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Abstract

A dedicated, non-symbolic, system yielding imprecise representations of large quantities (Approximate Number System, or ANS) has been shown to support arithmetic calculations of addition and subtraction. In the present study, 5-7-year-old children without formal schooling in multiplication and division were given a task requiring a scalar transformation of large approximate numerosities, presented as arrays of objects. In different conditions, the required calculation was doubling, quadrupling, or increasing by a fractional factor (2.5). In all conditions, participants were able to represent the outcome of the transformation at above-chance levels, even on the earliest training trials. Their performance could not be explained by processes of repeated addition, and it showed the critical ratio signature of the ANS. These findings provide evidence for an untrained, intuitive process of calculating multiplicative numerical relationships, providing a further foundation for formal arithmetic instruction.

Core Multiplication in Children

Adults possess an intuitive mathematical system for representing magnitudes in an approximate, abstract fashion (Barth, Kanwisher, & Spelke, 2003; Cordes, Gelman, Gallistel & Whalen, 2001). This approximate number system (ANS) is shared by many other species, including rats, pigeons, and non-human primates (Brannon, Wusthoff, Gallistel & Gibbon, 2001; Cantlon & Brannon, 2006; Capaldi & Miller, 1998). The discriminability of two numbers represented by the ANS is determined by their ratio and not their absolute difference; 10 and 20 are as discriminable as 100 and 200. There is evidence that young infants and unschooled children possess this same system, for they readily discriminate between different numerosities from early in development (Brannon, 2002; McCrink & Wynn, 2007; Xu & Spelke, 2000), and perhaps even at birth (Izard, Sann, Spelke, & Streri, 2009), with a characteristic ratio limit (Barth, LaMont, Lipton & Spelke, 2005; Lipton & Spelke, 2004). The ANS is very noisy in neonates who detect numerical differences in a 3:1 ratio (Izard et al., 2009), sharpens rapidly during infancy (Lipton & Spelke, 2004), and gradually reaches adult-like levels of ~1.15:1 (Halberda & Feigenson, 2008; Izard & Dehaene, 2008; Pica, Lemer, Izard & Dehaene, 2004).

These ANS representations can be used in the arithmetic operations of ordering (Brannon, 2002), addition, and subtraction (Barth et al., 2005, 2006; Cordes, Gallistel, Gelman & Latham, 2007; McCrink, Dehaene, & Dehaene-Lambertz, 2007; McCrink & Wynn, 2004, 2009). Children demonstrate early informal understanding of certain types of arithmetic logic, including inversion and commutativity (Baroody, 1999; Gilmore & Spelke, 2008) that are likely supported by this core numerical system. Nevertheless, the origins of other mathematical operations, particularly multiplication and division, are unknown. There is mixed theoretical and empirical evidence as to whether the ANS can support these operations.

Although symbolic multiplication is taught to school children on the basis of addition (by the operation of repeated addition), its formal and conceptual definitions in mathematics are independent of repeated addition and apply to any real numbers. Multiplication involves a scalar transformation of one magnitude into another (James & James, 1976). Thus, in order to determine whether this scaling process we term *core multiplication* is present as an additional untrained numerical operation, we must look for two key features. First, children must be able to perform a multiplicative scaling directly, without relying on repeated addition as a heuristic. Second, children must do so by relying on their core system of number- the ANS- and scale over number *per se,* not over perceptual variables commonly confounded with number, such as area.

The fact that core multiplication can be defined independently of addition does not imply that the two operations are psychologically distinct or equally primitive. Indeed, both studies of adults and studies of children suggest that addition and subtraction are more accessible than multiplication. Many studies of adults show a dissociation in the neural systems underlying multiplication, relative to subtraction. Subtraction problems, such as 13-5, likely involve non-verbal quantitative processing of magnitudes, for they are supported bilaterally by the intraparietal sulcus (Chochon, Cohen, van de Moortele, & Dehaene, 1999; Simon, Cohen, Mangin, Bihan, & Dehaene, 2002). Multiplication, as it is studied in the neuropsychology literature, likely involves automatic retrieval of verbally encoded facts, for it is is supported by the left angular gyrus (Delazer, Domahs, Lochy, Bartha, Brenneis, & Trieb, 2004; Lee, 2000). Neuropsychological studies reveal a double dissociation between impairments in subtraction and multiplication (Dagenbach & McCloskey, 1992; Dehaene & Cohen, 1997; Lampl, Eshel, Gilad, & Sarova-Pinas, 1994; Lee, 2000), and case studies on semantic dementia find impaired multiplication performance (on rote, exact problems such as 4 x 5) but spared non-verbal subtraction abilities (Lemer, Dehaene, Spelke, & Cohen, 2003; Zamarian, Karner, Benke, Donnemiller, & Delazer, 2006). All these findings suggest that the ANS directly supports subtraction but provide no evidence that it supports multiplication. Because these studies are conducted on adults who have rote-learned their multiplication tables, however, it is possible that a core multiplicative process exists but is overshadowed by later-learned verbal processes.

Studies of infants' sensitivity to ratios, and of preschool children's proportional reasoning, suggest that a scaling process could very well exist in children. These studies have probed children's extraction of a scaling factor to determine proportional equivalence. To our knowledge, only one study has found such scaling in the purely numerical realm (McCrink & Wynn, 2007). In this study, six-month-old infants were habituated to a series of slides displaying large numbers of objects of two types in a constant ratio. The infants subsequently looked longer at slides that presented the object types in a different numerical ratio, relative to slides that contained the familiar ratio, even though the absolute numbers of objects varied across trials and their novelty was equated at test (McCrink & Wynn, 2007). A number of other researchers have also observed sensitivity to proportional relationships in preschoolers and young children, but only under conditions in which continuous variables such as area vary (Goswami, 1992;

Jeong, Levine, & Huttenlocher, 2007; Mix, Levine, & Huttenlocher, 1999; Park & Nunes, 2001; Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1991, Vasilyeva & Huttenlocher, 2004). In all of these successful studies, the children intuited a common *invariant relationship* between two continuous variables, and used this relationship to guide their performance. The extraction of this invariant is inherently multiplicative: The only way to know that the relationship between 4 and 8 is somehow similar to the relationship between 10 and 20 is to have a mental tag of "twice as much." Piaget (1965) (followed by Vergnaud, 1988), termed this a *schema of correspondence*, in which participants detect the scaling invariant of a one-to-many relationship (for example, a distribution of x flowers per vase) and make inferences from that invariant detection.

In a recent study, Barth, Baron, Spelke and Carey (2009) investigated whether young children can perform multiplicative scalar transformations over numerical and continuous values prior to instruction on these operations. In one condition, the authors presented children with a set of dots, occluded the stimuli, and performed a "magic" transformation with musical notes that resulted in a doubled amount. Children were asked to compare this represented amount to a separate, visible array of dots. The children performed at above-chance levels on this task, but subsequent analyses cast doubt on the thesis that children engaged in genuine multiplication. As the authors noted, children's performance on the multiplication task was consistent with learning of a response strategy that was independent of multiplication and depended only on the range of numerical magnitudes presented in the comparison array. Children reliably judged that the comparison array was larger than the product when it was especially large and smaller than the product when it was especially small, irrespective of the actual size of the product. On critical trials when the comparison array was intermediate in size but differed from the product by the same ratio as on the other trials, however, children performed at chance. Barth et al. (2009) therefore provided little evidence that children were capable of the simplest multiplicative scale transformation: doubling. Nevertheless, it is possible that children are able to perform this operation but were led, by training, to pursue an alternative, range-based comparison strategy. Accordingly, Experiment 1 uses a different training procedure to test whether children are sensitive to doubling. Unlike Barth et al. (2009), we tested children's sensitivity to doubling not only after training but on every training trial including the first, so as to investigate whether children spontaneously evoke a scaling process.

