Anabelian Intersection Theory

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Abstract

Let F be a field finitely generated and of transcendence degree 2 over $\overline{\mathbb{Q}}$. We describe a correspondence between the smooth algebraic surfaces X defined over $\overline{\mathbb{Q}}$ with field of rational functions F and Florian Pop's geometric sets of prime divisors on $\mathrm{Gal}(\overline{F}/F)$, which are purely group-theoretical objects. This allows us to give a strong anabelian theorem for these surfaces. As a corollary, for each number field K, we give a method to construct infinitely many profinite groups Γ such that $\mathrm{Out}_{\mathrm{cont}}(\Gamma)$ is isomorphic to $\mathrm{Gal}(\overline{K}/K)$, and we find a host of new categories which answer the Question of Ihara/Conjecture of Oda-Matsumura.

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Acknowledgements

The embryo of the idea for this thesis was conceived at a Trivial Notions talk by Anand Patel, developed limbs on the streets of Hồ Chí Minh City, and gestated at the Hausdorff Institute in Bonn, the Băleanu Institute for Advanced Studies in București, and the Subotić Center for Mathematics in Beograd. Its birth was a long and painful process, which culminated at Harvard and in Turkey. I thank my hosts for their hospitality during my visits, and recognize that this work might not have arrived without the stimulating environment they provided.

I was supported financially by Harvard University, an NSF GRFP stipend, and an ASEE NDSEG fellowship.

Florian Pop explained to me his theory of geometric sets until I could see mold it to "see geometry," and encouraged this project. Ethan Street helped me translate my intuition for non-transverse intersections into language topologists can understand, and David Massey showed me why Reeves' lemma is obvious. Dialogue about valuations with Adam Topaz helped immensely while I was developing this theory. I have also benefited from discussions with Sinan Ünver, David Geraghty, Glenn Stevens, Kürşat Aker, Matt Emerton, Chris Skinner, Adrian Ioviță, Henri Darmon, Tamás Szamuely, Jon Hanke, Kirsten Wickelgren, and Alex Suciu.

I am lucky to have friends whose nonmathematical involvement in my life kept me breathing. Nikola Kamburov has been my roommate through the writing of this thesis, and he has tolerated my focus like a champion. I have been talking to Dan Grin about math, science, career, and life since I was very young, and this dialogue and his friendship has been tremendously important in my development. The opportunity to play in the Dudley Big Band provided Mandatory Fun once a week, and I thank Mike Heller and Jean-François Charles for organizing it throughout my time in graduate school. I have traveled with Roja Najafi, Hossein Namazi, Nikola Kamburov, Sara Silberstein, Rachel Paiste, Steven Sivek, Ana Caraiani, Vašek Cviček, Diana Negoescu, Thomas

Koberda, Antoni Rangachev, Andrei Neguț, Ștefania Bercu, Cristina Popa, Alex Petrescu, Tamara Băleanu, George Zamfir, Simion Filip, Corina Panda, and Sorina Vasile, and being on the road were some of my best memories from this period of my life. Mathematical life in the time I spent in București was lively because of Alex Popa, Vicențiu Pașol, Nicu Beli, and Adrian Diaconu. I am especially thankful to Grandma Eva and Grandpa Fred; the Napa Bronks; the Woodside Bronks; the Mirmonsef-Takloo-Bighash family; the Dağdelen family; the Băleanu family; the Bercu family; the Petrescu family; the Subotić family; the Proistosescu family; the Păcurar family; the Papaioannou family; Đỗ Hoàng Kim; the Körner family; the Grin-Tarlow family; Andreea Nicoara; the Cviček family, and its Los Angelesian extension; the Panda family; the Negoescu family; and the Kamburov family for making me feel at home in so many places around the globe. I have been lucky to have the Boston Bronks — Pete, Susan, Gabe and Johanna - nearby. I entered graduate school with Thomas Koberda, Jack Huizenga, Katy Körner, Tanya Kobylyatskaya, Ivana Bozić, Ana Caraiani, Jonathan Barlev, Ethan Street, Daniel Kane, Yi Li, and Yu-Shen Lin, and I could not have asked for a better cohort. Jam sessions with Adnan, cooking Ethiopian food with Andreea, HONK! with Corina, long walks with Veronika in Cambridge and Sorina in București, Asia and Chevalley-Weil with Thomas, Kyoto with Adam, Andrew, Ascher, Sarah, and Chris, and duck parties with my cohort remain highlights of graduate school. Esmé Wang has been a constant font of support. To anyone else not mentioned here, but who played a part in my life during graduate school: thank you!

Mathematically, I was fortunate to learn groups from Thomas Koberda; algebraic geometry from Jesse Kass and David Smyth; manifolds from John Mather; analysis from H.-T. Yau, Eli Stein, and Paul Hagelstein; number theory from Glenn Stevens, Ramin Takloo-Bighash, Jay Pottharst, Andrew Wiles, Chris Skinner, Djordje Milićević, and Lior Silberman; categories from Cameron Freer; representation theory from Dick Gross; topology from Bill Browder, Sam Isaacson, and Vladimir Voevodsky; and a whole bunch of stuff that I have a hard time quantifying from Curt McMullen. If I omitted your name here, please forgive me.

The students I have been able to work with while at Harvard — especially Krishna Dasaratha, Lucia Mocz, Eva Belmont, Michael Jemison, and John Sheridan — kept me on my toes.

Barry Mazur and Sophie Morel read my thesis and their comments and attention proved invaluable in ensuring its legibility and correctness.

Florian Pop has been both the single most important mathematical influence on my life and a dear friend, and I am lucky to have met him.

Dick Gross made sure I made certain correct choices at Harvard, and I am continually grateful for his wisdom.

Mark Goresky has provided support at every step of my career, and his mathematical influence on me cannot be underestimated.

I am in mathematics because of Glenn Stevens, and he has been a fantastic mentor and a true friend since I was first interested in mathematics. I learned much of the basic mathematics I know from David Fried and Steve Rosenberg. The friends I made through PROMYS — especially Dustin Clausen, Dan Jerison, Agnès Beaudry, Claudia Scheimbauer, Ila Varma, Kevin Hughes, David Coley, Jon Hanke, Ander Steele, Big Papa, Maya Vinokour, Cameron Freer, Lucas Culler, Seo Hyung Kim, and Jay Pottharst — have a very special place in my heart. Jay Pottharst and Cameron Freer, especially, have been of great mathematical and moral support to me while this thesis was being written.

Finally, I dedicate this thesis to my family — my parents, Joanie and Rich, and my brother and sister, Brandon and Sara — whose moral and material support allowed me to become a mathematician.

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Chapter 1

Introduction

1.1 Grothendieck, Torelli, and Schottky

Since the dawn of what we now know as algebraic geometry, it has been recognized that there are two types of fundamental invariants of algebraic varieties: "discrete" invariants, and "continuous" invariants. Implicit in the work of Riemann is a phenomenon that has dictated the shape of algebraic geometry:

1. Discrete invariants are topological.

2. Continuous invariants are geometric, analytic, or algebraic.

These discrete invariants are usually given by the machinery of homology/cohomology and fundamental groups. The continuous invariants are then realized as **structures** on these objects; one of the most productive, for instance, is a Hodge structure.

Nowhere is this more evident than in the theory of projective algebraic smooth curves C over the complex numbers $\mathbb{C}$:

1. The complex points $C(\mathbb{C})$ endowed with the complex topology form an oriented, compact,

topological surface. There is a single "discrete invariant" of a smooth, projective curve C: a positive integer, the **genus** of the underlying compact, topological surface $C(\mathbb{C})$:

$$g_C = \frac{1}{2}\dim H_1(C(\mathbb{C}),\mathbb{R}).$$

If C_1, C_2 are two algebraic curves with the same genus,

(a) For any two points $p_1 \in C_1(\mathbb{C}), p_2 \in C_2(\mathbb{C}), \pi_1(C_1, p) \simeq \pi_1(C_2, p')$; in fact, $C_1(\mathbb{C})$ and $C_2(\mathbb{C})$ are Eilenberg-Mac Lane spaces, and are diffeomorphic.

(b) There exists a smooth, but not necessarily projective, algebraic curve C', two points p_1, p_2 on C', and a smooth, algebraic variety $\mathcal{X}$ with a morphism $\varphi : \mathcal{X} \to C'$ such that

   i. $\varphi^{-1}(p_1) \simeq C_1$ and $\varphi^{-1}(p_2) \simeq C_2$.

   ii. φ is, topologically, a fibration.

   Here, the "discrete invariant" traditionally understood as a number is fundamentally used in its guise as a discrete group.

2. To determine a specific algebraic curve from all the algebraic curves sharing its genus (and thus fundamental group), we only need to look at the Hodge structure on the Lie algebra of the maximal 2-step nilpotent quotient of its topological fundamental group; this is the classical Torelli theorem [And58]. Classically, this would be said as: the **periods** of an algebraic curve — a set of complex numbers, which can vary continuously — determine the algebraic curve.

3. There are criteria by which we can determine whether a Hodge structure on such an abstract Lie algebra comes from a curve, but this is a much more open-ended problem. This is the Schottky problem [vdG85].

This gives rise to two general classes of problems. The first, which we call the **Torelli problem**, is:

Problem 1. *Given two algebraic varieties X_1, X_2 with a given set of "discrete invariants," how much structure do we need to add to these invariants in order to determine whether $X_1 \simeq X_2$?*

On the other hand, consider the problem of Schottky:

Problem 2. *Given a class of algebraic varieties $\{X_i\}$ which share a topological invariant Γ, how do we know when an algebraically induced structure on Γ comes from some X_i?*

These are two central questions in algebraic geometry, and I do not believe there currently exists a satisfactorily general and precise formulation of either of them.

In his *Esquisse d'un Programme* [Gro97b] and letter to Faltings [Gro97a], Grothendieck put forward a program for translating questions in arithmetic and algebraic geometry into questions about profinite groups via $\pi_1^{\text{ét}}$, the étale fundamental group.

In general, let X be a geometrically integral, algebraic K-variety, $X_{\overline{K}}$ its base-change to $\overline{K}$, the algebraic closure of K, and $\overline{x}$ a geometric point of X. Then $\pi_1^{\text{ét}}$ sits in a canonical short exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\overline{K}}, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x}) \to \pi_1^{\text{ét}}(\operatorname{Spec} K, \overline{x}) \to 1.$$

Since $\pi_1^{\text{ét}}(\operatorname{Spec} K, \overline{x}) \simeq G_K$, the absolute Galois group of K, conjugation by the quotient gives an outer action of G_K on $\pi_1^{\text{ét}}(X_{\overline{K}}, \overline{x})$.

The "yoga" of Grothendieck's anabelian geometry is that if the fundamental group $\pi_1^{\text{ét}}(X)$ is "rich enough", then it should encode much of the information about X as a variety; such varieties X should be called **anabelian in the sense of Grothendieck**. Some of the questions central to anabelian geometry are:

1. What kind of information about X is captured by $\pi_1^{\text{ét}}(X, \overline{x})$ and, in particular, which X are anabelian?
2. What group-theoretic properties characterize the étale fundamental groups among all profinite groups?
3. Which maps between étale fundamental groups $\pi_1^{\text{ét}}(X, \overline{x}) \to \pi_1^{\text{ét}}(X', \overline{x}')$ come from maps of their underlying varieties?

Indeed, Grothendieck was inspired by arithmetic; Hodge theory can be mimicked through Galois actions, and he thought that an étale fundamental group being "rich enough" involved an arithmetic Galois action. In short, fundamental groups — broadly interpreted — are complicated/interesting enough in arithmetic settings that they can incorporate both topological and continuous invariants. In fact, proving a variety is anabelian proves that it satisfies a generalized Torelli problem. The second of the above questions is a strong Schottky problem; there is very little progress on this problem, however.

1.2 Reconstruction Techniques for Fields: Valuations, and Galois Theory

As we will use it, a **reconstruction technique** is an algorithm for recovering an algebro-geometric object from its Galois or fundamental group by using only group-theoretic recipes on that group. The goal of this thesis is to describe a new reconstruction technique for function fields of surfaces, which makes fundamental use of classical algebraic geometry.

We start out with some reconstruction techniques which arise from valuation theory. The idea that valuations in an arithmetical context could be used to understand the relationship between a field and its absolute Galois group goes back to Neukirch, who gave the recipe for recovering decomposition and inertia groups of valuations on number fields from their absolute Galois groups [Neu69]. This was then developed in [Uch77] and [Ike77] (and Iwasawa, in unpublished work) to give the following theorem (which is also exposited in [Neu77]):

Theorem 3. *Let K_1, K_2 be number fields. Then $\mathrm{Isom}(K_2, K_1)$ is in natural bijection with $\mathrm{Isom}^{\mathrm{Out}}(G_{K_1}, G_{K_2})$, the quotient of the isomorphism set between their absolute Galois groups by conjugation by G_{K_2}.*

There is a corresponding theorem for global fields of positive characteristic, but it is slightly more complicated and we will not be dealing with the effects of positive characteristic at all in this thesis. Pop [Pop94, Sza04], finally, extended these techniques to show

Theorem 4. *Let K_1, K_2 be finitely generated fields of characteristic zero. Then $\mathrm{Isom}(K_2, K_1)$ is in natural bijection with $\mathrm{Isom}^{\mathrm{Out}}(G_{K_1}, G_{K_2})$, the quotient of the isomorphism set between the two groups by conjugation by G_{K_2}.*

In the early 1990s, Bogomolov suggested that birational anabelian phenomena should manifest also for higher-dimensional function fields over algebraically closed basefields, thus in total absence of arithmetic Galois actions [Bog91]. The strongest evidence of this is Bogomolov's idea that **commuting liftable pairs** in Galois groups of certain fields originate from a special class of valuations. The ideas in valuation theory are developed in [Bog91], [BT11], [Pop10], and [Pop11a]. We will take the recent preprint of Topaz [Top12] as the definitive reference, as it encompasses, generalizes, and unifies all previous work on the valuation-theoretic side of this program.

Let now F be a field, finitely-generated over $\overline{\mathbb{Q}}$, and of transcendence degree 2. We will construct a natural equivalence of categories

$$\mathcal{M} : \mathfrak{G}\mathfrak{Bir}(F) \to \mathfrak{Bir}(F),$$

where $\mathfrak{Bir}(F)$ is the category whose objects are smooth varieties with function field F and whose morphisms are birational maps between such varieties; and $\mathfrak{GBir}(F)$ is a category whose objects are constructed and whose morphisms are defined by purely group-theoretical recipes from G_F, the absolute Galois group of F. We do not merely give the category $\mathfrak{GBir}(F)$; from the group theory, given an object $\mathfrak{X}$ of $\mathfrak{GBir}(F)$ and its corresponding model $X = \mathcal{M}(\mathfrak{X})$, we have explicit recipes to:

1. Determine if X is proper.

2. Compute Betti numbers for X and if X is proper, compute Hodge numbers.

3. Compute $\mathrm{NS}(X)$, the Néron-Severi group of X, if X is proper.

4. Compute the closed points of X.

5. Identify the prime divisors on X and compute local and global intersection numbers of divisors.

6. Calculate global sections of line bundles associated to divisors on X if X is proper.

Arithmetic reconstruction techniques rely, essentially, on the theory of curves. These approaches make a lot of sense from both Grothendieck's original perspective — in which anabelian varieties should be fibrations of curves, and in which the Galois action is fundamental. As a Galois action on a fundamental group can imitate Hodge theory, one point of view is that one could try to push through a Torelli theorem for curves given this Galois action; see [Moc99]. On the other hand, from a field-theoretic perspective, the relative valuation theory of a field extension of transcendence degree 1 is immeasurably simpler than the relative valuation theory of a field extension of transcendence degree > 1 (see, for instance, [Pop94]). The proof of birational theorems is an induction process, where the base case is the theorem of Neukirch-Uchida-Ikeda-Iwasawa

(Kronecker dimension 1), and the induction step on Kronecker dimension uses the theory of relative curves.

