On Newforms for Split Special Odd Orthogonal Groups

On Newforms for Split Special Odd Orthogonal Groups

A dissertation presented

by

Pei-Yu Tsai

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

April 2013

Dissertation Advisor: Professor Benedict H. Gross | Pei-Yu Tsai

On Newforms for Split Special Odd Orthogonal Groups

Abstract

The theory of local newforms has been studied for the group of PGL_n and recently PGSp_4 and some other groups of small ranks. In this dissertation, we develop a newform theory for generic supercuspidal representations of SO_{2n+1} over non-Archimedean local fields with odd characteristic by defining a family of open compact subgroup $\mathrm{K}(\mathfrak{p}^m)$, $m \geq 0$ (up to conjugacy) which are analogous to the groups $\Gamma_0(\mathfrak{p}^m)$ in the classical theory of modular forms. We give lower bounds on the dimension of the fixed subspaces of $\mathrm{K}(\mathfrak{p}^m)$ in terms of the conductor of the generic representation, and give a conjectural description of the space of old forms. These results generalize the known cases for $n = 1, 2$ by Casselman [**4**] and Roberts and Schmidt [**23**].

Contents

Acknowledgements vi

Chapter 1. Introduction 1

1.1. Historical background 1

1.2. Statement of the main results 1

Part 1. p-adic groups 6

Chapter 2. Structure theory 7

2.1. Notations 7

2.2. Compatible good basis 10

2.3. The groups SO_{2n+1}, SO_{2n}, and GL_n 12

2.4. Parabolic subgroups Q, P 15

2.5. Parahoric subgroups G_{x_i}, H_{x_i} 17

Chapter 3. Generic representations 20

3.1. Admissible representations 20

3.2. Whittaker linear forms 25

3.3. Modules of the mirabolic group P_{n+1} 27

3.4. A Lemma 31

3.5. Hecke algebras 34

Chapter 4. Local factors of generic representations 41

4.1. Standard L-function for $\mathrm{SO}_{2n+1}(k)$ 41

4.2. ε-factor and conductor 47

4.3. Rankin-Selberg convolutions for $\mathrm{SO}_{2n+1}(k) \times \mathrm{GL}_n(k)$ 50
4.4. Rankin-Selberg L-function of $\pi \times \tau$ 54

Chapter 5. The Fourier transform $\Psi(v, X; X_1, X_2, ..., X_n)$ 59
5.1. Spherical Whittaker functions on $\mathrm{GL}_n(k)$ 60
5.2. Fourier transforms of Whittaker functions 64
5.3. Actions of Hecke operators 69
5.4. Fourier transform Ψ and Jacquet's polynomial Ω 73

Part 2. Test vectors 75

Chapter 6. Review for cases of lower rank 76
6.1. Rank 1: $\mathrm{SO}_3(k) \simeq \mathrm{PGL}_2(k)$ 76
6.2. Rank 2: $\mathrm{SO}_5(k) \simeq \mathrm{PGSp}_4(k)$ 79

Chapter 7. Open compact subgroups and their fixed vectors 90
7.1. Definition of $\mathrm{K}(\mathfrak{p}^m)$, $m \geq 0$ 90
7.2. $\mathrm{K}(\mathfrak{p}^m)$ with $m = 0, 1$ 94
7.3. Existence of Fix vectors 96
7.4. Fixed vectors at the level equal to the conductor 100

Chapter 8. Action of the Hecke operators 103
8.1. Level raising operators 104
8.2. Hecke operators 108
8.3. Hecke eigenvectors 115
8.4. Minimal level 125

Chapter 9. Main Theorems 129
9.1. New vectors and old vectors 129

Bibliography 133

Acknowledgements

I want to give my highest gratitude to my advisor Benedict H. Gross, whose constant inspiration and support makes this work finally possible. I am so grateful to be one of his students and be nurtured to be a mathematician. Not only his insight in math, but also his care put on his student as a teacher both in math and in life. I am thankful to be introduced this subject with profound beauty and to many discussions between him and me during me graduate life. To name more, I would like to thank Wee Teck Gan, Atsushi Ichino and Jiu-Kang Yu for their valuable comments in the Theta Festival 2011 in Postech, Korea. Many ideas was obtained from discussion with them during this time. I would also like to thank Dihua Jiang, Jiu-Kang Yu, and Lei Zhang for their patient guidance and selfless instruction in helping me gain acquaintance to this field. Many technical issues were solved from conversation with them.