Even if evidence for sensitivity to doubling were to be obtained, it would be difficult to draw conclusions about multiplication from such evidence, because a doubling operation can be effected by a process of repeated addition. Instead of extracting a multiplicative factor, children could have mentally duplicated the set and then added the original set and its duplicate together (e.g., Barth et al., 2005). As a more direct test of sensitivity to scale transformations, Barth et al. (2009) ran a halving condition and found that children were able to halve the occluded array successfully, without relying on range-based strategies. This finding leaves open the question whether children can perform scalar transformations on numerical quantities, however, because number was confounded in this condition with continuous quantitative variables. In Barth et al's (2009) halving experiments, children were presented on each trial with an array of elements that was constant not only in number but also in element size and display area. They subsequently compared half of that array to a second array on the basis of number, because the comparison array presented elements at different sizes and densities, specifically chosen so as to control for non-numerical quantitative variables. Nevertheless, the halving operation itself was performed on the original, unchanging array, whose numerical and non-numerical properties therefore were confounded. Although children's performance on the halving test indicated that they were able to mentally bisect this array, it does not reveal whether the quantity on which they based their bisection was number, display area, summed surface area, or some other continuous variable. Barth et al.'s (2009) experiments therefore reveal that children are capable of halving arrays of discrete elements, but they do not reveal whether the process by which they do so involves a numerical scale transformation.

These initial findings open the door to a systematic exploration of the multiplication abilities of the ANS. Using a method adapted from Barth et al. (2009), we tested whether children with no formal education in multiplication or division were able to transform arrays by a multiplicative factor of 2 (Experiment 1), 4 (Experiment 2), or 2.5 (Experiment 3). Children first were introduced to a magic wand that effected a single doubling or quadrupling operation on a single visible object, or that transformed 2 simultaneously visible objects into 5 objects. Crucially, each of the objects in the array continuously changed in size, orientation and position, so that none of these factors or the continuous quantitative variables to which they give rise were correlated with number. On each of a series of trials, we then presented children with a numerical array, occluded the array, waved the wand while the array remained occluded, presented a visible comparison array, and asked children to judge which array was more numerous: the occluded product or the new visible array. The comparison arrays were either distant from, or near to, the correct product. If children can detect an invariant multiplicative factor across arrays, then they should learn this transformation and generalize it across arrays. If this process builds on representations of approximate number, then their performance should show the signature ratio limit of the ANS.

EXPERIMENT 1: TIMES 2

Method

Participants. Sixteen 5-7-year-old children (8 females, 8 males; age range of 60 months 8 days to 83 months 25 days, mean age of 71 months) were recruited via a large mailing database in the greater Boston area. Participants were divided into older (72-84 months) and younger (60-72 months) age groups. None of the participants in the final sample had formal education in multiplication or division. One additional child was excluded from the final dataset for attending a school that had already started teaching multiplication, and two additional children refused to complete the experiment.

Displays & Procedure.

Multiplicative factor introduction. The child and experimenter were seated together in a quiet testing room at a large table, and watched the videos on a Macintosh laptop. The children first viewed a video consisting of a single blue rectangle, which grew and shrank for several seconds before becoming stationary. As the rectangle remained on the screen, an animated wand appeared from off-screen left and waved over the rectangle while making a “magical” twinkling sound. After several seconds of waving, the rectangle broke into two pieces, and the experimenter exclaimed “Look! It’s our magic multiplying wand. It made more. There used to be one blue rectangle, and now there are two. It doesn't matter if the rectangles are big or small. The wand takes one rectangle, and makes it two." [1] After this video, the child saw a follow-up transformation video that was identical to the previous video, but now the rectangle was occluded during the waving of the wand. The experimenter paused the movie after the wand waved, and asked the child how many rectangles there were behind the screen. Once the child was able to answer this question correctly (on the first trial for 12/16 children and on the second trial for the rest), the experiment moved onto the training trials.

Training block. In the training videos, children saw an array of blue rectangles on the left side of the screen. During the first training video, the experimenter pointed to this array, and said "Now we have **this** many rectangles. There's too many to count, so we're going to concentrate, and use our imagination. So it's not a counting game, it's an imagination game, and we just have to think really hard." After five seconds, an occluder came up from off-screen and occluded the initial array. The multiplying wand then came out and waved over the occluded array, while the experimenter exclaimed "Look! They're getting multiplied." A comparison array composed of pink rectangles came down on the right side of the screen. These rectangles were of identical size (1 cm^2) and density (~15 objects per 25 cm^2) across all training trials. The children were asked to choose where they thought there were more rectangles.

[1] In devising these events, we aimed to present a process akin to the growth of cell colonies: like multiplying bacteria, the rectangle divided into two, but the subsequent independent size and movement of the outcome rectangles established the existence of two distinct entities- double the initial array. Although most children were not familiar with the term "multiply" prior to the experiment, children accepted our terminology.

To control for experimenter bias, the experimenter a) sat slightly behind the child and several feet from the screen, b) phrased the question neutrally ("Where do you think there are more?", "Which side of the screen do you think has more?"), and c) looked at the child, and not the screen, until he or she provided an answer. After answering left/right, or pointing to the right or left side of the screen, the experimenter recorded the child's answer, pressed a button to drop the occluder and reveal the transformed array, and provided feedback to the children as to whether his or her response was correct or incorrect. The outcome array had identical area to that of the initial array and was created by carving each initially-presented rectangle into two rectangles. In this way, the children received evidence that the relevant multiplicative variable was number (which was now greater by a factor of two), and not summed area (which remained the same.) The next training movie was then presented. After 12 training trials were completed, children had a small break in which they stretched and moved around the room. After this break, the experiment moved on to the test trials.

Testing block. The 16 test trials were similar to the training trials: an array was presented, occluded, transformed by a stroke of the wand, and then a comparison array came down from offscreen. The child made the 'more' judgment just as in training. However, there are three differences between training and testing. First, the children did not see the final product of the multiplication behind the occluder (it remained covered even after the child answered) and they were not told whether they answered correctly or incorrectly; instead, the experimenter provided uniformly positive feedback on each trial ("Great job! You're doing so well. Let's do another!"). Second, the comparison arrays, which had previously been controlled for density and item size, now were controlled for area and contour length. If children had come to rely on area and contour length during the training trials as the sole cues to numerosity of the comparison array (higher area and contour length equaled higher number), they would perform at chance on the test trials. Additionally, the overall area and contour length were equidistant from both the very large and very small visualized outcome arrays. Even though the children never saw the outcome arrays, this control discourages a strategy of comparing the area or contour length of the imagined outcome array to the comparison array. Thus, children must attend to numerical values, and not perceptual variables that co-vary with number, to succeed on the task. Third, new numerical values were used in testing so that children's performance could not reflect rote learning of the values in the training trials. Figure 1 presents a schematic of the procedure and stimuli.

Design. There were 12 training trials and 16 test trials. On equal numbers of trials in each category, the comparison array differed from the outcome by a Distance factor of /2 (i.e., the correct outcome to the multiplication problem divided by 2), /1.5, *1.5, or *2. For the exact values used in this experiment and the other experiments in this paper, see Table 1.

Results

Each participant was given an average score composed of their performance on trials whose comparison arrays were a particular distance from the correct outcome (/ 2, / 1.5, *1.5, or *2.) Overall performance during the training block (85%) and testing block (79%) was significantly above chance (one-sample t-tests; ts (15) = 18.26, 10.00, both ps < .001, two-tailed.) (See Figure 2.) There was no difference between performance during training and testing (t (15) = 1.99, p = .07, two-tailed.) Using a test proportion of

67.8% (19/28 trials with an alpha level of .05), we also calculated the number of children whose individual performance was above chance; 16 of 16 children met this criteria (16 observed successes with a total N of 16, binomial sign test with an alpha of .05, $p<.001$). An ANOVA with block (training block, testing block) and distance (/2, /1.5, *1.5, and *2) as within-subject factors and gender (male, female) and age (old, young) as between-subject factors revealed no main effects of block ($F(1,12) = 4.10$, $p = .07$), gender ($F(1,12) = .03$, $p = .87$), or age ($F(1,12) = .01$, $p = .97$). There was a main effect of distance ($F(3,36) = 7.18$, $p = .001$); follow-up Bonferroni-corrected pairwise comparisons revealed significantly lower performance in the *1.5 Distance trials (67%), which differed from the /2 (89%) and *2 Distance trials (91%; both ps < .05), but not the /1.5 trials (79%; $p = .78$). There was a significant interaction between block and distance ($F(3,36) = 5.98$, $p = .002$); performance "dipped" at slightly different Distance trials for the training block (worst performance at Distance *1.5) and testing block (worst performance at Distance /1.5.) With the exception of the Distance *1.5 training trials (58% correct), children performed significantly better than chance on all testing and training trial types (one-sample t-tests, all ps<.05). This may be due to a tendency to overestimate outcomes during training trials.

