On the Equations Defining Abelian Varieties. III

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On the Equations Defining Abelian Varieties. III *

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§ 10. Non-Degenerate Theta Functions

The third part of this paper is devoted (1) to a complete description of the boundary of the moduli space for abelian varieties described in § 9, and (2) to connecting our theory with the classical theory of theta functions. We begin by defining a theta function in a coordinate-free manner and investigating how and under what non-degeneracy restrictions we can construct a tower of abelian varieties having this as its theta function. Our goal is to find an inverse to the moduli map Θ described in § 9. Fix

- o) an algebraically closed field k , $\text{char}(k) \neq 2$;
- i) a $2g$ -dimensional vector space V over $\mathbb{Q}_2$;
- ii) a skew-symmetric bi-multiplicative map:

$$e: V \times V \rightarrow \{2^n\text{-th roots of 1 in } k\},$$

i.e.,

$$e(\alpha, \alpha) = 1$$

$$e(\alpha + \beta, \gamma) = e(\alpha, \gamma) \cdot e(\beta, \gamma)$$

$$e(\alpha, \beta + \gamma) = e(\alpha, \beta) \cdot e(\alpha, \gamma);$$

- iii) a maximal isotropic lattice $A \subset V$ (i.e., a compact, open subgroup such that $e(\alpha, \beta) = 1$, all $\alpha, \beta \in A$, maximal with this property);
- iv) a quadratic character

$$e_*: \frac{1}{2}A/A \rightarrow \{\pm 1\}$$

such that

$$e_*(\alpha + \beta) e_*(\alpha) e_*(\beta) = e(\alpha, \beta)^2,$$

all $\alpha, \beta \in \frac{1}{2}A$.

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We assume, however, that via a suitable isomorphism $V \cong Q_2^{2g}$, $A \cong Z_2^{2g}$, and e, e_* have the form defined in § 9. In fact, this is nearly always the case: if we write

$$e_*(\alpha) = (-1)^{Q(\alpha)}$$

where Q is a quadratic form on $\frac{1}{2}A/A$ with values in the field $F_2 = \{0, 1\}$, then Q has an Arf invariant $\Delta(Q) \in F_2$. It is not hard to show that (V, A, e, e_*) has the required form only if $\Delta(Q) = 0$. We leave this point to the reader.

Definition 1. A *theta-function* Θ on V is a map $\Theta: V \rightarrow k$ satisfying

- i) $\Theta(\alpha + \beta) = e_*(\beta/2) \cdot e(\beta/2, \alpha) \Theta(\alpha)$, all $\alpha \in V$, $\beta \in A$,
 - ii) $\Theta(-\alpha) = \Theta(\alpha)$, all $\alpha \in V$,
 - iii) $\prod_{i=1}^4 \Theta(\alpha_i) = 2^{-g} \sum_{\eta \in \frac{1}{2}A/A} e(\gamma, \eta) \cdot \prod_{i=1}^4 \Theta(\alpha_i + \gamma + \eta)$
- if $\gamma = -\frac{1}{2} \sum_{i=1}^4 \alpha_i$, $\alpha_1, \dots, \alpha_4 \in V$ arbitrary.

If we let

$$S_0 = \{\alpha \mid \Theta(\alpha) \neq 0\} = \text{support}(\Theta),$$

then S_0 is a union of cosets of A . The structure of S_0 is a “fine” property of Θ , so we introduce:

Definition 2. The *coarse support* S_1 of Θ is:

$$S_1 = \{\alpha \mid \Theta(\alpha + \eta) \neq 0, \text{ for some } \eta \in \frac{1}{2}A\}.$$

We will see in § 11 that the coarse support S_1 of a theta function is either all of V , or $\frac{1}{2}A + W$ where $W \subset V$ is a proper subvectorspace. This is the essential difference between good and bad theta functions.

Note that $S_0 = -S_0$ and $S_1 = -S_1$. We always assume, in what follows, that $\Theta \neq 0$, i.e., $S_0 \neq \emptyset$.

- 1. If $x_1 \notin S_1$, $x_2, x_3, x_4 \in S_0$, then $2x_1 + x_2 + x_3 + x_4 \notin S_0$.

Proof. Use the quartic relation on Θ , with $\alpha_1 = 2x_1 + x_2 + x_3 + x_4$, $\alpha_2 = x_2$, $\alpha_3 = x_3$, $\alpha_4 = x_4$, $\gamma = -x_1 - x_2 - x_3 - x_4$. *Q.E.D.*

- 2. $0 \in S_1$.

Proof. Assume $0 \notin S_1$. Take any $y \in S_0$. Apply (1.) with $x_2 = x_3 = y$, $x_4 = -y$ and we get a contradiction. *Q.E.D.*

- 3. $x, y \in S_0 \Rightarrow \frac{1}{2}(x + y) \in S_1$.

Proof. Apply (1.) with $x_1 = \frac{1}{2}(x + y)$, $x_2 = x$, $x_3 = -y$ and $x_4 = -x$. *Q.E.D.*

Because of (2.), there is an $\eta_0 \in \frac{1}{2}A$ such that $\Theta(\eta_0) \neq 0$. Fix one such η_0 .

$$4. (0) \subseteq (S_0 + \eta_0) \subseteq (2S_0 + A) \subseteq (4S_0 + A) \subseteq \dots$$

Proof. By (3), if $x \in S_0$, then $\frac{1}{2}(x + \eta_0) \in S_1$, so $x + \eta_0 \in 2S_0 + A$. This gives the 1st inclusion. This also shows that $2x \in 4S_0 + A$. Hence if $y \in 2^k S_0$, so $y = 2^k \cdot x$, $x \in S_0$, then $2^k \cdot x \in 2^{k+1} S_0 + A$. This gives the rest of the inclusions. *Q.E.D.*

Definition 3.

$$S_\infty = \bigcup_{k \geq 1} [2^k S_0 + A].$$

5. S_∞ is a group.

Proof. Let $x, y \in S_\infty$. Now $x, y \in (2^l \cdot S_0 + A)$ for some $l \geq l_0$. Then $x = 2^l \cdot x_0 + \eta$, $y = 2^l \cdot y_0 + \zeta$, $x_0, y_0 \in S_0$ and $\eta, \zeta \in A$. Therefore by (3), $\frac{1}{2}(x_0 + y_0) \in S_1$, hence $2^l(x_0 + y_0) \in 2^{l+1} \cdot S_0 + A$. Therefore $x + y \in (2^{l+1} S_0 + A) \subset S_\infty$. *Q.E.D.*

6. $S_\infty = W + A$, for some subvector space $W \subset V$.

Proof. This is easily seen to be equivalent to asserting that S_∞/A is a divisible subgroup of V/A . But if $x \in 2^k \cdot S_0 + A$, then $x = 2^k \cdot x_0 + \eta$, $x_0 \in S_0$, $\eta \in A$, hence $x - \eta \in 2\{2^{k-1} S_0\} \subset 2 \cdot S_\infty$, i.e., the image of x in S_∞/A is divisible by 2. *Q.E.D.*

Definition 4. A theta function is *non-degenerate* if equivalently:

$$(a) \quad S_\infty = V.$$

$$(a') \quad S_\infty \supset \frac{1}{2}A.$$

$$(a'') \quad \text{For all sufficiently large } n, 2^n \cdot S_0 + A \supset \frac{1}{2}A.$$

(a''') For all sufficiently large n , and $\alpha \in 2^{-n-1}A$, there is an $\eta \in 2^{-n}A$ such that $\Theta(\alpha + \eta) \neq 0$.

The next step is to form, via the function Θ , a sequence of graded rings:

Definition 5. If M is a vector space of k -valued functions on V , let

$$\mathcal{S}(M) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n(M),$$

where $\mathcal{S}_0(M) = k$, $\mathcal{S}_1(M) = M$, and $\mathcal{S}_n(M)$, for $n \geq 2$, is the vector space of functions on V spanned by the products $f_{i_1} \dots f_{i_n}$, ($f_{i_j} \in M$, all j). Another convenient notation is the following:

$$M^* = \left\{ \begin{array}{l} \text{set of functions } \alpha \mapsto f(\alpha/2), \\ \text{all } f \in M \end{array} \right\}.$$

In particular, let

$$M_{2k} = \text{span of the functions } \Theta_{[\beta]}, \quad \text{all } \beta \in 2^{-k}A$$

where

$$\Theta_{[\beta]}(\alpha) = e(\beta/2, \alpha) \cdot \Theta(\alpha - \beta).$$

The corresponding rings $\mathcal{S}(M_{2k})$ will be the heart of our analysis. These are only half of the rings we need, however. To define the others, choose a decomposition:

$$\Lambda = \Lambda_1 \oplus \Lambda_2$$

such that $Q_2 \cdot \Lambda_i = V_i$ is an isotropic subspace under e , and such that $e_*(\alpha/2) = 1$ for all $\alpha \in \Lambda_1$ or Λ_2 . This exists because if we choose coordinates $V \cong Q_2^{2g}$ such that Λ, e, e_* take their standard forms, then $\Lambda_1 = \mathbb{Z}_g^2 \times \{0\}$, $\Lambda_2 = \{0\} \times \mathbb{Z}_g^2$ have these properties. In terms of Λ_1 and Λ_2 , we now define a kind of “dual” theta-function ϕ . It is to satisfy the equations:

$$\sum_{\zeta \in \frac{1}{2}\Lambda_1/\Lambda_1} e(\alpha, \zeta) \cdot \Theta(\alpha + \beta + \zeta) \cdot \Theta(\alpha - \beta + \zeta) = \phi(\alpha) \cdot \phi(\beta)$$

all $\alpha, \beta \in V$. In fact, if we let $\Phi(\alpha, \beta)$ denote the left-hand side of this equation, then the quartic equations on Θ are equivalent to:

$$\Phi(\alpha, \beta) \cdot \Phi(\gamma, \delta) = \Phi(\alpha, \delta) \cdot \Phi(\gamma, \beta)$$

for all $\alpha, \beta, \gamma, \delta \in V$ (cf. proof of Lemma 2, § 8). This, plus the elementary fact $\Phi(\alpha, \beta) = \Phi(\beta, \alpha)$ implies that one and (up to scalars) only one such ϕ exists. Notice that ϕ satisfies the equations:

(i) $\phi(\alpha + \beta) = f_*(\beta) \cdot e(\beta, \alpha) \cdot \phi(\alpha)$, for all $\alpha \in V$, $\beta \in \frac{1}{2}\Lambda_1 + \Lambda_2$, if $f_*(\frac{1}{2}\beta_1 + \beta_2) = e(\frac{1}{2}\beta_1, \beta_2)$ ($\beta_i \in \Lambda_i$).

(ii) $\phi(-\alpha) = \phi(\alpha)$, all $\alpha \in V$,

as well as certain quartic equations. Now let

$$M_{2k+1} = \text{span of the functions } \phi_{[\beta]}, \quad \beta \in 2^{-k-1} \cdot \Lambda$$

where

$$\phi_{[\beta]}(\alpha) = e(\beta, \alpha) \cdot \phi(\alpha - \beta).$$

Proposition 1. 1. $\mathcal{S}_2(M_{2k}) \subseteq M_{2k+1}$, equality holding if and only if for all $\beta \in 2^{-k-1}\Lambda$, $\exists \gamma \in 2^{-k}\Lambda$ such that $\phi(\beta + \gamma) \neq 0$.

