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Superstrings and Topological Strings at Large N

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Abstract

We embed the large N Chern-Simons/topological string duality in ordinary superstrings. This corresponds to a large N duality between generalized gauge systems with $N = 1$ supersymmetry in 4 dimensions and superstrings propagating on non-compact Calabi-Yau manifolds with certain fluxes turned on. We also show that in a particular limit of the $N = 1$ gauge theory system, certain superpotential terms in the $N = 1$ system (including deformations if spacetime is non-commutative) are captured to all orders in $1/N$ by the amplitudes of non-critical bosonic strings propagating on a circle with self-dual radius. We also consider D-brane/anti-D-brane system wrapped over vanishing cycles of compact Calabi-Yau manifolds and argue that at large N they induce a shift in the background to a topologically distinct Calabi-Yau, which we identify as the ground state system of the Brane/anti-Brane system.

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1. Introduction

The idea that large N gauge theories should have a phase described by perturbative strings, set forth by ‘t Hooft [1], has been beautifully realized by various examples. The first example of this kind was found by Kontsevich [2], which relates the bosonic string theory coupled to certain matter $((1, 2)$ minimal model, which is equivalent to pure topological gravity formulated by Witten [3]), to a matrix integral with cubic interaction (which can be viewed as a particular gauge theory in zero dimensions)¹. Many more examples were also found in the context of non-critical bosonic strings. For example, it was found [4] that bosonic strings propagating on a circle with self-dual radius is equivalent to Penner matrix model [5].

More recently it was recognized that ‘t Hooft’s conjecture is also realized even for much more complicated and physically more interesting gauge theories [6][7][8]. In particular certain gauge theories at large N are equivalent to superstrings propagating on AdS backgrounds. Another example of a string/large N duality was discovered in [9], where it was shown that large N limit of Chern-Simons gauge theory on S^3 is equivalent to topological strings on a non-compact Calabi-Yau threefold which is a blow up of the conifold (given by $O(-1)+O(-1)$ bundle over $\mathbf{P}^1$). This duality was tested to all orders in the $1/N$ expansion including checks at the level of Wilson Loop observables of the Chern-Simons theory [10][11]. It is also known [12] that in some limit (large N , fixed Chern-Simons coupling k) this theory has the same partition function as bosonic strings at the self-dual radius.

This paper was motivated by trying to connect the duality discovered in [9] with the dualities discovered in the context of AdS/CFT correspondences. The basic idea is to consider type IIA superstring propagating in the conifold background (which is symplectically the same as T^*S^3) in the presence of N D6 branes wrapped around S^3 and filling the spacetime. It has been known [13] that the topological string amplitudes for the internal theory on the non-compact Calabi-Yau compute superpotential terms on the left-over R^4 worldvolume of the D6 brane. On the other hand it is also known that the internal topological string theory with N D-branes wrapped on S^3 is equivalent to Chern-Simons gauge theory on S^3 [14]. Thus the duality found in [9] suggests that type IIA string on the conifold with N D6 branes is equivalent to the blown up version of the conifold with no branes

¹ This is not the same as the old matrix model which discretizes the worldsheet—rather it is the target space description exactly in line with ‘t Hooft’s conjecture.

left over. At first sight this sounds strange, because having no D-branes left would naively suggest a theory with $N = 2$ supersymmetry rather than $N = 1$. Moreover Ramond fluxes should also be turned on in the blown up geometry corresponding to the flux generated by the D6 brane. The main puzzle was why in the dual topological string theory discovered in [9] there is no mention of RR fluxes? Indeed it is an ordinary topological string (the A-model) on the blown up conifold.

The resolution turns out to be that turning on the RR flux does not affect the topological string amplitudes, and the dual string theory *does* involve RR fluxes. Turning on RR flux, however, does generate an $N = 1$ superpotential term [15][16], which can be computed in terms of the topological string amplitudes. Thus the duality found in [9] can be viewed as an all order check in the $1/N$ expansion for the $N = 1$ superpotential computations in the context of this type IIA superstring/gauge theory duality. One can also consider the mirror symmetry acting on all these statements, which as noted in [9] give rise to similar dualities. In the superstring realization, the mirror case (in a certain limit) would correspond to considering type IIB string on the blow up of the conifold with N D5 branes wrapped on $\mathbf{P}^1$ and we end up with type IIB on deformed conifold geometry T^*S^3 but with RR flux turned on.

One can also consider wrapped D-brane in the context of compact Calabi-Yau manifolds. However in this case we also need to put anti-D-branes, in order to have no net D-branes. In this case we conjecture that the large N limit will correspond to having a new Calabi-Yau with fluxes, which can decay as discussed in [17] to a theory with no fluxes left-over and with supersymmetry increased to $N = 2$. The effect of the non-BPS states has been to shift the background to a new background. This is a novel way of deforming backgrounds, and as we will suggest later in the paper may have many interesting extensions.

The organization of this paper is as follows: In section 2 we review aspects of topological string amplitudes and what they compute in the corresponding superstring theory. In section 3 we revisit the duality of [9] and embed it in the context of Type IIA superstrings. In section 4 we apply mirror symmetry to the statements in section 3 and discuss the equivalent Type IIB superstring theory. In section 5 we discuss possible applications of $c = 1$ non-critical bosonic strings to the question of generation of superpotential in the large N limit of $N = 1$ supersymmetric gauge theory. In section 6 we discuss wrapped brane/anti-brane systems in the context of compact Calabi-Yau manifolds and use the

above duality to make new predictions about the shift in the background. In section 7 we discuss some generalizations of this work.

While preparing this paper, three papers appeared which have overlaps with different aspects of our work. In particular [18][19] have some overlap with our work in the context of large N duals of $N = 1$ gauge theories in the context of type IIB strings, which we will briefly comment on in section 4. Also the same configuration of wrapped D-branes/anti-D-branes considered in section 6 was also studied in [20] in a different context.

2. Topological Strings and Superstrings

In this section we discuss aspects of topological strings and their relevance for superpotential computations in the corresponding superstring compactifications. We will divide our discussion to two parts: Closed string case (i.e. without D-branes) and open string case (i.e. including D-branes). We also point out the relevance of topological string amplitudes for $N = 1$ superpotential computations when RR-fluxes are turned on.

2.1. Closed topological string and superstring amplitudes in $4d$

Consider A-model topological strings on a Calabi-Yau manifold K (similar remarks apply to the mirror B-model). For simplicity of notation let us assume that the CY manifold has only one Kahler class, parameterized by the complexified Kahler parameter t . Then closed topological string amplitude on K is given by

$$F(t, \lambda_s) = \sum_g \lambda_s^{2g-2} F_g \quad (2.1)$$

$$F_g = \sum_d F_{d,g} e^{-dt}$$

where, roughly speaking $F_{d,g}$ denote the “numbers” (Gromov-Witten invariants) of genus g curves in class d . The topological strings compute certain amplitudes in the corresponding type IIA superstring compactifications on the Calabi-Yau [13][21][22]. In particular they compute terms in the action of the form

$$\int d^4\theta \mathcal{W}^{2g} F_g(t) = g R^2 F^{2g-2} F_g(t) + \dots$$

where $\mathcal{W}_{\alpha\beta}$ denotes the graviphoton field strength multiplet, R^2 and F^{2g-2} denote certain contractions of the self-dual part of the Riemann tensor and of the gravi-photon field

strength, and t denotes the vector superfield with the vev of the lowest component being the Kahler parameter t . One way to derive this formula is to note that with $2g-2$ insertions of the spin operator, needed to compute the amplitude involving the F^{2g-2} , the ordinary sigma model is topologically twisted. At genus 0 what one gets is

$$\int d^4\theta F_0(t) = \partial^2 F_0(t) F^t \wedge F^t + \dots$$

where F^t denotes the (self-dual part of the) $U(1)$ field strength in the same multiplet as t . In the type IIA this arises from the 4-form field strength G by setting it to

$$G = F^t \wedge \omega_t$$

where ω_t denotes the Kahler form associated to t .

