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Citation

Elkies, Noam D. 2006. Points of low height on elliptic curves and surfaces I: Elliptic surfaces over P^1 with small d . Lecture Notes in Computer Science 4076: 287-301.

Published Version

http://dx.doi.org/10.1007/11792086_21

Permanent link

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Points of low height on elliptic curves and surfaces

I: Elliptic surfaces over $\mathbf{P}^1$ with small d

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Abstract. For each of $n = 1, 2, 3$ we find the minimal height $\hat{h}(P)$ of a nontorsion point P of an elliptic curve E over $\mathbf{C}(T)$ of discriminant degree $d = 12n$ (equivalently, of arithmetic genus n), and exhibit all (E, P) attaining this minimum. The minimal $\hat{h}(P)$ was known to equal $1/30$ for $n = 1$ (Oguiso-Shioda) and $11/420$ for $n = 2$ (Nishiyama), but the formulas for the general (E, P) were not known, nor was the fact that these are also the minima for an elliptic curve of discriminant degree $12n$ over a function field of any genus. For $n = 3$ both the minimal height ($23/840$) and the explicit curves are new. These (E, P) also have the property that that mP is an integral point (a point of naïve height zero) for each $m = 1, 2, \dots, M$, where $M = 6, 8, 9$ for $n = 1, 2, 3$; this, too, is maximal in each of the three cases.

1. Introduction.

1.1 Statement of results. Let K be a function field of a curve C of genus g over a field k of characteristic zero,¹ and E a nonconstant elliptic curve over K . Let d be the degree of the discriminant of E (considered as a divisor on C), a natural measure of the complexity of E ; and let $\hat{h} : E(K) \rightarrow \mathbf{Q}$ be the canonical height. Necessarily $12|d$; in fact it is known that $d = 12n$ where n is the arithmetic genus of the elliptic surface $\mathcal{E}$ associated with E . It is not hard to show that, given d , the set of numbers H that can occur as the canonical height of a rational point on E is discrete. In particular, for each $d = 12n$ there is a minimal positive height $\hat{h}_{\min}(d)$, and also a minimal positive height $\hat{h}_{\min}(g, d)$ for elliptic curves over function fields of genus g (except for $g = d = 0$, when E is a constant curve over $\mathbf{P}^1$ and thus has no points of positive height). It is thus a natural problem to compute or estimate these numbers $\hat{h}_{\min}(d)$ and $\hat{h}_{\min}(g, d)$. This paper is the first of a series concerned with different aspects of this problem.

In this paper we determine $\hat{h}_{\min}(12n)$ for $n = 1, 2$ and $\hat{h}_{\min}(0, 12n)$ for $n = 1, 2, 3$. Since we are working in characteristic zero, we may assume $k = \mathbf{C}$, when every genus-zero curve is isomorphic to $\mathbf{P}^1$ and its function field is isomorphic to $\mathbf{C}(T)$.

¹ One can also usefully define the canonical height etc. in positive characteristic, but we need to use the ABC conjecture for K and thus must assume that K has characteristic zero.

Theorem 1. *i) (Oguiso-Shioda [7]) $\hat{h}_{\min}(0, 12) = 1/30$.
ii) $\hat{h}_{\min}(12) = 1/30$. Moreover, let E be an elliptic curve with $d = 12$ over a complex function field K , and $P \in E(K)$. Then the following are equivalent:
(a) $\hat{h}(P) = 1/30$; (b) Each of $P, 2P, 3P, 4P, 5P, 6P$ is an integral point on E ;
(c) $K \cong \mathbf{C}(T)$, and (E, P) is equivalent to the curve*

$$E_1(q) : Y^2 + (s' - (q+1)s)XY + qss'(s-s')Y = X^3 - qss'X^2 \quad (1)$$

over the $(s : s')$ line with the rational point $P : (X, Y) = (0, 0)$, for some $q \in \mathbf{C}$ other than 0 or 1.

Theorem 2. *i) (Nishiyama [6]) $\hat{h}_{\min}(0, 24) = 11/420$.
ii) $\hat{h}_{\min}(24) = 11/420$. Moreover, let E be an elliptic curve with $d = 24$ over a complex function field K , and $P \in E(K)$. Then the following are equivalent:
(a) $\hat{h}(P) = 11/420$; (b) mP is an integral point on E for each $m = 1, 2, \dots, 8$;
(c) $K \cong \mathbf{C}(T)$, and (E, P) is equivalent to the curve*

$$\begin{aligned} E_2(u) : Y^2 + (r^2 - r'^2 + (u-2)rr')XY \\ - r^2r'(r+r')(r+ur')(r+(u-1)r')Y \\ = X^3 - rr'(r+r')(r+ur')X^2 \end{aligned} \quad (2)$$

over the $(r : r')$ line with the rational point $P : (X, Y) = (0, 0)$, for some $u \in \mathbf{C}$ other than 0, 1.

Theorem 3. *i) $\hat{h}_{\min}(0, 36) = 23/840$.
ii) Let $E/\mathbf{C}(T)$ be an elliptic curve with $d = 36$, and P a rational point on E . Then the following are equivalent: (a) $\hat{h}(P) = 23/840$; (b) mP is an integral point on E for each $m = 1, 2, \dots, 9$; (c) (E, P) is equivalent to the curve*

$$\begin{aligned} E_3(A) : Y^2 + (At^3 + (1-2A)t^2t' - (A+1)tt'^2 - t'^3)XY \\ - t^3t'(t+t')(At+t')(At+(1-A)t')(At^2+tt'+t'^2)Y \\ = X^3 - tt'(t+t')(At+t')(At^2+tt'+t'^2)Y \end{aligned} \quad (3)$$

over the $(t : t')$ line with the rational point $P : (X, Y) = (0, 0)$, for some $A \in \mathbf{C}$ other than 0, 1.