2. $\mathcal{S}_2(M_{2k+1})^* \subseteq M_{2k+2}$, equality holding if and only if for all $\beta \in 2^{-k-1}\Lambda$, $\exists \gamma \in 2^{-k}\Lambda$ such that $\Theta(\beta + \gamma) \neq 0$.

Proof. To compute $\mathcal{S}_2(M_{2k})$, note that it is spanned by the functions:

$$f(\alpha) = \sum_{\eta \in \frac{1}{2}\Lambda_1/\Lambda_1} e\left(\eta, \frac{\beta_1 + \beta_2}{2}\right) \cdot \Theta_{[\beta_1 - \eta]}(\alpha) \cdot \Theta_{[\beta_2 - \eta]}(\alpha)$$

where $\beta_i \in 2^{-k}A$. But

$$\begin{aligned} f(\alpha) &= e\left(\frac{\beta_1 + \beta_2}{2}, \alpha\right) \cdot \sum_{\eta \in \frac{1}{2}A_1/A_1} e\left(\alpha - \frac{\beta_1 + \beta_2}{2}, \eta\right) \times \\ &\quad \times \Theta(\alpha - \beta_1 + \eta) \Theta(\alpha - \beta_2 + \eta) \\ &= e\left(\frac{\beta_1 + \beta_2}{2}, \alpha\right) \cdot \phi\left(\alpha - \frac{\beta_1 + \beta_2}{2}\right) \cdot \phi\left(\frac{\beta_1 - \beta_2}{2}\right) \\ &= \phi_{\left[\frac{\beta_1 + \beta_2}{2}\right]}(\alpha) \cdot \phi\left(\frac{\beta_1 - \beta_2}{2}\right) \in M_{2k+1}. \end{aligned}$$

We get every $\phi_{[\gamma]}$, $\gamma \in 2^{-k-1}A$, in this way, if and only if every such γ can be written:

$$\gamma = \frac{\beta_1 + \beta_2}{2}, \quad \beta_i \in 2^{-k}A$$

such that

$$\phi\left(\frac{\beta_1 - \beta_2}{2}\right) \neq 0.$$

This is exactly the condition in (1). To prove (2), first notice the identity:

$$\begin{aligned} (\alpha) \quad & \sum_{\zeta \in \frac{1}{2}A_2/A_2} e(\alpha, \zeta)^2 \cdot \phi(\alpha + \beta + \zeta) \cdot \phi(\alpha - \beta + \zeta) \\ &= \sum_{\substack{\zeta \in \frac{1}{2}A_2/A_2 \\ \eta \in \frac{1}{2}A_1/A_1}} e(\alpha, \zeta)^2 \cdot e(\alpha + \beta + \zeta, \eta) \cdot \Theta(2\alpha + 2\zeta + \eta) \cdot \Theta(2\beta + \eta) \\ &= \sum_{\eta \in \frac{1}{2}A_1/A_1} \Theta(2\alpha + \eta) \cdot \Theta(2\beta + \eta) \cdot e(\alpha + \beta, \eta) \cdot \left[\sum_{\zeta \in \frac{1}{2}A_2/A_2} e(2\zeta, \eta) \right] \\ &= 2^g \cdot \Theta(2\alpha) \cdot \Theta(2\beta). \end{aligned}$$

Now $\mathcal{S}_2(M_{2k+1})^*$ is spanned by the various functions:

$$f(\alpha) = \sum_{\eta \in \frac{1}{2}A_2/A_2} e(\eta, \beta_1 + \beta_2) \cdot \phi_{[\beta_1 - \eta]}(\alpha/2) \cdot \phi_{[\beta_2 - \eta]}(\alpha/2)$$

where $\beta_i \in 2^{-k-1}A$. But this f comes out as:

$$f(\alpha) = 2^g \cdot \Theta_{[\beta_1 + \beta_2]}(\alpha) \cdot \Theta(\beta_1 - \beta_2) \in M_{2k+2}.$$

(2) now follows just like (1). *Q.E.D.*

Corollary. *If Θ is non-degenerate, then for all $k \geq 0$,*

$$\begin{aligned} \mathcal{S}_2(M_{2k}) &= M_{2k+1} \\ \mathcal{S}_2(M_{2k+1})^* &= M_{2k+2}. \end{aligned}$$

Proof. The 2nd equality is clear, by condition (a''') of the definition of non-degenerate. As for the first, note that by formula (α) in the proof of the Proposition,

$$2^g \Theta(\alpha)^2 = \sum_{\zeta \in \frac{1}{2} A_2 / A_2} e(\alpha, \zeta) \cdot \phi(\alpha + \zeta) \cdot \phi(\zeta).$$

Therefore, $[\Theta(\alpha) \neq 0] \Rightarrow [\phi(\alpha + \zeta) \neq 0, \text{ some } \zeta \in \frac{1}{2} A_2]$. Thus the non-degeneracy of Θ implies the same for ϕ , and the 1st equality follows too. *Q.E.D.*

In the following discussion, we shall assume that Θ is non-degenerate. As usual, if $R = \Sigma R_n$ is a graded ring, then $R(2)$ is the graded ring ΣR_{2n} . The Corollary shows that there exists a k_0 such that for all $k \geq k_0$,

$$(\beta) \quad \mathcal{S}(M_k)(2) \cong \mathcal{S}(M_{k+1}).$$

In particular, the corresponding schemes

$$X = \text{Proj}(\mathcal{S}(M_k)),$$

for $k \geq k_0$, are all canonically isomorphic. We shall prove eventually that this X is an abelian variety.

So far, we know that $\mathcal{S}(M_k)$ is finitely generated over k . Moreover, it has no nilpotents: if it did, it would have a homogeneous nilpotent element $f \in \mathcal{S}_n(M_k)$. Then $f \neq 0 \Rightarrow f(\alpha) \neq 0$, some $\alpha \in V \Rightarrow f^N(\alpha) \neq 0$, all $N \Rightarrow f^N \neq 0$ in $\mathcal{S}_n(M_k)$. Therefore, X is a reduced algebraic scheme over k . In fact, we can map

$$V/\Lambda \rightarrow X$$

by evaluating functions in $\mathcal{S}(M_k)$ at points of V . To be more precise, for all $\alpha \in V$, define a homogeneous prime ideal $P(\alpha) \subset \mathcal{S}(M_{2k})$ [resp. $P(\alpha) \subset \mathcal{S}(M_{2k+1})$] by:

$$P(\alpha) = \sum_n P_n(\alpha)$$

$$\begin{aligned} P_n(\alpha) &= \{f \in S_n(M_{2k}) \mid f(2^k \alpha) = 0\} \\ \text{resp.} \quad &= \{f \in S_n(M_{2k+1}) \mid f(2^k \alpha) = 0\}. \end{aligned}$$

It is easy to check that for all k , if the $P(\alpha)$ in $\mathcal{S}(M_k)$ is intersected with $\mathcal{S}(M_k)(2)$, the resulting ideal is equal to the $P(\alpha)$ in $\mathcal{S}(M_{k+1})$ under the isomorphisms (β) . For this reason, we omit a k in the notation $P(\alpha)$. Thus $P(\alpha)$ gives a well-defined point $\bar{P}(\alpha) \in X$. It follows easily from the definition that:

- a) $\bar{P}(\alpha)$ is a k -rational point of X ,
- b) $\bar{P}(\alpha + \beta) = \bar{P}(\alpha)$, if $\beta \in \Lambda$.

Moreover:

c) $\{\bar{P}(\alpha) \mid \alpha \in V\}$ is dense in X .

Proof of c. Take $2k \geq k_0$. If (c) were false, for large n , there would be a non-zero function $f \in \mathcal{S}_n(M_{2k})$ that vanished at all $\bar{P}(\alpha)$'s. But $f(\bar{P}(\alpha)) = 0 \Leftrightarrow f(2^k \alpha) = 0$, so f would vanish everywhere on V , hence $f = 0$. *Q.E.D.*

One can do even more: for $\alpha \in V$, I claim that there is an automorphism $T_\alpha: X \rightarrow X$ such that $T_\alpha(\bar{P}(\beta)) = \bar{P}(\alpha + \beta)$, all $\beta \in V$. To construct T_α , let k_1 be the least integer such that $2^{k_1} \alpha \in A$. Define

$$\begin{aligned} T_\alpha^*: \mathcal{S}(M_{2k}) &\rightarrow \mathcal{S}(M_{2k}) \\ \text{resp.} \quad \mathcal{S}(M_{2k+1}) &\rightarrow \mathcal{S}(M_{2k+1}) \end{aligned}$$

by:

$$\begin{aligned} T_\alpha^* f(\beta) &= e(\beta, 2^{k-1} \alpha)^n \cdot f(\beta + 2^k \alpha), & \text{all } f \in \mathcal{S}_n(M_{2k}) \\ \text{resp.} \quad &= e(\beta, 2^k \alpha)^n \cdot f(\beta + 2^k \alpha), & \text{all } f \in \mathcal{S}_n(M_{2k+1}) \end{aligned}$$

(where we assume $k \geq k_1$). To check that this is, indeed, an automorphism of $\mathcal{S}(M_{2k})$ [resp. $\mathcal{S}(M_{2k+1})$], it suffices to check that $T_\alpha^* \theta_{[\gamma]} \in M_{2k}$, all $\gamma \in 2^{-k} A$; and $T_\alpha^* \phi_{[\gamma]} \in M_{2k+1}$, all $\gamma \in 2^{-k-1} A$. But, in fact, one computes:

$$\begin{aligned} T_\alpha^* \theta_{[\gamma]} &= e_*(2^{k-1} \alpha) \cdot e(\gamma, 2^k \alpha) \cdot \theta_{[\gamma]} \\ (\gamma) \quad T_\alpha^* \phi_{[\gamma]} &= f_*(2^k \alpha) \cdot e(\gamma, 2^{k+1} \alpha) \cdot \phi_{[\gamma]}. \end{aligned}$$

Moreover, one finds that T_α^* , acting on $\mathcal{S}(M_k)$, induces the same automorphism on $\mathcal{S}(M_k)$ (2) that you get by considering the T_α^* acting on $\mathcal{S}(M_{k+1})$ and carrying it across via the isomorphisms (β) of $\mathcal{S}(M_k)$ (2) and $\mathcal{S}(M_{k+1})$. Therefore, the T_α^* 's all define one and the same automorphism T_α of X . Note that:

$$d) (T_\alpha^*)^{-1}(P(\beta)) = P(\alpha + \beta).$$

Proof. If $f \in \mathcal{S}_n(M_{2k})$ or $\mathcal{S}_n(M_{2k+1})$, then

$$T_\alpha^* f \in P(\beta) \Leftrightarrow T_\alpha^* f(2^k \beta) = 0 \Leftrightarrow f(2^k \alpha + 2^k \beta) = 0 \Leftrightarrow f \in P(\alpha + \beta),$$

hence

$$d') T_\alpha(\bar{P}(\beta)) = \bar{P}(\alpha + \beta).$$

One checks also (via (γ) if you like) that:

$$e) T_{\alpha_1 + \alpha_2} = T_{\alpha_1} \circ T_{\alpha_2},$$

$$f) T_\alpha = \text{id.} \Leftrightarrow \alpha \in A,$$

so that T is a faithful action of the group V/A on the scheme X .