It is natural to ask what changes in the closed topological string computations when we turn on some RR flux in the target space. The choices are² the 2-form field strength in the internal space F , 4-form field strength G_{int} along the internal CY directions and the G along the spacetime directions G_4 , which we equivalently study in terms of the dual 6-form field strength $G_6 = *G_4$. It turns out that the topological string amplitudes in the presence of RR fields is not modified at all! This is particularly simple to show in the Berkovits formalism [23][22][24]. Instead of demonstrating it in this way we follow a related idea, which we will need later in this paper, by studying the generation of $N = 1$ superpotential terms in the presence of RR fluxes, which we will discuss next.

2.2. Generation of superpotential due to internal field strength

RR fluxes have been studied in the context of CY compactifications [25][26][15][16]. In particular it has been shown in [15][16] that turning on internal field strength in the CY leads to generation of superpotential terms in 4d $N = 1$ theory (see also similar situations considered in [27][28]). In the context of type IIA theory with RR fluxes corresponding to F and G_{int} and G_6 discussed above, the superpotential is given by

$$\lambda_s W = \int F \wedge k \wedge k + i \int G \wedge k + \int G_6 \quad (2.2)$$

² We can also include the 0-form field strength dual to 10 form field strength in type IIA, but since we will not deal with it in this paper we will not discuss it. It will give rise to an $N = 1$ superpotential of the form $\int G_0 \wedge k^3$.

where k is the complexified Kahler class. To see this one considers the BPS charge in the presence of BPS domain walls which may be partially wrapped over the CY. For example, considering a D6 brane wrapped over 4-cycles of CY gives a domain wall with BPS tension $\frac{1}{\lambda_s} \int k \wedge k$ integrated over the internal part of the 6-brane. This in turn shifts the dual F by one unit. This BPS formula should be captured by a ΔW and we can see from the above form of (2.2) that the first term above precisely captures this term. More precisely what we mean by the formula (2.2) is the *worldsheet quantum corrected* formula for the kahler forms (as is well known in the context of mirror symmetry the mass of the D-branes receives corrections by the worldsheet instantons). In particular if t denotes the complexified area of the basic 2 cycle, then the volumes of the 0, 2, 4 and 6 cycles are given by

$$1, t, \frac{\partial F_0}{\partial t}, 2F_0 - t \frac{\partial F_0}{\partial t}$$

where F_0 is the genus zero topological string amplitude. So in particular suppose we have N units of the F flux through the basic 2-cycle, where t denotes the complexified area of this 2-cycle. Then the first term in (2.2) is equivalent to

$$\int F \wedge k \wedge k = N \frac{\partial F_0}{\partial t}$$

Similarly if we considered $D4$ branes wrapped over 2 cycles and $D2$ branes with no wrappings, we deduce the existence of the second and third term in (2.2). In particular if we denote the fluxes of F , G_{int} , G_6 by integers N, L, P relative to integral 2, 4 and 6 cycles, we have

$$\lambda_s W = N \frac{\partial F_0}{\partial t} + itL + P \quad (2.3)$$

Note that equation (2.2) can also be written in the form

$$\lambda_s W = \int (F + i * G) \wedge k + \int G_6 \quad (2.4)$$

where again here by $*$ we mean the worldsheet quantum corrected $*$ operation.

Now we come to the discussion of why turning on RR fluxes should not modify the topological amplitudes. We will concentrate on genus 0 amplitudes (similar arguments can be advanced for the higher genus amplitudes as well). The vector superfield with bottom component t has an auxiliary field in the superspace of the form

$$t + \theta^2 (F + *iG) + \dots$$

where F and G are the usual RR fluxes of the internal Calabi-Yau³. In the usual supersymmetric background they are set to zero. Now suppose we wish to turn them on. Suppose for example we wish to turn on N units of F . Consider the topological string amplitude $F_0(t)$. We claim that this already yields the correct structure for the generation of $N = 1$ superpotential precisely if F_0 is *unmodified in the presence of RR flux*. To see this note that using the expansion of t in terms of the RR field strength auxiliary fields we have

$$\int d^4\theta F_0(t) = \int d^2\theta N \frac{\partial F_0}{\partial t}$$

which is exactly the expected answer if F_0 is unmodified. Similarly turning on the G_{int} flux and using (2.4) we see that the term in (2.2) involving G_{int} will also have the correct structure if F_0 is unmodified.

There is another auxiliary field in the vector multiplet which come from the NS-NS sector which is relevant for us. This corresponds to the field strength associated with the lack of integrability of complex structure. In particular if we write $\overline{D} = \overline{\partial} + A\partial$, where A is an anti-holomorphic one form taking values in the tangent bundle, then

$$\overline{D}^2 = (\overline{\partial}A + [A, A])\partial = \mathcal{F}\partial$$

where $\mathcal{F}$ is an anti-holomorphic 2-form with values in the tangent bundle which is equivalent, by lowering the vector index by the three form, to a $(2, 2)$ form. If this is non-vanishing it also corresponds to making the $(3, 0)$ form in the CY not to be annihilated by $\overline{\partial}$. These turn out effectively to add to F and G_{int} complex pieces of the form iF^{NS}/λ_s and iG_{int}^{NS}/λ_s . A similar NS auxiliary field gives rise effectively to the complex part of $*G_6$.⁴ In other words, even with these fields turned on, the formula (2.3) remains correct but now N, L, P also include imaginary pieces of the form $iN_I/\lambda_s, iL/\lambda_s, iP/\lambda_s$. We will continue to denote the superpotential as (2.3) and keep in mind that N, L and P can have complex pieces given by an integer over λ_s .

Turning these vevs on breaks the $N = 2$ supersymmetry to $N = 1$. The field t is now the bottom component of an $N = 1$ chiral multiplet whose auxiliary field descends from another auxiliary field (which also comes from the NS sector) in the original $N = 2$ multiplet which is not turned on.

³ I have greatly benefited from discussions with Nathan Berkovits in connection with the auxiliary field structure of the superfields.

⁴ Turning these fields on is mirror to turning on H_{NS} on the mirror CY.

Note also that the higher genus topological amplitudes also give rise to certain $N = 1$ superpotential terms when the auxiliary field of the $N = 2$ multiplet t takes a vev. In particular with N units of RR flux for F we get

$$\int d^4\theta \mathcal{W}^{2g} F_g(t) \rightarrow N \int d^2\theta \mathcal{W}^{2g} \frac{\partial F_g}{\partial t} \quad (2.5)$$

where we continue to denote by $\mathcal{W}_{\alpha\beta}$ the reduction of the $N = 2$ multiplet to an $N = 1$ multiplet with the self-dual part of the graviphoton field strength as its bottom component.

So in conclusion we have learned that the topological string amplitudes are not sensitive to turning on RR field strengths, but they are useful in determining the superpotential terms that will be generated once certain RR and NS fields take a vev. This is captured by equation (2.3).

