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DYNAMICS OF ASTROPHYSICAL BUBBLES AND BUBBLE-DRIVEN SHOCKS: BASIC THEORY, ANALYTICAL SOLUTIONS, AND OBSERVATIONAL SIGNATURES

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ABSTRACT

Bubbles in the interstellar medium are produced by astrophysical sources, which continuously or explosively deposit large amounts of energy into the ambient medium. These expanding bubbles can drive shocks in front of them, the dynamics of which is markedly different from the widely used Sedov–von Neumann–Taylor blast wave solution. Here, we present the theory of a bubble-driven shock and show how its properties and evolution are determined by the temporal history of the source energy output, generally referred to as the *source luminosity law*, $L(t)$. In particular, we find the analytical solutions for a driven shock in two cases: the self-similar scaling law, $L \propto (t/t_s)^p$ (with p and t_s being constants) and the finite activity time case, $L \propto (1 - t/t_s)^{-p}$. The latter with $p > 0$ describes a finite-time-singular behavior, which is relevant to a wide variety of systems with explosive-type energy release. For both luminosity laws, we derived the conditions needed for the driven shock to exist and predict the shock observational signatures. Our results can be relevant to stellar systems with strong winds, merging neutron star/magnetar/black hole systems, and massive stars evolving to supernovae explosions.

Key words: ISM: bubbles – ISM: jets and outflows – shock waves

Online-only material: color figure

1. INTRODUCTION

Astrophysical bubbles are formed around sources with outflows in the form of winds, electromagnetic radiation, Poynting flux, etc., when outflows are strong enough to sweep up much material in the ambient medium. Examples include supernova remnants, pulsar wind nebulae, stellar-wind-driven bubbles around early-type stars, Wolf-Rayet stars or star clusters (Gaensler & Slane 2006; Crowther 2007; Maeder & Meynet 2012), as well as some models of gamma-ray bursts (GRBs; Lyutikov & Blandford 2003). In a separate paper (Medvedev & Loeb 2013), we demonstrate that close binaries of compact objects (neutron stars (NSs), magnetars, and black holes) can produce Poynting-flux-driven bubbles during their final inspiral and merger (McWilliams & Levin 2011; Etienne et al. 2012). The bubbles can be accompanied by shocks, which are known to be efficient accelerators of cosmic rays. Observationally, the shocks are detected via synchrotron radiation from the accelerated population of electrons in the ambient or self-generated magnetic fields, or by Compton upscattering of lower-energy photons by the energetic electrons. Theoretical understanding of the dynamics of the bubbles and the associated shocks combined with multi-wavelength telescope observations, gravitational wave search (e.g., with Advanced LIGO), and neutrino detection experiments (with KamiokaNDE, IceCube, and others) can provide valuable information about the nature of the central engines.

In this paper, we consider systems in which energy is continuously pumped into the medium by a central source. This energy produces excess pressure which pushes on the surrounding plasma and results in the formation of an expanding bubble (or a cavity). If the bubble expansion velocity exceeds the sound speed in the ambient medium, then a shock forms ahead of

the bubble, as shown in Figure 1. Such a shock wave is different from the Sedov–von Neumann–Taylor blast wave (Sedov 1946) produced by a point-like instantaneous explosion and freely expanding into the interstellar medium (ISM). Instead, the shock is continuously driven by the ever-increasing pressure inside the bubble, rendering the classical Sedov–von Neumann–Taylor solution inapplicable.

Formation and dynamics of astrophysical bubbles has been a subject of intense research (see, e.g., the review by Ostriker & McKee 1988; Koo & McKee 1992a, 1992b). However, analytical studies have mostly been limited to self-similar solutions $R(t) \propto t^\alpha$, $v(t) \propto t^{\alpha-1}$ with R and v being the size and the expansion speed, and α being a constant. This requires the energy deposition rate into the bubble, which we colloquially refer to as the “source luminosity” $dE_{\text{bubble}}/dt \equiv L(t)$, to be a power law in time, $L(t) \propto t^p$, where p is a constant. This is a very restricting assumption because many astrophysical sources exhibit an explosive behavior: they are nearly constant-luminosity sources for a long time followed by a rapid—“explosive”—increase in energy output, formally tending to infinity within a finite time:³

$$L(t) = L_s(1 - t/t_s)^{-p}, \quad (1)$$

where t_s is the time during which the source is active, i.e., its “lifetime,” L_s is the source luminosity at early times $t \ll t_s$, and p is the luminosity index, determining the rate of energy deposition in the bubble. Such a luminosity law with the finite-time singularity (FTS) is not rare in nature. For example, a binary consisting of two compact objects, i.e., an NS, a magnetar, or a black hole (least one of the companions should be magnetized) is producing a bubble filled with material with the relativistic equation of state (Medvedev & Loeb 2013) and its luminosity is described by Equation (1) with the index p being around 3/2; the

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³ The singular (explosive) behavior occurs if $p > 0$. However, our analysis below makes no assumption about the value of p .

Figure 1. Schematic representation of the system. The central source outflow produces a bubble, which creates a shock in the ambient medium and the shell of the shock-heated gas.

(A color version of this figure is available in the online journal.)

exact value depends on specific details of the energy extraction. The merger of two NSs is considered to be the conventional model of short GRBs. However, the formation of the bubble and the associated driven shock and their observational signatures have not been addressed. The bubble-driven shock can, however, produce a unique precursor to the main short GRB event and serve as a benchmark of the NS–NS progenitor. Combining these results with the detection of gravitational waves accompanied by a neutrino signal measured, respectively, by Advanced LIGO and other gravitational wave detectors (during the last moments of the binary inspiral) and by KamiokaNDE, IceCube, and other neutrino detectors (during the merger event itself) will also greatly advance our understanding. Thus, the development of the theory of the bubble and shock dynamics in such systems is of great importance for astrophysics.

The paper is organized as follows. In Section 2, we develop the theoretical model of a bubble and the bubble-driven shock and obtain the master equation describing the system evolution for an arbitrary luminosity law $L(t)$. In Section 3, we find analytical solutions for the master equation for both the self-similar and singular (i.e., FTS) luminosity laws and explore the conditions under which the driven shock solutions exist. We also determine the conditions under which the exact FTS solutions (for the radius and velocity) exist. However, realistic systems, such as the NS binaries, do not fall into that category. Nevertheless, we are able to find simple analytical approximate FTS solutions for them as well. Finally, we demonstrate that the obtained analytical solutions agree very well with the full numerical solutions in the appropriate limits. Finally, in Section 4 we present conclusions.

2. THE BUBBLE+SHOCK MODEL

The system at hand is depicted in Figure 1. For simplicity, we consider spherically symmetric systems. The bubble is filled with some material with pressure p_{bubble} and the equation of state parameterized by the adiabatic index γ_b , which depends on the bubble nature and composition. If the bubble is driven by Poynting flux (e.g., in a form of a strongly magnetized relativistic e^-e^+ wind) as in pulsar wind nebulae, or double NS or magnetar binaries, then the value of the index is $\gamma_b = 4/3$. If the bubble is driven by a non-relativistic strong stellar wind, then $\gamma_b = 5/3$.

The bubble surface acts as a piston and pushes the ambient plasma outward to produce an outgoing shock wave. Is the outgoing shock relativistic? Let us assume, for example, that the total energy $E \sim 10^{46}$ erg is deposited into the bubble over some time t_{source} during which the source is active. If the shock is relativistic, it should expand with speed $\sim c$ and mean Lorentz factor of the order of

$$\Gamma \sim \frac{E}{Mc^2} \sim \frac{E}{(4\pi/3)(n_{\text{ISM}}m_p)(ct_{\text{source}})^3c^2} \gg 1, \quad (2)$$

where M is the mass of the ISM material swept by the shock over time t_{source} and m_p is the proton mass (assuming purely hydrogen ISM). For the ISM with a uniform density $n_{\text{ISM}} \sim 1 \text{ cm}^{-3}$, we have a constraint on the source lifetime:

$$t_{\text{source}} \ll 4 \times 10^5 E_{46}^{1/3} n_{\text{ISM},0}^{-1/3} \sim 4.5 E_{46}^{1/3} n_{\text{ISM},0}^{-1/3} \text{ days}, \quad (3)$$

where $E_{46} = E/(10^{46} \text{ erg})$, $n_{\text{ISM},0} = n_{\text{ISM}}/(1 \text{ cm}^{-3})$ and similarly for other quantities henceforth. Thus, if the typical source activity is much longer than a few days, the shock can be assumed to be non-relativistic. This constraint is even less restrictive for lower-energy sources.