A remarkable consequence of all this is:

Proposition 2. *If θ is non-degenerate, then $\mathcal{S}(M_k)$ is an integral domain, for all k .*

Proof. We show first that $\mathcal{S}(M_k)$ is a domain if $k \geq k_0$. Since $\mathcal{S}(M_k)$ has no nilpotents, this is equivalent to showing that X is irreducible. Now V/A acts on X , so it permutes the various components of X , i.e., we have a homomorphism:

$$V/A \rightarrow S = \left\{ \begin{array}{l} \text{gp. of permutations} \\ \text{of components of } X \end{array} \right\}.$$

But S is a *finite* group and V/A is a *divisible* group. So V/A must map each component X_i into itself. On the other hand, the collection of points $\{\bar{P}(\alpha)\}$ forms a single orbit of the action of V/A on X . Therefore, all these points $\{\bar{P}(\alpha)\}$ belong to a single component of X . Since they are also dense in X , X can have only a single component. Therefore $\mathcal{S}(M_k)$ is a domain if $k \geq k_0$.

In general, suppose some $\mathcal{S}(M_k)$ were not a domain. Then there would be homogeneous elements $f \in \mathcal{S}_n(M_k)$, $g \in \mathcal{S}_m(M_k)$ such that $f \cdot g = 0$, $f \neq 0$, $g \neq 0$. Now f^2 and g^2 can be considered as elements of $\mathcal{S}(M_{k+1})$. Since $f \cdot g = 0$, we still have $f^2 \cdot g^2 = 0$. Also, since $\mathcal{S}(M_k)$ has no nilpotents, $f^2 \neq 0$ and $g^2 \neq 0$. Therefore $\mathcal{S}(M_{k+1})$ is not a domain either. Continuing in this way, we find that $\mathcal{S}(M_l)$ is not a domain for all $l \geq k$, which contradicts the first part of the proof. *Q.E.D.*

Corollary 1. *The following are equivalent:*

- i) Θ is non-degenerate,
- ii) $S_1 = V$, i.e., for all $\alpha \in V$, $\exists \eta \in \frac{1}{2}A$ such that $\Theta(\alpha + \eta) \neq 0$.
- iii) For all $\alpha \in \frac{1}{4}A$, $\exists \eta \in \frac{1}{2}A$ such that $\Theta(\alpha + \eta) \neq 0$.

Proof. Clearly (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Now assume (i) holds. If $\Theta(\alpha + \eta) = 0$, all $\eta \in \frac{1}{2}A$, then it would follow from the definition of ϕ that $\phi(\alpha + \beta) \times \phi(\beta) = 0$, all $\beta \in V$. But this means that $\phi_{[-\alpha]} \cdot \phi_{[0]} = 0$, i.e., one of the rings $\mathcal{S}(M_{2k+1})$ is not domain. This contradicts the Prop., so (ii) must hold. *Q.E.D.*

Corollary 2. $\mathcal{S}(M_k)(2) \cong \mathcal{S}(M_{k+1})$, for all $k \geq 2$.

Proof. In view of Prop. 1, this follows from Cor. 1 provided that we check: $\forall \alpha \in V$, $\exists \eta \in \frac{1}{2}A$ such that $\phi(\alpha + \eta) \neq 0$. Looking back at the proof of the Cor. to Prop. 1, you see that this too follows from Cor. 1. *Q.E.D.*

To show that X is actually an abelian variety, we could either define the group law explicitly, using the addition formula of § 2, or else we can use only the action of V/A on X and combine this with general structure theorems on the automorphisms of a variety. Although the former is more elementary, we follow the latter approach as it is quicker.

X is given to us together with a projective embedding. For example, $X = \text{Proj}(\mathcal{S}(M_2))$, so

$$X \subset \mathbf{P}(M_2).$$

Let L_2 be the invertible sheaf induced on X via this embedding. If, via the isomorphism $X \cong \text{Proj } (\mathcal{S}(M_k))$, we embed X in $P(M_k)$, the induced sheaf L_k is just:

$$L_k \cong L_2^{2^{k-2}}.$$

Let $\mathcal{P}$ denote the family of all invertible sheaves algebraically equivalent to L_2 . We shall use the fact that $\text{Aut}(X, \mathcal{P})$, the group of automorphisms of the pair $X, \mathcal{P}$, is an algebraic group (MATSUSAKA [14], GROTHENDIECK [15], p. 221–20). For all $\alpha \in V/\Lambda$, if $2^k \alpha \in \Lambda$, then T_α is induced by an automorphism T_α^* of $\mathcal{S}(M_{2k})$; therefore $T_\alpha^*(L_{2k}) \cong L_{2k}$; therefore $T_\alpha^*(L_2)$ differs from L_2 by an invertible sheaf of finite order; therefore $T_\alpha^{-1}(\mathcal{P}) = \mathcal{P}$. In other words, the action of V/Λ on X factors through an injective homomorphism:

$$V/\Lambda \rightarrow \text{Aut}(X, \mathcal{P}).$$

Let A be the Zariski-closure of V/Λ in $\text{Aut}(X, \mathcal{P})$. Then A is connected since V/Λ is divisible and dense in A (cf. proof of Prop. 2), and A is commutative since V/Λ is commutative and dense in A . Moreover, since the V/Λ -orbit of $\bar{P}_0$ is dense in X , the A -orbit of $\bar{P}_0$ must be an open dense set in X , i.e., A acts generically transitively on X . In fact, the morphism

$$\begin{aligned} \psi: A &\rightarrow X \\ \sigma &\mapsto \sigma(\bar{P}_0) \end{aligned}$$

is an open immersion of A in X . This follows since the image $\psi(A)$ is always isomorphic to A/H , H = the stabilizer of $\bar{P}_0$; and since A is commutative and acting faithfully on X , all stabilizers are trivial.

Next, we want to compute the dimension of X . I claim that the Hilbert polynomial of (X, L_2) is given by:

Proposition 3. $\chi(L_2^n) = 4^g \cdot n^g$.

Proof. For k large,

$$\begin{aligned} \chi(L_2^{2^{2k}}) &= \dim(S_{2^{2k}}(M_2)) \\ &= \dim(M_{2+2k}). \end{aligned}$$

Now $M_{2(k+1)}$ is, by definition, the span of the $2^{2g(k+1)}$ functions $\Theta_{[\beta]}$, where β runs over cosets of $2^{-k-1}\Lambda/\Lambda$. But these functions are linearly independent. To see this, look at the automorphisms T_α^* of $\mathcal{S}(M_{2(k+1)})$, where $\alpha \in 2^{-k-1}\Lambda$. Use formulae (γ) above and note that each $\Theta_{[\gamma]}$ gives rise to a distinct set of eigenvalues for the T_α^* 's. Therefore, the $\Theta_{[\gamma]}$'s could not be dependent unless one were identically zero, and this is not the case. Therefore

$$\dim M_{2(k+1)} = 4^g \cdot (2^{2k})^g.$$

This shows that $\chi(L_2^n)$ and $4^g \cdot n^g$ agree for an infinite set of values of n . Since both are polynomials, they are always equal. *Q.E.D.*

Corollary. $\dim X = g$.

Returning to A , we find that A is a commutative g -dimensional algebraic group containing a subgroup isomorphic to $(\mathbb{Q}_2/\mathbb{Z}_2)^{2^g}$. From well-known structure theorems on algebraic groups, the only such A 's are abelian varieties. Therefore A is complete, hence $A = X$, hence:

(I) X is an abelian variety.

Moreover, in the course of proving this, we have also found that V/Λ is acting on X via translations, hence (comparing orders) we find:

(II) $\alpha \mapsto \bar{P}(\alpha)$ is a group isomorphism of V/Λ with $\text{tor}_2(X)$.

Up to this point, identifying the various $\text{Proj}(\mathcal{S}(M_k))$'s has been useful. But to go further, it is more convenient now to drop these identifications. Therefore, now let

$$X_n = \text{Proj}(\mathcal{S}(M_{2^n})).$$

This is a family of isomorphic abelian varieties. However, the most natural maps between them are given by the inclusions:

$$\begin{aligned} M_{2^n} &\subset M_{2^{n+2}} \\ \mathcal{S}(M_{2^n}) &\subset \mathcal{S}(M_{2^{n+2}}) \end{aligned}$$

inducing finite morphisms:

$$X_n \xleftarrow{p} X_{n+1}.$$

To check that p is defined, we must know that $\mathcal{S}(M_{2^{n+2}})$ is integrally dependent on $\mathcal{S}(M_{2^n})$. But I claim:

$$\Theta(\gamma)^2 \cdot \Theta_{[\beta]}^2 = 2^{-g} \cdot \sum_{\eta \in \frac{1}{2}\Lambda/\Lambda} e(\eta, \gamma) \Theta(\eta)^2 \cdot \Theta_{[\beta+\gamma-\eta]} \cdot \Theta_{[\beta-\gamma+\eta]}.$$

$$[\text{Proof. } \Theta(\gamma)^2 \cdot \Theta_{[\beta]}^2(\alpha)^2 = e(\beta, \alpha) \Theta(\gamma) \Theta(\gamma) \Theta(\beta - \alpha) \Theta(\alpha - \beta).$$

By the quartic relations on Θ , we get

$$\begin{aligned} &= 2^{-g} e(\beta, \alpha) \sum_{\eta} e(-\gamma, \eta) \Theta(\eta)^2 \Theta(\beta - \alpha - \gamma + \eta) \Theta(\alpha - \beta - \gamma + \eta) \\ &= 2^{-g} \sum_{\eta} e(\eta, \gamma) \Theta(\eta)^2 \cdot \Theta_{[\beta+\gamma-\eta]}(\alpha) \cdot \Theta_{[\beta-\gamma+\eta]}(\alpha). \quad \text{Q.E.D.} \end{aligned}$$

Choose $\gamma \in \beta + \frac{1}{2}\Lambda$ so that $\Theta(\gamma) \neq 0$. Then if $\beta \in 2^{-n-1}\Lambda$, this equation shows that $\Theta_{[\beta]}^2 \in \mathcal{S}(M_{2^n})$. This proves that p is a finite morphism. Since X_n and X_{n+1} are abelian varieties, p must be an isogeny.