The values of $\hat{h}_{\min}(12)$ and $\hat{h}_{\min}(24)$ are new. Note that we do not claim to determine $\hat{h}_{\min}(36)$. As indicated, the values of $\hat{h}_{\min}(0, 12)$ and $\hat{h}_{\min}(0, 24)$ (the first parts of Theorems 1 and 2) were already known, but were obtained using techniques that are specific to the geometry of rational and K3 elliptic surfaces and do not readily generalize past $n = 2$. Our approach lets us treat all three cases uniformly, and in principle lets us determine $\hat{h}_{\min}(0, 12n)$ for any n , though the computations rapidly become infeasible as n grows beyond 3. The minimizing (E, P) had not been previously exhibited, except for a single case of a rational

elliptic surface with a section of height $1/30$ obtained by Shioda in a later paper [11], which we will identify with $E_1(4/5)$.

The connections with integral multiples of P (see statement (b) of part (ii) of each Theorem) are also new. We do not expect them to persist past $n = 3$, and in fact find that for $n = 4$ the largest number of consecutive integral multiples occurs for (E, P) with $\hat{h}(P) = 19/630$ or $13/360$, whereas $\hat{h}_{\min}(0, 48) \leq 41/1540 < 19/630 < 13/360$. We shall say more about integrality later; for now we content ourselves with the following remarks. A point on an elliptic curve over a function field $k(C)$ is said to be integral if it is a nonzero point whose naïve height vanishes. Geometrically, if we regard E as an elliptic surface $\mathcal{E}$ over C , and a rational point $P \in E(K)$ as a section s_P of $\mathcal{E}$, this means that s_P is disjoint from the zero-section s_0 of $\mathcal{E}$. Since $g = 0$ in our case, we can give an explicit algebraic characterization of integrality. Write E in extended Weierstrass form as

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6 \quad (4)$$

where each a_i is a homogeneous polynomial of degree $i \cdot n$ in two variables. Then a rational point (X, Y) is integral if X, Y are homogeneous polynomials of degrees $2n, 3n$ respectively. The equation (4) depends on the choice of coordinates X, Y on E ; replacing X, Y by

$$\delta^2(X + \alpha_2), \quad \delta^3(Y + \alpha_1X + \alpha_3) \quad (5)$$

(some α_i and nonzero δ) yields an isomorphic curve. If moreover $\delta \in \mathbf{C}^*$ and each α_i is a homogeneous polynomial of degree $i \cdot n$ then the new equation for E has the same discriminant degree and the same integral points.

1.2 Outline of this paper. For each $n = 1, 2, 3$ we prove Theorem n , except for the implications (a),(b) $\Rightarrow$ (c) of part (ii), which require different methods that we defer to a later paper. Our proofs use the following ingredients:

- $\hat{h}(mP) = m^2\hat{h}(P)$ for all $m \in \mathbf{Z}$.
- If $mP \neq 0$ then

$$\hat{h}(mP) = h(mP) + \sum_v \lambda_v(mP), \quad (6)$$

where $h(\cdot)$ is the naïve height and the sum extends over all places $v \in C(\mathbf{C})$ lying under singular fibers E_v of E . (All places of K are of degree 1 thanks to our use of the algebraically closed field $\mathbf{C}$ for k .) The local corrections $\lambda_v(mP)$ are described further below.

- The naïve height takes values in $\{0, 2, 4, 6, \dots\}$, and satisfies $h(m'P) \leq h(mP)$ for any integers m, m' such that $m' | m$ and $mP \neq 0$.
- Each local correction $\lambda_v(mP)$ depends only on the Kodaira type of the fiber E_v and on the component of E_v meeting P . We shall call this component c_v . The values of $\lambda_v(\cdot)$ are known explicitly for all Kodaira types and each possible component, see for instance [13, Thm. 5.2].

- Finally, the condition that E have discriminant degree $d = 12n$ imposes two conditions on the Kodaira types of the singular fibers. The first condition is

$$d = \sum_v d_v, \quad (7)$$

where d_v is the local discriminant degree of E_v . This allows only finitely many collections of fiber types. The second condition follows from an inequality due to Shioda [9, Cor. 2.7 (p.30)], and eliminates some of these collections that have too few fibers. According to this condition, if a nonconstant elliptic curve of discriminant degree d over a function field $K = \mathbf{C}(C)$ has a nontorsion point then the conductor degree of the curve strictly exceeds $(d/6) + \chi(C)$. Here $\chi(C) = 2 - 2g$ is the Euler characteristic of C . The conductor degree may be defined as the number of multiplicative fibers plus twice the number of additive fibers; thus it is also a sum of invariants of the singular fibers. When $(g, d) = (0, 12n)$ we have $\chi(C) = 2$ and $d/6 = 2n$, so the conductor degree is at least $2n + 3$.

We shall refer to these constraints as the “combinatorial conditions” on $\hat{h}(P)$, $h(mP)$, and the collection of (E_v, c_v) that arise for (E, P) . (For other uses of such conditions to obtain lower bounds on heights, see for instance [3, 14] and work referenced in these sources.) In general the combinatorial conditions yield only a lower bound on $\hat{h}_{\min}(0, 12n)$, because they allow some possibilities that do not actually occur for any (E, P) . But for each of $n = 1, 2$, and 3 this lower bound turns out to be attained by some (E, P) over $\mathbf{C}(T)$, namely those exhibited in statement (c) of part (ii) of Theorem n . (Note that we do not yet need to derive the formulas for these (E, P) , nor to prove that they are the only ones possible.) Moreover, using (6) we can check that $\hat{h}(P) = \hat{h}_{\min}(0, 12n)$ if and only if the naïve height $h(mP)$ vanishes for all m up to 6, 8, or 9 respectively.