To determine if children needed a good deal of practice before learning this multiplicative relationship, we tabulated separately each participant's percentage correct for the very first and last trials at each Distance trial type (correct/2, /1.5, *1.5, *2) during the training block. Performance was above chance for both the first and the last trials (84% and 89% respectively; one-sample t-tests against chance yield ts (15) = 8.88, 9.93, both ps < .001, two-tailed). There is no evidence that children required much training to understand this concept; on the first training trial they performed at 84%, and on the last training trial 89% (paired-sample t-test t (15) = -.76, p = .46, two-tailed.)

Discussion

The results from this experiment indicate that children are able to double a represented numerical amount. They do so using only their core approximate number system, and do not require formal schooling or knowledge about multiplication to perform this calculation. Furthermore, this multiplicative relationship seems to be relatively transparent; children were able to perform this core multiplication after seeing just one simple introductory movie in which a single rectangle turned into two rectangles. In accord with other research on the ANS (Pica et al., 2004; Barth et al., 2005; Lipton & Spelke, 2004), children's performance in discriminating the outcome of the operation from the comparison array was modulated not by the absolute difference of two amounts, but rather by their ratio; children performed significantly better when the comparison array differed from the product by a larger ratio (2:1) compared to a smaller ratio (1.5:1).

How did children perform this doubling operation: by multiplication or by repeated addition? Repeated addition is a heuristic used by many elementary mathematics textbooks in their initial lessons on multiplication (Watanabe, 2003), and it is seen as a first step to reasoning about multiplication in many educational settings (e.g., Fischbein, Deri, Nello, & Marino, 1985; Greer, 1988). Further, both children and infants are able to add and subtract using only the ANS (Barth et al., 2005, 2006; McCrink & Wynn, 2004, 2009.) The capacity for repeated addition, however, is not the focus of this paper. Rather, we are interested in determining if children can draw on an intuitive, core scaling function from one set of representations from the ANS into a second set of such representations, without any intermediate operations of addition.

To address this question, we conducted a new experiment involving a Times 4 transformation. This simple change from doubling to quadrupling makes the repeated-addition strategy hard to achieve, because it requires the creation and maintenance of at least five distinct numerical magnitudes: the four subsets and the superset that they combine to form. Even educated adults find it difficult to maintain and update representations of four numerical magnitudes during a single task (Halberda, Sires, & Feigenson, 2006). Halberda et al. (2006) found that most adults can encode the numerosity of only three sets of objects (two subsets, plus the larger superset), and their performance declines steeply if additional sets must be encoded. If children were using a repeated addition strategy in Experiment 1, therefore, they should fail to learn the quadrupling relation in Experiment 2.

EXPERIMENT 2: TIMES 4

Participants. Sixteen 5-7-year-old children (8 females, 8 males; age range of 60 months 23 days to 81 months 16 days, mean age of 69 months) were included in the final sample, and were recruited from the same database as in Experiment 1. None of the participants had formal education in multiplication or division. Two additional children showed a side bias and selected the same array on every trial; they were replaced by children of the same gender and age.

Procedure, Stimuli and Design. This experiment was identical to Experiment 1, except that the introduction movie, training block, and testing block all portrayed or tested a Times 4 relationship instead of a Times 2 relationship. During the multiplicative factor introduction movie, the participants saw one rectangle break into 4 rectangles. As in Experiment 1, the comparison arrays were equated for item size and density during training, and area and contour length during testing. For the particular values used in this Experiment, see Table 1.

Results. Again, each participant was given an average score composed of their performance on trials whose comparison arrays were a particular distance from the correct outcome (/ 2, / 1.5, *1.5, or *2.). Two participants (5-year-old males) had what appeared to be response biases and answered "Right" for all the trials. In all of the following analyses, these children were replaced by two children of the same age and gender who did not show a perseverative side bias. [2]

Overall performance in the unbiased dataset during training (72%) was similar to performance during testing (66%; paired-sample t-test, t (15) = 1.49, p =.16, two-tailed), and both sessions were significantly above chance (ts (15) = 5.89 and 4.69, respectively; both ps <.001, two-tailed). (See Figure 3.) As in Experiment 1, using a test proportion of 67.8% (19/28 trials with an alpha level of .05), we also calculated the number of children whose individual performance was above chance; 9 of 16 children met this criteria (9 observed successes with a total N of 16, binomial sign test with an alpha of .05, p<.001). An ANOVA was conducted with block (training block, testing block) and distance (/2, /1.5, *1.5, and *2) as within-subject factors and gender (male, female) and age (60-72 months, 72-84 months) as between-subject factors. There were no main effects of block (F (1,12) = 2.57, p = .14), gender (F (1,12) = .63, p = .45), or age (F (1,12) =.22, p = .65).

[2] Overall performance during training (76%) and testing (63%) was significantly above chance even with those subjects whose responses were biased (one-sample t-tests; ts (15)= 6.18, 3.85, both ps <.01, one-tailed). Thus, overall performance by the group was significant even with the inclusion of these inattentive children.

Surprisingly, there was also no main effect of distance (F (3,36) = .91, p = .45), but distance interacted with other variables. There was a significant interaction of distance and age (F (3,36) = 2.90, p = .04); younger children were better on Distance trials /1.5 and /2 (69% and 73%, respectively) compared to Distance trials *1.5 and *2 (64% and 64%.) Older children, on the other hand, were better at Distance trials *1.5 and *2 (72% and 75%, respectively) than Distance trials /1.5 and /2 (53% and 64%). This pattern suggests that the younger children were overestimating the outcome of the multiplying operation, and older children were underestimating the outcome. There also was a significant interaction between session and distance (F (3,36) = 3.01, p = .04). Just as in the Times 2 condition, performance "dipped" at slightly different Distance trials for the training (worst performance at Distance *1.5) and testing (worst performance at Distance /1.5) sessions. With the exception of the Distance /1.5 testing trials, children performed significantly better than chance on all testing and training trial types (one-sample t-tests, all ps<.05). As noted above, this may be due to a tendency by the older children to underestimate outcomes during testing trials.

To test whether children needed extensive practice before learning this multiplicative relationship, we tabulated each participant's percentage correct for the very first and last trials at each Distance trial type (correct/2, /1.5, *1.5, *2) during the training block, as in Experiment 1. Performance was above chance for both the first and last trials (64% and 71% respectively; one-sample t-tests against chance yield ts (15) = 2.06, 4.34, both ps < .05, one-tailed). There is no evidence that children required much training to understand this concept; on the first training trial they performed at 64%, and on the last training trial 71% (paired-sample t-test t (15) = -.92, p =.37, two-tailed.)

Discussion. The children in this experiment were able to infer the outcome of a non-symbolic quadrupling problem. These results reveal that core multiplying abilities extend beyond the case of simple doubling. Because of the difficulty of maintaining four separate arrays in memory (Halberda et al., 2006), they suggest that the use of a repeated-addition strategy does not drive the performance of children in this task.

Children's overall quadrupling performance resembled their doubling performance in Experiment 1, but also differed in important ways. Just as in Experiment 1, children learned the quadrupling calculation rapidly; they were able to infer the invariant multiplicative factor after seeing a single example of one rectangle turning into four rectangles, and they generalized this relationship to a large array of rectangles on the first training trials. In contrast to Experiment 1, however, this transformation was, overall, more difficult for the children. This is unsurprising when one considers that these numerical transformations of Times 4 would introduce significantly more variance to the outcome than numerical transformations of Times 2, with double the variance in the scaling factor. Also in contrast to Experiment 1, the impact of the distance of the comparison array from the correct outcome on performance was negligible. Children did not perform significantly better when the ratio of comparison array to the correct outcome was larger (2:1) than when it was smaller (1.5:1), as would be predicted by previous work on the nature of the ANS (e.g, Pica et al., 2004). Instead, the variability in performance came mainly from what appears to be over- or under-estimation of the outcomes depending on the block and the child's age. Overall, the children tended to overestimate outcomes during training and underestimate during testing, perhaps overcompensating after receiving feedback during training. Younger children (< 6 years old) tended to overestimate the outcomes to the problems, and older children tended to underestimate outcomes. Regardless of difficulty relative to Experiment 1, we still see a pattern of overall above-chance performance during both training and testing that reflects an underlying core multiplication calculation.

