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**ASYMMETRIC ORBIFOLDS:
PATH INTEGRAL AND OPERATOR FORMULATIONS**

K.S. Narain, M.H. Sarmadi

Theory Division, CERN

CH-1211 Geneva 23, Switzerland

and

C. Vafa

Lyman Laboratory of Physics, Harvard University

Cambridge, MA 02138, USA

ABSTRACT

We discuss some of the details involved in asymmetric orbifold construction. In particular, the path integral formalism is developed and shown to be consistent with operator interpretation. Conditions of one-loop modular invariance are discussed in detail. We also describe how one can construct correlation functions involving twisted states.

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1. Introduction

One of the most beautiful aspects of string theory is the way it makes contact and modifies our notion of classical geometry. String geometry has a far richer structure than the ordinary Riemannian geometry. Consider for instance toroidal compactification [1]. In this case by tuning the moduli of the torus we can obtain enhanced gauge symmetries, a phenomenon unheard of for ordinary point particle theory. Also these theories exhibit a striking isomorphism between the large radius and small radius tori (known as duality). Other examples of modification of geometry involve heterotic geometry [2], in which the space in which the string moves has an ambivalent nature: the left-movers are on one space and the right-movers on another. Again there is no classical geometrical interpretation. Another class of examples is provided by (symmetric) toroidal orbifolds [3,4] where one considers string propagation on the quotient of a torus by a discrete isometry. In this case again our notions of classical geometry must be modified. There are harmless singularities on the orbifold which do not destroy consistent string propagation, as they would in the case of point particles. In [5] we constructed 'asymmetric orbifolds' which combines the heterotic and orbifold constructions by left-right asymmetric quotients of toroidal compactifications. This construction is related to the fermionic construction and covariant lattice approach [6]. In the construction of asymmetric orbifolds we briefly mentioned how in principle one computes all the physical amplitudes of interest. The details were left for future work. In this paper we fill this gap.

There are two aspects to this problem. The first is whether it is possible even to define the amplitudes consistently and the second question is whether one can compute them explicitly. We should note that the first problem is not a trivial one as there is no left-right asymmetric notion of a non-linear sigma model. This is in contrast to some other known solutions of string theory corresponding to strings propagating on classical manifolds such as Calabi-Yau manifolds [7], which, even though one may not know all the amplitudes explicitly, are a priori well defined through the definition of a non-linear sigma model. The sec-

ond problem, the explicit computation of amplitudes, is possible here because of the underlying toroidal theory being a simple conformal theory. We should point out that there are ways to construct asymmetric orbifolds even in the context of Calabi-Yau manifolds. This is done through the realization of Calabi-Yau manifolds at special moduli by conformal field theories [8] represented by superconformal Landau-Ginzburg theories and modding the resulting theory by symmetries of the superpotential *with the inclusion of discrete torsion* [9]. In these cases the geometry of the underlying manifold is not as simple to 'visualize' as that of the asymmetric toroidal orbifolds, which has a deep connection with the geometry of self-dual even Lorentzian lattices. It is this underlying rich lattice which will play a prominent role in all our discussions in this paper, and restores a certain degree of geometricity to asymmetric toroidal orbifolds.

The proposal made in [5] to compute physical amplitudes for asymmetric toroidal orbifolds was based on doubling the degrees of freedom and embedding the asymmetric orbifold in a symmetric one for which there are well-defined rules of computation of amplitudes, and then taking the square root of the result in a careful manner. This is precisely what we shall do in this paper. In particular we develop the path integral formalism on the torus and higher genus Riemann surfaces and relate it to the operator formalism for asymmetric orbifolds. This discussion will necessarily include a discussion of modular invariance. In addition we compute correlation functions for twisted states.

The organization of this paper is as follows.

In Section 2 we discuss the path integral for twisted sectors. We define the chiral boson partition function as the holomorphic square root of the path integral for non-chiral bosons. While the path integral over oscillator modes factorises in the usual way, enabling one to take the square root of the integral, factorisation of the instanton contribution (i.e., the lattice part) is more tricky. The latter requires the introduction of a Z_2 phase which depends on the windings around different cycles of the Riemann surface (this phase is essentially the generalisation of the one appearing in ref [10], namely the exponential of an antisymmetric tensor

field which is a mod 2 reduction of the metric). We show that the path integral factorises for arbitrary genus surfaces, provided the Z_2 phase satisfies a cocycle condition (eq. (2.10a) below). Thus one can define the square root in each twisted sector up to overall constant phases.

In Section 3, we discuss modular invariance conditions. First, we derive the relations between the Z_2 phase in different twisted sectors resulting from the one- and two-loop modular invariance conditions and show that they imply consistent operator interpretation. Next we show that provided the level matching condition is satisfied, the constant overall phases in each twisted sector can be chosen to satisfy modular invariance at least at the 1-loop level.

In Section 4, we discuss the operator interpretation of the overall constant appearing in each twisted sector at genus-1. In the $(g, 1)$ sector, this constant represents the number of copies of g -twisted Hilbert spaces and in ref [5] this was shown to be equal to the dimension of the irreducible representation of the vertex operators corresponding to untwisted states. More precisely, these vertex operators comprise two parts; one which acts within a given twisted Hilbert space and the other which mixes different copies of the twisted Hilbert space (recall that there is one copy for each fixed point) and thus acts on the fixed points. If we denote the number of fixed points for the g -twisted sector as D_g^2 , then it turns out that the D_g^2 dimensional representation splits into D_g copies of D_g , dimensional irreducible representations under the action of vertex operators (the fact that D_g is an integer is non-trivial and is a consequence of the self-duality of the lattice Γ describing the initial torus). We can pick any one of these irreducible representations and the interactions will close. D_g was also the number appearing in the $(g, 1)$ sector (by doing either the path integral or by modular transformation of the $(1, g)$ sector) thus indicating the consistency of modular invariance with the Hilbert space interpretation. In Section 4, we show that the constant appearing in the (g, h) sector, up to the undetermined phase, is just the trace of h in the D_g -dimensional irreducible representation as it should be in order to have an operator interpretation. We also show that the relative phases appearing between different

momentum states have the correct operator interpretation.

In Section 5, we show how one can construct correlation functions for twisted fields. We construct four-point functions on the sphere involving g, h, g^{-1} and h^{-1} twisted states in the path integral formalism. Holomorphic factorisation of the instanton contribution again requires a phase that obeys the cocycle condition (2.10a) (the cocycle in fact is related to the Lepowsky cocycle [11]). The resulting correlation function satisfies crossing symmetry. In particular, factorising in the channel g -twisted $\times g^{-1}$ twisted states gives rise to an untwisted intermediate channel and agrees with the results calculated directly in the operator formalism (ref [5]) while factorising in the channel g -twisted $\times h$ -twisted gives rise to gh -twisted intermediate states as expected. This way one can in principle get all the operator algebra coefficients for the asymmetric orbifold.

In Appendix A, we discuss the sublattices spanned by the allowed momenta for each sector, both at 1 loop and higher. Appendix B contains a discussion of the overall constant appearing in each sector. In Appendix C, we derive an equality (eq. (3.28) below) which we will use in Section 3.

2. Path integral formalism for asymmetric orbifolds

We begin by briefly reviewing the asymmetric toroidal orbifold construction of [5]. Our starting point is a toroidal compactification of the heterotic string in $(10-d)$ space-time dimensions. Thus the compact space is a torus of dimension $d+(16+d)$ which can be represented as

$$T^{d+(16+d)} = R^{d+(16+d)} / \Gamma_{d,16+d} \quad (2.1)$$

As was shown in ref [1], modular invariance requires that the lattice $\Gamma_{d,16+d}$ describing the torus $T^{d+(16+d)}$ be even and self-dual with respect to a metric of signature $(d, 16+d)$, where the right-movers and the left-movers propagate in the first d and the second $(16+d)$ dimensions respectively. Let G be a discrete symmetry group of the torus that does not mix the left- and right-movers. Let (g, δ_g)

denote an element of G , where g is a rotation on $R^{d+(16+d)}$ that does not mix left- and right-movers and δ_g is a translation. Since G is a symmetry of the torus it follows that $g, \Gamma = \Gamma$ (for simplicity we will suppress the subscript $(d, 16+d)$ from Γ). The asymmetric orbifold is now obtained by modding out the torus by the group G . There are two types of string states on the orbifold, the untwisted ones that correspond to strings that are closed on the torus and the twisted ones that are not closed on the torus but are closed on the orbifold. The boundary condition for the latter is given by

$$X(\sigma + 1, t) = gX(\sigma, t) + \delta_g \quad (2.2)$$

for every non-trivial element of G . In order to preserve the world-sheet supersymmetry, the right-moving fermions also have to be twisted simultaneously

$$\psi(\sigma + 1, t) = g_R \psi(\sigma, t) \quad (2.3)$$

where g_R is the part of g acting on the right-moving directions. It is clear that for each non-trivial (g, δ_g) there are twisted states (we shall in the following denote this element by g ; by the context it will become clear whether g refers just to the rotation part or to the full element). We shall denote by $\mathcal{H}_g$ the corresponding Hilbert space of states ($\mathcal{H}_1$ will denote the untwisted Hilbert space).

However not all the states of $\mathcal{H}_1$ are relevant states for the orbifold. Since one is modding out the torus by G , the operators in the theory are G -invariant operators. $\mathcal{H}_1$ splits into different subspaces that are irreducible representations of G -invariant operators and these different subspaces are characterised by the action of G . In particular the G -invariant subspace $\mathcal{H}_1^{\text{inv}}$ is one such irreducible representation which contains $(10-d)$ -dimensional gravitons. In ref [5] it was shown that this implied that in order to satisfy modular invariance we should include all the twisted Hilbert spaces (each of them appearing with a certain multiplicity) and that each $\mathcal{H}_g$ must also be projected to C_g -invariant states, where C_g is the maximal subgroup of G that commutes with g . In path integral language, this means that we must include all pairs of commuting boundary conditions (g, h)

for the one-loop diagram, where g represents the twist along σ and h along t directions. The fact that one must include all twists along the σ direction is clear as it amounts to taking traces over all twisted Hilbert spaces. The twists along t directions appear due to the fact that one must take traces only over the C_g -invariant subspaces. The latter can be ensured by inserting a projection operator inside the trace. The projection operator is just $\frac{1}{|C_g|} \sum_{h \in C_g} h$ where $|C_g|$ is the order of C_g . In the path integral formalism, insertion of h in the trace amounts to twisting the boundary condition along t by h . This explains the fact that we must include all the commuting pairs of boundary conditions (g, h) .

In this section we will define these path integrals precisely. The basic problem is that one is dealing with chiral bosons. The standard approach to defining path integrals for chiral theories is to double the variables: for every left-mover we introduce an identical right-mover and vice-versa. As a result, the variables become non-chiral and one can calculate the path integrals. The integral for the chiral theory is then defined up to a phase as the square root of the above result. On the torus, the contribution of the oscillator part factorises naturally into left and right parts, thus enabling one to take the square root; however, the difficulty is with respect to the instanton contribution. If this procedure is to make sense, the latter must be mod. square of $\sum_{(P_L, P_R)} q^{P_L^2/2} \bar{q}^{P_R^2/2}$, where the sum is over an appropriate lattice. For example in the $(1,1)$ sector, this lattice must be Γ and it is known [6,10] that the instanton contribution is indeed of this form, provided one introduces a suitable antisymmetric tensor field B . First we have to define this “doubled” variable X , which must coordinate a $16 + 2d$ dimensional torus. As discussed in ref [5] we take the corresponding lattice to be $\frac{1}{2}\Gamma_{16+d}$ (the dot products between different v and w belonging to this lattice will always be taken as $v_L w_L - v_R w_R$ where the L and R parts are the first $16+d$ and the second d components of the vectors). In the following we shall also need Euclidean dot products between vectors. For this we define the following mapping: every $v = (v_L, v_R) \in \Gamma$ is mapped to $\bar{v} = (v_L, -v_R) \in \bar{\Gamma}$. Then the Euclidean dot product between $v, w \in \Gamma$ is the same as the dot product between v and $\bar{w}$ as

$v\bar{w} = v_L w_L + v_R w_R$. Let v and w be the winding vectors along σ and t directions; then a constant B field gives rise to a phase $\varepsilon(v, w)$ in the path integral. Thus the instanton contribution to the path integral in the $(1,1)$ sector is:

$$Z_{\text{inst}} = A^{d/2} \sum_{v, w \in \frac{1}{2}\Gamma} \varepsilon(v, w) e^{-2\pi i v \bar{v} - 2\pi i \frac{1}{2}(w + \tau_1 v)(\bar{w} + \tau_1 \bar{v})} \quad (2.4)$$

where $\tau_1 + i\tau_2 = \tau$ is the Teichmüller parameter of the world-sheet torus, A is the area of the surface representing the normalisation of the zero mode and d is the dim. of Γ . As shown in ref [10], the appropriate phase in this sector is $\varepsilon(v, w) = e^{4i\pi v \cdot w}$. By doing a Poisson resummation over w we obtain

$$\begin{aligned} \frac{Z_{\text{inst}}}{(A i Y)^{d/2}} &= \sum_{\substack{k \in 2\Gamma^* \\ v \in \frac{1}{2}\Gamma}} q^{\frac{1}{2}(\frac{1}{2}k + v + v)(\frac{1}{2}k + \bar{v} + v)} \frac{1}{q^{\frac{1}{2}(\frac{1}{2}k + \bar{v} - v)(\frac{1}{2}k - \bar{v} + v)}} \\ &= \left| \sum_{P \in \Gamma} q^{\frac{1}{2}P^2} \frac{1}{q^{\frac{1}{2}P^2}} \right|^2 \end{aligned} \quad (2.5)$$

where $iY = \tau - \bar{\tau}$.

Now let us consider the twisted sector (g, h) . To simplify the discussion let us first set the shifts $\delta_L = \delta_R = 0$ so that g and h consist of only rotations. Later we will see how one can handle the shifts. A general X will be of the form

$$X = x + L_1 \sigma + L_2 t + \text{oscillator modes} \quad (2.6)$$

Boundary conditions relevant for this sector imply

$$(1-g)x = -L_1 + v \quad (2.7a)$$

$$(1-h)x = -L_2 + w \quad (2.7b)$$

$$(1-g)L_1 = (1-h)L_1 = (1-g)L_2 = (1-h)L_2 = 0 \quad (2.7c)$$

for some $v, w \in \frac{1}{2}\Gamma$.

In order to analyse these conditions let us define the following sublattices:

$$I_g = \{u \in \Gamma \mid gu = u\} \quad (2.8a)$$

$$N_g = \{u \in \Gamma \mid u \cdot v = 0 \text{ for all } v \in I_g\} \quad (2.8b)$$

We can decompose Γ in terms of the above normal directions. We denote these projections of a given vector v as $v|_I$ and $v|_N$, respectively and the lattices formed by these projections when v varies over entire Γ as $\Gamma|_I$ and $\Gamma|_N$. We also define the sublattice I of Γ which is invariant under both g and h and let N be the sublattice normal to I . Then eq. (2.7c) implies that $L_1|_N = L_2|_N = 0$. Together with eqs. (2.7a,b) this means that:

$$L_1 = v|_I, L_2 = w|_I \quad (2.9a)$$

$$(1-g)w = (1-h)v \quad (2.9b)$$

Thus given a pair $v, w \in \frac{1}{2}\Gamma$ satisfying eq. (2.9b), we can find L_1, L_2 and x that satisfy the boundary conditions. Note that while $x|_I$ is arbitrary $x|_N$ is determined by eqs (2.7a,b) in terms of v and w , and hence is discrete. Each such allowed $x|_N$ thus defines an I -dimensional fixed plane (more precisely a fixed torus corresponding to the lattice $\frac{1}{2}I$). Let $n = \dim N$ and $m = \dim I$ and $\frac{1}{2}\Gamma^0$ be the $2m + n$ dim. sublattice of $(\frac{1}{2}\Gamma)^2$ consisting of vectors satisfying eq. (2.9b). Let Q denote the group of transformations on $\frac{1}{2}\Gamma^0$ defined as $v \rightarrow v + (1-g)p$ and $w \rightarrow w + (1-h)p$ for any p such that $(1-g)p$ and $(1-h)p \in \frac{1}{2}\Gamma$. Then from eq. (2.7a,b) only the zero mode transforms as $x|_N \rightarrow (x+p)|_N$. However not all the allowed $x|_N$ define distinct fixed planes - indeed if two $x|_N$'s differ by a vector of N^* ($\star$ will always denote the dual lattice), then the two fixed planes are identical. Let $\tilde{Q}$ denote the subgroup of Q obtained by restricting $p|_N \in \frac{1}{2}N^*$, then $\tilde{Q}$ leaves the instanton configuration invariant. The total number of distinct fixed planes is therefore the order of the set $[Q/\tilde{Q}]$ and the distinct instanton configurations (modulo the shifts in the fixed planes) are labelled by the $2m$ -dimensional lattice $\frac{1}{2}\tilde{\Gamma} = (\frac{1}{2}\Gamma^0/Q)$ which can be identified with the set of vectors $v|_I$ and $w|_I$.

