Generalizing \(J_2\) Flow Theory: Fundamental Issues in Strain Gradient Plasticity

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Abstract It has not been a simple matter to obtain a sound extension of the classical J_2 flow theory of plasticity that incorporates a dependence on plastic strain gradients and that is capable of capturing size-dependent behaviour of metals at the micron scale. Two classes of basic extensions of classical J_2 theory have been proposed: one with increments in higher order stresses related to increments of strain gradients and the other characterized by the higher order stresses themselves expressed in terms of increments of strain gradients. The theories proposed by Muhlhaus and Aifantis in 1991 and Fleck and Hutchinson in 2001 are in the first class, and, as formulated, these do not always satisfy thermodynamic requirements on plastic dissipation. On the other hand, theories of the second class proposed by Gudmundson in 2004 and Gurtin and Anand in 2009 have the physical deficiency that the higher order stress quantities can change discontinuously for bodies subject to arbitrarily small load changes. The present paper lays out this background to the quest for a sound phenomenological extension of the rate-independent J_2 flow theory of plasticity to include a dependence on gradients of plastic strain. A modification of the Fleck-Hutchinson formulation that ensures its thermodynamic integrity is presented and contrasted with a comparable formulation of the second class wherein the higher order stresses are expressed in terms of the plastic strain rate. Both versions are constructed to reduce to the classical J_2 flow theory of plasticity when the gradients can be neglected and to coincide with the simpler and more readily formulated J_2 deformation theory of gradient plasticity for deformation histories characterized by proportional straining.

Keywords: Strain gradient plasticity. Deformation plasticity. J_2 flow theory of plasticity

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1 **Introduction**

A wide array of micron scale experiments have revealed strong size-dependent strengthening associated with plastic deformations involving gradients of strain. In parallel, a large theoretical literature has appeared seeking to encapsulate strain gradient effects into a theory of micron scale plasticity. Some of theory has been conducted within the context of a single crystal framework, but, equally, there has been interest in developing simple phenomenological extensions of the classical J_2 flow theory of plasticity. Indeed, many of the relevant experiments have been conducted on small grained polycrystalline materials, and most of the attempts to correlate theory with these experiments have been made using phenomenological isotropic theories. It is now generally accepted that these theories must be higher order, not only by incorporation strain gradients but also in having higher order stresses that are work conjugate to the strain gradients. Such theories open up the possibility of modelling extra boundary conditions outside the scope of conventional theory. An insightful critical overview of the status of these theories as of 2004 was given by Gudmundson [1]

One of the most widely used phenomenological extensions of rate-independent J_2 theory is that of Fleck and Hutchinson [2] which has features in common with an earlier version proposed by Muhlhaus and Aifantis [3]. The simplest version introduces only a single new material length parameter. Moreover, the form of the theory lends itself nicely to numerical implementation. However, Gudmundson [1] and Gurtin and Anand [4] noted that there exist strain histories for which this theory, as formulated, does meet thermodynamic restrictions related to the requirement of non-negative plastic dissipation—clearly unacceptable for a basic theory. A second class of basic phenomenological theories free of this thermodynamic deficiency was proposed by Gudmundson [1] and Gurtin and Anand [5]. These authors circumvented the dissipation problem by expressing the higher order stresses in terms of the increments of plastic strain and its gradient. An unintended consequence of this new formulation has been highlighted by the work of Fleck and Willis [6], who formulated variational principles for incremental boundary value problems based on this class of theories. The expression of higher order stresses in terms of *increments* of strain and strain gradients leads to the possibility of discontinuous temporal changes in the higher order stresses. Specifically, a change in the direction of loading on a body will generally give rise to finite changes in the higher order stresses within the body, i.e., finite stress changes due to infinitesimally small loading changes.

While the current understanding of the connection between higher order stresses and dislocation distributions is incomplete, finite changes in stress due to infinitesimal changes in strains are not likely to be acceptable from a physical point of view. Thus, as will be argued later in this paper, it is likely that this second class of theories will need to be modified in some manner to rectify this physical deficiency.

In Section 3, a relatively simple modification of the theory of Fleck and Hutchinson [2] is proposed to correct the thermodynamic deficiency noted above. Section 4 presents and discusses the corresponding generalization of J_2 flow theory for the second class of theories. This paper limits attention to the simplest extensions of J_2 plasticity, in part, because of the ubiquitous role that classical J_2 theory plays in describing bulk plasticity of solids and, in part, to expose in the clearest possible manner the issues that arise in creating the extensions. The issues are not confined to the phenomenological theories. They arise as well in the continuum formulations of single crystal plasticity that depend of gradients of plastic slip.

The objectives in generalizing the J_2 theory are as follows.

1) To construct a phenomenological isotropic theory of plasticity that incorporates a dependence on the gradients of plastic strain in a simple meaningful manner and that reduces to the classical J_2 flow theory in the limit the gradients are sufficiently small.
2) To have as inputs the isotropic moduli, Young's modulus E and Poisson's ratio ν, the uniaxial tensile relation between stress and plastic strain, $\sigma_0(\varepsilon_P)$, and one or more material length parameters, ℓ, characterizing the gradient dependence. The tensile relation, $\sigma_0(\varepsilon_P)$, is arbitrary but monotonically increasing representing a hardening solid.
3) To coincide with the J_2 deformation theory with the same inputs for proportional straining based on the reasoning given in Section 2.

Similar objectives have been pursued in formulating lower order strain gradient plasticity theories that employ only the Cauchy stress by Acharya and Bassani [7], Chen and Wang [8], and Huang et al. [9].

2 Strain gradient version of J_2 deformation theory

Deformation, or total, theories of plasticity are a special class of path-independent nonlinear elasticity theories, while flow theories are incremental and inherently path-dependent. Classical J_2 deformation theory and J_2 flow theory are linked by the fact that they coincide when the deformation involves proportional straining, given that both theories have been fit to the same tensile stress-strain data. Here, following Fleck and Hutchinson [2,10], a strain gradient version of deformation theory will be introduced at the start. It will be used as a template for the flow theory in the sense that the flow theory will be constructed to coincide with the deformation theory for proportional straining histories. Deformation theory can be used to play this fundamental role, as it does in conventional plasticity theory, because for proportional straining histories the material can be modelled as being nonlinear elastic. The clarity provided by that framework can be brought to bear on the incorporation of strain gradient effects.

