Dynamics on Algebraic Surfaces

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MPI Arbeitstagung 2007

Curtis T. McMullen

In this talk we discuss connections between algebraic integers and automorphisms of compact complex surfaces.

Integers. Conjecturally, the smallest algebraic integer $\lambda > 1$ is the root $\lambda_{\text{Lehmer}} = 1.1762808\ldots$ of *Lehmer's polynomial*,

$$P(x) = 1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}.$$

By *smallest* we mean the Mahler measure, given by the product of the conjugates of λ outside the unit disk, is minimized. In fact λ_{Lehmer} is a *Salem number*: it is the unique root of $P(x)$ outside the disk, so it is its own Mahler measure.

Lehmer's polynomial has several geometric manifestations; especially, it is the characteristic polynomial of the Coxeter element for the Weyl group W_{10} with Coxeter diagram E_{10}:

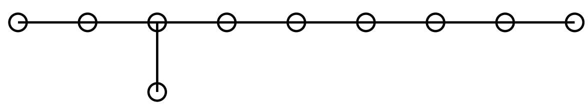

Figure 1. The E_{10} Coxeter graph.

In [Mc1] we use the Hilbert metric and the enumeration of minimal hyperbolic Coxeter diagrams to show that Lehmer's number has the minimal Mahler measure for algebraic integers arising from reflections groups.

Theorem 1 *The spectral radius of any element w in a Coxeter reflection group satisfies $\rho(w) = 1$ or $\rho(w) \geq \lambda_{\text{Lehmer}}$.*

An earlier result on pretzel knots by E. Hironaka motivated the theorem above [Hir].

K3 surfaces. We now turn to the problem of constructing dynamically interesting automorphisms $f : X \to X$ of compact complex surfaces.

The log of the spectral radius of the action of f^* on $H^*(X, \mathbb{Z})$ agrees with the *topological entropy* $h(f)$ and is a measure of its dynamical complexity. By a theorem of Cantat, if $h(f) > 0$ then (up to a 2-fold cover) X must be birational to a complex torus, a K3 surface or the projective plane.

The automorphisms of $\mathbb{P}^2$ and of $\mathbb{C}^2/\Lambda$ are essentially linear, but automorphisms of K3 surfaces can be more complicated: over the reals, they exhibit all the rich behavior of a generic area-preserving map. KAM theory shows $f : X(\mathbb{R}) \to X(\mathbb{R})$ can have orbits trapped in elliptic islands. Can similar invariant islands exist for the complex points of X?

The answer is yes. Let us say $U \subset X$ is a *Siegel disk* if $f|U$ is holomorphically conjugate to an irrational rotation. In [Mc2] we show:

Theorem 2 *There exists K3 surface automorphisms of positive entropy with Siegel disks.*

Such automorphisms can be *synthesized* from suitable Salem polynomials $S(x)$ of degree 22. One begins by constructing an automorphism f^* of the even unimodular lattice L of type $(3, 19)$, such that $S(x) = \det(xI - f^*|L)$ [GM]. Note that L is isomorphic to the middle-dimensional cohomology $H^2(X, \mathbb{Z})$ of a K3 surface. Choosing an f^*-invariant Hodge structure and applying the Torelli theorem, one obtains a K3 surface X and an automorphism $f : X \to X$ realizing f^*. Theorems of Lefschetz and Atiyah-Bott then allow one to show that f has a unique fixed point $p \in X$, and that its derivative Df_p is an irrational rotation. Finally results from transcendence theory and analytic dynamics show f (near p) is conjugate to its linear part Df_p, providing the desired Siegel disk.

Unfortunately, these Siegel disks are invisible to us: they live on *non-projective* K3 surfaces, and we can only detect them implicitly, through Hodge theory and cohomology.

What about the projective case? The answer is not known, but it is possible that every positive entropy automorphism of a *projective* K3 surface has a dense orbit.

Rational surfaces. While $\mathbb{P}^2$ has no interesting automorphisms, Cantat's theorem actually admits the possibility of interesting dynamics on *blowups* of the projective plane at n points.

In this case $H^2(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{1,n}$ with the Minkowski intersection form. The sublattice orthogonal to the canonical class $k_n = (-3, 1, 1, \ldots, 1)$ is a copy of the E_n lattice, and we obtain a natural action of the Weyl group W_n on $H^2(X, \mathbb{Z})$. By a theorem of Nagata, we have $f^* \in W_n$ for any $f \in \mathrm{Aut}(X)$. Using Theorem 1 on Coxeter groups, we can conclude:

Theorem 3 *If $f : X \to X$ is a positive entropy automorphism of a compact complex surface, then $h(f) \geq \log \lambda_{\mathrm{Lehmer}}$.*

Realization. One can then ask: which elements $w \in W_n$ can be *realized* by an automorphism of X, if the n points to be blown up are suitably chosen? In [Mc3] we use configurations of points along a cuspidal cubic to show:

Theorem 4 *For $n \geq 10$, every Coxeter element $w_n \in W_n$ can be realized by a rational surface automorphism f_n with entropy $h(f_n) = \log \rho(w_n) > 0$.*

(For $n \leq 9$, all automorphisms have zero entropy.)

Since $\rho(w_n) = \lambda_{\text{Lehmer}}$ for $n = 10$, we find:

Corollary 1 *The map f_{10} is a surface automorphism with the smallest possible positive entropy.*

This automorphism first appears in the Appendix to [BK1].

In fact we obtain a different automorphism for each of the algebraic conjugates of λ_n; using conjugates on the unit circle, we finally obtain examples of invariant islands for automorphisms of projective surfaces.

Corollary 2 *There are infinitely many rational surfaces admitting automorphisms of positive entropy with Siegel disks.*

Invariant curves. Since we blow up points on a cubic curve in the constructions above, we obtain an effective anticanonical divisor $Y \subset X$, which is invariant under f. The space $X - Y$ behaves like an open K3 surface, providing a bridge between these two types of examples. Similar considerations apply to the blowup of the 10 double points of a generic rational plane sextic, studied classically by Coble [Cob].

Very recently Bedford and Kim have shown there are rational surface automorphisms of positive entropy with *no* invariant curves [BK2]. It is a challenge to construct such examples synthetically.

Note. The papers [Mc1], [Mc2], [GM] and [Mc3] contain more details and references, and are available at `http://math.harvard.edu/~ctm/papers`.

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Mathematics Department, Harvard University, Cambridge, MA