For each fixed plane the instanton contribution can now be computed. Just as in the untwisted case we will need to introduce a phase $\varepsilon_{(g,h)}(v, w)$ in order to factorise the instanton contribution. We demand that this phase satisfy the

This means that there exists a vector $\delta_{(g,h)} \in \frac{1}{2}S^*/S^*$ (this is because the ϵ 's are Z_2 phases) such that for $v|_I = 0$,

$$\epsilon_{(g,h)}(v, w) = e^{-4i\pi\delta_{(g,h)} \cdot w|_I} \quad (2.13b)$$

Let us now fix an L_1 in (2.11) and sum over all allowed L_2 's. Pick an L_2^0 such that (L_1, L_2^0) is an allowed pair. Then we can write $L_2 = L_2^0 + \ell$ where $\ell \in \frac{1}{2}S$ and

$$\sum_{(L_1, L_2)} = \sum_{L_1} \sum_{\ell \in \frac{1}{2}S}$$

Here the sum over L_1 ranges over all allowed vectors for arbitrary L_2 satisfying eqs. (2.9a,b). In Appendix A, we show that the sublattice spanned by all the allowed L_1 's is exactly $\frac{1}{2}S^*$. Using this, eqs. (2.10a,b) and (2.13a,b) and doing Poisson resummation over ℓ we get

$$\frac{Z_{\text{inst}}}{(A \cdot iY)^{m/2}} = F[S/I] \sum_{L_1 \in \frac{1}{2}S^*} \sum_{k \in 2S^*} \epsilon(v, w) \cdot e^{4i\pi(\frac{1}{2}k + L_1 + \delta_{(g,h)}) \cdot L_2^0} q^{\frac{1}{2}(\frac{1}{2}k + L_1 + L_1 + \delta_{(g,h)})^2} \frac{1}{q^{\frac{1}{2}(\frac{1}{2}k + L_1 - L_1 + \delta_{(g,h)})^2}} \quad (2.14)$$

where $v, w \in \frac{1}{2}\Gamma$ such that $v|_I = L_1$ and $w|_I = L_2^0$, and we have used the symbol $iY = \tau - \bar{\tau} = 2i\tau_2$. Let $k = 2\bar{P}$ and $L_1 = \frac{1}{2}(P' - P)$ where $P, P' \in S^*$. Choose some W and W' such that (P, W) and (P', W') are allowed pairs and $W' - W = 2L_2^0$. Now using eq. (2.10) we obtain:

$$\frac{Z_{\text{inst}}}{(A \cdot iY)^{m/2}} = F[S/I] \sum_{P \in S^*} q^{\frac{1}{2}(P + \delta_{(g,h)})^2} q^{\frac{1}{2}(P' + \delta_{(g,h)})^2} \epsilon_{(g,h)}(p/2, w/2) e^{i\pi(P + 2\delta_{(g,h)}) \cdot W} \quad (2.15)$$

where $p, w \in \Gamma$ satisfy eq. (2.9c) and $p|_I = P$ and $w|_I = W$. This expression is again in the factorised form so that one can take the square root of it to define the chiral partition function (up to an overall phase). Note that the combination $\epsilon_{(g,h)}(p/2, w/2) e^{i\pi(P + 2\delta_{(g,h)}) \cdot W}$ depends only on P due to eq. (2.10). Also from eq.

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following condition:

$$\epsilon_{(g,h)}(v + v', w + w') = \epsilon_{(g,h)}(v, w) \cdot \epsilon_{(g,h)}(v', w') \cdot e^{4i\pi \int dX \wedge dX'} \quad (2.10a)$$

where X and X' are instanton configurations with windings (v, w) and (v', w') respectively. Note that the cocycle $4 \int dX \wedge dX' = 4(v \cdot gw' - w \cdot hv') = 4(L_1 \cdot L_2' - L_2 \cdot L_1')$ is an integer, which can be easily proven by using eq. (2.9a,b). Thus we can choose a Z_2 phase satisfying eq. (2.10a). Since $\tilde{Q}$ is an invariance of the instanton configuration we demand that for every $q \in \tilde{Q}$,

$$\epsilon_{(g,h)}(qv, qw) = \epsilon_{(g,h)}(v, w) \quad (2.10b)$$

Actually we shall demand eq. (2.10b) for all $q \in Q$ as otherwise the partition function would vanish and the question of holomorphic factorisability would not arise. Indeed, as the action is invariant under Q , summing over distinct fixed planes gives a contribution $\sum_{q \in (Q/\tilde{Q})} \epsilon(qv, qw) = 0$ unless eq. (2.10b) is satisfied for all $q \in Q$. We shall discuss the modular invariance condition on this phase in Section 3.

The instanton contribution in the (g, h) sector is:

$$Z_{\text{inst}} = A^{m/2} F[\sqrt{I^*/I}] \cdot \sum_{(L_1, L_2)} \epsilon_{(g,h)}(v, w) \cdot e^{-2\pi\tau_2(L_1)^2 - 2\pi\frac{1}{2}(L_2 + \tau_1 L_1)^2} \quad (2.11)$$

where the subscript E denotes the Euclidean dot product, $[\text{set}]$ denotes the order of the set, F the number of distinct fixed planes (i.e., distinct $x|_N$) and the factor $\sqrt{[I^*/I]}$ is the volume factor of each fixed plane (this appears due to the integration over $x|_I$ which coordinates a torus corresponding to the lattice I and also explains the presence of $A^{m/2}$ where $m = \dim$ of I and A is the area of the surface representing the normalisation of the zero mode).

Let us define a sublattice S of $\Gamma|_I$ as

$$S = \{2w|_I \mid (v, w) \text{ is an allowed pair for some } v \in \Gamma \text{ such that } v|_I = 0\} \quad (2.12)$$

Note that S contains I . It follows from eq. (2.10) that if there are two allowed pairs (v, w) and (v', w') such that $v|_I = v'|_I = 0$ (therefore $w|_I, w'|_I \in S$) then

$$\epsilon_{(g,h)}(v + v', w + w') = \epsilon_{(g,h)}(v, w) \epsilon_{(g,h)}(v', w') \quad (2.13a)$$

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(2.13a,b), $\delta_{(g,h)}$ is defined only modulo S^* . Indeed one can verify that the square root of eq. (2.15) is invariant up to an overall phase upon changing $\delta_{(g,h)}$ by an element of S^* .

To get the full partition function, we must also integrate over the oscillator modes. This is done in the standard way and the result can be expressed in terms of Θ -functions. The overall constant (i.e., the coefficient of the lowest order term in q and $\bar{q}$) appearing in the oscillator mode contribution can be easily shown to be $(\det'(1-h)|I_g)^{-1}$ where the notation means that the determinant is to be taken only in g -invariant directions over the non-zero eigenvalues of $(1-h)$. Combining this with the contribution from Z_{inst} we get an overall constant $C_{(g,h)}$ which represents the degeneracy of the ground state in (g,h) sector:

$$C_{(g,h)} = F[S/I](\det'(1-h)|I_g)^{-1} \quad (2.16)$$

In Appendix B we simplify this expression and show that it is equal to the order of the set

$$\frac{N_g}{(1-g)N_g^* + (1-h)N_g}.$$

In fact $C_{(g,h)}$ just gives the number of fixed planes in the g -twisted sector (i.e., $C_{(g,1)}$) that are h -invariant and hence has a clear operator interpretation of the insertion of h in the trace over $C_{(g,1)}$ fixed planes. One would like to show for the consistency of the asymmetric orbifold that after taking the square root, $\sqrt{C_{(g,h)}}$ is again the trace of h over the $\sqrt{C_{(g,1)}}$ dimensional irreducible representation of the vertex operators (the fact that the latter is an integer and gives the dimension of the irreducible representation was shown in ref [5]). This will be done in Section 4.

Holomorphic factorisation on higher genus surfaces.

In the remaining part of this section we will describe how the holomorphic factorisation works for higher genus surfaces (at least for commuting twists). For notational simplicity we will demonstrate this for genus 2 surfaces but the generalisation to arbitrary genus surfaces is straightforward. Let us choose a particular marking on the genus 2 surface (see fig. 1). Let $\{a_i, b_i; i = 1, 2\}$ denote the cycles

and $\{g_i, h_i; i = 1, 2\}$ denote the twists along these cycles. We will assume that all g 's and h 's commute with each other (i.e., we assume that the orbifold is an Abelian one). A general instanton configuration will have the following boundary conditions for X along different cycles

$$X(P_2) = h_1 X(P_1) + w_1 \quad (2.17)$$

$$X(P_3) = g_1^{-1}(X(P_2) - v_1) \quad (2.18)$$

Similarly one defines the boundary conditions along the second loop with windings w_2 and v_2 . As X lies on the torus, v_i and $w_i \in \frac{1}{2}\Gamma$. The fact that X must be single-valued on the cut Riemann surface (Fig. 1) implies that

$$\sum_i c_i \equiv \sum_i (g_i h_i)^{-1} [(1-h_i)v_i - (1-g_i)w_i] = 0 \quad (2.19)$$

Note that $c_1 = X(P_4) - X(P_0)$ and it represents the winding of X along the dividing geodesic $a_1 b_1 a_1^{-1} b_1^{-1}$.

To analyse these equations let us introduce the following sublattices. Let I be the sublattice invariant under all these twists, I_i the sublattices invariant under g_i and h_i for each i , and N and N_i be the sublattices orthogonal to them respectively. Then $I = I_1 \cap I_2$ and $\dim(N) = \dim(N_1 + N_2)$. Let n denote $\dim(N)$ and m denote $\dim(I)$. Then the constraint eq. (2.19) gives n conditions. Given $\{v_i, w_i\} \in (\frac{1}{2}\Gamma)^4$ satisfying the constraint there exists an instanton configuration. Let us denote by $\frac{1}{2}\Gamma^0$ the $(4m+3n)$ -dimensional lattice of such vectors. However not all the vectors of Γ^0 give distinct instanton configurations. Let Q be the group of transformations $(v_i, w_i) \rightarrow (v_i + (1-g_i)p; w_i + (1-h_i)p)$ where p is an arbitrary n -dim. vector such that for each i , $(1-g_i)p$ and $(1-h_i)p \in N$ then from (2.17,18) this just shifts $X \rightarrow X - p$ and hence does not change the instanton action. Moreover if $p \in \frac{1}{2}\Gamma$ then it does not alter the instanton configuration. Let $\tilde{Q}$ be the subgroup of Q when p is restricted to $\frac{1}{2}\Gamma$. Distinct instanton configurations are labelled by the $(4m+2n)$ -dimensional lattice $\frac{1}{2}\tilde{\Gamma} = \frac{1}{2}\Gamma^0 / Q$ up to a constant shift in X .

Holomorphic factorisation again requires a Z_2 phase $\varepsilon_{(g_i, h_i)}(\{v_i, w_i\})$ which satisfies the condition:

$$\varepsilon(\{(v + v'), (w + w')\}) = \varepsilon(\{v_i, w_i\})\varepsilon(\{v'_i, w'_i\}).e^{4i\pi \int dX \wedge dX'} \quad (2.20)$$

where we have suppressed the subscripts in ε . The cocycle can be expressed in terms of windings (2.17,18) as

$$\int dX \wedge dX' = \sum_i [w_i h_i (v'_i + c'_i) - v_i g_i (w'_i + h_i c'_i)] + \sum_{i=1}^{\ell-1} \sum_{j>i} c_i c'_j \quad (2.21)$$

Further we assume that for every $q \in Q$,

$$\varepsilon(\{qv_i, qw_i\}) = \varepsilon(\{v_i, w_i\}) \quad (2.22)$$

The fact that we should require this condition for $q \in \tilde{Q}$ is obvious from the preceding discussion. Here we require a stronger condition for all $q \in Q$ as otherwise Z_{nat} would vanish. What this condition says is that the zero mode of X decouples from the functional integral. Because of eq. (2.22) ε is a map from $\frac{1}{2}\tilde{\Gamma}$ to Z_2 (or more generally when the group G includes also translations to S^1). Of course the cocycle (2.21) is a map $\frac{1}{2}\tilde{\Gamma} \otimes \frac{1}{2}\tilde{\Gamma} \rightarrow Z_2$. When $c_i = 0$, the cocycle factorises into the sum of two 1-loop cocycles and in order to ensure the correct degeneration limit we demand that when $c_i = 0$,

$$\varepsilon(\{v_i, w_i\}) = \varepsilon_{(g_1, h_1)}(v_1, w_1)\varepsilon_{(g_2, h_2)}(v_2, w_2) \quad (2.23)$$

In particular when $c_i = v_i|_{I_i} = 0$,

$$\varepsilon(\{v_i, w_i\}) = e^{-2i\pi \sum_i \delta_{(g_i, h_i)} w_i |_{I_i}} \quad (2.24)$$

Since all the g 's and h 's commute we can simultaneously diagonalise them. Let U be a unitary $n \times n$ matrix acting on $X|_N$ so that $U g_i U^{-1}$ and $U h_i U^{-1}$ are diagonal. Let $X^\mu, \mu = 1, \dots, n$ denote the components of X (along N directions) in this basis (note that X^μ are complex; however, they define an n -dim. real subspace

in C^n). It is clear that U is block-diagonal in the sense that it does not mix the $(N \cap I_i)$ -directions with the rest.

Along the I -directions the instanton configuration is

$$X(P) = x + \sum_{i=1,2} [L_i \cdot \int_O^P \omega_i(z) + \bar{L}_i \cdot \int_O^P \bar{\omega}_i(\bar{z})] \quad (2.25)$$

where ω_i are the two Abelian differentials normalised as $\int_{a_i} \omega_j = \delta_{i,j}$ and the period matrix $\tau_{ij} = \int_{b_i} \omega_j$. By considering the boundary conditions of X along the different cycles exactly as in the 1-loop case we find the following relations on L_i

$$L_i = (iY)^{-1}_{ij} (\bar{\tau}_{jk} v_k|_I - w_j|_I) \quad (2.26)$$

where v_i and $w_i \in \frac{1}{2}\Gamma$ and $iY_{ij} = (\bar{\tau} - \tau)_{ij}$. The treatment of these untwisted directions is standard. We will focus here on the twisted directions.

To express the configuration along N -directions we need to use the twisted 1-differentials. For each μ -direction there exists a twisted holomorphic 1-differential Ω^μ with twists by g_i and h_i along a_i and b_i cycles. (Since μ is in one of the N -directions not all the eigenvalues of g_i and h_i are one and hence Ω^μ is not a section of the canonical line bundle.) By the Riemann-Roch theorem there exists only one such linearly independent differential on genus-2 surfaces (for genus ℓ there are precisely $(\ell - 1)$ such independent differentials). It is convenient to introduce an $n \times n$ matrix $\Omega = U \Omega_{(\text{diag})} U^{-1}$ where $\Omega_{(\text{diag})\mu\nu} = \delta_{\mu\nu} \Omega^\mu$. Note that $g_i \Omega g_i^{-1} = h_i \Omega h_i^{-1} = \Omega$ for each i . As $\int \Omega$ along the boundary of fig.1 must vanish, we have the condition

$$\sum_i [(1 - h_i) A_i - (1 - g_i) B_i] = \sum_i C_i = 0 \quad (2.27)$$

where $A_i = \int_{a_i} \Omega$ and $B_i = g_i \int_{b_i} \Omega$. Using Ω , we can express the instanton configuration (2.17,18) in the N directions:

$$X(P) = x + \left(\int_{P_0}^P \Omega(z) \bar{\ell} + c.c \right) \quad (2.28)$$

where

$$\ell = (iY'')^{-1} \sum_i [B_i^T g_i^{-1} v_i - A_i^T h_i^{-1} w_i] \quad (2.29)$$

where $iY' = \int \Omega^T \wedge \tilde{\Omega}$. It is easy to see that Y' is a Hermitian invertible matrix. It is useful for later purposes to give an explicit expression for Y' :

$$iY' = \sum_i [A_i^\dagger h_i^{-1} (C_i + B_i) - B_i^\dagger (g_i^{-1} A_i - C_i)] - C_i^\dagger C_i \quad (2.30)$$

To proceed further we have to distinguish between two cases. Case (1) is for directions along $(I_i \cap N)$ where $g_i = h_i = 1$ and Case (2) is for the remaining N -directions (i.e., along $N_1 \cap N_2$ directions). As mentioned above, U is block-diagonal and hence the A_i , B_i and Y' matrices all are block-diagonal. We shall denote these different blocks by superscripts i along $I_i \cap N$ -directions and by c along the orthogonal directions $N_1 \cap N_2$.

Case (1)

Let $g_1 = h_1 = 1$ (i.e., we are considering $I_1 \cap N$ -directions). This means that $C_1 = C_2 = c_1 = c_2 = 0$. Since this is an N -direction both g_2 and h_2 cannot be equal to one. Thus we can invert either $(1 - h_2)$ or $(1 - g_2)$ in each direction to eliminate $A_2^{(1)}$ or $B_2^{(1)}$ respectively. Substituting this in eq. (2.30) we find $iY'^{(1)} = (A_1^{(1)} B_1^{(1)} - \text{Hermitian conjugate})$. We choose the normalisation $A_1^{(1)} = 1$ and define $\tau'^{(1)} = B_1^{(1)}$. Using the fact that $\int \Omega^T \wedge \Omega = 0$ one finds that $(\tau'^{(1)})^T = \tau'^{(1)}$, thus $iY'^{(1)} = \tau'^{(1)} - \bar{\tau}'^{(1)}$ and $\ell^{(1)} = (iY'^{(1)})^{-1} (\tau' v_1 - w_1)^{(1)}$. Note that v_2 and w_2 do not appear in $\ell^{(1)}$. Indeed they satisfy the 1-loop condition, eq. (2.9b). Different v_2 and w_2 however give rise to different fixed points x (modulo lattice shifts of x) and summing them up just gives the number of fixed points.

Case(2)

Let $h_1 \neq 1$ and $h_2 \neq 1$. Then substituting $A_i^{(c)} = (1 - h_i)^{-1} (C_i^{(c)} + (1 - g_i) B_i^{(c)})$ and choosing the normalisation $C_1^{(c)} = 1$ we obtain $iY'^{(c)} = \tau'^{(c)} - \bar{\tau}'^{(c)}$ where

$$\tau'^{(c)} = (1 - h_2)^{-1} B_2^{(c)} - (1 - h_1)^{-1} B_1^{(c)} - (1 - h_1)^{-1} (1 - h_2)^{-1} \quad (2.31)$$

and from (2.29)

$$\ell^{(c)} = (iY'^{(c)})^{-1} (\tau' c - u)^{(c)} \quad (2.32)$$

where

$$u = (1 - h_1)^{-1} w_1^{(c)} - (1 - h_2)^{-1} w_2^{(c)} + (1 - h_1)^{-1} (1 - h_2)^{-1} h_1 h_2 c \quad (2.33)$$

and $c = c_1^{(c)}$ is the winding through the dividing geodesic. For other cases (eg. when $h_1 = 1$ and $g_1 \neq 1$), we can again go through the above analysis by eliminating appropriate variables and obtain eq. (2.31-33) with $\tau'^{(c)}$ and u appropriately defined.