The shock is propagating into the unperturbed unmagnetized or weakly magnetized ISM with mass density ρ_{ISM} and the adiabatic index $\gamma = 5/3$. For simplicity, we assume the ISM to be cold, so that we can neglect its pressure; hence the shock is a high-Mach-number strong shock. Besides, weak shocks are less interesting from the observational point of view in any case: being just a mild perturbation to the medium, their observational signatures are hard to detect. Finally, between the shock and the bubble lies the shell of the shocked gas, whose mass density and pressure, ρ_{shell} and p_{shell} , are determined by the Rankine–Hugoniot shock jump conditions. The pressure equilibration time behind the shock is assumed to be fast enough to establish pressure equilibrium throughout the system, hence the bubble–shell interface plays a role of a contact discontinuity and $p_{\text{bubble}} = p_{\text{shell}}$.

We assume the central engine deposits energy (e.g., electromagnetic, kinetic, etc.) in the bubble with luminosity $L(t) \equiv dE/dt$, which is a function of time. This power goes into the internal energies (1) of the bubble, dU_{bubble} and (2) of the shocked gas shell, dU_{shell} , (3) the change of the kinetic energy of the bulk motion of the shell, dK_{shell} , assuming its swept-up mass $M_{\text{swept}} = \text{const}$, (4) the change of the kinetic energy, $dK_{\text{@shock}}$, of the newly swept gas, dM_{swept} , and also (5) heating of this shocked gas, $dU_{\text{@shock}}$, to satisfy the Rankine–Hugoniot relations. We can neglect the $p dV$ work due to the expansion because the external pressure is assumed to be vanishingly small in the cold ISM. We also neglect radiative losses (see Section 2.5). Finally, we assume that the matter inside the bubble (ejecta) carries negligible momentum. This is a good approximation if

the bubble pressure is dominated by the magnetic pressure. The alternative case of a finite speed ejecta has been considered elsewhere, e.g., by Koo & McKee (1992b). Thus, the master equation is

$$L(t) dt = dU_{\text{bubble}} + dU_{\text{shell}} + dK_{\text{shell}}|_{M_{\text{swept}}} + dU_{\text{@shock}} + dK_{\text{@shock}}. \quad (4)$$

In addition to this equation, the Rankine–Hugoniot relations for a strong shock should be used. The continuity equation yields

$$\frac{\rho_{\text{shell}}}{\rho_{\text{ISM}}} = \frac{u_1}{u_2} = \frac{v_{\text{shock}}}{v_{\text{shock}} - v_{\text{shell}}(R_{\text{shock}})} \simeq \frac{\gamma + 1}{\gamma - 1} \equiv \kappa \quad (5)$$

and

$$\frac{v_{\text{shell}}(R_{\text{shock}})}{v_{\text{shock}}} = 1 - \kappa^{-1}, \quad (6)$$

where κ is the constant compression ratio of a strong nonradiative shock, u_1 and u_2 are the upstream and post-shock velocities in the shock comoving frame which transform to the lab frame as $v_{\text{shock}} = u_1$ and $v_{\text{shell}}(R_{\text{shock}}) = u_1 - u_2$. From the momentum conservation, $p_{\text{ISM}} + \rho_{\text{ISM}} u_1^2 = p_{\text{shell}} + \rho_{\text{shell}} u_2^2$, for a cold ISM (i.e., neglecting p_{ISM}), one obtains

$$p_{\text{shell}} = \rho_{\text{ISM}} v_{\text{shock}}^2 (1 - \kappa^{-1}). \quad (7)$$

We now calculate the terms in Equation (4).

2.1. Internal Energies

The internal energies of the bubble and the post-shock gas shell are those of an ideal gas:

$$dU_{\text{bubble}} = d \left(\frac{1}{\gamma_b - 1} p_{\text{bubble}} V_{\text{bubble}} \right), \quad (8)$$

$$dU_{\text{shell}} = d \left(\frac{1}{\gamma - 1} p_{\text{shell}} V_{\text{shell}} \right), \quad (9)$$

where $V_{\text{bubble}} = (4\pi/3)R_{\text{bubble}}^3$ and $V_{\text{shell}} = V_{\text{shock}} - V_{\text{bubble}}$ are the volumes of the bubble and the shell, respectively, and $V_{\text{shock}} = (4\pi/3)R_{\text{shock}}^3$ and recall that $p_{\text{bubble}} = p_{\text{shell}}$. The swept-up ISM mass occupies the post-shock shell, $M_{\text{swept}} = \rho_{\text{ISM}} V_{\text{shock}} = \rho_{\text{shell}} V_{\text{shell}}$, hence one has the relations:

$$\frac{V_{\text{shell}}}{V_{\text{shock}}} = \kappa^{-1}, \quad \frac{V_{\text{bubble}}}{V_{\text{shock}}} = 1 - \kappa^{-1}. \quad (10)$$

Therefore,

$$dU_{\text{bubble}} = \frac{(1 - \kappa^{-1})^2}{\gamma_b - 1} \rho_{\text{ISM}} d(v_{\text{shock}}^2 V_{\text{shock}}), \quad (11)$$

$$dU_{\text{shell}} = \frac{(1 - \kappa^{-1})\kappa^{-1}}{\gamma - 1} \rho_{\text{ISM}} d(v_{\text{shock}}^2 V_{\text{shock}}). \quad (12)$$

2.2. Kinetic Energy

To calculate the shell kinetic energy, one needs to know the post-shock velocity profile $v_{\text{shell}}(R)$. The post-shock gas is hot, hence thermal conduction is rapid and the temperature can be taken uniform throughout the shell. Since both

temperature and pressure are uniform, then density is uniform as well, $\rho(R) = \rho_{\text{shell}} = \rho_{\text{ISM}}\kappa = \text{const}$. The continuity equation of an incompressible gas in a steady state, $\nabla \cdot \mathbf{v} = \partial_R(R^2 v_{\text{shell}}(R)) = 0$, together with the boundary condition $v(R_{\text{shock}}) = v_{\text{shell}}(R_{\text{shock}}) = v_{\text{shock}}(1 - \kappa^{-1})$ yields

$$v_{\text{shell}}(R) = v_{\text{shell}}(R_{\text{shock}}) \left(\frac{R_{\text{shock}}}{R} \right)^2. \quad (13)$$

The kinetic energy of the shocked gas (see Figure 1) is

$$K_{\text{shell}} = \int_0^{M_{\text{swept}}} \frac{dM v_{\text{shell}}(R)^2}{2} = \frac{M_{\text{swept}} v_{\text{shock}}^2}{2} \xi, \quad (14)$$

where

$$\begin{aligned} \xi &= 3r(1 - \kappa^{-1})^2 [(1 - \kappa^{-1})^{-1/3} - 1] \\ &\simeq \frac{12}{\gamma^2 - 1} \left[\left(\frac{\gamma + 1}{2} \right)^{1/3} - 1 \right] = \frac{9}{4} (6^{2/3} - 3) \end{aligned} \quad (15)$$

and we have used that $dM = (\rho_{\text{ISM}}\kappa) 4\pi R^2 dR$ and $R_{\text{bubble}}/R_{\text{shock}} = (V_{\text{bubble}}/V_{\text{shock}})^{1/3} = (1 - \kappa^{-1})^{1/3}$. Finally, the change of the shell kinetic energy due to acceleration/deceleration is

$$dK_{\text{shell}}|_{M_{\text{swept}}} = \rho_{\text{ISM}} (4\pi/3) R_{\text{shock}}^3 \xi d(v_{\text{shock}}^2/2). \quad (16)$$

