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Compactifications of F-Theory on Calabi–Yau Threefolds – I

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We study compactifications of F-theory on certain Calabi–Yau threefolds. We find that $N = 2$ dualities of type II/heterotic strings in 4 dimensions get promoted to $N = 1$ dualities between heterotic string and F-theory in 6 dimensions. The six dimensional heterotic/heterotic duality becomes a classical geometric symmetry of the Calabi–Yau in the F-theory setup. Moreover the F-theory compactification sheds light on the nature of the strong coupling transition and what lies beyond the transition at finite values of heterotic string coupling constant.

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1. Introduction

In constructing compact examples of D-manifolds for type IIB strings some evidence has emerged for the existence of a 12 dimensional formulation of type IIB strings, the ‘F-Theory’ [1]. The existence of a 12 dimensional viewpoint has been suspected from various different viewpoints over the years [2,3] and also more recently [4–10]. Let us briefly recall the setup in [1]. The proposal there was that there is a (10,2) theory underlying the type IIB theory. Upon compactifying 1 space and 1 time coordinate, the physical degree of freedom of this compactification is characterized by a complex structure τ of a torus. Thus compactifications of F-theory can also be viewed as compactification on manifolds which admit an elliptic fibration—in this way the compactified manifold behaves *as if* it is Euclidean, but via the map discussed in [1] one can translate the geometry to that of a (10,2) manifold. In discussing compactifications it is most natural to take this into account and consider the manifold as if it is a Euclidean manifold with elliptic fibers. To be more precise one is considering an elliptically fibered manifold *together* with the choice of an embedded base manifold. Mathematically this means that we have an elliptically fibered manifold together with a choice of a section¹.

Upon compactification of F-theory to 10 dimensions on T^2 we get type IIB theory. Upon compactification on elliptic $K3$ we get, as proposed in [1], a model dual to heterotic strings on T^2 . It is natural to consider F-theory compactifications on other manifolds, and the simplest next case is on a Calabi–Yau threefold leading to $N = 1$ theory in 6 dimensions. The most general manifold in this class to consider is a Calabi–Yau threefold which admits an elliptic fibration. We will divide these into two natural classes: In the first class we study elliptic Calabi–Yau manifolds which in addition admit a $K3$ fibration; in other words we consider the case where the $K3$ fiber itself is elliptically fibered. This would be useful for dualities with heterotic strings. The next class would be elliptic fibrations which do not admit a compatible $K3$ fibration. (An example of this was studied in [1]—the model there cannot be dual to a heterotic string compactification as it has more than one tensor multiplet.) We consider the first case in this paper. The second type as well as certain aspects of the first type will be discussed in a forthcoming paper [11].

¹ We will assume here that the base intersects the fiber at one point. It would be interesting to see whether or not multiple intersections make sense. If they do, they would correspond already in 10 dimensions to type IIB-like theories which have a U -duality group corresponding to subgroups of $SL(2, \mathbb{Z})$.

The outline of this paper is as follows. In section 2 we consider general aspects of heterotic compactifications on $K3$ and further compactification down to 4 dimensions on T^2 . We note some of the type II duals proposed for these models [12] and show how they lead to natural F-theory duals already in 6 dimensions. We also note the strong coupling problem pointed out in [13] in this context. In section 3 we discuss certain mathematical facts for the elliptic Calabi–Yau threefolds. In particular, just assuming that there is a dual F-theory compactifications we can *derive* the Calabi–Yau manifold needed to compactify the dual F-theory; the manifolds we find agree with some of those proposed in [12]. Our derivation uses results from algebraic geometry for elliptic manifolds going back to the work of Kodaira [14] and others. The present method goes far beyond checking the spectrum of massless particles, and in fact can be used as a systematic method to *construct* type II duals for heterotic string compactification on $K3$ or $K3 \times T^2$. We will see examples of this in the present paper, postponing a more complete analysis to [11].

In section 4 we discuss the physics that these dualities teach us. This will include geometrizing the heterotic/heterotic duality recently proposed in [13] as well as shedding light on what lies beyond the strong coupling transition noted there. It turns out that this transition gets mapped on the F-theory side to crossing a wall of the Kähler cone. Luckily these kind of situations have been extensively studied in the context of 2d conformal field theory and mirror symmetry in [15,16].

There is some overlap between our work and a concurrently released paper of Aspinwall and Gross [17].

2. Heterotic Compactifications on $K3$

In considering compactifications of the heterotic string on a manifold M we have to choose which gauge group ($SO(32)$ or $E_8 \times E_8$) and what type of bundle to use. One requirement is that the first Pontryagin number of the bundle should be the same as that of the manifold:

$$\frac{1}{2}p_1(V) = \frac{1}{2}p_1(M).$$

For example if we consider compactification on $M = K3$ and take into account the fact that half the Pontryagin number of $K3$ is 24, we learn that we have to choose instanton number 24 for the gauge bundle. An instanton number 24 gauge bundle generically breaks $SO(32)$ to $SO(8)$, so in this case for generic choices of gauge bundle we expect an $SO(8)$ gauge symmetry in 6 dimensions. In the case of $E_8 \times E_8$ the generic gauge symmetry we

get depends on how we distribute the instanton numbers (k_1, k_2) among the two E_8 's. If they are given by $(k_1, k_2) = (12, 12)$ one can show [12] that generically one does not obtain a gauge symmetry and the instantons break both the E_8 's completely. There are some other simple cases that we consider for the sake of examples in this paper including:

$$(k_1, k_2) = (24, 0) \rightarrow G = E_8$$

$$(k_1, k_2) = (20, 4) \rightarrow G = E_7$$

$$(k_1, k_2) = (18, 6) \rightarrow G = E_6$$

$$(k_1, k_2) = (12, 12) \rightarrow G = \{0\}.$$

In all these cases we can determine which group we end up with by Higgsing as much as possible as was done in [12]. Moreover in the above examples for the resulting gauge group G we are left with no charged matter, which is somewhat special. In all of the above cases, except for the last one, there was a puzzle raised in [13]: $N = 1$ theory in 6d is chiral and has potential anomalies. The anomaly 8-form factorizes as [18,19]