The instanton action now becomes

$$S = -4\pi [(V^T \bar{\tau}' - W^T) (iY')^{-1} (\tau' V - W) + \sum_{i,j,k,l=1}^2 (v_i \bar{\tau}'_{ij} - w_j) (iY')_{jk}^{-1} (\tau_k v_l - w_k)]_I \text{ (Eucl)} \quad (2.34)$$

where $V^{(c)} = c$, $W^{(c)} = u$, $V^{(i)} = v_i|_{I_i \cap N}$ and $W^{(i)} = w_i|_{I_i \cap N}$. Note that V , W , $v_i|_I$ and $w_i|_I$ are unchanged under the transformation $v_i \rightarrow v_i + (1 - g_i)p$ and $w_i \rightarrow w_i + (i - h_i)p$, as this transformation just shifts the centre-of-mass position of X and hence does not enter into the action. The cocycle (2.21) becomes

$$4 \int dX \wedge dX' = 4(V.W' - W.V') + \sum_i (v_i w'_i|_I - w_i v'_i|_I) \in Z \quad (2.35)$$

This cocycle has the following important property which is crucial in proving the holomorphic factorisation of the path integrals. Let L denote $2\pi + 4m$ dimensional vector (where m is the dimension of I) formed by $(V, v_i|_I, v_2|_I, W, w_i|_I, w_2|_I)$. Then due to the fact that these vectors are invariant under Q , L is an explicit parametrisation of $\frac{1}{2}\tilde{\Gamma}$. Indeed $\frac{1}{2}\tilde{\Gamma}$ is the lattice formed by all the vectors L as v_i and w_i span $\frac{1}{2}\Gamma$ subject to the constraint (2.19). Let us further define the following sublattices of $\tilde{\Gamma}$. Let a denote an arbitrary n dim. vector, and b, c denote m dim. vectors. Then define:

$$S = \{(a, b, c) \mid (0, 0, 0; a, b, c) \in \tilde{\Gamma}\}$$

right-moving parts respectively. Also in the argument of ε we have only indicated the variables corresponding to $\tilde{\Gamma}$, as by (2.22) ε is invariant under Q for every $v_i, w_i \in \frac{1}{2}\Gamma$ satisfying the constraint (2.19). Finally $\tilde{F}$ in the above expression is a constant given by:

$$\tilde{F} = \frac{[Q/\tilde{Q}]\sqrt{[I^*I]}\sqrt{[S/(S \cap S^*)]}}{\sqrt{[S^*/(S \cap S^*)]}} \quad (2.38)$$

where $[Q/\tilde{Q}]$ is just the number of different fixed m-dim. tori ($= R^m / I$) for a fixed element in $\frac{1}{2}\tilde{\Gamma}$ (i.e., it counts the number of distinct $x|_N$), $\sqrt{[I^*I]}$ is the volume of each of these fixed tori (the result of $x|_I$ integration, which also explains the presence of A^m in eq. (2.37) where A is the area of the Riemann surface representing the normalisation of the zero mode) and the remaining factor in eq. (2.38) is the inverse of the volume of the basic cell of S and results from the Fourier transformation involved in Poisson resummation.

The equation (2.37) is holomorphically factorised except for the terms involving $\det(\imath Y')$, $\det(\imath Y)$ and A . In fact these terms combine with the integration over the oscillator modes (non-zero modes) of the bosons. The latter integration gives $\det'(\tilde{\partial}^\dagger \tilde{\partial})^{-1/2}$ for each twisted and untwisted boson. It is convenient to go to the μ -basis defined earlier where g_i and h_i are diagonal by means of the unitary matrix U . Then $\det(\imath Y') = \prod_\mu (\Omega^\mu, \bar{\Omega}^\mu)$. In the μ th direction let $g_i = e^{2i\pi\theta_i}$ and $h_i = e^{2i\pi\phi_i}$. In ref [12], $\det'(\tilde{\partial}^\dagger \tilde{\partial})$ for different line bundles are expressed in terms of that for the untwisted degree-0 line bundle. Using eq. (5.4) and (5.6) of the first ref in [12], we obtain:

$$\frac{\det' \tilde{\partial}^\dagger \tilde{\partial}_\mu}{(\Omega^\mu \bar{\Omega}^\mu)} = C \left(\frac{\det' \tilde{\partial}^\dagger \tilde{\partial}}{\det(\imath Y) \cdot A} \right) \left(\frac{N(\mu \times B(-P_1) \times B_A^{-1})}{N(B(-P_1 - P_2 + Q) \times B_A)} \right) \left(\frac{\prod_i G(P_i, Q)^2}{G(P_1, P_2)^2} \right) \left(\frac{|\det(\omega_i(P_j))|^2}{|\Omega(P_1)|^2} \right) \quad (2.39)$$

where C is a constant, P_i ($i = 1, 2$) and Q are arbitrary points on the surface, μ refers to the twisted line bundle along the μ -th direction, B_A is the preferred spin bundle associated with the given choice of marking, $B(D)$ is the line bundle corresponding to the divisor D , $\log G$ is the scalar Green function (see ref [12] for

$$S' = \{(a, b, c) \mid (a, b, c; 0, 0, 0) \in \tilde{\Gamma}\}$$

$$S^* = \{(a, b, c) \mid (a, b, c; a', b', c') \in \tilde{\Gamma} \text{ for some } (a', b', c')\}$$

$$S'^* = \{(a, b, c) \mid (a', b', c'; a, b, c) \in \tilde{\Gamma} \text{ for some } (a', b', c')\} \quad (2.36)$$

Then the cocycle (2.35) defines a mapping $S^* \times S \rightarrow Z$ and similarly for the primed spaces. Clearly S^* is contained in the dual of S with respect to the cocycle. In appendix A we prove a lemma for an arbitrary genus. Specialising it to genus 2, we have

Lemma 1: With respect to cocycle (2.21) $\tilde{\Gamma}$ is self-dual and S^* is the dual of S .

Now we can proceed to prove the holomorphic factorisation of the instanton contribution. Fix an element $L_1 = (V, v_1|_I, v_2|_I) \in \frac{1}{2}S^*$ and choose an element $L_2 = (W^0, w_1^0|_I, w_2^0|_I)$ such that $(L_1; L_2) = (V, v_1|_I, v_2|_I; W^0, w_1^0|_I, w_2^0|_I) \in \frac{1}{2}\tilde{\Gamma}$. Then an arbitrary element of $\frac{1}{2}\tilde{\Gamma}$ for a fixed L_1 is given by $(L_1; L_2 + \ell)$ where ℓ spans $\frac{1}{2}S$. Now we can carry out the Poisson resummation over ℓ by doing a Fourier transformation. We multiply the instanton sum by $\exp(2i\pi k \cdot \ell)$ where $k \in \text{dual}(\frac{1}{2}S) = 2S^*$ (by lemma 1) and integrate over ℓ using the cocycle property of ε , eq. (2.20). The steps are exactly as in the 1-loop case. The final result is:

$$\frac{Z_{\text{inst}}}{\det(\imath Y')^{1/2} (A \cdot \det(\imath Y))^{m/2}} = \tilde{F} \left| \sum_{(P, p_1, p_2)} \exp \left(i\pi (P^T \tau' P + p_i^T \tau_{ij} p_j)_L \right) \exp \left(-i\pi (P \tilde{\tau}' P + p_i^T \tilde{\tau}_{ij} p_j)_R \right) \right| e^{i\pi ((P + \delta') \cdot W + (p_i + \delta_i) \cdot w_i)} \varepsilon \left(\frac{1}{2} (P - \delta', p_1 - \delta_1, p_2 - \delta_2; W, w_1, w_2) \right) \quad (2.37)$$

where (P, p_1, p_2) span the shifted lattice $S^* + (\delta', \delta_1, \delta_2)$, W is an n dim. vector and w_1 and w_2 are m -dim. vectors such that $(P - \delta', p_1 - \delta_1, p_2 - \delta_2; W, w_1, w_2) \in \tilde{\Gamma}$. δ' is an n -dim. vector such that $\delta'(c) = 0$ and $\delta'(i) = \delta_{(g_i, h_i)}|_{N \cap I}$; and $\delta_i = \delta_{(g_i, h_i)}|_I$ (this follows from using eq. (2.24)). Note that the above expression is independent of the choice of (W, w_1, w_2) . Here summation over repeated indices i, j is understood and the subscripts L and R refer to the projection to left- and

the precise definition) and N is

$$N(\psi) = e^{-4i\pi y \cdot y^{-1} \cdot y} |\Theta(z, \tau)|^2 \quad (2.40)$$

where ψ is any degree-0 line bundle and z is the Abel map from ψ to the Jacobi variety, and $y = \text{Im } z$. The right-hand side of eq. (2.39) is independent of the choice of insertion points P_i and Q . In particular taking the limit $Q \rightarrow P_2$ and using the Riemann vanishing theorem we see that N in the denominator of (2.39) vanishes at the same rate as $G(P_2, Q)$ yielding a finite limit. Further, using the fact that the Abel map of the line bundle $(\mu \times B_A^{-2})$ is $(\phi + \tau\theta)$ where ϕ and θ denote column vectors with components ϕ_i and θ_i respectively, and using the definitions of Θ functions we can simplify the above expression to

$$\frac{\det' \bar{\partial}_\mu^\dagger \bar{\partial}_\mu}{\Omega^\mu \bar{\Omega}^\mu} = C \left(\frac{\det' \bar{\partial}_1^\dagger \bar{\partial}_1}{\det(\bar{\partial}_1^\dagger \bar{\partial}_1) \cdot A} \right) \left| \frac{\det(\omega_i(P_j))}{\Omega(P_1)} \frac{\Theta[\frac{\theta}{\omega_i}(P_2) \partial_{-i} \Theta(z, \tau)]}{\sum_i \omega_i(P_2) \partial_{-i} \Theta(z, \tau)} \right|^2 \quad (2.41)$$

where $z = (z_1, z_2)$ is the Abel map of the line bundle $B_A \times B(-P_1)$. Eq.(2.41) shows that up to the determinant over the untwisted line bundle the result is holomorphically factorised. A few comments are in order. First note that the expression inside mod square is independent of the choice of P_1 and P_2 . Secondly, the constant C , which is independent of the twists θ_i and ϕ_i (so long as not all are zero) can be determined by degenerating the surface and demanding that one reproduces the one-loop result. This in fact fixes C to be one. Indeed, consider the case when $\theta_1 = \phi_1 = 0$. Let us now pinch the surface along the dividing geodesic with P_1 on the first loop and P_2 on the second loop. In this limit $\Omega(P_1) = \omega_1(P_1) = \omega_2(P_2) = 1$ (in the standard coordinates for the torus; the first relation is a consequence of the fact that we have chosen the normalisation for Ω so that $\int_{a_1} \Omega = 1$) and $\omega_1(P_2) = \omega_2(P_1) = 0$. The genus-2 Θ function factorises into a product of genus-1 Θ functions as $\Theta[\frac{\theta}{\omega_i}](z, \tau) \rightarrow \Theta[\frac{\theta_1}{\omega_1}](z_1 + \frac{1}{2} + \frac{1}{2}\tau_1) \Theta[\frac{\theta_2}{\omega_2}](\frac{1}{2} + \frac{1}{2}\tau_2)$ where $z_1 = \int_{P_0}^{P_1} \omega_1$ and P_0 is the pinching point. For the case at hand, the first Θ function cancels between the numerator and denominator of (2.41) and the second just gives the answer for the twisted genus-1 contribution including the

$\det'(1 - h_2)|_{I_2}$ - the coefficient multiplying the lowest energy state. For case 2 (i.e., when at least one of g_i and h_i is not one for each i), $\Omega(P_1)$ is a meromorphic twisted differential having a simple pole at the pinching point P_0 with residue 1 (as we have normalised Ω so that $C_1 = 1$ for this case). Now taking the limit $P_1 \rightarrow P_0$ the product $\Omega(P_1) \Theta(z_1 + \frac{1}{2} + \frac{1}{2}\tau_1) \rightarrow \partial\Theta(\frac{1}{2} + \frac{1}{2}\tau_1)$, and once again we obtain the genus-1 result including the coefficient $\prod_i \det'(1 - h_i)|_{I_{g_i}}$.

Similarly one can relate the determinant over the twisted spin bundles to that of the untwisted spin bundle times the mod square of a holomorphic expression involving Θ functions. Combining all the contributions we obtain the untwisted partition function times a holomorphically factorised mod square term. Thus we can define the square root of the twisted partition function unambiguously up to an overall phase for each twist. Although here we have shown this for genus-2, the generalisation to an arbitrary genus is straightforward, as a result of the lemma proved in Appendix A and the definitions (2.20) and (2.22) for ϵ .

So far we have discussed the situation when the orbifold group G consists only of rotations. To include the shifts is straightforward. Let δ_{g_i} be the shift associated with the element g_i and δ_{h_i} with h_i . Then defining new ϵ as

$$\epsilon(\{g_i, h_i\})(\{v_i, w_i\}) \rightarrow \epsilon(\{g_i, h_i\})(\{v_i, w_i\}) \cdot e^{4i\pi \int dX \wedge dX_i} \quad (2.42)$$

where dX_i is the sum of twisted meromorphic and antimeromorphic differentials with at most first order poles, and is characterised by the windings $v_i' \in \delta_{g_i} + \Gamma$ and $w_i' \in \delta_{h_i} + \Gamma$. The new ϵ again satisfy eq. (2.20). $\bar{Q}$ -invariance of ϵ implies that $\sum_i c_i' \in \Gamma$, which is just the condition that as we go around the boundary of the cut Riemann surface $\prod_i (g_i h_i g_i^{-1} h_i^{-1}) = 1$ (here g and h refer to the full group element (g, δ_g) etc. with the obvious product rule; note that a shift by a vector of Γ is identity). Finally Q invariance of ϵ implies that $\sum_i c_i' \in \sum_i ((1 - g_i)\Gamma + (1 - h_i)\Gamma)$ which implies that there exist windings v_i' and w_i' in the shifted lattices indicated above, satisfying the constraint $\sum_i c_i' = 0$ (otherwise as expected the partition function vanishes). In particular for genus 1, this means that $\delta_{g,h} \rightarrow \delta_{g,h} + \delta_{g|h}$. Thus shifts can be incorporated in the previous discussion by generalising ϵ to

be arbitrary phases (and not just Z_2) satisfying eq. (2.20) and (2.22). In the following we shall always consider this more general situation.

3. Modular invariance

As we saw above, the path integral in each sector factorises holomorphically allowing one to take the square root, which is defined up to an overall phase. This was achieved by introducing a phase ε dependent on the lattice momenta and subject to condition (2.10a,b). In proving modular invariance one has thus to discuss two issues: first that there exist ε such that the path integral is modular invariant (before taking the square root) which would ensure modular invariance of the square roots up to overall (momentum independent) phases, and secondly to show that the latter phases exist consistent with modular invariance. We discuss these two issues in the following two sub-sections.

3a Modular invariance on ε

In each twisted sector (g, h) we can choose a phase $\varepsilon_{(g,h)}$, satisfying eq. (2.10a,b) allowing holomorphic factorisation of the instanton sum. But in order to have modular invariance, ε 's in different twisted sectors in the same modular orbit must be related. Thus if under modular transformation $(g, h) \rightarrow (g', h')$ and $(v, w) \rightarrow (v', w')$ then we have the relation

$$\varepsilon_{(g',h')}(v', w') = \varepsilon_{(g,h)}(v, w) \quad (3.1)$$

This relation essentially says that ε should be independent of the choice of marking on the torus. Is eq. (3.1) consistent with the other requirements on ε 's? First we note that the condition (2.10a) is clearly consistent with (3.1) as the cocycle $\int_{T^2} dX \wedge dX'$ is independent of the choice of marking. To show that (3.1) is consistent we must also analyse the modular transformations which bring the (g, h) sector back to itself. Let g and h be two independent generators of G . If the order

of g is n and that of h is m , then we can without loss of generality assume that m divides n . Then such a transformation is $\tau \rightarrow (a\tau + b)/(c\tau + d)$ where $a = 1 \bmod m$, $b = 0 \bmod n$, $c = 0 \bmod m$ and $d = 1 \bmod n$ and $ad - bc = 1$. Under this transformation $v|_I \rightarrow (dv|_I + cv|_I)$ and $w|_I \rightarrow (bw|_I + aw|_I)$. It is easy to see that $((a-1)w|_I + bv|_I) \in \frac{1}{2}S$. Thus using (2.10a) and (2.13a,b), modular invariance of ε implies:

$$e^{4i\pi(v+\delta_{(g,h)})|_I \cdot ((a-1)w+bv) - (w+\delta_{(h,g^{-1})})|_I \cdot ((d-1)v+cw) + ((d-1)v+cw)|_I \cdot ((a-1)w+bv)} = 1 \quad (3.2)$$

Here we have used the fact that $\varepsilon_{(g,h)}(v, w) = \varepsilon_{(h,g^{-1})}(w, -gv)$ which follows from (3.1). Now from eqs. (2.9a,b) it follows that $nv|_I, mw|_I \in \frac{1}{2}I$. This implies that $4nv|_I^2$ and $4mw|_I^2$ are integers, and in fact even integers if the orders n and m are odd. However, if for example n is even then $4nv|_I^2$ need not be even. In ref [5] we required that this be even also for even n . This is however not a restriction on the choice of orbifold, as in the cases when $4nv|_I^2$ is odd, we can double the order from n to $2n$, and include all the twisted sectors $(1, g, \dots, g^{2n-1})$. Thus without losing generality, we can set:

$$4nv|_I^2 \in 2Z \quad (3.3)$$

whenever $v \in \frac{1}{2}I$ and $nv|_I \in \frac{1}{2}I_g$, by if necessary doubling the order of the group. Using (3.3), we see that all the terms in the exponential in (3.2), except the ones involving δ 's, give rise to a unit phase. In the following we will discuss these remaining terms in (3.2), but at present we just note that setting $a = d = 1$ and $b = n, c = 0$ (or $b = 0, c = m$), eq. (3.2) implies:

$$\begin{aligned} n\delta_{(g,h)} &\in S_{(g,h)} \\ m\delta_{(h,g^{-1})} &\in S_{(h,g^{-1})} \end{aligned} \quad (3.4)$$

Let us now consider the 2-loop modular invariance and loop factorisation, as this is expected to put further restrictions on ε 's. Consider a 2-loop twisted sector $(g_1, h_1; g_2, h_2)$ and windings $(v_1, w_1; v_2, w_2)$ along $(a_1, b_1; a_2, b_2)$ cycles respectively. The windings now satisfy eq. (2.19) to ensure that X is single-valued as one goes

around $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$. However in general the winding along the dividing geodesic $a_1 b_1 a_1^{-1} b_1^{-1}$ will not vanish in the twisted directions and is defined in (2.19) as c_1 . This physically represents the flow of a soliton state through the dividing geodesic and reflects the fact that in the twisted sector a fixed point x_1 can go to a fixed point $x_2 = x_1 \bmod \frac{1}{2}\Gamma$ (which is the same fixed point) by emitting a soliton state. When we factorise the 2-loop integral along the dividing geodesic, the contribution of such soliton states would go like $t^{L_0} \bar{t}^{L_0}$ where L_0 and $\bar{L}_0$ are the left- and right-moving energies of the state and t is the pinching parameter. In the $t \rightarrow 0$ limit, therefore, only those terms would survive for which $c_1 = 0$, which is exactly eq. (2.9b) for 1 loop.