On the other hand, in the geometric situation, the reduction to curves is not possible, as the absolute Galois group of the function field of a curve over an algebraically closed field of characteristic zero depends only on the cardinality of the basefield, and in particular does not carry any information — geometric or otherwise — about the curve. The strategy used in, for instance [Pop10], is to recover the function field directly from its absolute Galois group, by first recovering the group of principal divisors of the field, and then interpreting this as the projectivization of the additive group of the function field. Once we have this, we may deduce, by using the fundamental theorem of projective geometry, the full field structure.

We suggest in this thesis a paradigm shift: we look for varieties inside of birational fundamental groups, rather than their function fields. Underlying this is the fundamental conviction that the algebraic geometry encoded in the Galois groups we consider is essentially classical in nature, and that the dictionary between Geometry and Galois Theory is more reminiscent of the development of algebraic geometry in [Wei46] than of [Gro60]. Following this viewpoint, we take a variety as being an object defined algebraically as a collection of subrings of its function field, so our Galois-theoretic analogues of varieties will be objects derived from the absolute Galois group of such a function field. As the algebra of the field holds all the rings of functions on the affine patches of a variety together, we will define "affine patches" group-theoretically, which are "held together" by the absolute Galois group of the function field. The idea that fundamental groups "see" intersection theory is implicit in early work of Zariski [Zar29]. On the other hand, our reconstruction techniques fundamentally rely on what one might consider traditionally anabelian varieties — iterated fibrations of curves. In working in this larger group — the Galois group of the function field — however, we have a single object from which we can encode not only all anabelian varieties birational to a given one, but how they fit together to form arbitrary varieties, which are not necessarily anabelian.

The second chapter of the thesis reviews the work already done to complete Bogomolov and Pop's program to recover valuations for $\overline{\mathbb{Q}}$. We do not give proofs or develop the theory of commuting liftable pairs as in [Pop11b] and [Top12]; rather, we take these procedures as a black box for the rest of the reconstruction. In particular, given the function field F of a smooth surface X defined over $\overline{\mathbb{Q}}$, we need to define a certain class of valuations on F — the **Parshin chains** — which have geometric meaning. The inspiration — the classical (partial) Parshin chains — are defined on a variety X over an algebraically closed field K, and are defined as chains

$$X = D_0 \supset D_1 \supset D_2 \supset D_3 \cdots \supset \dots D_n$$

where $n \leq \dim X$, and this n is called the **rank** of the chain. Each D_i is a divisor on D_{i-1}. The valuations we are interested in are birational versions of the classical Parshin chains. These are Parshin chains on a variety birationally equivalent to X. This definition has, then a completely field-theoretic description in terms of valuations. A Weil prime divisor on a smooth variety birationally equivalent to X gives a valuation v on $K(X)$, discrete, trivial on K, and with no transcendence defect: the transcendence degree of the residue field of v over K is one less than the transcendence degree of $K(X)$ over K. Such a valuation is called a **prime divisor** of $K(X)$. Conversely, for each prime divisor of $K(X)$ there is a Weil prime divisor on a smooth variety birationally equivalent to X. We define the rank one birational Parshin chains to be the prime divisors on $K(X)$. The birational notion of a prime divisor v_2 "on a prime divisor v_1" — to make longer Parshin chains — will be a prime divisor v_2 on the residue field of v_1. A birational Parshin chain will be defined to be a composite $v_n \circ v_{n-1} \circ \cdots \circ v_1$ of valuations, where v_i is a prime divisor on the residue field of v_{i-1}. Since we work exclusively with birational Parshin chains, we drop the word "birational," and we will refer to the valuation-theoretic notion only as a Parshin chain.

Each valuation on $K(X)$ has a decomposition and inertia group in the absolute Galois group $G_{K(X)}$ of $K(X)$. In the case of surfaces, we describe the geometric and algebraic interpretations of the decomposition and inertia groups corresponding to a Parshin chain. We then recall that there is a recipe to determine, purely from the group theory of $G_{K(X)}$, whether a given subgroup is the decomposition or inertia group of a Parshin chain.

For any model X' of the function field $K(X)$, the corresponding set $\mathcal{S}_{X'}$ of all the prime divisors on $K(X)$ (that is, considered as valuations), whose centers are codimension 1 on X', is called the **geometric set** of prime divisor defined by X'. In general, we say that a set of prime divisors is a geometric set if it is equal to $\mathcal{S}_{X'}$ for some smooth model X'. We recall that there is a group-theoretic algorithm to determine whether a set of subgroups is exactly the set of inertia groups corresponding to the elements of a geometric set for some model X' of $K(X)$.

The third chapter concerns the reconstruction procedure itself, which goes in roughly four steps:

1. Let D_1, D_2 be divisors on a smooth surface X over $\overline{\mathbb{Q}}$ with function field F, and let p be a point on D_1. Hence, D_2 determines a rank-1 Parshin chain on F; and the pair (D_1, p) determines a rank-2 Parshin chain on F. We show how to compute a *group-theoretic* local intersection number of D_1 and D_2 at p using the inertia groups of the Parshin chains corresponding to D_2 and (D_1, p) and the geometric set $\mathcal{S}_X$ as structures on G_F, which agrees with the geometrically defined local intersection number $(D_1, D_2)_p$ as defined in, for instance, [Ful98], subject to certain global geometric conditions.

2. Using Hodge theory, we show that there is a group-theoretical recipe to determine when these global geometric conditions hold, so we can determine when a group-theoretically defined intersection number means what it is supposed to, as long as it is defined at a point

on the interior of the maximal smooth model.

3. We then use this intersection theory to detect when the maximal smooth model of a geometric set is a fibration of curves by curves.

   To see one reason this is subtle, consider the following question, to which I suspect a positive answer, but which I cannot produce: Let K be a countable, algebraically closed field of characteristic zero. Let X be a quasi-projective, algebraic surface over K (hence reduced, separated, etc.). Let $\{C_i\}_{i \in \mathbb{N}}$ be a set of prime divisors of X, such that:

   (a) $\bigcup_{i \in \mathbb{N}} C_i(K) = X(K)$.

   (b) $C_i(K) \cap C_j(K) = \emptyset$ for $i \neq j$.

   (c) All C_i have the same genus and number of punctures (that is, are in the same connected component of the moduli of curves).

   Do these C_i then form an algebraic family within X?

4. We then can finish the reconstruction, in two different ways. We take a geometric set whose maximal smooth model is proper, determine which rank-1 Parshin chains on it correspond to very ample divisors, and we use the Fundamental Theorem of Projective Geometry, as in [Art88], to reconstruct the projective coordinate ring of that maximal smooth model group-theoretically. On the other hand, we can cover the variety with fibrations of curves over curves as in the last step, in order to construct a topological model — and in fact, to reconstruct its étale site — as a space glued out of open sets that are $K(\pi, 1)$'s.

One end result of this is the **Birational Anabelian Theorem for Surfaces over $\overline{\mathbb{Q}}$**:

Theorem 5 (The birational anabelian theorem for surfaces over $\overline{\mathbb{Q}}$)**.** *Let F_1 and F_2 be two fields, finitely-generated and of transcendence degree 2 over $\overline{\mathbb{Q}}$. Then the natural map*

$$\varphi : \mathrm{Isom}(F_2, F_1) \to \mathrm{Isom}^{\mathrm{Out}}_{\mathrm{cont}}(G_{F_1}, G_{F_2})$$

is a bijection (where Isom *denotes all field-theoretic isomorphisms, and not merely the $\overline{\mathbb{Q}}$-isomorphisms); and in particular, F_1 and F_2 are isomorphic as fields if and only if their absolute Galois groups are isomorphic as topological groups.*

This result goes beyond the (unpublished) [Pop11a], where the Birational Anabelian Theorem is proven for transcendence degree ≥ 3. Our result can also be used as the base case in an induction procedure to reprove the Birational Anabelian Theorem for higher-dimensional varieties; we will not do that in this thesis, however.

Recall that a variety X over a field k is **birationally rigid** if $k(X)$ has no nontrivial field automorphisms, and **geometrically, birationally rigid** if there are no nontrivial $\overline{k}$-automorphisms of $\overline{k}(X)$. Note that if X is birationally rigid and geometrically, birationally rigid, then $\mathrm{Aut}(\overline{k}(X)) = G_k$. We have then as an immediate corollary of Theorem 5.

Theorem 6. *Let k be a number field. Let X be a birationally rigid and geometrically, birationally rigid surface over k. Then* $\mathrm{Out}_{\mathrm{cont}}(G_{\overline{k}(X)}) \simeq G_k$.

The (short) fourth chapter involves writing down explicitly surfaces which satisfy the hypotheses of Theorem 6.

The conjecture of Ihara/Oda-Matsumoto is:

Conjecture 7. *Let $\mathcal{V}_{/k}$ be the category of geometrically-based varieties over a number field k. Let $\pi_1^{\text{ét}}$ be the étale fundamental group functor*

$$\pi_1^{\text{ét}} : \mathcal{V}_k \to \mathbf{ProfGrp},$$

taking values in the category of profinite groups. Then

$$G_k \simeq \mathrm{Out}(\pi_1^{\text{ét}}),$$

where Out *is the automorphisms of the functor modulo the automorphisms gotten by compatible systems of inner automorphisms.*

For a proof of Conjecture 7 and its history, see [Pop11b]. We may deduce, by replacing the partial birational reconstruction in [Pop11b] with Theorem 5, a refinement of Conjecture 7 (which, by [Pop11b] proves Conjecture 7):

Theorem 8. *Let X be a birationally rigid and geometrically, birationally rigid surface over k, and let $\mathcal{V}$ be any full subcategory of $\mathcal{V}_k$ so that for any open subvariety $U \subseteq X$, there is an open subset $U' \subseteq U \subseteq X$ such that U' is an object of $\mathcal{V}$. Then*

$$G_k \simeq \mathrm{Out}(\pi_1^{\text{ét}}|_{\mathcal{V}}).$$

As a corollary of geometric reconstruction, we may obtain a result in this direction which we cannot access using birational reconstruction.

Theorem 9. *Let $\mathcal{V}$ be the full category containing all the open subsets of $\mathcal{M}_{0,5}$, the moduli space of 5 points in $\mathbb{P}^1$, as in [HS00], and enough maps $\varphi : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$ to kill all automorphisms of the full subcategory of $\mathcal{V}$ containing $\mathcal{M}_{0,5}$ and $\mathcal{M}_{0,4}$. Then*

$$G_{\mathbb{Q}} \simeq \mathrm{Out}(\pi_1^{\text{ét}}|_{\mathcal{V}}).$$

This theorem implies the main result of [Pop11b].

Chapter 2

An Anabelian Cookbook

2.1 The Geometric Interpretation of Inertia and Decomposition Groups of Parshin Chains

Definition 10. *Let F be a field finitely generated over some algebraically closed field K of characteristic zero; such a field will be called a **function field**. Note that the field K is determined by F (for instance, its multiplicative group is the set of all divisible elements in the multiplicative group of F); it will be denoted by $K(F)$, and be called the **field of constants** of F. The transcendence degree of F over $K(F)$ will be called the **dimension** of F. We will denote by G_F the absolute Galois group of F and $\overline{F}$ the algebraic closure of F.*

For general theory of valuations, including proofs of the algebraic theorems cited without proofs, see [FJ08]. We will also use results from [Ser56] and [Gro03] with impunity.

Definition 11. *A **valuation** v on F is an ordered group $(vF, \leq)$, called the **value group** of v, along with a surjective map*

$$v : F \to vF \cup \{\infty\} \tag{2.1}$$

which satisfies

1. $v(x) = \infty$ *if and only if* $x = 0$.

2. $v(xy) = v(x) + v(y)$ *(here, we define* $\infty + g = \infty$ *for all* $g \in vF \cup \{\infty\}$*.)*

3. $v(x + y) \geq \min\{v(x), v(y)\}$, *where we extend the ordering* $\leq$ *to* $vF \cup \{\infty\}$ *by* $g \leq \infty$ *for all* $g \in vF$.

A valuation v *gives rise to a* ***valuation ring*** $\mathcal{O}_v$, *which is the set of all* $x \in F$ *such that* $v(x) \geq 0$. $\mathcal{O}_v$ *is integrally closed in* F *and local, and we call its maximal ideal* $\mathfrak{m}_v$*. We then define the* ***residue field*** *to be*

$$Fv = \mathcal{O}_v/\mathfrak{m}_v, \tag{2.2}$$

A subring $R \subseteq F$ *is a valuation ring* $\mathcal{O}_w$ *for some valuation* w *if and only if for every* $x \in F^\times$ *either* $x \in R$ *or* $x^{-1} \in R$*. Therefore, equivalently, we may define a valuation* v *by its* ***place***

$$p_v : F \to Fv \cup \{\infty\} \tag{2.3}$$

where $\mathcal{O}_v$ *is mapped to its reduction mod* $\mathfrak{m}_v$ *and* $F \setminus \mathcal{O}_v$ *is mapped to* ∞*. Any map from a field* F *to another field* L *which is a ring homomorphism "with* ∞*" thus gives rise to a valuation on* F.

Definition 12. *Two valuations will be called* ***equivalent*** *if and only if they have the same valuation ring.*

Definition 13. v *is a valuation on* F *and* w *is a valuation on* Fv*, then we may define a valuation* $w \circ v$ *on* F *by considering the composition*

$$p_w \circ p_v : F \to (Fv)w \cup \{\infty\} \tag{2.4}$$

*as a place map on F. This valuation is called the **composition** of w with v.*

Definition 14. *A **model** X of a function field F is a smooth, connected $K(F)$-scheme of finite-type with a map*

$$s_X : \operatorname{Spec} F \to X, \tag{2.5}$$

*the **structure map** of the model, which identifies F with the field of rational $K(F)$-functions on X. A **normal model** is a model with the requirement of smoothness replaced by normality.*

Definition 15. *By virtue of the structure map, the models form a full subcategory of the category of schemes under $\operatorname{Spec} F$. We define an F**-morphism** of models to be a morphism of varieties under $\operatorname{Spec} F$, and $\mathfrak{Bir}(F)$ the full subcategory of varieties under $\operatorname{Spec} F$ whose objects are precisely the models of F.*

Definition 16. *We say that a valuation v on F **has a center** or **is centered** on X if X admits an affine open subset $\operatorname{Spec} A$ such that $A \subset \mathcal{O}_v$, the valuation ring of v. Let $R \subset F$ be a subring giving an affine open $\operatorname{Spec} R \subseteq X$. Then the **center** of v on $\operatorname{Spec} R$ is the Zariski closed subset of X defined as $Z(\mathfrak{m}_v \cap R)$; the center of v on X is the union of the centers of v on $\operatorname{Spec} R$ as $\operatorname{Spec} R$ ranges over all affine opens of X. We denote by $|v|$ the center of v.*

Definition 17. *A **prime divisor** v on F is a discrete valuation trivial on $K(F)$ such that*

$$\text{tr. deg.}_{K(F)} Fv = \text{tr. deg.}_{K(F)} F - 1. \tag{2.6}$$

*(This condition is very important in birational anabelian geometry in general, and we say that v has **no transcendence defect**; see [Pop94] for more details.)*

Definition 18. *A **rank-**1 **Parshin chain** for F is a prime divisor. A **rank-**n **Parshin chain** is a composite $w \circ v$, where v is a rank-$n-1$ Parshin chain, and w is a prime divisor on Fv.*

Definition 19. *We denote by* $\mathrm{Par}_i(F)$ *the collection of* i*-Parshin chains for* F*. Given a rank-*k *Parshin chain* v *we denote by* $\mathrm{Par}_i(v)$ *the collection of rank-*i *Parshin chains of the form* $w \circ v$*.* $\mathrm{Par}_i(v)$ *is empty if* $i \leq k$*; is* $\{v\}$ *if* $i = k$*; and is infinite if* $i \geq k$*. If* $S \subseteq \mathrm{Par}_i(F)$ *we define*

$$\mathrm{Par}_i(S) = \bigcup_{v \in S} \mathrm{Par}_i(v). \tag{2.7}$$

We also let

$$\mathrm{Par}(S) = \bigcup_i \mathrm{Par}_i(S) \textit{ and } \mathrm{Par}(F) = \bigcup_i \mathrm{Par}_i(F). \tag{2.8}$$

Example 20. *To describe the rank-*1 *Parshin chains on* F*, where* F *is the function field of a* ***surface*** *over* $\overline{\mathbb{Q}}$*, we consider the set of all pairs* (X, D) *where* X *is a* ***proper*** *model of* F *and* D *is a prime divisor on* X*. On this collection, we have an equivalence relation generated by the relation*

$$(X, D) \sim (X', D')$$

if there exists a dominant map

$$\varphi : X \to X'$$

respecting the structure maps of the models X *and* X' *such that* D *is mapped birationally to* D' *by* φ*.*

The rank-2 *Parshin chains are then equivalence classes* (X, D, p) *where* D *is a prime divisor on* D *and* p *is a smooth point on* D*, and the equivalence relation is now generated by*

$$(X, D, p) \sim (X', D', p')$$

if

$$\varphi : X \to Y$$

so that D is mapped birationally to D' by φ and $\mathfrak{p}$ is mapped to $\mathfrak{p}'$ by φ.