2.3. Open Topological Strings and $N=1$ amplitudes in $4d$

In the A-model, the open topological string corresponds to studying holomorphic maps from worldsheet with boundaries to the target space where the boundary lies on a 3-dimensional Lagrangian subspace of the CY, i.e. the 3-dimensional topological version of D-brane [14]. Moreover, it was shown that topological string field theory in this case is just the Chern-Simons gauge theory on the corresponding Lagrangian submanifold (possibly corrected with non-trivial worldsheet instantons). The implications of these theories for superstring amplitudes has been studied as well. In particular if we consider type IIA superstring in the presence of a CY with N D6 branes wrapping a lagrangian 3-cycle of CY and filling the rest of the spacetime we get an $N = 1$ gauge theory with $SU(N)$ gauge group in 4d. Then it was shown in [13] that for example the genus 0 open topological string amplitudes compute corrections of the form

$$\lambda_s W = \sum_h \int d^2\theta F_{0,h}[Nh S^{h-1}] + \alpha S + \beta \quad (2.6)$$

where $F_{0,h}$ is the partition function of the topological string at genus 0 with h holes, and $S = \lambda_s Tr W^2$ where W_α is the chiral superfield with gaugino as its bottom component⁵.

⁵ The coefficient of Nh in front arises because, as discussed in [13], we have to choose $h - 1$ holes to put the $tr W^2$ fields and this can be done in h ways and also the trace over the hole without a field gives a factor of N . Note also that $Tr W^2$ which is a fermion bilinear is nilpotent, in the sense that $(Tr W^2)^k = 0$ for $k > N^2$. It is relevant for us precisely because we are considering a large N limit. Thus in the large N limit the gaugino bilinear can even have a classical vev.

Here we have shown explicitly the contribution coming from $h = 2$ in the form of αS . This term, coming from annulus, is typically divergent, signifying the RG flow of the coupling constant of the gauge theory and needs regularization. Also we have added a constant β to remind us that we cannot fix that from open topological string considerations. Here h is the number of holes on the sphere and $F_{0,h}$ denotes the topological amplitude on the sphere with h holes. If the target space has some Kahler moduli t they will correspond to chiral fields in the $N = 1$ theory in 4 dimensions and $F_{0,h}$ will depend on t . The case of $h = 1$ in the above formula was recently discussed in [29][30][10]. Some superstring implications of higher genus open topological strings, i.e. $F_{g,h}$ with arbitrary g , has also been noted in [10]. In particular they compute terms of the form $\int d^2\theta F_{g,h} \mathcal{W}^{2g} [N h S^{h-1}]$.

Let us define the open topological string amplitude summed over all holes at a fixed genus by

$$F_g^{open}(r) = \sum_h F_{g,h} r^h.$$

Then, for example the genus 0 open topological string amplitude computes the following correction to the superpotential

$$\lambda_s W = N \int d^2\theta \frac{\partial F_0^{open}(S)}{\partial S} + \alpha S + \beta$$

This is strikingly similar to the form obtained in (2.3) in the context of closed topological string amplitudes. Similarly the higher genus correction computes terms of the form

$$N \int d^2\theta \mathcal{W}^{2g} \frac{\partial F_g^{open}(S)}{\partial S}$$

which is also similar to the higher genus correction obtained in the closed string context in the presence of flux (2.5). The main difference being that S is an operator for the open string amplitudes but t is a parameter in the closed string setup. Nevertheless we will see in the next section why this is not an accidental similarity and provides the superstring interpretation of the duality found in [9], when S takes an expectation value equal to t .

3. Embedding Large N Topological String duality in Superstrings

Consider Type IIA strings in a non-compact CY 3-fold geometry of the form of the conifold times the Minkowski space M^4 : The internal geometry is given by

$$f = x_1^2 + \dots + x_4^2 = \rho$$

where each x_i parameterizes $\mathbf{C}$. The real subspace of the above geometry is S^3 (for real ρ) and the imaginary directions sweep the cotangent direction of S^3 . The volume of S^3 in string units is given by ρ (here we are taking the canonical 3-form $\Omega = \prod dx_i/df$, which scales as ρ to give the volume). Thus symplectically the conifold is T^*S^3 . Consider N D6 branes wrapped over the S^3 of the conifold and filling the rest of the spacetime. On the uncompactified worldvolume of the D-brane we have an $N = 1$ supersymmetric $SU(N)$ gauge theory. Note that to leading order the action on the uncompactified worldvolume of the D-branes is given by the superpotential

$$\frac{1}{\lambda_s} \int d^2\theta \, SY \quad (3.1)$$

where $S = \lambda_s \text{Tr} W^2$ and where Y denotes the $N = 1$ chiral superfield with its bottom component given by $iC + \frac{\rho}{\lambda_s}$, where C is the vev of the 3-form gauge field on IIA (normalized with periodicity 2π) and plays the role of the theta angle for the gauge theory and ρ denotes the volume of the S^3 .

The choice of this type IIA geometry is based on the desire to utilize the topological open/closed string duality. In particular as discussed in the previous section the open topological string in this case computes corrections to the superpotential of the form

$$\frac{N}{\lambda_s} \int d^2\theta \, \partial_S F_0^{open}(S)$$

The topological A-model is insensitive to complex structure. In particular F^{open} is independent of ρ except for a linear terms in S (coming from the annulus) written in (3.1), which is related to the ambiguity of open topological string at the level of annulus. There is also a divergence of the annulus amplitude corresponding to the running of the gauge coupling constant, which, in the regularized form, can be viewed as addition of a linear term in S . The corrections above to the simple $YS = Y \text{Tr} W^2$ involve higher dimension operators (more powers of S) and are captured by the open string amplitude which coincide with the large N expansion of the Chern-Simon amplitudes on S^3 . Note also that the fact that they are independent of Y implies that they survive no matter what the size of the S^3 is.

Now we wish to consider the limit where we consider the $N \rightarrow \infty$ limit keeping $N\lambda_s$ fixed. In this limit, the analog of 't Hooft coupling for the gauge system is given by

$$\frac{1}{Ng_{YM}^2} \rightarrow \frac{Y}{N\lambda_s}.$$

which remains fixed in this limit. We would like to consider the gravity dual of this gauge system. In the spirit of AdS/CFT correspondence we will have to consider the near horizon geometry. What the precise notion of “near” horizon geometry in this case should be is more subtle because the expectation value of Y undergoes an RG flow, as noted above, and it will depend at which scale we are probing it. In other words we have to readjust the size of Y depending on how close we are approaching the branes. The limit should be such that the S^3 has zero size when we probe it in the UV of the gravitational side but finite size in the IR. To avoid such subtleties we try to look for a consistent gravitational background which the branes create. In particular we should find an S^2 of finite size emerging, surrounding the S^3 , with the D-branes completely disappeared and replaced by the corresponding fluxes. In the case at hand, since we have N $D6$ branes wrapping the S^3 in the geometry after transition we should get N units of the 2-form RR flux F through the dual S^2 . We will now turn to studying this geometry.

3.1. Type IIA Superstring on the blown up geometry

We thus seek the dual large N stringy description of the above gauge system, in the form of the Type IIA background with the blown up conifold geometry, i.e. the geometry corresponding to $O(-1) + O(-1)$ bundle over $\mathbf{P}^1$, with N units of 2-form F flux through $\mathbf{P}^1$. However we must also have internal 4-form and 6-form fluxes (in the form of NS and RR fields discussed before).