2.3. Shock Contributions

As the shock propagates, it heats up the cold ISM gas of mass $dM_{\text{swept}} = \rho_{\text{ISM}} dV_{\text{shock}}$ to the post-shock temperature $T_{\text{shell}} = p_{\text{shell}}/n_{\text{shell}} = m_p v_{\text{shock}}^2 (1 - \kappa^{-1})\kappa^{-1}$ and also accelerates it to the post-shock velocity, $v_{\text{shell}}(R_{\text{shock}}) = v_{\text{shock}}(1 - \kappa^{-1})$. Thus, at the shock

$$dU_{\text{@shock}} = \frac{(1 - \kappa^{-1})\kappa^{-1}}{\gamma - 1} \rho_{\text{ISM}} dV_{\text{shock}} v_{\text{shock}}^2, \quad (17)$$

$$dK_{\text{@shock}} = \rho_{\text{ISM}} dV_{\text{shock}} \frac{v_{\text{shock}}^2}{2} (1 - \kappa^{-1})^2. \quad (18)$$

2.4. Final Analysis

Equation (4) together with Equations (11), (12), and (16)–(18) determines the evolution of a shock driven by the pressure inside a cavity. Given the luminosity, $L(t)$, it allows us to determine $R_{\text{shock}}(t)$, because the shock velocity is $v_{\text{shock}} = dR_{\text{shock}}/dt \equiv \dot{R}_{\text{shock}}$ and the shocked volume is $V_{\text{shock}} = (4\pi/3)R_{\text{shock}}^3$. All other quantities, e.g., R_{bubble} , p_{shell} , etc., follow straightforwardly from the equations above. Hereafter, we will often omit the subscript “shock” wherever it does not cause confusion. Upon the substitutions, Equation (4) becomes

$$\frac{L(t)}{(4/3)\pi\rho_{\text{ISM}}} = \eta \frac{d(R^3 \dot{R}^2)}{dt} + \frac{\xi}{2} R^3 \frac{d\dot{R}^2}{dt} + \zeta \dot{R}^2 \frac{dR^3}{dt}, \quad (19)$$

where

$$\begin{aligned} \eta &= \frac{(1 - \kappa^{-1})^2}{\gamma_b - 1} + \frac{(1 - \kappa^{-1})\kappa^{-1}}{\gamma - 1} \\ &\simeq \frac{2(\gamma_b + 1)}{(\gamma + 1)^2(\gamma_b - 1)} = \frac{63}{32}, \end{aligned} \quad (20)$$

$$\zeta = 3 \left[\frac{(1 - \kappa^{-1})\kappa^{-1}}{\gamma - 1} + \frac{(1 - \kappa^{-1})^2}{2} \right] \simeq \frac{12}{(\gamma + 1)^2} = \frac{27}{16}. \quad (21)$$

Here, the first term represents the internal energies of the bubble and shell, the second term is the bulk kinetic energy of the shell, and the last term is the contribution of the shock. Hereafter we use, for concreteness, that the bubble is filled with relativistic material (e.g., a relativistic plasma, a highly magnetized wind, or electromagnetic radiation) with $\gamma_b = 4/3$, the ambient gas is non-relativistic with $\gamma = 5/3$ and the compression ratio is $\kappa \simeq 4$.

Master equation (19) is an inhomogeneous second-order nonlinear differential equation, which can further be simplified to yield

$$\frac{L(t)}{(4\pi/3)\rho_{\text{ISM}}} = (3\eta + \zeta)R^2\dot{R}^3 + (2\eta + \xi)R^3\dot{R}\ddot{R}. \quad (22)$$

This is the main equation of our analysis. For any function of the source emission luminosity, $L(t)$, pumping energy into the bubble, this equation describes the evolution of the shock radius, $R_{\text{shock}} \equiv R(t)$, and the associated parameters, e.g., the shock velocity, $v(t) = \dot{R}(t)$, the size of the bubble, $R_{\text{bubble}}(t) = (1 - \kappa^{-1})^{1/3}R(t) \simeq (3/4)^{1/3}R(t)$, etc. We stress that this equation is applicable to bubbles of various origin, provided the assumptions made in the analysis are satisfied (see Section 2.5). For example, it describes a Poynting-flux-driven bubble formed by an inspiraling binary; in this case $\gamma_b = 4/3$. It can also describe a bubble blown by a strong stellar wind with the kinetic energy luminosity given by $L(t)$ and the adiabatic index of the gas in the bubble $\gamma_b = 5/3$.

2.5. Comments on Assumptions

First, we assumed pressure equilibrium throughout the system. This assumption holds if the timescale for pressure equilibration exceeds the dynamic timescale of the bubble. For the shocked shell gas, pressure equilibrates on the sound crossing time, as measured in the frame of the post-shock medium (in the frame of the ISM gas, the pressure equilibration speed is $c_s + u$, where c_s and u are the post-shock sound speed and bulk gas velocities, respectively). For convenience, we choose the post-shock gas frame here, $u = 0$. The sound speed inside the shocked shell is of the order of the shock velocity, $c_{s,\text{shell}} \sim v_{\text{shock}} \sim R_{\text{shock}}/t$, where t is the dynamical time. The sound crossing time for the shell of thickness $\Delta R_{\text{shell}} = R_{\text{shock}} - R_{\text{bubble}} = [1 - (3/4)^{1/3}]R_{\text{shock}} \sim 0.1R_{\text{shock}}$ is estimated to be $\Delta R_{\text{shell}}/c_{s,\text{shell}} \sim 0.1t$. Thus, the pressure equilibration time in the shell is always much shorter than the dynamical time; hence the assumption is valid. The pressure equilibration speed, c_{eq} , in the bubble depends on what dominates its pressure. For example, c_{eq} is the sound speed for non-relativistic gases, the Alfvén speed (or other wave speeds) for bubbles dominated by magnetic pressure or turbulent pressure by Alfvénic turbulence (or by other types of wave turbulence, respectively), the speed of light for the bubbles driven by relativistic matter, Poynting flux, cosmic-ray pressure, etc. The pressure equilibrium assumption requires the above equilibration speed to exceed the shock velocity. Indeed, $R_{\text{bubble}}/c_{\text{eq}} \gtrsim R_{\text{shock}}/v_{\text{shock}}$ and $R_{\text{bubble}} \sim R_{\text{shock}}$, hence $c_{\text{eq}} \gtrsim v_{\text{shock}}$. This is easily satisfied, for example, for bubbles blown by a very hot gas, purely magnetic pressure, Poynting flux, or cosmic rays.

Second, we assumed rapid thermal conduction in the shocked gas shell. The energy equilibration between the ions and electrons is a rather complicated process which can involve various plasma instabilities. Detailed studies of this process go beyond the scope of this paper. Here, we assume that the ions and electrons maintain equipartition, i.e., $T_e \simeq T_i$. The mean free path of the electrons is of the order of

$$\lambda_{\text{mfp},e} \sim (\sigma n)^{-1} \sim (4\pi d_c^2 \ln \Lambda n)^{-1} \sim \frac{m_e^2 v_e^4}{4\pi \ln \Lambda e^4 n}, \quad (23)$$

where σ is the cross-section, $d_c \sim e^2/m_e v_e^2$ is the distance of the closest approach in a collision, v_e is the electron velocity, $\ln \Lambda \sim 20$ is the Coulomb logarithm, and we assumed a fully ionized hydrogen plasma, $n_p = n_e \equiv n$. Since $\lambda_{\text{mfp},e} \propto m_e^2 v_e^4$ and equipartition implies $m_e \bar{v}_e^2 \sim m_p \bar{v}_p^2$ (where $\bar{v}_e$ and $\bar{v}_p$ are the thermal speeds of the two species), the mean free paths of the electrons and ions (protons) are comparable, $\lambda_{\text{mfp},p} \sim \lambda_{\text{mfp},e} \equiv \lambda_{\text{mfp}}$, and very large

$$\lambda_{\text{mfp}} \sim 2 \times 10^{16} v_{\text{shock},9}^4 n_{\text{ISM},0}^{-1}, \quad (24)$$