Lastly, I would like to thank the most important component of my graduate life, my best friends and the whole Math Department members. Susan Gilbert is the best secretary ever and there is nothing which can not be solved by her. I love all of the fourth floor fellows–Anand, Bao, Cheng-Chiang, Eric, Gabriel, Hansheng, Henry, Jameel, Jeff, Jerry, Ji Oon, Nasko, Oleg, Oliver, Peter, Robin, Stergios, Suh Hyun, Yu-Jong, Yunqing, too many to be named all–whose company makes my life beautiful and colorful. I am especially thankful to my friends Yunqing Tang, Cheng-Chiang Tsai and Xiaoheng Jerry Wang for their support, suggestions and hassles to comment on me and many versions of this thesis.

To my dear mom.

CHAPTER 1

Introduction

1.1. Historical background

The theory of newforms is a central topic in the classical theory of holomorphic modular forms. The Fourier coefficients of a newform encode a great deal of arithmetic information and the local theory of newforms gives a dictionary from the classical theory of modular forms to the modern theory of automorphic forms on GL(2). The local Langlands correspondence predicts that the invariants of local Galois representations, such as L-function and ε-factor, should match the corresponding analytic invariants of a local representation π of p-adic algebraic groups. The ε-factor determines the conductor $a_\pi \geq 0$ and the root number ε_π, which for representations of PGL(2) is equal to ± 1.

The theory of local newforms was developed for PGL(2) by Casselman [**4**] in 1970s and was generalized to PGL(n) by Jacquet, Piatetski-Shapiro and Shalika [**14**] in 1980. Recently a local newform theory has been established for PGSp(4) by Roberts and Schmidt [**23**], for $\mathrm{U}(1,1)$ by Lansky and Raghuram [**16**], and for unramified $\mathrm{U}(2,1)$ by Miyauchi [**20**] [**18**] [**19**]. In a letter to Serre in 2010, Gross conjectured that it holds in general for $\mathrm{SO}(2n+1)$. The goal of this work is to establish a local newform theory for generic representations of $\mathrm{SO}(2n{+}1)$ over non-Archimedean fields.

1.2. Statement of the main results

Assume that k is a non-Archimedean local field and the characteristic of k is not equal to 2. Let V be a split quadratic space of dimension $2n+1$ over k with even quadratic form q and discriminant 2. Let $\mathrm{SO}_{2n+1} = \mathrm{SO}(V)$ be the special orthogonal group of V. Denote by $\langle\ ,\ \rangle$ the associated bilinear form $\langle v, w\rangle = \frac{1}{2}[q(v+w) - q(v) - q(w)]$ on V. We fix a canonical basis of V in Section 2.2

$$\{e_1, e_2, ..., e_n, v_0, f_n, ..., f_2, f_1\}$$

under which the Gram matrix of $\langle\ ,\ \rangle$ is equal to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix}.$$

Let H be the subgroup of $\mathrm{G} = \mathrm{SO}(V)$ which fixes the anisotropic vector v_0. Then H is isomorphic to the special even orthogonal group SO_{2n} and is reductive.