The theories in this paper will be restricted to small strain, rate-independent behaviour. As noted above, the material inputs are the isotropic elastic properties, the uniaxial relation, $\sigma_0(\varepsilon_P)$, and, in this paper, a single material length parameter, ℓ. The length parameter is the only parameter not present in the classical theory. For all these theories, u_i is the displacement vector, $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ is the strain, ε'_{ij} is its deviator, σ_{ij} is the symmetric Cauchy stress, s_{ij} is its deviator, and the effective stress is $\sigma_e = \sqrt{3 s_{ij} s_{ij}/2}$. Throughout, $m_{ij} = 3 s_{ij}/(2\sigma_e)$ is a dimensionless deviator tensor co-directional with the deviator stress.

For the deformation theory, the "plastic strain" is given by $\varepsilon_{ij}^P = \varepsilon_P m_{ij}$ where ε_P is the magnitude, $\varepsilon_P = \sqrt{2\varepsilon_{ij}^P \varepsilon_{ij}^P / 3}$. The Cauchy stress is given by

$$\sigma_{ij} = 2\mu\varepsilon_{ij}^{e\prime} + \lambda\varepsilon_{kk}^{e}\delta_{ij} \ , \qquad \varepsilon_{ij}^{e} = \varepsilon_{ij} - \varepsilon_{ij}^{P} \tag{1}$$

with $\varepsilon_{ij}^{e\prime}$ as the deviator of the "elastic strain" ε_{ij}^{e} and with $\mu = E/[2(1+\nu)]$, $\lambda = E/[3(1-2\nu)]$ and δ_{ij} as the Kronecker delta. In the simplest strain gradient deformation theory of plasticity of the various versions considered in [2], the spatial gradient, $\varepsilon_{P,i}$, is used as the measure of the plastic strain gradients. A gradient enhanced effective plastic strain,

$$E_P = \sqrt{\varepsilon_P{}^2 + \ell^2 \varepsilon_{P,i}\varepsilon_{P,i}} \ , \tag{2}$$

is introduced to capture the combined effect of the plastic strain and strain gradients with ℓ ensuring dimensional consistency. The strain energy density of the solid is taken to be

$$U(\varepsilon_{ij}^{e},\varepsilon_{P},\varepsilon_{P,i}) = \mu\varepsilon_{ij}^{e\prime}\varepsilon_{ij}^{e\prime} + \frac{1}{2}\lambda\varepsilon_{kk}^{e\ 2} + U^{P}(E_{P}) \tag{3}$$

where $U^{P}(E_{P})$ is defined in terms of the tensile stress-plastic strain curve of the material by

$$U^{P}(E_{P}) = \int_{0}^{E_{P}} \sigma_{0}(\varepsilon_{P})d\varepsilon_{P} \tag{4}$$

The replacement of ε_{P} by E_{P} in $U^{P}(\varepsilon_{P})$ above reveals the essence of the role of the plastic strain gradient in this phenomenological theory. In words, the plastic work needed to deform the material element in the presence of strain gradients under proportional straining as measured by E_{P} is taken equal to that at the same strain, $\varepsilon_{P} = E_{P}$, in the absence of gradients, consistent with the notion that the gradient contribution to E_{P} accounts for the additional stored geometrically necessary dislocations.

The stress quantities that are work conjugate to the strain quantities follow as [2]

$$\dot{U}(\varepsilon_{ij}^{e},\varepsilon_{P},\varepsilon_{P,i}) = \frac{\partial U}{\partial \varepsilon_{ij}^{e}}\dot{\varepsilon}_{ij}^{e} + \frac{\partial U}{\partial \varepsilon_{P}}\dot{\varepsilon}_{P} + \frac{\partial U}{\partial \varepsilon_{P,i}}\dot{\varepsilon}_{P,i} = \sigma_{ij}\dot{\varepsilon}_{ij}^{e} + Q\dot{\varepsilon}_{P} + \tau_{i}\dot{\varepsilon}_{P,i} \tag{5}$$

with σ_{ij} given by (1) and

$$Q = \sigma_{0}(E_{P})\frac{\varepsilon_{P}}{E_{P}}, \quad \tau_{i} = \ell^{2}\sigma_{0}(E_{P})\frac{\varepsilon_{P,i}}{E_{P}} \tag{6}$$

The incremental form of (6) will be important in the sequel:

$$\dot{Q} = C\dot{\varepsilon}_{P} + C_{j}\dot{\varepsilon}_{P,j} \quad \& \quad \dot{\tau}_{i} = C_{i}\dot{\varepsilon}_{P} + C_{ij}\dot{\varepsilon}_{P,j} \tag{7}$$

with

$$\begin{aligned} C &= \frac{\partial^{2}U^{P}}{\partial \varepsilon_{P}^{\ 2}} = \left(\frac{\sigma_{0}(E_{P})}{E_{P}} + \frac{d}{dE_{P}}\left(\frac{\sigma_{0}(E_{P})}{E_{P}}\right)\frac{\varepsilon_{P}^{\ 2}}{E_{p}}\right) \\ C_{j} &= \frac{\partial^{2}U^{P}}{\partial \varepsilon_{P}\partial \varepsilon_{P,j}} = \ell^{2}\frac{d}{dE_{P}}\left(\frac{\sigma_{0}(E_{P})}{E_{P}}\right)\frac{\varepsilon_{P,j}\varepsilon_{P}}{E_{P}} \\ C_{ij} &= C_{ji} = \frac{\partial^{2}U^{P}}{\partial \varepsilon_{P,i}\partial \varepsilon_{P,j}} = \ell^{2}\left(\frac{\sigma_{0}(E_{P})}{E_{P}}\delta_{ij} + \ell^{2}\frac{d}{dE_{P}}\left(\frac{\sigma_{0}(E_{P})}{E_{P}}\right)\frac{\varepsilon_{P,i}\varepsilon_{P,j}}{E_{P}}\right) \end{aligned} \tag{8}$$

In the limit $\ell \to 0$, corresponding to no gradient dependence, this constitutive model reduces to classical J_2 deformation theory with $Q = \sigma_e$.