Still, already at $n = 1$ we see some redundancy. The combinatorial conditions allow $\hat{h}(P) = 1/30$ to be attained in any of five ways, four of which are realized by the curves $E_1(q)$ of Theorem 1 for suitable choices of q . Shioda’s $E_1(4/5)$ has singular fibers of types I_5, I_3, I_2 , and II . (We specify the components c_v later in the paper.) The fibers of $E_1(-1)$ have types I_5, IV, I_2 , and I_1 , while those of $E_1(4)$ have types I_5, I_3, III , and I_1 . In all other cases, the fibers of $E_1(q)$ have types I_5, I_3, I_2, I_1, I_1 : the first three at $s = 0, s' = 0, s' = s$, and the last two at the roots of the quadratic $(q + 1)^3 s^2 = (11q^2 - 14q + 2)ss' + (q - 1)s'^2$. When $q = 4/5$, these roots coincide and the two I_1 fibers merge to form a II ; likewise at $q = -1$ or $q = 4$, one of the I_1 fibers merges with the I_3 or I_2 fiber to form a IV or III respectively. (The one merger that does not occur is $I_1 + I_1 \rightarrow I_2$.) But none of these degenerations changes $\hat{h}(P)$, nor any $h(mP)$, nor the conductor degree N . In fact a fiber of type II, III , or IV contributes as much to our formulas for $\hat{h}(P), h(mP), N$ as a pair of fibers of types I_1 and I_ν ($\nu = 1, 2$, or 3). Thus it is enough to minimize $\hat{h}(P)$ under the further assumption that no fibers of type II, III , or IV occur. We find similar replacements for all components of fibers of the remaining additive types $I_\nu^*, II^*, III^*, IV^*$. See Proposition 2. This simplifies

the computation of the combinatorial lower bound on $\hat{h}_{\min}(0, 12n)$: instead of an exhaustive search over all combinations of (E_ν, c_ν) , we need only try those for which each E_ν is multiplicative (of type I_ν for $\nu = d_\nu$).

We programmed the search over all partitions $\{d_\nu\}$ of $12n$ in GP [8] and ran it on a Sun Ultra 60. This took only a fraction of a second for $n = 1$, five seconds for $n = 2$, and five minutes for $n = 3$. It took about an hour to carry out the same computation for $n = 4$, and about 20 hours for $n = 5$; but the resulting bounds are probably not attained: as we shall see in a later paper, the required (E_ν, c_ν) data impose more conditions than the number of parameters needed to specify (E, P) . We do produce explicit (E, P) that show $\hat{h}_{\min}(0, 48) \leq 41/1540$ and $\hat{h}_{\min}(0, 60) \leq 261/10010$, and conjecture that these are the correct values of $\hat{h}_{\min}(0, 12n)$ for $n = 4, 5$. We have not attempted to extend the computation past $n = 5$.

1.3 Coming attractions. Happily, the computation of the surfaces (1,2,3) not only completes the proofs of Theorems 1 through 3 but also points the way to further results and connections. We outline these here, and defer detailed treatment to a later paper in this series. In each step of the computation we in effect obtain a new birational model for the moduli space, call it $\mathcal{X}$, of pairs (E, P) consisting of an elliptic curve and a point on it. Our new parametrizations of this rational surface $\mathcal{X}$ have several other applications. One is a geometric interpretation of Tate's method for exhibiting the generic elliptic curve with an N -torsion point: we readily locate the modular curves $X_1(N)$ ($N \leq 16$) on $\mathcal{X}$, together with nonconstant rational functions of minimal degree that realize each $X_1(N)$ as an algebraic curve of genus ≤ 2 . Arithmetically, we can use our parametrizations of $\mathcal{X}$ to find (E, P) over $\mathbf{Q}$ (or over some other global field) such that P is a nontorsion point with small $\hat{h}(P)$, and/or with many integral multiples in the minimal model of E . For instance, we prove that there are infinitely many $(E, P)/\mathbf{Q}$ such that mP is integral for each $m = 1, 2, \dots, 11, 12$. Our numerical results for a isolated curves (E, P) over $\mathbf{Q}$ may be found on the Web at http://www.math.harvard.edu/~elkies/low_height.html. They include new records for consecutive integral multiples and for the Lang ratio $\hat{h}(P)/\log |\Delta_E|$. We have mP integral for each $m = 1, 2, \dots, 13, 14$ for

$$E : Y^2 + XY = X^3 - 139761580X + 1587303040400, \quad (8)$$

an elliptic curve of conductor $1029210 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 29$, and P the nontorsion point $(X, Y) = (11480, 1217300)$; and we find the curve

$$Y^2 + XY = X^3 - 161020013035359930X + 24869250624742069048641252 \quad (9)$$

of conductor $3476880330 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 31 \cdot 2111$ with the nontorsion point $(-296994156, 6818852697078)$ of canonical height² $\hat{h}(P) = .0190117 \dots <$

² There are two standard normalizations, differing by a factor of 2, for the canonical height of a point on an elliptic curve over $\mathbf{Q}$. We use the larger one, which is the one consistent with our formulas for function fields.

$1.691732 \cdot 10^{-4} \log |\Delta_E|$. The curves (8,9) are the specializations of our formula (3) with $(A, t/t') = (35/32, -8/15), (33/23, 115/77)$.

Our simplified formula for $\hat{h}(mP)$ (Proposition 2) also bears on the asymptotic behavior of $\hat{h}_{\min}(g, 12n)$ for fixed g as $n \rightarrow \infty$. Hindry and Silverman [3] used the combinatorial conditions (except for the condition: $h(m'P) \leq h(mP)$ if $m'|m$) to show that there exists $C > 0$ such that

$$\hat{h}(g, 12n) \geq Cn - O_g(1), \quad (10)$$

This proved the function-field case of a conjecture of Lang [4, p.92]. The error terms $O_g(1)$ are effectively computed, and can be omitted entirely if $g \leq 1$. Hindry and Silverman also produce an explicit constant C , but it is quite small: about $7 \cdot 10^{-10}$. Their approach requires a point meeting every additive fiber in its identity component, which they achieved by working with $12P$ instead of P , at the cost of a factor of $1/12^2$ in C . Our results here let one apply the same methods directly to P , thus saving a factor of 12^2 and raising C to about 10^{-7} . In a later paper we show how to gain another factor of approximately 5000, raising the lower bound on $\liminf_n \hat{h}(g, 12n)/n$ to $1/2111$. This is within an order of magnitude of the correct value: for all $n \equiv 0 \pmod{5}$ we obtain $\hat{h}_{\min}(0, 12n) \leq 261n/50050$ via base change from our $n = 5$ example.