$$I_8 = \frac{1}{16(2\pi)^4} (tr R^2 - v tr F^2)(tr R^2 - \tilde{v} tr F^2)$$

Moreover it has been shown in [20] that this implies that the gauge kinetic term will contain a term of the form

$$\mathcal{L} \propto (ve^{-\phi} + \tilde{v}e^{\phi}) tr F^2,$$

where $\exp(2\phi) = \lambda^2$ is the heterotic string coupling constant in 6 dimensions. In all the above examples—except the completely Higgsed case of $(12, 12)$ where $\tilde{v} = 0$ —one finds that v and $\tilde{v}$ are both non-zero and have opposite signs. This in particular implies that for finite values of heterotic string coupling constant the gauge kinetic term becomes zero at

$$\exp(-2\phi) = \frac{-\tilde{v}}{v} \tag{2.1}$$

suggesting a phase transition. We will be able to shed light on this phase transition once we construct the F-theory duals of these heterotic vacua. For later use let us list the values of $\tilde{v}/v$ that we find for the cases above

$$SO(32) \rightarrow \frac{-\tilde{v}}{v} = 2$$

$$(24, 0) \rightarrow \frac{-\tilde{v}}{v} = 6$$

$$(20, 4) \rightarrow \frac{-\tilde{v}}{v} = 4$$

$$(18, 6) \rightarrow \frac{-\tilde{v}}{v} = 3.$$

Upon further compactification on T^2 we get a theory in $d = 4$ with $N = 2$ supersymmetry. For this case there are a number of examples where a dual string theory has been proposed corresponding to type II compactification on Calabi–Yau threefolds. For example, for the $SO(32)$ case the dual proposed is [12,21] the Calabi–Yau manifold defined by

$$M_{SO(32)} = (WP_{1,1,4,12,18}^4, 36).$$

Among the other $E_8 \times E_8$ examples mentioned above for the first and the last one there were duals proposed in [12]; a counting² also suggests duals in the other two cases in terms of Calabi–Yau’s which are $K3$ fibrations [22]:

$$M(24, 0) = (WP_{1,1,12,28,42}^4, 84)$$

$$M(20, 4) = (WP_{1,1,8,20,30}^4, 60)$$

$$M(18, 6) = (WP_{1,1,6,16,24}^4, 48)$$

$$M(12, 12) = (WP_{1,1,2,8,12}^4, 24).$$

All of these proposed duals are hypersurfaces in weighted projective spaces with the weights (as subscripts) and degree as given above. The last example has been studied extensively in the Coulomb phase in [23].

It is natural to expect that a duality between heterotic strings and type II strings in 4 dimensions should lead to some statement in the decompactification limit of T^2 and thus to a statement about a 6-dimensional duality of strings. In fact, as suggested in [1], there is such a duality in terms of F-theory. Consider the case where the manifold M admits an elliptic fibration. Then type IIA on M is on the same moduli as M-theory on $M \times S^1$ and this in turn is on the same moduli as F-theory on $M \times S^1 \times S^1$. So one would expect that the six-dimensional heterotic string on $K3$ in the limit in which T^2 decompactifies corresponds to the F-theory compactified on M . This can actually also be argued using adiabatic

² This counting was done jointly with Shamit Kachru.

arguments [21]: Start from the compactification of F-theory on elliptic $K3$ which is dual to heterotic compactification on T^2 . Upon further compactification on S^1 we would get the duality between M-theory on $K3$ to heterotic strings on T^3 [24]. Upon compactification on T^2 we would get the duality between type IIA strings on $K3$ and heterotic strings on T^4 [25,26]. However, we can do another thing: We can consider a one parameter family of the eight dimensional dual theories, parametrized by $\mathbb{P}^1$, in such a way that on the F-theory side we get a Calabi–Yau with an elliptic $K3$. This automatically implies that on the heterotic side we get an elliptic $K3$ compactification with an appropriate gauge bundle. Note that on the F-theory side we have an elliptic fibration over a $\mathbb{P}^1$ -bundle over yet another $\mathbb{P}^1$. In other words the base B of the Calabi–Yau threefold is given by a $\mathbb{P}^1$ -bundle over $\mathbb{P}^1$ with the fiber being T^2 . Compactifications of this system down from 6 dimensions to 5 and 4 on S^1 and T^2 respectively will give the chain of dualities suggested above.

So given the dualities proposed in [12] which admit elliptic fibrations (see also other examples in [27]) we are led to a conjectured duality between F-theory on the same Calabi–Yau threefold and heterotic string on $K3$.

Let the Hodge numbers of the $K3$ -fibered Calabi–Yau manifold which also admits an elliptic fibration be given by (h_{11}, h_{12}) . Let us count the multiplets in the 6 dimensional sense. Since a tensor multiplet and a vector multiplet in $N = 1, d = 6$ will both lead to vector multiplets in $N = 2, d = 4$ and the T^2 compactification gives rise to 2 additional vector multiplets, we learn that $r(V) + T = h_{11} - 2$, where by $r(V)$ here we mean the rank of the vector multiplets and by T the number of tensor multiplets. Moreover since the number of hypermultiplets are the same in 6 and in 4 dimensions we learn that we have $h_{21} + 1$ hypermultiplets in 6 dimensions. Out of these, h_{21} (together with certain other modes) correspond to complex moduli of Calabi–Yau. It is a fact that the number of complex deformation of an elliptic Calabi–Yau is the same as that of general Calabi–Yau in accordance with this count of hypermultiplets. (This will be explained in the next section.) The geometric origin of the last hypermultiplet will be discussed below.

In the context of dualities with the heterotic string we know that $T = 1$, that is, there is only one tensor multiplet upon compactification of the heterotic string on $K3$. In particular the scalar component of this tensor multiplet is the heterotic string coupling constant. Note that there are two Kähler modes of F-theory on a manifold whose base is a $\mathbb{P}^1$ -bundle over $\mathbb{P}^1$, corresponding to the two Kähler classes of the $\mathbb{P}^1$'s. Let k_f, k_b

correspond to the Kähler classes of the fiber and base $\mathbb{P}^1$'s respectively. Then the six dimensional heterotic string coupling constant is identified with

$$\frac{1}{\lambda^2} = \exp(-2\phi) = \frac{k_b}{k_f} \quad (2.2)$$

The other combination $k_f k_b$ is part of the extra hypermultiplet left out in the count above. To see (2.2) note that upon toroidal compactification to eight dimensions the heterotic string coupling constant λ^2 is identified with k_f [1]. Since we are now considering a $\mathbb{P}^1$ family of them, and the six dimensional coupling gets rescaled as usual by the volume $\lambda^2 \rightarrow \lambda^2/k_b$, we find the relation (2.2).