for the fiducial values used later in this paper, $\bar{v}_p \sim v_{\text{shock}} \sim 10^9 \text{ cm s}^{-1}$, cf. Equations (34) and (61), and $\bar{v}_e \sim (m_p/m_e)^{1/2} \bar{v}_p \gg \bar{v}_p$. Here, $v_{\text{shock},9} = v_{\text{shock}}/(10^9 \text{ cm s}^{-1})$ and $n_{\text{ISM},0} = n_{\text{ISM}}/(1 \text{ cm}^{-3})$. Since $\lambda_{\text{mfp}} \gtrsim R_{\text{shock}}$, thermal conduction proceeds by free streaming of particles (i.e., in the saturated regime) with the characteristic velocity being the thermal velocity of the electrons if both species are thermally coupled, with the thermal velocities of the protons and electrons for each of the species, if they are thermally decoupled. However, in both cases, the temperature equilibration time is comparable to or (much) shorter than the pressure equilibration time in the shell, which occurs at approximately the same speed $c_s \sim \bar{v}_p \sim v_{\text{shock}}$. At smaller shock speeds, $\lambda_{\text{mfp}} \lesssim R_{\text{shock}}$ and this regime is difficult to treat analytically. At even smaller speeds, when $\lambda_{\text{mfp}} \ll R_{\text{shock}}$, thermal conduction is mostly by the electrons and the thermal conductivity takes the Spitzer value, $\kappa_{\text{Sp}} \sim \lambda_{\text{mfp}} \bar{v}_e$. However, the energy equilibration between the electrons and protons will not be instantaneous but will occur over some distance. We neglect this process for simplicity and assume instantaneous equilibration. Then, the isothermal approximation requires the thermal conduction time, $\tau_{\text{cond}} \sim \Delta R_{\text{shell}}^2/\kappa_{\text{Sp}}$, to be shorter than the dynamical time, $t \sim R_{\text{shock}}/v_{\text{shock}}$, thus

$$\lambda_{\text{mfp}} \gtrsim 0.01 (m_p/m_e)^{-1/2} R_{\text{shock}} \sim 10^{-4} R_{\text{shock}}, \quad (25)$$

where we used that $\bar{v}_e \sim (m_p/m_e)^{1/2} v_{\text{shock}}$ and $\Delta R_{\text{shell}} \sim 0.1 R_{\text{shock}}$. Magnetic fields of the ambient medium will also affect thermal conduction. Regular large-scale fields yield anisotropic thermal conduction. However, thermal conductivity is suppressed only perpendicular to the field but occurs at the unattenuated rate along the field lines, thus rendering our isothermal assumption generally valid. Tangled field lines, in contrast, yield isotropic but suppressed thermal conductivity with the suppression factor being rather uncertain, though not very large if the field is turbulent (Narayan & Medvedev 2001). Such a general problem of thermal evolution of a magnetized shocked gas goes beyond the scope of the present paper. We should comment, however, that thermal conduction is strong only in magnetically connected plasmas. Therefore, thermal conduction across the shell-bubble interface can safely be omitted, unless there is strong reconnection at this interface.

Third, we assumed that radiative losses from the system are negligible. The dipole energy loss per unit time is $\dot{E}_{\text{EM}} = 2|\dot{\mathbf{d}}|^2/3c^3$, where $\mathbf{d}$ is the dipole moment, $|\dot{\mathbf{d}}| \sim ed_c/(d_c/v_e)^2 \sim ev_e^2/d_c$, and d_c/v_e is the characteristic collision time. Thus, the bremsstrahlung energy loss by an electron per strong collision (i.e., in which the momentum change of an electron is of the order of its momentum, $\Delta(m_e v_e) \sim m_e v_e$) is

$$\Delta E_{\text{EM}} \sim \dot{E}_{\text{EM}} d_c/v_e \sim m_e v_e^2 (v_e/c)^3, \quad (26)$$

that is, the electron loses about $(v_e/c)^3$ of its energy per collision. The average number of collisions grows with time as $N \sim v_e t / \lambda_{\text{mfp}} \sim (m_p/m_e)^{1/2} R_{\text{shock}} / \lambda_{\text{mfp}}$ and only about $(\ln \Lambda)^{-1} \sim 20^{-1}$ of them are strong collisions (weak collisions deflect electrons slightly and do not contribute much to radiation). Hence, approximately, $N_{\text{Brems}} \sim (m_p/m_e)^{1/2} (\ln \Lambda)^{-1} R_{\text{shock}} / \lambda_{\text{mfp}} \sim R_{\text{shock}} / \lambda_{\text{mfp}}$ and the upper limit on the total fractional energy loss per particle is

$$\Delta E_{\text{Brems}} / m_e v_e^2 \sim \Delta E_{\text{EM}} N_{\text{Brems}} \sim 0.5 R_{\text{shock},16} v_{\text{shock},9}^{-1} n_{\text{ISM},0}, \quad (27)$$

because it assumes that all the energy is in the electrons (or there is instantaneous energy transfer from the ions to the electrons, which is not generally the case at high thermal velocities). Bremsstrahlung from the protons is even weaker because of their large mass. Note that the shock velocity cannot be too small, $v_{\text{shock}} \gtrsim 10^8 \text{ cm s}^{-1}$, as follows from Equation (25). Similar estimates apply to the radiative cooling from the bubble. If it is filled with non-relativistic gas, the pressure equilibrium assumption implies that the gas thermal speed is $v_{\text{th}} \sim c_s \gtrsim v_{\text{shock}}$; hence even weaker radiation losses are expected in this case, because they scale as $\propto v_{\text{th}}^{-1}$. If the bubble is driven by Poynting flux, it is filled with cold, relativistically outflowing, strongly magnetized plasma. Since the energy density is dominated by the magnetic field and the plasma is cold, radiation losses can be safely neglected as well, unless the field dissipation is somehow happening very rapidly. In general, for other systems one has to check that the above assumptions are satisfied; radiative shocks in supernova remnants, for example, are a classical counterexample (Raymond et al. 2001; Slavin et al. 2004).

3. RESULTS AND OBSERVATIONAL PREDICTIONS

To proceed further, one needs to specify the *source luminosity law*, $L(t)$. Below, we will consider the two most common ones: the self-similar law $L(t) \propto t^p$ and the “explosive” FTS law $L(t) \propto (t_s - t)^{-p}$ with $p > 0$.

3.1. Self-similar Solution

3.1.1. Structure

We first look for a self-similar solution. Let us assume that the source luminosity and the shock position are power-law functions of time

$$L(t) = L_s (t/t_s)^p, \quad R(t) = R_s (t/t_s)^\alpha, \quad (28)$$

where R_s and α are constants to be determined and L_s , t_s , and p are known constants set by the source physics. This self-similar solution is valid for the duration of the source activity, i.e., while the bubble pressure is high enough to push the shock; at

(much) later times, the shock dynamics should asymptote the Sedov–von Neumann–Taylor solution.

The self-similar solution (28) to Equation (22) is

$$\alpha = \frac{p+3}{5}, \quad R_s = \left(\frac{3}{4\pi} \frac{L_s t_s^3}{\rho_{\text{ISM}} A} \right)^{1/5}, \quad (29)$$

where

$$A = (3\eta + \zeta)\alpha^3 + (2\eta + \xi)\alpha^2(\alpha - 1) \\ \simeq \frac{9}{4}\alpha^2 \left[\frac{5}{4} - 6^{2/3} + \alpha \left(\frac{17}{8} + 6^{2/3} \right) \right]. \quad (30)$$

The scalings given in Equations (28) and (29) agree with those in Ostriker & McKee (1988). The above solution is meaningful if $R > 0$, i.e., $A > 0$, and hence

$$\alpha > \alpha_{\text{crit}} = \frac{6^{2/3} - 5/4}{6^{2/3} + 17/8} \simeq 0.378 \quad (31)$$

and

$$p > p_{\text{crit}} = 5\alpha_{\text{crit}} - 3 \simeq -1.11. \quad (32)$$

This condition means that the energy injection in the system cannot be too slow, otherwise the shock would move too fast for the contact discontinuity to catch up with it and the assumption of the pressure equilibration breaks down.

