers

$$c_n = \mathcal{L}(x^n) = \int_a^b x^n d\mu$$

is called the sequence of *moments* for the measure μ.

Given μ, a useful collection of data is its sequence of orthogonal polynomials, which provide a convenient basis for the set of integrable functions on $[a,b]$.

Definition A.2. A sequence of *monic orthogonal polynomials* for the measure μ is a sequence of polynomials $\{p_n(x)\}$ satisfying

(i) p_n is monic and $\deg(p_n) = n$

(ii) $\langle p_n, p_m \rangle_\mu = 0$ for $n \neq m$ and $\|p_n\|_\mu > 0$ for all n.

It is very easy to prove:

Lemma A.3. *Given a weighting function $w(x)$ and associated measure μ, a sequence of monic orthogonal polynomials $\{p_n(x)\}$ exists and is unique.*

We will thus refer to *the* sequence of monic orthogonal polynomials for a given measure μ. The theory of orthogonal polynomials is old and well-understood. One has the following famous result, known as the three-term recurrence:

Theorem A.4. *([**2**], Theorem 5) For the measure μ, the monic orthogonal polynomials $p_n(x)$ satisfy the following recurrence relation:*

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x) \tag{A.1}$$

where we initialize $p_0(x) = 1$, $p_{-1}(x) = 0$, and where α_n and β_n are constants depending on μ.

In fact, the numbers α_n and β_n can be computed as follows:

$$\alpha_n = \frac{\langle x p_n(x), p_n(x) \rangle_\mu}{\|p_n(x)\|_\mu}, \quad \beta_n = \frac{\|p_n(x)\|_\mu}{\|p_{n-1}(x)\|_\mu}$$

There is a fascinating relationship between the collection of moments $\{c_n\}$ and the coefficients α_n and β_n.

Theorem A.5. *Let $F_\mu(x)$ be the ordinary generating function for the moments $c_n = \mathcal{L}_\mu(x^n)$:*

$$F_\mu(x) = \sum_{n=0}^{\infty} c_n x^n$$

Then the coefficients α_n and β_n from (A.1) and $F_\mu(x)$ satisfy the continued fraction identity:

$$F_\mu(x) = \cfrac{c_0}{1-\alpha_0 x - \cfrac{\beta_1 x^2}{1-\alpha_1 x - \cfrac{\beta_2 x^2}{1-\alpha_2 x - \cfrac{\beta_3 x^2}{1-\alpha_3 x - \cdots}}}} \tag{A.2}$$

The Euler Numbers. The numbers E_n are defined via:

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n. \tag{A.3}$$

Since $\operatorname{sech}(z)$ is an even function, we have $E_n = 0$ for n odd. The sequence has first few terms (beginning with the 0th term):

$$\{E_n\} = 1, 0, -1, 0, 5, 0, -61, 0, 1385, \ldots$$

where the nonzero entries are alternatively positive and negative. It is shown in [**10**] that the ordinary generating function for the sequence $\{|E_{2n}|\}$ of absolute values of nonzero Euler numbers has the following remarkable continued fraction expansion:

$$E(z) = \sum_{n=0}^{\infty} |E_{2n}|\, z^{2n} = \cfrac{1}{1 - \cfrac{1^2 z^2}{1 - \cfrac{2^2 z^2}{1 - \cfrac{3^2 z^2}{1 - \ldots}}}} \tag{A.4}$$

Here we must use the absolute value sign on the Euler numbers to agree with the conventions of [**10**].

We now tackle our main problem, which is understanding the relations in the cohomology ring of the moduli space $\mathcal{R}_{0,2m+1}$ from Chapter 1. There, we showed that there is a relation $r_{0,2m+1}(\alpha, \beta)$ in the generators α and β which is of degree m in α and $\lfloor m/2 \rfloor$ in β. Set

$e = m - 2\lfloor m/2 \rfloor$, the remainder 0,1 of m when divided by 2. As in the proof of Proposition 1.6.1, if we write

$$r_{0,2m+1}(\alpha, \beta) = A_m \alpha^m + A_{m-2}\alpha^{m-2}\beta + \ldots + A_e \alpha^e \beta^{\lfloor m/2 \rfloor}$$

then the $(\lfloor m/2 \rfloor + 1)$-vector $(A_m, A_{m-2}, \ldots, A_e)$ is in the kernel of the matrix E_{ij} for $i = 0, \ldots, \lfloor m/2 \rfloor - 1$ and $j = 0, \ldots, \lfloor m/2 \rfloor$ with

$$E_{ij} = |E_{2e+2i+2j}| \,. \tag{A.5}$$

Let us take this as a definition of $r_{0,2m+1}(\alpha, \beta)$, under the additional constraint that we take $r_{0,2m+1}(\alpha, \beta)$ to be monic in α.

Theorem A.6. *The polynomials $r_{0,2m+1}(\alpha, \beta)$ satisfy the recurrence (1.6.1).*

Proof. Suppose that μ is a measure on an interval $[a, b]$ whose moments are given by

$$\mathcal{L}_\mu(x^n) = |E_n| \,.$$

Define the de-homogenized single-variable polynomial:

$$s_m(x) := r_{0,2m+1}(x, 1).$$

Evaluating the matrix E_{ij} on the $(\lfloor m/2 \rfloor + 1)$-vector $(A_m, A_{m-2}, \ldots, A_e)$ gives the $\lfloor m/2 \rfloor$-vector

$$\left(\langle x^e, s_m(x) \rangle_\mu , \langle x^{e+2}, s_m(x) \rangle_\mu , \ldots , \langle x^{m-2}, s_m(x) \rangle_\mu \right),$$

which we assume vanishes. Hence, the pairing of $s_m(x)$ (which is degree m in x) with any polynomial in x of lower degree having only terms with the same degree parity is zero. Of course, pairing with monomials with the opposite parity gives 0 as well. This implies $s_m(x)$ is orthogonal to $s_k(x)$ for $k < m$, and so by induction the polynomial sequence $\{s_m(x)\}$ is monic orthogonal for μ. By Theorem A.5, we see that $s_m(x)$ satisfies the recurrence

$$s_{m+1}(x) = (x - \alpha_m) s_m(x) - \beta_m s_{m-1}(x)$$

where α_m and β_m are the coeffiecients in the continued fraction A.2. But by definition, the generating function to use is given by $E(z)$ as in (A.4), and so we see that $\alpha_m = 0$ and $\beta_m = -m^2$. Hence:

$$s_{m+1}(x) = xs_m(x) - m^2 s_{m-1}(x),$$

and since $r_{0,2m+1}(\alpha, \beta)$ is homogeneous in α and β (with these variables assigned degrees 2 and 4, respectively), we obtain the recursion (1.6.1). $\square$

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