3. Mathematical Aspects of Elliptic Calabi–Yau Threefolds

In this section we discuss some mathematical aspects of elliptic Calabi–Yau threefolds. We will mainly concentrate on the case where the base manifold B is a $\mathbb{P}^1$ -bundle over $\mathbb{P}^1$; as explained in the previous section this would be the class of interest in constructing heterotic duals. Some of our remarks are however valid for arbitrary base manifold B .

In constructing heterotic duals the following observations prove to be very crucial: Suppose we know a heterotic string has a gauge symmetry G with no matter (for simplicity). Then the proposed Calabi–Yau dual must have a singularity of type G [28–31]. What this means, recalling the relation between the elliptic fiber and the 7-brane, is the following: The regions where the elliptic modulus $\tau \rightarrow \infty$ correspond to the worldvolume of the 7-branes, which consists of a surface Σ sitting in the base B together with the 6-dimensional spacetime. Depending on how the torus degenerates as we approach $\Sigma \subset B$, we get various type of gauge groups (characterized in the simplest cases of degeneration by A-D-E). The regions where the torus degenerates may have several components Σ_i which would be identified as (part of) several 7-brane worldvolumes. Since the c_1 of the Calabi–Yau is zero it relates a particular linear combination of the classes $[\Sigma_i]$ with the canonical class of the base B . This is the generalization to the threefold of the similar statement for elliptic $K3$'s. In that case the regions where the torus degenerates correspond to points on the base $\mathbb{P}^1$ and the condition for vanishing c_1 when the fibration is generic is simply that we have 24 of them. (The condition is more complicated for non-generic fibrations and will be discussed below.) Similarly in the case of elliptic Calabi–Yau threefolds, the linear combinations of the classes $[\Sigma_i]$ will depend on what type of singularities Σ_i correspond to. This will prove very powerful in constructing the Calabi–Yau duals for the heterotic

models studied in the previous section and allows one to have a constructive method for finding the dual Calabi–Yau.

Note that effectively what we have learned via F-theory, is that we *can* talk about compactifications of type IIB strings on certain manifolds with $c_1 > 0$, and construct a compact version of D-manifolds [29], with appropriate 7-brane skeletons arranged to cancel the c_1 . This comment applies to the case of compactification of F-theory in all dimensions; note that the worldvolume of the 7-brane intersects the base in complex codimension 1 which is the correct dimension to cancel the c_1 of the manifold. For instance for compactifications of F-theory on Calabi–Yau 4-fold (leading to $N = 1$ in $d = 4$) the 7-brane skeleton inside the Calabi–Yau lives on a 4 dimensional space, i.e. in complex codimension 1 on the base.

In this section we first discuss the relation between the classes $[\Sigma_i]$ and the canonical class of the base. We then talk about aspects of $\mathbb{P}^1$ -bundles over $\mathbb{P}^1$. We then apply this technology to construct the duals for the heterotic side and recover the predictions of the previous section. In the next subsection we discuss the condition of having elliptic fibration for Calabi–Yau’s. It turns out that not all the $K3$ -fibered Calabi–Yau’s admit an elliptic fibration. Finally we explain why in the elliptic fibrations of threefolds the number of complex deformations is the same as that with relaxing the condition on elliptic fibration.

3.1. Relation between the Geometry of the Base and the 7-Brane Worldvolume

The types of singular fibers which can occur on a nonsingular elliptic surface with no “exceptional curves of the first kind”³ were classified long ago by Kodaira [14]. All of the components of these fibers are $\mathbb{P}^1$ ’s (possibly with singularities), and the ways in which they can be joined together are quite constrained. For many of the fibers, the $\mathbb{P}^1$ ’s are all nonsingular, and their points of intersection all take the form of two $\mathbb{P}^1$ ’s meeting transversally. In this case, it is convenient to represent the fiber by means of a so-called dual graph whose vertices correspond to the components of the fiber and whose edges indicate which components meet. The list of such dual graphs is precisely the list of Dynkin diagrams for the simply-laced affine Lie algebras, and we use the standard notation for such diagrams (i.e., $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$) to indicate the type of fiber. We display

³ Surfaces with $c_1 = 0$ —the case of primary interest for us—have no such “exceptional curves of the first kind.”

Kodaira's results in the first two columns of Table 1.⁴ The fibers which are not associated

to simply-laced affine Lie algebras are displayed in Figure 1.

Kodaira notation	Singular fiber	Weierstrass singularity	a_i
I_1	Fig. 1	none	$\frac{1}{12}$
$I_b, b \geq 2$	$\tilde{A}_b$	A_b	$\frac{b}{12}$
II	Fig. 1	none	$\frac{1}{6}$
III	Fig. 1	A_1	$\frac{1}{4}$
IV	Fig. 1	A_2	$\frac{1}{3}$
$I_b^*, b \geq 0$	$\tilde{D}_{b+4}$	D_{b+4}	$\frac{1}{2} + \frac{b}{12}$
II*	$\tilde{E}_8$	E_8	$\frac{5}{6}$
III*	$\tilde{E}_7$	E_7	$\frac{3}{4}$
IV*	$\tilde{E}_6$	E_6	$\frac{2}{3}$

Table 1

⁴ We have omitted the cases on Kodaira's list which correspond to so-called multiple fibers,

since these do not occur when there is a section of the fibration (as we are assuming).

Figure 1.

Kodaira found a formula for the canonical bundle of the total space S which takes the form

$$K_S = \pi^*(K_C + \sum a_i P_i), \quad (3.1)$$

where the sum is over points P_i in the base C over which the fibers are singular, and the coefficients a_i are determined by the type of singular fiber as specified in the last column of Table 1. Thus, in order to obtain a $K3$ surface from this construction we must have

$$K_C = -\sum a_i P_i. \quad (3.2)$$

This generalizes the “generic” case discussed above, since if all of the singular fibers are of type I_1 , then each coefficient a_i is equal to $\frac{1}{12}$ and we find that there must be precisely 24 of them since K_C has degree -2 .