That there should be an NS 4-form flux corresponds to the fact that the size of the S^3 is changing (i.e. that Ω is no longer closed and $\rho = \int_{S^3} \Omega$ changes), inducing a running of the gauge coupling constant. Moreover to preserve $N = 1$ supersymmetry for a finite value t of the complexified area of the blown up $\mathbf{P}^1$ we need both 4-form as well as 6-form fluxes. In fact, as discussed in the previous section and summarized in equation (2.3) we have a superpotential of the form

$$W = N\partial_t F_0(t) + itL + P \tag{3.2}$$

where for the geometry at hand $F_0(t)$ is, up to a cubic polynomial, the tri-logarithm function, given by

$$F_0(t) = \frac{1}{6}t^3 - \sum_{n>0} \frac{e^{-nt}}{n^3} + P_2(t). \tag{3.3}$$

(where $P_2(t)$ is a polynomial of order 2 in t and is somewhat ambiguous). Similarly for higher genus F_g we have

$$F_g(t) = \frac{B_{2g}}{2g(2g-2)!} \sum_{n>0} n^{2g-3} e^{-nt} + \frac{B_{2g}B_{2g-2}}{2g(2g-2)(2g-2)!} \quad g > 1$$

$$F_1(t) = \frac{t}{24} + \frac{1}{12} \log(1 - e^{-t})$$

where B_{2g} denotes the Bernoulli numbers. The terms involving e^{-nt} in the above formula reflects the corrections due to worldsheet instantons wrapping n times over the $\mathbf{P}^1$.

The content of the duality obtained in [9] is that

$$F_g^{Open}(S) = F_g(t)$$

for all g if we set $S = t$. We now try to interpret this statement in the superstring context. For this we need to study solutions to the gravitational equations.

Typically in physics and mathematics when we try to solve some system of equations, there are topological obstructions that have to be shown to be absent. Once they are shown to be absent then a solution exists. For example, when we are trying to find Ricci-flat metrics on Kahler manifolds we need the first chern class of the manifold to be zero. In fact this is also sufficient for being able to find a Calabi-Yau metric. Of course explicit solution for the metric has not been possible in almost all cases and in fact the Ricci-flat metric is only an approximate metric which gives rise to a conformal worldsheet theory. In a sense the topological condition, guaranteeing the existence of the solution is more fundamental than the solution itself.

Now we come to the case at hand. To preserve $N = 1$ supersymmetry we need $W = dW = 0$. Once these are satisfied, we expect physically that there must be a solution to the gravitational system. In fact a very similar example with the same number of supercharges (namely 4) was already studied from this point of view. Namely if we consider M-theory on Calabi-Yau 4-fold with G-flux turned on, the gravitational equations have been studied in [31]. The topological conditions they find for the existence of the gravitational solution has been shown to be identical to the condition that $W = dW = 0$ [27].

Of course the low energy gravitational equation in the present case can also be studied similar to what was done in [31] and will involve warped geometries mixing the spacetime with the Calabi-Yau. Even though solving the gravity equations would be interesting, we

have to remember that due to worldsheet instantons wrapping the $\mathbf{P}^1$ there are important corrections to the gravity equations, and so at best we can trust the low energy gravity description in the limit of large t . Nevertheless, as already noted above, the superpotential terms *including* the corrected string geometry, can be incorporated to all orders in the W which is computable by topological string amplitudes.

Before even solving the conditions $W = dW = 0$ we can already interpret the duality of [9] in the superstring context. If we compare the equation (3.2) with the superpotential given in the gauge theory side namely $W = N\partial_S F_0^{open}(S) + \alpha S + \beta$ we see that they are identical in form with an appropriate identification of α and β with L and P . Therefore, since the vacuum in the gauge theory side, as well as the moduli in the gravity side correspond to $W = dW = 0$, and W has the same form for the gauge as well as the gravitational system, this will identify

$$\langle S \rangle = \langle \lambda_s Tr W^2 \rangle = t \quad F_g^{open}(S) = F_g(t) \quad (3.4)$$

Thus the condition of vacuum configuration which sets $\langle S \rangle = t$ also translates the duality found in [9] to the match between amplitudes in the gravity side and the gauge theory side to all orders in $1/N$ at least as far as superpotential terms are concerned!

Note that the idea that $\langle S \rangle \neq 0$, i.e. that we have gaugino condensation, is very natural for the open string theory under discussion as it does have an $N = 1$ Yang-Mills theory associated with it. Part of the above check involves, on the gauge theory side, the statement that gaugino condensation generates superpotential terms captured through topological open string amplitudes and in fact this was already pointed out in [13]. Of course here we have a more refined gauge theory than just the $N = 1$ Yang-Mills theory and in particular we have in the open string system, also the higher dimension operators present, which are captured by the topological string amplitude. At any rate, the result of [9] strongly suggests not only the existence of a large N duality involving this $N = 1$ brane system with this closed string background, but also that the gaugino condensation takes place.

We now come to finding solutions to the equations $W = dW = 0$ on the gravity side, which should guarantee the existence of a solution. There are four parameters under control: the modulus t and the fluxes N, L, P . The two equations $W = dW = 0$

$$\partial_t W = 0 \rightarrow N F_0'' + iL = 0 \rightarrow L = iN F_0''$$

$$W = 0 \rightarrow P = -NF'_0 + NtF''_0$$

imply that two of these four quantities are fixed in terms of the other two. The N is of course fixed for us by the number of D6 branes. As is clear from our description of the dual gauge system the choice of a shift in L is related to a shift in the bare coupling constant of the gauge system. In particular in order to agree with the bare coupling constant of the gauge theory $iL = \rho/\lambda_s$, where ρ is the volume of the S^3 where the D6 branes are wrapped around. Thus the value of t (and also of P) is fixed and from (3.3) and $\partial_t W = 0$ the solution for t is given by

$$[c(e^t - 1)]^N = \exp(-\rho/\lambda_s) \quad (3.5)$$

The constant c depends on the ambiguities hidden in the $P_2(t)$. As we will argue from the dual gauge theory description, it should be fixed in our case (by a suitable regularization of the one loop divergence of the gauge theory) to be $c \sim N\lambda_s$.

Next we turn to the question of how the dynamics of the gauge system is reflected in the W and the other superpotential terms. What do we expect for the dynamics of the $N = 1$ supersymmetric theory living on the D6 brane? If we ignore the higher powers of S in the superspace integral, i.e. if we ignore the higher order operators, as already discussed, the leading term with the lowest number of derivatives, is given by the superpotential term

$$\frac{1}{\lambda_s} \int d^2\theta \, SY$$

where $S = \lambda_s Tr W^2$ and $Y = iC + \frac{\rho}{\lambda_s}$ where ρ denotes the size of the S^3 in the string frame. In the usual geometric engineering of standard $N = 1$ gauge theories, and in particular the ones discussed in [32] one considers the limit where ρ is large, in which case the field Y gets demoted to a parameter in the lagrangian (the corresponding D-term involving $Y\bar{Y}$ becomes very large). However here we are not necessarily interested only in a regime where Y is very large. In other words we consider the field Y to be a dynamical field. Thus we have a non-standard $N = 1$ supersymmetric gauge theory with its coupling constant as a dynamical field. Even though it is somewhat unconventional, as we will now argue some of the basic features of this theory are similar to that of $N = 1$ QCD, in the limit where we ignore the higher derivative terms of the form $\int d^2\theta [tr W^2]^k$. In other words, if we consider the field space where $S = tr W^2 \ll 1$ (in string units) we have a theory which is more or less similar to $N = 1$ supersymmetric Yang-Mills theory. Even though we do not have to restrict our attention only to this limit, and the duality with gravity holds regardless

of which field configuration in S we consider, it is first instructive to consider the small S region to gain intuition for what this theory is.