Following a suggestion of A. Brumer, we define the open compact subgroups $\mathrm{K}(\mathfrak{p}^m)$ of $\mathrm{G}(k)$ as follows:

For $m \geq 0$, let $\mathbb{L}_m$ be the quadratic lattice

$$(\bigoplus_{i=1}^{n} \mathfrak{o}e_i \oplus \mathfrak{p}^m f_i) \oplus \mathfrak{p}^m v_0$$

with associated bilinear form $\varpi^{-m}\langle\ ,\ \rangle$, where ϖ is a uniformizer of $\mathfrak{o}$. The Gram matrix for $\mathbb{L}_m$ is

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2\varpi^m & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix}.$$

This endows a quadratic form on $\mathbb{L}_m/\mathfrak{p}\mathbb{L}_m$ over the residue field $\mathfrak{f}$, which is nondegenerate for $m = 0$ and degenerate for $m \geq 1$. The reductive quotient $\mathrm{SO}(\mathbb{L}_m/\mathfrak{p}\mathbb{L}_m)$ is hence $\mathrm{SO}_{2n+1}(\mathfrak{f})$ for $m = 0$ and $\mathrm{O}_{2n}(\mathfrak{f})$ for $m \geq 1$.

Definition 1.2.1. For $m \geq 0$, let $\mathrm{J}(\mathfrak{p}^m)$ denote the subgroup $\mathrm{SO}(\mathbb{L}_m)(k)$ of $\mathrm{G}(k)$. Define $\mathrm{K}(\mathfrak{o}) = \mathrm{J}(\mathfrak{o})$ which is the hyperspecial maximal compact subgroup $\mathrm{G}(\mathfrak{o})$. For

$m \geq 1$, define the open compact subgroup $\mathrm{K}(\mathfrak{p}^m)$ as the kernel of the composite map

$$\mathrm{SO}(\mathbb{L}_m)(k) \xrightarrow{\text{mod } \mathfrak{p}} \mathrm{O}_{2n}(\mathfrak{f}) \xrightarrow{det} \{\pm 1\}.$$

Then $\mathrm{K}(\mathfrak{p}^m)$ is a normal subgroup of $\mathrm{J}(\mathfrak{p}^m)$ of index 2.

An important property of the open compact subgroups $\mathrm{K}(\mathfrak{p}^m)$ is that $\mathrm{H}_{x_m} := \mathrm{K}(\mathfrak{p}^m) \cap \mathrm{H}(k)$ is a hyperspecial maximal compact subgroup of H. When $n = 1$, these are the subgroups $\Gamma_0(\mathfrak{p}^m)$ in $\mathrm{PGL}_2(k)$ and H_{x_m} is $\mathrm{GL}_1(\mathfrak{o})$.

Assume π is an irreducible generic supercuspidal representation of G. We introduce the local zeta integral of π in Chapter 4 and defined the conductor a_π and the root number ε_π by the functional equation of the zeta integrals in Section 4.2. Note that $\mathrm{K}(\mathfrak{p}^m)$ contains H_{x_m}. We discuss the Rankin-Selberg convolutions for $\mathrm{SO}_{2n+1}(k) \times \mathrm{GL}_n(k)$ in Section 4.4. By using the Rankin-Selberg convolutions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_n$ with unramified second factor, we then study properties of vectors in the subspaces $V_\pi^{\mathrm{H}_{x_m}}$ which later play the central role in studying vectors in the fixed spaces of $\mathrm{K}(\mathfrak{p}^m)$. The spherical Hecke algebra of $\mathrm{GL}_n(k)$ is isomorphic to $\mathbb{C}[T_1, T_2, ..., T_n, T_n^{-1}]$ under the Satake isomorphism where T_i is the i^{th} elementary symmetric polynomial in variables X_1, X_2,...,X_n. This leads to the following proposition in Section 5.4:

Proposition 1.2.2. *There is an injective $\mathbb{C}$-linear map Ω from the subspace $\pi^{\mathrm{H}_{x_m}}$ to the ring $\mathbb{C}[T_1, T_2, ..., T_n, T_n^{-1}]$. Moreover, we can put a $\mathcal{H}(\mathrm{H}(k), \mathrm{H}_{x_m})$-module structure on the fixed subspace $\pi^{\mathrm{H}_{x_m}}$ such that Ω is also a $\mathcal{H}(\mathrm{H}(k), \mathrm{H}_{x_m})$-module homomorphism.*

Here ω_m is a certain lift of a special Weyl element of $\mathrm{O}_{2n}(\mathfrak{f})$ to $\mathrm{J}(\mathfrak{p}^m)$. This proposition will give us a nice way to distinguish different $\mathrm{K}(\mathfrak{p}^m)$-fixed vectors and puts conditions on the dimension of the fixed spaces. Moreover, it also proves us the existence of nonzero vectors that are fixed by $\mathrm{K}(\mathfrak{p}^m)$ for some m.

Definition 1.2.3. A nonzero vector in $\pi^{K(\mathfrak{p}^m)}$ is called a *fixed vector of level* m. In particular, a fixed vector v level a_π is called a *new vector* of π.

Our main theorem is that the open compact subgroups $\mathrm{K}(\mathfrak{p}^m)$ determine the local invariants a_π and ε_π. This is implied by the following Main Theorems.

Theorem 1.2.4. *The fixed subspace of* π *of the open compact subgroup* $\mathrm{K}(\mathfrak{p}^m)$ *is nonzero if and only if* $m \geq a_\pi$.

Theorem 1.2.5. *The subspace* $\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ *is a line generated by the new vectors and the group* $\mathrm{J}(\mathfrak{p}^{a_\pi})/\,\mathrm{K}(\mathfrak{p}^{a_\pi})$ *of order* 2 *acts on this line by the quadratic character* ε_π. *Moreover, the Whittaker functional* ℓ_θ *with respect to the given generic data* $(\mathrm{B}, \mathrm{T}, \theta)$ *is nontrivial on this line.*

In other words, the conductor a_π is the minimal level for which a fixed vector exist and such a fixed vector, called a new vector, of level a_π is unique up to scaling. Moreover, the root number ε_π can be read off form the action of $\mathrm{J}(\mathfrak{p}^{a_\pi})$ on the new vectors.

To prove the two main theorems above, we use Hecke eigenvalues and Fourier coefficients. This idea follows the method in classical theory of modular forms and Roberts-Schmidt's proof in the case $n = 2$. To do so, we make use of the zeta integrals of π and work out the Hecke eigenvalues in Chapter 8. Although we believe that the arguments in this thesis can be completed to provide a full proof, at the moment the proof of the multiplicity one statement is heuristic.

Similar to classical holomorphic form we have the level raising operators and can talk about oldforms. The level raising operators θ_0, θ_0^* and η_λ are defined in Section 8.1. Moreover, combining with the result from Ω in Section 5.4 we can also obtain a lower bound on the dimension of fixed spaces of higher levels. We expect that this is the exact dimension. When the equality holds, we can obtain an oldform theory which says all fixed vectors are old vectors.

Definition 1.2.6. A nonzero fixed vector is an *old vector* if it is the image of the new vector under a composition of some of the level raising operators θ_λ and η_λ.

Theorem 1.2.7. $\dim \pi^{\mathrm{K}(\mathfrak{p}^m)} \geq \binom{n+\lfloor\frac{m-a_\pi}{2}\rfloor}{n} + \binom{n+\lfloor\frac{m-a_\pi+1}{2}\rfloor-1}{n}.$

Conjecture 1.2.8. *The lower bound of* $\dim \pi^{\mathrm{K}(\mathfrak{p}^m)}$ *is the exact dimension and all nonzero fixed vectors of level greater than* a_π *are old vectors.*