Holomorphic factorisation of the instanton contribution again requires a phase $\varepsilon_{(g_1, h_1; g_2, h_2)}(v_1, w_1; v_2, w_2)$, which satisfies eq. (2.20). As a result this equation is again 2-loop modular invariant. If $c_1 = 0$, then the 2-loop factorisation along the dividing geodesic demands that:

$$\varepsilon_{(g_1, h_1; g_2, h_2)}(v_1, w_1; v_2, w_2) = \varepsilon_{(g_1, h_1)}(v_1, w_1) \varepsilon_{(g_2, h_2)}(v_2, w_2) \quad (3.5)$$

The requirement that upon modular transformation eq. (3.5) should still hold, provided the transformed $c_1 = 0$, gives non-trivial conditions. The non-trivial modular transformation is the one which mixes the two loops. Let us specialise to the case $g_1 = g_2 = g$ and $h_1 = h$ and $h_2 = h'$. Consider the modular transformation:

$$\begin{aligned} a_1 &\rightarrow b_2 a_2^{-1} b_2^{-1} a_1 b_2 a_2 b_2^{-1} \\ b_1 &\rightarrow b_1 a_1^{-1} b_2 a_2 b_2^{-1} \\ a_2 &\rightarrow a_2 \\ b_2 &\rightarrow b_2 a_2^{-1} b_2^{-1} a_1 b_2 \end{aligned} \quad (3.6)$$

Under this transformation

$$(g, h; g, h') \rightarrow (gh, 1; g, hh') \quad (3.7)$$

Assuming that initially

$$c_1 = g^{-1} h^{-1} [(1-h)v_1 - (1-g)w_1] = 0 \quad (3.8)$$

the transformed c_1 is given by

$$c'_1 = g^{-1}(1 - gh)(v_1 - v_2) \quad (3.9)$$

When c'_1 is also zero (as this is the case of interest) the windings transform as:

$$\begin{aligned} v'_1 &= v_1 + gw_1 \\ w'_1 &= -g^{-1}(v_1 - v_2) \\ v'_2 &= v_2 \\ w'_2 &= w_1 + hw_2 + g^{-1}(v_1 - v_2) \end{aligned} \quad (3.10)$$

Thus the two-loop factorization implies:

$$\begin{aligned} \varepsilon_{(g, h)}(v_1, w_1) \varepsilon_{(g, h')}(v_2, w_2) &= \varepsilon_{(g, h, 1)}(v_1 + gw_1, -g^{-1}(v_1 - v_2)) \\ &\quad \varepsilon_{(g, h, h')}(v_2, w_1 + hw_2 + g^{-1}(v_1 - v_2)) \end{aligned} \quad (3.11)$$

when (v_i, w_i) satisfy eqs. (2.19), (3.8) and (3.9). This equation can be further simplified by noting that given a pair (v_1, w_1) satisfying (3.8) and a v_2 such that (3.9) is satisfied, there is also a pair $(v_2, w_1 + h(v_1 - v_2))$ which is an allowed pair for (g, h) sector. Thus substituting $v_1 \rightarrow v_2$ and $w_1 \rightarrow w_1 + h(v_1 - v_2)$ in eq. (3.11) we obtain (denoting $v_2 = v$, $w_1 + h(v_1 - v_2) = w$ and $w_2 = w'$):

$$\varepsilon_{(g, h)}(v, w) \varepsilon_{(g, h')}(v, w') = \varepsilon_{(g, h, h')}(v, w + hw') \varepsilon_{(g, h, 1)}(v + gw, 0) \quad (3.12)$$

Setting $g = h = 1$, $v = 0$ and using the fact that $\varepsilon_{(1, 1)}(., 0) = \varepsilon_{(1, 1)}(0, .) = 1$ we see that

$$\varepsilon_{(1, h')}(0, w) = \varepsilon_{(h', 1)}(w, 0) = 1 \quad (3.13)$$

for all w and h' (this is indeed necessary from the requirement of the operator interpretation for the untwisted sector). Substituting this back in (3.11) we obtain:

$$\varepsilon_{(g, h)}(v, w) \varepsilon_{(g, h')}(v, w') = \varepsilon_{(g, h, h')}(v, w + hw') \quad (3.14)$$

This is just the generalisation of the condition for the case of discrete torsion [13] to the present case, where the group is the semidirect product of rotations and the

lattice translations. An element $\mathbf{g}$ of the group is a pair (g, v) with the obvious product rule. ε assigns a phase for every commuting pair $(\mathbf{g}, \mathbf{h})$ and the above relation is simply:

$$\varepsilon(\mathbf{g}, \mathbf{h}) \varepsilon(\mathbf{g}, \mathbf{h}') = \varepsilon(\mathbf{g}, \mathbf{h} \cdot \mathbf{h}') \quad (3.15)$$

where $\varepsilon(\mathbf{g}, \mathbf{h})$ denotes $\varepsilon_{(g, h)}(v, w)$. Let us consider a few consequences of eq. (3.14). Setting $h' = 1$, $v = 0$ and using the linearity for $\varepsilon_{(g, h)}(0, \dots)$ we get:

$$\varepsilon_{(g, h)}(0, w) = \varepsilon_{(g, 1)}(0, w) \quad (3.16)$$

for all $w \in \frac{1}{2}\Gamma$ such that $w|_{I_g} \in \frac{1}{2}I_g$. This equation relates $\delta_{(g, 1)}$ to $\delta_{(g, h)}$ for all h :

$$\delta_{(g, 1)} = \delta_{(g, h)} \mod I_g^* \quad (3.17)$$

which will be important when we discuss the operator interpretation in the following. Eq.(3.17) implies using eq. (3.4) for the $(g, 1)$ sector and (3.3):

$$n\delta_{(g, h)} \in I_{(g, h)} \quad (3.18a)$$

$$n\delta_{(g, h)}^2 = n\delta_{(g, 1)}^2 \mod 2Z \quad (3.18b)$$

where n is the order of g . Note that eq. (3.18a) is stronger than the condition (3.4). Using the fact that $\varepsilon_{(gh, h)}(0, w) = \varepsilon_{(g, h)}(-gw, w)$ by eq. (3.1) and applying this for $w \in I_{(g, h)}$ we find that

$$\delta_{(gh, h)} = \delta_{(g, h)} + \delta_{(h, g^{-1})} \mod I_{(g, h)}^* \quad (3.19a)$$

$$n'\delta_{(gh, h)}^2 = n'(\delta_{(g, h)} + \delta_{(h, g^{-1})})^2 \mod 2Z \quad (3.19b)$$

where n' is divisible by n and m (the order of h). Eqs. (3.18) and (3.19) will play an important role in Sect. 3b.

By setting $h' = h$ and repeatedly applying eq. (3.14) we obtain:

$$(\varepsilon_{(g, h)}(v, w))^m = \varepsilon_{(g, 1)}(v, (1 + h + \dots + h^{(m-1)})w) \quad (3.20)$$

It follows, by using eq. (2.10) and the fact that $mw|_{I_{(g, h)}} \in \frac{1}{2}I_{g, h}$ and eq. (2.13a, b), that for every $v, w \in \frac{1}{2}\Gamma$ such that (v, w) is an allowed pair (i.e., satisfies (2.9c):

$$e^{4i\pi m(\delta_{(h, g^{-1})} - (m-1)w)|_{I_{(g, h)}}} = 1 \quad (3.21a)$$

and similarly

$$e^{4i\pi n(\delta_{(g, h)} - (n-1)v)|_{I_{(g, h)}}} = 1 \quad (3.21b)$$

Substituting these in eq. (3.2) and using elementary properties of a, b, c, d one can verify that the 1-loop modular invariance condition (3.2) is satisfied.

3b. Modular invariance and overall phases.

Choosing ε satisfying conditions (2.10a), (3.1) and (3.14) ensures that up to overall phases in each sector, the partition function is modular invariant. We can choose these phases by demanding that $\sum_{g, h} Z(g, h)$ is modular invariant. However for this to be consistent we need to check that for those modular transformations which take a sector (g, h) to itself, $Z(g, h)$ is unchanged and there is no extra phase. For example if n is the order of g , then $\tau \rightarrow \tau + n$ is such a transformation, and we should demand:

$$e^{2i\pi n(E_L - E_R)} = 1 \quad (3.22)$$

In a diagonal basis for g , if the eigenvalues of g_L and g_R are given by $e^{2i\pi r_i/n}$ and $e^{2i\pi r_a/n}$ respectively, then (3.22) gives the following level matching condition:

$$n = \text{odd} : \sum_i \tau_i^2 - n^2 \delta_g^2 = 0 \mod n \quad (3.23a)$$

$$n = \text{even} : \sum_i \tau_i^2 - n^2 \delta_g^2 = 0 \mod 2n \text{ and } \sum_a r_a = 0 \mod 2 \quad (3.23b)$$

Here we have used eq. (3.18b) and we denote δ_g for $\delta_{(g, 1)}$. Now we need to show that there are no new conditions arising when one considers all the other

transformations which do not change the boundary conditions. The level matching condition being both necessary and sufficient in the case of left-right symmetric Abelian orbifolds has been discussed in refs. [13,14]. In our bosonic formulation of left-right asymmetric orbifolds we have to discuss phases coming from the oscillator part (which is similar to refs. [13,14]), as well as the instanton part. Let us first consider the instanton contribution. This was obtained in Section 2 (the square root of eq. (2.15)). Note that the combination

$$\hat{\varepsilon}_{(g,h)}(P/2, W/2) \equiv \varepsilon_{(g,h)}(P/2, W/2) e^{-i\pi P W} e^{2i\pi(\delta_{(g,h)} W - \delta_{(h,g^{-1})} P)}$$

depends only on the conjugacy class of P in S^*/I (actually it depends only on the class S^*/S' but we will not need this in the following). Thus the square root of eq. (2.15) can be written as:

$$Z_{\text{inst}} = \sum_{P_0 \in S^*/I} \varepsilon\left(\frac{P_0}{2}, \frac{W}{2}\right) \sum_{P \in P_0 + I} q^{\frac{1}{2}(P+\delta)_R^2} \frac{1}{2} \bar{q}^{\frac{1}{2}(P+\delta)_L^2} e^{2i\pi P \cdot \delta'} \quad (3.24)$$

where P_0 is a representative in a given class, δ and δ' denote $\delta_{(g,h)}$ and $\delta_{(h,g^{-1})}$ respectively and we have dropped the subscripts g and h . Let us fix a conjugacy class $P_0 \in S^*/I$ and perform an arbitrary modular transformation $\tau \rightarrow \tau' = \frac{a\tau+b}{c\tau+d}$. The result is:

$$\begin{aligned} & \sum_{P \in I} e^{i\pi\tau(P+P_0+\delta)_L^2 - i\pi\tau'(P+P_0+\delta)_R^2 + 2i\pi(P+P_0)_L \cdot \delta'} \\ &= [I^*/I]^{-1/2} (ic(c\tau+d))^{-r/2} (-ic(c\bar{\tau}+d))^{-s/2} e^{-i\pi(bd\delta^2 + a\delta'^2 - 2(ad-1)\delta \cdot \delta')} \\ & \sum_{P'_0 \in I^*/I} \sum_{P' \in I/cI} e^{-i\pi \frac{d}{c} (P' + P_0 - aP'_0)^2} e^{-2i\pi b P_0 \cdot P'_0 + i\pi a b P_0'^2} \\ & \sum_{k \in I} e^{i\pi\tau'(k+P'_0 - c\delta' + d\delta)_L^2 - i\pi\tau'(k+P'_0 - c\delta' + d\delta)_R^2 - 2i\pi(k+P'_0)_L \cdot (b\delta - a\delta')} \end{aligned} \quad (3.25)$$

In deriving the above, one first writes $\tau' = \frac{a}{c} - \frac{1}{c(c\tau+d)}$ and then replaces the summation over $P \in I$ by $(cP + P')$ where $P \in I$ and $P' \in I/cI$, and finally one performs a Poisson resummation (for more details on this and some of the steps below see ref [15]). The integers r and s are respectively the positive and negative eigenvalues in the signature of I .

Let us consider the sector (g, h) , where g and h have respectively the orders n and m and m is a multiple of n . The transformations that do not change this boundary condition are of the type

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \bmod m & m \\ 0 \bmod m & 1 \bmod n \end{pmatrix} \quad (3.26)$$

What is the extra phase in (3.25) for transformations of the type (3.26)? First of all, because $cP \in I$ for $P \in I^*$ and $cP^2 \in 2Z$ (eq. (3.3)), one can easily show:

$$\sum_{P' \in I/cI} e^{-i\pi \frac{d}{c} (P' + P_0 - aP'_0)^2} = \begin{cases} \sum_{P' \in I/cI} e^{-i\pi \frac{d}{c} P'^2} & \text{if } P_0 - aP'_0 \in I \\ \text{otherwise} & \end{cases} \quad (3.27)$$

As $a = 1 \bmod m$, this implies that a non-zero contribution appears in eq. (3.25) only when P'_0 and P_0 are in the same conjugacy class I^*/I . In fact, because $m\delta' \in I$ and $n\delta \in I$ (see eq. (3.18a)), we see that for transformations of the type (3.26), we arrive at the same shifted lattice but with some additional factors which we now consider. First, following ref [15], in Appendix C we show that

$$[I^*/I]^{-1/2} (ic)^{-r/2} (-ic)^{-s/2} \sum_{P' \in I/cI} e^{-i\pi \frac{d}{c} P'^2} = 1 \quad (3.28)$$

Therefore the only additional factor besides $(c\tau + d)^{-r/2} (c\bar{\tau} + d)^{-s/2}$ which is handled as in the usual case, is the following overall phase:

$$e^{i\pi a c \delta'^2 - i\pi b d \delta^2 + 2i\pi(a-1)\delta \cdot \delta'} \quad (3.29)$$

Let us now see what phases come from the oscillator parts. A pair of complex fermions with boundary conditions $e^{2i\pi\chi}$ and $e^{-2i\pi\chi}$ along σ and $e^{2i\pi\nu}$ and $e^{-2i\pi\nu}$ along t make a contribution to the partition function which has the form

$$q^{\frac{1}{24} - \frac{1}{2}\chi(1-\chi)} \prod_{m=0}^{\infty} (1 - e^{-2i\pi\nu} q^{m+\chi})(1 - e^{2i\pi\nu} q^{m+1-\chi}) \quad (3.30)$$

The contribution of the oscillator part of the bosons is the inverse of this expression. In the (g, h) sector where χ and ν are fractions such that $n\chi$ and $m\nu$ are integers, after a modular transformation of the type (3.26) we get the following phase:

$$e^{i\pi(ac(\nu^2 - \nu) + (a-1)(c+1)\nu - bd(\chi^2 - \chi) - (d-1)(b+1)\chi - 2(a-1)\chi\nu)} \quad (3.31)$$

Now we can see that upon combining the phases (3.29) and (3.31) there is no overall phase for the following reasons:

$$e^{-i\pi\delta d(\sum_i(x_i-x_i^2)+\delta^2)} = 1$$

$$e^{i\pi ac(\sum_i(v_i-v_i^2)+\delta^2)} = 1$$

due to the level matching conditions in the $(g, 1)$ and $(h, 1)$ sectors respectively, while

$$e^{2i\pi(g-1)(\sum_i x_i v_i + \delta \delta')} = 1$$

due to the level matching condition in the $(gh, 1)$ sector (here we have used eqs. (3.19b) and (3.18b)). Also $e^{i\pi(g-1)(c+1)\nu} = 1$. For m odd, this is obviously true, and for m even it is true upon using the level matching condition in the $(h, 1)$ sector. Similarly $e^{-i\pi(d-1)(b+1)\chi} = 1$.

To summarise, we have shown that starting from any sector in a given modular orbit we can assign phases to all other sectors in that orbit such that the full partition function is 1-loop modular invariant, provided the level matching condition is satisfied.