In the dynamics of $N = 1$ supersymmetric gauge theory, a prominent role is played by instantons. Here a similar effect exists: In particular if we consider Euclidean D2 brane instantons wrapping the S^3 the superpotential gets corrected. Moreover this can also be viewed as point-like instantons for the $SU(N)$ gauge theory. To have the right number of fermionic zero modes to lead to a chiral superspace potential we need $1/N$ -th of this instanton. Since the action for this instanton is e^{-Y} , the term that can appear in the action is $e^{-Y/N}$. The coefficient in front of it should be of order N^2 (as argued in [33]). So we must have the effective superpotential given by

$$W = \int d^2\theta \left(\frac{1}{\lambda_s} SY + iN^2 \alpha e^{-Y/N} \right) \quad (3.6)$$

where the constant α , by a shift in $Y \rightarrow Y + Y_0$, can be identified with a shift in the bare coupling constant of S , i.e.,

$$\alpha = e^{-Y_0/N} \quad (3.7)$$

(The choice of α is also related to how we regularize the one-loop divergence which corrects the action with a term $a \int d^2\theta S$).

This effective superpotential has the same structure as that encountered in the proof of mirror symmetry in 2 dimensions [34]. This same superpotential structure was encountered in [10] in the context of $N = 1$ domain walls in 4d, which we will also need in this paper, and we will discuss further below. Notice that here since Y is a dynamical field, we can integrate it out by setting

$$\partial_Y W = 0 \rightarrow \frac{1}{\lambda_s} S = iN\alpha e^{-Y/N}$$

which leads to

$$Y = \log\left(\frac{S}{iN\alpha\lambda_s}\right)^{-N}$$

plugging it back to the superpotential gives the effective superpotential for S :

$$W_{eff}(S) = \frac{1}{\lambda_s} \left[S \log\left(\frac{S}{iN\alpha\lambda_s}\right)^{-N} + NS \right]$$

This is the familiar effective superpotential expected for the gaugino bilinear S in the standard $N = 1$ supersymmetric gauge theory. Indeed setting $\partial_S W = 0$ leads to

$$\left[\frac{S}{iN\alpha\lambda_s}\right]^N = 1 \rightarrow S = iN\alpha\lambda_s e^{2\pi il/N} = iN\lambda_s e^{(-Y_0 + 2\pi il)/N} \quad (3.8)$$

Note that we see the N vacua of $SU(N)$ Yang-Mills, in the standard way.

Let us compare the vev we found for $S = \lambda_s Tr W^2$ with the gaugino condensate for standard $N = 1$ Yang-Mills, which is of the form

$$Tr W^2 = iN\Lambda^3 e^{[\frac{-1}{Ng_{YM}^2}] + \frac{2\pi i l}{N}}$$

In comparison with what we have above, note that this is in perfect agreement with (3.8), where Λ corresponds to the string scale and $1/g_{YM}^2 \rightarrow Y_0$.

Note that the effective superpotential we have found for S , for small S , also follows from either the open topological string amplitudes in the limit $S \rightarrow 0$, or the dual closed topological string amplitude in the limit $S = t \rightarrow 0$ which is given by

$$F_0(t) \rightarrow -\frac{1}{2}t^2 \log t + at^2 + bt + c$$

and so

$$W(S) = \frac{1}{\lambda_s} [N\partial_S F_0(S) + \alpha S + \beta] = \frac{1}{\lambda_s} (S \log S^{-N} + N \cdot \text{const.} S + N \cdot \text{const.})$$

in perfect agreement with expectations based on the gauge theory analysis as well as with the contribution of the Euclidean D2 brane instantons in the string context. This comparison with gauge theory and recalling that $t \leftrightarrow \lambda_s Tr W^2$, also fixes the value c in (3.5) to be $c \sim N\lambda_s$. Note that the choice of α , the linear terms in S , on the gravity side is controlled by the 4-form fluxes dual to the $\mathbf{P}^1$, as discussed before.

Having discussed the geometry of the vacua of $N = 1$ theory, we now turn to another important feature of $N = 1$ theories, namely the domain walls interpolating between various vacua.

3.2. Domain Walls

$N = 1$ Yang-Mills theory admits BPS domain walls interpolating between various vacua. As noted in [33] at large N they behave as D-branes for QCD string. In particular their tension is of the order of N . Since in the present context the QCD string is realized by the fundamental string, ordinary D-branes of string theory should play the role of domain walls. This is indeed the case: On the gravity side we have a blown up $\mathbf{P}^1$. If we consider D4 branes wrapped over $\mathbf{P}^1$ they correspond to domain walls. Their tension goes as

$$T \sim \frac{1}{\lambda_s} |t| = N \frac{|t|}{N\lambda_s}$$

As discussed before $|N\lambda_s| \sim |t|$ so we obtain the expected behavior. For the QCD domain wall the phase of the S field should change as we go from one vacuum to another. In particular it should shift by $\exp(2\pi i/N)$ for domain walls interpolating adjacent vacua. Let us see how this is realized in the gravity setup. Since we have identified the domain wall with $D4$ brane wrapped over S^2 we should note that the value of the G flux shifts as we cross the domain wall. Consider in particular the imaginary part of the Y field introduced earlier, which was identified with

$$ImY = C_{S^3}$$

i.e. the vev of the C field along the S^3 . We now discuss how this changes from the left-side of the domain wall to the right-side. Since the G flux should be equal to one for the $D4$ brane, it implies that ImY should shift by 2π , i.e.

$$Y \rightarrow Y + 2\pi i$$

as we go across the domain wall. In fact we can find the geometry of the BPS domain walls by the usual technique of the LG theory in 2d with $N = 2$ susy [35]. In fact for the case at hand similar BPS domain walls were considered in [36]. These domain walls also featured in the discussion of $N = 1$ generation of superpotential in [10]. Note that since we have

$$S = \lambda_s N \exp(-(Y + Y_0)/N)$$

this implies that the phase of S changes by $\exp(-2\pi i/N)$ as expected. Of course this is suppressed at large N , in agreement with the fact that classically the wrapped D-brane does not change the value of t .

It is also easy to see from the form of the action (3.6) that the BPS tension, which is given by ΔW is given by

$$\Delta W = \frac{1}{\lambda_s} S \Delta Y$$

Since S is identified with t , this corresponds to

$$\Delta W = \frac{2\pi i}{\lambda_s} t$$

as expected for the tension of the BPS wrapped $D4$ brane.

3.3. Subleading Corrections in the $1/N$ Expansion

So far we have concentrated on the interpretation of the leading corrections in large N . In the context of topological strings also the subleading terms to all orders in $1/N$ were found to agree between the Chern-Simons gauge theory and the closed topological string expansion. What is the interpretation of these higher terms for the gauge theory system?

In the limit of small t the topological string amplitudes is given by

$$F(t) = \sum_g F_g \lambda_s^{2g-2} t^{2-2g}$$

where $F_g = \frac{B_{2g}}{2g(2g-2)}$ and B_{2g} are the Bernoulli numbers (F_g turns out to be equal to the Euler characteristic of the moduli space of genus g Riemann surfaces). In this limit the topological string partition function coincides with that of non-critical bosonic strings on a circle with self-dual radius (this connection is well understood and will be reviewed in section 5). The $N = 2$ amplitude that this computes is given by

$$\int d^4\theta \mathcal{W}^{2g} F_g t^{2-2g} = g R^2 F^{2g-2} F_g t^{2-2g} + \dots \quad (3.9)$$

This correction has been physically understood by considering turning on constant graviphoton field strength in the Minkowski space and computing the effect of wrapped D2 branes on $\mathbf{P}^1$ to the R^2 term [37]. In the present context the wrapped D2 branes correspond to the baryon vertex, as in the usual AdS/CFT correspondence [38]. The Baryon fields are charged under the graviphoton field with charge proportional to their BPS mass t . Thus turning on graviphoton F effectively turns on a background field strength for F_v , i.e. the $U(1) \subset U(N)$ living on the D6 branes, which can be identified with a global $U(1)$ symmetry (the ‘Baryon number’ symmetry). Let us try to see how this can come about from the gauge theory side.