3.1.2. Estimates

We have obtained that, given the luminosity of the source, $L(t) = L_s (t/t_s)^p$, the shock evolution is given by $R_{\text{shock}}(t) = R_s (t/t_s)^\alpha$ and $v_{\text{shock}}(t) = (\alpha R_s / t_s) (t/t_s)^{\alpha-1}$ with $\alpha = (p+3)/5$. The value of R_s is rather insensitive to the numerical value of A (unless α is very close to the critical value), so, $R_s \sim (L_s t_s^3 / \rho_{\text{ISM}})^{1/5}$ is a good order-of-magnitude estimate. More accurately, assuming the source activity to be of the order of a hundred days, $t_s \sim 10^7 \text{ s}$, and the luminosity $L_s \sim 10^{39} \text{ erg s}^{-1}$, we estimate that

$$R_{\text{shock}}(t_s) = R_s \simeq 4.3 \times 10^{16} A^{-1/5} L_{s,39}^{1/5} t_{s,7}^{3/5} n_{\text{ISM},0}^{-1/5} \text{ cm}, \quad (33)$$

$$v_{\text{shock}}(t_s) = \alpha R_s / t_s \\ \simeq 4.3 \times 10^9 \alpha A^{-1/5} L_{s,39}^{1/5} t_{s,7}^{-2/5} n_{\text{ISM},0}^{-1/5} \text{ cm s}^{-1}. \quad (34)$$

If $\alpha < 1$, then the shock velocity $\propto t^{\alpha-1}$ is decreasing with time, thus at some earlier time it was $\sim c$, so the non-relativistic approximation was invalid. For the solution to be valid at early times for the duration of the source activity, the shock velocity should be increasing with time, i.e., α should be greater than unity, hence the luminosity index must be $p > 2$.

3.1.3. Observable Signature

We assume that the shock accelerates electrons via the Fermi process and generates/amplifies a magnetic field. The relativistic electrons in magnetic fields produce synchrotron radiation which can be observed. We assume that the electrons and magnetic fields carry, respectively, fractions ϵ_e and ϵ_B of the internal energy density of the shocked gas, cf. Equation (12),

$$u_{\text{shell}} = \frac{1 - \kappa^{-1}}{\gamma - 1} \rho_{\text{ISM}} v_{\text{shock}}^2, \quad (35)$$

that is

$$\bar{\gamma}_e m_e c^2 n_{e,\text{shell}} = \epsilon_e u_{\text{shell}}, \quad B^2/8\pi = \epsilon_B u_{\text{shell}}, \quad (36)$$

where $\bar{\gamma}_e m_e c^2$ is the average energy of an electron and $n_{e,\text{shell}} = \kappa n_{e,\text{ISM}} \simeq \kappa n_{\text{ISM}}$ is the number density of electrons in the shocked gas shell. Numerically, the average Lorentz factor of accelerated electrons is

$$\begin{aligned} \bar{\gamma}_e(t) &= \epsilon_e \frac{\kappa - 1}{\kappa^2(\gamma - 1)} \frac{m_p}{m_e} \left(\frac{v_{\text{shock}}}{c} \right)^2 \\ &\simeq 11 \epsilon_e \alpha^2 A^{-2/5} L_{s,39}^{2/5} t_{s,7}^{-4/5} n_{\text{ISM},0}^{-2/5} t_7^{2(\alpha-1)}, \end{aligned} \quad (37)$$

so the bulk electrons are mildly relativistic, $\bar{\gamma}_e \sim 3$ for a typical acceleration efficiency of $\epsilon_e = 0.3$. If the radiating electrons are distributed in energy as a power law with index s as $dn_e/d\gamma \propto \gamma^{-s}$ with a minimum Lorentz factor γ_m , then $\gamma_m = \bar{\gamma}_e(s-2)/(s-1)$. The magnetic field strength is

$$\begin{aligned} B(t) &= \left(\epsilon_B \frac{\kappa - 1}{\kappa(\gamma - 1)} 8\pi m_p n_{\text{ISM}} v_{\text{shock}}^2 \right)^{1/2} \\ &\simeq 3.0 \times 10^{-2} \epsilon_B^{1/2} \alpha A^{-1/5} L_{s,39}^{1/5} t_{s,7}^{-2/5} n_{\text{ISM},0}^{3/10} t_7^{\alpha-1} \text{G}, \end{aligned} \quad (38)$$

that is, B is of the order of a milligauss for a nominal $\epsilon_B \sim 10^{-3}$, which is much larger than the typical microgauss fields in the ISM. This means that the field should be generated or amplified at the shock by an instability (Weibel, Bell, cyclotron, etc.), or by MHD turbulence, or by some other mechanism.

Relativistic electrons emit synchrotron radiation. The peak frequency of the radiation spectrum is in the radio band,

$$\begin{aligned} \nu_s(t) &= (2\pi)^{-1} \gamma_m^2 \frac{eB}{m_e c} \simeq 9.3 \times 10^6 \epsilon_e^2 \epsilon_B^{1/2} \alpha^5 A^{-1} \\ &\times [(s-2)/(s-1)]^2 L_{s,39}^{-2} t_{s,7}^{-1/2} n_{\text{ISM},0}^{5(\alpha-1)} \text{Hz}. \end{aligned} \quad (39)$$

The total emitted power by a relativistic electron is $P = (4/3)\sigma_T c \bar{\gamma}_e^2 (B^2/8\pi)$, where σ_T is the Thompson cross-section. Thus, the spectral power at the peak (measured in $\text{erg s}^{-1} \text{Hz}^{-1}$) is $P_{\nu,\text{max}}(t) \approx P/\nu_s \simeq (\sigma_T m_e c^2/3e)B$. The observed spectral peak flux from a source located in our galaxy at the distance $D = 10^{23} \text{ cm}$ (i.e., $\sim 30 \text{ kpc}$) is $F_{\nu,\text{max}} = N_e P_{\nu,\text{max}}/(4\pi D^2)$, where $N_e = n_{e,\text{ISM}} V_{\text{shock}}$ is the total number of emitting electrons, hence

$$\begin{aligned} F_{\nu,\text{max}}(t) &= \frac{1}{4\pi D^2} \left(\frac{4\pi}{3} R_{\text{shock}}^3 n_{\text{ISM}} \right) \frac{\sigma_T m_e c^2}{3e} B \\ &\simeq 3.0 \times 10^3 D_{23}^{-2} \epsilon_B^{1/2} \alpha A^{-4/5} L_{s,39}^{4/5} t_{s,7}^{7/5} n_{\text{ISM},0}^{7/10} t_7^{4\alpha-1} \text{Jy}. \end{aligned} \quad (40)$$

Although the peak frequency can fall in the self-absorbed part of the spectrum, as is typical of supernova shocks (e.g., Waxman & Loeb 1999), the peak flux above is still useful for the normalization of the spectrum

$$\begin{aligned} F_\nu(\nu, t) &= F_{\nu,\text{max}}(t) \left(\frac{\nu}{\nu_s(t)} \right)^{-(s-1)/2} \\ &\propto \nu^{-(s-1)/2} t^{(3-5s+\alpha(3+5s))/2} \\ &\propto \nu^{-0.75} t^{7.75\alpha-4.75}, \end{aligned} \quad (41)$$

where the latter scaling corresponds to the nominal value of $s = 2.5$. We recall that the index α is related to that of the energy injection $L(t) \propto t^p$ as $\alpha = (p+3)/2$.

Let us write $F_\nu(\nu, t) \propto \nu^a t^b$, where a and b are spectral and temporal indexes determined from observations. Then one can infer the (effective) energy injection index as

$$p = (1 - 5a + b)/(4 - 5a). \quad (42)$$

Also, for a non-self-similar luminosity law, one can define the effective index as

$$p_{\text{eff}} = d \log L(t)/d \log t, \quad (43)$$

which can be compared with observations.