We give some backgrounds on p-adic groups and generic representations in Chapter 2 and 3 of Part 1. In Chapter 4, we write down the local factors and the Rankin-Selberg convolutions for $\mathrm{SO}_{2n+1}(k) \times \mathrm{GL}_n(k)$. Most of the tools used in proving the main theorems will be given in Chapter 5 of Part 1 where we discuss the invariant subspace $\pi^{\mathrm{H}_{x_m}}$ that contains $\pi^{\mathrm{K}(\mathfrak{p}^m)}$. Starting from Part 2, we start to talk about the fixed vectors of $\mathrm{K}(\mathfrak{p}^m)$ from various aspects. We first briefly review the lower rank case with $n = 1, 2$ in Chapter 6 which are proved by Casselman and Roberts-Schmidt but now in the form of $\mathrm{SO}_3(k)$ and $\mathrm{SO}_5(k)$. Then we introduce the open compact subgroup $\mathrm{K}(\mathfrak{p}^m)$ for general rank n in Chapter 7. Chapter 8 is devoted to the Hecke actions and the proof of Theorem 1.2.4. Finally in Chapter 9, we prove all the theorems stated above.

Notation 1.2.9. We warm that in this thesis, the notations denoted in roman font are fixed through out the whole thesis while the italic ones are floating and depend on the local content.

Part 1

p-adic groups

CHAPTER 2

Structure theory

Let k be a non-Archimedean local field of residue characteristic p with ring of integers $\mathfrak{o}$. Let $\mathfrak{p} = (\varpi)$ denote the unique maximal ideal $\mathfrak{p}$ where ϖ is some fixed uniformizer. Let $|\cdot| : k \to \mathbb{R}$ be the valuation on k normalized such that $|\varpi| = q$ where q is the cardinality of the residue field $\mathfrak{f} = \mathfrak{o}/\mathfrak{p}$. Fix a unitary additive character $\psi : k^+ \to \mathrm{S}^1$, $\mathrm{S}^1 = (\mathbb{C}^\times)$ with norm 1, with conductor $\mathfrak{o}$. Assume $\mathrm{char}(k) \neq 2$.

2.1. Notations

Let $\underline{G}$ be a reductive group scheme and let G denote its generic fiber. We abuse the notation and denote the R-points $\underline{G}(R)$ of $\underline{G}$ by $G(R)$. We assume that G is split over k. There exits a k-rational Borel subgroup, say B, of G and a k-split maximal torus, say T, contained in it. Assume we fix $T \subset B \subset G$ defined over $\mathfrak{o}$. Denote by $\mathrm{X}^\bullet(T) = \mathrm{Hom}_k(T, \mathbb{G}_m)$ and $\mathrm{X}_\bullet(T) = \mathrm{Hom}_k(\mathbb{G}_m, T)$ the character group and co-character group of T respectively. Let $\langle\ ,\ \rangle$ denote the natural perfect pairing

$$\mathrm{X}_\bullet(T) \otimes_{\mathbb{Z}} \mathrm{X}^\bullet(T) \to \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m).$$

The root system of G is denoted by $\Phi_G \subset \mathrm{X}^\bullet(T)$. We shall sometimes denote by ϖ^λ the image of ϖ in $T(k)$ under some co-character $\lambda \in \mathrm{X}_\bullet(T)$.

The Bruhat-Tits building of G over k is denoted by $\mathcal{B}(G)$. The (affine) apartment of T in $\mathcal{B}(G)$, which is the underlying affine space of $E = \mathrm{X}_\bullet(T) \otimes_{\mathbb{Z}} \mathbb{R}$, is denoted by $\mathcal{A}(G)$. For convenience, we shall identify $\mathcal{A}(G)$ with E using $0 \in \mathcal{A}(G)$ as a base point. The root system Φ_G gives a hyperplane structure by the affine hyperplanes

$\{H_{\alpha+n}\}_{\alpha\in\Phi,n\in\mathbb{Z}}$ of $\mathcal{A}(G)$ by the affine linear functionals $\alpha+n : x \mapsto \langle x,\alpha\rangle + n$. The group G acts on the Bruhat-Tits building $\mathcal{B}(G)$ and the stabilizer of a building point x is a parahoric subgroup of G, which we shall denote by G_x.