2. The naïve and canonical heights.

We collect here the facts we shall use about elliptic curves E over function fields K in characteristic zero, the associated elliptic surface $\mathcal{E}$, and the naïve and canonical height functions on $E(K)$.

2.1 The naïve height. The *naïve height* $h(P)$ of a nonzero $P \in E(K)$ can be defined using intersection theory on the elliptic surface $\mathcal{E}$ associated to some model of E . Let s_0 be the zero-section of the elliptic fibration $\mathcal{E} \rightarrow C$, and s_P the section corresponding to P . Then $h(P) := 2s_P \cdot s_0$. Since we assumed that $P \neq 0$, the sections s_0, s_P are distinct curves on $\mathcal{E}$. Hence their intersection number $s_P \cdot s_0$ is a nonnegative integer, and $h(P)$ is a nonnegative even integer. Moreover $h(P) = 0$ if and only if s_P is disjoint from s_0 , in which case we say that P is an *integral point* on E .

When $C = \mathbf{P}^1$, we can give an equivalent algebraic definition of $h(P)$ in terms of a Weierstrass equation of E . This definition emphasizes the analogy with the canonical height in the more familiar case of an elliptic curve over $\mathbf{Q}$. Recall that each coefficient a_i in the Weierstrass equation (4) is a homogeneous polynomial of degree $i \cdot n$ in the projective coordinates on $\mathbf{P}^1$. Then the coordinates x, y of a nonzero $P \in E(K)$ are homogeneous rational functions of degrees $2n, 3n$. If x, y are written as fractions “in lowest terms”, as quotients of coprime homogeneous polynomials, then the denominators are (up to scalar multiple) the square and cube of some polynomial ζ . The roots of ζ , with multiplicity, are the images on $\mathbf{P}^1$ of the intersection points of s_0 and s_P . Hence $s_P \cdot s_0 = \deg \zeta$. Therefore $h(P)$ is the degree of the denominator ζ^2 of x , which is also the number of poles

of x counted with multiplicity. An integral point is one for which ζ is a nonzero scalar and thus x, y are homogeneous polynomials of degrees $2n, 3n$.

For an arbitrary base curve C , the coefficients a_i are global sections of $\mathcal{L}^{\otimes i}$ for some line bundle $\mathcal{L}$ on C , and x, y are meromorphic sections of $\mathcal{L}^{\otimes 2}, \mathcal{L}^{\otimes 3}$. The pole divisors of x, y are $2Z, 3Z$ for some effective divisor Z on C , whose degree is $s_P \cdot s_0$; thus again $h(P)$ is the degree of the pole divisor $2Z$ of x , and P is integral iff $Z = 0$ iff x, y are global sections of $\mathcal{L}^{\otimes 2}, \mathcal{L}^{\otimes 3}$. A linear change of coordinates according to (5) yields the same notion of integrality if and only if $\delta \in \mathbf{C}^*$ and $\alpha_i \in \Gamma(\mathcal{L}^{\otimes i})$ for each i .

We shall need one more property of the naïve height beyond its relation with the canonical height and the fact that $h(mP) \in \{0, 2, 4, 6, \dots\}$ ($mP \neq 0$):

Lemma 1. *Let P be a point on an elliptic curve over $k(C)$, and let m, m' be any integers such that $m' \mid m$ and $mP \neq 0$. Then $h(m'P) \leq h(mP)$.*

Proof: Each point of $s_{m'P} \cap s_0$ is also a point of intersection of s_{mP} with s_0 , to at least the same multiplicity. Hence $s_{m'P} \cdot s_0 \leq s_{mP} \cdot s_0$, so

$$h(m'P) = 2s_{m'P} \cdot s_0 \leq 2s_{mP} \cdot s_0 = h(mP)$$

as claimed. □

Remarks:

1. We could also state the result as: The naïve height of a point is less than or equal to the naïve height of any of its multiples that is not the zero point. This is a more natural formulation (the first point does not have to be written as $m'P$), but less convenient for our purposes.
2. In the proof, “at least the same multiplicity” can be strengthened to “exactly the same multiplicity” in our characteristic-zero setting. In general $h(mP)$ may strictly exceed $h(m'P)$ because $s_{mP} \cap s_0$ may also contain points where $m'P$ reduces to a nontrivial (m/m') -torsion point.

The naïve height satisfies further inequalities along the lines of Lemma 1, for instance

$$h(6P) + h(P) \geq h(2P) + h(3P). \quad (11)$$

Lemma 1 suffices for the proofs of Theorems 1–3 in the genus-zero case, but inequalities such as (11) are sometimes needed to exclude possible configurations with positive g , as we shall see for $d = 24$. The strongest such inequality we found is:

Lemma 2. *Let P be a point on an elliptic curve over $k(C)$, and let m be any integer such that $mP \neq 0$. Then*

$$\sum_{m' \mid m} \mu(m/m') h(m'P) \geq 0. \quad (12)$$

Proof: The left-hand side can be interpreted as twice the number of points of C , counted with multiplicity, at which $mP = 0$ but $m'P \neq 0$ for each proper factor m' of m . $\square$

Inequality (11) is the special case $m = 6$ of this Lemma. The sum in (12) may be considered as an analogue of the formula $\prod_{m'|m} (x^{m'} - 1)^{\mu(m/m')}$ for the m -th cyclotomic polynomial. We recover Lemma 1 by summing the inequality (12) over all factors of m , including m itself but not 1, to obtain $h(mP) \geq h(P)$, which is equivalent to Lemma 1 by the first Remark above.

2.2 Local invariants, and Shioda's inequality. To go from the naïve to the canonical height we must use the minimal model of E for the elliptic surface $\mathcal{E}$. We next describe this model, collect some known facts on the singular fibers of $\mathcal{E}$, and give Shioda's lower bound on the conductor degree.