Finally, we estimate the self-absorption frequency as the frequency at which the optical depth of the emitting region of thickness $R_{\text{shock}} - R_{\text{bubble}}$ is of the order of unity, $\tau_\nu = \kappa_\nu(R_{\text{shock}} - R_{\text{bubble}}) \sim 1$, where κ_ν is the self-absorption coefficient (Rybicki & Lightman 1979):

$$\kappa_\nu = -\frac{1}{8\pi m_e \nu^2} \int d\gamma P_\nu(\gamma) \gamma^2 \frac{\partial}{\partial \gamma} \left(\frac{1}{\gamma^2} \frac{dn_e}{d\gamma} \right). \quad (44)$$

A power-law distributed electrons $dn_e/d\gamma = (n_{e,\text{shell}}/\gamma_m) (\gamma/\gamma_m)^{-s}$ produce the spectrum (per electron) $P_\nu(\gamma) \simeq P_{\nu,\text{max}} f(\nu/\nu_c)$, where $f(x)$ is the function $F(x)$ of Rybicki & Lightman (1979) involving the integral of a modified Bessel function, up to a numerical factor. Using a dimensional analysis, noting that $n_{e,\text{shell}}(R_{\text{shock}} - R_{\text{bubble}}) \simeq n_{\text{ISM}} R_{\text{shock}}/3$ and absorbing all numerical factors into a constant $K \sim 10^{-2}$, one obtains the absorption frequency, ν_a ,

$$\nu_a \simeq (K \sigma_T c \bar{\gamma}_e n_{\text{ISM}} R_{\text{shock}}/m_e)^{2/(s+4)} \nu_s^{(s-2)/(s+4)}, \quad (45)$$

which depends very weakly on the constant K , the density, and the shock radius and is nearly independent of the synchrotron frequency for typical spectral indices $s \gtrsim 2$. For typical values of the parameters, ν_a is of the order of $\sim 10^8$ – 10^9 Hz . Note that this analysis assumes that all the electrons are relativistic; otherwise, if only a fraction $\eta_{\text{rel}} < 1$ of them is relativistic, the self-absorption frequency is lower by $\sim \eta_{\text{rel}}^{2/(s+4)}$.

3.2. Solution with a Finite-time Singularity

3.2.1. General Structure

Traditionally in astrophysics, one looks for a self-similar solution, which is presented above. However, for some astrophysical systems, self-similarity may not be a good approximation. For example, in the paper (Medvedev & Loeb 2013) we have shown that the evolution of the binary system is described by an FTS solution, because the inspiral takes a finite time. We have found that the luminosity is

$$L(t) = L_s (1 - t/t_s)^{-p_*}, \quad (46)$$

where L_s , t_s and p_* are known “source” constants set by its physics and initial conditions. To be precise, this scaling should break down at some earlier time $t_m < t_s$, otherwise the total energy diverges if $p_* > 1$, which is the case of an NS binary, for example. It also breaks down on physics grounds because general relativistic and other effects, as well as finite object sizes, were omitted from consideration. The above form of $L(t)$ suggests looking for a solution in the form

$$R(t) = R_{s,*} (1 - t/t_s)^{-\alpha_*}, \quad (47)$$

where R_s and α_* are constants to be determined. Such a solution is formally valid at times $t < t_s$. At late times $t \gg t_s$, there is no energy injection, therefore the shock dynamics should asymptote the Sedov–von Neumann–Taylor solution.

The FTS solution, Equation (47), to the master equation (22) is

$$\alpha_* = \frac{p_* - 3}{5}, \quad R_{s,*} = \left(\frac{3}{4\pi} \frac{L_s t_s^3}{\rho_{\text{ISM}} A_*} \right)^{1/5}, \quad (48)$$

where

$$\begin{aligned} A_* &= (3\eta + \zeta)\alpha_*^3 + (2\eta + \xi)\alpha_*^2(\alpha_* + 1) \\ &\simeq \frac{9}{4}\alpha_*^2 \left[-\frac{5}{4} + 6^{2/3} + \alpha_* \left(\frac{17}{8} + 6^{2/3} \right) \right]. \end{aligned} \quad (49)$$

The solution is physically meaningful if $R > 0$, hence $A_* > 0$,

$$\alpha_* > \alpha_{*,\text{crit}} = \frac{5/4 - 6^{2/3}}{17/8 + 6^{2/3}} \simeq -0.378 \quad (50)$$

and

$$p_* > p_{*,\text{crit}} = 5\alpha_{*,\text{crit}} + 3 \simeq -1.11. \quad (51)$$

This condition constrains the energy injection index: if $p_* < p_{*,\text{crit}}$ then the energy injection rate is not enough for the contact discontinuity to catch up with the shock.

Unlike the self-similar solution, this solution with an FTS has another constraint. The physically meaningful solution (in this particular class of solutions) is the one of the *expanding* shock; hence

$$\alpha_* > 0 \text{ and } p_* > 3. \quad (52)$$

We emphasize that the above scalings are applicable for the duration of the central engine activity only, $t < t_s$. Moreover, the solution discussed in the previous subsection does not have much physical meaning, because the system size increases to infinity in a finite time. Perhaps it can make sense at times substantially smaller than t_s , when relativistic and other effects are negligible.

As we mentioned earlier, some astrophysical systems, such as the NS and magnetar binaries (Medvedev & Loeb 2013), have luminosity indexes $p_* = 3/2$ or $7/4$, depending on the electromagnetic energy extraction mechanism. For such p_* values from in the range, $p_{*,\text{crit}} < p_* < 3$, the bubble-driven shock exists, but the solution given by Equation (47) describes an unphysical collapsing shock with $R(t) \rightarrow 0$ as $t \rightarrow t_s$. This result is independent of most of the assumptions and easily follows from the energetics considerations or just the dimensional analysis, cf. Equation (22),

$$\begin{aligned} (1 - t/t_s)^{-p_*} &\propto L(t) \sim d(Mv^2)/dt \\ &\sim \rho d(R^3 \dot{R}^2)/dt \propto (1 - t/t_s)^{-5\alpha_* - 3}, \end{aligned} \quad (53)$$

which yields the relation $p_* = 5\alpha_* + 3$. On physics grounds, for values $p_{*,\text{crit}} < p_* < 3$, the shock radius tends to a constant, $R \rightarrow R_{\text{max}}$ as $t \rightarrow t_s$, but the form $(1 - t/t_s)^{-\alpha_*}$ does not have such an asymptotic for any $\alpha_* \neq 0$. A different form is needed, but it seems unlikely that one can find a general or partial solution to Equation (22) directly, because the solution should be of the form $\int dt \sqrt{(1 - t)^{-p_*+1} + C}$, which cannot be expressed in elementary functions for an arbitrary p_* , though it can be expressed in elliptic functions for some rational p_* . Below we obtain an approximate solution for such a regime.

3.2.2. Approximate Realistic FTS Solution

In the previous section, we have found that the bubble-driven shock can exist in systems with luminosity index $p_* > p_{*,\text{crit}}$, but if p_* is in the range $p_{*,\text{crit}} < p_* < 3$, the physically meaningful solutions are not described by the pure FTS solutions. Instead, the physical FTS solution should have the property that $R_{\text{shock}} \rightarrow R_{\text{max}} = \text{const}$, but $v_{\text{shock}} \propto (1 - t/t_s)^{-\alpha_* - 1} \rightarrow \infty$ as $t \rightarrow t_s$, with α_* being a new constant. The general and/or exact analytical solution of this kind does not exist, so we construct here a composite approximate solution.