Let $\Psi(G,B,T) = (\mathrm{X}^\bullet(T), \Phi_G^+, \mathrm{X}_\bullet(T), \check{\Phi}_G^+)$ be the based root datum of G, where $\Phi_G^+ \subset \Phi_G$ is the set of positive roots of G determined by the Borel subgroup B and $\check{\Phi}_G^+$ is the corresponding set of co-roots. Denote by Δ_G the set of simple roots in Φ_G^+, by $\Lambda(G)$ the co-weight lattice and by $\Lambda(G)_r$ the co-root lattice in E. Let $n = \dim E$ denote the rank of G and write $\Delta_G = \{\alpha_1, \alpha_2, ..., \alpha_n\}$. Let β_G be the highest root in the set of positive roots Φ_G^+. Then the $n+1$ basic affine roots are

$$\{\psi_0 = -\beta + 1, \psi_1 = \alpha_1, ..., \psi_n = \alpha_n\}.$$

The region $C = \{x \in \mathcal{A}(G) \mid \psi_i(x) \geq 0, i = 0, 1, ..., n\}$ is the closure of the fundamental alcove and the region $P^+ = \{x \in \mathcal{A}(G) \mid \psi_i(x) \geq 0, i = 1, 2, ..., n\}$ is the closure of the fundamental Weyl chamber with respect to the polarization Φ_G^+ in $\mathcal{A}(G)$.

Denote by $(W_G)_{\text{aff}}$ the affine Weyl group of G, which is the Coxeter group generated by reflection maps $s_{\alpha+n}$ on the apartment $\mathcal{A}(G)$ with respect to the affine hyperplanes $H_{\alpha+n}$ respectively. It acts transitively on the set of alcoves in $\mathcal{A}(G)$ and C is a fundamental domain of its action on $\mathcal{A}(G)$. The Weyl group W_G of G is the Coxeter group generated by the reflections s_α with $\alpha \in \Phi_G$ and P^+ is a fundamental domain of its action on $\mathcal{A}(G)$. $(W_G)_{\text{aff}}$ can be viewed as a semi-direct product of W_G with the co-root lattice $\Lambda(G)_r$. The groups W_G, $(W_G)_{\text{aff}}$ preserve the affine apartment of T and can be lifted to the subgroup $N_G(T)$ of normalizers of T in G. The group $N_G(T)/T(\mathfrak{o}) \simeq W_G \ltimes \mathrm{X}_\bullet(T)$ is the extended affine Weyl group, denoted $\tilde{W}_G$. We have $W_G = N_G(T)/T$ and $(W_G)_{\text{aff}} \subset \tilde{W}_G$. There exists a cyclic abelian group Ω_G such that $\tilde{W}_G = (W_G)_{\text{aff}} \rtimes \Omega_G$. $(W_G)_{\text{aff}}$ are Coxeter groups and admit a Bruhat order $\geq$ and a length function ℓ with respect to the generators $\{s_{\alpha_i}\}_{i=1,2,...,n}$ and $\{s_{\psi_i}\}_{i=0,1,...,n}$. These extends to a partial order $\geq$ on $\tilde{W}_G$ such that for $\sigma_1 = s_1 \cdot \tau_1, \sigma_2 = s_2 \cdot \tau_2 \in (W_G)_{\text{aff}} \rtimes \Omega_G$, $\sigma_1 \geq \sigma_2 \iff s_1 \geq s_2, \ell(s_1) = \ell(s_2)$ and $\tau_1 = \tau_2$, and a length function ℓ such that $\ell(\sigma) = \ell(s)$ for $\sigma = s \cdot \tau \in (W_G)_{\text{aff}} \rtimes \Omega_G$.