Within regions with non-zero ε_P, the principle of virtual work for a body with volume V and surface S is

$$\int_V \left(\sigma_{ij} \delta\varepsilon_{ij}^e + Q\delta\varepsilon_P + \tau_i \delta\varepsilon_{P,i} \right) dV = \int_S \left(T_i \delta u_i + t\delta\varepsilon_P \right) dS \tag{9}$$

with T_i as the surface traction and t is the higher order traction that works through ε_P at the surface. Body forces are omitted. The equations of equilibrium are

$$\sigma_{ij,j} = 0 \quad \text{and} \quad \sigma_e = Q - \tau_{i,i} \tag{10}$$

and, on the boundary with n_i as the outward unit normal,

$$T_i = \sigma_{ij} n_j \quad \text{and} \quad t = \tau_j n_j \tag{11}$$

If plastic deformation begins at zero stress, i.e., $\sigma_0(0) = 0$, then (9)-(11) apply throughout the body. However, if $\sigma_0(0) > 0$, Q, τ_i and t all vanish within any elastic region for which $\varepsilon_P = 0$. At an internal boundary between an elastic region and a plastic region, $\varepsilon_P \to 0$ as the boundary is approached from the plastic side with $\varepsilon_{P,i} \neq 0$. As a result, by (6), $Q \to 0$ at the boundary but t will generally not vanish at the plastic side of the boundary. It is assumed that the elastic region can support the non-zero t acting across the boundary, analogous to what one would assume for a boundary between a plastically deforming region and a rigid material or an elastic material with higher yield strength.

The potential energy functional for a deformation theory solid with volume V and surface S is

$$PE(\mathbf{u}, \varepsilon_P) = \int_V U(\varepsilon_{ij}^e, \varepsilon_P, \varepsilon_{P,i})\, dV - \int_{S_T} \left(T_i u_i + t\varepsilon_P \right) dS \tag{12}$$

where T_i and t are prescribed on the portion of the surface S_T. Among all admissible fields, (u_i, ε_P), with $\varepsilon_P \geq 0$, the potential energy is minimized by the solution, assuming the tensile input, $\sigma_0(\varepsilon_P)$, is monotonically increasing. If ε_P is unconstrained on the portion of the boundary, then $t = 0$ on that boundary, while, if ε_P is constrained to be zero, t will generally be non-zero at the boundary. An internal elastic-plastic boundary in a homogeneous material must be located as part of the minimization process, and the condition that ε_P vanish as the boundary is approached on the plastic side must be imposed. This principle reduces to the corresponding minimum principle for the classical theory J_2 deformation theory in the limit $\ell\sqrt{\varepsilon_{P,i}\varepsilon_{P,i}} << \varepsilon_P$.

2.1 Proportional straining

Consider the highly restricted set of fields, referred to as proportional straining, which increase according to

$$\varepsilon_{ij} = \zeta\,\overline{\varepsilon}_{ij}\,,\quad \varepsilon_P = \zeta\,\overline{\varepsilon}_P\,,\quad \varepsilon_{P,i} = \zeta\,\overline{\varepsilon}_{P,i} \tag{13}$$

with ζ as a load parameter which increases monotonically from zero. The barred quantities may vary in space but they are independent of ζ. For proportional straining, Q and τ_i are given by (6) and it is readily shown that the increments in (7) satisfy

$$\dot{Q} = \frac{d\sigma_0(E_P)}{dE_P}\dot{\varepsilon}_P\,,\quad \dot{\tau}_i = \ell^2\frac{d\sigma_0(E_P)}{dE_P}\dot{\varepsilon}_{P,i} \tag{14}$$

with $E_P = \zeta\,\overline{E}_P = \zeta\sqrt{\overline{\varepsilon}_P{}^2 + \ell^2\overline{\varepsilon}_{P,i}\overline{\varepsilon}_{P,i}}$.

The flow theories constructed below will be required to coincide with this deformation theory for proportional straining. The rationale for this requirement is similar to that for the coincidence of the conventional versions of J_2 flow and deformation theory. Using the invariants chosen to formulate the theories (in this paper, $\varepsilon_{ij}^{e\prime}\varepsilon_{ij}^{e\prime}$, ε_P and $\varepsilon_{P,i}\varepsilon_{P,i}$), one can model the solid as a small strain, nonlinear elastic solid if the straining histories are proportional. The straightforward and unambiguous derivation above which uncovers the new stress quantities, Q and τ_i, provides a valuable constraint and template for the flow theory version for proportional straining. As noted in the Introduction the versions of the theories discussed in this paper are based on the simplest choice of invariant of the gradient of plastic strain. The process given below for constructing the flow theories can be extended to other choices of invariants, such as those detailed in [2].

3 Strain gradient version of J_2 flow theory #1—increments of higher order stresses dependent on increments of strain gradients

In this section a constitutive relation is proposed relating increments of the Cauchy stress and increments of the new stresses, $\dot{Q}$ and $\dot{\tau}_i$, to increments of strain, plastic strain and strain gradient. The relation will be constructed such that it coincides with the deformation version in the previous section under situations in which the straining is proportional. An alternative version will be presented in the next section in which the stresses themselves, Q and τ_i, are specified in terms of the increments of the plastic strain gradient, following the construction suggested by Gudmundson [1] and Gurtin and Anand [5]. Both of these versions employ ε_P and $\varepsilon_{P,i}$ as the measures of plastic strain and strain gradient, along with the additional stress quantities, Q and τ_i.

In both flow theories, the plastic strain rate (not the plastic strain) is constrained to be co-directional to s_{ij},

$$\dot{\varepsilon}_{ij}^{P} = \dot{\varepsilon}_P m_{ij}, \tag{15}$$

where $m_{ij} = 3s_{ij}/(2\sigma_e)$, as before, with $\dot{\varepsilon}_P = \sqrt{2\dot{\varepsilon}_{ij}^{P}\dot{\varepsilon}_{ij}^{P}/3}$. The effective plastic strain is updated as an integral over the history of deformation, $\varepsilon_P = \int \dot{\varepsilon}_P dt$, with $\varepsilon_{P,i} = \left(\int \dot{\varepsilon}_P dt\right)_{,i} = \int \dot{\varepsilon}_{P,i} dt$. The principle of virtual work (9) applied to the incremental problem is

$$\left.\begin{aligned} &\int_V \left(\dot{\sigma}_{ij}\delta\dot{\varepsilon}_{ij}^{e}\right)dV = \int_S \left(\dot{T}_i\delta\dot{u}_i\right)dS && \text{elastic regions} \\ &\int_V \left(\dot{\sigma}_{ij}\delta\dot{\varepsilon}_{ij}^{e} + \dot{Q}\delta\dot{\varepsilon}_P + \dot{\tau}_i\delta\dot{\varepsilon}_{P,i}\right)dV = \int_S \left(\dot{T}_i\delta\dot{u}_i + \dot{t}\delta\dot{\varepsilon}_P\right)dS && \text{plastic regions}\end{aligned}\right\} \tag{16}$$

The associated incremental equilibrium conditions require $\dot{\sigma}_{ij,j} = 0$ throughout the body and $\dot{\sigma}_e = \dot{Q} - \dot{\tau}_{i,i}$ within the plastic regions. Boundary conditions involve specification of $\dot{T}_i = \dot{\sigma}_{ij}n_j$ or $\dot{u}_i$ on all boundaries and $\dot{t} = \dot{\tau}_i n_i$ or $\dot{\varepsilon}_P$ on boundaries bordering plastically deforming regions.