Thus far, we have evidence for a process of multiplication that operates over number *per se* and is independent from formal schooling and repeated-addition processes. However, the strongest test of this ability would come from a task in which repeated addition is not even a logically possible strategy: the task of multiplying by a factor that is not a whole number. Some findings in the literature on preschool mathematics suggest that fractions are more difficult for children to grasp than whole numbers (Behr, Wachsmuth, Post, & Lesh, 1984; Gelman, 1991; Mack, 1995). This difficulty likely occurs because children cannot rely upon their knowledge of counting, on the repeated-addition heuristic that is initially taught to solve multiplication, or on rote-memorized multiplication tables. In this next experiment, we extended the design of Experiments 1 and 2 to test whether the observed non-symbolic multiplication can be performed with a fractional multiplicative factor. If this process is truly multiplicative, and distinct from one that capitalizes on repeated addition, we should see above-chance performance and similar patterns of competency in this experiment as in Experiments 1 and 2.

EXPERIMENT 3: TIMES 2.5

Participants. Sixteen 5-7-year-old children (8 females, 8 males; age range of 60 months 15 days to 84 months 5 days, mean age of 70 months) were included in the final sample, and were recruited from the same database as in Experiments 1 and 2. None of the participants had formal education in multiplication or division, and all participants fully completed both testing and training.

Procedure, Stimuli and Design. This experiment was identical to Experiments 1 and 2, except that the introduction movie, training block, and testing block all portrayed or tested a Times 2.5 relationship. During the multiplicative factor introduction movie, the participants saw two rectangles turn into five rectangles (one rectangle became two, and the other rectangle became three.) As in the Experiments 1 and 2, the comparison arrays were equated for item size and density during training, and area and contour length during testing. For the particular values used in this experiment, see Table 1.

Results. As in the previous analyses, each participant was given an average score composed of their performance on trials whose comparison arrays were a particular distance from the correct outcome (/ 2, / 1.5, *1.5, or *2.) Overall performance during training (77%) and testing (73%) was significantly above chance (one-sample t-tests; *t*s (15) = 10.99, 6.90, both *p*s < .001, two-tailed) (See Figure 4.) Using a test proportion of 67.8% (19/28 trials with an alpha level of .05), we also calculated the number of children whose individual performance was above chance; 12 of 16 children met this criterion (12 observed successes with a total N of 16, binomial sign test with an alpha of .05, p<.001). An ANOVA with block (training block, testing block) and distance (/2, /1.5, *1.5, and *2) as within-subject factors and gender (male, female) and age (60-72 months, 72-84 months) as between-subject factors was performed over these scores. There were no main effects of block (F (1,12) = .86, p = .37), gender (F (1,12) = 1.14, p = .31), or age (F (1,12) = .80, p = .39). There was a main effect of distance (F (3,36) = 4.81, p = .006). Follow-up Bonferroni-corrected pairwise comparisons suggest this was due to significantly lower performance in the *1.5 Distance trials (63%), which differed from the *2 Distance trials (83%; $p < .05$), but not the /1.5 or /2 trials (72% and 85%, respectively; *p*s = .99 and .16). There was a significant interaction between session and distance ($F(3,36) = 12.99$, $p < .001$). Just as in the other conditions, performance "dipped" at slightly different Distance trials for the training (worst performance at Distance *1.5) and testing (worst performance at Distance /1.5) sessions, indicating a shift from overestimation of the outcome during training to underestimation of the outcome during testing. With the exception of the Distance *1.5 training, and the Distance /1.5 testing trials, children performed significantly better than chance on all testing and training trial types (one-sample t-tests, all *p*s<.05).

To test whether children learned this multiplicative relationship by watching a single introductory movie, we tabulated each participant's percentage correct for the first and last trials at each Distance trial type (correct/2, /1.5, *1.5, *2) during the training block. Performance was above chance for both the first and the last trials (73% and 78% respectively; one-sample t-tests against chance yield *t*s (15) = 4.39, 7.27, both *p*s < .001, one-tailed). There is no evidence that children required much training to understand this concept as they did not perform reliably better on the last than on the first training trial (paired-sample t-test $t(15) = -.64$, $p = .53$, two-tailed.)

Discussion. The results from Experiment 3 indicate that children are able to compute outcomes to complex multiplication problems with a fractional factor (2.5) as the multiplicand. As in the previous experiments, this core multiplication utilizes approximate magnitudes from the ANS, operates independently from non-numerical perceptual variables, and is present before formal schooling. The results of this experiment are especially convincing as to the existence of a core scaling process because it is impossible to effect this transformation through a process of counting or repeated addition.

Might children have performed this task in some other way? First, perhaps children literally applied the mechanics of the introduction movie to the training and testing trials, and instead of scaling by a multiplicative factor they doubled half the objects via repeated addition, tripled the other half of the objects via repeated addition, and then summed together the two intermediate sums to arrive at the correct outcome. One challenge to this explanation is that the process of initially halving the array is fundamentally related (via inversion) to the multiplicative doubling observed in Experiment 1; because the elements in this initial array were continuously changing in size, orientation and position, moreover, halving in this experiment would occur on the basis of number. But the other challenge comes from research on the limits of working memory for numerical sets (Halberda et al., 2006). The process described above would require a mental grasp of at least 8 arrays and their accompanying updating: one half the initial array, the other half of the initial array, the twin array of one half the initial array, two twin arrays of the other half of the initial array, the intermediate outcomes of each of these addition problems, and the final outcome of both the transformed-via-repeated-addition arrays. Since adults can represent only 3-4 sets at any given time, this strategy would impose impossible demands on children's working memory.

Second, perhaps the children in the current experiment used the heuristic of simply doubling the initial amount, or doubling and then and adding a bit more after this process. One version of this account is circular: if children were doubling and then adding half of the original array afterwards to arrive at an outcome, then they would be drawing on a process that calculates a multiplicative invariant. Two other versions, however, are testable in our data. First, if children were just implementing a process of roughly doubling, some of the Distance trials (/2 and /1.5) should be more difficult, yielding significantly worse performance at these Distances compared to identical Distance trial types in the Times 2 condition. Moreover, some of the Distance trials (*1.5 and *2) should be easier, yielding significantly better performance in present experiment. Contrary to these predictions, children performed non-significantly better in the doubling experiment for all types of trials (for distance /1.5, 80% in the Times 2 condition compared to 71% in the Times 2.5 condition; t (15) = 1.46, p = .16, two-tailed; for Distance /2, 90% vs. 85%; t (15)=1.28, p =.22, two-tailed; for Distance 1.5, 66% vs 63% (t (15) = .63, p = .54, two-tailed; for Distance2: 91% vs. 83% (t (15) = 1.95, p = .07, two-tailed.) Children's performance did not vary with Distance trial type in the manner that would be predicted if they solved the Times 2.5 task by simple doubling. Second, if children were doubling and then adding a small amount more, with the latter amount independent of the size of the initial amount, then performance in the Times 2.5 condition should be lower than that in the other two conditions, because of the error inherent in this addition strategy. This prediction can be tested by comparing children's accuracy across the three experiments, analyses to which we now turn.

Extended Analyses. A final series of analyses were conducted across the data from the three experiments to address five questions. First, did children truly compute the results of these transformations, or did their success depend on an alternative strategy of range-based comparison? Second, insofar as children truly engaged in multiplicative operations, were they aided by verbal counting? Third, was there an overall effect of ratio of comparison array: outcome, due to the approximate nature of the ANS? Fourth, did the three operations of doubling, multiplying by 2.5, and quadrupling differ in difficulty for children in a manner consistent with the Weber signature of the ANS? Finally, did these operations differ in difficulty in a manner consistent with a whole-number approximation strategy in the Times 2.5 condition? We consider each question in turn.