Let $x \in \mathcal{A}(G)$ be a building point and let W_x be the subgroup of W_{aff} generated by reflections $s_{\alpha+n}$ which fix x. In other words, x lies on the hyperplanes $H_{\alpha+n}$, for $s_{\alpha+n} \in W_x$. The action of G on $\mathcal{B}(G)$ depends only on the hyperplane structure hence we only care about the facet containing x. Let C_x be an alcove whose closure contains x. Let B_x be the subgroup of G that stabilizes C_x. Then the subgroup stabilizing x is the set $G_x = \sqcup_{s \in W_x} B_x w_s B_x$ where w_s is a lift of the affine Weyl element $s \in W_x$. These are the *parahoric subgroups* of G and B_x is called an *Iwahori subgroup*. The definition of G_x is independent of the choice of C_x. Let G_x^+ be the stabilizer of all such alcoves C_x. Then G_x normalizes G_x^+ and the quotient G_x/G_x^+ is a reductive group G_x. Let Φ_x be the set of α such that $s_{\alpha+n} \in W_x$ for some $n \in \mathbb{Z}$. Then Φ_x forms a root system of G_x. In particular, B_x/B_x^+ is toral. Furthermore, since G_x are stabilizers, we indeed have $G = \sqcup_{s \in W_x \backslash \tilde{W}_G / W_x} G_x w_s G_x$. In general, one can do

$$G = \sqcup_{s \in W_{x_1} \backslash \tilde{W}_G / W_{x_2}} G_{x_1} w_s G_{x_2} \tag{2.1.1}$$

as long as x_1, x_2 are contained in the closure of a same alcove. A point x is a special vertex if $\Phi_x \simeq \Phi_G$. Any building point in the co-weight lattice is a special vertex. A parahoric subgroup G_x stabilizing a special vertex x is hyperspecial and $\mathsf{G}_x \simeq G(\mathfrak{f})$.

Let U be the unipotent radical of B. The adjoint action of T on U (resp. its opposite $\overline{U}$) decomposes U (resp. $\overline{U}$) into root subgroups U_α (resp. $U_{-\alpha}$), where $\alpha \in \Phi_G^+$. For any $\alpha \in \Phi_G$, fix $x_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ a 1-parameter subgroup of G which satisfies

$${}^t x_\alpha(a) = x_\alpha(\alpha(t)a), \quad \forall a \in k, t \in \mathrm{T},$$

and let G_α be the Chevalley group generated by U_α and $U_{-\alpha}$. Denote by T_α the connected component of $\ker\alpha$ in T. There exists $n_\alpha \in N_{G_\alpha}(T_\alpha) - T_\alpha$, such that $n_\alpha^2 \in T_\alpha$ and

$$x_{-\alpha}(c^{-1}) = x_\alpha(c)\check{\alpha}(c)n_\alpha x_\alpha(c),\ c \in k^\times. \tag{2.1.2}$$

The element $n_\alpha \in G_\alpha$ normalizes T and is a lift of the reflection $s_\alpha \in W_G$ to $N_G(T)$. The equation (2.1.2) in SL_2 is famous identity: $\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}\begin{bmatrix} x^{-1} & \\ & x \end{bmatrix}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}$.

A rational character $\theta : U \to k^+$ of U is said to be *generic* if the stabilizer under the adjoint action of a maximal torus T lies in the center of G; equivalently, the restriction θ_α of θ to each of simple root subgroups U_α, $\alpha \in \Delta_G$, of U is nontrivial. If G is of adjoint type, any two generic characters are $T(k)$-conjugate.

A triple (B, T, θ) with a k-rational Borel B of G, a maximal k-split torus T of G contained in B and a generic rational character θ of the unipotent radical U of B is called a *generic data* of G. We shall abuse the notation and denote also by θ the composition $U \xrightarrow{\theta} k^+ \xrightarrow{\psi} \mathrm{S}^1$.