Whereas a naïve height could be defined for any model of E ,³ the canonical height requires the Néron minimal model. It is known that there exists a minimal line bundle $\mathcal{L}$ on C with the following property: let D be a divisor on C such that $\mathcal{O}(D) \cong \mathcal{L}$; then E is isomorphic to a curve with an extended Weierstrass equation (4) whose coefficients a_i are global sections of iD . In characteristic zero we can easily obtain D and $\mathcal{L}$ by putting E in narrow Weierstrass form $Y^2 = X^3 + a_4X + a_6$. Then D is the smallest divisor such that $(a_4) + 4D \geq 0$ and $(a_6) + 6D \geq 0$. In other words, we can regard a_4, a_6 as global sections of $\mathcal{L}^{\otimes 4}, \mathcal{L}^{\otimes 6}$ such that there is no point of C where a_4 and a_6 vanish to order at least 4 and 6 respectively. Once we have $a_i \in \Gamma(\mathcal{L}^{\otimes i})$, we can regard the Weierstrass equation (4) as a surface in the plane bundle $\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}$ over C . If all the roots of the discriminant $\Delta \in \Gamma(\mathcal{L}^{\otimes 12})$ are distinct then this surface is smooth and is the minimal model of E . Otherwise it has isolated singularities, which we blow up as many times as needed (we may follow Tate's algorithm [16]) to obtain the minimal model $\mathcal{E}$. This is a smooth algebraic surface of arithmetic genus $n = \deg \mathcal{L}$, equipped with a map to C with generic fiber E and $\omega_{\mathcal{E}/C} \cong \mathcal{L}$. See for instance [1, pp.149ff.].

We shall need much information about the singular fibers that can arise for the elliptic fibration $\mathcal{E} \rightarrow C$. We extract from Tate's table [16, p.46] the following local data for each possible Kodaira type of a singular fiber E_v : the discriminant degree d_v , the conductor degree N_v , and the structure of the group $E_v/(E_v)_0$ of multiplicity-1 components. We also list in each case the root lattice L_v that E_v contributes to the Néron-Severi lattice $\text{NS}(\mathcal{E})$ of $\mathcal{E}$. In each case, L_v has rank $d_v - N_v$, and $E_v/(E_v)_0 \cong L_v^*/L_v$ where $L_v^* \subset L_v \otimes \mathbf{Q}$ is the dual lattice. The lattice " A_0 " that appears for Kodaira types I_1 and II is the trivial lattice of rank zero. For Kodaira type I_ν^* , the group $E_v/(E_v)_0$ always has order 4, and has exponent 2 or 4 according as ν is even or odd. For positive ν of either parity, a fiber of type I_ν^* has a distinguished multiplicity-1 component of order 2 in $E_v/(E_v)_0$, namely

³ Two models may yield different heights h, h' , but $h' = h + O(1)$ holds for any pair of naïve heights on the same curve. It also follows that the property $\hat{h} = h + O(1)$ of the canonical height does not depend on the choice of naïve height h .

the one closest to the identity component. In the L_v picture, the distinguished component corresponds to the nontrivial coset of $D_{4+\nu}$ in $\mathbf{Z}^{4+\nu}$. When $\nu = 0$ there is no distinguished component: all three non-identity components of multiplicity 1 are equivalent, as are all three nontrivial cosets due to the triality of D_4 .

Kodaira type	$I_\nu(\nu > 0)$	II	III	IV	I_ν^*	IV^*	III^*	II^*
d_v	ν	2	3	4	$6 + \nu$	8	9	10
N_v	1	2	2	2	2	2	2	2
$E_v/(E_v)_0$	$\mathbf{Z}/\nu\mathbf{Z}$	$\{0\}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	$D_{4+\nu}^*/D_{4+\nu}$	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\{0\}$
root lattice	$A_{\nu-1}$	A_0	A_1	A_2	$D_{4+\nu}$	E_6	E_7	E_8

The discriminant and conductor degrees d, N of $\mathcal{E}$ are sums of the discriminant and conductor degrees of the singular fibers:

$$12n = d = \sum_v d_v, \quad N = \sum_v N_v. \quad (13)$$

Hence $d - N = \sum_v (d_v - N_v) = \sum_v \text{rk } L_v$ is the rank of the subgroup $\oplus_v L_v$ of $\text{NS}(\mathcal{E})$ due to the singular fibers. Shioda used this to prove [9, Cor. 2.7 (p.30)]:

Proposition 1. *Let E be a nonconstant elliptic curve over a function field $K = k(C)$ of genus g , with discriminant and conductor degrees $d = 12n$ and N . Then*

$$N \geq 2n + (2 - 2g) + r, \quad (14)$$

where r is the rank of the Mordell-Weil group $E(K)$.

Proof: Let $T \subseteq \text{NS}(\mathcal{E})$ be the subgroup spanned by s_0 , the generic fiber, and $\oplus_v L_v$. Then we have a short exact sequence (see for instance [10, Thm. 1.3]):

$$0 \rightarrow T \rightarrow \text{NS}(\mathcal{E}) \rightarrow E(K) \rightarrow 0, \quad (15)$$

where the map $\text{NS}(\mathcal{E}) \rightarrow E(K)$ is the sum on the generic fiber. Taking ranks, we find

$$\text{rk } \text{NS}(\mathcal{E}) = \text{rk } T + \text{rk } E(K) = 2 + (d - N) + r. \quad (16)$$

But $\text{NS}(\mathcal{E})$ embeds into $H^{1,1}(\mathcal{E}, \mathbf{Z})$, a group of rank $h^{1,1}(\mathcal{E}) = 10n + 2g$. Hence $\text{rk } \text{NS}(\mathcal{E}) \leq 10n + 2g$. Therefore

$$N \geq (d + 2 + r) - (10n + 2g) = 2n + (2 - 2g) + r,$$

as claimed. ■

Remarks:

1. Since $r \geq 0$ it follows that

$$N \geq 2n + (2 - 2g) = (d/6) + \chi \quad (17)$$

for any nonconstant elliptic surface. This weaker inequality is sufficient for most of our purposes, even though we are interested in curves with a non-torsion point, for which the strict inequality $N > (d/6) + \chi$ holds because $r > 0$.