Range-based strategies. Before we can conclude that children are truly multiplying, we must examine whether the children were using alternative strategies to solve these problems. Barth et al. (2009) found that children who performed doubling were sensitive to the relative extremity of comparison values, consistent with the use of a range-based strategy of judging that the comparison array was larger when it was especially large and smaller when it was especially small. That is, if children were tabulating the range of comparison arrays they see for the first few trials and, using the feedback given, they could infer that comparison arrays that are on the high end of the range are always the correct answer and comparison arrays on the low end are always the incorrect answer. This initial feedback could tip the children off that comparison arrays with extreme values, irrespective of what happened to the transformed initial value, are the correct or incorrect answer. To test whether the children in the present studies used this range-based strategy, we performed four sets of analyses.

First, children who followed a range-based strategy should perform at chance on the very first training trials, since they have not had experience with the range of values to be presented as comparison arrays. Moreover, children's performance should improve over the course of the training session. As seen in the sections above, neither prediction was confirmed since children in all experiments exhibited above-chance performance on the first set of training trials (84%, 64%, and 73%, respectively for Experiments 1, 2, and 3), and they showed no improvement over trials.

Second, a range strategy predicts chance performance on trials with comparison array values that lie in the middle of the range. To test this prediction, participants' scores were averaged for their performance on test trial types in the middle of the range (the 8 comparison array values which were not especially large or small). Performance at midrange was 72% for Experiment 1, (one-sample t (15) = 5.03, p <.001, two-tailed), 64% for Experiment 2 (one-sample t (15) = 3.44, p < .005, two-tailed), and 61% for Experiment 3 (one-sample t (15) = 2.21, p < .05, two-tailed). The above-chance performance observed on the early trials and the midrange trials in each of these experiments reveals that range-based strategies cannot entirely account for children's success on the multiplication task.

Third, if children are using range information only, there will be two specific problems per Experiment in which their performance should be *significantly below chance*. These problems are 4x2 vs. 12 and 4x2 vs. 16 (for Times 2), 4x4 vs. 24 and 4x4 vs. 32 (for Times 4), and 4x2.5 vs. 15 and 4x2.5 vs. 20 (for Times 2.5.) In all of these problems, the range-based strategy will result in the children choosing the incorrect answer. This did not happen in any of the critical cells; even when aggregated, performance on these trials is not below chance (χ^2 (1, N= 96) = .26, p = .31, one-tailed.)

Finally, children who used a range-based strategy should perform identically for identical comparison arrays: their performance should not be modulated by the distance of the comparison array from the correct outcome. To test this prediction, we analyzed performance for two 'yoked' comparison arrays during testing in which the numerical magnitude of the comparison array is the same (and therefore lies at the same point in the range of presented values) but the distance from the correct outcome differs. For example, in Experiment 1, the comparison array values of 12 and 48 each have one problem type where they differ from the correct outcome by a Distance factor of 1.5, and one problem where they differ from the correct outcome by a Distance factor of 2.0. Children were assigned a score for their overall performance across these two comparison array values for each factor (Distance 1.5, Distance 2.0.) A range-based strategy would result in no modulation in this score as a function of Distance, while a multiplicative process grounded in the approximate number system predicts poorer performance for the smaller Distance factor. Performance was higher overall for the comparison arrays which differed from the correct outcome by a factor of 2.0 (84%) than 1.5 (69%). In all three experiments, children performed better at the larger Distance factor (+25% for Experiment 1, +9% for Experiment 2, and +13% for Experiment 3.) As predicted from the core multiplication model, a repeated-measures ANOVA with Distance (Distance 1.5, Distance 2.0) as the within-subjects factor and multiplicative factor (Times 2, Times 4, or Times 2.5) as the between-subjects factor revealed a significant main effect of distance (F (1,45) = 5.09, p = .029), and no significant interaction with multiplicative factor (F (2,45) = .48, p = .63.)

Counting strategies. Through preschool and parental involvement, children have come to rely upon counting processes as a way to "get the right number." In the present experiments, we explicitly instructed children not to count, and their counting at this age is relatively slow, effortful, and obvious. Moreover, it is unclear how counting could help children in this task, because there is no countable magnitude that forms the multiplicative factor. Regardless, if counting was somehow helping the children in some way, children should perform better on trials with smaller initial arrays, which are more likely to be exactly enumerated via counting. To test this prediction, we compared each child's scores for test trials with different initial operands. An ANOVA with initial operand (4, 8, 12, or 16) as the within-subject factor and experiment as the between-subject factor revealed a linear trend (F (1,45) = 5.92, p = .019), in the *opposite* direction to the counting hypothesis. Children performed the worst on problems that had the smallest initial operand (4; 66%) compared to operands of 8 (74%), 12 (78%), or 16 (73%). If anything, counting may have been detrimental to children's performance.

Ratio-dependent performance. If children are utilizing their ANS representations during this task, their performance will decrease as the ratio between the comparison array and outcome decreases. To test for a main overall effect of ratio of comparison array : outcome, we collapsed performance across blocks and across the *2 and /2 trials and *1.5 and /1.5 trial types, yielding scores for each child on Factor 2 problems and Factor 1.5 problems. A paired-comparison t-test revealed a significant main effect of ratio (t(47)=6.03, p<.001); children responded correctly 82% of the time on Factor 2 trials and 69% of the time on Factor 1.5 trials. This pattern is also significant for both training and testing blocks individually (training: 84% Factor 2, 75% Factor 1.5, p<.001; testing: 80% Factor 2, 66% Factor 1.5, p<.001.)

Testing for the Weber signature of multiplication. A core process of numerical scaling over ANS representations would yield increased error and, subsequently, poorer performance, as the multiplicative factor increased in magnitude, from 2 to 2.5 to 4. A linear contrast analysis over percentage correct in each experiment, entered with the increasing scalar factor of Times 2, Times 2.5, and Times 4 reveals a significant linear trend: children perform at 82% overall in Times 2, 75% overall in Times 2.5, and 69% overall in Times 4 ($F(1,44)=14.77$, $p<.001$, $MSe=.13$). This performance profile provides further evidence for a core scaling process that operates over representations of approximate numerical magnitudes.

Testing for strategies of repeated addition and whole-number approximation. An account of the present findings which roots children's performance in processes of repeated addition and whole-number approximation predicts that performance should be lowest in the Times 2.5 condition, intermediate in the Times 4 condition, and highest in the Times 2 condition. To test this account, we performed a second linear contrast analysis, entered with the increasing repeated addition factor of Times 2, Times 4, and Times 2.5. This analysis revealed no significant linear trend ($F(1,44) = 3.66$, p = n.s., MSe =.03). Thus, children's performance profile across the three experiments was consistent with a process of direct scaling but not with processes of repeated addition and whole-number approximation.

General Discussion

The results from these experiments converge upon one central conclusion. Children possess a core multiplication ability that allows them to extract the invariant proportional relationship between two numerical magnitudes and then apply this relationship to new magnitudes. This ability depended on children’s approximate number system, for children's performance was modulated by the ratio between the product and the comparison array, and by the numerical magnitude of the scaling factor: two signatures of the ANS in a wealth of research on the approximate number system (Barth et al., 2005; Lipton & Spelke, 2004; Pica et al., 2004). Core multiplication came easily to the children, who inferred the correct outcome of a multiplicative transformation after seeing just a single exemplar of that transformation. Finally, children's multiplicative abilities were quite general: children learned fractional as well as whole number multiplicative factors, and they applied these factors to new numerical problems.

Because the children were 5-7 years of age, many experiences could have contributed to the abilities that they exhibited. Nevertheless, certain accounts of the development of multiplicative operations can be ruled out by the present findings. First, none of the children in the present experiments had received any formal instruction in multiplication. The intuitive multiplicative abilities that they exhibited therefore did not reflect direct instruction in the operation. Second, all of the children had learned to count and some had begun to receive instruction in addition, but several aspects of the data suggest that these experiences did not account for their success. Children performed as well in Experiment 3 as in Experiment 1. Because a strategy of repeated addition could not be applied to Experiments 2 or 3, their success provides evidence that their multiplication was not based on addition. Further, children performed better on problems with operands that were too large for counting than with operands that were small enough to be countable. Because counting-based strategies predict the opposite relationship, this finding suggests that counting had no positive effect on children's multiplication performance. We conclude that children's performance depended on an intuitive grasp of numerical proportion.