4. Operator interpretation.

In Section 2, we obtained the partition function for each twisted sector. The momenta in the (g, h) sector lie on a shifted lattice $S_{(g,h)}^* + \delta_{(g,h)}$. The phase multiplying each momentum state $P + \delta_{(g,h)}$, where $P \in S_{(g,h)}^*$ is

$$\varepsilon_{(g,h)}\left(\frac{P}{2}, \frac{W}{2}\right) e^{i\pi(P+2\delta_{(g,h)}) \cdot W}$$

up to an overall momentum-independent phase. Finally the overall constant multiplying the lowest energy state was $\sqrt{C_{(g,h)}}$. In order to have an operator interpretation, $Z(g, h)$ must be $\text{tr}(h q^{L_0})$ over the g -twisted Hilbert space $\mathcal{H}_g$. This means that $S_{(g,h)}^* + \delta_{(g,h)} \subset I_g^* + \delta_{(g,1)}$, the latter being the shifted momentum

lattice for the $(g, 1)$ sector. Moreover we need to show that $\sqrt{C_{(g,h)}}$ is the $\text{tr}(h)$ in the $\sqrt{C_{(g,1)}}$ -dimensional irreducible representation of the fixed point set (recall that the latter is an integer-a fact proven in ref [5], and represents the number of copies of $\mathcal{H}_g$ which gives an irreducible representation of the untwisted vertex operators). Finally if there is a state common to the sectors (g, h) , (g, h') and (g, hh') then we must show that the phase multiplying this state in the (g, hh') sector is the product of the phases in the (g, h) and (g, h') sectors. In this section we address these issues.

First we note that $S_{(g,h)}^*$ is the set of vectors $v|_{I_g}$, where $v \in \Gamma$ and $(1-h)v = (1-g)w$ for some $w \in \Gamma$. In particular this means that only those $P \in I_g^*$ survive which are invariant under h . This is to be expected as the other momentum states that are not invariant under h would drop out of the $\text{tr}(h)$. A given momentum state $P \in I_g^*$ appears with the fixed planes labelled by $x|_{N_g} \in \frac{(1-g)^{-1}(v|_{N_g} + N_g)}{N_g}$ where $v \in \Gamma$ such that $v|_{I_g} = P$. The above equation on v and w says that only those P 's survive for which $hP = P$ and whose associated fixed point set contains points which are invariant under the action of h . This is again expected from the operator interpretation. Finally from eq. (3.17) we see that there exist some $P \in I_g^*$ such that $P + \delta_{(g,1)} = \delta_{(g,h)}$, proving that the spectrum appearing in the (g, h) sector is contained in the one appearing in the $(g, 1)$ sector.

In Section 2 we saw that as a result of the path integral computation one gets $C_{(g,h)}$ as an overall constant in the (g, h) sector. In Appendix B, we show that this number is equal to the number of those fixed points, out of $C_{(g,1)} = \left[\frac{N_g}{(1-g)N_g^*}\right]$, which are invariant under h . Therefore $C_{(g,h)}$ is equal to the $\text{tr}(h)$ over the $D^2 (= C_{(g,1)})$ -dimensional representation. As we discussed in ref [5], this D^2 -dimensional representation of the vertex operators is reducible; it reduces to D copies of D -dimensional irreducible reps. and when considering the asymmetric case one should take one of these copies. Therefore we need to show that $\sqrt{C_{(g,h)}}$ is the trace of h over the D -dimensional representation, up to a phase (in the following we will show that h acts within each of these D -dim. irreps.). In fact this phase is different for different copies; if they were the same then the trace of

h over D -dim. rep. would have been $C_{(g,h)}/D$, which is not the case, as we shall see shortly.

First, recall from ref [5] that the vertex operator for an untwisted state with winding $p \in \Gamma$ when acting on g -twisted states contains a matrix e_p , which satisfies

$$e_p e_q = C(p, q) e_q e_p \quad (4.1)$$

where $p, q \in \Gamma$ and

$$C(p, q) = \prod_{\ell=1}^n (-\omega^\ell)^{p \cdot q} \quad (4.2)$$

where $\omega = e^{2i\pi/n}$, n being the order of g . Let ω_0 be the primitive m th root of unity where $m = n$, when n is even and $m = 2n$ when n is odd. Then there exists a unique central extension:

$$1 \rightarrow \omega_0 \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 1 \quad (4.3)$$

where ω_0 denotes the Z_m group generated by ω_0 . We shall in the following use hatted variables $\hat{p}, \hat{q}$ etc. to denote the elements of $\hat{\Gamma}$ and their projections to Γ will be denoted by p, q etc. We assume that the action of the orbifold group G can be lifted from Γ to $\hat{\Gamma}$ which we denote by $\hat{G}$. We will see later that this is always possible if G can be lifted to $\hat{\Gamma}$ for the untwisted sector (i.e., for $g=1$) where the cocycle $C(p, q)$ is simply $(-1)^{p \cdot q}$. Now we want to construct a $\hat{\Gamma}$ module on which $\hat{G}$ acts. We start by constructing the $\hat{N}_g$ module. Note that $\hat{N}_g$ only acts on the fixed point set and does not change the g -invariant winding states. Thus an irreducible $\hat{N}_g$ module would yield an irreducible fixed point set. For $p, q \in N_g$ the cocycle (4.2) becomes:

$$C(p, q) = e^{-2i\pi p \cdot (1-g)^{-1} q} \quad (4.4)$$

From eq. (4.4) it is easy to see that $\forall p \in N_g$, $C(p, q) = 1$ iff $q \in (1-g)N_g^*$. Thus the centre of $\hat{N}_g$ is $\hat{M}$ where $M = (1-g)N_g^*$. Let $\tau : \hat{M} \rightarrow \omega_0$ be a homomorphism. Then for each such τ we can construct a D -dimensional irreducible $\hat{N}_g$ module where ω acts as $e^{2i\pi/n}$, as follows. The group $\hat{N}_g/\text{Ker}\tau$ is

of order mD^2 and the order of its commutator subgroup $<\omega> = n$. Therefore the number of singlet representations is $\frac{m}{n}D^2$. These are in fact the ones obtained by setting $\omega = 1$ (ie, $\omega_0 = 1$ for n even and $\omega_0 = \pm 1$ for n odd). We can choose a section of $\hat{N}_g$ as $p \in N_g \rightarrow (\omega_0^r, p) \in \hat{N}_g$ such that $(1, p)(1, q) = (c(p, q), p+q)$ where $c(p, q) \in \omega_0$ and $\text{Ker}\tau = \{(1, p) \mid p \in (1-g)N_g^*\}$. Then these singlet representations are labelled by $q \in \frac{m}{M}$ and $(1, p)$ acts as $e^{2i\pi p(1-g)^{-1} q}$.

The number of conjugacy classes in $\hat{N}_g/\text{Ker}\tau$ is $\frac{m}{n}(n+D^2-1)$. This can be seen as follows. For $r = 1, \dots, m$, each ω_0^r is a single-element conjugacy class. There are m of these. For every non-trivial element $p \in N_g/M$ we can find a $q \in N_g/M$ such that $C(p, q)$ is non-trivial. If n is prime then this means that the conjugacy class of $\hat{p}$ contains precisely all the elements $\omega^r \hat{p}$, $r = 1 \dots n$. The number of such classes is $\frac{m}{n}(D^2-1)$. Using the fact that the number of irreducible representations of a finite group is equal to the number of its conjugacy classes, we see that the number of non-singlet irreps is $\frac{m}{n}(n-1)$. These are the ones where ω acts as a non-trivial n th root of unity. For n prime, the characters in each of these irreps is the same up to phases. Now the characters of any finite group K satisfies the relation

$$\sum_{\mu} |\chi_{\mu}(a)|^2 = \frac{|K|}{N(a)} \quad (4.5)$$

where $|K|$ is the order of K , μ labels the irrep and $N(a)$ is the number of elements in the conjugacy class of $a \in K$. Applying this formula to the identity element of the group $\hat{N}_g/\text{Ker}\tau$ and using the fact that there are $\frac{m}{n}D^2$ singlet irreps, we see that the dimension of non-singlet irreps is D . These irreps are labelled by $s = 1, \dots, n-1$, in which $\omega = e^{2i\pi s/n}$ and for all non-trivial $p \in N_g/M$ the character $\chi(1, p) = 0$.

It will be useful for the calculation of correlation functions in the next section to give an explicit construction of the D -dimensional representation in terms of the states in the D^2 -dimensional representation (ie. the fixed point set). Let $\hat{H}$ be a maximal Abelian subgroup of $\hat{N}_g/\text{Ker}\tau$. Then H is a maximal subgroup of $N_g/(1-g)N_g^*$ such that the cocycle (4.2) restricted on H is trivial. If d is the order of H then the order of $\hat{H}$ is md . We will now show that $d = D$. First note

that $\hat{H}$ is a normal Abelian subgroup of $\hat{N}_g/\text{Kerr}$. We can construct an induced representation of $\hat{N}_g/\text{Kerr}$ from a one-dimensional rep of $\hat{H}$ as follows. Let q_i for $i = 1, \dots, \frac{D^2}{(1-g)N_g}$ denote the cosets of $\frac{N_g}{(1-g)N_g}$ with respect to H . Thus any element $q \in \frac{N_g}{(1-g)N_g}$ can be expressed as $q = q_i + p$ for some i and some $p \in H$. Let us choose a section of $\hat{N}_g/\text{Kerr}$ such that c restricted on H is unity. Then $T(\omega^r, p) = e^{2i\pi r/n} e^{2i\pi q(1-g)^{-1}p}$, for $p \in H$ is a one-dimensional representation of $\hat{H}$ where ω acts as $e^{2i\pi r/n}$. For different choices of q_i we get different one-dimensional reps. Now the induced representation of $\hat{N}_g/\text{Kerr}$ obtained from any one of these one-dimensional reps of $\hat{H}$ has the dimension D^2/d . The matrix representing an element $\hat{q} \in \hat{N}_g/\text{Kerr}$ has a non-zero (ij) entry provided $q_i = q_j + q_i$, where $q = q_i + p$ for some $p \in H$, and in which case this entry is equal to $T(\hat{q}_i^{-1} \hat{q} \hat{q}_j)$. This induced representation is in fact an irrep. of $\hat{N}_g/\text{Kerr}$, the reason being that for every $q \notin H$ we can find a $p \in H$ such that the cocycle $C(p, q) \neq 1$ (as otherwise $\hat{H}$ would not be a maximal Abelian subgroup of $\hat{N}_g/\text{Kerr}$), proving that for every $q \notin H$, the (ω^r, q) -conjugated character of $\hat{H}$ is not equal to the character of $\hat{H}$ (see for example the proposition 3.2 of ref [16] on induced representations). As the non-singlet irreps. of $\hat{N}_g/\text{Kerr}$ are D -dimensional we have $D^2/d = D$ implying that the order of H is D .

To see more explicitly the connection between this induced representation and the D^2 -dimensional representation of $\hat{N}_g/\text{Kerr}$ consisting of the fixed point set, first construct a D -dim reducible representation of $\hat{H}$ labelled by $|p\rangle = e_p|0\rangle \forall p \in H$. Then define the state $|q\rangle = e_q|p\rangle$, where $q = p + q_i$. The state $|q\rangle$ can be identified with the fixed point $(1-g)^{-1}q$. It is clear that $\hat{N}_g/\text{Kerr}$ acting on $\{|q\rangle; q \in \frac{N_g}{(1-g)N_g}\}$ provides a D^2 -dimensional representation of $\hat{N}_g/\text{Kerr}$. To construct the above induced rep. of $\hat{N}_g/\text{Kerr}$, we first construct an irrep. of $\hat{H}$ out of the states $|p\rangle$. There are D such irreps. for a fixed non-trivial $\omega = e^{2i\pi r/n}$ for an integer $s \in [1, n-1]$. For example, the one-dimensional rep $T(\omega^r, p)$ of $\hat{H}$ given in the preceding paragraph is obtained by considering the action of $\hat{H}$ on the state $e_q|\phi_H\rangle$ where $|\phi_H\rangle = \sum_{p \in H} |p\rangle$. The D -dimensional induced representation of $\hat{N}_g/\text{Kerr}$ is obtained by its action on the states $\{e_q|\phi_H\rangle; i = 1, \dots, D\}$.

So far we have seen that for every choice of $\tau_g : \hat{M} \rightarrow <\omega_0>$ there is a unique D -dim irrep of $\hat{N}_g$ where $\omega = e^{2i\pi/n}$. Note that $e^{-i\pi p/n} \tau_g(\hat{p}^{-1}\hat{g}\hat{p})$ is a homomorphism from Γ to $<\omega>$. Thus for every $\hat{p} \in \Gamma$

$$\tau_g(\hat{p}^{-1}\hat{g}\hat{p}) = e^{i\pi p/n} e^{2i\pi \delta_g, p} \quad (4.6)$$

where δ_g is such that $\delta_g|_{N_g} = 0$ (this is to ensure that the trace of g is non-vanishing as will become clear in the following). δ_g in fact represents the shift in $\hat{g}$. In particular this implies that if $p \in I_g$ then $\hat{g}\hat{p} = e^{2i\pi \delta_g, p} \hat{p}$. Commutativity of $\hat{G}$ implies that

$$(1-h)\delta_g \in I_g^* \quad \forall h \in G \quad (4.7)$$

We label the corresponding D -dimensional irrep ($\omega = e^{2i\pi/n}$) as D_0 . Now we can construct a $\hat{\Gamma}$ -module by applying $\hat{v} \in \hat{\Gamma}$ on D_0 as follows

$$|D_{v_*}(v + \delta_g)|_{I_g} = \hat{v}|D_0, \delta_g\rangle$$

where D_{v_*} is the D -dimensional representation associated with τ_g given by

$$\tau_g^v(\hat{p}^{-1}\hat{g}\hat{p}) = \tau_g(\hat{v}^{-1}(\hat{p}^{-1}\hat{g}\hat{p})\hat{v}) = e^{i\pi p/n} e^{2i\pi(\delta_g + \delta_{v_*})p/n} \quad (4.8)$$

We wish to compute $\text{tr } \hat{h}q^{L_0}$ over the above $\hat{\Gamma}$ -module. Since L_0 depends only on the g -invariant part of the momentum, one needs to compute $\text{tr } \hat{h}$ over the states $|D_{v_*}(v + \delta_g)|_{I_g}$, for a fixed momentum $(v + \delta_g)|_{I_g}$. First note that in fact $\hat{h}$ acts within the $\hat{\Gamma}$ module. Under the action of $\hat{h}$ a state in $|D_{v_*}(v + \delta_g)|_{I_g}$ goes over to a state in $|D_{h v_*}, h(v + \delta_g)|_{I_g}$. Using eq. (4.7), we have $h(v + \delta_g)|_{I_g} = h v|_{I_g} + \delta_g \bmod I_g$, proving that this state is also in the same module. Now if $h(v + \delta_g)|_{I_g} \neq (v + \delta_g)|_{I_g}$ then clearly such a state does not contribute to $\text{tr } \hat{h}q^{L_0}$ as L_0 is the same for these two states. Therefore let us consider the states for a fixed $(v + \delta_g)|_{I_g}$ such that $(1-h)(v + \delta_g)|_{I_g} = 0$. Now for every $\hat{p} \in \hat{N}_g$ we have $\text{tr } (\hat{h}) = \text{tr } (\hat{p}^{-1}\hat{h}\hat{p})$, and moreover if $(1-h)p = (1-g)q$ for some $q \in \Gamma$ then we have $\text{tr } (\hat{p}^{-1}\hat{h}\hat{p}) = \tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) \text{tr } (\hat{h})$, from which it follows that for such an $\hat{p}$, if $\text{tr } \hat{h}$ is not zero then

$$\tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) = 1 \quad (4.9)$$

Let $N_{g,h}$ be the sublattice of N_g consisting of vectors p such that $(1-h)p \in (1-g)N_g^*$ and let $S'_{g,h}$ be the sublattice of Γ consisting of vectors p such that $(1-h)p \in (1-g)N_g$. Then it is easy to see, using the cocycle (4.4), that $\tau_g(\hat{p}^{-1}\hat{h}\hat{p})$ for $p \in N_{g,h} + S'_{g,h}$ is a homomorphism from the latter to $<\omega>$. Thus there exists a vector δ'_h such that $\tau_g(\hat{p}^{-1}\hat{h}\hat{p}) = e^{2i\pi\delta'_h \cdot p}$ when $p \in N_{g,h} + S'_{g,h}$. Without losing generality we can assume that $\delta'_h|_{I_g \cap N_h} = 0$. In particular when $p \in I_h$, $\hat{h}\hat{p} = e^{2i\pi\delta'_h \cdot p}$ implying that $\delta'_h|_{I_h} = \delta_h \bmod I_h^*$, where the latter is the shift associated with h . Returning to (4.9), since $\tau_g^*(\hat{p}^{-1}\hat{h}\hat{p}) = e^{2i\pi\delta'_h \cdot p} C(v, (1-h)p)$ for $p \in N_{g,h}$, we should therefore have

$$\exp[2i\pi p \cdot (\frac{1-h}{1-g}v - \delta'_h)]|_{N_g} = 1$$

Decomposing $p \in N_g$ as (p_1, p_2) where p_2 is the projection of p along h -invariant space and p_1 is the orthogonal projection of p , we see that $p \in (\frac{1-g}{1-h}R, x) \cap N_g$ where x is an arbitrary real vector in h -invariant subspace and $R = \{c \mid (c, 0) \in N_g^*\}$. Therefore the above phase is unity if $(\frac{1-h}{1-g}v - \delta'_h) \in N_g^* + \frac{1-h}{1-g}N_g$ where we have used the fact that $R^* = N_g|_1$ (see Appendix A). Thus the phase is unity iff there exist $w \in \Gamma$ and $v' \in \Gamma$ such that $v'|_{I_g} = v|_{I_g}$ and $(1-h)v'|_{N_g} = (1-g)(w + \delta'_h)$. In $\text{tr } \hat{h}$, therefore only those states appear for which $(v + \delta_g)|_{I_g}$ is such that $(1-h)(v + \delta_g) = (1-g)(w + \delta'_h)$ for some $w \in \Gamma$. We will show below that when this condition is satisfied, $\text{tr } \hat{h} \neq 0$.