On the worldvolume of the D6 branes we have terms of the form

$$\int_{R^4} [G_4 + F \wedge F_v] \left(\int_{S^3} [CS(\omega) - CS(A)] \right)$$

where ω denotes the spin connection on S^3 and A is the internal gauge field on S^3 . This term arises (by integrating by parts) from the usual inducement of brane charge by gravitational and gauge curvature on the brane (see [39] and references therein). Thus shifting F effectively shifts F_v ⁶.

⁶ Note that if we change the G_4 flux this is equivalent to turning on an internal Chern-Simons action for the supersymmetric system on the brane. It should be possible to derive directly the relation between generation of superpotentials on the brane and the Chern-Simons theory on S^3 from this fact.

There is another term that is also generated from (3.9) when we recall that t has some auxiliary field turned on. In particular this gives rise to the term

$$N \int d^2\theta \mathcal{W}^{2g} \partial_t F_g(t) = N \int d^2\theta F^{2g} \partial_t F_g(t) + \dots$$

Recalling that in the gauge theory setup t is replaced by S , the gaugino bilinear superfield, the above term corresponds to the superpotential term

$$\lambda_s W = N \int d^2\theta F^{2g} \partial_S F_g(S) = -N \int d^2\theta \frac{B_{2g}}{2g} F^{2g} S^{1-2g}$$

So turning on the (self-dual) graviphoton field strength in 4 dimensions deforms the superpotential. What is the gauge theory interpretation of this? As noted above turning on F has the effect of turning on the field strength F_v in the $U(1) \in U(N)$, which is also equivalent to turning on B field in spacetime. Thus this seems to be related to considering the non-commutative version of the above gauge system [40]. In particular considering a self-dual non-commutativity in spacetime presumably generates a superpotential, as is predicted from the above formula. Note that this is consistent with the fact that in the UV where S is smaller this modification of the superpotential is a more pronounced effect, and it disappears in the IR where S is larger. It would be interesting to derive this result directly in the context of the non-commutative $N = 1$ Yang-Mills theory. Moreover the dependence of the genus g partition function on the non-commutativity parameter is identical⁷ to that obtained in [41]. Namely, in the large N expansion, there is no modification at the level of planar diagrams, i.e. at $g = 0$. Moreover at genus g the amplitudes are expected (when we have a self-dual non-commutativity) to scale (in the leading order) as $\frac{1}{\theta^{4g}} \sim B^{4g}$ which in our case translates to an F^{4g} dependence. This is in agreement with the fact that $|\partial_S W|^2$ indeed scales as F^{4g} .

3.4. More General Values of S

So far, in the context of gauge theory discussion we mainly considered the limit where $\langle S \rangle$ is small compared to the string scale. However the duality we are proposing holds for arbitrary $\langle S \rangle$. If $\langle S \rangle$ is not small, on the gauge theory side we get modification to the form we have written above, which is computed by the Chern-Simons theory on S^3 . What kind of gauge theory does this correspond to? The gravity side provides a

⁷ We thank R. Gopakumar for pointing this out to us.

hint: If we consider wrapped D4 brane domain walls, we have infinitely many species of domain walls. The reason for this is that we can consider the bound state of n D2 branes with the D4 brane manifested through turning on n units of $U(1)$ flux through the S^2 part of the worldvolume of the D4 brane. This can also be viewed as the effect of changing the B-field on the $\mathbf{P}^1$ by $2\pi in$. The effect of such domain walls is thus shifting $t = S \rightarrow S \exp(2\pi i/N) + 2\pi in$. In other words we have the vev's of S , not only taking values around a circle about the origin, but also circles about $2\pi in$ for any integer n . Moreover the BPS tension for such domain walls is given by

$$\Delta W = \frac{1}{\lambda_s}(S + 2\pi in)$$

The geometry of these domain walls can be recovered from an enlarged field content [10]: We can introduce one variable Y_n for each n , capturing the corresponding domain wall by its shift in the argument, and consider the superpotential

$$W = \int d^2\theta \sum_n [(S + 2\pi in)Y_n + iN^2\alpha e^{-Y_n/N}] \quad (3.10)$$

the domain wall with 1 D4 brane wrapped over $\mathbf{P}^1$ bound to n D2 branes will now correspond to shifting $Y_n \rightarrow Y_n + 2\pi i$. Integrating the Y_n 's out will give

$$W = \frac{1}{\lambda_s} \sum_n (S + 2\pi in) \log(S + 2\pi in)^{-N} + a(S + 2\pi in) + b$$

which is indeed equal to

$$W = \int d^2\theta \frac{1}{\lambda_s} N \frac{\partial F_0(S)}{\partial S}.$$

The variables Y_n were introduced to incorporate the kinks, but their appearance on the original gauge theory side, except for Y_0 seems mysterious. It would be interesting to see if one can find a direct interpretation of all the Y_n 's. We expect that to be related to the possibility of doing large $SU(N)$ gauge transformations on the S^3 part of the worldvolume of D6 brane.

The higher genus corrections in the case of large S are also similar to the modification at the genus 0 case. In particular we get an infinite sum with S replaced by $S + 2\pi in$. This in particular is related to the fact that we can have a new baryon vertex for each wrapped D2 brane with D0 brane turned on [42].

3.5. Adding Matter

In the context of geometric engineering of $N = 1$ supersymmetric gauge theories realized as D6 branes wrapped around S^3 cycles of CY manifolds [32] matter can be realized as extra D6 branes wrapped around other S^3 's intersecting the gauge theory S^3 along a circle (where the vev of the Wilson line around the circle on the probe brane plays the role of mass for the matter). How does our duality extend to this case? In fact in the topological string the duality does extend to this case [10]. In particular in computation of Wilson loop observable for the Chern-Simons theory one adds extra topological branes intersecting the original S^3 along a knot, and it was shown that the closed topological string amplitudes agrees with the expected result for knot invariants for Chern-Simons theory. More checks have been made in [11] for a large number of distinct knots. In the context of embedding the topological string dualities in the superstring what this means is that the dual gravitational system will not only have a blown up S^2 but will also have additional D6 branes (which for algebraic knots will intersect the S^2 along a circle). The fact that the topological computations agree on both sides translates to the statement that the superpotential computations on both sides agree and is further evidence for this duality in the superstring context. Note that for each knot we obtain a different “matter” system for this generalized gauge theory, which in the limit of large Y give rise to the same low energy physics, but are distinct theories in the context of generalized gauge theories we have been considering. The gauge theoretic interpretation of these results is currently under investigation [43].

4. The Mirror Type IIB Description

As is well known, type IIA on a CY is equivalent to type IIB on a mirror CY. This implies that everything we have said above in the context of type IIA has a type IIB counterpart.