2. The inequality (17) is now usually known as the “Szpiro inequality”, but Shioda’s paper [9] predates Szpiro’s [15] by almost two decades (see also [12, p.114]). It is by now well-known that (17) can be proved by elementary means via Mason’s theorem [5] (the ABC inequality for function fields). Can one also give an elementary proof of Shioda’s inequality, or even of its consequence that $r = 0$ if $N = (d/6) + \chi$?
3. The requirement that E not be a constant curve is essential. There is an analogous statement for constant curves but many details must change. Suppose E is such a curve, that is, $\mathcal{E} = C \times E_0$ for some elliptic curve E_0/k . Then $E(K)$ is not finitely generated, because it contains a copy of $E_0(k)$. Still, $E(K)/E_0(k)$ is finitely generated, and identified with the group $\text{NS}(\mathcal{E})/T$. Again we call the rank of this group r . Since $n = d = N = 0$ in this setting, we obtain the inequality $r + 2 \leq h^{1,1}(C \times E_0) - 2$. But for a constant curve, $h^{1,1}(C \times E_0) = 2g + 2$, instead of the $2g$ that one would expect from the $10n + 2g$ formula. Hence $r \leq 2g$. This can also be proved using the identification of $E(K)/E_0(k)$ with $\text{End}(\text{Jac}(C), E_0)$, an approach that also yields the equality condition: clearly $r = 2g$ if $g = 0$; if $g > 0$ then $r = 2g$ if and only if E_0 has complex multiplication and $\text{Jac}(C)$ is isogenous with E_0^g . See for instance [2].
4. The hypothesis of characteristic zero, too, is essential here. In positive characteristic, one cannot decompose the second Betti number $b_2(\mathcal{E})$ as $h^{2,0} + h^{1,1} + h^{0,2}$, so one has only the weaker upper bound $b_2(\mathcal{E})$ on $\text{rk}(\text{NS}(\mathcal{E}))$. This upper bound exceeds the characteristic-zero bound by $2g$ for a constant curve and $2(n + g - 1)$ for a nonconstant one. For instance, a constant curve $C \times E_0$ has $r \leq 4g$, with equality if and only if either $g = 0$ or E_0 and $\text{Jac}(C)$ are both supersingular. In general $\mathcal{E}$ is said to be “supersingular” if $\text{NS}(\mathcal{E}) \cong \mathbf{Z}^{b_2(\mathcal{E})}$; such surfaces were studied and used in [10, 2].

2.3 Local height corrections. We next list the local height corrections $\lambda_v(mP)$ for each of the Kodaira types. For convenience we abuse notation by using mP to refer also to the section s_{mP} .

- If mP is on the identity component of E_v then

$$\lambda_v(mP) = d_v/6. \quad (18)$$

In particular this covers fibers of type II or II*.

- If E_v is of type I_ν and P passes through component $a \in \mathbf{Z}/\nu\mathbf{Z}$, let $x = \bar{a}/\nu$ for any lift $\bar{a}$ of a to $\mathbf{Z}$; then

$$\lambda_v(mP) = \nu B(mx), \quad (19)$$

where $B(\cdot)$ is the second Bernoulli function $B(z) := \sum_{n=1}^{\infty} \cos(2\pi n z)/(\pi n)^2$. Since B is $\mathbf{Z}$ -periodic, the choice of $\bar{a}$ does not matter. Likewise, since

$B(z) = B(-z)$ it does not matter that a cannot be canonically distinguished from $-a$. We have

$$B(z) = z^2 - z + \frac{1}{6} \quad (20)$$

for all $z \in [0, 1]$, so in particular $B(0) = 1/6$. Hence $\lambda_v(mP) = \nu/6$ if mP passes through the identity component of E_v , as also asserted by (18) in that case.

- If E_v is of type III, IV, I_0^* , III^* , or IV^* , and mP passes through a non-identity component of E_v , then $\lambda_v(mP) = 0$.
- Finally, suppose E_v is of type I_ν^* ($\nu > 0$) and that mP passes through a non-identity component. If that component is the distinguished one of order 2 then $\lambda_v(mP) = \nu/6$. Otherwise $\lambda_v(mP) = -\nu/12$. (We could have also allowed $\nu = 0$, when there is no distinction among the three non-identity components, but $\lambda_v(mP) = \nu/6 = -\nu/12 = 0$ for all of them.)

We record two applications of these formulas for future use:

Lemma 3. *Let E be an elliptic curve of discriminant degree $12n$ over a function field K , and P any nonzero point of $E(K)$. Then*

$$-n \leq \hat{h}(P) - h(P) \leq 2n. \quad (21)$$

Proof: For each v we have $-d_v/12 \leq \lambda_v \leq d_v/6$. Summing over v yields (21). $\square$

Lemma 4. *Let E be an elliptic curve of discriminant degree $12n$ over a function field K , and P any point of $E(K)$. If for some integer m the multiple mP is a nonzero integral point then $\hat{h}(mP) \leq 2n/m^2$.*

Proof: By our formulas for λ_v we have $\lambda_v(mP) \leq d_v/6$ for all v . Hence

$$m^2 \hat{h}(P) = \hat{h}(mP) = h(mP) + \sum_v \lambda_v(mP) \leq h(mP) + \sum_v d_v/6. \quad (22)$$

But $h(mP) = 0$ since mP is integral, and $\sum_v d_v/6 = d/6 = 2n$. Hence $m^2 \hat{h}(P) \leq 2n$, and the Lemma follows. $\square$