First, we notice that the dependence $L(t) = L_s(1 - t/t_s)^{-p_*}$ implies a constant-luminosity source at early times, $t \ll t_s$, that is $L(t) \simeq L_s(t/t_s)^0$. This case corresponds to the self-similar solution with $p = 0$ in Section 3.1 for which the solution exists:

$$R_{\text{shock}}(t) = R_s(t/t_s)^{3/5} \rightarrow R_s, \text{ as } t \rightarrow t_s, \quad (54)$$

where R_s is given by Equation (29) or (59). Second, we make substitutions

$$R = R_s, \quad \dot{R} \equiv v_{\text{shock}}(t) = v_s(1 - t/t_s)^{-\alpha_* - 1}, \quad (55)$$

where v_s is a constant to be determined. By examining the right-hand side of Equation (22), one sees that both terms are divergent but the last term dominates as $t \rightarrow t_s$, because $p_* < 3$ and hence $\alpha_* < 0$. Keeping the leading term, one has

$$\begin{aligned} \alpha_* &= \frac{p_* - 3}{2}, \quad v_s = \left(\frac{3}{4\pi} \frac{L_s t_s}{\rho_{\text{ISM}} (2\eta + \xi) R_s^3} \right)^{1/2} \\ &= \left(\frac{A}{2\eta + \xi} \right)^{1/2} \frac{R_s}{t_s}, \end{aligned} \quad (56)$$

where $A \simeq 4.3$ is given by Equation (30) with $\alpha = 3/5$ and the prefactor $[A/(2\eta + \xi)]^{1/2} \simeq 0.97$.

Thus, we have found an approximate solution describing evolution of the bubble-driven shock at late times:

$$v_{\text{shock}}(\Delta t) \simeq v_s(\Delta t/t_s)^{-(p_*-1)/2}, \quad (57)$$

$$R_{\text{shock}}(\Delta t) \simeq R_s(\Delta t/t_s)^0 = \text{const}, \quad (58)$$

where we introduced a new variable $\Delta t = (t_s - t) \rightarrow 0$, which is more convenient when $t \lesssim t_s$. Note that the obtained regime is rather interesting: the shock is rapidly accelerating but its size and swept-up mass remain nearly constant.

3.2.3. Estimates

The characteristic values of the shock radius and velocity in the solution above are given by Equations (57) and (58). However, these values follow from an approximate analytical analysis. Comparison with the exact numerical solutions (Section 3.3) yields ad hoc correction factors for R_s and v_s , see Equations (70) and (71), namely, $\chi_R \simeq 1.33$ and $\chi_v \simeq 0.51$. We use these values in the estimates below. We have

$$R_s \simeq 3.2 \times 10^{16} \chi_R L_{s,39}^{1/5} t_{s,7}^{3/5} n_{\text{ISM},0}^{-1/5} \text{ cm}, \quad (59)$$

$$v_s \simeq 3.1 \times 10^9 \chi_v L_{s,39}^{1/5} t_{s,7}^{-2/5} n_{\text{ISM},0}^{-1/5} \text{ cm s}^{-1}, \quad (60)$$

where we assumed $\gamma_b = 4/3$ and $\gamma = 5/3$, for concreteness. We thus have

$$\begin{aligned} v_{\text{shock}}(\Delta t) &\simeq 9.8 \times 10^{8+p_*/2} \chi_v L_{s,39}^{1/5} t_{s,7}^{-(9-5p_*)/10} n_{\text{ISM},0}^{-1/5} \\ &\quad \times \Delta t_6^{-(p_*-1)/2} \text{ cm s}^{-1} \\ &\simeq 5.5 \times 10^9 \chi_v L_{s,39}^{1/5} t_{s,7}^{-3/20} n_{\text{ISM},0}^{-1/5} \Delta t_6^{-1/4} \text{ cm s}^{-1}, \end{aligned} \quad (61)$$

where we used a nominal value $p_* \simeq 3/2$. Note that if $p_* > 1$ then $v_{\text{shock}} \propto (\Delta t/t_s)^{-(p_*-1)/2} \rightarrow \infty$, as $\Delta t \rightarrow 0$ and v_{shock} approaches the speed of light at times

$$\Delta t \lesssim 10^{7-2/(p_*-1)} t_{s,7} \left(\chi_v L_{s,39}^{1/5} t_{s,7}^{-2/5} n_{\text{ISM},0}^{-1/5} \right)^{2/(p_*-1)} \text{ s}, \quad (62)$$

that is, about 10^3 s before the “explosion” time t_s , for $p_* = 3/2$. At these times, our assumption of the non-relativistic shock breaks down and a different analysis is needed. Note also that at this time, the dynamical time of the bubble needed to establish pressure equilibrium throughout is longer, $R_s/c \sim 10^6$ s, so an accurate analysis should instead be involved to find a full dynamical solution. Such a consideration goes beyond the scope of the present paper.

Regardless of the model assumption used, plasma processes and details of particle acceleration impose additional constraints as follows. If the characteristic dynamical time of the system, which is $\sim \Delta t$, is longer than the inverse collisional frequency ν_{coll}^{-1} , then a collisional shock forms. Otherwise, when $\Delta t < \nu_{\text{coll}}^{-1}$ then the shock is collisionless. In this case, if Δt is (much) shorter than the Larmor frequency in the ambient field, the shock structure is not sensitive to the ISM field, but instead is determined by kinetic plasma instabilities (e.g., electrostatic Buneman or electromagnetic Weibel-like ones driven by particle anisotropy at the shock). The shortest associated timescale is the ion plasma time $\omega_{pi}^{-1} \sim 10^{-3} n^{-1/2}$ s and, moreover, it takes about a hundred ω_{pi}^{-1} seconds for an electrostatic shock (or $\omega_{pi}^{-1} (c/v_{\text{shock}})$ seconds for a Weibel shock) to form and respond to the changing conditions; it takes even longer for particle Fermi acceleration. Thus, the synchrotron shock model should be used with great caution for Δt as short as tens of milliseconds or less.

3.2.4. Observables

The above scalings can be used in equations for observables in Section 3.1.3 (note, most of the parameters turn out to be functions of the shock velocity alone). In particular, for $\Delta t \sim 10^6$ s before an explosion, we have

$$\begin{aligned} \bar{\gamma}_e(\Delta t) &\simeq 5.5 \times 10^{-1+p_*} \chi_v^2 \epsilon_e L_{s,39}^{2/5} t_{s,7}^{-9/5+p_*} n_{\text{ISM},0}^{-2/5} \Delta t_6^{-(p_*-1)} \\ &\simeq 17 \chi_v^2 \epsilon_e L_{s,39}^{2/5} t_{s,7}^{-3/10} n_{\text{ISM},0}^{-2/5} \Delta t_6^{-1/2}, \end{aligned} \quad (63)$$

$$\begin{aligned} B(\Delta t) &\simeq 6.7 \times 10^{-3+p_*/2} \chi_v \epsilon_B^{1/2} L_{s,39}^{1/5} t_{s,7}^{-(9-5p_*)/10} n_{\text{ISM},0}^{3/10} \\ &\quad \times \Delta t_6^{-(p_*-1)/2} \text{ G} \\ &\simeq 3.8 \times 10^{-2} \chi_v \epsilon_B^{1/2} L_{s,39}^{1/5} t_{s,7}^{-3/20} n_{\text{ISM},0}^{3/10} \Delta t_6^{-1/4} \text{ G}, \end{aligned} \quad (64)$$

$$\begin{aligned} \nu_s(\Delta t) &\simeq 5.7 \times 10^{3+5p_*/2} \chi_v^5 \epsilon_e^2 \epsilon_B^{1/2} [(s-2)/(s-1)]^2 \\ &\quad \times L_{s,39}^{-(9-5p_*)/2} t_{s,7}^{-1/2} n_{\text{ISM},0}^{-5(p_*-1)/2} \text{ Hz} \\ &\simeq 3.2 \times 10^7 \chi_v^5 \epsilon_e^2 \epsilon_B^{1/2} [(s-2)/(s-1)]^2 \\ &\quad \times L_{s,39}^{-3/4} t_{s,7}^{-1/2} n_{\text{ISM},0}^{-5/4} \Delta t_6^{-5/4} \text{ Hz}, \end{aligned} \quad (65)$$

Figure 2. Log-log plot of the shock radius driven by an expanding bubble as a function of time. The solid curve represents the full numerical solution of Equation (22) and the dashed curve is the self-similar solution given by Equations (28) and (29).

$$\begin{aligned} F_{v,\text{max}}(\Delta t) &\simeq 2.8 \times 10^{2+p_*/2} \chi_R^3 \chi_v D_{23}^{-2} \epsilon_B^{1/2} \\ &\quad \times L_{s,39}^{4/5} t_{s,7}^{(9+5p_*)/10} n_{\text{ISM},0}^{7/10} \Delta t_6^{-(p_*-1)/2} \text{ Jy} \\ &\simeq 1.6 \times 10^3 \chi_R^3 \chi_v D_{23}^{-2} \epsilon_B^{1/2} \\ &\quad \times L_{s,39}^{4/5} t_{s,7}^{33/20} n_{\text{ISM},0}^{7/10} \Delta t_6^{-1/4} \text{ Jy}, \end{aligned} \quad (66)$$

$$\begin{aligned} F_v(\Delta t) &\propto v^{-(s-1)/2} \Delta t^{-(5s-3)(p_*-1)/4} \\ &\propto v^{-(s-1)/2} \Delta t^{-(5s-3)/8}. \end{aligned} \quad (67)$$

Here, we recall that one should not use these scalings for too short Δt , as discussed at the end of the previous subsection.