Since our fibration has a section, we can form a “Weierstrass model” by blowing down all components of each fiber which do not meet the section [32]. This is the geometric model we should expect to use in F-theory, since there is only one “size” associated to the fiber. And in fact, the singularities which we introduce in this way are precisely of the type (so-called A-D-E singularities) which should lead to gauge symmetry enhancement in the type IIA theory compactified on $K3$ [24,33]. String/string duality predicts that the gauge group should be the one whose Dynkin diagram corresponds to the resolution graph of the singularity. The singularities we get in Weierstrass models are shown in the third column of Table 1.

Kodaira’s formula (3.1) was extended by Kawamata [34], Fujita [35], and Nakayama [36] to the case of an elliptic fibration over a base B of higher dimension, where it takes the form

$$K_M = \pi^*(K_B + \sum a_i[\Sigma_i]) + \text{error term.} \quad (3.3)$$

The coefficients a_i are again determined by the type of singular fiber at the general point of each component Σ_i of the locus within B on which the elliptic curve degenerates. The error term is present due to less accurate control over the birational geometry of these spaces in higher dimension; however, thanks to work of Grassi [37] we need not concern ourselves with this error term in studying elliptic Calabi–Yau threefolds if we choose our birational model correctly.⁵ In particular, in order to get K_M to vanish we require:

$$K_B = - \sum a_i[\Sigma_i]. \quad (3.4)$$

As in the $K3$ case, there is a “Weierstrass model” obtained by blowing down all components of fibers not meeting the section [36] (provided that we start with a good birational model [37]). This Weierstrass model will have curves of singularities, which should contribute an enhanced gauge symmetry group to the type IIA theory. In fact, the curves of singularities arising in Weierstrass models of elliptic fibrations were precisely the geometric tool used in [31] to study enhanced gauge symmetry.⁶ As explained there, in order to obtain models without charged matter, the components of the locus Σ must not meet each other. (Models with charged matter stemming from intersection points of components of Σ are discussed in [29,38].)

⁵ Grassi’s results require us to allow for the possibility of some mild singularities on the base B . However, these singularities will not be present in any of the examples discussed in this paper. (In particular, B has a smooth model if the threefold has a Weierstrass model.)

⁶ The earlier method of [28] based on $K3$ fibrations can also be used to study the models of the present paper.

3.2. $\mathbb{P}^1$ -Bundles over $\mathbb{P}^1$

The set of possible $\mathbb{P}^1$ -bundles over $\mathbb{P}^1$ can be described in a very concrete way. We begin with the base $\mathbb{P}^1$ represented in the form $\mathbb{C}^2 - \{(0, 0)\}/\mathbb{C}^*$. That is, we let $\lambda \in \mathbb{C}^*$ act on the homogeneous coordinates (s, t) by $(s, t) \mapsto (\lambda s, \lambda t)$; $\mathbb{P}^1$ is the quotient space by that action.

In order to form a $\mathbb{P}^1$ -bundle, we represent the fiber in a similar way, as the quotient by the action $(u, v) \mapsto (\mu u, \mu v)$ for $\mu \in \mathbb{C}^*$ (restricting to $(u, v) \neq (0, 0)$). In order to get a nontrivial bundle structure, we should take the homogeneous coordinates u and v of the fiber to transform as sections of a line bundle over the base $\mathbb{P}^1$. In other words, under λ these should transform as $(u, v) \mapsto (\lambda^n u, \lambda^m v)$. By changing the generators of our $\mathbb{C}^* \times \mathbb{C}^*$ action and switching u and v if necessary, we can assume that $m = 0$ and $n \geq 0$. The resulting quotient space is known as the minimal ruled surface $\mathbb{F}_n$; these are known to be all of the possible $\mathbb{P}^1$ -bundles over $\mathbb{P}^1$.

When n is even, this manifold is topologically a product of two S^2 's. The Kähler class on such a manifold can be specified by the areas of the two S^2 's, leading to two Kähler parameters k_f and k_b . Geometrically, k_b is the coefficient of the divisor $D_s = \{s = 0\}$ which is the fiber of the $\mathbb{P}^1$ -bundle. The other generating class k_f is trickier to compute, but turns out to be the coefficient of $D_v + (n/2)D_s$, where $D_v = \{v = 0\}$. (The cohomology ring is naturally generated by D_s , D_v , and $D_u = \{u = 0\}$, with a relation $D_u = D_v + nD_s$ and the intersection pairing determined by $D_s \cdot D_s = 0$, $D_s \cdot D_v = 1$ and $D_u \cdot D_v = 0$. It follows that the self-intersection of $D_v + (n/2)D_s$ is 0.) We note for later use that the canonical bundle of $B = \mathbb{F}_n$ is given by $K_B = -2D_v - (n + 2)D_s$.

A general Kähler class is now specified as $k_b D_s + k_f (D_v + (n/2)D_s)$. The area of the divisor D_v is then given by

$$\text{area}(D_v) = D_v \cdot \left(k_b D_s + k_f (D_v + \frac{n}{2} D_s) \right) = k_b - \left(\frac{n}{2} \right) k_f. \quad (3.5)$$

The fact that this area must be positive leads to the inequality

$$\frac{k_b}{k_f} \geq \frac{n}{2}, \quad (3.6)$$

which defines one of the boundaries of the Kähler cone of $\mathbb{F}_n$. The other boundary is given by the condition $k_f \geq 0$.

3.3. Constructing the F-Theory Duals

We now turn to the construction of elliptic Calabi–Yau threefolds (with a section) over a base B which is one of the surfaces $\mathbb{F}_n$. We wish to consider models with no charged matter, so we insist that the components of the locus Σ of degenerate elliptic curves should not meet each other.

The divisor D_v plays a special rôle on $\mathbb{F}_n$ in that $D_v \cdot D_v = -n$ which is negative (when $n > 0$); we also have $K_B \cdot D_v = n - 2$. If D_v is not one of the components of Σ , then by (3.4), $-K_B$ can be represented by a divisor with positive coefficients whose components do not include D_v . It follows that $-K_B \cdot D_v \geq 0$, which implies that $n \leq 2$.