2.4 Reduction to the semistable case. Recall that an elliptic curve is said to be *semistable* if all its singular fibers are of type I_ν for some ν . Suppose E/K is semistable and P is a nontorsion point in $E(K)$. We associate to (E, P) an element γ of the abelian group $\mathbf{G}$ of formal $\mathbf{Z}$ -linear combinations of orbits of $\mathbf{Q}$ under the infinite dihedral group D_∞ generated by $z \mapsto z + 1$ and $z \leftrightarrow 1 - z$. We denote by $[z]$ the generator of $\mathbf{G}$ corresponding to the orbit of z . Then γ is defined as a sum of local contributions $\gamma_v \in \mathbf{G}$ that record the types $\nu(v)$ of the singular fibers E_v and the component $c_v = a(v) \in \mathbf{Z}/(\nu(v))\mathbf{Z}$ of each fiber that contains P , as follows:

$$\gamma_v := \sum_v \gcd(a(v), \nu(v)) \cdot \left[\frac{a(v)}{\nu(v)} \right]. \quad (23)$$

Then each of the height corrections $\hat{h}(mP) - h(mP)$, as well as the discriminant degree, are images of γ under homomorphisms $\lambda_m, \mathbf{d}$ from $\mathbf{G}$ to $\mathbf{Q}$ or $\mathbf{Z}$, and the conductor is bounded above by the image of a homomorphism $\mathbf{N} : \mathbf{G} \rightarrow \mathbf{Z}$. We define these homomorphisms on the generators of $\mathbf{G}$ and extend by linearity. Suppose $\mathbf{Q} \ni z = a/b$ with $b > 0$ and $\gcd(a, b) = 1$. Note that b is an invariant of the action of D_∞ . Then we set

$$\lambda_m([z]) := b B_2(mz), \quad \mathbf{d}([z]) := b, \quad \mathbf{N}([z]) := 1. \quad (24)$$

Then our formulas (19,13) yield the identities

$$\hat{h}(mP) = h(mP) + \lambda_m(\gamma) \quad (m = 1, 2, 3, \dots), \quad 12n = d = \mathbf{d}(\gamma) \quad (25)$$

and the estimate

$$N \leq \mathbf{N}(\gamma). \quad (26)$$

(This last is an upper bound rather than an identity because each v contributes 1 to N and $\gcd(a(v), \nu(v)) \geq 1$ to $\mathbf{N}(\gamma)$.) It follows that

$$\mathbf{N}(\gamma) \geq N \geq (d/6) + (2 - 2g) + r \geq \frac{1}{6}\mathbf{d}(\gamma) + 3 - 2g. \quad (27)$$

The second step is Shioda's inequality (Prop. 1), and the third step uses the positivity of r , which follows from our hypothesis that P is nontorsion.

To generalize these formulas to curves that may not be semistable, it might seem that we would have to extend $\mathbf{G}$ with generators that correspond to Kodaira types other than I_ν . But we can associate to any additive fiber E_v an element of $\mathbf{G}$ whose images under λ_m and $\mathbf{d}$ coincide with $\lambda_v(mP)$ and d_v , and whose image under $\mathbf{N}$ is $\geq N_v$. (Note that we already did this for multiplicative fibers with $f = \gcd(a(v), \nu(v)) > 1$, replacing them in effect by f fibers with a, ν coprime and the same value of a/ν .) As in the multiplicative case, this element is positive, in the sense that it is a nonzero formal linear combination of elements of $\mathbf{Q}/D_\infty$ with nonnegative coefficients. Specifically, we have:

Proposition 2. *Let E be an elliptic curve over a function field K of genus g , and $P \in E(K)$ a nontorsion point. Define for each singular fiber E_v a positive $\gamma_v \in \mathbf{G}$, depending on (E_v, c_v) as follows:*

- If E_v is multiplicative, γ_v is defined by (23).
- If c_v is the identity component then $\gamma_v := d_v [0]$.
- If c_v is a non-identity component of a fiber E_v of type III, IV, IV*, or III* then γ_v is respectively

$$[1/2] + [0], \quad [1/3] + [0], \quad 2 \cdot [1/2] + 2 \cdot [0], \quad 3 \cdot [1/3] + 3 \cdot [0].$$

- If c_v is a distinguished component of a fiber E_v of type I_ν^* then

$$\gamma_v := 2 [1/2] + (\nu + 2) [0].$$

- If c_v is a non-distinguished, non-identity component of a fiber E_v of type I_ν^* then

$$\gamma_v := (\mu + 2) [1/2] + 2 [0]$$

if $\nu = 2\mu$, and

$$\gamma_v := [1/4] + (\mu + 1) [1/2] + [0]$$

if $\nu = 2\mu + 1$ for some integer μ .

Then:

- i) $\lambda_v(mP) = \lambda_m(\gamma_v)$ for each $m = 1, 2, 3, \dots$;
- ii) $d_v = \mathbf{d}(\gamma_v)$; and
- iii) $N_v \leq \mathbf{N}(\gamma_v)$.

Thus (25,26,27) hold for $\gamma := \sum_v \gamma_v$. Equality in (iii) holds if and only if E_v is either a multiplicative fiber with $\gcd(a, \nu) = 1$, a fiber of type III or IV with c_v a non-identity component, or a fiber of type II.

[Note that, as was true for the λ_v formulas, the first two formulas in Prop. 2 overlap in the case of a multiplicative fiber with $a(v) = 0$, but give the same answer in this case. Here both prescriptions yield $\gamma_v = \nu(v) \cdot [0]$ for such v .]

Proof: The multiplicative case was seen already. For each of the other Kodaira types, it is straightforward to verify that $\lambda_v(mP) = \lambda_m(\gamma_v)$ for each nonnegative m less than the exponent of the finite group $E_v/(E_v)_0$ (which is at most 4), and to check that $d_v = \mathbf{d}(\gamma_v)$, and that $N_v \leq \mathbf{N}(\gamma_v)$, with strict inequality except in the three cases listed. We recover (25,26,27) by summing over v . ■

3. The values of $\hat{h}_{\min}(0, 12n)$ for $n = 1, 2, 3$, and consecutive integral multiples.

For each n we can use the formulas and results above to obtain a lower bound on $\hat{h}_{\min}(g, 12n)$. When $g = 0$ and $n = 1, 2, 3$ we also show that this bound is attained if and only if mP is integral for $m \leq M = 6, 8, 9$, and verify that the (E, P) exhibited in Theorem n satisfy those conditions.