Given an algebraic extension $L|F$ every valuation extends to L, though not necessarily uniquely.

Definition 21. *We define $\mathcal{X}_v(L|F)$ to be the set of valuations on L which restrict to v on F. If $L|F$ is Galois, then $\mathrm{Gal}(L|F)$ acts transitively on $\mathcal{X}_v(L|F)$. For any Galois extension $L|F$ and $\tilde{v} \in \mathcal{X}_v(L|F)$ we define the* ***decomposition group***

$$D_{\tilde{v}}(L|F) = \{\sigma \in \mathrm{Gal}(L|F) \mid \sigma(\mathcal{O}_{\tilde{v}}) = \mathcal{O}_{\tilde{v}}\}\,. \tag{2.9}$$

Each $D_{\tilde{v}}(L|F)$ has a normal subgroup, the ***inertia group*** *$T_{\tilde{v}}(L|F)$, defined as the set of elements which act as the identity on $L\tilde{v}$.*

We have a short-exact sequence, the **decomposition-inertia exact sequence**

$$1 \to T_{\tilde{v}}(L|F) \to D_{\tilde{v}}(L|F) \to \mathrm{Gal}(L\tilde{v}|Fv) \to 1. \tag{2.10}$$

If $\tilde{v}_1, \tilde{v}_2 \in \mathcal{X}_v(L|F)$ and $\tilde{v}_1 = \sigma\tilde{v}_2$ for some $\sigma \in \mathrm{Gal}(L|F)$, then

$$D_{\tilde{v}_1}(L|F) = \sigma^{-1} D_{\tilde{v}_2}(L|F)\sigma \text{ and } T_{\tilde{v}_1}(L|F) = \sigma^{-1} T_{\tilde{v}_2}(L|F)\sigma.$$

Thus, all $D_{\tilde{v}}(L|F)$ and $T_{\tilde{v}}(L|F)$, respectively, are conjugate for a given v, and when the lift is not important, we denote some element of the conjugacy class of subgroups by $D_v(L|F)$ and $T_v(L|F)$, respectively. We define D_v and T_v, respectively, to be $D_v(\overline{F}|F)$ and $T_v(\overline{F}|F)$.

For any Galois extension $L|F$, a valuation v on F and a valuation w on Fv, we may choose $\tilde{v} \in \mathcal{X}_v(L|F)$ and $\tilde{w} \in \mathcal{X}_w(L\tilde{v}|Fv)$. There is then a natural short exact sequence, the **composite**

inertia sequence:

$$1 \to T_{\tilde{v}}(L|F) \to T_{\tilde{w}\circ\tilde{v}}(L|F) \to T_{\tilde{w}}(L\tilde{v}|Fv) \to 1, \tag{2.11}$$

where $T_w \subseteq G_{Fv}$, for any composite of valuations. Thus, if $T_{\tilde{v}}(L|F)$ is trivial,

$$T_{\tilde{w}} \simeq T_{\tilde{w}\circ\tilde{v}}.$$

We will use three different types of fundamental groups, with the following notation:

1. π_1^{top} is the topological fundamental group, the fundamental group of a fiber functor on the category of topological covers of a topological space.
2. $\hat{\pi}_1$ is the profinite completion of π_1^{top}, the fundamental group of a fiber functor on the category of *finite* topological covers of a topological space.
3. $\pi_1^{\text{ét}}$ is the étale fundamental group.

For a normal variety X over an algebraically closed subfield of $\mathbb{C}$, one has an equivalence between the category of finite, étale covers of X and the finite, unramified covers of X^{an}, its corresponding analytic space over $\mathbb{C}$, by [GR57]. This leads immediately to the

Theorem 22 (Comparison Theorem)**.** *Let X be a normal variety over $\mathbb{C}$ and $x \in X(\mathbb{C})$. There is a canonical isomorphism*

$$\hat{\pi}_1(X^{\text{an}}, x) \simeq \pi_1^{\text{ét}}(X, x). \tag{2.12}$$

Every known computation of nonabelian fundamental groups of varieties factors through this comparison theorem; in characteristic p, for instance, this is combined with Grothendieck's specialization theorem [Gro03, X.2.4] to obtain information about fundamental groups.

Let now $K = \mathbb{C}$ and F be a function field over $\mathbb{C}$. Then we have the following interpretation of D_v for a prime divisor. First, v is the valuation associated to a Weil prime divisor on some normal model X of F — that is, a normal variety with function field F, considered as a $\mathbb{C}$-scheme. Given X, there is a corresponding normal analytic space X^{an}. Let X be a model of F on which $|v|$ is a prime divisor.

Example 23. *The exceptional divisor E on $\mathrm{Bl}_p(X)$, the blowup at some closed point p of X, gives a prime divisor on F but its center on X is not codimension 1 and so is not centered as a prime divisor on X.*

Let $D' \subseteq X$ be the nonsingular locus of $|v|$; notice that the underlying topological space of D' is connected, as resolution of singularities gives an isomorphism with it and an open subset of a smooth algebraic curve.

Definition 24. *Let $\mathcal{N}$ then be a normal disc bundle for (equivalently, a tubular neighborhood of) D' and*

$$\mathcal{T} = \mathcal{N} \setminus D'$$

the complement of D' in its normal disc bundle $\mathcal{N}$, which admits the ***normal bundle fiber sequence***

$$1 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} D' \longrightarrow 1, \tag{2.13}$$

where $\mathcal{F}$ is one of the fibers.

Let p be a point on $\mathcal{F}$. Note that $\mathcal{F}$, like all fibers of π, is a once-punctured disk and thus is homotopy equivalent to a circle. There is a surjection

$$\rho : G_F \to \pi_1^{\text{ét}}(X, p) \simeq \hat{\pi}_1(X^{\mathrm{an}}, p) \tag{2.14}$$

(proof: each normal étale cover of X gives a normal extension of its field of functions) whose kernel we will define to have fixed field F_X, and the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\pi}_1(\mathcal{F}, p) & \xrightarrow{\iota} & \hat{\pi}_1(\mathcal{T}, p) & \xrightarrow{\pi} & \hat{\pi}_1(D', \pi(p)) & \longrightarrow & 1 \\ & & & \searrow & \downarrow & & & & \\ & & & & \pi_1^{\text{ét}}(X, p). & & & & \end{array} \tag{2.15}$$

Then we have

Proposition 25 (The Geometric Theory of Decomposition and Inertia Groups)**.** *In the short exact sequence 2.15:*

1. *The top row is a central extension of groups, as the normal bundle is complex-oriented.*

2. *The image of $\hat{\pi}_1(\mathcal{F}, p)$ in $\pi_1^{\text{ét}}(X, p)$ is a $T_v(F_X|F)$.*

3. *The image of $\hat{\pi}_1(\mathcal{T}, p)$ in $\pi_1^{\text{ét}}(X, p)$ is a $D_v(F_X|F)$.*

4. *$\hat{\pi}_1(D', \pi(p))$ is a quotient of G_{F_v}, corresponding to covers of D' pulled back from covers of X.*

We will denote by t_v a generator of $\pi_1^{\text{top}}(\mathcal{F}, p) \subseteq \hat{\pi}_1(\mathcal{F}, p)$, as well as any of its images in $\pi_1^{\text{ét}}(X, p)$ or $T_v(F_X|F)$.

Definition 26. *We refer to such a t_v as a* ***meridian*** *of v.*

Each meridian is almost unique — its inverse also gives a meridian of v, albeit "in the opposite direction". This should be viewed as a "loop normal to $|v|$". Its image generates $\hat{\pi}_1(\mathcal{F}, p)$. In general, if we are working in a situation in which we do not specify a basepoint, the meridian becomes defined only up to conjugacy.

Definition 27. *Let Γ be a subgroup of a group Π. Then the abelianization functor gives a map*

$$\mathrm{ab} : \Gamma^{\mathrm{ab}} \to \pi_1^{\mathrm{ab}}. \tag{2.16}$$

We denote by Γ^a the image of ab. In particular, given v a valuation, and Π a quotient of G_F or $\pi_1^{\mathrm{top}}(X)$ for some model of X, we will denote by T_v^a and D_v^a the images of inertia and decomposition, respectively, in Π^{ab}, which will sometimes appear in the sequel as H_1. We let t_v^a be the image of a meridian in Π^{ab}.

We can also define the meridian of a valuation v on a model X if $|v|$ is smooth, and extend the definition to non-smooth $|v|$ as follows. We resolve the singularities of $|v|$ on X to get a birational map

$$\eta : \tilde{X} \to X$$

such that

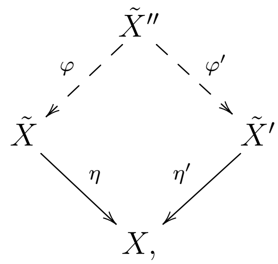
1. η is an isomorphism outside of $|v|$.

2. $|v| \subseteq \tilde{X}$ is smooth.

Then we define a meridian t_v on X to be

$$t_v = \eta_*(t_v) \tag{2.17}$$

where t_v is a meridian on $\tilde{X}$. To see this is well-defined, if we have two such maps η, η' as in the following diagram:



(2.18)

we may always construct φ and φ' birational morphisms so that:

1. The above diagram commutes.

2. $\varphi, \varphi', \eta \circ \varphi, \eta' \circ \varphi'$ are isomorphisms outside of $|v|$.

3. $|v|$ is smooth in $\tilde{X}''$.

In this case, the meridians in X defined by η' and η are the image of a meridian in $\tilde{X}''$ under $\eta \circ \varphi$ and $\eta' \circ \varphi'$ so, by commutativity of the diagram, the two meridians are the same.

We now describe the relation with composites of valuations. Let v be a prime divisor on F and Fv its residue field. Let w be then a prime divisor on Fv. A meridian t_w in $G(L\tilde{v}|Fv)$ for some model C of Fv then for each lift $\tilde{w}$ of w to $L\tilde{v}$ thereto a unique $t_{w \circ v}$ in $T_{\tilde{w} \circ \tilde{v}}(L|F)$ and in any model is a loop normal to $|w|$ lying completely on $|v|$.

Definition 28. *A **meridian of a rank-**2 **Parshin chain** $t_{w \circ v}$ for $L|F$ is defined to be such a lift. As per Definition 27, any image in an abelianization will be denoted $t^a_{w \circ v}$.*

Let F have dimension n. Then if v is an $n-1$-dimensional Parshin chain, Fv is the function field of a curve over K. It is equipped with a fundamental, birational invariant: its **unramified genus** $g(v)$. We can compute this as follows:

$$g(v) = \mathrm{rk}_{\hat{\mathbb{Z}}}\, Dv^a / \langle Tv^a, T^a_p \rangle_{p \in \mathrm{Par}_n(v)}.$$

2.2 Geometric Sets and the Maximal Smooth Model

Definition 29. *We say that a set $\mathcal{S}$ of prime divisors of a function field F is a* ***geometric set (of** F**)*** *if and only if there exists a normal model X of F such that $\mathcal{S}$ is precisely the set of valuations with centers Weil prime divisors on X. In this case, we write*

$$\mathcal{S} = \mathcal{D}(X).$$

If X is smooth, we say X is a ***model*** *of $\mathcal{S}$.*

Theorem 30 (Pop)**.** *If F is a function field with $K(F) = \overline{\mathbb{Q}}$, and let $\Gamma \subseteq G_F$ be a closed subgroup, up to conjugacy. Then there is a topological group-theoretic criterion, given one of the representatives of Γ to determine whether there exists i and $v \in \mathrm{Par}_i(F)$ such that $\Gamma = T_v$ or $\Gamma = D_v$.*

This theorem is proven with G_F replaced by the pro-ℓ completion of G_F in [Pop11a]. To see this for G_F as a whole, we apply Key Lemma 5.1 of [Pop11b]. Pop also proved [Pop11b]:

Theorem 31 (Pop)**.** *Given a geometric set $\mathcal{S}$ of prime divisors on F,*

1. *If $\mathcal{S}$ is a geometric set of prime divisors on F, then a (possibly different) set $\mathcal{S}'$ of prime divisors on F is a geometric set if and only if it has finite symmetric difference with $\mathcal{S}$.*

2. *There exists a group-theoretic recipe to recover*

$$\mathfrak{Geom}(F) = \{\{(T_v, D_v) \mid v \in \mathcal{S}\} \mid \mathcal{S} \text{ a geometric set}\}.$$

Definition 32. *If $\mathcal{S}$ is any set of prime divisors on F, we define the* ***fundamental group of** $\mathcal{S}$* *to be:*

$$\Pi_{\mathcal{S}} = G_F / \langle T_v \rangle_{v \in \mathcal{S}}.$$

Here, $\langle T_v \rangle_{v\in\mathcal{S}}$ is the smallest closed, normal subgroup of G_F which contains every element in every conjugacy class in each T_v. If $\mathcal{T} \subseteq \mathcal{S}$ is a subset, then we denote by

$$\rho_{\mathcal{T}\mathcal{S}} : \Pi_{\mathcal{T}} \to \Pi_{\mathcal{S}}$$

the restriction map, and drop subscripts when they are unambiguous.

Given a geometric set $\mathcal{S}$, there are many possible X such that $\mathcal{D}(X) = \mathcal{S}$; for instance, any model less a finite subset of points has the same set of prime divisors. We now define the maximal model on which we will be able to effect our intersection theory.