Define prime ideals:

$$\begin{aligned} &P^{(k)}(\alpha) \subset \mathcal{S}(M_{2k}) \\ \text{via} \quad &P^{(k)}(\alpha) = \sum_n P_n^{(k)}(\alpha) \\ &P_n^{(k)}(\alpha) = \{f \in \mathcal{S}_n(M_{2k}) \mid f(\alpha) = 0\}. \end{aligned}$$

Then $P^{(k)}(\alpha)$ defines a k -rational point $\psi_k(\alpha) \in X_k$. We have

- (a) $p(\psi_{k+1}(\alpha)) = \psi_k(\alpha)$.
- (b) $\alpha \mapsto \psi_k(\alpha)$ defines an isomorphism

$$V/2^k A \xrightarrow{\approx} \text{tor}_2(X_k).$$

(b) here follows from conclusion (II) above, noticing how we have reinterpreted the ideal $P(\alpha)$. In fact, if we call X the common abelian variety to which all the X_k 's were previously identified, then $\bar{P}(\alpha) \in X$ corresponds exactly to $\psi_k(2^k \alpha) \in X_k$. Therefore $\psi_k(\alpha) = 0 \Leftrightarrow \bar{P}(2^{-k} \alpha) = 0 \Leftrightarrow 2^{-k} \alpha \in A$. Moreover, this shows that via these identifications, we get a morphism:

$$\begin{array}{ccc} X & & \bar{P}(\alpha) \\ \wr \parallel & & \downarrow \\ X_{k+1} & & \psi_{k+1}(2^{k+1} \alpha) \\ p \downarrow & & \downarrow \\ X_k & & \psi_k(2^{k+1} \alpha) \\ \wr \parallel & & \downarrow \\ X & & \bar{P}(2\alpha) = 2\bar{P}(\alpha). \end{array}$$

This map, from X to X , agrees with 2δ at all points $\bar{P}(\alpha)$. Therefore it is equal to 2δ . In particular:

(c) The degree of p is 2^{2g} and $\text{Ker}(p) = \text{Ker}(2\delta)$. It follows that all the X_n 's generate a single 2-tower. Call this $X = \{X_\alpha\}_{\alpha \in S}$, and let $X_n = X_{\alpha_n}$, $\alpha_n \in S$. Moreover, these α_n 's are a cofinal set in S , by (c). In view of (a)

$$\alpha \mapsto \{\psi_k(\alpha)\}$$

defines a homomorphism

$$\psi: V \rightarrow V(X),$$

and (b) implies that ψ is an isomorphism. More, (b) shows that the compact open subgroups $2^k A$ and $T(\alpha_k)$ correspond to each other under ψ .

This 2-tower is polarized too. Let L_k be the sheaf $\mathcal{O}(1)$ on X_k coming from its presentation as $\text{Proj}(\mathcal{S}(M_{2k}))$. Since the p 's comes from gradation preserving homomorphisms of the $\mathcal{S}(M_{2k})$'s it follows that $p^*(L_k) \cong L_{k+1}$. To check that L_k is totally symmetric, we need the inverse on X_k :

Let $\iota^*(f)(\alpha) = f(-\alpha)$, all $f \in \mathcal{S}(M_{2k})$.

Then ι^* defines an involution

$$\iota: X_k \rightarrow X_k$$

such that $\iota(\psi_k(\alpha)) = \psi_k(-\alpha)$.

Therefore ι agrees with the inverse of X_k on all points $\psi_k(\alpha)$, hence $\iota = \text{inverse of } X_k$.

Since ι is induced at all by an automorphism ι^* of $\mathcal{S}(M_{2k})$, it follows that L_k is at least a symmetric sheaf. Since

$$\{\psi_k(\alpha) \mid \alpha \in 2^{k-1} \Lambda / 2^k \Lambda\} = \text{Kernel of } 2\delta \text{ in } X_k,$$

L_k is totally symmetric if and only if ι^* is the identity in $\mathcal{S}(M_{2k})/P^{(k)}(\alpha)$, all $\alpha \in 2^{k-1} \Lambda$. This means that for all $f \in M_{2k}$, $\iota^* f - f \in P_1^{(k)}(\alpha)$, i.e., $f(\alpha) = f(-\alpha)$. But M_{2k} is spanned by $\Theta_{[\beta]}$'s, $\beta \in 2^{-k} \Lambda$, and if $\beta \in 2^{-k} \Lambda$, $\alpha \in 2^{k-1} \Lambda$, then:

$$\Theta_{[\beta]}(-\alpha) = e\left(\frac{\beta}{2}, -\alpha\right) \Theta(-\alpha - \beta) = e\left(\frac{\beta}{2}, \alpha\right) \Theta(\alpha - \beta) = \Theta_{[\beta]}(\alpha).$$

Therefore all the L_n 's are totally symmetric and $\{X_n, L_n\}$ extends to a polarized 2-tower $\mathcal{T} = \{X_\alpha, L_\alpha\}$. We shall leave it to the reader to check the key fact that ψ is symplectic:

$$(d) e_\lambda(\psi\alpha, \psi\beta) = e(\alpha, \beta), \text{ all } \alpha, \beta \in V.$$

Recapitulating this whole section so far, we have defined an arrow:

$$\Xi: \left\{ \begin{array}{l} \text{Given a non-degenerate} \\ \text{theta function } \Theta \text{ on } V \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{construct a polarized} \\ \text{2-tower } \mathcal{T} = \{X_\alpha, L_\alpha\}, \\ \text{plus a symplectic isomorphism} \\ \psi: V \xrightarrow{\sim} V(X) \end{array} \right\}.$$

Now, on V we have the vector space of functions spanned by all the $\Theta_{[\beta]}$'s. On $V(X)$, we have the vector space of all theta functions $\mathfrak{g}[\Gamma(\mathcal{T})]$ of the tower $\mathcal{T}$.

Proposition 4. *Via ψ , these vector spaces are equal:*

$$\text{Span of } \Theta_{[\beta]} \text{'s} = \{\mathfrak{g}_{[s]} \circ \psi \mid s \in \Gamma(\mathcal{T})\}.$$

Moreover, Θ itself is the unique function f (up to scalars) of the form $\mathfrak{g}_{[s]} \circ \psi$ satisfying the functional equation:

$$f(\alpha + \beta) = e_*(\beta/2) \cdot e(\beta/2, \alpha) \cdot f(\alpha), \quad \text{all } \alpha \in V, \beta \in \Lambda.$$

Key Corollary 1. *If $V = Q_2^2$, $\Lambda = Z_2^2$, and e, e_* have the standard forms of § 9, then Θ is exactly the theta function $\mathfrak{g}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \circ \psi$ associated to the*

triple $(X, \mathcal{T}, \psi^{-1})$ in § 9. In other words, Ξ is an inverse to the map Θ of § 9.

Proof of Prop. 4. Let $\alpha \in 2^{-k_1} \Lambda$ and let $k \geq k_1$. Define $T_\alpha^*: \mathcal{S}(M_{2k}) \rightarrow \mathcal{S}(M_{2k})$ slightly differently from before:

$$T_\alpha^* f(\beta) = e \left(\beta, \frac{\alpha}{2} \right)^n \cdot f(\beta + \alpha), \quad \text{all } f \in S_n(M_{2k}).$$

Note $T_\alpha^{*-1}(P^{(k)}(\beta)) = P^{(k)}(\alpha + \beta)$. Let $T_\alpha: X_k \rightarrow X_k$ be the automorphism induced by T_α^* . Then $T_\alpha(\psi_k(\beta)) = \psi_k(\alpha + \beta)$, hence T_α is translation by the point $\psi_k(\alpha)$, i.e.,

$$T_\alpha = T_{\psi_k(\alpha)}.$$

Moreover, T_α^* also induces a compatible isomorphism:

$$g_k(\alpha): L_k \xrightarrow{\sim} T_{\psi_k(\alpha)}^* L_k.$$

For all $k \geq k_1$, these are compatible, so the totality of pairs

$$g(\alpha) = \{(\psi_k(\alpha), g_k(\alpha)) \mid k \geq k_1\}$$

is a point of $\mathcal{G}(\mathcal{T})$.

(*) $g(\alpha) = \sigma[\psi(\alpha)]$, i.e., $g(\alpha)$ is the canonical element of $\mathcal{G}(\mathcal{T})$ over the point $\psi(\alpha)$ in $V(X)$.

*Proof of *.* This requires checking 2 things: (i) $g(\alpha)$ is a symmetric element of $\mathcal{G}(\mathcal{T})$, i.e., $\delta_{-1} g(\alpha) = g(\alpha)^{-1}$, and (ii) $g(2\alpha) = g(\alpha)^2$. In terms of T_α^* , this is the same as:

$$(i) \iota^* \circ T_\alpha^* = (T_\alpha^*)^{-1} \circ \iota^*.$$

$$(ii) T_{2\alpha}^* = T_\alpha^* \circ T_\alpha^*.$$

These are both immediate. *Q.E.D.*

Next, notice that $M_{2k} \cong \Gamma(X_k, L_k)$. In fact, there is a canonical map $M_{2k} \rightarrow \Gamma(X_k, L_k)$; it is injective, since the ring $\mathcal{S}(M_{2k})$ has no nilpotents, and only nilpotent elements of $\mathcal{S}_n(M_{2k})$ define trivial sections of L_k^n ; but it is easy to check that both $\dim M_{2k}$ and $\dim \Gamma(X_k, L_k)$ are equal to $2^{2k}g$; therefore $M_{2k} \cong \Gamma(X_k, L_k)$. Therefore,

$$\Gamma(\mathcal{T}) = \varinjlim_k \Gamma(X_k, L_k) \cong \bigcup_k M_{2k} = \left\{ \begin{array}{l} \text{Span of all the} \\ \text{functions } \Theta_{[\beta]} \\ \beta \in V \end{array} \right\}.$$

Now let f be some linear combination of the $\Theta_{[\beta]}$. Say $f \in M_{2k_1}$. Let f define $s \in \Gamma(X_{k_1}, L_{k_1})$. I claim that:

$$(*) \quad f(\alpha) = \mathfrak{g}_{[s]}(\psi \alpha), \quad \text{all } \alpha \in V.$$

Taking a larger k_1 if necessary, we may suppose that $\alpha \in 2^{-k_1} \Lambda$. By definition, $\mathfrak{g}_{[\alpha]}$ at $\psi\alpha$ is the “value” at the origin of X_{k_1} of the section of L_{k_1} obtained via the map:

$$\Gamma(X_{k_1}, L_{k_1}) \xrightarrow{\sim_{g_{k_1}(-\alpha)}} \Gamma(X_{k_1}, T_{\psi_{k_1}(-\alpha)}^* L_{k_1}) \xrightarrow{\sim_{T_{\psi_{k_1}(\alpha)}^*}} \Gamma(X_{k_1}, L_{k_1}).$$

This means that we simply apply the automorphism $(T_{-\alpha}^*)^{-1}$ of M_{2k} to f , and take the value at the origin. But $T_{-\alpha}^* = T_{\alpha}^{*-1}$, and $(T_{\alpha}^* f)(0) = f(\alpha)$, so $(*)$ is proven. Thus the span of the $\Theta_{[\beta]}$'s is the same as the space of functions $\mathfrak{g}_{[\alpha]} \circ \psi, s \in \Gamma(\mathcal{T})$.