On the other hand, if D_v is one of the components of Σ and has coefficient $a_v > 0$ in the canonical bundle formula (3.4), then

$$-2 = (K_B + D_v) \cdot D_v = ((1 - a_v)D_v + \text{other terms}) \cdot D_v = (1 - a_v)D_v \cdot D_v.$$

(The last equality follows from the fact that D_v is disjoint from the other divisors appearing in the formula.) This determines the value of n as being $n = 2/(1 - a_v)$, if we know the singularity type over D_v (and hence the value of a_v). Thus, to produce a theory with no gauge group we should work with $n \leq 2$, while if we want $G = SO(8)$, E_6 , E_7 , or E_8 we find $a_v = \frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, or $\frac{5}{6}$ so we should use $n = 4, 6, 8$, or 12 , respectively.

Each Weierstrass model we are seeking can be described as a hypersurface within a bundle over B [39,32,36]. The fibers of this bundle will be weighted projective spaces⁷ $WP_{1,2,3}^2$, which we represent in terms of homogeneous coordinates z, x, y (not all zero) with an action by $\nu \in \mathbb{C}^*$ of $(z, x, y) \mapsto (\nu z, \nu^2 x, \nu^3 y)$. The Weierstrass equation will take the form

$$y^2 = x^3 - f(s, t, u, v) x z^4 - g(s, t, u, v) z^6, \tag{3.7}$$

where f and g depend on the coordinates s, t, u, v of B . As in our constructions of bundles in the previous subsection, the coordinates z, x and y should be taken as sections of some line bundles over the base, and we can specify which bundles we are using by means of their transformation properties under $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$ (the group used to construct $\mathbb{F}_n$). By an appropriate change of basis of $(\mathbb{C}^*)^3$ we can assume that the action of (λ, μ) on z is trivial. Moreover, the actions on x and on y are constrained by the fact that y^2 and

⁷ It is more common in the mathematics literature to represent the fibers as ordinary projective spaces, but the weighted projective space representation is equivalent.

x^3 both occur in the equation of the hypersurface, so must transform the same way. The upshot is that the transformation by (λ, μ) must take the form

$$(z, x, y) \mapsto (z, \lambda^{2\alpha} \mu^{2\beta} x, \lambda^{3\alpha} \mu^{3\beta} y).$$

It is convenient at this point to summarize our description in the following way. We have homogeneous coordinates s, t, u, v, x, y, z on which (λ, μ, ν) act with exponents as specified in the following table:

	s	t	u	v	x	y	z
λ	1	1	n	0	2α	3α	0
μ	0	0	1	1	2β	3β	0
ν	0	0	0	0	2	3	1.

Our Calabi–Yau threefold is obtained by starting with these homogeneous coordinates, removing the loci $\{s = t = 0\}$, $\{u = v = 0\}$, $\{x = y = z = 0\}$, taking the quotient by $(\mathbb{C}^*)^3$, and restricting to the solution set of (3.7).

In fact, we have just given a description of this Calabi–Yau as a hypersurface in a toric variety, and all of the powerful techniques of toric geometry can be brought to bear on these examples. (See [16] for a review in the physics literature.) Alternatively, we can use this data to give a linear sigma model description of the theory [15]. From either point of view, there are conditions which must be satisfied in order to get a Calabi–Yau manifold from this data (see [40] for a summary in physical terms). The first of these is that the monomial $stuvxyz$ should have the same weight as terms appearing in the equation (3.7). This condition immediately lets us solve for the unknown exponents, and we find $\alpha = n + 2$, $\beta = 2$. The remaining conditions (which, in the language of toric geometry, state that a certain polyhedron should be “reflexive”) lead to the restriction $n \leq 12$; if we also demand that the components of Σ be disjoint from one another (so that there is no charged matter) then we are limited to the cases $n = 0, 1, 2, 3, 4, 6, 8, 12$.

Notice that we can generically use the second and third $\mathbb{C}^*$ ’s to set the values of v and z to be 1. This leaves us with 5 homogeneous coordinates and a single $\mathbb{C}^*$; in fact, we have mapped our Calabi–Yau to a hypersurface of degree $6n + 12$ in $WP^4_{1,1,n,2n+4,3n+6}$. For the cases of $n = 4, 6, 8, 12$, this reproduces the conjectured heterotic duals with gauge groups $SO(8)$, E_6 , E_7 and E_8 !⁸ Note that this also explains the coincidence observed in [12] that

⁸ The case $n = 3$ corresponds to an $SU(3)$ gauge symmetry with no matter on the heterotic side. It is very likely that this is dual to the heterotic model with instanton numbers $(9, 15)$, as there is a branch of this model with generic unbroken $SU(3)$ [13]. The value of heterotic string coupling constant where there is phase transition is also consistent with this identification.

there seems to be a chain of dualities obtained by shifting the weights of the projective space by multiples of $(0, 0, 2, 4, 6)$ (similar remarks appear in [27]). Note that in the above we have also understood why not all the multiples of $(0, 0, 2, 4, 6)$ appear (i.e. why there is a gap and a bound for values of n). The cases with $n = 0, 1, 2$ involve some interesting features; we will discuss them in the next section.

3.4. The $(3, 243)$ Models

The construction of Weierstrass models over $\mathbb{F}_n$ which we have given above leads, for $n = 0, 1, 2$, to three families of Calabi–Yau threefolds with Hodge numbers $(3, 243)$, all equipped with $K3$ fibrations as well as elliptic fibrations [36]. On the other hand for $E_8 \times E_8$ heterotic strings there are three classes of instanton numbers where complete Higgsing is possible leading to the same Hodge numbers [12, 27, 13]: $(12, 12), (11, 13), (10, 14)$. It is natural to try and match the above choices of n with these three cases. It was conjectured in [12] that the $(12, 12)$ is dual to the Calabi–Yau given by $n = 2$ above. Subsequently it was conjectured in [27] that $(10, 14)$ also lies on the same moduli. If these conjectures are both true, one would thus expect that two of the above choices of n are in fact connected. This turns out to be the case. As we will see below $n = 0$ and $n = 2$ are connected and represent the same Calabi–Yau.

The case $n = 2$ can be represented in terms of hypersurfaces of degree 24 in $WP_{1,1,2,8,12}^4$. Of the 243 complex structure moduli of this family, only 242 are realized as deformations of the equation of the hypersurface. In other words, there is one “non-polynomial” deformation, reminiscent of the examples studied in [31].