Theorem 33. *Let $\mathcal{S}$ be a geometric set for a function field F of dimension 2. There exists a unique model $\mathcal{M}(\mathcal{S})$ of F such that the following holds:*

1. *$\mathcal{D}(\mathcal{M}(\mathcal{S})) = \mathcal{S}$.*
2. *$\mathcal{M}(\mathcal{S})$ is smooth.*
3. *$\hat{\pi}_1(\mathcal{M}^{\mathrm{an}}, p) \simeq \pi_1^{\text{ét}}(\mathcal{M}(\mathcal{S}), p) \simeq \Pi_{\mathcal{S}}$.*
4. *If X is any other smooth model of F which satisfies $\mathcal{S} = \mathcal{D}(X)$, then there exists a unique F-morphism $X \to \mathcal{M}(\mathcal{S})$, and this is a smooth embedding.*

Proof. Let U be a model of $\mathcal{S}$, and let X be a smooth compactification of U. Let

$$\partial = \mathcal{D}(X) \setminus \mathcal{D}(U)$$

be the collection of field-theoretic prime divisors in the boundary of U in X. This is a finite set, as the boundary divisor is itself a finite union of prime divisors. We now define a sequence of pairs (X_i, ∂_i) of varieties X_i and finite sets of divisors $\partial_i \subset \mathcal{D}(X)$ inductively as follows:

1. Let $X_1 = X, \partial_1 = \partial$.

2. We now construct $(X_{i+1}, \partial_{i+1})$ from (X_i, ∂_i). First, take the collection $\{v_j\} \subseteq \partial_i$ such that each $|v_j|$ is a -1-curve such that no other $|v'|$ that intersects it in the boundary is a -1-curve, and blow down. Set X_{i+1} to be this blowdown, and $\partial_{i+1} = \partial_i \setminus \{v_j\}$.

As ∂_1 is finite, at some point, this sequence becomes stationary — let's say at (X_n, ∂_n). Then we define

$$U_{\max} = X_n \setminus \bigcup_{v \in \partial_n} |v|. \tag{2.19}$$

To prove that this satisfies property 4, let U' be another model and X' a smooth compactification of it. Run the algorithm on X' to get a pair $(X'_{n'}, \partial'_{n'})$ Then by strong factorization for surfaces (see Corollary 1-8-4, [Mat02]), there exists a roof

$$\begin{array}{ccc} & Y & \\ \swarrow^{\delta} & & \searrow^{\upsilon} \\ X'_{n'} & & X_n, \end{array}$$

where δ and υ are both sequences of blowups. For any morphism

$$\varphi : Z \to Z'$$

of varieties we may define the **exceptional locus**

$$\mathcal{E}(\varphi) = \left\{ p \in Z' \mid \dim(\varphi^{-1}(p)) \geq 1 \right\}$$

Let $p \in \mathcal{E}(\upsilon) \cap U_{\max}$. Then $\upsilon^{-1}(p)$ is connected, so $\delta(\upsilon^{-1}(p))$ is also connected. It is proper, so is either a union of divisors or a point. If it is a union of divisors and one of these divisors were in $U'_{\max}$, this divisor would be contracted, so $U_{\max}$ and $U'_{\max}$ would not have the same codimension

1 theory. We may argue the same way for δ. Thus, $\delta \circ \upsilon^{-1}|_{U_{\max} \setminus \mathcal{E}(\upsilon)}$ is well-defined and regular outside codimension 2, so extends to a morphism

$$\delta \circ \upsilon^{-1} : U_{\max} \to U'_{\max},$$

injective on closed points. By the same argument we may produce the inverse

$$\upsilon \circ \delta^{-1} : U'_{\max} \to U_{\max},$$

so we have that the maximal smooth model is indeed unique up to isomorphism.

To prove property 3, we note that the map

$$U \to U_{\max}$$

gives a natural equivalence of the category of étale covers of U with the category of étale covers of $U_{\max}$, by the Nagata-Zariski purity theorem [Gro05, X.3.4], and thus gives an isomorphism on fundamental groups by Theorem 22.

□

Corollary 34. *Let U be an affine or projective smooth variety with function field F. Then $U = \mathcal{M}(\mathcal{D}(U))$.*

Proof. If U is proper, the algorithm in the proof of Theorem 33 terminates immediately, so $U_{\max} = U$.

If U is affine, let $\iota : U \to U_{\max}$ be the embedding of U as an affine open of $U_{\max}$. Assume there were a a closed point $x \in U_{\max} \setminus U$. Then there is an affine neighborhood $U' \subset U_{\max}$ such that $x \in U'$. Then $U' \cap U$ is affine. Its complement $U' \setminus (U' \cap U)$ must then contain a divisor, which contains x, and is not contained in U. Thus, $U_{\max}$ has a different codimension-1 theory from U, which gives a contradiction. $\square$

Now, let F be a function field over $\overline{\mathbb{Q}}$, and fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. Then if $\mathcal{X}_F$ is the inverse system of all smooth models of F,

$$G_F \simeq \lim_{X \in \mathcal{X}_F} \hat{\pi}_1((X \times_{\overline{\mathbb{Q}}} \operatorname{Spec} \mathbb{C})^{\mathrm{an}}),$$

where we will leave the notion of basepoint ambiguous (as we never need to specify it); this isomorphism is highly noncanonical.

Chapter 3

The Anabelian Intersection Theory

3.1 The Local Theory: The Intersection Theorem

The following theorem shows how the fundamental group detects intersections in the best-case scenario.

Theorem 35 (The Local Anabelian Intersection Theorem)**.** *Let X be a smooth (not necessarily proper) surface over $\mathbb{C}$, and let C_1 be a unibranch germ of an algebraic curve at a point $p \in X$ and C_2 an irreducible, reduced algebraic curve on X (a prime divisor), with distinct branches γ_j at p, with C_1 distinct from each branch of C_2. Let $Y = X \setminus C_2$ be the complement of C_2 in X; this is an open subvariety of X. Let $w \circ v_1$ be the rank-two Parshin chain corresponding to C_1 at p, and let v_2 be the Weil prime divisor corresponding to C_2. Then, we may choose meridians $t^a_{v_2}$ and $t^a_{w \circ v_1}$ in $\pi_1^{\text{ét}}(X \setminus C_2)^{\text{ab}}$*

$$t^a_{w \circ v_1} = i(p, C_1 \cdot C_2; X)\, t^a_{v_2}, \tag{3.1}$$

where $i(p, C_1 \cdot C_2; X)$ is the local intersection number as defined in Fulton [Ful98, Definition

7.1]. Then

$$[T_{v_2}(F_{X\setminus C_2}|F)^a : T_{w\circ v_1}(F_{X\setminus C_2}|F)^a] | i(p, C_1 \cdot C_2; X) \tag{3.2}$$

with equality if $T_v(F_{X\setminus C_2}|F)^a$ *is infinite in* $\pi_1^{\text{ét}}(X \setminus C_2)^{\text{ab}}$.

Lemma 36 (Reeve's Lemma)**.** *We use the notation of Theorem 35. There is a neighborhood* $B \subseteq X$ *of* p *biholomorphic to a unit ball* $B \subseteq \mathbb{C}^2$ *about the origin so that* p *corresponds to* $(0,0)$*. Let* ∂B *be the boundary of* B*, which is homeomorphic to a* 3*-sphere. Then for small enough such* B*,*

1. $H_1(\partial B \setminus (C_2 \cap \partial B), \mathbb{Z}) \simeq \bigoplus_j \mathbb{Z} t^a_{\gamma_j}$*, the direct sum generated by the meridian around each branch, where each curve goes in the counterclockwise direction after identifying the normal bundle with an open disc in* $\mathbb{C}$*.*

2. C_1 *intersects* ∂B *in a loop* ℓ*, which can be given an orientation by the complex orientation on* X*. Then in* $H_1(\partial B \setminus (C_2 \cap \partial B), \mathbb{Z})$*, we have*

$$\ell = \sum_j i(p, C_1 \cdot \gamma_j; X) t^a_{\gamma_j}. \tag{3.3}$$

We include a simple proof of this lemma (communicated to us by David Massey), which seems to have been first written down in the difficult-to-access Reeve [Ree55], and it seems to be sufficiently well-known to experts that it does not occur elsewhere in the literature.

Proof. We prove the two assertions in order.

1. Embedded algebraic curves have isolated singularities, so each γ_j intersects ∂B in a smoothly embedded S^1 (by the inverse function theorem), and each of these circles are disjoint. The Mayer-Vietoris sequence gives the first claim immediately.

2. Let

$$f_j(x, y) = 0 \tag{3.4}$$

be a local equation for γ_j, and let

$$u \mapsto (x(u), y(u)) \tag{3.5}$$

be a local parameterization of the normalization $\tilde{C}_1$ of C_1 at p. By [Ful98, Example 1.2.5b]

$$i(p, C_1 \cdot \gamma_j; X) = v_u(f_j(x(u), y(u)), \tag{3.6}$$

the u-adic valuation — or order of vanishing — of $f_j(x(u), y(u)) \in \mathbb{C}[[u]]$.

Then the intersection with ∂B of C_1 can be taken to be a path $\eta : [0, 1] \longrightarrow \mathbb{C}$ with winding number one around the origin so that

$$\ell(t) = (x(\eta(t)), y(\eta(t))). \tag{3.7}$$

Let

$$\pi_j : H_1(\partial B \setminus (C_2 \cap \partial B), \mathbb{Z}) \longrightarrow \mathbb{Z}t^a_{\gamma_j} \tag{3.8}$$

be the projection given by the direct sum decomposition as above. Consider the differential

1-form $d \log f_j(x, y)$ restricted to ∂B. Then

$$\pi_j(\ell(t)) = \frac{1}{2\pi i} \int_{\ell(t)} d \log f_j(x, y),$$

on B, as $d \log f_j(x, y)$ is holomorphic away from the zero-set of f_j and otherwise measures the winding of a loop around the zero-set. But this pulls back to the contour integral

$$\frac{1}{2\pi i} \int_{\eta(t)} d \log f(x(u(t)), y(u(t)))$$

which evaluates exactly to $v_u(f_j(x(u), y(u)))$ by the Residue Theorem.

□

Proof of Theorem 35. Let ∂B be as in Lemma 36. Then given the map

$$\iota : \partial B \setminus (C_2 \cap \partial B) \longrightarrow X, \tag{3.9}$$

we must compute

$$\iota_* : H_1(\partial B \setminus (C_2 \cap \partial B), \mathbb{Z}) \longrightarrow H_1(X, \mathbb{Z}). \tag{3.10}$$

But from Proposition 25, for each j,

$$i_*(t^a_{\gamma_j}) = t^a_{v_2}$$

and in the notation of Proposition 25,

$$\ell = t^a_{w \circ v_1}.$$

Thus by linearity, we have

$$t^a_{w\circ v_1} = \sum_j i(p, C_1 \cdot \gamma_j; X)\, t^a_{v_2} = i(p, C_1 \cdot C_2; X)\, t^a_{v_2}.$$

□

3.2 The Global Theory I: Points and Local Intersection Numbers

Definition 37. *Let $\mathcal{S}$ be a geometric set.*

1. *Let $v \in \mathcal{S}$. We define for every rank 2 Parshin chain $p \circ v$ the subset*

$$\Delta(p\circ v) = \left\{ w \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall Y \subset \mathcal{S} \text{ finite, s.t. } w \in Y \text{ and } T_w \text{ is torsion-free in} \\ \Pi^{\mathrm{ab}}_{\mathcal{S}\setminus Y}, T^a_{p\circ v} \neq \{0\} \text{ in } \Pi^{\mathrm{ab}}_{\mathcal{S}\setminus Y} \end{array} \right\}. \quad (3.11)$$

2. *We say $p \circ v \sim p' \circ v'$ if and only if $\Delta(p \circ v) = \Delta(p' \circ v')$, and the equivalence class of rank-two Parshin chains will be denoted by $[p \circ v]$ and called a* ***point****. Given such a point $[p \circ v]$, we will use $\Delta([p \circ v])$ to denote, for any $p' \circ w \in [p \circ v]$, $\Delta(p' \circ w)$.*

3. *We say that a prime divisor $w \in \mathcal{S}$* ***intersects*** *a point $[p \circ v]$ if $w \in \Delta([p \circ v])$.*

4. *The set of points on $\mathcal{S}$ is denoted $\mathcal{P}(\mathcal{S})$.*

5. *If $v \in \mathcal{S}$ then $\mathcal{P}(v)$ will be the set of points which contain an element of the form $p \circ v$.*

6. *We denote by $\mathcal{M}(\mathcal{S})$ the closed K-points of the maximal smooth model $\mathcal{M}(\mathcal{S})$ of $\mathcal{S}$.*

Definition 38. *Let $Y \subset \mathcal{S}$ be a finite subset. Then we say that Y* ***recognizes*** *the intersection of v at p if T_v^a is torsion-free in $\Pi^{\mathrm{ab}}_{\mathcal{S}\setminus Y}$ and $Y \cap \Delta(p) = \{v\}$.*

Definition 39. *Let X be a smooth surface over an algebraically closed field of characteristic zero. Let D_1, D_2 be two divisors on X. We define the* ***total intersection product***

$$(D_1 \cdot D_2) = \sum_{p \in D_1} i(p, D_1 \cdot D_2; X). \tag{3.12}$$

Theorem 40 (The Algebraic Inertia Theorem)**.** *In this theorem (and its proof), all homology will be taken with integral coefficients. Let $\{D_i\}_{i\in I}$ be a finite collection of smooth, distinct prime divisors on a smooth, proper, complex, algebraic surface X such that*

$$D = \bigcup_{i \in I} D_i \tag{3.13}$$

is simple normal crossing. Let

$$\eta_1 : \mathrm{NS}(X) \to \bigoplus_{i\in I} \mathbb{Z} t^a_{v_i} \tag{3.14}$$

be given by

$$\eta_1(D) = \bigoplus_{i \in I} (D \cdot |v_i|) t^a_{v_i}. \tag{3.15}$$

Then there is a short-exact sequence

$$\mathrm{NS}(X) \xrightarrow{\partial} \bigoplus_{i\in I} \mathbb{Z} t^a_{v_i} \longrightarrow \langle t^a_{v_i} \rangle_{i\in I} \subseteq H_1(X \setminus D) \longrightarrow 0. \tag{3.16}$$

Proof. For each $i \in I$ let

$$\begin{array}{ccc} \mathcal{N}_i & \longrightarrow & X \\ \big\downarrow{\scriptstyle \pi_i} & \nearrow & \\ D_i & & \end{array} \tag{3.17}$$

be an open normal open square bundle of D_i in X with projection to its central divisor. Then

$$\mathcal{T}_i = \mathcal{N}_i \setminus D_i \tag{3.18}$$

is an open subset of X, as is $\mathcal{N}_i$. For each $i \in I$ let

$$P_i = \left(\bigcup_{j \in I, j \neq i} D_i \cap D_j \right) \subset D_i \text{ and } P = \bigcup_{i \in I} P_i. \tag{3.19}$$

We further shrink the square bundles so that each component of each intersection between $\mathcal{N}_i$'s contains exactly one point $p \in P$. Call each component $\mathcal{N}_p$, identified by the point of intersection it contains. We can then further shrink the square bundles so that

$$p \neq q \implies \mathcal{N}_p \cap \mathcal{N}_q = \emptyset. \tag{3.20}$$

Let also

$$D_{i,p} = \pi_i(\mathcal{N}_p). \tag{3.21}$$

By the assumption that D is simple normal crossings, and $p \in D_i \cap D_j$ where $i \neq j$, we can assume that there is an analytic isomorphism

$$\eta_p : \mathcal{N}_p \simeq \{(x, y) \in \mathbb{C}^2 \mid \Re x, \Re y, \Im x, \Im y \in (-1, 1)\}, \tag{3.22}$$

so that

$$\eta_p^{-1}(\{0\} \times (-1, 1)) = D_{j,p} \text{ and } \eta_p^{-1}((-1, 1) \times \{0\}) = D_{i,p}. \tag{3.23}$$

and so that the diagrams

$$\begin{array}{ccc} \mathcal{N}_p & \xrightarrow{\eta_p} & \{(x,y) \in \mathbb{C}^2 \mid \Re x, \Re y, \Im x, \Im y \in (-1,1)\} \\ \downarrow{\scriptstyle \pi_i|_{N_p}} & & \downarrow{\scriptstyle \pi_1} \\ D_{i,p} & \xrightarrow{\eta_p} & \{x \in \mathbb{C} \mid \Re x, \Im x \in (-1,1)\}, \end{array} \tag{3.24}$$

and

$$\begin{array}{ccc} \mathcal{N}_p & \xrightarrow{\eta_p} & \{(x,y) \in \mathbb{C}^2 \mid \Re x, \Re y, \Im x, \Im y \in (-1,1)\} \\ \downarrow{\scriptstyle \pi_j|_{N_p}} & & \downarrow{\scriptstyle \pi_2} \\ D_{j,p} & \xrightarrow{\eta_p} & \{x \in \mathbb{C} \mid \Re x, \Im x \in (-1,1)\} \end{array} \tag{3.25}$$

commute, where π_1 and π_2 are projection onto the first and second coordinates, respectively. We then set

$$D_i' = D_i \setminus P_i. \tag{3.26}$$

and let $\mathcal{T}_i'$ be defined as a fiber bundle over D_i' with structure map π_i' by the pullback square

$$\begin{array}{ccc} \mathcal{T}_i' & \longrightarrow & \mathcal{T}_i \\ \downarrow{\scriptstyle \pi_i'} & & \downarrow{\scriptstyle \pi_i|_{\mathcal{T}_i}} \\ D_i' & \longrightarrow & D_i. \end{array} \tag{3.27}$$

We now define for each $p \in P$ with $p \in D_i \cap D_j$ and $i \neq j$

$$\mathcal{T}_p = \mathcal{N}_p \setminus (D_{i,p} \cup D_{j,p}). \tag{3.28}$$

We now define for each $i \in I, p \in P_i$:

1. A set $D_{i,p}^+$, which is a small, open neighborhood of $D_{i,p}$ in D_i of which $D_{i,p}$ is a deformation retract, each taken so it does not intersect any other $D_{j,p'}^+$, unless $p = p'$ in which case it

can intersect at p.