The success of children on these experiments cannot be explained by a process of repeated addition. First, the children were able to successfully multiply by a factor of 2.5 and a factor of 4.0. In order to use repeated addition, children would need to mentally represent 8 arrays (for Times 2.5) and 5 arrays (for Times 4); both of these amounts exceed the number of arrays even adults can hold in working memory (Halberda et al., 2006). Second, even if they were somehow able to use repeated addition with this many arrays, this account predicts that performance would be lowest on the Times 2.5 condition, which is not the case.

Our data instead point to an account of multiplicative scaling of numerical variables represented by the approximate number system. This is clearly demonstrated by the presence of two signature Weber properties in the data. First, overall performance was better for trials in which the ratio between the outcome and comparison array was larger. Second, as the scaling factor increased from 2 to 2.5 to 4, so did the error (as indicated by poorer overall performance). Children performed best in Times 2, at an intermediate level in Times 2.5, and worst in Times 4. The fact that children are not performing repeated addition, but *are* dependent on a scaling function when mapping one ANS representation to another, is evidence for an operation that should be added to the canon of addition, subtraction, and ordering: core multiplication. This core multiplication process speaks directly to the conceptual definition as noted by mathematicians (James & James, 1976).

Our findings help to shed light on the conflicting literature on multiplication, and serve to highlight the distinction between the core multiplication process as it is formulated in mathematics and tested here and the constructed process of repeated addition emphasized in many elementary mathematics curricula. As noted in the introduction, previous work with infants and young children provides evidence for an intuitive, unlearned process of scalar multiplication (Barth et al., 2009; McCrink & Wynn, 2007; Nunes, Schliemann, & Carraher, 1993), whereas previous work with adults often suggests that multiplication depends entirely on rote-learned symbolic knowledge (e.g., Lee, 2000; Lemer et al., 2003). This discrepancy is likely due both to the way that multiplication is taught in schools, and to the particular nature of the administered tasks.

The core multiplicative process revealed by these experiments contrasts markedly with the multiplication processes that children are taught in school. Starting in early primary school, children are drilled with multiplication tables to learn their single-digit multiplication facts, and then they move to procedural strategies for solving multi-digit multiplication problems. Neither of these techniques encourages the use of children's intuitive sense of numerical magnitudes. Even if children initially use their ANS and core multiplication abilities to reason about multiplicative relationships, the problems likely become stored as facts over the course of arithmetic instruction, and are then accessed via a cognitive pathway that is not reliant upon this quantity-based processing. Consistent with this suggestion, a recent study of adults' learning of new multiplication facts has shown such a process happening in real time during a short fMRI session. At the start of training on the multiplication problems, the intraparietal sulcus showed greater activation, indicating that the core magnitude estimation system was at work.

Later in training, however, the left angular gyrus showed greater activation- a signature of the use of verbal retrieval processes (Delazer et al., 2004).

In addition, the experiments showing use of verbal processes in multiplication generally use designs which push the subjects towards the verbal, automatic, and procedural strategies they learned in school. For example, Dehaene and Cohen (1997) tested their patients with multiplication facts involving single-digit operands, which comprise the bulk of the rote multiplication table. This is an understandable design feature; after all, part of being an educated adult is learning to automatize useful arithmetic facts, store and retrieve them in an exact fashion, and apply effective procedures that will yield exact answers. It is undoubtedly critical to have knowledge about these processes. However, the tendency of adults to rely upon rote multiplication in these studies may obscure their core multiplicative abilities. In support of this idea, there are patients who have impaired multiplication abilities on small-number multiplication problems such as 5 * 4, but who respond correctly to division problems such as 20 / 5 (Dehaene & Cohen, 1997). Adults may recruit their core number knowledge in division problems to detect the relationship between the two amounts, because they have minimal stored verbal knowledge of these problems compared to the multiplication problems. To test this interpretation, one could administer the present task to patients who have shown impaired symbolic multiplication in the neuropsychological work on arithmetic operations (Dagenbach & McCloskey, 1992; Dehaene & Cohen, 1997; Lampl, Eshel, Gilad, Sarova-Pinas, 1994; Lee, 2000). We predict that such patients will be able to perform this task, because it uses nonsymbolic numerical arrays that bypass the symbolic, fact-based, verbal multiplication system.

Core multiplication, as assessed in the present experiments, differs in other ways from symbolic multiplication as taught in school. Symbolic multiplication problems involve three explicitly represented numerical values: the two operands and the product. In our experiments, like those of Barth et al. (2009), only two numerical magnitudes are presented on each trial. When a child sees that 16 objects are covered and transformed into 32 objects, she may encode the cardinal value '2' and then combine 16 and 2 multiplicatively. Alternatively, she may represent the outcome array as 'twice as much' as the input array, an inherently relational representation. The present experiments do not distinguish between these alternative interpretations of children's performance. Within-subject designs, providing cardinal values on some trials and proportional numerical relationships on other trials, may help to reveal the types of representations that children form.

The present findings accord with research on infants, providing evidence for sensitivity to proportional relationships in arrays with two types of elements (McCrink & Wynn, 2007). They also speak to research on non-human animals, who apportion their foraging time between distinct locations in accordance with their relative expected rate of payoff, suggesting that animals can multiply continuous quantities such as volume of foodstuff (e.g., Harper, 1982; Leon & Gallistel, 1998; see Gallistel, 1990, for review). In one relevant study, Harper (1982) found that foraging ducks apportion their time between two food sources thrown onto a lake in accord with the volume of food that appears per unit time, not the number of food items. Because past research on animals examines the role of continuous quantitative variables, such as the magnitude of foodstuff or the rate of food distribution, it would be interesting to conduct the present experiments with a non-human species to see if number *per se* contributes to foraging calculations above and beyond overall amount, when these cues are not in conflict.

Finally, the present findings accord with research on humans living in diverse cultures. In one set of studies, Nunes et al. (1993) found that Brazilian fishermen without formal education were able to calculate proportions by establishing the correspondence (the number of objects per reference unit) between pairs of variables, and then manipulate these variables while keeping the correspondence between them constant, an ability very similar to the one utilized in the present study. Again, it would be interesting to test this population with the present experimental controls for alternative strategies and non-numerical quantitative variables.

The present findings diverge, however, from those of Barth et al. (2009), who found that in some instances children used range-based strategies as an alternative to the multiplicative operation of doubling, and Jeong et al. (2007), who found that children were unable to perform proportional analogies when given discrete units. There are two reasons as to why we observed above-chance performance in the current study, with no apparent evidence of range-based strategy and using discrete stimuli. First, Barth et al. (2009) presented practice trials that established a range of values for the comparison arrays that was perfectly translatable to the experimental portion of the study; they suggest that this design may have obscured the children's ability to respond appropriately to the doubling transformations, because the range information was so salient and useful. Second, a procedural change between previous studies (Barth et al., 2009; Jeong et al., 2007) and the current study may have had major conceptual ramifications. Although Barth et al. (2009) presented children with many pairs of initial and doubled arrays, they never showed children the splitting of one object into two. Jeong et al. (2007) also did not appeal to a splitting relationship between discrete units. In the current studies, in contrast, we used the introduction movie to show the actual, literal, splitting of a rectangle into parts. Thus, we explicitly provided children with what Confrey (1994) calls the "basic conceptual primitive" of splitting. Confrey (1994) suggests that the emphasis on counting and repeated addition for learning multiplication shifts much-needed attention away from what is a very natural way for children to reason about invariant, multiplicative, relationships like the ones tested in the present study. The authors advocate for an emphasis during early primary school on this conceptual primitive, and say it "can establish a more adequate and robust approach to such traditionally thorny topics as ratio and proportion, multiplicative rate of change, [and] exponential functions." (Confrey, 1994, p. 298)

As Confrey's argument suggests, an exploration of core multiplication may have wide practical utility for elementary instruction in mathematics. There has been a spate of recent work showing that the ANS is heavily involved in the steps children take towards mastering arithmetic. For example, children who have mastered verbal counting, and are on the threshold of arithmetic instruction, can utilize their non-symbolic number system to perform symbolic addition and subtraction (Gilmore, McCarthy & Spelke, 2007). Furthermore, a child's level of general ANS acuity is associated with their level of mathematical achievement (Halberda, Mazzocco, & Feigenson, 2008; Ramani & Siegler, 2008; Wilson, Revkin, Cohen, Cohen, & Dehaene, 2006). In a retrospective study, Halberda et al. (2008) found that individual differences in ANS acuity at age 14 were significantly associated with symbolic math achievement back to kindergarten. In an intervention study, Wilson et al. (2006) found that a computer training program which improved ANS acuity also improved certain arithmetic skills in a school setting. Together with the present findings, these experiments suggest that core multiplication ability may provide further means to increase children's learning of school mathematics. Combining intuition with education may lead to enhanced conceptual understanding of a central and challenging part of the elementary school mathematics curriculum.