The relation between the Cauchy stress and the elastic strains in (1) also continues to apply with incremental form

$$\dot{\sigma}_{ij} = 2\mu\dot{\varepsilon}_{ij}^{e\prime} + \lambda\dot{\varepsilon}_{kk}^{e}\delta_{ij} \tag{17}$$

where $\dot{\varepsilon}_{ij}^{e} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^{P}$ and $\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2$.

The version of the incremental higher order stress theory proposed by Fleck and Hutchinson [2] employs (14) together with (17) as the incremental constitutive relation for plastic loading. The resulting theory coincides with the version of the J_2 deformation prescribed in the previous section for proportional straining, but, as noted by Gudmundson [1] and Gurtin and Anand [4], it can violate thermodynamic restrictions on non-negative plastic dissipation. Specifically, in the formulation in [2], $Q\dot{\varepsilon}_P + \tau_i \dot{\varepsilon}_{P,i}$ is regarded as the plastic dissipation, but the requirement,

$$Q\dot{\varepsilon}_P + \tau_i \dot{\varepsilon}_{P,i} \geq 0 \,, \tag{18}$$

will be violated for certain non-proportional strain histories. For the special case for which the input tensile curve has a constant tangent modulus (as considered, for example, by Muhlhaus and Aifantis [3]), the requirement of positive plastic dissipation can be met by interpreting the gradient contributions as recoverable, or energetic in the terminology of Gurtin and Anand [4], and not dissipative. However, a constant tangent modulus is not a realistic restriction for a general plasticity model.

In what follows, an incremental constitutive relation is proposed which meets thermodynamic restrictions and retains the property that it coincides with the deformation theory for proportional straining. For plastic loading, the construction includes two types of contributions to the higher order stress quantities: recoverable and dissipative according to

$$(\dot{Q}, \dot{\tau}_i) = (\dot{Q}^{rec} + \dot{Q}^{dis}, \dot{\tau}_i^{rec} + \dot{\tau}_i^{dis}) \tag{19}$$

The recoverable contributions, together with the Cauchy stress, are derived from a free energy taken as

$$\psi(\varepsilon_{ij}^e, \varepsilon_P, \varepsilon_{P,i}) = \psi^e(\varepsilon_{ij}^e) + \psi^p(\varepsilon_P, \varepsilon_{P,i}) \tag{20}$$

with

$$\psi^e(\varepsilon_{ij}^e) = \mu \varepsilon_{ij}^{e\prime} \varepsilon_{ij}^{e\prime} + \frac{1}{2}\lambda \varepsilon_{kk}^{e\,2} \quad \text{and} \quad \psi^p(\varepsilon_P, \varepsilon_{P,i}) = U^P(E_P) - U^P(\varepsilon_P) \tag{21}$$

where E_P is again defined by (2) and U^P is given by (4). For this definition, the plasticity contribution to the free energy vanishes in the absence of a gradient of the plastic strain as measured by $\varepsilon_{P,i}$. Thus, ψ^p models the energy associated with the plastic strain gradients as recoverable. The model is consistent with the notion that $\varepsilon_{P,i}$ is employed as the measure of stored geometrically necessary dislocations whose energy, in principle, can be released by eliminating the gradients. For formulations which include the strain gradients as a quadratic contribution to the energy, such as those of Muhlhaus and Aifantis[3] and Bittencourt et al.[11] for single crystal slip, ψ^P is simply that contribution, i.e., $\psi^P \approx \varepsilon^P_{,i}\varepsilon^P_{,i}$ in the present variables.

By (21), $\sigma_{ij} = \partial\psi^e / \partial\varepsilon^e_{ij}$, giving (1), and

$$Q^{rec} = \frac{\partial\psi^P}{\partial\varepsilon_P} = \frac{\sigma_0(E_P)\varepsilon_P}{E_P} - \sigma_0(\varepsilon_P), \quad \tau_i^{rec} = \frac{\partial\psi^P}{\partial\varepsilon_{P,i}} = \ell^2\,\frac{\sigma_0(E_P)\varepsilon_{P,i}}{E_P} \tag{22}$$

The incremental form of these relations have (17) for the Cauchy stress rate and

$$\dot{Q}^{rec} = \frac{\partial^2\psi^P}{\partial{\varepsilon_P}^2}\dot{\varepsilon}_P + \frac{\partial^2\psi^P}{\partial\varepsilon_P\partial\varepsilon_{P,i}}\dot{\varepsilon}_{P,i} = \left(C - \frac{d\sigma_0(\varepsilon_P)}{d\varepsilon_P}\right)\dot{\varepsilon}_P + C_i\dot{\varepsilon}_{P,i} \tag{23}$$

$$\dot{\tau}_i^{rec} = \frac{\partial^2\psi^P}{\partial\varepsilon_P\partial\varepsilon_{P,i}}\dot{\varepsilon}_P + \frac{\partial^2\psi^P}{\partial\varepsilon_{P,i}\partial\varepsilon_{P,j}}\dot{\varepsilon}_{P,j} = C_i\dot{\varepsilon}_P + C_{ij}\dot{\varepsilon}_{P,j} \tag{24}$$

where C, C_i, and C_{ij} are given in (8).

The dissipative contribution is taken as

$$\dot{Q}^{dis} = \frac{d\sigma_0(\varepsilon_P)}{d\varepsilon_P}\dot{\varepsilon}_P, \quad \dot{\tau}_i^{dis} = 0 \tag{25}$$

The higher order stresses are updated according to $Q^{rec} = \int \dot{Q}^{rec} dt$, $\tau_i^{rec} = \int \dot{\tau}_i^{rec} dt$, or, equivalently, in integrated form by (22) in terms of ε_P and $\varepsilon_{P,i}$, and by $Q^{dis} = \int \dot{Q}^{dis} dt = \sigma_0(\varepsilon_P)$ and $\tau^{dis} = 0$. Prior to any plastic deformation, $Q^{rec} = 0$, $Q^{dis} = \sigma_0(0)$, the initial yield stress, and $\tau_i^{rec} = 0$. The Cauchy stress is given by (1) with the plastic strains integrated according to $\varepsilon^P_{ij} = \int \dot{\varepsilon}^P_{ij} dt = \int \dot{\varepsilon}_P m_{ij} dt$. In general, $\varepsilon_P \neq \sqrt{2\varepsilon^P_{ij}\varepsilon^P_{ij}/3}$ except for proportional straining. It is easily verified that the incremental relation coincides with (14) for the J_2 deformation theory for proportional straining. In addition, it is straightforward to see that the theory reduces to the classical J_2 flow theory in the limit when gradients effects are unimportant. In that limit, the above constitutive relation produces the classical relation: $\dot{Q} = \dot{\sigma}_e = (d\sigma_0(\varepsilon_P)/d\varepsilon_P)\dot{\varepsilon}_P$ and $\dot{\tau}_i = 0$.