2. $\mathcal{T}_p^+ = \pi_i'^{-1}(D_{i,p}^+) \cup \pi_j'^{-1}(D_{j,p}^+)$. Note that $\mathcal{T}_p \subset \mathcal{T}_p^+$ is an open subset. Again, we modify our $D_{i,p}^+$ so that the distinct $\mathcal{T}_p^+$ are mutually disjoint.

3. $\mathcal{U}_i = \mathcal{T}_i \setminus \left(\bigcup_{p \in P_i} (\pi_i \mid_{\mathcal{T}_i})^{-1} (\overline{D_{i,p}}) \right)$.

4. $D_{i,p}^- = D_{i,p}^+ \setminus \overline{D_{i,p}}$ and $\mathcal{T}_{i,p}^- = \pi_i'^{-1}(D_{i,p}^-)$.

Then formula 3.22 gives us a pushout-pullback square of open subsets

$$\begin{array}{ccc} \coprod_{i \in I, p \in P_i} \mathcal{T}_{i,p}^- & \xrightarrow{\iota_1} & \coprod_{p \in P} \mathcal{T}_p^+ \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} \\ \coprod_{i \in I} \mathcal{U}_i & \xrightarrow{\iota_4} & \mathcal{T}. \end{array} \tag{3.29}$$

We choose a fiber $\mathcal{F}_i$ of each $\mathcal{T}_i'$ and let

$$\epsilon_i : \mathcal{F}_i \longrightarrow \mathcal{T}_i \tag{3.30}$$

be its inclusion, and note that

$$H_1(\mathcal{F}_i) = \mathbb{Z} t_{v_i}^a. \tag{3.31}$$

We now set

$$\mathcal{T} = \bigcup_{i \in I} \mathcal{T}_i \text{ and } \mathcal{N} = \bigcup_{i \in I} \mathcal{N}_i, \tag{3.32}$$

and we have a pushout-pullback square of open subsets

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\iota_1} & \mathcal{N} \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} \\ X \setminus D & \xrightarrow{\iota_4} & X \end{array} \tag{3.33}$$

Let

$$\eta_1 : H_2(X) \longrightarrow \bigoplus_{i \in I} H_1(\mathcal{F}_i) \text{ and } \eta_2 : \bigoplus_{i \in I} H_1(\mathcal{F}_i) \longrightarrow H_1(\mathcal{T}) \tag{3.34}$$

be given by

$$\eta_1(\gamma) = \bigoplus_{i \in I} (\gamma \cap D_i) t_{v_i} \text{ and } \eta_2 = \bigoplus_{i \in I} \epsilon_{i*}. \tag{3.35}$$

We have a diagram:

$$\begin{array}{ccccccc} & \overset{\eta_1}{\nearrow} & \bigoplus_{i \in I} H_1(\mathcal{F}_i) & & & & \\ & & \downarrow \eta_2 & & & & \\ H_2(X) & \xrightarrow{\partial} & H_1(\mathcal{T}) & \xrightarrow{\iota_{1*} + \iota_{2*}} & H_1(\mathcal{N}) \oplus H_1(X \setminus D) & \xrightarrow{\iota_{3*} - \iota_{4*}} & H_1(X). \end{array} \tag{3.36}$$

The bottom row is exact, by the Mayer-Vietoris sequence applied to diagram 3.33. We prove the following, the combination of which implies the exactness of the short exact sequence 3.16:

1. We can apply the Mayer-Vietoris sequence to the diagram 3.29, and we get

$$\bigoplus_{i \in I, p \in P_i} H_1(\mathcal{T}_{i,p}^-) \xrightarrow{\iota_{1*} + \iota_{2*}} \Big(\bigoplus_{p \in P} H_1(\mathcal{T}_p) \Big) \oplus \big(\bigoplus_{i \in I} H_1(\mathcal{U}_i) \big) \xrightarrow{\iota_{3*} - \iota_{4*}} H_1(\mathcal{T}) \xrightarrow{\partial_{\mathcal{T}}} \tag{3.37}$$

$$\bigoplus_{i \in I, p \in P_i} H_0(\mathcal{T}_{i,p}^-)$$

   This allows us to compute $H_1(\mathcal{T})$ as follows.

   (a) The Künneth formula gives us that

$$H_1(\mathcal{T}_{i,p}^-) = \mathbb{Z} t_{v_i}^a \oplus H_1(D_{i,p}^-) \tag{3.38}$$

and if $p \in P_j$ then we may use Lemma 36 to give us

$$H_1(D^-_{i,p}) = \mathbb{Z}t^a_{v_j}. \tag{3.39}$$

(b) The bundle $\mathcal{T}_i'$ to D_i' is complex-oriented so has trivial monodromy action. But

$$H^2(D_i') = 0, \tag{3.40}$$

so every such bundle must be trivial. $\mathcal{U}_i$ is thus a trivial bundle over $D_i \setminus \left(\bigcup_{p\in P_i} \overline{D_{i,p}}\right)$ which is a deformation retract of D_i'. Thus, by the Künneth formula again,

$$H_1(\mathcal{U}_i) = \mathbb{Z}t^a_{v_i} \oplus H_1(D_i') \tag{3.41}$$

(c) Let $p \in P_i \cap P_j$ where $i \neq j$. Then $\mathcal{T}_p$ is a product, by diagram 3.22, and by the Künneth formula again:

$$H_1(\mathcal{T}_p) = \mathbb{Z}t^a_{v_i} \oplus \mathbb{Z}t^a_{v_j}. \tag{3.42}$$

We may run a similar — but simpler — analysis for $\mathcal{N}$. Plugging this information into the Mayer-Vietoris sequence 3.37, we get a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \left(\bigoplus_{i\in I} H_1(D_i)\right) \oplus \left(\bigoplus_{i\in I} \mathbb{Z}t_{v_i}\right)/R & \xrightarrow{\alpha_{\mathcal{T}}} & H_1(\mathcal{T}) & \xrightarrow{\partial_{\mathcal{T}}} & \bigoplus_{i\in I, p\in P_i} H_0(\mathcal{T}^-_{i,p}) \\ & \downarrow \Phi & & \downarrow \iota_{1*} & & \Vert \\ 0 \longrightarrow & \left(\bigoplus_{i\in I} H_1(D_i)\right) & \xrightarrow{\alpha_{\mathcal{N}}} & H_1(\mathcal{N}) & \xrightarrow{\partial_{\mathcal{N}}} & \bigoplus_{i\in I, p\in P_i} H_0(\mathcal{T}^-_{i,p}) \end{array} \tag{3.43}$$

with exact rows, where Φ is the map which sends all meridians to zero, and R is the module of relations

$$R = \left\langle \sum_{\substack{p \in P_i \\ p \in D_i \cap D_j}} t_{v_j} \right\rangle_{i \in I} . \tag{3.44}$$

Chasing the diagram, we see that

$$\operatorname{im} \alpha_{\mathcal{T}} \left(\left(\bigoplus_{i \in I} \mathbb{Z} t_{v_i} \right) / R \right) = \operatorname{im} \eta_2. \tag{3.45}$$

2. We now wish to show that there exists η_1 such that

$$\partial = \eta_2 \circ \eta_1, \tag{3.46}$$

and that η_1 is defined as in formula 3.15. We note that if indeed η_1 satisfies these properties, then $R \subseteq \operatorname{im} \eta_1$, so these are all the relations given in 3.16. But by Diagram 3.43,

$$\ker \iota_{1*} \cap \ker \iota_{2*} \subseteq \ker \iota_{1*} \subseteq \operatorname{im} \eta_2. \tag{3.47}$$

Thus, ∂ factors through the image of η_2, and we call this map η_1.

To compute η_1, we first use the orientation class $[\omega] \in H_4(X)$ to give the intersection $\cap$-product pairing

$$\cap : H_2(X) \bigotimes H_2(X) \to \mathbb{Z}. \tag{3.48}$$

Let $c \in H_2(X)$ have realization a connected, closed topological surface $|c|$, which intersects each D_i transversely in finitely many points $\{Q_j\}_{j \in J}$ with orientations

$$o(j) = \pm 1 \tag{3.49}$$

away from P_i. Such elements generate H_2, so we only need to verify the identity for them. We recall that for a space Y, $C_i(Y)$ is the $\mathbb{Z}$-module of singular chains of Y with coefficients in $\mathbb{Z}$. We then have, from the proof of the Mayer-Vietoris sequence, a diagram

$$\begin{array}{ccccccc} & & C_2(\mathcal{N}) \oplus C_2(X \setminus D) & \xrightarrow{\iota_{3*}-\iota_{4*}} & C_2(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_1 \oplus \partial_2} & & \downarrow{\scriptstyle \partial_3} & & \\ C_1(\mathcal{T}) & \xrightarrow{\iota_{1*}+\iota_{2*}} & C_1(\mathcal{N}) \oplus C_1(X \setminus D) & \xrightarrow{\iota_{3*}-\iota_{4*}} & C_1(X) & \longrightarrow & 0 \end{array} \tag{3.50}$$

Let $\mathcal{N}_i^0 \subset \mathcal{N}_i$ be a slightly smaller tubular neighborhood of D_i, and let

$$\mathcal{N}^0 = \bigcup_{i \in I} \mathcal{N}_i^0. \tag{3.51}$$

We recall the definition of the boundary map ∂ in the Mayer-Vietoris long exact sequence: we take $\tilde{c}$ to be a preimage of the cycle representing c by $\iota_{3*} - \iota_{4*}$, which is surjective. Then $\partial_1(\tilde{c})$ lies in the kernel of $\iota_{3*} - \iota_{4*}$ by commutativity of the diagram, so lies in the image of $\iota_{1*} + \iota_{2*}$, because $\partial_3(c) = 0$. We can then choose the lift so the assertion is clear. We can take

$$\tilde{c} = (|c| \cap \overline{\mathcal{N}^0}) \oplus -(|c| \cap (X \setminus \mathcal{N}^0)), \tag{3.52}$$

where these intersections stand for their representations as 2-cycles. But $\partial_1(|c| \cap \overline{\mathcal{N}^0})$ is a boundary, as $|c| \cap \overline{\mathcal{N}^0}$ is a union of closed disks. Thus, the only contribution in homology comes from ∂_2. But we can compute

$$\partial(|c| \cap (X \setminus \mathcal{N}^0)) = -\sum_{j \in J, Q_j \in D_i} o(j) t_{v_i}. \tag{3.53}$$

This is exactly

$$\sum_{i\in I} -(c\cap D_i)t_{v_i}. \tag{3.54}$$

However, as each D_i is algebraic, the Hodge-Lefschetz $(1,1)$-theorem says that this map factors through $\mathrm{NS}(X)$, which finally proves the exactness of the sequence 3.16.

□

Corollary 41 (The Separation of Inertia Criterion)**.** *We preserve the notation of Theorem 40. Suppose that, in addition to the hypotheses, for each $i, j \in I$ there does not exist $E \in \mathrm{NS}(X)$ s.t. for every $k \in I \setminus \{i, j\}$,*

$$(E\cdot D_k) = 0,\ \textit{but}\ (E\cdot D_i) \neq 0\ \textit{or}\ (E\cdot D_j) \neq 0. \tag{3.55}$$

For each $i \in I$, let v_i be the prime divisor on $\mathbb{C}(X)$ associated to D_i. Then in $H_1(X \setminus D)$,

1. *$T^a_{v_i} \simeq \mathbb{Z}$.*
2. *For $j \in I, j \neq i$,*

$$T^a_{v_i} \cap T^a_{v_j} = 0. \tag{3.56}$$

Proof. We apply the hypothesis in 3.55 to 3.16 to see that there are no elements in the image of ∂ none which are zero at all but one component to prove the first claim, and which are zero at all but two components to prove the second claim. □

Lemma 42 (Divisor Existence Lemma)**.** *Let $D_1, \ldots, D_j$ be a finite set of divisors on a smooth surface, and $p_1, \ldots, p_n$ be a finite set of points. Then there are infinitely (in fact "generically") many prime divisors which intersect each of the D_i but not at the p_i.*

Proof. Choose a very ample divisor C. Then $C \cdot D_i > 0$ for all i. Then having an intersection at p_i is a closed condition (since C is basepoint-free), and so the divisors which do not intersect at those points form an open, nonempty subset of the linear system $|C|$, which is then infinite. $\square$

Corollary 43. *Let $\mathcal{S}$ be a geometric set and let $P \in \mathcal{M}(\mathcal{S})$. Let $v \in \mathcal{S}$ and let $p \circ v$ be a rank-2 Parshin chain such that*

$$|p \circ v| = P. \tag{3.57}$$

Then

1. $w \in \Delta(p \circ v)$ *if and only if* $P \in |w|$.
2. *If* $w \in \Delta(p \circ v)$ *then there exists a set* $Y \subset \mathcal{S}$ *which recognizes the intersection of* w *and* $[p \circ v]$.
3. *If* $v' \in \mathcal{S}$ *and* $p' \circ v'$ *is not centered on* $\mathcal{M}(\mathcal{S})$, *then* $\Delta(p' \circ v') \neq \Delta(p \circ v)$.

Then there is a canonical map

$$\iota : \mathcal{M}(\mathcal{S}) \longrightarrow \mathcal{P}(\mathcal{S}), \tag{3.58}$$

given by

$$\iota(x) = \{v \in \mathcal{S} \mid x \in |v|\}. \tag{3.59}$$

This map is always an injection, and is a bijection if and only if $\mathcal{M}(\mathcal{S})$ is proper.