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Figure Captions

*Table 1.*Exact operand and comparison array values used in Experiments 1, 2, and 3.

Figure 1. Schematic of the multiplication videos presented to the children.

Figure 2. Performance for testing trials of Experiment 1 (multiplicative factor = 2). Error bars represent *SEM*, and the dotted line indicates chance (50%). Asterisks indicate a <.05 level of significance.

Figure 3. Performance for testing trials of Experiment 2 (multiplicative factor = 4). Error bars represent *SEM*, and the dotted line indicates chance (50%). Asterisks indicate a <.05 level of significance.

Figure 4. Performance for testing trials of Experiment 3 (multiplicative factor = 2.5). Error bars represent *SEM*, and the dotted line indicates chance (50%). Asterisks indicate a <.05 level of significance.

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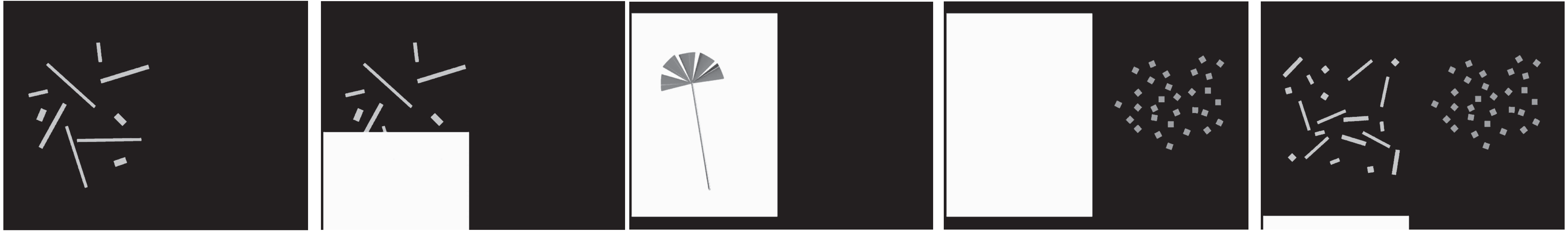
Table 1

Experiment:		
Times 2	**Times 4**	**Times 2.5**
Training	**Training**	**Training**
6 * 2 vs. 6	6 * 4 vs. 12	6 * 2.5 vs. 8
6 * 2 vs. 8	6 * 4 vs. 16	6 * 2.5 vs. 10
6 * 2 vs. 18	6 * 4 vs. 32	6 * 2.5 vs. 23
6 * 2 vs. 24	6 * 4 vs. 48	6 * 2.5 vs. 30
10 * 2 vs. 10	10 * 4 vs. 20	10 * 2.5 vs. 13
10 * 2 vs. 13	10 * 4 vs. 27	10 * 2.5 vs. 17
10 * 2 vs. 30	10 * 4 vs. 30	10 * 2.5 vs. 38
10 * 2 vs. 40	10 * 4 vs. 40	10 * 2.5 vs. 50
14 * 2 vs. 14	14 * 4 vs. 28	14 * 2.5 vs. 18
14 * 2 vs. 19	14 * 4 vs. 37	14 * 2.5 vs. 23
14 * 2 vs. 42	14 * 4 vs. 84	14 * 2.5 vs. 53
14 * 2 vs. 56	14 * 4 vs. 112	14 * 2.5 vs. 70
Testing	**Testing**	**Testing**
4 * 2 vs. 4	4 * 4 vs. 8	4 * 2.5 vs. 5
4 * 2 vs. 5	4 * 4 vs. 11	4 * 2.5 vs. 7
4 * 2 vs. 12	4 * 4 vs. 24	4 * 2.5 vs. 15
4 * 2 vs. 16	4 * 4 vs. 32	4 * 2.5 vs. 20
8 * 2 vs. 8	8 * 4 vs. 16	8 * 2.5 vs. 10
8 * 2 vs. 11	8 * 4 vs. 21	8 * 2.5 vs. 13
8 * 2 vs. 24	8 * 4 vs. 48	8 * 2.5 vs. 30
8 * 2 vs. 32	8 * 4 vs. 64	8 * 2.5 vs. 40
12 * 2 vs. 12	12 * 4 vs. 24	12 * 2.5 vs. 15
12 * 2 vs. 16	12 * 4 vs. 32	12 * 2.5 vs. 20
12 * 2 vs. 36	12 * 4 vs. 72	12 * 2.5 vs. 45
12 * 2 vs. 48	12 * 4 vs. 96	12 * 2.5 vs. 60
16 * 2 vs. 16	16 * 4 vs. 32	16 * 2.5 vs. 20
16 * 2 vs. 21	16 * 4 vs. 43	16 * 2.5 vs. 27
16 * 2 vs. 48	16 * 4 vs. 96	16 * 2.5 vs. 60
16 * 2 vs. 64	16 * 4 vs. 128	16 * 2.5 vs. 80

Figure 1



An array of objects is presented to the child.

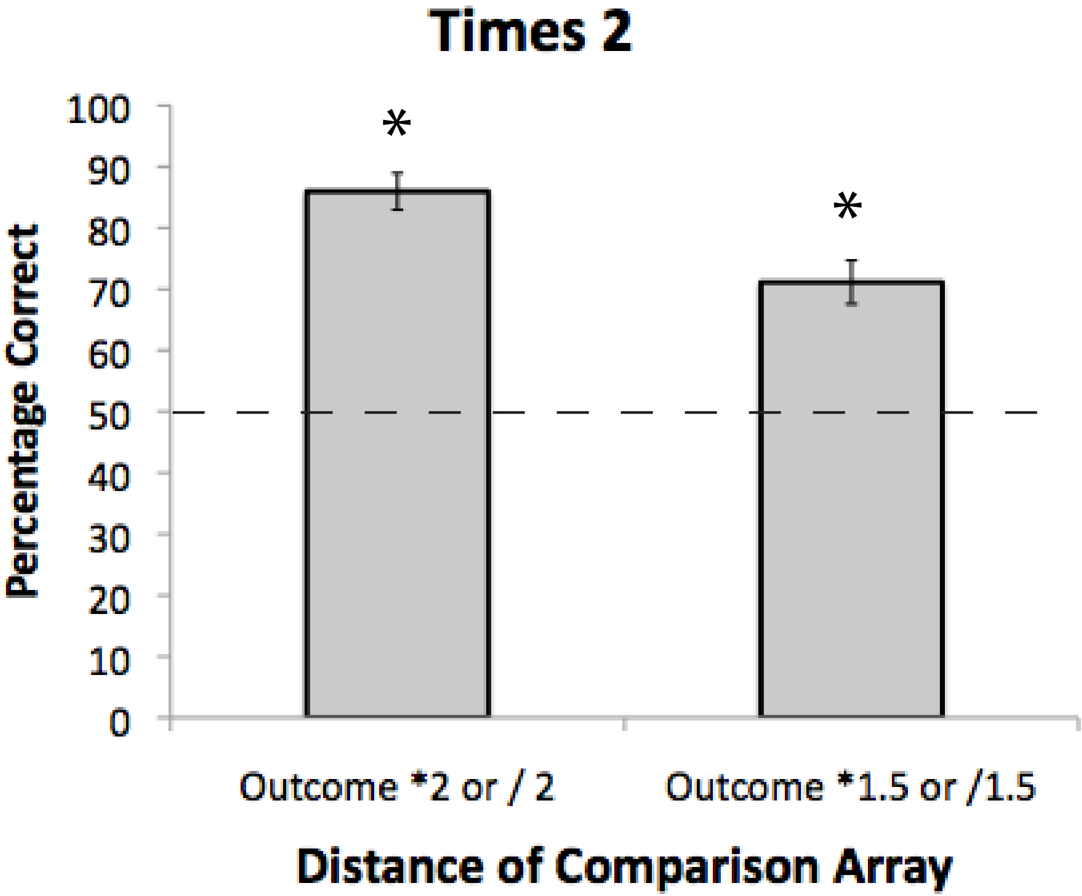
The array is covered by the occluder.