For example instead of D6 branes of type IIA wrapped around S^3 we consider D5 branes of type IIB wrapped around S^2 . Also turning on even-form fluxes in type IIA is mirror to turning on 3-form H_{RR} and H_{NS} flux in the type IIB side and the superpotential that gets generated in this context is given by

$$W = \frac{1}{\lambda_s} \int \Omega \wedge [H_{RR} + \tau H_{NS}]$$

where τ is the complex coupling constant of type IIB, and Ω is the holomorphic 3-form of the CY. The above integral can be done and yields the formula in terms of the prepotential of the corresponding $N = 2$ theory, as discussed in the Type IIA case. Note however, that the type IIB system is simpler in that by mirror symmetry the worldsheet instantons that were relevant in the context of type IIA theory in computing the prepotential, are absent for the type IIB case, and classical geometry already captures these corrections. In particular the B-topological theory (known as the Kodaira-Spencer theory of gravity) simply involves aspects of complex geometry of Calabi-Yau.

So as far as writing a classical gravitational background, the type IIB description would be more useful because the worldsheet instanton effects are absent. However, as far as the conformal theory on the string worldsheet, i.e. the large N expansion description of the gauge system, the type IIA and type IIB theories are of course identical.

In the above context we would need to know the mirror of local CY: $O(-1) + O(-1) \rightarrow \mathbf{P}^1$. The mirror of this is known and it is essentially the conifold with one subtlety [36]: The conifold has only one compact 3-cycle, whereas $O(-1) + O(-1) \rightarrow \mathbf{P}^1$ has two compact even cycles, namely 0 and 2 cycle. As was noted in [36] in the limit where the Kahler class of $\mathbf{P}^1$ approaches zero, i.e., $t \rightarrow 0$, the mirror becomes effectively the conifold (the actual mirror differs from the conifold by having some variables being $\mathbf{C}^*$ variables rather than $\mathbf{C}$ variables)⁸. Similar observations were made in [44]. Even though in principle we can consider the full mirror geometry, since the complex geometry of the conifold is more familiar and better studied we restrict our attention to this case⁹. This will correspond to a particular limit of our Type IIA theory, where we consider only the small $< S >$ region. Recall that this was the regime where the theory retained only the leading dimension operators in the action and led to a theory which was similar to the standard $N = 1$ supersymmetric gauge theory.

We will be brief for this case, as most of the discussion can be literally borrowed from our discussion in the previous section. We start with N D5 branes wrapped over the $\mathbf{P}^1$

⁸ The actual mirror is given by $x_1 + x_2 + x_1 x_2 e^{-t} + 1 - uv = 0$ where x_1, x_2 are $\mathbf{C}^*$ variables and u, v are $\mathbf{C}$ variables.

⁹ In this limit the internal topological theory corresponds to a G/G model on S^2 (coming from the holomorphic Chern-Simons theory on S^2 [14][45]) which should also be equivalent, by mirror symmetry, to the large N fixed k limit of the Chern-Simons theory. This topological theory should also be equivalent to the Penner Matrix model. It would be interesting to verify these equivalences among these topological gauge theories more directly.

in the $O(-1) + O(-1) \rightarrow \mathbf{P}^1$ geometry. The large N limit of this, in the limit of shrinking $\mathbf{P}^1$ corresponds to blowing up an S^3 with N units of H_{RR} flux through the S^3 . Let us write the conifold geometry as

$$z_1 z_2 - z_3 z_4 = \mu$$

Then the genus 0 prepotential is given as

$$F_0(\mu) = \frac{-1}{2} \mu^2 \log \mu + P_2(\mu)$$

where $P_2(\mu)$ is an undetermined polynomial of degree 2 in μ . Now we consider turning on fluxes: The mirror of turning on NS 4-form flux corresponds here to turning on H_{NS} and in the cycle dual to S^3 . Thus as far as the superpotential is concerned we have

$$W = \frac{1}{\lambda_s} [N \partial_\mu F_0(\mu) + M \mu] = \frac{1}{\lambda_s} [-N \mu \log \mu + a \mu + b]$$

where $M = M_1 + \tau M_2$, and the discussion reduces to the small S limit of the discussion in the previous section.

While this paper was being prepared two papers [18][19] appeared which are related to this type IIB construction. In particular (among other things) they consider the gravitational background corresponding to D5 branes wrapped on 2-cycles of CY and their results are consistent with the superpotential analysis here.

5. c=1 Non-critical Bosonic String and N=1 Superpotentials at Large N

As discussed above the type IIA or type IIB near a conifold background with some fluxes turned on can be interpreted as large N limit of certain $N = 1$ supersymmetric gauge theories. In particular the string expansion is equivalent to the large N expansion of a gauge theory. Moreover certain superpotential corrections of the gauge theory can be viewed as computations of the corresponding topological strings in the CY background. These are readily computed and thus carry a large amount of information to all orders in $1/N$, for the gauge theories in question. In particular here we will explain how the Type IIA superstring near the small blow up of conifold, or equivalently the Type IIB superstrings in the conifold geometry relate the non-critical bosonic string amplitudes with that of the superpotential computations at the large N limit of the corresponding $N = 1$ gauge systems.

It was shown in [46] that the conformal theory near the conifold is given by the same system found in [47] in connection with non-critical bosonic strings on a circle of self-dual radius. This conformal theory is that of a supersymmetric Kazama-Suzuki coset construction

$$SL(2)/U(1)$$

at level $k = 3$, and the relation with non-critical bosonic strings is that the topological twisting of this system is equivalent to considering bosonic string propagating on a circle of self-dual radius with the fermions of the coset model playing the role of the ghosts in the bosonic string. This relation between bosonic string on a self-dual circle and the superconformal theory of a conifold is in agreement with the fact [48] that the ground ring of the bosonic string for this background is isomorphic to the holomorphic function on the conifold (which is generated by z_1, z_2, z_3, z_4 subject to the relation $z_1 z_2 - z_3 z_4 = \mu$) where the cosmological constant of the bosonic string is mapped to the deformation parameter of the conifold. Moreover the observables of the $c = 1$ theory are mapped to deformations of the conifold geometry:

$$\sum_n \epsilon_n(z_1, z_2, z_3, z_4) + z_1 z_2 - z_3 z_4 = \mu$$

where ϵ_n is a polynomial of degree n in z_i . These deformation parameters get mapped to states of the bosonic string which are indexed by a representation of $SU(2)_L \times SU(2)_R$ of this system, viewing z_i as entries of a 2×2 matrix M with the conifold being defined as $\det M = \mu$ and where the $SU(2)_L$ and $SU(2)_R$ are realized by left and right multiplication of M with $SU(2)$. In particular the degree n polynomial ϵ_n decomposes into representation of spin $(j_L, j_R) = (n/2, n/2)$ with $|m_L, m_R| \leq n/2$. Let us denote the totality of these parameters by μ_i (except for μ). The bosonic string amplitudes compute topological B-twisting of the deformed conifold¹⁰. For various aspects of $c = 1$ non-critical bosonic string see [50]. As already discussed the genus g partition function will be a function $F_g(\mu, \mu_i)$ of these parameters deforming the conifold background. Recall that in the gauge theory context μ is identified with S and we will thus denote $F_g(S, \mu_i)$. The topological string computes, at genus g , the term in the effective action given by

$$\frac{N}{\lambda_s} \int d^2\theta [\mathcal{W}^2]^g \partial_S F_g(S, \mu_i) = \int d^2\theta F^{2g} \partial_S F_g(S, \mu_i) + \dots$$

¹⁰ Aspects of this relation has recently been verified and certain results of bosonic strings have been recovered directly using the Calabi-Yau picture and the Kodaira-Spencer theory [49].