In fact, as in [31], there is a natural locus of enhanced gauge symmetry for these models: it corresponds to blowing down the curve D_v in $\mathbb{F}_2$, and the corresponding surface lying above it in the Calabi–Yau. Doing so produces a singular Calabi–Yau space with a genus 1 curve of A_1 singularities. The results of [29, 30, 31] then suggest that we should see an enhanced $SU(2)$ gauge symmetry with 1 adjoint of matter along this locus. Indeed, to produce the singularities on the Calabi–Yau space, we have had to restrict to codimension 1 in Kähler moduli, but also to codimension 1 in complex structure moduli (the codimension 1 space of those complex structures which can be represented within $WP_{1,1,2,8,12}^4$).

As in the case of the examples in [31], it is possible to see the full complex structure moduli by using a complete intersection model rather than a hypersurface model. The

space $\mathbb{F}_2$ with the curve D_v blown down to a point can be represented as a quadric hypersurface in $\mathbb{P}^3$ with a singularity, given by, say

$$\xi\eta + \zeta^2 = 0, \quad (3.8)$$

with the homogeneous coordinates being (ξ, η, ζ, τ) ; the polynomials f and g used to describe the Weierstrass model can then be represented as polynomials in ξ, η, ζ and τ of degrees 8 and 12. If we perturb (3.8) to

$$\xi\eta + \zeta^2 = c\tau^2, \quad (3.9)$$

then we get the “non-polynomial” deformations of the Weierstrass model.

When $c \neq 0$, (3.9) defines a surface isomorphic to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In fact, for generic moduli we have reproduced a Weierstrass model over $\mathbb{F}_0$, the $n = 0$ case mentioned above. Thus we see that the $n = 0$ and $n = 2$ cases of our construction actually form part of the same family. In the next section we connect the cases with $n > 0$ with strong coupling phase transition in the heterotic string [13]. Note that this observation is consistent with the recent observation in [41] that if we consider a special subspace of $(10, 14)$ heterotic vacuum, the strong coupling puzzle in [13] is avoided in this case by Higgsing⁹. In the above picture we have drawn, this is the same as making a smooth deformation from the $n = 2$ case to the more generic $n = 0$ case by deforming to the more generic moduli.

The $n = 1$ case would appear to be different, however, and it is natural to identify it with the $(11, 13)$ $E_8 \times E_8$ heterotic vacuum¹⁰. We can try to use the methods of the previous section and map the Calabi–Yau to a hypersurface in $WP_{1,1,1,6,9}^4$, but the Hodge numbers associated to that space are $(2, 272)$. In fact, our Calabi–Yau maps to a hypersurface within $WP_{1,1,1,6,9}^4$ which is so singular that its Hodge numbers, which are again $(3, 243)$, differ from that of the generic Calabi–Yau hypersurface in that space.

⁹ One can also check, as in section 4 that the value of heterotic string coupling constant where a problem is expected to develop is $1/\lambda^2 = n/2 = 1$ and is consistent with this picture.

¹⁰ This is also supported by the fact that, as in discussion in section 4, one can check that the strong coupling transition occurs at $1/\lambda^2 = n/2 = 1/2$.

3.5. Condition for Existence of Elliptic Fibration

Before considering the implications of the above dualities let us note that we do not expect that all $N = 2$ dualities between type IIA and heterotic strings in 4 dimensions come from $N = 1$ duality between F-theory and the heterotic string in 6 dimensions. The reason for this is that some of the dualities may come from first compactifying on T^2 and then using special moduli of T^2 to get enhanced gauge symmetry and then freezing these moduli by turning on such gauge fields on the $K3$. For example the first main example in [12] was of this type. For such cases we thus expect that there should be no elliptic fibration. This is not in contradiction with the fact that the corresponding Calabi–Yau manifold has a $K3$ fibration, and that $K3$ admits elliptic fibrations. The reason for that is that not all the $K3$ ’s admit elliptic fibrations and this in particular prevents the first model considered in [12] from having an F-theory dual in 6 dimensions.

In fact, the conditions for a Calabi–Yau threefold to admit either an elliptic fibration or a $K3$ fibration are quite easy to state [42]. In order to have an elliptic fibration, there must be an effective divisor D such that (1) $D \cdot \Gamma \geq 0$ for all curves Γ , (2) $D^3 = 0$, and (3) $D^2 \cdot F \neq 0$ for some other divisor F .¹¹ In order to have a $K3$ fibration, there must be an effective divisor $\hat{D}$ such that (1) $\hat{D} \cdot \Gamma \geq 0$ for all curves Γ , and (2) $\hat{D}^2 \cdot F = 0$ for all divisors F (which implies that $\hat{D}^3 = 0$). For a Calabi–Yau with both kinds of fibration, the two fibrations will be compatible (i.e., the $K3$ ’s will themselves be elliptically fibered) if $D^2 \cdot \hat{D} = 0$.

It is easy to see from the properties of the intersection ring of the first model of [12] that it does not have divisors of the type required for an elliptic fibration. That ring was calculated in [43] to be generated by classes H and L with intersection numbers $H^3 = 4$, $H^2L = 2$, $HL^2 = 0$, $L^3 = 0$. The only effective divisors whose triple self-intersection is 0 are multiples of L (which define the $K3$ fibration), and multiples of $3H - 2L$. But for the latter class, $(3H - 2L) \cdot \ell = -2$, where ℓ is the class of a particular $\mathbb{P}^1$ on the Calabi–Yau; thus, this divisor does not meet part (1) of the stated condition.

¹¹ We are omitting one condition here: in order to obtain an elliptic fibration with a *section* as in this paper, we need to demand that D and F can be chosen so that $D^2 \cdot F$ is a small number.

3.6. Counting of Complex Structure of Elliptic Calabi–Yau’s

There is one major difference between the $K3$ and Calabi–Yau cases of F-theory compactification which may appear rather surprising at first sight. In the $K3$ case, it is only at special values of the moduli that an elliptic fibration structure can be found. However, for Calabi–Yau’s, if one member of a family contains an elliptic fibration then the general member of that family will also contain such a structure. We want to explain how this comes about.