The multiplying wand comes out and waves over the occluded array.

A comparison array comes down next to the occluded array, and the child is asked to choose which side has more.

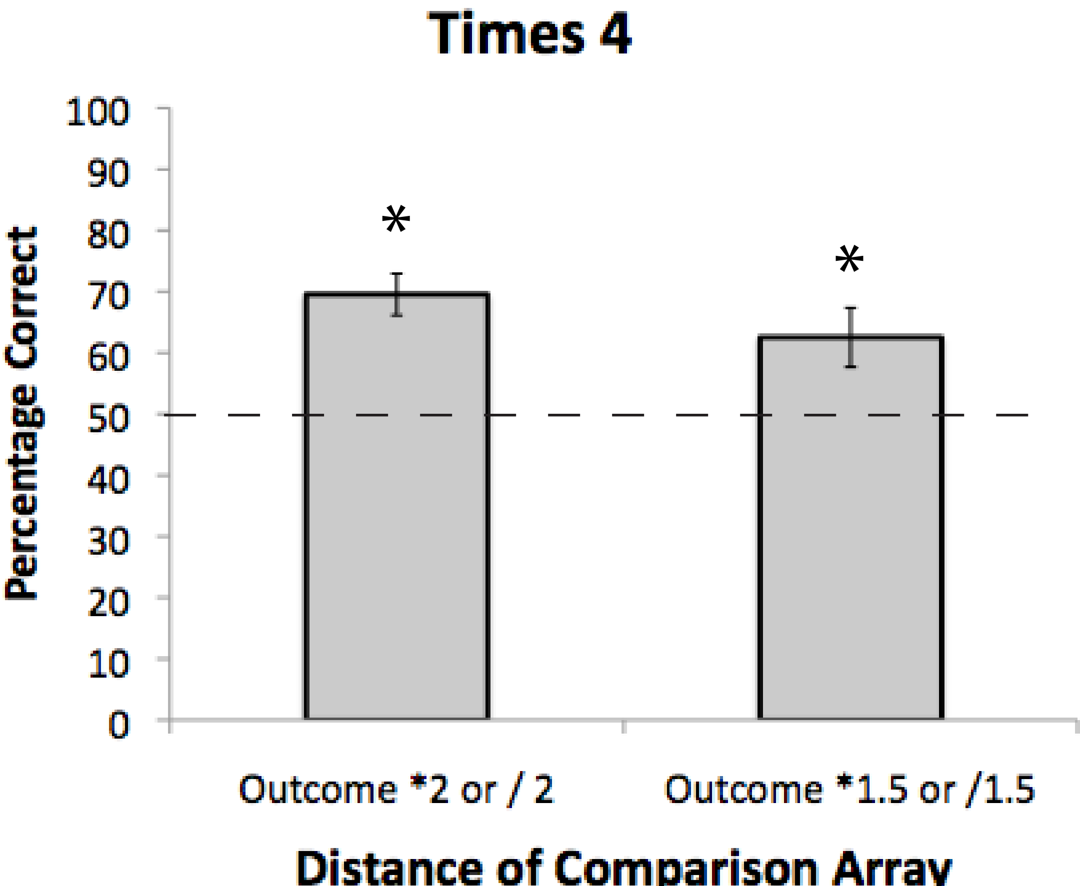
In training, the occluder drops to reveal the outcome, and the child is told if they were correct or incorrect. (In testing, there is no revelation of the outcome.)

Figure 2



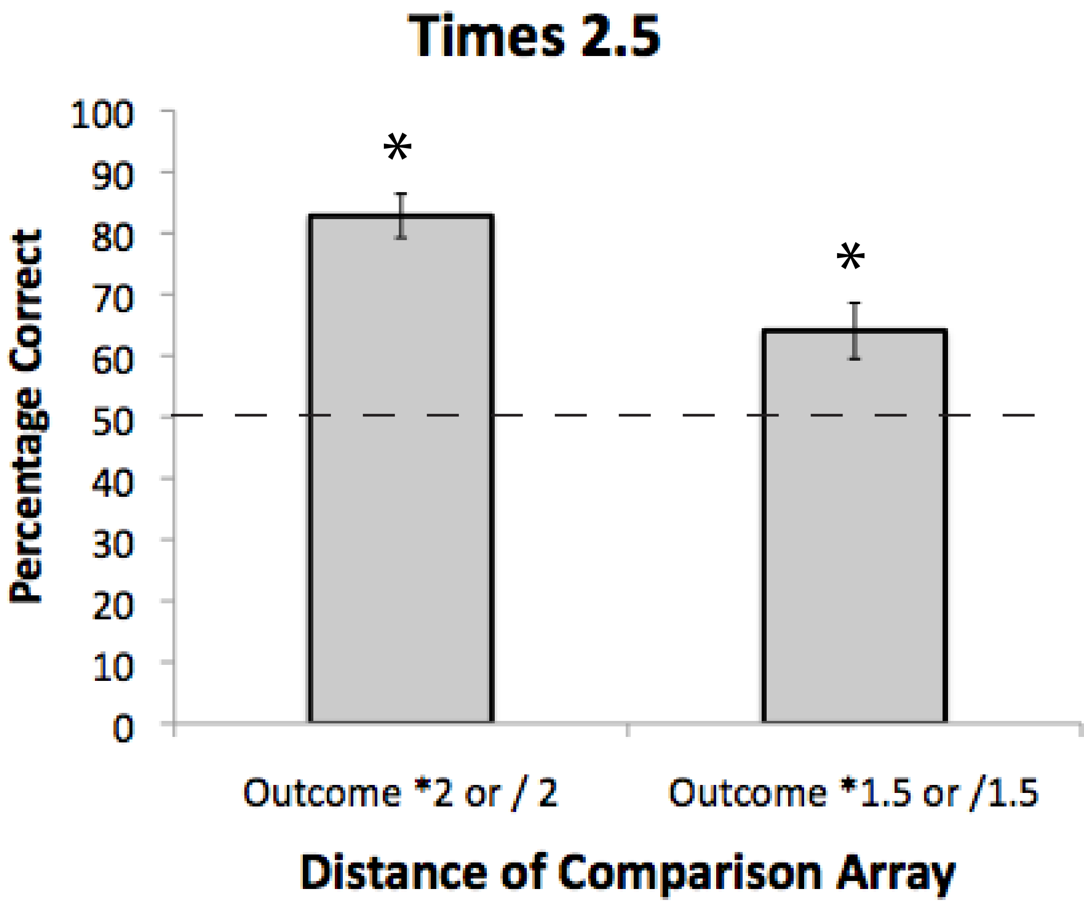
Times 2
Percentage Correct
100
90
80
70
60
50
40
30
20
10
0
*
*
Outcome *2 or / 2
Outcome *1.5 or /1.5
Distance of Comparison Array

Figure 3



Times 4
100
90
80
70
60
50
40
30
20
10
0
Percentage Correct
*
*
Outcome *2 or / 2
Outcome *1.5 or /1.5
Distance of Comparison Array

Figure 4


Times 2.5
Percentage Correct
100
90
80
70
60
50
40
30
20
10
0
*
*
Outcome *2 or / 2
Outcome *1.5 or /1.5
Distance of Comparison Array