Conditions for plastic loading and elastic unloading will be introduced in the next sub-section. Anticipating that plastic loading requires $\dot{\varepsilon}_P > 0$, it follows that Q^{dis} is positive and monotonically increasing because $d\sigma_0(\varepsilon_P)/d\varepsilon_P > 0$, by assumption. Thus, the plastic dissipation rate, $Q^{dis}\dot{\varepsilon}_P$, is never negative. Although not a thermodynamic requirement, $Q\dot{\varepsilon}_P = (Q^{rec} + Q^{dis})\dot{\varepsilon}_P \geq 0$ is also always met because Q is also positive. The contribution, $\tau_i^{rec}\dot{\varepsilon}_{P,i}$, is positive for proportional straining but it can be negative for strongly non-proportional histories when the stored energy associated with the plastic gradients is being released.

3.1 Conditions for plastic loading and elastic unloading

Insufficient attention has been given to conditions for plastic yielding and elastic unloading for the strain gradient theories. It is useful to begin by reviewing these conditions for the classical J_2 flow theory. The condition for yield is $\sigma_e = \sigma_Y$, where during plastic loading the yield stress, σ_Y, evolves according to $\dot{\sigma}_Y = \dot{\sigma}_e$. For elastic increments, σ_Y remains unchanged and σ_e must satisfy $\sigma_e \leq \sigma_Y$. The initial yield stress is $\sigma_Y = \sigma_e(0)$. Given yield is satisfied, i.e., $\sigma_e = \sigma_Y$, the conditions for plastic loading and elastic unloading for the next incremental step are

$$\dot{\varepsilon}_P > 0 \ \ \& \ \ m_{ij}\dot{\varepsilon}_{ij} > 0 \quad \text{(loading)}, \quad \dot{\varepsilon}_P = 0 \ \ \& \ \ \dot{\sigma}_e = 2\mu m_{ij}\dot{\varepsilon}_{ij} \leq 0 \quad \text{(unloading)} \qquad (26)$$

Now consider the strain gradient version. The two branches of the incremental constitutive model are specified by (17), (23)-(25) for plastic loading and by (17) (with $\dot{\varepsilon}_P = 0$, $\dot{\varepsilon}_{P,i} = 0$, $\dot{Q} = 0$ and $\dot{\tau}_i = 0$) for elastic unloading. A criterion for switching from one branch to the other is required with the constraint that it reduces to the classical criterion (26) when strain gradients play no role. It is important to note that, of all the stress quantities, only σ_{ij} changes when the solid is deforming elastically—Q and τ_i change only when plastic straining occurs. Thus, only the Cauchy stress, σ_{ij}, can be used to characterize whether the state of stress lies inside the yield surface and whether the stress re-attains yield following an elastic excursion. For the generalization of J_2 flow theory proposed above, a criterion consistent with the formulation and with the observations just noted is the criterion for the conventional theory specified by the same yield condition and (26). Thus, initial yield requires $\sigma_e = \sigma_0(0) = \sigma_Y$ and yield following plastic straining requires $\sigma_e = \sigma_Y$, where the yield stress evolves according to $\dot{\sigma}_Y = \dot{\sigma}_e$ during plastic yielding. As in the conventional theory, the plastic strain increment, $\dot{\varepsilon}_{ij}^P$, is normal to the current yield surface specified by $\sigma_e = \sigma_Y$. For the conventional J_2 flow theory the evolution of yield stress can be integrated to give $\sigma_Y = \sigma_e(\varepsilon_P)$. For the gradient version, the yield stress can be integrated to give

$$\sigma_Y = Q - \tau_{i,i} = \sigma_0(E_P)\frac{\varepsilon_P}{E_P} - \left(\ell^2\sigma_0(E_P)\frac{\varepsilon_{P,i}}{E_P}\right)_{,i} \tag{27}$$

In a numerical implementation of the theory it will generally be preferable to use $\dot{\sigma}_Y = \dot{\sigma}_e$ when plastic loading occurs to update σ_Y because this requires evaluation of only the first gradients of $\dot{\varepsilon}_P$. For the gradient theory, $\dot{\sigma}_e$ can be negative for plastic loading and, thus, for some deformation histories σ_Y can undergo a decrease.

It can be noted in passing that one can show that the higher order stress quantities introduced above always satisfy the equation

$$\sqrt{Q^2 + \ell^{-2}\tau_i\tau_i} = \sigma_0(E_P) \tag{28}$$

It might be tempting to regard this as a yield condition, but it is not. This equation is a consequence of the postulated constitutive relation, and it remains in force even when the solid has unloaded elastically and is not at yield. As already noted, only the Cauchy stress changes when the straining is elastic and only it can be used to characterize the elastic region.

The yield condition and the associated criteria for loading/unloading (26) are compatible with the equilibrium equations relating the stress quantities. Specifically, in regions of plastic loading in the incremental boundary value problem, satisfaction of $\sigma_e = Q - \tau_{i,i}$ is ensured given that $\dot{\sigma}_e = \dot{Q} - \dot{\tau}_{i,i}$ and given the previously stipulation for updating the stress quantities. Furthermore, prior to any plastic deformation, $Q = \sigma_Y = \sigma_e(0)$ and $\tau_i = 0$ such that at initial yield $\sigma_e = Q = \sigma_Y$. In elastic regions of the incremental boundary value problem, the second equilibrium equation (8), $\sigma_e = Q - \tau_{i,i}$, will generally not be satisfied, but re-activation of this equation occurs continuously with reloading for the yield condition chosen because Q and τ_i do not change for elastic deformations and because σ_e reassumes $\sigma_e = \sigma_Y$ when yielding last occurred.