As for the final assertion of the Proposition, on the one hand, Θ does satisfy the functional equation there; and, from the general theory of the space $\mathfrak{g}[\Gamma(\mathcal{T})]$ in § 8, we know that this functional equation has only a 1-dimensional set of solutions in $\mathfrak{g}[\Gamma(\mathcal{T})] \circ \psi$. *Q.E.D.*

Corollary 2. *All g -dimensional principally polarized abelian varieties X are isomorphic to $\text{Proj}(\mathcal{S}(M_2))$, where M_2 is the span of the $\Theta_{[\beta]}$'s, $\beta \in \frac{1}{2}\Lambda$, for some non-degenerate theta function Θ on V .*

Proof. Just take Θ to be the $\mathfrak{g} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ attached to X as in § 9, and carried over to a function on V by a suitable isomorphism of V and $V(X)$. *Q.E.D.*

Corollary 3. *The open set $M_{\infty} \subset \overline{M}_{\infty}$, which in § 9 represents the moduli functor $\mathcal{M}_{\infty}$, is the open set whose geometric points represent non-degenerate theta functions, i.e.,*

$$E = \left\{ \text{set of all systems of coset representatives} \right\} \\ r: \frac{1}{4} \mathbf{Z}_2^{2g} / \frac{1}{2} \mathbf{Z}_2^{2g} \rightarrow \frac{1}{4} \mathbf{Z}_2^{2g}$$

For all $r \in E$, let

$$U_r = \left\{ \text{open set in } \overline{M}_{\infty} \text{ defined by} \right\} \\ \left\{ X_{\alpha} \neq 0, \text{ all } \alpha \in \text{Image}(r) \right\}.$$

Then

$$M_{\infty} = \bigcup_{r \in E} U_r.$$

§ 11. Satake's Compactification

In this section, I want to analyze the degenerate theta functions Θ on V , in the sense of § 10. In particular, they all come from lower dimensional non-degenerate theta-functions via “cusps”. This will show that the whole moduli scheme $\overline{M}_{\infty}$ is a disjoint union of copies of the M_{∞} 's for dimensions g and lower i.e., that $\overline{M}_{\infty}$ is the Satake compactification of M_{∞}^1 .

¹ *Added in Proof.* A closer study has shown that $\overline{M}_{\infty}$ is not normal along $\overline{M}_{\infty} - M_{\infty}$. Its normalization is Satake's compactification.

Return to the discussion at the beginning of § 10: let V, Λ, e, e^* be given as before. First, I want to describe a way of forming degenerate theta functions on V out of theta functions on lower dimensional spaces.

Definition 1. A *cuspidal* is a subspace $W \subset V$ such that $W^\perp \subset W$, i.e., if $\alpha \in V$ has the property $e(\alpha, \beta) = 1$, all $\beta \in W$, then $\alpha \in W$.

Given a cuspidal W , let:

$$\tilde{V} = W/W^\perp$$

$$\tilde{\Lambda} = \Lambda \cap W / \Lambda \cap W^\perp$$

$$\tilde{e} = \text{induced skew-symmetric pairing, } \tilde{V} \times \tilde{V} \rightarrow k^*.$$

Lemma. $\tilde{\Lambda}$ is a maximal isotropic lattice in $\tilde{V}$, (for $\tilde{e}$).

Proof. Notice that $\Lambda/\Lambda \cap W$ is a free $\mathbb{Z}_2$ -module. Therefore the sequence:

$$0 \rightarrow \Lambda \cap W \rightarrow \Lambda \rightarrow \Lambda/\Lambda \cap W \rightarrow 0$$

splits, and $\Lambda = \Lambda_1 \oplus (\Lambda \cap W)$ for some sub $\mathbb{Z}_2$ -Module Λ_1 . Let $V_1 = Q_2 \cdot \Lambda_1$, so $V = V_1 \oplus W$. Now I claim:

$$(*) \quad (\Lambda \cap W)^\perp = \Lambda + W^\perp.$$

[In fact, let $\alpha \in V$ satisfy $e(\alpha, \beta) = 1$, all $\beta \in \Lambda \cap W$. Since V_1 and W are dual vector spaces via e , there is a $\gamma \in W^\perp$ such that $e(\alpha, \beta) = e(\gamma, \beta)$ all $\beta \in V_1$. But then $\alpha - \gamma$ is orthogonal to both V_1 and $\Lambda \cap W$, hence orthogonal to Λ , hence $\alpha - \gamma \in \Lambda$. Thus $\alpha \in W^\perp + \Lambda$.]

Now to show $\tilde{\Lambda}$ is maximal isotropic, let $\alpha \in W$ have an image $\tilde{\alpha}$ in $\tilde{V}$ perpendicular to $\tilde{\Lambda}$, i.e., $\alpha \in (W \cap \Lambda)^\perp$. By (*), $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in \Lambda$, $\alpha_2 \in W^\perp$. But then $\alpha_1 = \alpha - \alpha_2 \in W$. Therefore $\alpha_1 \in W \cap \Lambda$ so $\tilde{\alpha} = \tilde{\alpha}_1 \in \tilde{\Lambda}$. *Q.E.D.*

Definition 2. A *cuspidal with origin* is a cuspidal $W \subset V$, plus an element $\eta_0 \in \frac{1}{2}\Lambda$ such that

$$\text{i) } e_*(\alpha) = e(\alpha, \eta_0)^2, \text{ all } \alpha \in W^\perp \cap (\frac{1}{2}\Lambda).$$

$$\text{ii) } e_*(\eta_0) = 1.$$

It is not hard to check that every cuspidal has at least one origin: we leave this to the reader. Given a cuspidal with origin, look at the map

$$\alpha \mapsto e_*(\alpha) \cdot e(\alpha, \eta_0)^2$$

where $\alpha \in \frac{1}{2}\Lambda \cap W$. If $\beta \in \frac{1}{2}\Lambda \cap W^\perp$, then

$$\begin{aligned} e_*(\alpha + \beta) \cdot e(\alpha + \beta, \eta_0)^2 &= e_*(\alpha) \cdot e_*(\beta) \cdot e(\alpha, \beta)^2 \cdot e(\alpha, \eta_0)^2 \cdot e(\beta, \eta_0)^2 \\ &= e_*(\alpha) \cdot e(\alpha, \eta_0)^2. \end{aligned}$$

Thus there is a quadratic form $\tilde{e}_*: \frac{1}{2}\tilde{A}/\tilde{A} \rightarrow \{\pm 1\}$ such that

$$(*) \quad \tilde{e}_*(\tilde{\alpha}) = e_*(\alpha) \cdot e(\alpha, \eta_0)^2, \quad \text{all } \alpha \in \frac{1}{2}A \cap W.$$

It is not hard to check that the new data $(\tilde{V}, \tilde{A}, \tilde{e}, \tilde{e}_*)$ has the standard form required in § 10 (i.e., that the associated Arf-invariant is 0). We leave this to the reader also.

Now let $\tilde{\Theta}$ be a theta-function on $\tilde{V}$.

Definition 3. For all $\alpha \in V$, let

$$T_{W, \eta_0} \Theta(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \eta_0 + W + A \\ e_*\left(\frac{\eta_1}{2}\right) e\left(\frac{\eta_1}{2}, \eta_0\right) e\left(\frac{\eta_0 + \eta_1}{2}, \alpha\right) \tilde{\Theta}(\tilde{\alpha}_0) & \text{if } \alpha = \eta_0 + \eta_1 + \alpha_0, \eta_1 \in A, \alpha_0 \in W. \end{cases}$$

Proposition 1. *The above $T_{W, \eta_0} \tilde{\Theta}$ is well-defined (note that the $\alpha \in V$ may be decomposed in more than way as $\alpha = \eta_0 + \eta_1 + \alpha_0$), and is a theta-function on V .*

The proof of this Proposition is a ghastly but wholly straightforward set of computations. It took me several hours to do every bit and as I was no wiser at the end — except that I knew the definition was correct — I shall omit details here. Our main result is:

Theorem. *Let Θ be any theta-function on V , and let W be the subspace of V such that $S_\infty = W + A$ (cf. § 10). Then W is a cusp, and if η_0 is any origin for W , Θ is equal to $T_{W, \eta_0} \tilde{\Theta}$ for some non-degenerate theta-function $\tilde{\Theta}$ on $\tilde{W}$. In particular, W is characterized by:*

$$\text{coarse support } (\Theta) = W + \frac{1}{2}A.$$

The proof of this theorem will be based on the $\Theta \leftrightarrow \mu$ correspondence, given in Lemma 1, § 8. Before taking up the proof of the Theorem, we want to give this correspondence a more intrinsic formulation. Let $V = W_1 \oplus W_2$, where W_i are maximal isotropic subspaces, such that

- i) $A = A_1 \oplus A_2$, $A_i = A \cap W_i$.
- ii) $e_*(\alpha/2) = 1$, all α in A_1 or in A_2 .

Then

- a) Define a measure μ on W_1 , from a theta function Θ on V via

$$\mu(\alpha_1 + 2^n A_1) = 2^{-ng} \sum_{\alpha_2 \in 2^{-n} A_2 / A_2} e\left(\alpha_1, \frac{\alpha_2}{2}\right) \cdot \Theta(\alpha_1 + \alpha_2).$$

- b) Define a theta function Θ on V , from a measure μ on W_1 , via

$$\Theta(\alpha_1 + \alpha_2) = e\left(\alpha_1, \frac{\alpha_2}{2}\right) \int_{\alpha_1 + A_1} e(\alpha_2, \beta) \cdot d\mu(\beta).$$

Our proof will be based on the fact that any finitely additive measure μ (on the algebra of compact open subsets of W_1) has a *support*, i.e., a smallest closed set S such that:

$$\mu(U)=0, \quad \text{all compact open } U\text{'s in } W_1 - S.$$

Proof. Say S_A and S_B are closed sets such that $\mu(U)=0$ if $U \subset W_1 - S_A$ or $U \subset W_1 - S_B$. Then let $U \subset W_1 - (S_A \cap S_B)$ be a compact open set. We must decompose U into $U_A \cup U_B$, where $U_A \subset W_1 - S_A$, and $U_B \subset W_1 - S_B$, and U_A and U_B are compact and open. For all $x \in U \cap S_A$, note that $x \notin S_B$, so we can find a compact, open neighborhood U_x of x such that

$$U_x \subset U \cap (W_1 - S_B).$$

Since $U \cap S_A$ is compact, it can be covered by a finite set of these U_x 's: say

$$U \cap S_A \subset [U_{x_1} \cup \dots \cup U_{x_n}].$$

Let $U_B = U_{x_1} \cup \dots \cup U_{x_n}$. By construction $U_B \subset U \cap (W_1 - S_B)$ and U_B is compact and open. Let $U_A = U - U_B$. Then U_A is also compact and open and since $U_B \supset U \cap S_B$, it follows that $U_A \subset U \cap (W_1 - S_B)$. By assumption on S_A and S_B , we have $\mu(U_A)=0$ and $\mu(U_B)=0$. Therefore $\mu(U)=0$. This shows that the family of sets:

$$\mathcal{S} = \{S \text{ closed in } W_1 \mid \mu(U)=0 \text{ for all compact open sets } U \subset W_1 - S\}$$

is closed under finite intersections. Now let

$$S^* = \bigcap_{S \in \mathcal{S}} S.$$

I claim $S^* \in \mathcal{S}$ too. Let $U \subset W_1 - S^*$ be a compact open set. Since

$$W_1 - S^* = \bigcup_{S \in \mathcal{S}} (W_1 - S),$$

it follows that U is covered by the open sets $U \cap (W_1 - S)$, where $S \in \mathcal{S}$. Since U is compact, it can be covered by a finite number of such open sets:

$$U \subset (W_1 - S_1) \cup \dots \cup (W_1 - S_n)$$

where $S_1, \dots, S_n \in \mathcal{S}$. Now let $T \in \mathcal{S}$ be a closed set contained in all these S_i . Then $U \subset W_1 - T$. But $T \in \mathcal{S}$ means that this implies $\mu(U)=0$. So $\mu(U)=0$ whenever $U \subset W_1 - S^*$, i.e., $S^* \in \mathcal{S}$ too. *Q.E.D.*

Proposition. Let μ be a non-zero even Gaussian measure on W_1 (i.e., μ has the property (A) of Lemma 1, § 8). Then the support S of μ is a sub-vector space of W_1 .