Proof. Let X be a smooth compactification of $\mathcal{M}(\mathcal{S})$ with only simple normal crossings at the boundary and let

$$\partial\mathcal{S} = \mathrm{Div}(X) \setminus \mathcal{M}(\mathcal{S}) \subset \mathrm{Par}_1(F) \tag{3.60}$$

be the prime divisors supported as prime divisors on the boundary of X.

1. Let $P \in |w|$ and $\Upsilon \subset \mathcal{S}$ a cofinite subset which does not contain w and so that T_w^a is torsion-free in $\Pi_\Upsilon^{\mathrm{ab}}$. Then Lemma 36 implies that the projection of $T_{p\circ v}^a$ to the T_w^a-part of the direct summand is torsion-free in T_w^a in any $\Pi_\Upsilon^{\mathrm{ab}}$, so must be torsion-free itself, so w satisfies the properties of an element of $\Delta(p \circ v)$.

2. Lemma 42 shows that there exists a finite set $\{\beta_i\}_{i\in I}$ of prime divisors such that $\{|\beta_i|\}_{i\in I} \cup \{|w|\} \cup \{|\beta|\}_{\beta\in\partial\mathcal{S}}$ satisfies the hypotheses of Corollary 41, and so $\{\beta_i\}_{i\in I} \cup \{w\}$ recognizes the intersection of w with $p \circ v$.

3. This follows immediately from the fact that there exists a prime divisor in $\mathcal{M}(\mathcal{S})$ which intersects the boundary at $|p' \circ v'|$ and does not intersect $|p \circ v|$, so the valuation associated to this divisor will be in $\Delta(p' \circ v')$ but not in $\Delta(p \circ v)$.

The existence and injectivity of the map ι is now straightforward. The bijectivity in case of properness is the valuative criterion for properness. $\square$

Definition 44. *Let $\mathcal{S}$ be a geometric set, $v, w \in \mathcal{S}$, and $p \circ w$ a rank-2 Parshin chain. Then we define the* ***local intersection number*** $(p, v \cdot w; \mathcal{S})$ *as follows:*

1. *If there does not exist a set Y which recognizes the intersection of v at p (and this includes the case where $v \notin \Delta(p)$, then we define*

$$(p, v \cdot w; \mathcal{S}) = 0. \tag{3.61}$$

2. *Otherwise, let Y recognize the intersection of v at p. Then we define*

$$(p, v \cdot w; \mathcal{S}) = [T_v : T_{p\circ w}]_{\Pi_{\mathcal{S}\setminus Y}}.$$

Theorem 45. *Let $\mathcal{S}$ be a geometric set, $p \in \mathcal{P}(\mathcal{S})$ a point with a center $|p| \in \mathcal{M}(\mathcal{S})$ and $v \in \mathcal{S}$. Then*

$$\sum_{p' \circ w \in p, w \in \mathcal{S}} (p', v \cdot w; \mathcal{S}) = i(|p|, |v| \cdot |w|; \mathcal{M}(\mathcal{S})). \tag{3.62}$$

Proof. By Corollary 43, either both sides of Equation 3.62 are zero, or there exists a geometric set which recognizes the intersection of p with v. Now, each $p' \circ w \in p$ represents a branch of $|w|$ at p, and we will call this germ $\xi_{p' \circ w}$. We then have by the Local Anabelian Intersection Formula

$$i(|p|, |v| \cdot |w|; \mathcal{M}(\mathcal{S}) = \sum_{|p' \circ w| \in p} i(|p|, |v| \cdot \xi_{p' \circ w}; \mathcal{M}(\mathcal{S})) = \sum_{|p' \circ w| \in p} (p', v \cdot w; \mathcal{S}). \tag{3.63}$$

□

Definition 46. *Let $p \circ v$ be a rank-2 Parshin chain with $v \in \mathcal{S}$. Then we say that $p \circ v$ is a* ***nonnodal chain*** *for $\mathcal{S}$ if $[p \circ v]$ is distinct from every other $[p' \circ v]$, and we say that $p \circ v$ is a* ***noncuspidal chain*** *if there exists a $v' \in \mathcal{S}$ with $(p, v' \cdot v; \mathcal{S}) = 1$. A rank-2 Parshin chain which is both nonnodal and noncuspidal will be called a* ***smooth chain****.*

Proposition 47. *If $|p \circ v| \in \mathcal{M}(\mathcal{S})$ then $p \circ v$ is nonnodal (resp. noncuspidal) if and only if $|v|$ is not nodal (resp. cuspidal) at $|p \circ v|$.*

Proof. This follows directly from Corollary 43 and the definition of a nodal, resp. cuspidal, point on a curve. □

Recall that $\mathcal{P}(\mathcal{S}) \subset 2^{\mathrm{Par}_2(F)}$; thus, each point of each geometric set is a subset of a larger set, and we will consider them as sets in the next sequence of definitions.

Definition 48. *Let $\mathcal{S}'$ be a geometric set, and let $p \in \mathcal{P}(\mathcal{S}')$. Then we define the* ***$\mathcal{S}'$-limits of p in $\mathcal{S}$*** *by*

$$\mathrm{Lim}_{\mathcal{S}}(p) = \{\pi \in \mathcal{P}(\mathcal{S}) \mid \pi \cap p \neq \emptyset\}. \tag{3.64}$$

Definition 49. *If $v \in \mathcal{S}'$, then we define the **$\mathcal{S}'$-limits of** v **in** $\mathcal{S}$ to be*

$$\mathrm{Lim}_{\mathcal{S}}(v) = \bigcup_{p \circ v \in \mathrm{Par}_2(v)} \mathrm{Lim}_{\mathcal{S}}([p \circ v]). \tag{3.65}$$

Definition 50. *Let $\mathcal{S}'$ be a geometric set so that $\mathcal{S} \subset \mathcal{S}'$. Then we define the **boundary points of** $\mathcal{S}$ **relative to** $\mathcal{S}'$ as*

$$\partial_{\mathcal{S}'}(\mathcal{S}) = \bigcup_{v \in \mathcal{S}' \setminus \mathcal{S}} \mathrm{Lim}_{\mathcal{S}}(v), \tag{3.66}$$

*and the **interior points of** $\mathcal{S}$ **relative to** $\mathcal{S}'$ as*

$$\mathcal{P}_{\mathcal{S}'}(\mathcal{S}) = \mathcal{P}(\mathcal{S}) \setminus \partial_{\mathcal{S}'}(\mathcal{S}). \tag{3.67}$$

Definition 51. *We say that a point $[p \circ v]$ is **absolutely uncentered** on $\mathcal{S}$ if $T_{p' \circ v'}$ is nontrivial in the total fundamental group $\Pi_{\mathcal{S}}$ of S for some $p' \circ v' \in [p \circ v]$. We define*

$$a(\mathcal{S}) = \{[p \circ w] \in \mathcal{P}(\mathcal{S}) \mid [p \circ w] \textit{ absolutely uncentered}\}.$$

*We define the **candidate points of** $\mathcal{S}$ by*

$$\mathcal{A}(\mathcal{S}) = \mathcal{P}(\mathcal{S}) \setminus a(\mathcal{S}),$$

and these are the points which are not absolutely uncentered.

We have immediately:

Proposition 52. *For any geometric set $\mathcal{S}$,*

$$\mathcal{M}(\mathcal{S}) \subseteq \mathcal{A}(\mathcal{S}).$$

That is, absolutely uncentered points of $\mathcal{S}$ do not have centers on $\mathcal{M}(\mathcal{S})$, and candidate points have a chance.

The converse to this proposition is false in general:

Example 53. *Let $F = \overline{\mathbb{Q}}(x, y)$, and $X = \operatorname{Spec} \overline{\mathbb{Q}}[x, y]$ and $\mathcal{S} = \operatorname{Div}(X)$. If $p \circ v$ is not centered on X, then the algebraic inertia theorem (accounting for resolution of singularities) shows that $T^a_{p \circ v} = \mathbb{Z}$ in any divisor complement, so*

$$\Delta(p \circ v) = \mathcal{S}. \tag{3.68}$$

But as $\Pi_{\mathcal{S}}$ is trivial,

$$\mathcal{A}(\mathcal{S}) = \mathcal{P}(\mathcal{S}). \tag{3.69}$$

Thus,

$$\mathcal{A}(\mathcal{S}) = X \cup \{\infty\}. \tag{3.70}$$

However, for the "visible affine opens" we define below, the converse is true. The first goal of Anabelian Intersection Theory is to identify these special geometric sets, and use this to construct a salvage of the converse.

3.3 The Global Theory II: Visible Affines and Properness

In this section, we fix a two-dimensional function field F.

Definition 54. *Let U be a model of F which admits a surjective map*

$$\pi : U \longrightarrow B$$

*to a hyperbolic curve B, with smooth, hyperbolic fibers of the same genus with at least three punctures. We call U a **visible affine of** F (this is a topological fibration, if not a Zariski fibration). There is then a **horizontal-vertical decomposition***

$$\mathcal{D}(U) = \mathcal{H} \cup \mathcal{V}$$

*into **horizontal** divisors (the members of $\mathcal{H}$) and **vertical** divisors (the members of $\mathcal{V}$), where the vertical divisors are given as the fibers of π. This π determines the horizontal-vertical decomposition, and a horizontal-vertical decomposition determines π up to automorphisms of the base.*

Proposition 55. *Let U be a visible affine of F with horizontal-vertical decomposition*

$$\mathcal{D}(U) = \mathcal{H} \cup \mathcal{V}. \tag{3.71}$$

Then

1. $\iota(\mathcal{P}(U)) = \mathcal{A}(\mathcal{D}(U))$.

2. *For any $v, v' \in \mathcal{V}$, $D_v = D_{v'}$ in $\Pi_{\mathcal{D}(U)}$.*

3. *For any $v \in \mathcal{V}$, $\pi_1^{\text{ét}}(B) = \Pi_{\mathcal{D}(U)}/D_v$.*

4. *Let*

$$\partial B = \overline{B} \setminus B. \tag{3.72}$$

Then for each $h \in \mathcal{H}$ and $p \in \partial B$ there exists $q \circ h \in \mathrm{Par}_2(h)$ such that $0 \neq \pi_*(T^a_{q\circ h}) \subseteq T^a_p$, the closure of a group generated by a meridian around p in B. In particular, let

$$I^h_U = \langle T^a_p \rangle_{p \in \mathrm{Par}_2(h)}$$

be the closure of the subgroup of $\pi_1^{\text{ét}}(U)^{\text{ab}}$ generated by all inertia of rank-2 valuations, and let I_B be the divisible hull of $\pi_*(I^h_U) \subseteq \pi_1^{\text{ét}}(B)^{\text{ab}}$. Then I_B is independent of choice of h, and

$$g(B) = \mathrm{rk}_{\hat{\mathbb{Z}}}(\pi_1^{\text{ét}}(B)^{\text{ab}}/I_B), \tag{3.73}$$

where $g(B)$ is the unramified genus of B.

Proof. 1. By Corollary 43, $\iota(U) \subseteq \mathcal{A}(\mathcal{D}(U))$. Let C be any smooth, hyperbolic, possibly open curve. Then for any choice of basepoint $p \in C, \pi_2(C,p) = 0$ and $\pi_1^{\text{top}}(C,p)$ is residually finite. Thus, for $v \in \mathcal{V}$ and $p \in |v|$ there is a short-exact fiber sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(|v|, p) \longrightarrow \pi_1^{\text{ét}}(U,p) \longrightarrow \pi_1^{\text{ét}}(B, \pi(p)) \longrightarrow 1. \tag{3.74}$$

Let $q \circ v$ be a rank-2 Parshin chain that is not centered on U. If $w \in \mathcal{V}, T_{q\circ w}$ is a nontrivial subgroup of the first term of the short exact sequence; otherwise, $T_{q\circ w}$ projects to a nontrivial subgroup of $\pi_1^{\text{ét}}(B)$. In either case, $T_{q\circ w}$ is nontrivial, so $\mathcal{A}(\mathcal{D}(U)) \subseteq \iota(U)$.

2. The fibration short exact sequence and Proposition 25 gives for any $v \in \mathcal{V}$ that

$$D_v = \ker \pi_* : \pi_1^{\text{ét}}(U) \longrightarrow \pi_1^{\text{ét}}(B). \tag{3.75}$$

3. By Proposition 25,

$$\pi_1^{\text{ét}}(|v|, p) = D_v \tag{3.76}$$

in the short exact sequence 3.74, and the desired statement follows.

4. As $\pi|_{|h|}$ is nonconstant, we can complete and we get a diagram

$$\begin{array}{ccc} H & \longrightarrow & \overline{H} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} \\ B & \longrightarrow & \overline{B}. \end{array} \tag{3.77}$$

   where $\overline{\pi}$ is surjective, and branch points are isolated. If $p \in \partial B$ then t_p has inverse image a disjoint union of loops in $|h|$ and a choice of one such loop for each $p \in \partial B$ provides the necessary meridians of rank-2 Parshin chains by Proposition 25.

□

We can similarly define I_B and $g(B)$ for any geometric set having a horizontal-vertical decomposition.

Theorem 56. *Let $\mathcal{S}$ be a geometric set of F with a disjoint union decomposition*

$$\mathcal{S} = \mathcal{H} \cup \mathcal{V},$$

*where we call $\mathcal{H}$ the **horizontal** and $\mathcal{V}$ the **vertical fibers**. Then $\mathcal{M}(\mathcal{S})$ is a visible affine of F with horizontal-vertical decomposition $\mathcal{H} \cup \mathcal{V}$ if and only if it satisfies the following properties:*

1. ***Fullness**. Let $v \in \mathrm{Par}_1(F)$ and $v \notin \mathcal{S}$. Then either $\partial_{\mathcal{S}\cup\{v\}}(\mathcal{S}) \subseteq a(\mathcal{S})$ or $\#\partial_{\mathcal{S}\cup\{v\}}(\mathcal{S}) = 1$.*

2. ***Homeomorphicity of Fibers**. For $v \in \mathcal{S}$ let*

$$a(v) = \mathcal{P}(v) \cap a(\mathcal{S}) \text{ and } \mathcal{A}(v) = \mathcal{P}(v) \cap \mathcal{A}(\mathcal{S}). \tag{3.78}$$

Then for any $v_1, v_2 \in \mathcal{V}$,

$$\#a(v_1) = \#a(v_2) \tag{3.79}$$

and

$$g(v_1) = g(v_2) \tag{3.80}$$

3. ***Disjointness of Fibers***. $\mathcal{A}(\mathcal{S}) = \coprod_{v \in \mathcal{V}} \mathcal{A}(v)$, *(a disjoint union). Furthermore, there exists a geometric set* $\mathcal{S}' \supseteq \mathcal{S}$ *such that in* $\mathcal{S}'$, *for any* v_1 *and* v_2 *distinct elements of* $\mathcal{V}$, *and any* $p_1 \in \mathrm{Par}_2(v_1)$ *and* $p_2 \in \mathrm{Par}_2(v_2)$,

$$[p_1] \neq [p_2]. \tag{3.81}$$

Such an $\mathcal{S}'$ *is called* ***fiber-separating***.

4. ***Hyperbolicity of Base***. *The base has at least three punctures; that is, the* $\hat{\mathbb{Z}}$*-rank of* I_B *is* ≥ 2.

5. ***Numerical Equivalence of Fibers***. *For* $h \in \mathcal{H}$ *and* $v \in \mathcal{V}$, *let*

$$S_h(v) = \sum_{\substack{p \circ v \in \mathrm{Par}_2(v) \text{ s.t.} \\ [p \circ v] \in \mathcal{A}(\mathcal{S})}} (p, h \cdot v; \mathcal{S}).$$

For every $h \in \mathcal{H}$ *there exists* $n_h \in \mathbb{N}$ *and a finite subset* $\Sigma_h \subseteq \mathcal{V}$ *such that for all* $v \in \mathcal{V}$,

$$S_h(v) \leq n_h$$

with inequality strict only at Σ_h, *and*

$$\bigcap_{h \in H} \Sigma_h = \emptyset$$

for any cofinite subset $H \subseteq \mathcal{H}$.