3.2 Summary of incremental equations, convexity, minimum principles and uniqueness

The constitutive relation is summarized as follows. With yield satisfied, i.e., $\sigma_e = \sigma_Y$, plastic loading requires $m_{ij}\dot{\varepsilon}_{ij} > 0$ and $\dot{\varepsilon}_P > 0$ with $\dot{\varepsilon}_{ij}^P = \dot{\varepsilon}_P m_{ij}$ and $m_{ij} = 3s_{ij}/(2\sigma_e)$. The stress increments for plastic loading are

$$\dot{\sigma}_{ij} = 2\mu\dot{\varepsilon}_{ij}^{e\prime} + \lambda\dot{\varepsilon}_{kk}^{e}\delta_{ij}, \quad \dot{Q} = C\dot{\varepsilon}_P + C_i\dot{\varepsilon}_{P,i}, \quad \dot{\tau}_i = C_i\dot{\varepsilon}_P + C_{ij}\dot{\varepsilon}_{P,j} \tag{29}$$

with C, C_i, and C_{ij} given by (8), and where the recoverable and dissipative stresses have been combined. For plastic loading the yield stress evolves as $\dot{\sigma}_Y = \dot{\sigma}_e$ with $\varepsilon_P = \int \dot{\varepsilon}_P dt$ and E_P defined in (2). If $\sigma_e < \sigma_Y$ or, if $\sigma_e = \sigma_Y$, with $m_{ij}\dot{\varepsilon}_{ij} \le 0$ and $\dot{\varepsilon}_P = 0$, the incremental response is elastic with $\dot{\sigma}_{ij} = 2\mu\dot{\varepsilon}_{ij}^{e\prime} + \lambda\dot{\varepsilon}_{kk}^{e}\delta_{ij}$. For elastic increments, $\dot{\sigma}_Y = 0$.

The incremental equations for $\dot{Q}$ and $\dot{\tau}_i$ for plastic loading are identical to those of the deformation theory (7). It follows that Q and τ_i can be integrated and expressed in terms of ε_P and $\varepsilon_{P,i}$ by (6); (28) also holds. Just as in conventional J_2 flow theory, history dependence in this theory arises through ε_{ij}^P and σ_{ij} which are strongly path-dependent owing to the normality condition, $\dot{\varepsilon}_{ij}^P = \dot{\varepsilon}_P m_{ij}$, and the constraint $\dot{\varepsilon}_P \ge 0$. Thus, while Q and τ_i are given in integrated form by (6), they are nevertheless path-dependent through the path-dependence of ε_P and $\varepsilon_{P,i}$.

Next consider the incremental boundary value problem. Let

$$\begin{aligned}\Phi(\dot{\boldsymbol{\varepsilon}}_e, \dot{\varepsilon}_P, \dot{\varepsilon}_{P,i}) &= \frac{1}{2}\left(\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{e} + \dot{Q}\dot{\varepsilon}_P + \dot{\tau}_i\dot{\varepsilon}_{P,i}\right) \\ &= \frac{1}{2}\left(2\mu\dot{\varepsilon}_{ij}^{e\prime}\dot{\varepsilon}_{ij}^{e\prime} + \lambda\dot{\varepsilon}_{kk}^{e\,2}\right) + \frac{1}{2}\left(C\dot{\varepsilon}_P^{\,2} + 2C_i\dot{\varepsilon}_{P,i}\dot{\varepsilon}_P + C_{ij}\dot{\varepsilon}_{P,i}\dot{\varepsilon}_{P,j}\right)^*\end{aligned} \tag{30}$$

where $\dot{\varepsilon}_{ij}^{e} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_P m_{ij}$. The terms in the brackets $(\ )^*$ are set to zero with $\dot{\varepsilon}_P = 0$ if $\sigma_e < \sigma_Y$, or if $\sigma_e = \sigma_Y$ and $m_{ij}\dot{\varepsilon}_{ij} \le 0$; otherwise they are included with $\dot{\varepsilon}_P \ge 0$. With the set of strain increments denoted by $\dot{\mathbf{E}} \equiv (\dot{\boldsymbol{\varepsilon}}_e, \dot{\varepsilon}_P, \dot{\varepsilon}_{P,i})$ and the stress increments denoted by $\dot{\mathbf{S}} \equiv (\dot{\boldsymbol{\sigma}}, \dot{Q}, \dot{\tau}_i)$, $\Phi = \dot{\mathbf{S}}\bullet\dot{\mathbf{E}}/2$ and $\dot{\mathbf{S}} = \partial\Phi/\partial\dot{\mathbf{E}}$. One can prove that Φ is convex. That is, for all pairs generated by (30), $(\dot{\mathbf{E}}^{(1)}, \dot{\mathbf{S}}^{(1)})$ and $(\dot{\mathbf{E}}^{(2)}, \dot{\mathbf{S}}^{(2)})$, we have shown that

$$\Phi(\dot{\mathbf{E}}^{(2)}) - \Phi(\dot{\mathbf{E}}^{(1)}) - \dot{\mathbf{S}}^{(1)}\bullet(\dot{\mathbf{E}}^{(2)} - \dot{\mathbf{E}}^{(1)}) \ge 0 \tag{31}$$

where the equality holds if and only if $\dot{\mathbf{E}}^{(2)} = \dot{\mathbf{E}}^{(1)}$.

For a body with volume V and surface S, define a functional F of the incremental displacement fields, $\dot{u}_i$ and $\dot{\varepsilon}_P(\mathbf{x})$, with $\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2$, by

$$F(\dot{\mathbf{u}}, \dot{\varepsilon}_P) = \int_V \Phi(\dot{\boldsymbol{\varepsilon}}_e, \dot{\varepsilon}_P, \dot{\varepsilon}_{P,i}) dV - \int_{S_T} \left(\dot{T}_i \dot{u}_i + \dot{t} \dot{\varepsilon}_P \right) dS \tag{32}$$

where $\dot{T}_i$ and $\dot{t}$ are prescribed on the portion of the surface S_T and the dependence on $\dot{\varepsilon}_{P,i}$ in Φ is evaluated as the gradient of $\dot{\varepsilon}_P(\mathbf{x})$. It follows directly from the convexity of Φ that any solution to the incremental boundary value problem minimizes F among all admissible fields satisfying prescribed $\dot{u}_i$ and $\dot{\varepsilon}_P$ on the portions of the surface other than S_T. Moreover, if a solution exists, it is unique. Existence of a solution has not been established.