What is the interpretation of this for the gauge theory? As in the usual AdS/CFT correspondences, we would expect that the μ_i will be related to operators on the gauge theory side, deforming the gauge theory action by terms $\mu_i \mathcal{O}_i$. In fact, in the context of 3-brane probes of the conifold [51] aspects of such deformations for the gravity side have been studied in [52] and a similar analysis should be extendable to the case at hand. Thus the topological strings compute the response of the system upon such deformations. In particular at genus 0, turning on the μ_i modifies the superpotential for the gaugino superfield. Also turning on F will give rise to $1/N$ correction to the superpotential. It would be extremely interesting to understand the source of these corrections on the gauge theory side.

It would also be interesting to find the conformal theory associated with the RR and NS fluxes turned on in the conifold geometry. This is very interesting in view of the fact that before turning fluxes on we have an exactly solvable conformal theory given by $SL(2)/U(1)$ KS model. It would be very interesting to find the deformation of this theory. It is likely to involve ingredients similar to the ones encountered in [53][54].

6. Wrapped D-branes and Compact CY

Consider type IIA superstrings compactified on a Calabi-Yau threefold. In the above, we considered a situation where we take a large number of $D6$ branes wrapped over an S^3 in the CY, and taking the analog of the near horizon geometry, to decouple the gravity, and then proposing a dual gravity description for the gauge system.

If we wish to repeat what we did in the previous sections, by considering $D6$ branes wrapped on some S^3 , which is part of a compact Calabi-Yau, and filling the rest of the spacetime, we immediately run into a problem. We cannot wrap a $D6$ brane over a 3-cycle as there would be nowhere for the flux to go for compact internal CY. However suppose we consider a CY manifold with some number of S^3 's and we wrap $D6$ -branes and anti- $D6$ branes over them, in such a way that the net $D6$ brane charge is zero. This is of course a non-supersymmetric situation.¹¹ We expect that the branes will eventually annihilate each other leaving us with an $N = 2$ background. If the S^3 's are rigid then this annihilation process takes some time, because there is a potential barrier for the wrapped $D6$ branes

¹¹ We can also consider the type IIB mirror of this where we consider $D5$ /anti- $D5$ branes wrapped around vanishing S^2 's of the CY.

to move in the CY (i.e. there is a potential for the scalar corresponding to moving them in the normal direction in the CY).

We now wish to apply the considerations of this paper and propose a large N dual in this context. For the considerations of the gauge theory to be applicable we have to consider the limit where S^3 has shrunk to zero size. This is the analog of $Y_0 = 0$ in the formula in section 3). What we will find is that taking the large N limit induces a transition to a topologically distinct CY, with some fluxes turned on. Moreover the fluxes can disappear as in [17] leaving us with an $N = 2$ supersymmetric vacuum. Thus the effect of the brane annihilation at large N has been to shift the background.

Consider a Calabi-Yau with R vanishing S^3 cycles $[C_i]$ which span a $K < R$ dimensional subspace of H_3 . In other words assume

$$\sum Q_{ji}[C_i] = 0 \quad \text{for } j = 1, \dots, R - K$$

for some integral matrix Q . Let us consider N_i D6 branes partially wrapped around C_i , where we allow some N_i to be negative, in which case we mean the number of the corresponding anti-branes. The condition that the net D6 brane flux is zero implies that

$$N_i = l_j Q_{ji}$$

for $R - K$ integers l_j . Now, let us consider the limit where the S^3 's are vanishingly small. In this limit, applying the discussion of the near horizon geometry, we are naturally led to consider the S^2 blown up geometry where the blow up parameter for the i -th sphere is given by

$$t_i = i\lambda_s N_i = i\lambda_s l_j Q_{ji}$$

Notice that not all the t_i are independent. In particular there are only $R - K$ independent parameters l_j which determine them. This is exactly as it should be for the local geometry to have a blowup¹². In other words the blow up geometry is a CY with K less dimension of $h^{2,1}$ but $R - K$ more dimensions of $h^{1,1}$. Moreover the condition on the various Kahler classes of the $\mathbf{P}^1$'s is exactly the same as that found above for the t_i . This gives further support for the conjecture that the large N limit of the wrapped brane-anti-brane geometry,

¹² In fact this shift in the hodge numbers can be understood from the viewpoint of inverse process of Higgsing of $U(1)^{R-K}$ by R charged fields [55] where the charged fields can be viewed as wrapped D2 branes in the blown up geometry.

when we have no net branes is equivalent to a blown up CY with the Kahler parameters for the blown up spheres given as above. However, here we will also have RR fluxes through the S^2 's. In this case the supersymmetry is completely broken [25][26] by the RR fluxes. The fluxes can disappear as recently studied in [17].

It would be interesting to check this generalized conjecture in the topological string setup: Namely this suggests that the topological open string amplitudes in the context of compact CY manifolds when there is no net topological D-brane, and when the D-branes are wrapped over spheres is easily computable by a related closed string theory computation on the blown up CY with different Hodge numbers.

7. Generalizations

There are many natural generalizations of this work. In particular it is natural to consider transitions among topologically distinct manifolds, going through vanishing cycles, and find a large N brane system/gravity duals, where the large N gauge system will lie on one side of the transition and the dual gravitational system will lie on the other side (this was in fact the philosophy advocated in [9])¹³. In other words the large N brane systems can be viewed as inducing transitions in the background geometry. For examples there are transitions in the CY which involve the vanishing of certain 3-cycles and blowing up 4 manifolds, such as Del-Pezzo manifolds. In this context it is natural to conjecture the existence of a duality involving a large N limit of wrapped D6 branes about the 3-cycles in the context of type IIA with the 4 manifold resolution of the singularity on the gravity side, with certain fluxes turned on¹⁴. Or in the context of M-theory on 4-folds it is natural

¹³ If we consider a Morse function f on a manifold the critical points of it encode certain topological aspects of the manifold. Near a critical point with p positive and q negative eigenvalues for $\partial_i \partial_j f$, for $f \neq f_{critical}$ the manifold near the critical point has the geometry of a filled $S^{p-1} \times S^{q-1}$. For $f > f_{critical}$ the S^{q-1} is filled and for $f < f_{critical}$ the S^{p-1} is filled. This is the general kind of transition expected for large number of branes replaced by fluxes. If we consider two manifolds in the same cobordism class, and consider a Morse function on the interpolating manifold the above picture suggests that branes can induce the transition. So if the cobordism classes are trivial and we have suitable branes we can interpolate between any two manifolds in this way.

¹⁴ In fact there is already evidence for some such cases based on quotienting the Chern-Simons duality on S^3 by finite groups on both sides [56]. For example Chern-Simons on S^3/Z_2 should be equivalent to $\mathbf{P}^1 \times \mathbf{P}^1$ blow up inside a Calabi-Yau. Some aspects of this predictions have already been checked.

to look for transitions involving shrinking S^3 's and growing 4-cycles, where we consider a large number of $M5$ branes wrapped over the S^3 's and filling the 3-dimensional spacetime, which should be dual to the geometry involving the blowup of the 4-cycle with some G flux turned on (in fact in this context the gravity solutions are already worked out in [31]).

It is also natural to extend our results for the case of $SU(N)$ systems to SO and Sp groups by including orientifolds. In fact it has been shown in [57] that the large N duality of Chern-Simons theory for $SU(N)$ groups extend to the SO and Sp case as well.

Finally, the idea that studying BPS/anti-BPS systems are important for a more fundamental understanding of basic degrees of freedom for string theory as advocated by Sen is in line with the example we have found: We can describe one string background in terms of the ground state of a different one in the presence of D-brane/anti-D-brane systems. In a sense, this idea, combined with the idea that various transitions among manifolds can be induced by large N limit of brane systems, suggests that if we start with any background in string theory, and consider complicated enough configurations of branes and anti-branes, we can effectively be discussing arbitrary backgrounds of string theory.

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