6. ***Triviality of Monodromy***. *Let $v \in \mathcal{V}$. Then T_v^a is torsion-free in $\Pi^{\text{ab}}_{\mathcal{S}\setminus\{v\}}$, and the action of T_v by conjugation on any $D_{v'}$ for $v' \in \mathcal{V}$ is inner in $\Pi_{\mathcal{S}\setminus\{v\}}$.*

7. ***Inheritance***. *Let $\mathcal{V}'$ be any cofinite subset of $\mathcal{V}$. Then all the above properties hold for $\mathcal{V}' \cup \mathcal{H}$.*

In this case, $\mathcal{S}$ will be called a ***visible affine geometric set***.

Proof. That every visible affine satisfies the hypotheses is straightforward. Let

$$\overline{\mathcal{S}} = \mathcal{D}(\overline{\mathcal{M}(\mathcal{S})})$$

where $\overline{\mathcal{M}(\mathcal{S})}$ is a smooth compactification of $\mathcal{M}(\mathcal{S})$ which is also fiber-separating. Such an $\overline{\mathcal{S}}$ exists by applying resolution of singularities to a compactification of $\mathcal{M}(\mathcal{T})$ for any fiber-separating $\mathcal{T}$; such a $\mathcal{T}$ exists by Disjointness of Fibers.

We define two subsets of $\mathcal{P}(\mathcal{S})$:

$$\beta_i = \partial_{\overline{\mathcal{S}}}(\mathcal{S}) \cap a(\mathcal{S}) \text{ and } \beta_e = \partial_{\overline{\mathcal{S}}}(\mathcal{S}) \cap \mathcal{A}(\mathcal{S}). \tag{3.82}$$

We define the **vertical support** of $\partial_{\overline{\mathcal{S}}}(\mathcal{S})$ to be

$$\Delta(\beta_e) = \bigcup_{p \in \beta_e} \Delta(p) \cap \mathcal{V}. \tag{3.83}$$

By fullness, $\Delta(\beta_e)$ is finite.

We define

$$\mathcal{V}' = \mathcal{V} \setminus \Delta(\beta_e). \tag{3.84}$$

Then $\mathcal{S}' = \mathcal{V}' \cup \mathcal{H}$ is a geometric set with horizontal-vertical decomposition, by Inheritance.

If C, D and E are divisors on $\overline{\mathcal{M}(\mathcal{S})}$, we define

$$C \sim_E D \text{ if and only if } (C \cdot E) = (D \cdot E), \tag{3.85}$$

where we use the total intersection product as in Definition 39. If $E_1, \ldots, E_n$ generates $\mathrm{NS}(X)$, then we see that C is numerically equivalent to D if and only if $C \sim_{E_i} D$ for every $i = 1, \ldots, n$. If C and D are numerically equivalent, effective and pairwise disjoint, they are algebraically equivalent by [Ful98][19.3.1].

Thus, to prove that $\mathcal{M}(\mathcal{S})$ is a visible affine open, we need to prove that there exists a cofinite subset $\mathcal{V}'' \subseteq \mathcal{V}'$ and a finite set $\{h_1, \ldots, h_n\} \subset \mathcal{H}$ such that:

1. $|h_i|$ generate $\mathrm{NS}(\overline{\mathcal{M}(\mathcal{S})}) \otimes \mathbb{Q}$.

2. For all $v_1, v_2 \in \mathcal{V}''$, we have

$$|v_1| \sim_{|h_i|} |v_2| \tag{3.86}$$

   for each $1 \leq i \leq n$.

Once we know this, we will know that the $\mathcal{M}(\mathcal{V}'' \cup \mathcal{H})$ is indeed an affine open, for the divisors each vary in an algebraic family. Triviality of Monodromy allows us to use [Tam97, Theorem 0.8] to "plug the holes" and deduce that $\mathcal{M}(\mathcal{S})$ itself is indeed a visible affine open. It is here that we use homeomorphicity of fibers to make sure we're "plugging the holes" with the right divisors.

$\mathrm{NS}(\overline{\mathcal{M}(\mathcal{S})}) \otimes \mathbb{Q}$ is spanned by very ample, prime divisors. As the $|v|$ with $v \in \mathcal{V}'$ are mutually disjoint, they cannot be very ample, as each very ample divisor intersects every other divisor. Thus, all very ample divisors must be horizontal. We thus can choose $h_1, \ldots, h_n \in \mathcal{H}$ so that $\{|h_i|\}$ generates $\mathrm{NS}(\overline{\mathcal{M}(\mathcal{S})}) \otimes \mathbb{Q}$. For each h_i we have

$$\left(|h_i| \cdot \bigcup_{b \in \overline{\mathcal{S}} \setminus \mathcal{S}} |b| \right) < \infty. \tag{3.87}$$

Thus, by separation of points, there is at most a finite subset $\sigma_i' \subset \mathcal{V}'$ for which if $s \in \sigma_i'$ there is an intersection between $|s|$ and $|h_i|$ at a point in β_i, and we may take

$$\sigma_i = \sigma_i' \cup (\Sigma_{h_i} \cap \mathcal{V}').$$

But for each $v_1, v_2 \in \mathcal{V}' \setminus \sigma_i$, we have by Theorem 35,

$$|v_1| \sim_{h_i} |v_2|.$$

Thus, we may take

$$\mathcal{V}'' = \mathcal{V}' \setminus \bigcup_{i=1}^{n} \sigma_i. \tag{3.88}$$

□

Proposition 57. *Let U be a visible affine and X a maximal smooth model of F. Then there is an open immersion*

$$U \longrightarrow X$$

under $\operatorname{Spec} F$ *if and only if*

$$\mathcal{D}(U) \subseteq \mathcal{D}(X) \tag{3.89}$$

and

$$\mathcal{A}(\mathcal{D}(U)) = \mathcal{P}_{\mathcal{D}(X)}(\mathcal{D}(U)). \tag{3.90}$$

Proof. There is a birational map

$$U \dashrightarrow X \tag{3.91}$$

defined outside a set of codimension 2. The minimal such exceptional set is, however, exactly $\partial_{\mathcal{D}(X)}(\mathcal{D}(U))$, so this arrow extends to a regular map if and only if $\partial_{\mathcal{D}(X)}(\mathcal{D}(U))$ is empty or, equivalently, $\mathcal{A}(\mathcal{D}(U)) = \mathcal{P}_{\mathcal{D}(X)}(\mathcal{D}(U))$. □

As immediate corollaries we have:

Corollary 58. *Let $\mathcal{S}$ be a geometric set. Then a point $p \in \mathcal{P}(\mathcal{S})$ is in the image of ι if and only if there is a visible affine geometric set $\mathcal{S}'$ such that $p \in \mathcal{P}_{\mathcal{S}}(\mathcal{S}')$. This is a group-theoretic criterion, and we will call these points, as group-theoretic objects,* ***geometric points*** *and denote the collection of all of them by $\mathcal{P}^{\mathrm{geom}}(\mathcal{S})$; $\mathcal{P}^{\mathrm{geom}}(\mathcal{S})$ is identified by ι with $\mathcal{M}(\mathcal{S})$.*

Corollary 59. *Given $\mathcal{S}$, there is a group-theoretical recipe to determine whether $\mathcal{M}(\mathcal{S})$ is proper.*

Definition 60. *A geometric set $\mathcal{S}$ such that $\mathcal{M}(\mathcal{S})$ is proper will be called itself* ***proper****.*

Definition 61. *We define a partial ordering $\preceq$ on $\mathfrak{Geom}(F)$ by saying that*

$$\mathcal{S} \preceq \mathcal{S}'$$

if the following two conditions hold:

1. $\mathcal{S} \subseteq \mathcal{S}'$.

2. $\mathcal{P}^{\mathrm{geom}}(\mathcal{S}) \subseteq \mathcal{P}_{\mathcal{S}'}(\mathcal{S})$.

The category formed by this partial ordering (so a morphism $\varphi : \mathcal{S} \longrightarrow \mathcal{S}'$ is the relation $\mathcal{S} \preceq \mathcal{S}'$) is denoted by $\mathfrak{GBir}_{\max}(F)$. The maximal smooth model $\mathcal{M}$ thus extends uniquely to a functor

$$\mathcal{M} : \mathfrak{GBir}_{\max}(F) \longrightarrow \mathfrak{Bir}(F)$$

and the set of prime divisors likewise extends

$$\mathcal{D} : \mathfrak{Bir}(F) \longrightarrow \mathfrak{GBir}_{\max}(F).$$

Corollary 62. *$\mathcal{M}$ is fully faithful. The functors*

$$\mathfrak{GBir}_{\max}(F) \underset{\mathcal{D}}{\overset{\mathcal{M}}{\rightleftarrows}} \mathfrak{Bir}(F)$$

form an adjoint pair, with $\mathcal{M}$ right-adjoint to $\mathcal{D}$.

3.4 Algebraic, Numerical, and Linear Equivalence of Divisors

In this section, $\mathcal{S}$ will denote a *proper* geometric set. The **divisor group** $\mathrm{Div}(\mathcal{S})$ is defined to be the free abelian group generated by $\mathcal{S}$.

Definition 63. *1. We call an element $\sum a_i v_i \in \mathrm{Div}(\mathcal{S})$ **effective** if and only if each $a_i \geq 0$, and we denote this by $D \geq 0$. If $D \geq 0$ and $D \neq 0$ then we write $D > 0$. We also define a preorder on the divisors by:*

$$D \geq (\textit{resp.}\ >) D' \Longleftrightarrow D - D' \geq (\textit{resp.}\ >)\, 0.$$

2. *The **support** of a divisor D, denoted $\mathbf{supp}(D)$, is the collection of $v \in \mathcal{S}$ such that the coefficient of v in D is nonzero.*

3. *Given a divisor $D \in \mathrm{Div}(\mathcal{S})$ we may write D uniquely as*

$$D = D_+ - D_-$$

where D_+ and D_- are effective divisors, and $\mathbf{supp}(D_+) \cap \mathbf{supp}(D_-) = \emptyset$.

It is clear that

Proposition 64. *The map*

$$\mu : \mathrm{Div}(\mathcal{S}) \longrightarrow \mathrm{Div}(\mathcal{M}(\mathcal{S}))$$

given by

$$\mu\left(\sum_i a_i v_i\right) = \sum_i a_i |v_i|$$

is an isomorphism.

Let v_1 and v_2 be two distinct prime divisors. We define the **intersection pairing** to be

$$(v_1 \cdot v_2) = \sum_{p \circ v_2 \in \mathrm{Par}_2(v_2)} (p, v_1 \cdot v_2; \mathcal{S}),$$

By Theorem 35,

Proposition 65. *The intersection pairing* $(v_1 \cdot v_2)$ *coincides under pushforward with the intersection pairing on* $\mathcal{M}(\mathcal{S})$ *when* $v_1 \neq v_2$ *and otherwise extends by linearity to give self-intersection on* $\mathrm{Div}(\mathcal{S})$.

Let $\mathcal{F} \subset \mathcal{S}$ be a visible affine with horizontal-vertical decomposition $\mathcal{F} = \mathcal{H} \cup \mathcal{V}$, and

1. Let p be a puncture of the base, and let T_p be its inertia group; this is the divisible hull of the image of a corresponding inertia group in $\mathcal{F}$. Then if $v \in \mathcal{S} \setminus \mathcal{F}$, we say its **multiplicity at** p is the index

$$m_p(v) = \begin{cases} [T_p^a : \pi(T_v^a)] & \text{if } T_p^a \cap T_v^a \neq \{0\} \\ 0 & \text{otherwise} \end{cases}$$

in $(\Pi_{\mathcal{F}}/D_v)^{\mathrm{ab}}$, with this equal to zero if the two groups are disjoint.

2. The **complete family** of $\mathcal{F}$ will be the subset

$$\mathbf{Fam}(\mathcal{F}) \subset \mathrm{Div}(\mathcal{S})$$

given by

$$\mathcal{V} \cup \{\sum_{v \in \mathcal{S}} m_{p_i}(v)v \mid p_i \text{ a puncture of the base}\}.$$

3. We define **group-theoretical algebraic equivalence** to be the equivalence relation on $\mathrm{Div}(\mathcal{S})$ generated by $\mathbf{Fam}(\mathcal{F})$ for all visible affines $\mathcal{F}$ and denote this by $\sim_{\text{alg}}$. We define **group-theoretical linear equivalence** to be the equivalence relation generated by $\mathbf{Fam}(\mathcal{F})$ for all visible affines with base having trivial unramified fundamental group (that is, for genus 0 base) and denote this by $\sim_{\text{lin}}$. Two divisors D_1 and D_2 are said to be **group-theoretically numerically equivalent** if and only if for any divisor E we have $(D_1 \cdot E) = (D_2 \cdot E)$. This equivalence relation is denoted by $\sim_{\text{num}}$.

4. Let $D \in \mathrm{Div}(\mathcal{S})$ and $D > 0$. Then we define $|D|$ to be the set of effective divisors linearly equivalent to D.

5. Let $D \in \mathrm{Div}(\mathcal{S})$ and let

$$D = E - E' \tag{3.92}$$

be some expression of D as a difference of two effective divisors.

Then we define the **group-theoretical complete linear system** to be

$$|D| = \{D' - E' \mid D' \in |E| \text{ and } D' - E' \geq 0\}. \tag{3.93}$$

We see immediately:

Proposition 66. *Let $\mathcal{S}$ be a proper geometric set of prime divisors. Then the pushforward of group-theoretical linear (respectively, algebraic and numerical) equivalence on* $\mathrm{Div}(\mathcal{S})$ *by* μ *induces linear (respectively, algebraic and numerical) equivalence on* $\mathrm{Div}(\mathcal{M}(\mathcal{S}))$.

Corollary 67. *The group-theoretical complete linear systems* $|D|$ *coincide with complete linear systems* $|\mu(D)|$ *on* $\mathcal{M}(\mathcal{S})$ *and form finite-dimensional projective spaces over* $\overline{\mathbb{Q}}$*, and the lines in this projective space are given by linear families.*

In particular, the Picard and Néron-Severi groups of $\mathcal{M}(\mathcal{S})$ are group-theoretical invariants of $(G_F, \mathcal{S})$.

3.5 Local Geometry: Tangent Spaces

Let $\mathcal{S}$ be a geometric set on a two-dimensional function field F. As we work locally, we do not need properness.

Definition 68. *Let* $p \in \mathcal{P}^{\mathrm{geom}}(\mathcal{S})$ *be a point. Then*

$$\Delta^s(p) = \{v \in \Delta(p) \mid v \textit{ is smooth at } p\}. \tag{3.94}$$

Definition 69. *Let* $w \in \mathcal{S}$*, smooth at a rank-*2 *Parshin chain* $q \circ w$ *such that* $[q \circ w] \in \mathcal{P}^{\mathrm{geom}}(\mathcal{S})$ *and let* $p \in \Delta^s([q \circ w])$*. Then we say that* v *and* w *are* ***tangent to order*** n *at* p *if and only if the local intersection number*

$$(p, v \cdot w; \mathcal{S}) \geq n + 1. \tag{3.95}$$

Because $|v|$ and $|w|$ are actually tangent to order n at p, tangency to order n forms an equivalence relation, which we call $\sim_{n-\tan}$, and we thus recover the projectivized jet space

$$\mathbb{P}\mathcal{J}_p^n = \Delta^s(p)/\sim_{n-\tan},$$

at p. In particular,

$$\mathbb{P}\mathcal{J}_1 = \mathbb{P}T_p,$$

the projectivized tangent space to $\mathcal{M}(\mathcal{S})$ at p. As $\mathcal{M}(\mathcal{S})$ is a smooth surface, $\mathbb{P}T_p$ is a projective line.