As in any incremental plasticity problem, the location of the boundary between the regions which undergo elastic and plastic increments is unknown and depends on the current state and the imposed incremental boundary conditions. For the incremental problem for a homogeneous material, $\dot{\varepsilon}_P$ is not constrained at the elastic-plastic boundary, assuming dislocations can flow through the boundary. Thus, by the incremental principle of virtual work (16), $\dot{t} = \dot{\tau}_i n_i = 0$ on the plastic side of the boundary with n_i as its normal. This implies that a plastic region encroaching into a virgin elastic region has $t = 0$ and $Q = \sigma_0(0)$ at the boundary and, therefore, Q and t are continuous across the boundary. The situation is different at the boundary between two materials with differing yield strengths, one deforming plastically and the other deforming only elastically. On the plastic side of the boundary the constraining effect of the abutting higher strength material can be modelled by taking $\dot{\varepsilon}_P = 0$ at the boundary. Then, generally, $t = \tau_i n_i$ will not vanish at the boundary on the plastic side as plastic deformation proceeds. In this theory, it is assumed that the abutting elastic material can support the higher order traction, t, exerted on it.

4 Strain gradient version of J_2 flow theory #2—higher order stresses dependent on increments of strain gradients

The alternative version of the theory given in this section follows the procedure used by Gundmunsen [1] and Gurtin and Anand [5] to construct the constitutive model ensuring that the plastic deformation is entirely dissipative. The theory in this section employs the same measures introduced for the other flow theory in Section 3, i.e., $\dot{\varepsilon}_P$, $\dot{\varepsilon}_{P,i}$ with $\dot{\varepsilon}_{ij}^P = \dot{\varepsilon}_P m_{ij}$ and $\dot{\varepsilon}_P \geq 0$. The increments of the Cauchy stress increments are again given by (17) with $\dot{\varepsilon}_P > 0$ for plastic loading and $\dot{\varepsilon}_P = 0$ for elastic unloading.

For plastic loading, let $\boldsymbol{\Sigma} = \left(Q, \ell^{-1}\tau_i\right)$, $\dot{\mathbf{E}}_\mathbf{P} = \left(\dot{\varepsilon}_P, \ell\dot{\varepsilon}_{P,i}\right)$ and note the following

$$\boldsymbol{\Sigma}\bullet\dot{\mathbf{E}}_\mathbf{P} = Q\dot{\varepsilon}_P + \tau_i\dot{\varepsilon}_{P,i}\,,\ \Sigma = \left|\boldsymbol{\Sigma}\right| = \sqrt{Q^2 + \ell^{-2}\tau_i\tau_i}\,,\ \dot{E}_P = \left|\dot{\mathbf{E}}_\mathbf{P}\right| = \sqrt{\dot{\varepsilon}_P{}^2 + \ell^2\dot{\varepsilon}_{P,i}\dot{\varepsilon}_{P,i}} \qquad (33)$$

In this version, $E_P = \int \dot{E}_P dt$ is different from the definition (2) used in Section 3; the two definitions only coincide for proportional straining. The crucial step in constructing the class of constitutive relations of Gudmundson [1] and Gurtin and Anand [5] is to choose $\boldsymbol{\Sigma}$ co-directional with $\dot{\mathbf{E}}_\mathbf{P}$ so as to ensure that the plastic work rate, $\boldsymbol{\Sigma}\bullet\dot{\mathbf{E}}_\mathbf{P}$, is always positive. Here the specific choice of Fleck and Willis [6] is adopted because it has been formulated to coincide with the J_2 deformation theory in Section 2:

$$\boldsymbol{\Sigma} = \sigma_0(E_P)\frac{\dot{\mathbf{E}}_\mathbf{P}}{\dot{E}_P}\,,\quad \text{or}\quad Q = \sigma_0(E_P)\frac{\dot{\varepsilon}_P}{\dot{E}_P}\quad \text{and}\quad \tau_i = \ell^2\sigma_0(E_P)\frac{\dot{\varepsilon}_{P,i}}{\dot{E}_P} \qquad (34)$$

It follows immediately that $\boldsymbol{\Sigma}\bullet\dot{\mathbf{E}}_\mathbf{P} = \sigma_0(E_P)\dot{E}_P \geq 0$ and, also, that $\Sigma = \sigma_0(E_P)$. Constitutive models of this class have also been considered by Reddy [12].

Unlike the theories in Section 3, the stress quantities, Q and τ_i, in this class of theories are not known in the current state; only the Cauchy stress, σ_{ij}, is known. Here, Q and τ_i depend on the solution to the incremental boundary value conditions imposed on current state of a body. Thus, Q and τ_i, will, in general, change discontinuously when the boundary conditions for the incremental problem change the direction of loading. Specifically, if changes are made to the prescribed traction increments, $\dot{T}_i$ and $\dot{t}$, on S_T, and/or to prescribed values of $\dot{u}_i$ and $\dot{\varepsilon}_P$ on the remaining portion of the boundary, then Q and τ_i will usually change discontinuously throughout the body. Physical implications of such discontinuous behaviour will be discussed later.

Owing to the fact that the higher order stresses are expressed in term of the increments of strain and strain gradient, the incremental boundary value is not standard. Fleck and Willis [6] have formulated minimum principles for the incremental boundary value problem for this class of theories which determine the distribution of $\dot{\mathbf{E}}_{\mathbf{P}}$ and $\boldsymbol{\Sigma}$ throughout the body in terms of prescribed increments of boundary loads or displacements. They have called attention to the fact that this class of formulation has parallels to the classical theory of rigid-plasticity in the sense it too has the feature that the stress is a function of the plastic strain increment and, therefore, depends on the solution to the incremental boundary value problem itself.

A full description of conditions for plastic yield, plastic loading and elastic unloading has not yet been presented for this class of theories. For the version specified by (34), Fleck and Willis [6] have noted that

$$\Sigma = \sqrt{Q^2 + \ell^{-2}\tau_i\tau_i} = \sigma_0(E_P) \tag{35}$$

has the appearance of a yield condition in the sense that it is satisfied for any plastic loading increment. Moreover, $\dot{\mathbf{E}}_{\mathbf{P}}$ is normal to the surface specified by $\Sigma = \sigma_0(E_P)$. The correct way to think of (35) is that the stress, $\boldsymbol{\Sigma}$, locates its position on the surface such that $\dot{\mathbf{E}}_{\mathbf{P}}$ is aligned with the normal and not vice versa.