One can reverse the argument here and ask: what physical parameters of the system can be inferred from observations? Obviously, if the spectral slope and the light-curve indexes of the flux

$$F_v(t) \propto v^{\beta_v}(\Delta t)^{\beta_t} \quad (68)$$

are measured in observations, one can readily determine the energy injection index p_* :

$$p_* = \frac{1 - 5\beta_v - 2\beta_t}{1 - 5\beta_v}, \quad (69)$$

where $\beta_v \neq 1/5$, otherwise $s = 3/5$ and $\beta_t = 0$.

3.3. Comparison with Full Numerical Solution

The full numerical solution of Equation (22) and the analytical solutions described in previous sections are shown in Figures 2 and 3. Figure 2 shows the shock radius as a function of time. The solid curve is the exact numerical solution with $p = 7/4$ and the dashed curve represents the approximate self-similar solution given by Equations (28) and (29) with index $p = 0$, which corresponds to the early-time asymptotic of the realistic evolution (see the discussion in Section 3.2.2). The agreement of the self-similar solution with the exact one is remarkable. A noticeable deviation occurs only at the very late time, just before the coalescence time, but even then the difference is within a factor of order unity. Thus, the assumption of $R_{\text{shock}} \rightarrow \text{const}$ as $t \rightarrow t_s$, used in Section 3.2.2 to derive the FTS solution, is justified. We determine numerically that the analytical solution

Figure 3. Log-log plot of the driven-shock velocity as a function of time: t/t_s (left panel) and $1 - t/t_s$ (right panel); note that in the right panel, time evolution is from right to left: the merger occurs at $\Delta t/t_s \rightarrow 0$. The solid curve represents the full numerical solution of Equation (22), the dashed curve is the self-similar solution given by Equations (28) and (29) and $v_{\text{shock}} = \dot{R}(t)$, and the dot-dashed curve is the approximate finite-time-singular solution given by Equations (56) and (57) with a correction factor included (see Equation (71)).

underestimates the final radius (at $t = t_s$) of the shock by a factor of

$$R_{s,\text{exact}}/R_{s,\text{selfsim}} \equiv \chi_R \sim 1.33 \quad (70)$$

which we have included in the estimates of shock parameters and observables.

Figure 3 shows the shock velocity as a function of time: t/t_s (left panel) and of the time before singularity $\Delta t/t_s = (t_s - t)/t_s$ (right panel). The former elucidates the agreement of the exact numerical solution with the self-similar solution and the latter with the FTS analytical solution. Here, the solid curve is the exact numerical solution, the dashed curve represents the approximate self-similar solution given by Equations (28) and (29) with index $p = 0$, and the dot-dashed curve represents the approximate FTS solution. This analytical solution given by Equations (56) and (57) is found to overestimate the velocity by a factor of two, i.e.,

$$v_{s,\text{exact}}/v_{s,\text{FTS}} \equiv \chi_v \sim 0.51. \quad (71)$$

Thus, the analytical solution represented in this figure is the one given by Equation (57) with v_s replaced with $\chi_v v_s$. Note the remarkable agreement between the exact numerical and analytical solutions.

4. CONCLUSIONS

In this paper, we considered the formation and evolution of an astrophysical bubble and the bubble-driven shock propagating in an ambient ISM of uniform density under the assumptions of spherical symmetry of the pressure balance throughout the system. The equation of the dynamics of the shock (and all other parameters) has been accurately derived for an arbitrary energy rate output by the central source, which we colloquially refer to as the “luminosity law,” $L(t)$. Furthermore, we derived the analytical solutions for two special cases: the self-similar scaling, $L(t) \propto t^p$, and the FTS case, $L(t) \propto (t_s - t)^{-p}$, where t_s is the source lifetime, and p is a constant index. The latter “explosive” law can represent, for example, the energy output from a merging NS or magnetar binary. The analytical solution of the FTS type has not been derived before.

We have found that the dynamics of the bubble-driven shock is markedly different from the classical Sedov–von

Neumann–Taylor solution even in the case of the “explosive” FTS behavior. The driven shock solutions only exist if the energy injection rate is not too low, namely, the index p exceeds some critical value, $p > p_c$, which depends on the equations of state of the bubble and the ISM. In particular, for the standard ISM with adiabatic index $\gamma = 5/3$ and the bubble filled with the material with the relativistic equation of state (magnetized plasmas, electron–positron plasmas, and electromagnetic radiation), $\gamma_b = 4/3$, the critical value of p_c is -1.11 for both types of solutions. Otherwise, if $p < p_c$, the bubble expansion is too slow to catch up with the outgoing shock. For $p > p_c$, the self-similar and FTS solutions for the shock radius and velocity are

$$R_{\text{selfsim}}(t) \propto t^{(p+3)/5}, \quad v_{\text{selfsim}}(t) \propto t^{(p-2)/5}. \quad (72)$$

$$R_{\text{FTS}}(t) \propto (t_s - t)^{(p-3)/5}, \quad v_{\text{FTS}}(t) \propto (t_s - t)^{(p-8)/5}. \quad (73)$$

This FTS solution is physical only if $p > 3$; otherwise it is unphysical because it describes a converging shock. For $-1.11 < p < 3$, we have found a physically meaningful approximate solution

$$R_{\text{FTS}}(t) \propto (t_s - t)^0, \quad v_{\text{FTS}}(t) \propto (t_s - t)^{-(p-1)/2}. \quad (74)$$

Interestingly, the derived solutions are also applicable to the class of systems with finite-time but non-singular luminosity laws: $L(t) \propto (t_s - t)^{-p}$ with $-1.11 < p < 0$. These are systems which have an energy deposition rate that declines with time and which are active for a finite time t_s . In systems with $p > 1$, the shock is accelerating as $t \rightarrow t_s$. Therefore, the bubble–shell interface may become Rayleigh–Taylor unstable if a less dense plasma of the bubble is pushing on denser ambient medium. Strong mixing is expected in this case. The overall dynamics may be somewhat affected, but the salient features should remain the same because the assumption of the pressure balance still holds. Another limitation of our analysis is the neglect of relativistic effects, which may be important if the shock velocity is singular at $t \rightarrow t_s$. We have also used the strong shock approximation throughout the analysis, and assumed spherical symmetry. Many processes may destroy such a symmetry or even lead to shell fragmentations, e.g., Richtmyer–Meshkov,

Kelvin–Helmholtz, Vishniac, thermal instabilities, as well as the asymmetry of the outflow from the central source, presence of large-scale magnetic fields in the ISM, turbulent motions and velocity shear in the ambient gas, and clumpiness of the ambient medium. The study of all these processes goes far beyond the scope of the present paper and should be addressed with higher-dimensional numerical simulations.

Finally, we made observational predictions. We calculated the emission light curves of the bubble+shock systems for both self-similar and FTS cases. We predicted that one can deduce the luminosity law index p from the spectral and temporal indexes (see Equations (68) and (69)). Our results may be relevant to stellar systems with strong winds, merging NS/magnetar/black hole systems, as well as massive stars evolving to supernovae explosions.

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