Proof. Notice that if μ_1, μ_2 are 2 measures on W_1 , and $\mu_1 \times \mu_2$ is the induced measure on $W_1 \times W_1$, then

$$\text{Support}(\mu_1 \times \mu_2) = \text{Support}(\mu_1) \times \text{Support}(\mu_2).$$

Let $\xi: W_1 \times W_1 \rightarrow W_1 \times W_1$ be the map $\xi((x, y)) = (x + y, x - y)$. By definition, a Gaussian measure μ is associated to a second measure ν such that

$$\xi_*(\mu \times \mu) = \nu \times \nu.$$

Therefore, if $S' = \text{Support}(\nu)$, it follows that $\xi(S \times S) = S' \times S'$. In particular

$$\begin{aligned} \alpha \in S &\Leftrightarrow (\alpha, \alpha) \in S \times S \\ &\Leftrightarrow (2\alpha, 0) = \xi((\alpha, \alpha)) \in S' \times S'. \end{aligned}$$

Since S is non-empty, $0 \in S'$, and $\alpha \in S \Leftrightarrow 2\alpha \in S'$, i.e., $S' = 2S$. Therefore $0 \in S$ too, and we find:

$$\begin{aligned} \alpha \in S &\Leftrightarrow (\alpha, 0) \in S \times S \\ &\Leftrightarrow (\alpha, \alpha) = \xi((\alpha, 0)) \in S' \times S' \\ &\Leftrightarrow \alpha \in S'. \end{aligned}$$

Therefore $S = S'$ also. Finally,

$$\begin{aligned} \alpha, \beta \in S &\Rightarrow (\alpha, \beta) \in S \times S \\ &\Rightarrow (\alpha + \beta, \alpha - \beta) \in S' \times S' \\ &\Rightarrow \alpha + \beta, \alpha - \beta \in S' = S. \end{aligned}$$

Thus S is a closed subgroup of W_1 , such that $S = 2S$. Therefore S is a subvectorspace over $\mathcal{Q}_2$. *Q.E.D.*

Corollary. For all $\gamma_2 \in W_2$, all theta functions Θ on V ,

$$\text{Support}(\Theta) \subset \{\alpha \mid e(\alpha, \gamma_2) = 1\} \Rightarrow \Theta(\alpha + \lambda \gamma_2) = e\left(\alpha, \frac{\lambda \gamma_2}{2}\right) \Theta(\alpha),$$

all $\lambda \in \mathcal{Q}_2$.

Proof. The assumption on the support of Θ implies (cf. (a) above) that $\mu(\alpha_1 + 2^n \Lambda_1) = 0$ if $e(\alpha_1, \gamma_2) \neq 1$. Therefore,

$$\text{Support}(\mu) \subset \{\alpha_1 \in W_1 \mid e(\alpha_1, \gamma_2) = 1\}.$$

Since this support is a vector space,

$$\text{Support}(\mu) \subset W_1 \cap (\mathcal{Q}_2 \cdot \gamma_2)^\perp.$$

Let H denote the hyperplane $W_1 \cap (Q_2 \cdot \gamma_2)^\perp$. Then

$$\Theta(\alpha_1 + \alpha_2) = e\left(\alpha_1, \frac{\alpha_2}{2}\right) \int_{(\alpha_1 + A_1) \cap H} e(\alpha_2, \beta) \cdot d\mu(\beta).$$

Thus

$$\Theta(\alpha_1 + \alpha_2 + \lambda \gamma_2) = e\left(\alpha_1, \frac{\alpha_2 + \lambda \gamma_2}{2}\right) \int_{(\alpha_1 + A_1) \cap H} e(\alpha_2 + \lambda \gamma_2, \beta) \cdot d\mu(\beta)$$

and since $e(\lambda \gamma_2, \beta) = 1$ when $\beta \in H$, this comes out

$$\begin{aligned} &= e\left(\alpha_1, \frac{\lambda \gamma_2}{2}\right) \cdot \left\{ e\left(\alpha_1, \frac{\alpha_2}{2}\right) \int_{(\alpha_1 + A_1) \cap H} e(\alpha_2, \beta) \cdot d\mu(\beta) \right\} \\ &= e\left(\alpha_1, \frac{\lambda \gamma_2}{2}\right) \cdot \Theta(\alpha_1 + \alpha_2). \quad Q.E.D. \end{aligned}$$

In fact, I claim that the same Corollary holds *for all* $\gamma \in V$, not just for $\gamma \in W_2$. This can be seen by noting that for any $\gamma \in V$, there is a symplectic automorphism $T: V \rightarrow V$ such that $T(A) = A$, i.e., $T \in \text{Sp}(V, A)$, such that $T^{-1}(\gamma) \in W_2$. Going back to the action of the symplectic group introduced in § 9, we see that:

$$\left\{ \begin{array}{l} \text{If } \Theta \text{ is a theta-function, then so is } \Theta', \text{ where} \\ \quad \Theta'(\alpha) = e(\eta/2, \alpha) \Theta(T\alpha - T\eta) \\ \text{where } \eta \in \frac{1}{2}A \text{ satisfies} \\ \quad e_*(\alpha/2) \cdot e_*(T\alpha/2) = e(\eta, \alpha), \quad \text{all } \alpha \in A. \end{array} \right.$$

Now assume $\text{Supp}(\Theta) \subset \{\alpha \mid e(\alpha, \gamma) = 1\}$. Then

$$\begin{aligned} \text{Supp}(\Theta') &= \eta + T^{-1}(\text{Supp}(\Theta)) \\ &\subset \eta + \{\alpha \mid e(\alpha, T^{-1}\gamma) = 1\} \\ &\subset \{\alpha \mid e(\alpha, 2^n T^{-1}\gamma) = 1\} \quad (\text{if } n \gg 0). \end{aligned}$$

Therefore, by the Corollary

$$\Theta'(\alpha + \lambda T^{-1}\gamma) = e\left(\alpha, \frac{\lambda T^{-1}\gamma}{2}\right) \Theta'(\alpha), \quad \text{all } \lambda \in Q_2,$$

from which

$$\Theta(\alpha + \lambda \gamma) = e\left(\alpha, \frac{\lambda \gamma}{2}\right) \cdot \Theta(\alpha)$$

follows immediately. We are now ready for the Proof itself:

Proof of Theorem. We know that the support of Θ meets $\frac{1}{2}A$ (cf. § 10): choose $\eta_0 \in \text{Supp}(\Theta) \cap \frac{1}{2}A$. Then:

$$\text{Supp}(\Theta) + \eta_0 \subseteq W + A$$

(§ 10, assertion (4.) at the beginning). Therefore, if $\gamma \in W^\perp \cap (2A)$ it follows that $e(\alpha, \gamma) = 1$, all $\alpha \in \text{Supp}(\Theta)$. But then by Corollary above – as generalized –

$$\Theta(\alpha + \lambda \cdot \gamma) = e\left(\alpha, \frac{\lambda \gamma}{2}\right) \cdot \Theta(\alpha), \quad \text{all } \lambda \in \mathcal{Q}_2.$$

This shows that

$$(*) \quad \Theta(\alpha + \gamma) = e\left(\alpha, \frac{\gamma}{2}\right) \cdot \Theta(\alpha), \quad \text{all } \gamma \in W^\perp.$$

In particular, $\Theta(\eta_0 + \gamma) \neq 0$, all $\gamma \in W^\perp$, hence $W^\perp + \eta_0 \subseteq W + A + \eta_0$. Therefore $W^\perp \subseteq W$, i.e., W is a cusp.

Now suppose we take an arbitrary point α in the Support of Θ . We know that α can be written as:

$$\alpha = \eta_0 + \eta_1 + \alpha_0, \quad \eta_1 \in A, \alpha_0 \in W.$$

But then:

$$\begin{aligned} \Theta(\alpha) &= e_*\left(\frac{\eta_1}{2}\right) \cdot e\left(\frac{\eta_1}{2}, \eta_0 + \alpha_0\right) \cdot \Theta(\eta_0 + \alpha_0) \\ &= e_*\left(\frac{\eta_1}{2}\right) \cdot e\left(\frac{\eta_1}{2}, \eta_0\right) \cdot e\left(\frac{\eta_0 + \eta_1}{2}, \alpha\right) \cdot \left[e\left(\alpha, \frac{\eta_0}{2}\right) \cdot \Theta(\eta_0 + \alpha)\right]. \end{aligned}$$

Define a function $\tilde{\Theta}$ on W by

$$\tilde{\Theta}(\alpha) = e\left(\alpha, \frac{\eta_0}{2}\right) \cdot \Theta(\alpha + \eta_0).$$

If $\gamma \in W^\perp$, we compute (using (*)):

$$\begin{aligned} \tilde{\Theta}(\alpha + \gamma) &= e\left(\alpha + \gamma, \frac{\eta_0}{2}\right) \cdot \Theta(\alpha + \eta_0 + \gamma) \\ &= e\left(\gamma, \frac{\eta_0}{2}\right) \cdot e\left(\alpha + \eta_0, \frac{\gamma}{2}\right) \cdot e\left(\alpha, \frac{\eta_0}{2}\right) \cdot \Theta(\alpha + \eta_0) \\ &= \tilde{\Theta}(\alpha). \end{aligned}$$

This shows that $\tilde{\Theta}$ is, in reality, a function on $\tilde{V} = W/W^\perp$, and that Θ is exactly the function $T_{W, \eta_0} \tilde{\Theta}$ obtained from $\tilde{\Theta}$ via Definition 3.

To check that η_0 is an origin for W , look at (*) when $\gamma^\perp \in W \cap A$. Then:

$$e\left(\alpha, \frac{\gamma}{2}\right) \cdot \Theta(\alpha) = \Theta(\alpha + \gamma) = e_*\left(\frac{\gamma}{2}\right) \cdot e\left(\frac{\gamma}{2}, \alpha\right) \cdot \Theta(\alpha)$$

hence

$$e_*\left(\frac{\gamma}{2}\right) = e(\alpha, \gamma) \quad \text{if } \Theta(\alpha) \neq 0.$$

So

$$e_*\left(\frac{\gamma}{2}\right) = e(\eta_0, \gamma), \quad \text{all } \gamma \in W^\perp \cap A.$$

Moreover, using

$$\Theta(\eta_0) = \Theta(-\eta_0 + 2\eta_0) = e_*(\eta_0) \Theta(-\eta_0)$$

and

$$\Theta(-\eta_0) = \Theta(\eta_0) \neq 0,$$

we conclude that $e_*(\eta_0) = 1$ too.