As a yield condition, (35) is incomplete. Similarly to condition (28) for the other flow theory, (35) is a consequence of (34) and not an extra equation. Moreover, for the same reasons described for the other theory, (35) cannot characterize the elastic region within the yield surface or the condition for plastic re-loading if the solid has undergone excursions within the elastic region. As noted in Section 3, only the Cauchy stress changes when the solid deforms elastically and, consequently, the Cauchy stress must enter into any criterion characterizing elastic responses. The yield condition, together with the conditions for plastic loading and elastic unloading (26), proposed for the other flow theory can also be invoked for this version. The equation for the evolution of the yield stress under plastic loading, $\dot{\sigma}_Y = \dot{\sigma}_e$, again allows for the possibility that σ_Y may undergo a decrease for certain deformation histories.

4.1 Are discontinuous stress changes due to infinitesimal changes in boundary tractions physically acceptable?

As noted, one consequence of the constitutive equation (34) is a discontinuous change in the direction of the stress quantities, Q and τ_i, with a change in the "direction" of prescribed surface traction increments or displacement increments on the boundary of the solid body. A simple illustration would be a bar or tube stretched into the plastic range in tension and then subject to an increment of both tension and torsion. Fleck and Willis [6] have formulated minimum principles for the incremental boundary value problems for the class of theories of which (34) is perhaps the simplest example. Their work shows that the distributions of $\dot{\varepsilon}_P$ and $\dot{\varepsilon}_{P,i}$, depend on the boundary conditions posed for the incremental problem. If the incremental boundary conditions are changed, the distributions of $\dot{\varepsilon}_P / \dot{E}_P$ and $\dot{\varepsilon}_{P,i} / \dot{E}_P$ on the right hand side of (34) will generally change. In other words, infinitesimal changes in prescribed boundary tractions or displacement can result in finite changes in Q and τ_i.

Discontinuous stresses due to infinitesimal changes in strain are unusual for solids, although, as noted in [6], such discontinuous stress behaviour is characteristic of rigid-plastic solids for which elastic strains are neglected. Rigid-plasticity theory cannot be used to evaluate elastic strains or even plastic strain changes on the order of elastic strains under non-proportional straining. It has not been the intention of the developers of either class of gradient plasticity considered in this paper to neglect elastic strains. Indeed, incompatibility associated with gradients of plastic strain must be offset by gradients of elastic strains. Thus, one must ask if it is physically acceptable for the higher order stresses to undergo discontinuous changes in the manner described by (34).

At this stage in the development of higher order theories, a definitive answer to this question may not be possible because a widely accepted physical intuition of higher order stresses is not yet in place. Nevertheless, physical arguments for *continuous changes* in the higher order stress can be put forward. If an internal or external boundary in the solid has unit normal, $\mathbf{n}$, it is generally held that $t = \tau_i n_i$ constitutes a measure of local traction on the boundary associated with the local plastic strain gradient. Recent efforts to model transmission of plastic straining across boundaries have made use of this interpretation (Aifantis, et al. [13]). If this interpretation is correct, it is hardly acceptable that infinitesimal changes in boundary tractions or displacements could result in finite changes in local boundary tractions within the body. In addition, higher order stresses are believed to be directly related to the current dislocation distribution. Challenging as the problem is, efforts to quantitatively characterize this connection have been pursued (e.g., Groma, et al. [14]). From a physical standpoint, it seems highly unlikely that dislocation distributions would routinely undergo finite changes due to infinitesimal changes in boundary tractions or displacements. Thus, the existence of a connection between the higher order stress and the current dislocation distribution would also suggest that discontinuous changes in stress with infinitesimal changes in boundary loads are physically suspect.

5 Summary of the current status of a basic J_2 flow theory of strain gradient plasticity

Two simple extensions of the classical J_2 flow theory have been given. The inputs to these two versions, and to the J_2 deformation theory version to which they have been tied, are the same: the isotropic elastic moduli, the tensile stress-strain curve in the plastic range and a single material length parameter that sets the scale of the gradient effects. Both flow theory versions have been constructed to coincide with the deformation theory for proportional straining and both reduce to the classical J_2 flow theory when gradient effects become negligible. The two versions differ for non-proportional straining. Version #1 in Section 3 specifies increments of stress in terms of increments of strain, while Version #2 in Section 4 specifies the higher order stresses themselves in terms of increments of strain.

Nearly all the micron scale plasticity tests to date have been tests with monotonic loading and straining conditions that do not depart significantly from proportional straining. By the same token, the theoretical efforts employed to interpret and fit the existing experimental data have invoked solutions with monotonic loading and near-proportional straining. For such problems, little difference between the two flow theory versions in this paper should be expected. Indeed, for the same reasons, it has been justified to use J_2 deformation theory solutions in a number of these cases to compare theory and experiment. Apart from an effort to measure the Bauschinger effect under reversed loading in thin films (Xiang and Vlassak [15]), we are unaware of any micron scale experiments carried out to explicitly explore non-proportional straining effects. Thus, at this time, it is not possible to make use of experimental data to settle the issues related to stress continuity and non-proportionality raised in Section 4 in connection with version #2.

In Section 4.1 it has been argued that there are physical grounds for requiring any constitutive law to give rise to continuous temporal variations of the higher order stress even if the incremental boundary conditions undergo an abrupt change in loading direction. If this argument survives further scrutiny, it would mean that the type of constitutive relation represented in its simplest form by (34) is not physically acceptable. The inclusion of a rate-dependence in this class of theories, as in Gurtin and Anand [5] and Lele and Anand [16], can eliminate temporal stress discontinuities. However, at any abrupt change in direction of the boundary conditions for which the rate-independent limit undergoes stress jumps, the dependence on the parameter setting the rate-dependence will be exceptionally strong and difficult to justify physically. Thus, the incorporation of rate-dependence side steps the problem without resolving the fundamental physical issue. Lele and Anand [16] have explored the sensitivity of this class of strain gradient formulations to the level of rate-dependence for problems without abrupt changes in boundary conditions.

The formulation in Section 3 in which increments of stress are related to increments of strain can be extended to more complicated versions which make use of other invariants of the plastic strain rate such as those identified in [2]. Nevertheless, there is need for a more systematic approach to construct incremental constitutive relations for gradient plasticity theories which satisfy thermodynamic constraints. The modified version of the earlier Fleck-Hutchinson constitutive model presented in Section 3 meets these constraints by partitioning the rate of plastic work into recoverable and dissipative components. In the version put forward, the work associated with the higher order stress τ_i is taken to be recoverable. From a physical standpoint it seems likely that some of the work associated with τ_i should be non-recoverable. To our knowledge, a general systematic method to construct incremental constitutive relations for dissipative or non-recoverable gradient contributions is not available.

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