In both cases, we can characterize the existence of an elliptic fibration by means of the existence of certain effective divisors in the space. The cohomology class of this desired divisor would not move as we vary the moduli, but whether that class can be represented by an effective divisor *can* change. In the $K3$ case, this is very likely to change because for generic moduli the class no longer lives in $H^{1,1}$ but has acquired a component in the $H^{2,0}$ and $H^{0,2}$ directions—it can no longer be represented by any divisor, effective or not. In the Calabi–Yau case, the change if it occurs can not be for that reason.

The fact that the divisor we need persists at generic moduli in the Calabi–Yau case is essentially a consequence of the fact that not only do the divisor classes remain within $H^{1,1}$, but also [44] the Kähler cone is constant for generic moduli. The Kähler cone can shrink at special values of moduli, but when it shrinks, it does so away from the locus $D^3 = 0$, and so no new elliptic fibration is introduced by the shrinking.

4. Physical Implications of F-theory/Heterotic Duality

In this section we discuss the implications of the F-theory/ heterotic dualities constructed above.

4.1. The Strong Coupling Phase Transition

It was pointed out in [13] that for generic compactifications of the heterotic string, at finite values of the heterotic string coupling constant there will be an infinitely strong coupling for gauge fields. Using the relation between the heterotic string coupling constant and the ratio of k_b/k_f (2.2) and using (2.1), we learn that the expected singularities is when

$$\frac{k_b}{k_f} = \frac{-\tilde{v}}{v} \tag{4.1}$$

However as shown in the previous section we know that there is a bound for $\frac{k_b}{k_f}$ for a rational ruled surface:

$$\frac{k_b}{k_f} \geq \frac{n}{2} \quad (4.2)$$

where n defines the rational ruled surface $\mathbb{F}_n$ as described in the previous section. We are thus led to the identification of

$$\frac{-\tilde{v}}{v} = \frac{n}{2} \quad (4.3)$$

It is easy to see that in all the examples constructed above this is a correct identity. In fact the relation between instanton numbers (k_1, k_2) and $\frac{\tilde{v}}{v}$ seems to be very simple. For instanton number (k_1, k_2) with $k_1 \geq 12$, we have

$$\frac{-2\tilde{v}}{v} = k_1 - 12$$

This is true in all the above examples and we believe it to be of general validity (this is also related to the observations in [45]). We would thus expect the simple identification

$$n = k_1 - 12$$

Using this, and the observations in the previous section, we see that the manifold $M(k_1, k_2)$ dual to $E_8 \times E_8$ heterotic strings on $K3$ with instanton numbers (k_1, k_2) should be given as a hypersurface

$$M(k_1, k_2) \subset WP_{1,1,k_1-12,2k_1-20,3k_1-30}^4,$$

of degree $6k_1 - 60$, possibly with additional singularities. (For the cases $k_1 = 12, 13, 14$ see the discussion in the previous section). We conjecture that this identification is true for all the allowed instanton numbers (and not just the ones we have discussed explicitly). In a sense we have *derived* this using the duality of F-theory and heterotic strings in eight dimensions and by using adiabatic arguments which suggests that the Calabi-Yau manifold should be elliptically fibered over $\mathbb{F}_n$. This is also consistent with the well known-fact that there is no elliptically fibered Calabi-Yau threefold over $\mathbb{F}_n$ for $n > 12$ [36]. Note that in this list, the heterotic string with instanton numbers $(16, 8)$ also appears and gives the same manifold we have discussed for the $SO(32)$ heterotic string. We thus conjecture that $E_8 \times E_8$ heterotic string on $K3$ with instanton numbers $(16, 8)$ is on the same moduli as heterotic string (or Type I string) with $SO(32)$ gauge group on $K3$.

The identification of the n with $-2\tilde{v}/v$ implies that in all the above examples, except for the case of $(12, 12)$ compactification of heterotic string where $n = 0$, there is a bound

for the heterotic string beyond which there is a phase transition. What can we say about this phase transition using the duality we have found?

Luckily this type of singularity in the Kähler moduli of Calabi–Yau manifolds has been studied extensively before [15,16]. However the singularities studied there are in the context of sigma models. But in 6 dimensions, F-theory compactifications strictly speaking do not correspond to perturbative ‘string vacua’ as different types of (p, q) strings have been used in their construction. Another way of saying this is that the dilaton has been turned on and there are points where it gives a large value of the coupling constant. Moreover we have used non-perturbative U-dualities even to define the F-theory vacuum. This in particular means that we are not necessarily justified in using sigma model techniques to study F-theory compactifications. However we will consider the following: Compactify first on T^2 , in which case we get an ordinary type IIA compactification on the same manifold being dual to the heterotic string on $K3 \times T^2$. Then we can use the sigma model techniques to study these singularities. What we learn from this investigation is that the vacuum makes sense beyond this singularity but it ceases to have an interpretation as a compactification on a geometric manifold; in other words we lose the manifold description, but we can effectively talk about the vacuum beyond this point. There are in general a number of different phases of the theory which can be seen from this sigma model analysis. We expect that these different phases correspond to different string compactifications—we are studying this issue now in the examples discussed above. Moreover we can also use mirror symmetry to construct the type IIB duals which would be equivalent to these compactifications where we now encounter complex degeneration as opposed to Kähler degenerations. This is nicer to study because it implies that we can use the same mirror manifold to study beyond the transition and we just have to change the complex structure. Thus we will find a geometric description for this phase transition in this way. At any rate from this description it is clear that in the four dimensional version certain new modes become massless as we are crossing the transition point, just as was the case for the conifold point [46,47] and other types of singularities [28–31]. By considering the large volume limit of T^2 we can then get insight directly into the six-dimensional transition. We are currently studying this in detail in some of the models above. There is preliminary evidence of enhanced gauge symmetries at the transition point; the results will be reported in [11].