The fact that $\tilde{\Theta}$ is again a theta-function is simply a matter of applying the calculations of Prop. 1 in reverse and is quite straightforward. We omit this. The final point is that $\tilde{\Theta}$ is non-degenerate. But since $S_\infty \supseteq W$, we know that for all $\alpha \in W$, $\alpha = 2^k \beta + \eta_1$, where $\Theta(\beta) \neq 0$, $\eta_1 \in A$. Then $\beta = \eta_0 + \eta_2 + \beta_0$, $\eta_2 \in A$, $\beta_0 \in W$, and $\tilde{\Theta}(\beta_0) \neq 0$. Since

$$\alpha - 2^k \beta_0 = \eta_1 + 2^k \eta_0 + 2^k \eta_2 \in W \cap A,$$

this shows that for all $\alpha \in W$, $\alpha = 2^k \beta_0 + \eta_3$, where $\tilde{\Theta}(\beta_0) \neq 0$, $\eta_3 \in W \cap A$. This means exactly that the S_∞ for $\tilde{\Theta}$ is all of $\tilde{V}$, i.e., $\tilde{\Theta}$ is non-degenerate. *Q.E.D.*

The main Theorem can now be reformulated to give a Satake-like decomposition of $\bar{M}_\infty$. More precisely, for each integer $g \geq 0$, let $\bar{M}_\infty(g)$ = the Proj defined in § 9, Def. 3 with indices $\alpha \in \mathbb{Q}_2^{2g}$. $M_\infty(g)$ = the open set in $\bar{M}_\infty(g)$ whose geometric points are the non-degenerate theta functions.

If $h < g$, we define a vast number of closed immersions

$$i_W: \bar{M}_\infty(h) \rightarrow \bar{M}_\infty(g)$$

as follows: let $W \subseteq \mathbb{Q}_2^{2g}$ be a cusp such that $2h = \dim(W/W^\perp)$. For each such W , choose an origin $\eta_0 \in \frac{1}{2}\mathbb{Z}_2^{2g}$, and a symplectic isomorphism:

$$\phi: \mathbb{Q}_2^{2h} \xrightarrow{\sim} W/W^\perp$$

such that

$$\phi(\mathbb{Z}_2^{2h}) = W \cap A / W^\perp \cap A,$$

$$\chi(\frac{1}{2} {}^t a_1 \cdot a_2) = \tilde{e}_*(\frac{1}{2} \phi(a)), \quad \text{all } a \in \mathbb{Z}_2^{2h}.$$

Then i_W is defined by the homomorphism of the homogeneous coordinate ring:

$$i_W^*(X_\alpha^{(g)}) = \begin{cases} 0 & \text{if } \alpha \notin \eta_0 + W + \mathbb{Z}_2^{2g} \\ e_*\left(\frac{\eta_1}{2}\right) e\left(\frac{\eta_1}{2}, \eta_0\right) e\left(\frac{\eta_0 + \eta_1}{2}, \alpha\right) \cdot X_{\phi^{-1}(\alpha_0)}^{(h)} & \text{if } \alpha = \eta_0 + \alpha_0 + \eta_1, \alpha_0 \in W, \eta_1 \in \mathbb{Z}_2^{2g}. \end{cases}$$

(Here $X_a^{(g)}, X_a^{(h)}$ are the coordinates used to define $\bar{M}_\infty(g)$, $\bar{M}_\infty(h)$ respectively). Then we get the restatement:

Main Theorem.

$$\bar{M}_\infty(g) = \left\{ \begin{array}{l} \text{disjoint union of the locally} \\ \text{closed subschemes } i_W(M_\infty(h)) \end{array} \right\},$$

the union being taken over all cusps $W \subseteq Q_2^2$.

§ 12. Analytic Theta Functions

In this section, we work over the field C of complex numbers. We have 2 purposes: (a) to sketch an approach to the classical theory of Θ -functions, analogous to our theory of algebraic Θ -functions, and (b) to use this to compute our algebraic Θ -functions via the classical ones, when $k = C$.

We will make use of the following lemma:

Lemma 1. *Let X be a compact Kähler manifold. Then the operator*

$$\frac{1}{2\pi i} \partial \bar{\partial}$$

defines a surjection:

$$\left\{ \begin{array}{l} C^\infty \text{ real} \\ \text{functions on } X \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{real closed } C^\infty(1,1)\text{-forms } \Omega \text{ on } X, \\ \text{with 0 cohomology class} \end{array} \right\}$$

with kernel consisting only of constants.

Corollary. *Let L be an analytic line bundle on X . Let $c_1(L) \in H^2(X, C)$ be its first chern class. Then for all real closed $C^\infty(1,1)$ -forms Ω whose cohomology class equals $c_1(L)$, there is one and (up to a constant) only one Hermitian structure $\| \|$ on L whose associated curvature form is Ω .*

The lemma is standard and we omit the proof. The Corollary can be proven by choosing one Hermitian structure $\| \|_0$ on L : let Ω_0 be its curvature form. Then any other Hermitian structure on L is given by $\rho \cdot \| \|_0$, where ρ is a positive real C^∞ function on X : and its curvature form Ω is

$$\Omega = \frac{1}{2\pi i} \partial \bar{\partial} \log \rho + \Omega_0.$$

Now use the Lemma and everything comes out. *Q.E.D.*

In particular, when X is an abelian variety, an analytic line bundle L on X has one and (up to a constant) only one Hermitian structure $\| \|$ whose curvature form Ω is a translation-invariant $(1,1)$ -form. In what follows, we will always put this Hermitian structure on line bundles on abelian varieties. In this case, Ω is determined by its value at the origin.

Now let $\hat{X}$ be the universal covering space of X . $\hat{X}$ is a complex vector space, and if

$$p: \hat{X} \rightarrow X$$

is the canonical homomorphism, dp induces a canonical identification between $\hat{X}$ and the tangent space of X at the origin (or at any other point). Therefore, any translation-invariant real 2-form Ω on X defines and is defined by a real-linear skew-symmetric form:

$$E: \hat{X} \times \hat{X} \rightarrow \mathbb{R}.$$

E is a (1, 1)-form if and only if $E(ix, iy) = E(x, y)$, all $x, y \in X$. Moreover, let $\Lambda = \text{kernel}(p)$. Λ is a lattice in X , canonically isomorphic to $H_1(X, \mathbb{Z})$. Since the first chern class of a line bundle is integral, if E represents $c_1(L)$, then E must take integral values on $\Lambda \times \Lambda$:

$$E(\Lambda \times \Lambda) \subseteq \mathbb{Z}.$$

If we lift L to $\hat{X}$, we have a situation in which the following lemma applies:

Lemma 2. *Let Y be a complex vector space, and let L_1, L_2 be 2 analytic-Hermitian line bundles on Y . Then a holomorphic-unitary isomorphism $\phi: L_1 \xrightarrow{\sim} L_2$ exists if and only if the curvature forms of L_1, L_2 are equal; if so, ϕ is unique up to a scalar of absolute value 1.*

Proof. Standard methods.

In particular, let $Y = \hat{X}$, and let $M = p^*(L)$ be induced from an abelian variety. Give L and hence M the Hermitian structure with constant curvature form E . The above lemma has 2 applications:

(I) Construction of a nilpotent group $\mathcal{G}$: If $x \in X$, and T_x denotes translation by x , then the lemma shows that M and $T_x^* M$ are holomorphic-unitary isomorphic. If

$$\mathcal{G}(M) = \{(x, \Phi) \mid \Phi \text{ a holo.-unit. isom. of } M \text{ with } T_x^* M\},$$

then $\mathcal{G}(M)$ is, as before, a group lying in an exact sequence:

$$1 \rightarrow C_1^* \rightarrow \mathcal{G}(M) \rightarrow X \rightarrow 0$$

(C_1^* = complex numbers of absolute value 1).

(II) Construction of canonical "trivialization" of M : Let $\mathbf{1}$ denote the trivial analytic line bundle over X with canonical section 1. To put a Hermitian structure on $\mathbf{1}$, we may set $\|1\| =$ any positive real C^∞ -function. For example, let

$$\|1\|(x) = e^{-\pi/2H(x,x)}$$

where H is a Hermitian form on X . The corresponding curvature form $E: \hat{X} \times \hat{X} \rightarrow \mathbf{R}$ is easily checked to equal $\text{Im}(H)$. But

$$H \mapsto E = \text{Im}(H)$$

sets up an isomorphism:

$$\left\{ \begin{array}{l} \text{hermitian} \\ \text{forms on } X \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{real skew-symmetric forms } E \text{ on } X \\ \text{such that } E(ix, iy) = E(x, y) \end{array} \right\},$$

so for each L on X with translation-invariant curvature form, we have a unique Hermitian structure on L of the above type so that $L \cong L$. In particular, we get a canonical

$$L \cong M.$$

We can now develop a theory along similar lines to our algebraic theory. For example, if H is positive definite, then let:

$\mathcal{H}$ = Hilbert space of L^2 -holomorphic sections of M over $\hat{X}$.

Then $\mathcal{G}(M)$ has a natural unitary representation on $\mathcal{H}$, it is irreducible, and it turns out to be the only irreducible unitary representation of $\mathcal{G}(M)$ in which $C_1^* \subset \mathcal{G}(M)$ acts by its natural character. This is the situation described by CARTIER [2], and studied by CARTIER and many others, e.g., MACKAY, FOCK, WEIL etc. Exactly as in § 1, $\mathcal{G}(M)$ governs the “descent” of the Hermitian bundle M to the abelian variety X , (or to other ones $X' = [\hat{X}/\text{another lattice}]$), and the “descent” of holomorphic sections of M to holomorphic sections of its descended form. Thus we get:

Proposition 1. *There is a 1–1 correspondence between*

1. *Hermitian-analytic line bundles L' on X such that $p^*L' \cong M$,*
2. *subgroups $K \subset \mathcal{G}(M)$, such that $K \cap C_1^* = \{1\}$ whose image in $\hat{X}$ is $A = \ker(p: \hat{X} \rightarrow X)$.*

Moreover, the holomorphic sections of M of the form $p^(s')$, $s' \in \Gamma(X, L')$, are exactly those sections s which are invariant under K , i.e.,*

$$s = T_x^*(\phi(s)), \quad \text{all } (x, \phi) \in K.$$

Proof. Straightforward.

Finally, via the canonical trivialization of M , holomorphic sections of M correspond to holomorphic functions on $\hat{X}$: thus each section $s \in \Gamma(X, L)$ defines a holomorphic function on $\hat{X}$. These are the classical theta-functions.

As far as moduli are concerned, the simplest and most basic result is the following: we set out to classify triples consisting of –

1. a complex vector space Y , of dimension 2;
2. an analytic, Hermitian line bundle M on Y , with curvature form $E = \text{Im } H$, H positive definite.
3. Parametrized lattices in Y , i.e., monomorphisms

$$\alpha: \mathbf{Z}^{2g} \rightarrow Y$$

such that

$$E(\alpha x, \alpha y) = {}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1$$

if

$$x = (x_1, x_2), \quad y = (y_1, y_2).$$

Such triples arise if we start with a principally polarized abelian variety (X, L) , together with a symplectic isomorphism:

$$\beta: \mathbf{Z}^{2g} \xrightarrow{\sim} H_1(X, \mathbf{Z}).$$

Namely, let $Y = \hat{X}$, $M = p^*L$ with canonical Hermitian structure, and let β define α via the natural maps $H_1(X, \mathbf{Z}) \cong \text{Ker}(p: \hat{X} \rightarrow X) \subset \hat{X}$. Conversely, the triple (Y, M, α) determines X and β , and L up to replacing L by T_x^*L , some $x \in X$.

Let $\mathfrak{H} = \text{SIEGEL'S } g \times g \text{ upper half-plane}$. Then the moduli result is:

Proposition 2. *There is a natural bijection between the set of isomorphism classes of triples (Y, M, α) and $\mathfrak{H}$. In this bijection, $\tau \in \mathfrak{H}$ corresponds to*

$$Y = \mathbf{C}^g,$$

$$M = \mathbf{1} \quad \text{with hermitian structure} \quad \|1\|(x) = e^{-\frac{\pi}{2} {}^t x \cdot B \cdot \bar{x}},$$

$$\alpha((x_1, x_2)) = x_1 + \tau \cdot x_2$$

where $B = (\text{Im } \tau)^{-1}$.