Suppose E is an elliptic curve over $\mathbf{C}(T)$ with discriminant degree $12n$. Let P be a nontorsion rational point on E , and γ the associated element of $\mathbf{G}$. From γ and $\hat{h}(P)$ we can recover all the naïve heights $h(mP)$ from the first formula in (25): $h(mP) = m^2 \hat{h}(P) - \lambda_m(\gamma)$. Given n and an upper bound H on $\hat{h}(P)$, there are only finitely many candidates for the pair $(\gamma, \hat{h}(P))$: there are finitely many $\gamma > 0$ with $\mathbf{d}(\gamma) = 12n$, and for each one there are only finitely many possible choices for $h(P)$ consistent with $h(P) + \lambda_1(\gamma) = \hat{h}(P) \in (0, H]$. For each candidate $(\gamma, \hat{h}(P))$ we can check the condition $m' | m \Rightarrow h(mP) \geq h(m'P) \geq 0$. Only finitely many m need be checked for each $(\gamma, \hat{h}(P))$: by Lemma 3 we know that $h(mP) \geq 0$ once $m^2 \hat{h}(P) \geq n$, and $h(mP) \geq h(m'P)$ for each $m' | m$ once $m^2 \hat{h}(P) \geq 4n$. The minimal $\hat{h}(P)$ among the $(\gamma, \hat{h}(P))$ that pass these tests is then our lower bound on $\hat{h}_{\min}(g, 12n)$. [We could also test the more complicated

inequality of Lemma 2, which may further improve the bound; instead we checked that inequality after the fact when necessary.]

We wrote a GP program to compute this bound by exhaustive search, and ran it with $H = 2n/M^2$ for $n = 1, 2, 3$. We chose this upper bound H to ensure that, by Lemma 4, we would also find all feasible $(\gamma, \hat{h}(P))$ such that $h(mP) = 0$ for each $m = 1, 2, 3, \dots, M$. For $n = 1$, we found that the minimum occurs for

$$\gamma = [1/5] + [1/3] + [1/2] + 2[0], \quad \hat{h}(P) = 1/30, \quad (28)$$

and is the unique $(\gamma, \hat{h}(P))$ such that $h(mP) = 0$ for each $m \leq 6$. For $n = 2$, we found that the minimum occurs for

$$\gamma = [1/11] + 2[2/5] + [1/3], \quad \hat{h}(P) = 4/165; \quad (29)$$

but this is not feasible because $h(mP) = 0, 2, 2, 2$ for $m = 2, 4, 6, 12$, so inequality (11) is violated when $m = 2$. Our lower bound on $\hat{h}_{\min}(g, 24)$ is thus the next-smallest value, which occurs for

$$\gamma = [1/7] + [2/5] + [1/4] + [1/3] + [1/2] + 3[0], \quad \hat{h}(P) = 11/420, \quad (30)$$

and is the unique $(\gamma, \hat{h}(P))$ such that $h(mP) = 0$ for each $m \leq 8$.

On the other hand, the $(\gamma, \hat{h}(P))$ pairs of (28,30) are also those associated with the curves and points E, P exhibited in (1,2). Hence those E, P attain our lower bounds $1/30, 11/420$ on $\hat{h}_{\min}(12), \hat{h}_{\min}(24)$, as well as the upper bounds 6 and 8 on the number of consecutive integral multiples for $n = 1$ and $n = 2$. This proves all of Theorems 1 and 2 except for the claims that every (E, P) attaining those bounds is isomorphic with some $E_1(q)$ or $E_2(u)$.

For $n = 3$, we find that there is a unique $(\gamma, \hat{h}(P))$ such that $h(mP) = 0$ for each $m \leq 9$, namely

$$\gamma = [1/8] + [3/7] + [1/5] + [1/4] + 2[1/3] + [1/2] + 4[0], \quad \hat{h}(P) = 23/840. \quad (31)$$

Again these are the γ and $\hat{h}(P)$ for the (E, P) exhibited in the Introduction (formula (3)). But we do not claim that $\hat{h}_{\min}(36) = 23/840$: Lemma 2 eliminates the second-smallest pair

$$(\gamma, \hat{h}(P)) = ([1/13] + [3/8] + [3/7] + [1/5] + [1/3], 229/10920)$$

(which violates the inequality (11) in the same way that (29) did), but not several other possibilities with $\hat{h}(P) < 23/840$. We next list all these possibilities, in order of increasing $\hat{h}(P)$:

γ	$\hat{h}(P)$	
$[1/13] + [3/11] + [3/8] + 2[1/2]$	$23/1144 \approx .02010$	
$[1/13] + [3/8] + [2/7] + [1/4] + 2[1/2]$	$17/728 \approx .02335$	
$[1/11] + [4/9] + [2/7] + [1/4] + [1/3] + 2[0]$	$65/2772 \approx .02345$	
$[1/12] + [3/11] + [3/8] + 2[1/2] + [0]$	$7/264 \approx .02652$	
$[1/11] + [3/7] + 2[1/5] + [1/4] + 2[1/2]$	$41/1540 \approx .02662$	(32)

(For comparison, $229/10920 \approx .02097$ and $23/840 \approx .02738$.) We have $\mathbf{d}(\gamma) \leq 7$ for each entry in the table (32); therefore by Prop. 1 none of them can occur for an elliptic curve over $\mathbf{P}^1$. (Even the weaker inequality (17) would suffice here; either of those inequalities also excludes (29) for $n = 2$, and would thus be enough to obtain $\hat{h}_{\min}(0, 24)$, but the determination of $\hat{h}_{\min}(24)$ required a further argument.) Thus $\hat{h}_{\min}(0, 36) = 23/840$, proving Theorem 3 except for the claim that every (E, P) satisfying conditions (a) and (b) is of the form $E_3(A)$ for some A .

Acknowledgements. I thank J. Silverman and T. Shioda for helpful correspondence concerning their papers and related issues, and M. Watkins for carefully reading a draft of this paper. This work was made possible in part by funding from the Packard Foundation and the National Science Foundation.

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