3.6 Projective Embeddings and Projective Coordinate Rings

We start with some basic projective geometry, as in Artin [Art88]. Let (X, L) be an abstract projective space, given as its set of points X and a set of lines L.

1. A subset $Y \subseteq X$ is **linearly closed** if for any two points $P, P' \in Y$ the line $\overline{PP'} \subseteq Y$. The linearly closed sets are closed under intersection. The **linear closure** of Y in X is the intersection of all linearly closed spaces which contain Y. As X is linearly closed, the linear closure always exists. We denote the linear closure of the union of a collection of subsets $V_1, \ldots, V_n \subseteq X$ by $\overline{V_1 \cdots V_n}$.

2. A point $P \in X$ is said to be **linearly independent** of a subset Y if and only if $P \notin \overline{Y}$. In particular, we call a set $P_1, \ldots, P_n \in X$ **linearly independent** if for any subset $M \subseteq \{1, \ldots, n\}$ and any $k \notin M$ we have $\overline{(P_m)_{m\in M}} \subsetneq \overline{P_k (P_m)_{m\in M}}$.

3. The **dimension** of X is the cardinality of a maximal set of linearly independent points minus 1, and is denoted $\dim X$ (this is possibly infinite).

Let $\mathcal{S}$ be a proper geometric set on a function field F of dimension 2.

Definition 70. *We say a point* $p \in \mathcal{P}(\mathcal{S})$ *is* ***supported*** *on a divisor* $D \in \mathrm{Div}(\mathcal{S})$ *if and only if* $\Delta(p) \cap \mathbf{supp}(D)$ *is nonempty. We say a set* $X \subseteq \mathrm{Div}(\mathcal{S})$ ***separate points*** *if and only if for any two points* $p_1, p_2 \in \mathcal{P}(\mathcal{S})$ *there are two divisors* $D_1, D_2 \in X$ *such that* $p_1 \in \mathbf{supp}(D_1), p_1 \notin \mathbf{supp}(D_2)$ *and* $p_2 \notin \mathbf{supp}(D_1), p_2 \in \mathbf{supp}(D_2)$. *Given a set* $S \subseteq \mathrm{Div}(\mathcal{S})$*, we will define its* ***support***

$$\mathbf{supp}(S) = \bigcup_{E \in S} \mathbf{supp}(E).$$

Definition 71. *A point* $p \in \mathcal{P}(\mathcal{S})$ *is said to be in the* ***base locus*** *(and is called a* ***base point****) of* $X \subseteq \mathrm{Div}(\mathcal{S})$ *if* p *is supported on every element of* X*; a set without base locus is called* ***basepoint free***.

Separating points is strictly stronger than basepoint free.

Definition 72. *We say the linear system* $|D|$ ***separates tangent lines*** *at* p *if and only if for any* $\ell \in \mathbb{P}T_p, |D| \cap \ell$ *is nonempty.*

Definition 73. *We call a divisor* D ***very ample*** *if* $|D|$ *separates points and tangent vectors.*

As this coincides with the algebro-geometric definition of very ample, we may, in fact, give the following equivalent definition:

Proposition 74. *A divisor* D *is very ample if and only if for every* $p \in \mathcal{P}(\mathcal{S})$ *there exists a visible affine open* $\mathcal{S}' \subset \mathcal{S}$ *of which* p *is a geometric point, in which every vertical divisor is an element of* $|D|$.

For any divisor D and any effective divisor E there is an injective map

$$\alpha : |D| \longrightarrow |D + E|$$

given by adding E to each effective divisor in $|D|$. This invokes a map in general

$$\alpha : |D| \times |E| \longrightarrow |D + E|$$

and in our particular case,

$$\alpha^n : \mathrm{Sym}^n(|D|) \longrightarrow |nD|$$

as addition is symmetric. We will now call a divisor D **adequate** if the linear closure of $\alpha^n(\mathrm{Sym}^n(|D|))$ in $|nD|$ is all of $|nD|$. Any very ample divisor is adequate; indeed, a very ample divisor gives relations on the projective coordinate ring, which is generated in the first dimension.

Definition 75. *We define a* ***projectivizing datum*** *to be a quadruple* $(\mathcal{S}, D, V, \rho)$ *where:*

1. $\mathcal{S}$ *is a proper geometric set.*

2. D *is a very ample divisor.*

3. V *is a* $\overline{\mathbb{Q}}$*-vector space.*

4. $\rho : |D| \longrightarrow \mathbb{P}V$ *is an isomorphism.*

We know by the fundamental theorem of projective geometry that this ρ is determined up to a semilinear automorphism of V, and fixing a ρ rids us of the indeterminacy. Let $\mathbf{ProjData}(\mathcal{S})$ be the collection of projectivizing data for $\mathcal{S}$. The outer automorphisms of G_F act on the collection of all projectivizing data, by translation of the corresponding $\mathcal{S}$, D, and $|D|$.

Proposition 76. *Given a projectivizing data* $(\mathcal{S}, D, V, \rho)$*, the maps*

$$\mathrm{Sym}^n(|D|) \longrightarrow |nD|$$

induce a canonical isomorphism

$$\mathrm{Sym}^n(\rho) : \mathbb{P}(\mathrm{Sym}^n(V)/I_n) \simeq |nD|$$

compatible with all α^i, where $I_n \subset \mathrm{Sym}^n(V)$ is a vector subspace.

This immediately gives us:

Theorem 77 (Geometric Reconstruction)**.** *A projectivizing datum $(\mathcal{S}, D, V, \rho)$ gives $\mathcal{M}(\mathcal{S})$ uniquely the structure of a smooth, projective $\overline{\mathbb{Q}}$-variety, so that*

$$\mathcal{M}(\mathcal{S})_{/\overline{\mathbb{Q}}} \simeq \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathrm{Sym}^n(V)/I_n\right),$$

which induces an isomorphism

$$\eta : \mathrm{Frac}(\overline{\mathbb{Q}}(\mathcal{M}(\mathcal{S}))) \longrightarrow F,$$

and a corresponding isomorphism

$$\eta : G_F \longrightarrow G_{\overline{\mathbb{Q}}(\mathcal{M}(\mathcal{S}))}.$$

which respects inertia and decomposition groups of divisors.

Proof of Theorem 5. There is a canonical, injective map

$$\varphi : \mathrm{Aut}(F) \longrightarrow \mathrm{Out}_{\mathrm{cont}}(G_F), \tag{3.96}$$

and we construct the inverse

$$\psi : \mathrm{Out}_{\mathrm{cont}}(G_F) \to \mathrm{Aut}(F). \tag{3.97}$$

Choose a projectivizing datum $(\mathcal{S}, D, V, \rho)$ fixed by no non-trivial automorphisms of F. By Theorem 77, we have

$$\eta : \overline{\mathbb{Q}}[\mathcal{M}(\mathcal{S}) \setminus D] \to F \tag{3.98}$$

which gives an injection from a finitely generated ring to its field of fractions; the automorphisms of F then act simply transitively on this set, as $(\mathcal{S}, D, V, \rho)$ is fixed by no non-trivial automorphisms of F; however, as $(\mathcal{S}, D, V, \rho)$ are determined by group theory, $\mathrm{Out}_{\mathrm{cont}}(G_F)$ acts on this set and this gives our section ψ.

We must now prove that every continuous outer automorphism $\zeta \in \mathrm{Out}_{\mathrm{cont}}(G_F)$ for which $\psi(\zeta) = e$ is an inner automorphism. Choose ζ' to be a genuine continuous automorphism in the class of ζ.

Let $\{L_n\}$ be a sequence of finite, Galois extensions of F and

$$\Gamma_n = \mathrm{Gal}(\overline{F}|L_n)$$

which satisfy the following properties:

1. $\zeta(\Gamma_n) = \Gamma_n$.

2. $\bigcap_n \Gamma_n = \{e\}$.

That such a filtration exists comes from the fact that there are only finitely many translates of any finite-index, closed subgroup of G_F, which follows from the fact that there is a group-theoretic recipe to detect ramification information, and that there are only finitely many covers of a given degree with prescribed ramification (which in turn follows from the fact that geometric fundamental groups in question are finitely-presented). But we can reconstruct L_n from Γ_n, so any class of ζ then gives us an action of μ on $\bigcup_n L_n = \overline{F}$ trivial on F, which shows that its action on Γ_n is induced by conjugation by an element of G_F/Γ_n and as

$$G_F = \varprojlim G_F/\Gamma_n, \tag{3.99}$$

ζ is inner, and ψ is an isomorphism. □

3.7 The Topological Model, Hodge Numbers, Betti Numbers, and $\mathfrak{GBir}$

Let $\mathcal{S}$ be a geometric set on G_F.

Definition 78. *A* ***visible étale open*** *of $\mathcal{S}$ is a pair (Σ, η) where $\Sigma \subseteq \mathcal{S}$ is a visible affine and $\eta \subseteq \Pi_\Sigma$ is a finite-index subgroup.*

Definition 79. *We say that a collection $\{(\Sigma_i, \eta_i)\}_{i \in I}$ of visible étale opens are* ***compatible*** *if and only if for any i, j and any visible affine $\mathcal{U} \subseteq \Sigma_i, \Sigma_j$, we have*

$$\rho^{-1}_{\mathcal{U}\Sigma_i}(\eta_i) = \rho^{-1}_{\mathcal{U}\Sigma_j}(\eta_j).$$

We define the ***points*** *$\mathcal{P}(\{(\Sigma_i, \eta_i)\})$ of a compatible collection by*

$$\mathcal{P}(\{(\Sigma_i, \eta_i)\}) = \bigcup_i \mathcal{A}(\Sigma_i).$$

Definition 80. *We say a compatible collection of visible étale opens $\{(\Sigma_i, \eta_i)\}_{i \in I}$* ***dominates***

(respectively, ***covers*** *and* ***is equivalent to****)* $\{(\Sigma'_j, \eta'_j)\}_{j\in J}$ *if*

1. $\mathcal{P}(\{(\Sigma_i, \eta_i)\}_{i\in I}) \subseteq$ *(resp.,* $=$ *and* $=$*)*$\mathcal{P}(\{(\Sigma'_j, \eta'_j)\}_{j\in J})$*.*

2. *For each* $i \in I$*, there exists* $j \in J$ *and a visible affine* $\mathcal{U} \subseteq \Sigma_i, \Sigma'_j$ *such that*

$$\rho^{-1}_{\mathcal{U}\Sigma_i}(\eta_i) \subseteq \text{(resp., } \subseteq \text{ and } =)\rho^{-1}_{\mathcal{U}\Sigma'_j}(\eta'_j). \tag{3.100}$$

Definition 81. *We define the* ***fundamental group***

$$\Pi_{(\Sigma_i,\eta_i)} = \rho^{-1}_{\emptyset\Sigma_i}(\eta_i)/\langle T_v \cap \rho^{-1}_{\emptyset\Sigma_1}(\eta_i)\rangle_{v\in\bigcup_i \Sigma_i}.$$

Proposition 82. *If* $\{(\Sigma_i, \eta_i)\}$ *dominates* $\{(\Sigma'_j, \eta'_j)\}$ *then the domination induces a map*

$$\rho_{\{(\Sigma_i,\eta_i)\}\{(\Sigma'_j,\eta'_j)\}}\Pi_{\{(\Sigma_i,\eta_i)\}} \longrightarrow \Pi_{\{(\Sigma'_j,\eta'_j)\}}.$$

Definition 83. *The* ***étale site*** $\text{Ét}(\mathcal{S})$ *has as its objects compatible collections, and a morphism takes an étale open to one that it dominates*

$$\text{Hom}(\{(\Sigma_i, \eta_i)\}, \{(\Sigma'_j, \eta'_j)\}) = \begin{cases} \Pi_{\{(\Sigma'_j,\eta'_j)\}}/N(\Pi_{\{(\Sigma_i,\eta_i)\}}) & if \{(\Sigma_i, \eta_i)\} \textbf{ dominates } \{(\Sigma'_j, \eta'_j)\} \\ \emptyset & \text{otherwise,} \end{cases}$$

where the N in the quotient denotes the normalizer of a subgroup.

Proposition 84. $\text{Ét}(\mathcal{S})$ *is naturally equivalent to the full subcategory of the étale site* $\text{Ét}(\mathcal{M}(\mathcal{S}))$ *of finite (rather than quasi-finite) étale maps.*

The étale site computes the étale cohomology of the variety $\mathcal{M}(\mathcal{S})$, so these invariants are encoded in the group theory of the geometric set $\mathcal{S}$. A dévissage argument then gives the étale cohomology of any variety defined in this way. This gives a group-theoretic argument for the Betti numbers of $\mathcal{M}(\mathcal{S})$.

Let $\mathcal{S}$ now be proper. Knowing $h^{1,1}$ and all Betti numbers of $\mathcal{M}(\mathcal{S})$ determines its Hodge numbers, by the symmetry properties of the Hodge diamond. However, as

$$h^{1,1} = \dim_{\mathbb{Q}}(\mathrm{NS}(X) \otimes \mathbb{Q}), \tag{3.101}$$

this gives recipes to determine the Hodge numbers of $\mathcal{M}(\mathcal{S})$.

Definition 85. *We define $\mathfrak{GBir}(F)$ to be the category obtained as the preorder on compatible étale neighborhoods $\{(U_i, \Pi_{U_i})\}$ given by domination.*

Étale descent [Gro03, VIII.2] allows us to extend the functor $\mathcal{M}$ to this category, and as every smooth variety has a cover by visible affines, we get:

Theorem 86. *The functor*

$$\mathcal{M} : \mathfrak{GBir}(F) \longrightarrow \mathfrak{Bir}(F)$$

is a natural equivalence, and

$$\mathfrak{GBir}_{\max}(F)$$

is a full subcategory of $\mathfrak{GBir}(F)$.

Chapter 4

An Application to Galois Groups of Number Fields

Theorem 6 is now a straightforward consequence of Theorem 5. We now write down explicit examples of fields F which satisfy the hypotheses of Theorem 6. We recall the following adaptation of the main theorem from [Tur94]:

Theorem 87 (Turbek)**.** *Consider the affine curve*

$$C_\alpha : x^7 + y^{20} + \alpha xy + 1 = 0.$$

Then for all but finitely many $\alpha \in \overline{\mathbb{Q}}$, this curve is nonsingular, hyperbolic, and has trivial automorphisms over its field of definition, which is $\mathbb{Q}(\alpha)$.

Let C_1 and C_2 be two non-isomorphic, complete, hyperbolic curves over $\overline{\mathbb{Q}}$, with no $\overline{\mathbb{Q}}$-automorphisms, so that the compositum of their minimal fields of definition is k. $C_1 \times C_2$ is the image of any birational variety under the canonical map, so is a birational invariant, and as $C_1 \times C_2$ has no automorphisms over $\overline{\mathbb{Q}}$, there are no automorphisms of $\overline{\mathbb{Q}}(C_1 \times C_2)$ over

$\overline{\mathbb{Q}}$. Let $k = \mathbb{Q}(\alpha)$ (we may always write it this way by Steinitz's theorem). Then, the field $F = \overline{\mathbb{Q}}(C_1 \times C_\alpha)$ satisfies the hypotheses of Theorem 6, so

$$\mathrm{Out}(G_F) \simeq G_k.$$

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