As we saw above, $\text{Ker } \tau$ is invariant under $\hat{h}$ when $(1-h)(v + \delta_g)|_{I_g} = 0$. Therefore $\hat{h}$ is an automorphism of $\hat{N}_g / \text{Ker } \tau$. We can construct a semidirect product group defined as $\hat{G} = \{(\hat{p}, \hat{h}^r) \mid r = 1, \dots, k; \hat{p} \in \hat{N}_g / \text{Ker } \tau\}$ where $\hat{h}^k = 1$, with the product rule

$$(\hat{p}, \hat{h}^r)(\hat{q}, \hat{h}^s) = (\hat{p}\hat{h}^r\hat{q}, \hat{h}^{r+s}). \quad (4.10)$$

The order of $\hat{G}$ is $m k D^2$. The elements conjugate to $(\hat{p}, \hat{h})$ are given by

$$(\hat{q}, \hat{h}^k)(\hat{p}, \hat{h})(\hat{q}, \hat{h}^k)^{-1} = (\hat{q}(\hat{h}^k\hat{p})\hat{h}q^{-1}, \hat{h}). \quad (4.11)$$

Also define the group $\mathcal{G} = \hat{G} / <\omega>$, obtained by setting $\omega = 1$. The order of $\mathcal{G}$ is $\frac{m}{n} D^2$. The representations of $\hat{G}$, where $\omega = 1$, are also representations of $\mathcal{G}$.

The conjugacy classes in $\hat{\mathcal{G}}$ are as follows:

- i) The elements $(\omega_0^r, 1)$ for $r = 1, \dots, m$ are each a conjugacy class of one element and there are m of them, whereas in $\mathcal{G}$ the corresponding one for even n is $(1, 1)$ and for odd n , $(\pm 1, 1)$. Thus the number of such classes in $\mathcal{G}$ is $\frac{m}{n}$.
- ii) The elements in the same conjugacy class as $(\omega_0^r, \hat{h})$ are of the type $(\omega_0^r \hat{q} \hat{h} \hat{q}^{-1}, \hat{h})$. When $\hat{q}$ is such that $(1-h)q \in (1-g)N_g^*$, then from (4.9), $\tau_g^*(\hat{q}\hat{h}\hat{q}^{-1}) = 1$ (otherwise $\text{tr } (\hat{h})$ in D_h would be zero) therefore there are m of these classes for $r = 1, \dots, m$. Similarly the elements in the conjugacy class of $(\omega_0^r, \hat{h}^s)$ are of the form $(\omega_0^r \hat{q}(\hat{h}^s \hat{q}^{-1}), \hat{h}^s)$. If $\hat{q}$ is such that $(1-h^s)q \in (1-g)N_g^*$, then $\tau_g^*(\hat{q}\hat{h}^s\hat{q}^{-1}) = C(q, (1+h+..+h^{s-1})q)$. One can prove that the latter is unity if n, k or s is odd, and if all n, k and s are even then it can be ± 1 . For simplicity let us first assume that it is unity for all s (we will comment later when this is not the case following eq. (4.14)). Then there are $m(k-1)$ of these classes for $s = 1, \dots, k-1$ and $r = 1, \dots, m$, and the number of elements in each is equal to the order of the set $\left(\frac{(1-h^s)N_g}{(1-g)N_g^* \cap (1-h^s)N_g} \right)$, which is the same as the order of the set $\left(\frac{(1-g)N_g^* \cap (1-h^s)N_g}{(1-g)N_g^*} \right)$. The corresponding number of classes in $\mathcal{G}$ is $\frac{m}{n}(k-1)$ and they have the same number of elements.
- iii) The rest of the conjugacy classes contain elements of the type $(\hat{p}, \hat{h}^s)$ where $p \notin (1-h)N_g$ and $s = 1, \dots, k$. Using eq. (4.11) for $\hat{h}^k = 1$ we see that $(\hat{p}, \hat{h}^s)$ and $(C(q, p)\hat{p}(\hat{h}^s \hat{q}^{-1}), \hat{h}^s)$ are in the same conjugacy class. One can find a q such that $(1-h^s)q \in (1-g)N_g^*$ and $C(q, p)$ is non-trivial. This can be shown as follows. Let $B = \{q \in N_g \mid (1-h^s)q \in (1-g)N_g^*\}$. Then the vectors which give a trivial cocycle C with all the elements of B must lie in $(1-g)B^*$. As we saw above, $(1-g)B^* = (1-g)N_g^* + (1-h^s)N_g$. Thus for every non-trivial $p \in N_g / ((1-g)N_g^* + (1-h^s)N_g)$, there exists a $q \in B$ such that the cocycle $C(q, p)$ is non-trivial and therefore, for prime n all the powers of ω will appear in the same class. This means that such classes are equal in number for $\hat{\mathcal{G}}$ and $\mathcal{G}$ —the number of elements in each class for the former being n times the one for the latter.

Therefore the difference in the number of conjugacy classes in $\hat{\mathcal{G}}$ and $\mathcal{G}$ is $\frac{m}{n}(n-1)k$,

from which it follows that

$$(\text{number of irreps in } \hat{\mathcal{G}}) - (\text{number of irreps in } \mathcal{G}) = \frac{m}{n}(n-1)k \quad (4.12)$$

This means that there are $\frac{m}{n}(n-1)k$ irreps. for which $\omega \neq 1$. It is easy to see what they are. If $T(\hat{p}, \hat{h}^r)$ is a representation then so is $e^{2i\pi r/k} T(\hat{p}, \hat{h}^r)$ for $\ell = 1, \dots, k$. These reps. are inequivalent as the characters are different (note that not all $\chi(\omega^s, \hat{h}^r)$ for $r = 1, \dots, (k-1)$ and $s = 1, \dots, n$ can be zero). Similarly, since n is prime, choosing ω to be any of $n-1$ non-trivial roots of unity will give inequivalent irreps with the same dimension and characters that differ only by a phase.

Thus applying eq. (4.5) for $\hat{\mathcal{G}}$ and $\mathcal{G}$ and taking the difference one gets

$$\sum_{\omega \neq 1} |\chi(1, \hat{h})|^2 = \frac{m}{n}(n-1)k |\chi(1, \hat{h})|^2 = \frac{m(nkD^2 - kD^2)}{N(1, \hat{h})} \quad (4.13)$$

where we have used the fact that the number of elements in the conjugacy class $N(1, \hat{h})$ is the same for $\hat{\mathcal{G}}$ and $\mathcal{G}$. In particular when $\hat{h} = 1$ eq. (4.13) shows that the dimension of the representation where $\omega \neq 1$ is D . Using the fact that $N(1, \hat{h}) = \left(\frac{(1-g)N_g^* + (1-h)N_f}{(1-g)N_g^*} \right)$, in the D -dimensional irrep. we get the desired result:

$$|\text{tr}_{D_*}(\hat{h})|^2 = C_{(g, h)} \quad (4.14)$$

Therefore $\text{tr}_{D_*}(\hat{h}) = \sqrt{C_{(g, h)}}$ up to a phase.

If there exists an s (necessarily even) for even n and k such that $\tau_g^s(\hat{q}\hat{h}^s\hat{q}^{-1}) = -1$ for some $q \in N_{g, h^*}$, then $\text{tr}_{D_*}(\hat{h}^s) = 0$. However one can still prove as above that $\text{tr}_{D_*}(\hat{h}) = \sqrt{C_{(g, h)}}$ up to a phase. The difference now is that the conjugacy classes corresponding to $(\omega_0^s, \hat{h}^s)$ for such s are of type (iii) above (assuming n is prime, namely 2 in this case) whereas one can find an $p \in N_g$ such that the class of $(\omega_0^p, \hat{h}^p)$ is of type (ii). The rest of the steps go through just as above.

Finally we consider the relative phases between the $\text{tr } \hat{h}$ for different momentum states $|(v + \delta_g)I_g\rangle$ and $|(v + p + \delta_g)I_g\rangle$ where $p \in \Gamma$ is such that $(1-h)p = (1-g)q$ for some $q \in \Gamma$. We have the following relation

$$\text{tr}_{D_{++}}(\hat{h}) = \tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) \text{tr}_{D_*}(\hat{h}) \quad (4.15)$$

Now we can compare these relative phases with the ones in the expression (2.15) obtained from a path integral computation. First of all, since $(v + \delta_g)I_g \cap N_h = 0$ we can identify $\delta_{g, h} \equiv (v + \delta_g)I_g \cap I_h$ (note that this is consistent with the requirement (3.17)). Then expression (2.15) suggests the following identification for the relative phases between $\text{tr } \hat{h}$ in the states $|D_v, \delta_{g, h}\rangle$ and $|D_{v+a}, \delta_{g, h} + pI_g\rangle$:

$$\tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) = \varepsilon_{g, h} \left(\frac{p}{2}, \frac{q}{2} \right) e^{i\pi(p+2\delta_{g, h}) \cdot qI_g} \quad (4.16)$$

for $(1-h)p = (1-g)q$.

We now show that this identification is consistent with all the requirements on $\varepsilon_{g, h}$. If (p, q) and (p', q') are allowed pairs for the (g, h) sector then

$$\begin{aligned} \tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) \tau_g^v(\hat{p}'^{-1}\hat{h}\hat{p}') &= \tau_g^v((\hat{p}^{-1}\hat{h}\hat{p})(\hat{p}'^{-1}\hat{h}\hat{p}')) \\ &= e^{-2i\pi q \cdot p' I_g} \tau_g^v((\hat{p}\hat{p}')^{-1}\hat{h}(\hat{p}\hat{p}')) \end{aligned} \quad (4.17)$$

Substituting this in eq. (4.16) we see that $\varepsilon_{g, h}$ satisfies the condition (2.10a). If $p \in N_g$ then eq. (4.16) implies $\varepsilon_{g, h}(p, q) = e^{2i\pi \delta_{g, h} \cdot qI_g}$ as $\tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) = 1$. Finally if there is a state carrying the same momentum $(\bar{v} + \delta_g)I_g$ along the I_g direction in the (g, h) , (g, h') and (g, hh') sectors, then we can identify $\delta_{g, h} = \delta_{g, h'} = \delta_{g, hh'} = (v + \delta_g)I_g$. Now consider another state $(p + v + \delta_g)I_g$ common in the (g, h) and (g, h') sectors. This means that there exist $q, q' \in \Gamma$ such that $(1-h)p = (1-g)q$ and $(1-h')p = (1-g)q'$. Hence $(1-hh')p = (1-g)(q + h'q')$ implying that $|(p + v + \delta_g)I_g\rangle$ also appears in the sector (g, hh') . We saw earlier that the phase $\tau_g^v(\hat{p}^{-1}\hat{h}\hat{p})$ is the relative phase between $\text{tr } \hat{h}$ for the states $|(p + v + \delta_g)I_g\rangle$ and $|(v + \delta_g)I_g\rangle$. Since these appear in the sectors (g, h) , (g, h') and (g, hh') the relative phases in these three sectors will give a relation on ε 's for these sectors.

To see this, note that

$$\tau_g^v(\hat{h}'(\hat{p}^{-1}\hat{h}\hat{p})) = \tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) \quad (4.18)$$

if $(1-h)p$ and $(1-h')p \in (1-g)N_g^*$. From this it follows that

$$\tau_g^v(\hat{p}^{-1}\hat{h}\hat{p}) \tau_g^v(\hat{p}'^{-1}\hat{h}'\hat{p}') = \tau_g^v(\hat{p}^{-1}\hat{h}\hat{h}'\hat{p}) \quad (4.19)$$

Using this relation and the fact that $\delta_{g,A} = \delta_{g,A'} = \delta_{g,AA'}$, we obtain the relation

$$\varepsilon_{g,A}(\frac{p}{2}, \frac{q}{2})_{\varepsilon_{g,A'}}(\frac{p'}{2}, \frac{q'}{2}) = \varepsilon_{g,AA'}(\frac{p}{2}, \frac{q}{2} + \frac{hq'}{2}) \quad (4.20)$$

which is the two-loop factorisation condition, given in eq. (3.14).

So far the above discussion was for a particular twisted sector. Modular transformation would relate different twisted sectors. Thus one needs to consider all the twisted sectors simultaneously. It is convenient to work with sections for this purpose. Let k be the smallest positive even integer such that $g^k = 1$ for all $g \in G$ and let ω_0 be the primitive $2k$ th root of unity. Then for each g -twisted sector we define the product rule for every $p, q \in \Gamma$ as :

$$(\omega_0^r, p) \circ_g (\omega_0^s, q) = (\omega_0^{r+s}, c_g(p, q), p + q) \quad (4.21)$$

where

$$c_g(p, q) = c_1(p, q) \omega_0^{\sum_{i=0}^{n-1} (\frac{k}{2} + \frac{k}{n} \Omega_p g^i q)} \quad (4.22)$$

n being the order of g and $c_1(p, q)$ is ± 1 and is such that $c_1(p, q)/c_1(q, p) = (-1)^{p \cdot q}$ defines the product for the untwisted sector. $c_g(p, q)/c_1(p, q)$ is in fact $\sqrt{C_g(p, q)}$ and it is easy to see that $c_g(p, q)/c_g(q, p) = C_g(p, q)$. We assume that the orbifold group G can be lifted as an automorphism to the untwisted sector, that is

$$g((1, p) \circ_1 (1, q)) = g(1, p) \circ_1 g(1, q) \quad (4.23)$$

Then it is easy to see that the same action of G also defines an automorphism for all the twisted sectors. Let

$$(1, p)^{-1} \circ_1 g(1, p) = (\rho_g(p), -(1-g)p) \quad (4.24)$$

Here the inverse is defined with respect to the product rule for the untwisted sector. When $p \in I_g$, $\rho_g(p) = e^{2i\pi\delta_g \cdot p}$ defines the shift associated with g .

In the D_0 representation for the g -twisted sector we define the action of $(1 - g)N_g^*$ following eq. (4.6) as

$$(1, -(1-g)p)|_{D_0} = e^{2i\pi\delta_g \cdot p} / \rho_g(p) \quad (4.25)$$

For $p \in \Gamma$ such that $(1-h)p = (1-g)q$ for some $q \in \Gamma$,

$$\tau_g^v((1, p)^{-1} \circ_g h(1, p)) = \frac{\rho_h(p)}{\rho_g(q)} (-1)^{p \cdot q} e^{i\pi(p+2v+2\delta_g) \cdot q} l_g, \quad (4.26)$$

where the inverse is with respect to the product rule for g -twisted sector. Comparing with eq. (4.16) one gets the following identification:

$$\varepsilon_{g,A}(\frac{p}{2}, \frac{q}{2}) = (-1)^{p \cdot q} \frac{\rho_h(p)}{\rho_g(q)} \quad (4.27)$$

By using the properties of ρ implied by its definition (4.24), one can prove that ε defined above satisfies all the conditions described in Section 3 arising from modular invariance (including S transformation which changes the twisted sectors) and two-loop factorisation. This analysis proves that if the orbifold group can be lifted to $\tilde{\Gamma}_1$ for the untwisted sector, then $\varepsilon_{g,A}$ exists satisfying modular invariance conditions and gives rise to a correct Hilbert space interpretation.

5. Correlation Functions.

In this section we will show how one can compute correlation functions for twisted sectors. Consider a genus-zero correlation function of four twisted states. This type of correlation function for symmetric orbifolds has been discussed in refs. [17, 18]. The expression as computed in these refs. is the product of the contribution of oscillator modes and the soliton modes. For the asymmetric orbifold we will again follow the procedure of Sect. 2, namely double the variables in order to make them non-chiral and then take the holomorphic square root of the result. While the oscillator contribution factorizes into holomorphic times antiholomorphic pieces and therefore one can simply extract the contribution of these modes for asymmetric orbifolds, the factorisation of the solitonic contribution again requires a phase ε satisfying a cocycle condition (2.10a). In addition there is an extra subtlety due to the fact that the twisted states are in the D -dimensional irrep. of the fixed points. The states in the latter are linear combinations of the

fixed points (that define the D^2 -dim. reducible representation) as we saw in the previous Section. We will see below that the holomorphic factorisation of the soliton contribution requires taking precisely this linear combination of geometrical fixed points. For completeness, we will repeat the computation of the solitonic contribution of refs. [17,18].