The final topic I want to discuss is the relation between the classical and algebraic theories. Let's start with:

X = abelian variety;

L = symmetric, ample, degree 1 sheaf on X . [Assume for simplicity that L is so chosen among its translates T_x^*L , $x \in X_2$, that its unique section is *even*; equivalently, that the Arf invariant of Q , where $e_*^L(x) = (-1)^{Q(x)}$, is 0.]

Let

L = line bundle on X whose holomorphic sections are L ;

$\hat{X}$ = universal covering space of X ;

$V_2(X)$ = 2-Tate group of X .

Also, let A_2 = inverse image in $\hat{X}$ of $\text{tor}_2(X)$, i.e.,

$$\bigcup_n 2^{-n} \cdot A, \quad \text{if } A = \text{Ker}(p: \hat{X} \rightarrow X).$$

Then we have canonical maps:

Note that A_2 is dense in both $V_2(X)$ and X . We have “trivialized” L when it is pulled up to $V_2(X)$ or to X , in § 8 and just above. Thus we have 2 distinct trivializations of L on A_2 . The main result is that these differ by an elementary factor:

Theorem 3. *Let $\mathbf{1}$ denote the trivial complex line bundle on A_2 . Then the following diagram commutes:*

where $\alpha \in C^*$ and $E = \text{Im}(H)$ is the curvature form of L .

Proof. Let $M_i = p_i^* L$ = induced line bundle on $V_2(X)$ or $\hat{X}$. Let $\psi: M_2 \xrightarrow{\sim} \mathbf{1}$ be the classical trivialization. The algebraic trivialization of M_1 is based on finding a distinguished collection of isomorphisms

$$\varphi_a: M_1 \rightarrow T_a^* M_1,$$

all $a \in V_2(X)$. In fact, let ι = inverse map in all our groups, and let $\rho: M_i \xrightarrow{\sim} \iota^* M_i$ be the isomorphism induced by the symmetry of L . Then, for all elements $2a \in V_2(X)$, φ_{2a} is characterized by the existence of φ_a satisfying:

- i) $\varphi_{2a} = T_a^* \varphi_a \circ \varphi_a$,
- ii) $\iota^* \varphi_a \circ \rho = T_{-a}^* [\rho \circ \varphi_a^{-1}]$,
- iii) φ_a is induced by an algebraic isomorphism

$$\varphi'_a: (2^n \delta)^* L \xrightarrow{\sim} (2^n \delta)^* (T_{p_1(a)}^* L)$$

for some n , i.e., via the factorization:

$$\begin{array}{ccc} & X & \\ p_1 \circ 2^{-n} \nearrow & & \searrow 2^n \delta \\ V_2(X) & & X \\ & p_1 \searrow & \end{array}$$

But introduce, for all $a \in X$, isomorphisms ψ_a from M_2 to $T_a^* M_2$ via:

$$M_2 \xrightarrow[\psi]{\approx} \mathbf{1} \xrightarrow[\text{mult. by } f_a(x)]{\approx} T_a^* \mathbf{1} \xleftarrow[T_a^* \psi]{\approx} T_a^* M$$

where

$$f_a(x) = e^{\pi[H(x, a) + H(a, a)/2]}.$$

Also introduce

$$\rho': M_2 \xrightarrow[\psi]{\approx} \mathbf{1} \xrightarrow[\text{canonical identification}]{} \iota^* \mathbf{1} \xleftarrow[\iota^* \psi]{\approx} \iota^* M.$$

One checks easily that ψ_a and ρ' are holomorphic and unitary isomorphisms. Therefore ρ and ρ' can differ only by a constant: and since both are the identity at $0 \in X$, $\rho = \rho'$. Moreover, if $a \in 2^{-n} \Lambda$, then the algebraic isomorphism $\varphi'_a: (2^n \delta)^* L \xrightarrow{\sim} (2^n \delta)^* T_{p_2(a)}^* L$, referred to in (iii) above, induces an isomorphism $\varphi''_a: M_2 \rightarrow T_a^* M_2$ via the factorization

$$\begin{array}{ccc} & X & \\ p_2 \circ 2^{-n} \nearrow & & \searrow 2^n \delta \\ \hat{X} & & X \\ & p_2 \searrow & \end{array}$$

Since φ''_a is also holomorphic and unitary, it differs from ψ_a only by a constant. Next, note that $\{f_a\}$ satisfy the identities:

$$\text{i')} f_{2a}(x) = f_a(x+a) \cdot f_a(x),$$

$$\text{ii')} f_a(-x) = f_a(x-a)^{-1}.$$

These translate readily into the identities on the $\{\psi_a\}$:

$$\text{i'')} \psi_{2a} = T_a^* \psi_a \circ \psi_a.$$

$$\text{ii'')} \iota^* \psi_a \circ \rho = T_{-a}^* [\rho \circ \psi_a^{-1}].$$

Finally, i'', ii'', plus the fact that φ'_a induces ψ_a , shows that ψ_a and φ_a induce the *same isomorphism* of L on Λ_2 , with $T_a^*(L \text{ on } \Lambda_2)$, all $a \in \Lambda_2$.

Finally, to compare the 2 trivializations, start with the unit section 1 of $\mathbf{1}$ on Λ_2 . This goes over, via the algebraic trivialization, to a section s of L on Λ_2 such that, for all $a \in \Lambda_2$,

$$s(a) = \phi_a(0) [s(0)]$$

(i.e., $\phi_a(0)$ is the induced isomorphism from the fibre L_0 or $(M_1)_0$ to the fibre $L_{p_1(a)}$ or $(M_1)_a$) But under the classical trivialization ψ , $\psi_a(0)$ corresponds to the isomorphism of fibres:

$$\begin{array}{ccc} \mathbf{1}_0 & \xrightarrow{\text{mult. by } e^{\pi/2 H(a, a)}} & \mathbf{1}_0 \\ \parallel & & \parallel \\ C & & C. \end{array}$$

Therefore, the section s goes over, under the classical trivialization, to a section of $\mathbf{1}$ which, if it has value α at 0, has value

$$\alpha \cdot e^{\pi/2 H(a, a)}$$

at a . All in all, the section 1 of $\mathbf{1}$ has gone into the section

$$g(a) = \alpha \cdot e^{\pi/2 H(a, a)}$$

of $\mathbf{1}$. *Q.E.D.*

Corollary. *If the unique section s of L (up to scalars) defines*

- a) *the holomorphic function Θ_h on $\hat{X}$ via the classical trivialization,*
 - b) *the 2-adic theta-function Θ_a on $V_2(X)$ via the algebraic trivialization,*
- then*

$$\Theta_h(x) = \alpha \cdot e^{\frac{\pi}{2} H(x, x)} \cdot \Theta_a(x)$$

all $x \in \Lambda_2$.

To calculate Θ_h and hence Θ_a by analytic means, we must know the “descent data”

$$K \subset \mathcal{G}(M_2)$$

that defines L on X . Let $e_*: \frac{1}{2}\Lambda/\Lambda \rightarrow \{\pm 1\}$ be the quadratic character defined by L . Then, as we saw in § 8, the descent data for the pull-back M_1 of L is the group:

$$\{(x, \phi) \mid x \in \Lambda \cdot \mathbf{Z}_2, \phi = e_*\left(\frac{1}{2}x\right) \cdot \phi_x\}.$$

In view of the proof of the theorem, this implies that

$$K = \{(x, \psi) \mid x \in \Lambda, \psi = e_*\left(\frac{1}{2}x\right) \cdot \psi_x\}.$$

(Notation as in proof of Theorem). Now a K -invariant section s of M_2 is one which satisfies $T_a^*(s) = \phi(s)$, all $(a, \phi) \in K$. Going back to the definition of ψ_a , one sees that if $f = \psi(s)$ is the function on $\hat{X}$ corresponding to s , then f is K -invariant if and only if

$$(*) \quad f(x+a) = e_*\left(\frac{1}{2}a\right) f_a(x) \cdot f(x)$$

all $x \in \hat{X}$, $a \in \Lambda$. From this it follows that Θ_h must be the unique holomorphic function satisfying (*).

To go further and write down this Θ_h as an infinite series, it is convenient to introduce coordinates. Let

$$i: \mathbb{Z}^{2g} \xrightarrow{\sim} \Lambda \quad \text{be a symplectic isomorphism.}$$

Coordinatize $\hat{X}$ via

$$\hat{X} \cong \mathbb{C}^g$$

so that $i((n_1, 0)) = n_1$, and let τ be the $g \times g$ matrix defined by

$$i((0, n_2)) = \tau \cdot n_2.$$

Because of our assumption on e_*^L , hence on e_* , if we choose coordinates correctly, we can assume that

$$e_*[\tfrac{1}{2}i(n_1, n_2)] = (-1)^{n_1 \cdot n_2}.$$

As we saw in Prop. 2, if we now express:

$$H(z, z) = {}^t z \cdot B \cdot \bar{z}$$

then $B = (\text{Im } \tau)^{-1}$. Finally, set

$$\Theta_h(z) = e^{\frac{\pi}{2} {}^t z \cdot B \cdot z} \cdot \sum_{n \in \mathbb{Z}^g} e^{2\pi i [\frac{1}{2} {}^t n \cdot \tau \cdot n + {}^t n \cdot z]}.$$

It is easy to check that this is a holomorphic function satisfying (*). Therefore, this is the sought-for theta-function. Combining this with the Corollary, we find

$$\Theta_a(z) = e^{\frac{\pi}{2} {}^t z \cdot B \cdot (z - \bar{z})} \cdot \sum_{n \in \mathbb{Z}^g} e^{2\pi i [\frac{1}{2} {}^t n \cdot \tau \cdot n + {}^t n \cdot z]} \quad \text{all } z \in \bigcup_k 2^{-k} \Lambda.$$

If

$$z = i((\alpha_1, \alpha_2)), \quad \alpha_i \in \bigcup_k 2^{-k} \cdot (\mathbb{Z}^g),$$

then after rearranging, one finds

$$\Theta_a(\alpha_1, \alpha_2) = e^{-\pi i {}^t \alpha_1 \cdot \alpha_2} \cdot \sum_{n \in \alpha_2 + \mathbb{Z}^g} e^{2\pi i [\frac{1}{2} {}^t n \cdot \tau \cdot n + {}^t n \cdot \alpha_1]}.$$

The function so defined clearly extends to a locally constant function defined for all $\alpha_1, \alpha_2 \in \mathbb{Q}^{2g}$: it is the sought-for algebraic theta function defined in § 8. Comparing this with the formula in Lemma 1, § 8, expressing Θ_a in terms of the finitely additive measure μ on $\mathbb{Q}_2^g$, we also get an analytic description for μ :

$$\left\{ \begin{array}{l} \mu \text{ is countably additive,} \\ \mu = \sum_{x \in D} e^{\pi i {}^t x \cdot \tau \cdot x} \cdot \delta_x, \\ \delta_x = \text{delta measure at } x, \\ D = \bigcup_k 2^{-k} \mathbb{Z}^g. \end{array} \right.$$

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