4.2. Heterotic/Heterotic Duality

Soon after the $N = 2, d = 4$ dualities were proposed in [12] it was pointed out by Klemm, Lerche and Mayr [22] that there are other additional symmetries on the heterotic side of the type of $S - T$ exchange symmetry, which is implied by these dualities but remain unexplained. On the type II side, this symmetry is a classical geometric symmetry of the manifold. This is true for both of the main examples considered in [12]. These were further considered in [48,49]. One of these cases, the case of $(12, 12)$ imbedding of instantons, was recently studied in [13] in which a strong/weak self-duality in six dimensions was proposed. Upon further compactification on T^2 this in particular means, using well-known facts [50], the $S - T$ exchange symmetry. Given that the four dimensional duality in [12] is already geometric and given the correspondence we have found with the F-theory compactifications on the same manifold, we see that already in 6 dimensions we should be able to see the duality proposed in [13] as a geometric symmetry on the F-theory side.

This is essentially obvious given the construction we gave in the previous section: We found that the Calabi–Yau threefold can be described as an elliptic fibration over $\mathbb{P}^1 \times \mathbb{P}^1$ where the elliptic fibration is given (in affine coordinates) by

$$y^2 = x^3 - f(z_1, z_2)x - g(z_1, z_2)$$

where (z_1, z_2) are the coordinates of the $\mathbb{P}^1 \times \mathbb{P}^1$ and f is of degree 8 in each of the z_i and g is of degree 12 in each of them. Note that we have 243 complex deformation parameters given by $13 \times 13 + 9 \times 9 - 3 - 3 - 1 = 243$ defining the coefficients of the polynomials up to $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ action and rescaling of x, y . Clearly there is an exchange symmetry if we exchange the two $\mathbb{P}^1$'s and at the same time change the coefficients of the polynomials so that the coefficient of $z_1^k z_2^l$ is exchanged with that of $z_1^l z_2^k$ in each of the terms. Note that since the heterotic string coupling constant is given by the ratio of the Kähler classes of these two $\mathbb{P}^1$'s (2.2), exchanging them inverts the heterotic string coupling constant. We have thus geometrized the symmetry observed in [13] and at the same time understood its action on the (complex part of) hypermultiplets. It is amusing to see how some of the tests in [13] will come out here. Let us do the simplest case and ask on what subspace we will get an enhanced $SU(2)$ symmetry. This could have two types of origins on the heterotic side: perturbative or non-perturbative through small sized instantons [51]. Given the fact that on the F-theory side the strong/weak duality is manifest, it suffices to concentrate on one of them, say the perturbative side. For the heterotic side if we are interested in getting an

$SU(2)$ gauge symmetry perturbatively we have to imbed the instanton number 12 bundle in an $E_7 \subset E_8$, and this gives us a space with 116 real parameters smaller. Noting that for us half of the hypermultiplet space is geometrically realized as complex deformation parameters (with the other half coming from modes such as turning on the four form A^+) we should expect (assuming the enhanced gauge symmetry occurs on the subspace where these extra modes are zero) on a real codimension 58 subspace, or complex codimension 29.

Note that $SU(2)$ gauge symmetry corresponding to the perturbative symmetry of heterotic strings should come from an unbroken part of the gauge symmetry in 10 dimensions. This in particular means that even in the 8 dimensional limit of $K3$ we should see a gauge symmetry. Let (z_1, z_2) denote the coordinates of the fiber and base respectively. The above condition translates to having two 7-branes characterized by positions in z_1 coming together. In other words the base $\mathbb{P}^1$ would correspond to part of a 7-brane world-volume with an A_1 singularity (the non-perturbative one will correspond to the fiber $\mathbb{P}^1$ corresponding to a 7-brane with A_1 type singularity). Let us count how big this space is.

The 7-brane worldvolume (where the elliptic curve degenerates) is given by the vanishing locus of the discriminant

$$\Delta = 4f^3 - 27g^2.$$

What we want is for the discriminant to vanish to second order along some curve described by $z_1 = \lambda$ for some constant λ . Δ will vanish to second order along that curve if and only if both Δ and $\partial\Delta/\partial z_1$ vanish along $z_1 = \lambda$. To get Δ to vanish we must have

$$4f(\lambda, z_2)^3 = 27g(\lambda, z_2)^2.$$

To get that to happen for all z_2 we need $f(\lambda, z_2) = 3h(z_2)^2$ and $g(\lambda, z_2) = 2h(z_2)^3$ for some polynomial $h(z_2)$ of degree 4.

To get $\partial\Delta/\partial z_1$ to vanish along $z_1 = \lambda$ we need

$$12f(\lambda, z_2)^2 f'(\lambda, z_2) = 54g(\lambda, z_2)g'(\lambda, z_2)$$

where we are using f' and g' to denote derivative with respect to z_1 . Substituting the previous result we have

$$108h(z_2)^4 f'(\lambda, z_2) = 108h(z_2)^3 g'(\lambda, z_2).$$

The solution to this is $g'(\lambda, z_2) = h(z_2)f'(\lambda, z_2)$.

Now we count parameters, making Taylor expansions of f and g around $z_1 = \lambda$. The zeroth order terms are $f(\lambda, z_2)$ and $g(\lambda, z_2)$, polynomials of degree 8 and 12. For a general Calabi–Yau in our family this accounts for $9 + 13 = 22$ of the complex parameters. Similarly, the first order terms are $f'(\lambda, z_2)$ and $g'(\lambda, z_2)$ and for the general Calabi–Yau this accounts for $9 + 13 = 22$ additional parameters. However, along the locus of gauge symmetry enhancement, these two terms are specified by $h(z_2)$ and $f'(\lambda, z_2)$, of degrees 4 and 8, accounting for only $5 + 9 = 14$ parameters. There is one additional parameter given by λ (the location of the gauge symmetry enhancement). So the complex codimension in the full space is $44 - 15 = 29$, as expected.

Note that the strong/weak duality of heterotic strings which is realized on the F-theory side by the exchange of the base and fiber implies that the singularity occurs at a particular point on the base $\mathbb{P}^1$. Since the base is ‘visible’ to the heterotic side, because of the 8 dimensional duality of F-theory with heterotic strings on T^2 [1] we see that the dual to perturbative gauge symmetries on the heterotic side occurs at moduli where there are singularities of the bundle/ $K3$ at particular points on the base of the $K3$ on the heterotic side. This is in accord with the interpretation of them as heterotic instantons of zero size [13]. Actually using the above duality one can in principle analyze a whole class of various singularities corresponding to either $K3$ singularities and bundle singularities and translate that into statements about geometric singularities of the elliptic Calabi–Yau on the F-theory side. This would be interesting to develop further.

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