In order to simplify the discussion, we will first consider the situation where there is no invariant direction with two of the states being from the g -twisted sector (at points z_2 and z_4) and two from g^{-1} -twisted sector (at z_1 and z_3). Later we will consider the more general case. Using complex notation for the field $X^i(z, \bar{z})$, we can diagonalise g so that $gX^i(z, \bar{z}) = e^{2i\pi k/n} X^i(z, \bar{z})$ where n is the order of g . In Section 4 we gave an explicit construction of twisted states for asymmetric orbifolds. Let the four twisted states $e_{q_i}|\phi_H\rangle$, $i = 1, \dots, 4$, be at the points z_i respectively, where H is a maximal subset of $\frac{\Gamma}{(1-g)\Gamma}$ such that the restriction of the Lepowsky cocycle (4.2) is trivial (see Sect. 4 for notation). We assume that $q_{21} - q_{34} \in H$ (where q_{ij} denotes $q_i - q_j$) as otherwise the correlation function would be zero, as will become clear in the following. Using $SL(2, \mathbb{C})$ invariance, we can set $z_1 = 0$, $z_2 = x$, $z_3 = 1$ and $z_4 = \infty$. The solitonic mode of $X(z, \bar{z})$ is given by

$$dX(z, \bar{z}) = A\omega_k(z) + B\bar{\omega}_{n-k}(\bar{z})$$

where $\omega_k(z)$ is the holomorphic 1-differential for the world sheet,

$$\omega_k(z) = z^{-k/n}(x-z)^{-1+k/n}(1-z)^{-k/n},$$

A and B are constants to be determined from $\oint_{\mathcal{C}_1} dX = v_1 \in \frac{1}{2}\Gamma$ and $\oint_{\mathcal{C}_2} dX = v_2 \in \frac{1}{2}\Gamma$. The loops $\mathcal{C}_1$ and $\mathcal{C}_2$ are two closed loops shown in Fig. 2, $\mathcal{C}_1$ encircles the points z_1 and z_2 and $\mathcal{C}_2$ encircles the points z_2 and z_3 . It is convenient to normalise $\omega_k(z)$ such that $\oint_{\mathcal{C}_1} \omega_k(z) = 1$. Denoting $\tau_k \equiv \frac{1}{1-e^{2i\pi k/n}} \oint_{\mathcal{C}_2} \omega_k(z)$ we then have

$$\tau_k = \frac{1}{2i \sin(\pi k/n)} \times \frac{F(1-x)}{F(x)} \quad (5.1)$$

where $F(x)$ is the hypergeometric function $F(\frac{k}{n}, 1 - \frac{k}{n}, 1; x)$. Now using

$$\begin{aligned} \int \omega_k \wedge \bar{\omega}_k &= \frac{1}{1 - e^{-2i\pi k/n}} \oint_{\mathcal{C}_1} \omega_k \oint_{\mathcal{C}_2} \bar{\omega}_k - \frac{1}{1 - e^{2i\pi k/n}} \oint_{\mathcal{C}_1} \bar{\omega}_k \oint_{\mathcal{C}_2} \omega_k \\ &= \bar{\tau}_k - \tau_k \end{aligned}$$

one finds the contribution of solitonic modes to the action to be

$$S_0 = -4i\pi \left(\frac{1}{1-g} v_2 - \frac{1}{2}(\tau + \bar{\tau})v_1 \right) \frac{1}{\bar{\tau} - \tau} \left[\frac{1}{1-g} v_2 - \frac{1}{2}(\tau + \bar{\tau})v_1 \right] - \frac{1}{4} v_1(\bar{\tau} - \tau)v_1 \quad (5.2)$$

where τ is obtained by transforming τ_{diag} ($d \times d$ diagonal matrix with entries τ_k) back to the basis in which X is real. The contribution of these modes to the correlator is

$$\sum_{v_1, v_2} e^{-S_0 \varepsilon(v_1, v_2)}$$

the summation being over $v_1 = v_1^0 + \tilde{v}_1$ and $v_2 = v_2^0 + \tilde{v}_2$ where $2v_1^0 \in H + q_{21}$, $2v_2^0 \in H + q_{32}$, and $2\tilde{v}_1$ and $2\tilde{v}_2 \in (1-g)\Gamma$. To see that in fact v_1 and v_2 range over these sublattices, note that if we take the external states to be labelled by x_i , $i = 1 \dots 4$, belonging to the fixed point set (ie, the D^2 -dimensional rep) then the zero mode of $X(z, \bar{z})$ is

$$X(z, \bar{z}) = x_1 + A \int_{z_1}^z \omega_k + B \int_{z_1}^{\bar{z}} \bar{\omega}_k,$$

from which it follows that $(1-g)(x_2 - x_1) = v_1$ and $(1-g)(x_3 - x_2) = v_2$. Thus v_1^0 takes the fixed point x_1 to the fixed point x_2 and similarly v_2^0 takes x_2 to x_3 . Now x_i are defined modulo $\frac{1}{2}\Gamma$, therefore summing over equivalent x_i 's amounts to summing over $\tilde{v}_i$ above, and finally as we saw in the last section, the external states $q_i|\phi_H\rangle$ are to be identified with the linear combinations $\sum_{p \in H} |q_i + p \bmod \frac{1}{2}\Gamma\rangle$. Summing over corresponding to the fixed points $x_i = \frac{1}{2}(1-g)^{-1}(q_i + p) \bmod \frac{1}{2}\Gamma$. Summing over $p \in H$ gives rise to summation over v_1^0 and v_2^0 above. Note that x_i 's must satisfy the condition $(1-g)(x_2 - x_1) = (1-g)(x_3 - x_4)$ in order for the configuration $X(z, \bar{z})$ to exist. This can be seen by considering $\oint_{\mathcal{C}_1} dX$ and deforming the contour to the one encircling the points 1 and ∞ . This fact explains the selection rule mentioned earlier, namely $q_{21} - q_{34} \in H$.

The phase $\varepsilon(v_1, v_2)$ is necessary in order to get the holomorphic factorisation of the expression. As we saw in Sect. 2, this was also the case for the factorisation of the partition function. Analogous to (2.10a), the phase $\varepsilon(v_1, v_2)$ is required to satisfy the condition:

$$\varepsilon(v_1 + v'_1, v_2 + v'_2) = \varepsilon(v_1, v_2) \varepsilon(v'_1, v'_2) e^{4i\pi \int dX \wedge dX'} \quad (5.4)$$

for $v'_1, v'_2 \in \frac{1}{2}H$. The cocycle on the right-hand side is

$$e^{4i\pi \int dX \wedge dX'} = e^{4i\pi(v_1, \frac{1}{1-g}v'_2 - v'_1, \frac{1}{1-g}v_2)}$$

and when v_1, v_2 are also in $\frac{1}{2}H$ it is equal to ± 1 (this is necessary for the consistency of eq. (5.4)). Now carrying out a Poisson resummation over $\tilde{v}_2$ we get

$$\sum e^{i\pi(p-2v_1)\tau(p-2v_1)-i\pi p^0 p} \varepsilon(v_1, v_2^0) \exp[4i\pi(p-v_1)(1-g)^{-1}v_2^0] \quad (5.5)$$

where the summation is over $p \in \Gamma$, v_1 and v_2^0 . Writing $p = p^0 + \tilde{p}$, where $p^0 \in \frac{\Gamma}{(1-g)\Gamma}$ and $\tilde{p} \in (1-g)\Gamma$, and summing over $v_2^0 \in \frac{1}{2}(H + q_{32})$ we get the restriction $p^0 \in H$ (this is because by definition H is a maximal subset of $\frac{\Gamma}{(1-g)\Gamma}$ such that the restriction of the Lepowsky cocycle (4.2) on H is trivial). Thus the final expression is of the factorised form

$$G(\tau, q_{21}, q_{32}) \times \tilde{G}(\bar{\tau}, 0, 0) \quad (5.6)$$

where

$$G(\tau, q, q') = \varepsilon(q, q') e^{i\pi q(1-g)^{-1}q'} \sum_{p^0 \in H} \sum_{\tilde{p} \in (1-g)\Gamma} e^{2i\pi(\tilde{p}+p^0-q)(1-g)^{-1}q'} e^{i\pi(\tilde{p}+p^0-q)\tau(\tilde{p}+p^0-q)} \quad (5.7)$$

Thus the contribution of the solitonic modes to the correlator of the asymmetric orbifold is $G(\tau, q_{21}, q_{32})$. The crossing symmetry of this can be seen by noting that under $x \rightarrow 1-x$ we have $\tau_k \rightarrow \tau'_k = -\frac{1}{4\sin^2(\pi k/n)\tau_4}$ and upon performing a Poisson resummation we get $G(\tau', q_{32}, q_{21})$. Using the $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$ limits

of the function $F(x)$ given in ref [17] we find the expected behaviour of the above expression; that under $x \rightarrow 0$ (and $x \rightarrow 1$) the intermediate states are untwisted states and contribute x^Δ (and $(1-x)^\Delta$) to the corresponding expansions, and that under $x \rightarrow \infty$, upon which $\tau \rightarrow \text{const}$, the intermediate state is a g^2 -twisted state with no winding momentum.

In the above discussion we assumed there were no g -invariant directions. Let us see what the modifications are when there are invariant directions. In Sect. 4, when discussing the $\hat{\Gamma}$ -module, we saw that every state in this module is labelled by a vector $k \in \Gamma$ and denoted it by $|D_k, k|_{I_g}\rangle$. Let $k_i, i = 1 \dots 4$ label the states at z_i . The integration over the invariant directions, $X|_{I_g}$, gives trivially $x^{k_1, k_2|_{I_g}} (1-x)^{k_3, k_4|_{I_g}}$. The computation of the contribution of normal directions is the same as the above except that now in eq. (5.2) the vectors $v_1, v_2 \in \frac{1}{2}N_g^*$ are such that $v_1 = v_1^0 + \tilde{v}_1$ and $v_2 = v_2^0 + \tilde{v}_2$ where $2v_i^0 \in H + k_{21}^N, 2v_i^0 \in H + k_{32}^N$ and $\tilde{v}_1, \tilde{v}_2 \in \frac{1}{2}(1-g)N_g^*$. After performing a Poisson resummation over $\tilde{v}_2$ we get eq. (5.5) where now $p \in N_g$. Again writing $p = p^0 + \tilde{p}$ where $p^0 \in \frac{N_g}{(1-g)N_g^*}$ and $\tilde{p} \in (1-g)N_g^*$, and performing the finite summation over v_2^0 we get the restriction $p^0 \in H$ and the final expression factorizes as $G(\tau, k_{21}^N, k_{32}^N) \times \tilde{G}(\bar{\tau}, 0, 0)$.

Let us now consider an example in which the four twisted states at the points $z_1 = 0, z_2 = x, z_3 = 1$ and $z_4 = \infty$ respectively are from g, g^{-1}, h and h^{-1} -twisted sectors, and compute the contribution of the solitonic modes. Let H_g be a maximal subset of $\frac{N_g}{(1-g)N_g^*}$ on which the restriction of the Lepowsky cocycle (4.2) for g -twisted sector is trivial and let H_h be a similar maximal set for the h -twisted sector. Let the four states at the points z_i respectively be $e_g, |\phi_{H_g}\rangle$ for $i = 1, 2$, and $e_h, |\phi_{H_h}\rangle$ for $i = 3, 4$, where q_1, q_2 represent the cosets of $\frac{N_g}{(1-g)N_g^*}$ w.r.t. H_g and q_3, q_4 similar cosets with g replaced by h . This of course means that the fixed points x_i at the four points z_i on the sphere are such that $x_i \in \frac{1}{2}(1-g)^{-1}(q_i + H_g)$ for $i = 1, 2$ and in $\frac{1}{2}(1-h)^{-1}(q_i + H_h)$ for $i = 3, 4$ mod. $\frac{1}{2}\Gamma$. For a configuration $X(z, \bar{z})$ with the values x_i at z_i on the sphere to exist the following condition must be satisfied:

$$(1-g)(x_2 - x_1) = (1-h)(x_3 - x_4)$$

which can be seen by considering the integral $\oint_{C_1} dX$ and deforming the contour to the one encircling the points z_3 and z_4 . From this equation one gets the selection rule $(q_{21} + H_g) \cap (q_{43} + H_h) \neq 0$ as otherwise the correlation function would be zero. In the following we assume that this selection rule is satisfied and then without losing generality we can set $q_{21} = q_{34}$. Define $\oint_{C_1} dX = v_1$ and $\oint_{C_2} dX = v_2$ where the contour C_2 is obtained by going around z_2 and z_3 clockwise once, then returning to z_2 and going around z_2 and z_3 counterclockwise once. Thus $v_1 = (1-g)(x_2 - x_1)$ and $v_2 = (1-g)(1-h)(x_3 - x_2)$. It follows that $v_1 = v_1^0 + \tilde{v}_1$ and $v_2 = v_2^0 + \tilde{v}_2$ where $v_1^0 \in \frac{1}{2}H_g \cap \frac{1}{2}H_h + \frac{1}{2}q_{21}$, $\tilde{v}_1 \in \frac{1}{2}(1-g)\Gamma \cap \frac{1}{2}(1-h)\Gamma$, $v_2^0 \in \frac{1}{2}(1-h)H_g + \frac{1}{2}(1-g)H_h + \frac{1}{2}Q$, where $Q = (1-g)q_3 - (1-h)q_2$ and $\tilde{v}_2 \in \frac{1}{2}(1-g)(1-h)\Gamma$.

For this case also one can write a holomorphic 1-differential on the world sheet

$$\omega_{k,l}(z) = z^{-k/n}(x-z)^{-1+k/n}(1-z)^{-l/m},$$

where m is the order of h , and obtain the following integral over the world sheet:

$$\int \omega_{k,l} \wedge \bar{\omega}_{k,l} = \frac{1}{\gamma} \oint_{C_1} \omega_{k,l} \oint_{C_2} \bar{\omega}_{k,l} - \frac{1}{\gamma} \oint_{C_1} \bar{\omega}_{k,l} \oint_{C_2} \omega_{k,l} + (\bar{\alpha} - \alpha) \oint_{C_1} \omega_{k,l} \oint_{C_1} \bar{\omega}_{k,l}$$

where γ and α are respectively the eigenvalues of $(1-g)(1-h)$ and $\frac{1}{2}(\frac{1}{1-h} - \frac{1}{1-g})$. The closed loop C_2 defined above is such that $\oint_{C_2} \omega_{k,l} = \gamma \int_x^1 \omega_{k,l}$. Again normalizing $\omega_{k,l}$ such that $\oint_{C_1} \omega_{k,l} = 1$ and denoting $\tau_{k,l} = \int_x^1 \omega_{k,l} + \alpha$ then the above integral is $\bar{\tau}_{k,l} - \tau_{k,l}$.

We can repeat the computation as before. The phase $\varepsilon(v_1, v_2)$ satisfies eq. (5.4) for $2v_1' \in (H_g + (1-g)N_g^*) \cap (H_h + (1-h)N_h^*)$ and $2v_2' \in (1-h)H_g + (1-g)H_h + (1-g)(1-h)\Gamma$. The explicit form of the cocycle on the right-hand side is

$$\exp 4i\pi(v_1 \frac{1}{\gamma} v_2' - v_1' \frac{1}{\gamma} v_2 - 2v_1' \alpha v_1)$$

Note that when v_1 and v_2 are also in the same sublattices as v_1' and v_2' defined above then the cocycle is ± 1 - a necessary condition for the consistency of (5.4). After performing a Poisson resummation over $\tilde{v}_2$ and the finite summation over v_2^0 , we find that the expression is in the factorised form:

$$G(\tau, q_{21}, Q) \times G(\bar{\tau}, 0, 0)$$

where

$$G(\tau, q, q') = \sum_{p^0} \sum_{\bar{p}} \varepsilon(\frac{1}{2}(p^0 + \bar{p} - q), \frac{1}{2}q') e^{i\pi(p^0 + \bar{p} - q)\tau - i\pi q'} e^{i\pi(p^0 + \bar{p} - q)\tau(p^0 + \bar{p} - q)} \quad (5.8)$$

and the sums are over $p^0 \in H_g \cap H_h$ and $\bar{p} \in (1-g)\Gamma \cap (1-h)\Gamma$. This expression is again crossing symmetric under $x \rightarrow (1-x)$ and under factorisation limits $x \rightarrow 0$, $x \rightarrow 1$ and $x \rightarrow \infty$ give correct intermediate states; namely in the first case the intermediate states are untwisted states carrying momenta along $N_g \cap N_h$ directions ($\tau \rightarrow \log x$) and in the remaining cases the intermediate states are in gh or gh^{-1} -twisted sectors. Thus, factorising in these channels, one can in principle obtain all the operator algebra coefficients of the theory.

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Appendix A

In this appendix we show that the set of allowed L_1 's is equal to $\frac{1}{2}S^*$. As we saw, following eq. (2.14), this was necessary for holomorphic factorisation. We also prove the lemma in Sect. 2 that was necessary for the holomorphic factorisation for higher genus surfaces.

Let us first introduce the following notation. For $P \in \Gamma$ decompose $P = (P_1, P_2, P_3, P_4)$ where

- P_1 =projection of P along directions not invariant under both g and h ,
- P_2 =projection of P along directions that are invariant under h but not g ,
- P_3 =projection of P along directions that are invariant under g but not h ,
- P_4 =projection of P along directions invariant under both g and h .

It is clear from eqs (2.9a,b) that $v = (v_1, v_2, 0, v_3, L_1)$ and $w = (w_1, 0, w_3, L_2)$. We defined $\frac{1}{2}S^*$ as the set of allowed L_2 's when $L_1 \in \frac{1}{2}I$. But when $L_1 \in \frac{1}{2}I$ then $(v_1, v_2) \in \frac{1}{2}N_g$. Now define S_N as follows

$$S_N = \{a \in N^* \mid (a, b) \in \Gamma \text{ for some } b \in S\} \quad (\text{A.1})$$

Then it is clear from the definition of S that the allowed $w|_N$'s (for $(v_1, v_2) \in N_g$) span all the conjugacy classes $\frac{S_N}{N}$. (Note that not every vector of S_N is an allowed $w|_N$ but what is being asserted here is that there are vectors in each conjugacy class $\frac{S_N}{N}$ that solve for $w|_N$ in eq. (2.9b).) Thus using eq. (2.9b) we have

$$S_N = N^* \cap ((1-g)^{-1}(1-h)N_g|_1, 0, N^*|_3) + N \quad (\text{A.2})$$

and then

$$S_N^* = N^* \cap [N + ((1-h)^{-1}(1-g)(N_g|_1)^*, N^*|_2, (N^*|_3)^*)]. \quad (\text{A.3})$$

Let us denote the sublattice of N_g^* consisting of the vectors of the type $(*, 0)$ as R . In fact we have $R = (N_g|_1)^*$. To see this: i) $R \subset (N_g|_1)^*$ because $a \in R \Rightarrow (a, 0) \in N_g^* \Rightarrow a.b \in Z$ for $\forall b \in N_g|_1 \Rightarrow a \in (N_g|_1)^*$ and ii) $(N_g|_1)^* \subset R$ because

$a \in (N_g|_1)^* \Rightarrow (a, 0).b \in Z$ for $\forall b \in N_g \Rightarrow (a, 0) \in N_g^* \Rightarrow a \in R$. Similarly we have $(N^*|_3)^* \subset \{a \mid (0, 0, a) \in N^*\}$. Therefore

$$S_N^* = N + N^* \cap ((1-h)^{-1}(1-g)R, N^*|_2, 0) \quad (\text{A.4})$$

But the set of all allowed $v|_N$ is just $N^* \cap ((1-h)^{-1}(1-g)R, N^*|_2, 0)$. Therefore

$$S_N^* = N + \{\text{allowed } v|_N\} \quad (\text{A.5})$$

Let K be the set of allowed $v|_I$, then

$$K = \{a \mid a \in I^* \text{ and } (b, a) \in \Gamma \text{ for some } b \in S_N^*\}$$

Now we wish to show that $K = S^*$. Indeed if $a \in K$ then there exists $b \in S_N^*$ such that $(a, b) \in \Gamma$. Let c be an arbitrary vector of S , then there exists $d \in S_N$ such that $(d, c) \in \Gamma$. Since Γ is integral $b.d + a.c \in Z$, therefore $a.c \in Z$ proving that $K \subset S^*$. Similarly let $a \in S^*$, then there exists $b \in N^*$ such that $(b, a) \in \Gamma$. Let d be an arbitrary element of S_N , then there exists $c \in S$ such that $(d, c) \in \Gamma$. As Γ is integral and $a.c \in Z$, it follows that $b.d \in Z$ implying that $b \in S_N^*$ and therefore $a \in K$ proving that $S^* \subset K$.

The above proof is quite difficult to generalise to higher genus. Therefore to prove Lemma 1 in Sect. 2 we proceed somewhat differently. Consider a genus ℓ surface. Let (v_i, w_i) be the windings along (a_i, b_i) cycles with twists (g_i, h_i) for $i = 1, \dots, \ell$, where v_i and w_i are vectors in R^d , d being the dimension of Γ . Thus $(v_1, w_1, \dots, v_\ell, w_\ell)$ is a vector in $R^{2\ell d}$. They satisfy eq. (2.19) where the sum goes from 1 to ℓ . If $d - n$ is the dim. of the subspace of R^d invariant under all g_i and h_i , then this equation defines a subspace P^0 of $R^{2\ell d}$ where the dimension of P^0 is $2\ell d - n$. The two cocycle $C(X, X') \equiv \int dX \wedge dX'$ given in equation (2.21) (where again the sum goes from 1 to ℓ) defines a degenerate 2-form on P^0 . Let Q be the group of translations defined by $v_i \rightarrow v_i + (1 - g_i)p$ and $w_i \rightarrow w_i + (1 - h_i)p$ where p is an arbitrary n dim. vector then $C(qX, X') = C(X, X')$ for arbitrary $q \in Q$. Let $\hat{P} = P^0/Q$, then $\hat{P}$ is $2(\ell d - n)$ dimensional and the cocycle restricted

on it is non-degenerate. Let π denote the projection $\pi : P^0 \rightarrow \tilde{P}$. Let us also define the lattices Γ^0 as the set of vectors in P^0 obtained from v_i and $w_i \in \Gamma$ satisfying the constraint (2.19), and $\tilde{\Gamma} = \Gamma^0/Q^0$ where Q^0 is subgroup of Q that acts on Γ^0 . It is clear from eq. (2.21) that the cocycle is integral on $\tilde{\Gamma}$. Now we wish to show that in fact $\tilde{\Gamma}$ is self-dual w.r.t. the cocycle. Let $\alpha \in \tilde{P}$ such that $C(\alpha, \beta) \in Z$ for all $\beta \in \tilde{\Gamma}$. Let us coordinate the n dim. subspace $\pi^{-1}\alpha$ in P^0 by (v'_i, w'_i) satisfying the constraint (2.19). Arbitrary β 's are given by arbitrary elements $v_i, w_i \in \Gamma$ satisfying (2.19). If we relax (2.19) on β , then using self-duality of Γ , $C(\beta, \alpha) \in Z$ implies that v'_i and w'_i also belong to Γ . Now the directions in P^0 that are degenerate when β are restricted to Γ^0 are precisely given by the action of Q . Thus $\pi^{-1}\alpha$ should be contained in $Q\Gamma^0$ implying that $\alpha \in \tilde{\Gamma}$.

Now let us set $v_i|_{I_i} = 0$ and $c_i = 0$ for each i where $v|_{I_i}$ means the projection of v along the directions of R^d that are invariant under both g_i and h_i . Let S_R denote the subspace in $\tilde{P}$ satisfying these conditions and S the corresponding sublattice in $\tilde{\Gamma}$. From (2.21) it is clear that S_R is a Lagrangian subspace of $\tilde{P}$ w.r.t. the non-degenerate cocycle. One can also show easily that it is in fact the maximal Lagrangian subspace showing that $\dim(S) = \ell d - n$. The conjugate space is the space of all allowed $v_i|_{I_i}$ and c_i projected to $\tilde{P}$. Let us denote by S_R^* this space and S^* the corresponding sublattice in $\tilde{\Gamma}$. Now we want to show that S^* is dual of S w.r.t. the cocycle. Let $\alpha \in S_R^*$ such that $C(\alpha, \beta) \in Z$ for all $\beta \in S$. Then we can as before consider the fiber $\Pi^{-1}\alpha$ in P^0 where Π is the projection from P^0 to S_R^* (i.e., Π projects every point of P^0 to the set of data $(v_i|_{I_i}, c_i)$). β 's span the sublattice in Γ^0 which are projected by Π to zero vectors. Thus β 's are labelled by v'_i and w'_i such that $v'_i \in N_i$, $w'_i \in \Gamma$ subject to the condition $c'_i = 0$. The cocycle (2.21) becomes

$$C(\alpha, \beta) = \sum_i [w_i h_i v'_i - v_i g_i w'_i] \quad (\text{A.6})$$

where v_i and w_i satisfying the constraint (2.19) parametrise the fiber $\Pi^{-1}\alpha$. If we relax the constraints $c'_i = 0$ and demand that $C(\alpha, \beta) \in Z$, then $w_i|_{N_i} \in N_i^*$ and $v_i \in \Gamma$. The directions that are orthogonal to $c'_i = 0$ surface are just the directions

generated by the group $Q_i : (v_i, w_i) \rightarrow (v_i + (1 - g_i)p_i, w_i + (1 - h_i)p_i)$ for arbitrary vectors p_i . Thus $\Pi^{-1}\alpha$ in P^0 is parametrised by $(Q_i(v_i, w_i); i = 1, \dots, \ell)$ where $v_i \in \Gamma$ and $w_i|_{N_i} \in N_i^*$. Note that $w_i|_{I_i}$ are arbitrary vectors. However as $w_i|_{N_i} \in N_i^*$ there are an infinite number of points in $\Pi^{-1}\alpha$ corresponding to $w_i \in \Gamma$. Also note that $\Pi Q_i = \Pi$. This implies that $\alpha = \Pi(\Pi^{-1}\alpha) \in S^*$. Thus we have the following lemma.

Lemma: With respect to the cocycle (2.21) on $\tilde{P}$, $\tilde{\Gamma}$ is self-dual and S^* is the dual of S .

Appendix B

In this appendix we calculate the number of fixed points for (g, h) sector. The constants appearing in eq. (2.16) are:

- 1) F = number of distinct fixed points in (g, h) sector for fixed $L_1, L_2 \in \frac{1}{2}I$.
- 2) $\frac{F}{2} \mid \frac{F}{2} \mid$ where $\frac{1}{2}S = \{L_2 \mid L_1 \in \frac{1}{2}I\}$
- 3) $\det'(1 - h)|_{I_g}$ coming from the oscillator contribution and represents non-zero part of $\det(1 - h)$ in the space invariant under g .

Let us first calculate $F' \equiv F \mid \frac{F}{2} \mid$. From 1) and 2) it is clear that F' is the number of distinct fixed planes in (g, h) sector for $L_1 \in \frac{1}{2}I$ and L_2 being an allowed but arbitrary vector. Now, since $L_1 \in \frac{1}{2}I$ and $v_3 = 0$ (following the notation introduced in Appendix A), eqs (2.9a,b) imply that the set of allowed (v_1, v_2) is $N_g \cap ((1 - h)^{-1}(1 - g)R, N_g|_2)$. (R was defined in Appendix A as $R = \{a|(\alpha, 0) \in N_g^*\}$.) Then eq. (2.7a) implies that

$$\{(x_1, x_2)\} = (1 - g)^{-1}N_g \cap ((1 - h)^{-1}R, (1 - g)^{-1}N_g|_2) \quad (\text{B.1})$$

Let R' be the sublattice of N_g^* such that $R'|_1 = 0$. Then it is clear that for each choice of allowed (x_1, x_2) the allowed x_3 lies in a shifted sublattice $(1 - h)^{-1}R'$. The precise shift depends on the conjugacy class $\frac{R}{N_g \cap N_h}$, however we will not need this information here. x_4 on the other hand is completely arbitrary. Thus

an allowed (x_1, x_2, x_3) specifies an 1-dimensional fixed plane. However not all these fixed planes are distinct. Indeed fix (x_1, x_2) and let x_3 and x'_3 be two allowed values. Then $x_3 - x'_3 \in (1-h)^{-1}R'$. If $x_3 - x'_3 \in R'$ then we see that $(x_1, x_2, x_3, x_4) \sim (x_1, x_2, x'_3, x'_4)$ where $(0, 0, x_3 - x'_3, x_4 - x'_4) \in \Gamma$. The fact that such a x'_4 exists is a consequence of the fact that $(0, 0, x_3 - x'_3) \in N^*$. Now as x_4 spans the 1-torus so does x'_4 , therefore the fixed planes labelled by (x_1, x_2, x_3) and (x_1, x_2, x'_3) are identical. This means that the number of distinct fixed planes for a given (x_1, x_2) is equal to

$$\left| \frac{(1-h)^{-1}R'}{R'} \right| = \det'(1-h)|_{I_g} \quad (\text{B.2})$$

Next let (x'_1, x'_2) be such that $(x'_1, x'_2) - (x_1, x_2) \in N^*$ then, just as before, we can find (x'_3, x'_4) and (x_3, x_4) such that the two points x and x' are the same points on the torus. Once again as x_4 spans the 1-torus so does x'_4 . Thus any two allowed pairs (x_1, x_2) and (x'_1, x'_2) that differ by a vector in N_g^* define the same fixed planes. On the hand if (x_1, x_2) is allowed then so is (x'_1, x'_2) , if the difference is in N_g^* . In other words

$$(1-g)^{-1}N_g \cap ((1-h)^{-1}R, (1-g)^{-1}N_g|_2) \supset N_g^*$$

This is easily seen because $N_g \supset (1-g)N_g^*$ and from the definition of $R, R \supset (1-h)N_g^*|_1$. Thus the number of distinct (x_1, x_2) is

$$\left| \frac{(1-g)N_g \cap ((1-h)^{-1}R, (1-g)^{-1}N_g|_2)}{N_g^*} \right|. \quad (\text{B.3})$$

F' is just the total number of distinct fixed planes and hence is the product of (B.2) and (B.3). Thus the overall constant $C_{(g,h)}$ is

$$\begin{aligned} C_{(g,h)} &= F'(\det'(1-h)|_{I_g})^{-1} \\ &= \left| \frac{(1-g)^{-1}N_g \cap ((1-h)^{-1}R, (1-g)^{-1}N_g|_2)}{N_g^*} \right| \\ &= \left| \frac{N_g}{(1-g)N_g^* + ((1-h)R^*, (1-g)(N_g|_2)^*)} \right|. \end{aligned} \quad (\text{B.4})$$

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Here we have used the fact that $|\frac{A}{B}| = |\frac{B^*}{A^*}|$. Now using $R^* = N_g|_1$, which we showed in Appendix A, and the fact that $(0, (N_g|_2)^*) \subset N_g^*$, the above expression can be simplified to:

$$C_{(g,h)} = \left| \frac{N_g}{(1-g)N_g^* + (1-h)N_h} \right| \quad (\text{B.5})$$

Can we understand geometrically the meaning of $C_{(g,h)}$? Note that $C(g, 1) = \left| \frac{N_g}{(1-g)N_g^*} \right|$ is just the number of fixed planes in the g -twisted sector. One would expect that $C_{(g,h)}$ must be the number of those fixed planes that are h invariant, as the ones that are not h -invariant would drop out of the partition function in the (g, h) sector. (Recall that the (g, h) sector corresponds to taking the trace with the insertion of the h operator.) This expectation turns out to be true. Indeed a fixed plane in the g -twisted sector is given by $(x_1, x_2) \in (1-g)^{-1}N_g$ and (x_3, x_4) arbitrary. Consider now the action of $h : (x_1, x_2, x_3, x_4) \rightarrow (hx_1, x_2, hx_3, x_4)$. If it is to describe the same plane then there must exist (x_1, x_2, x'_3, x'_4) such that $((1-h)x_1, 0, x'_3 - hx_3, x'_4 - x_4) \in \Gamma$. This of course means that $x_1 \in (1-h)^{-1}R$. Thus in order to be h -invariant

$$(x_1, x_2) \in (1-g)^{-1}N_g \cap ((1-h)^{-1}R, (1-g)^{-1}N_g|_2)$$

and the number of distinct fixed planes in the g -twisted sector is just $C_{(g,h)}$.

Appendix C

In this appendix, following ref [15], we will show

$$\frac{1}{\sqrt{|I_g^*|(ic)^{r/2}(-ic)^{s/2}}} G(d, c) = 1 \quad (\text{C.1})$$

where

$$G(d, c) = \sum_{P \in \frac{I_g}{I_g}} e^{-in_g^2 P^2} \quad (\text{C.2})$$

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In (3.25) setting P_0 , δ and δ' to zero we have

$$\begin{aligned} \sum e^{i\pi\tau P_L^2 - i\pi\tau P_R^2} &= \left(\frac{1}{I}\right)^{-1/2} (ic(c\bar{\tau} + d))^{-\tau/2} (-ic(c\bar{\tau} + d))^{-s/2} \\ &\times \sum_{P'_0 \in \frac{1}{I}\mathbb{Z}} e^{-i\pi\frac{\delta}{2}(P'_0 - aP'_0)^2} e^{i\pi ab P'_0{}^2} \\ &\times \sum_{k \in I} e^{i\pi\tau'(k+P'_0)^2 - i\pi\tau'(k+P'_0)^2_R} \end{aligned} \quad (C.3)$$

From this we get

$$\begin{aligned} \frac{1}{\sqrt{|I^*|} |(ic)\tau/2(-ic)^{s/2}} G(d, c) &= \\ \lim_{\tau \rightarrow -\frac{\delta}{2}} \left[(c\bar{\tau} + d)^{\tau/2} (c\bar{\tau} + d)^{s/2} \sum_{P \in I} e^{i\pi\tau P_L^2 - i\pi\tau P_R^2} \right] \end{aligned} \quad (C.4)$$

Note that here we have not demanded the form (3.26) for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. After performing a Poisson resummation on the right-hand side of (C.3) and demanding $c \equiv 0 \pmod{m}$ and $a \equiv 1 \pmod{m}$ we get

$$\lim_{\tau \rightarrow -\frac{\delta}{2}} \left[(a\bar{\tau} + b)^{\tau/2} (a\bar{\tau} + b)^{s/2} \sum_{P \in I} e^{i\pi\tau P_L^2 - i\pi\tau P_R^2} \right] = \frac{1}{|I^*|/I|(-c)^{(r+s)/2}} G(d, c). \quad (C.5)$$

Combining (C.4) and (C.5) we have

$$\frac{G(d, c)}{\sqrt{|I^*|/I|}(ic)^{\tau/2}(-ic)^{s/2}} = \frac{G(b, a)}{a^{\tau/2+s/2}}. \quad (C.6)$$

Here, in deriving (C.6), we have only demanded $c \equiv 0 \pmod{m}$ and $a \equiv 1 \pmod{m}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. But also $\begin{pmatrix} a + \ell c & b + \ell d \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ satisfies the same conditions for $\ell \in \mathbb{Z}$ and the same steps go through. Therefore

$$\frac{G(b + \ell d, a + \ell c)}{(a + \ell c)^{\tau/2+s/2}} = \frac{G(b, a)}{a^{\tau/2+s/2}}. \quad (C.7)$$

If $c = m$ then clearly we can choose ℓ such that $a + \ell c = 1$ and then from (C.7) we would conclude $\frac{G(b, a)}{a^{\tau/2+s/2}} = 1$. If $c = jm$, for some integer j , then we can follow the same steps for $\begin{pmatrix} a & jb \\ m & d \end{pmatrix}$ and conclude $\frac{G(b, a)}{a^{\tau/2+s/2}} = 1$. But since $G(b, a) = G(1, a)$ we conclude that $\frac{G(b, a)}{a^{\tau/2+s/2}} = 1$ and by (C.6) we have shown (C.1).

References

1. K.S. Narain, Phys. Lett. 169B (1986) 41; K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369
2. D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253
3. L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678; B274 (1986) 285
4. L.E. Ibáñez, H.P. Nilles and F. Quevedo, Phys. Lett. B187 (1987) 25
5. K.S. Narain, M.H. Sarmadi and C. Vafa, Nucl. Phys. B288 (1987) 551
6. H. Kawai, D.C. Lewellen and S.H.H. Tye, Nucl. Phys. B288 (1987) 1; I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. B289 (1987) 87; W. Lerche, D. Lüst and A. Schellekens, Nucl. Phys. B287 (1987) 477
7. P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46
8. D. Gepner, Phys. Lett. B199 (1987) 380; Nucl. Phys. B269 (1988) 757; B.R. Green, C. Vafa and N. Warner, Nucl. Phys. B324 (1989) 380; E. Martinec, in V. Knizhnik Memorial Volume (World Scientific, Singapore, 1990)
9. A. Font, L.E. Ibáñez, F. Quevedo, A. Sierra, Nucl. Phys. B337 (1990) 119; K. Intriligator and C. Vafa, Nucl. Phys. B339 (1990) 95
10. P. Ginsparg, Harvard preprint HUTP-86/A090 (1986)
11. J. Lepowsky, Proc. Nat. Acad. of Sci. 82 (1985) 8295; I.B. Frenkel, J. Lepowsky and A. Meurman, "Vertex Operator Calculus", Preprint.
12. L. Alvarez-Gaumé, J.B. Bost, G. Moore, P. Nelson and C. Vafa, Commun. Math. Phys. 112 (1987) 503; E. Verlinde and H. Verlinde, Nucl. Phys. B288 (1987) 357
13. C. Vafa, Nucl. Phys. B273 (1986) 592
14. D.S. Freed and C. Vafa, Commun. Math. Phys. 110 (1987) 349
15. R.C. Gunning, "Lectures on Modular Forms", Princeton University Press (1962)

16. W. Ledermann, "Introduction to Group Characters". Cambridge University Press (1987)
17. L. Dixon, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B282(1987)13
18. S. Hamidi and C. Vafa, Nucl. Phys. B279(1987)465

Fig-1:

Genus 2- cut Riemann surface, with the cycles and the corresponding twists.

Fig-2:

The world-sheet with four twist states from the sectors g^{-1} twisted at z_1 and z_3 and g -twisted at z_2 and z_4 . The loops C_1 and C_2 are two closed loops.