2.2. Compatible good basis

We are interested in the orthogonal groups over k. To set up our groups, we introduce the quadratic space over k that defines the groups which is the standard representation of the orthogonal group.

Let n be a nonnegative integer. Let V be the split quadratic space over k of dimension $2n+1$ and discriminant 2 with even quadratic form $q : V \to k$. Let $\langle\ ,\ \rangle$ be the associated bilinear form defined by $\langle v, w\rangle = \frac{1}{2}[q(v+w) - q(v) - q(w)]$. For any operator A on V, denote by ${}^*A$ the adjoint operator of A on V with respect to $\langle\ ,\ \rangle$. We fix G to be the split special odd orthogonal group $\mathrm{SO}(V)$ of degree $2n+1$, more precisely

$$\mathrm{G} = \{A \in \mathrm{GL}(V) \mid {}^*A\,A = 1, \det A = 1\}.$$

We say an ordered basis $\{e_1, e_2, ..., e_n, e_{n+1} = v_0, f_n, ..., f_2, f_1\}$ of V is a *good basis* if it satisfies $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$, $\langle e_i, f_j\rangle = \delta_{ij}$ and $\langle v_0, v_0\rangle = 2$, for $1 \le i, j \le n$. For a given good basis, a group scheme $\mathrm{SO}(\mathbb{L})$ over $\mathfrak{o}$ is chosen such that G is its generic fiber, where $\mathbb{L}$ is the $\mathfrak{o}$-lattice in V generated by the good basis. Moreover, we choose a Borel subgroup B of G stabilizing the isotropic flag

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X,$$

with $X_i = ke_1 \oplus ke_2 \oplus \cdots \oplus ke_i$ for $1 \le i \le n$, and a maximal split torus T contained in B that stabilizes the lines $ke_1, ke_2, ..., ke_n, kv_0, kf_n, ..., kf_2, kf_1$. The groups $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$ are defined over $\mathfrak{o}$.

The character group $\mathrm{X}^\bullet(\mathrm{T})$ has a canonical basis $\epsilon_1, \epsilon_2, ..., \epsilon_n$ which are the restrictions of the actions to the lines $ke_1, ke_2, ..., ke_n$ respectively. Denote the dual basis of ϵ_i also by ϵ_i and these form a basis of the dual group $\mathrm{X}_\bullet(\mathrm{T})$. The root system Φ_{G} of G has a base

$$\Delta_{\mathrm{G}} = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, ..., \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_n\}.$$

Following the convention in [**14**], for a chosen good basis we fix a generic character $\theta : \mathrm{U} \to k^+$ of U which satisfies the following condition:

$$\theta_{\alpha_i}^{-1}(\mathfrak{o})e_{i+1} = \mathfrak{o}e_i,\ 1 \le i \le n \quad (*)$$

That is, every good basis determines a generic data $(\mathrm{B}, \mathrm{T}, \theta)$. Conversely, given any generic data $(\mathrm{B}, \mathrm{T}, \theta)$ of G the condition $(*)$ fixes a good basis $\{e_1, e_2, ..., e_n, v_0, f_n, ..., f_2, f_1\}$ up to scaling by $\mathfrak{o}^\times$. We have the following definition.

Definition 2.2.1. A good basis $\{e_1, e_2, ..., e_n, v_0, f_n, ..., f_2, f_1\}$ is said to be *compatible* with a generic data $(\mathrm{B}, \mathrm{T}, \theta)$ if the following three conditions hold: (1) ke_i is an eigenspace of T; (2) the orbit of e_{i+1} under the action of B is contained in $\oplus_{j=1}^{i+1} ke_j$; (3) $\theta_{\alpha_i}^{-1}(\mathfrak{o})e_{i+1} = \mathfrak{o}e_i$, for $1 \leq i \leq n$.

Remark 2.2.2. A generic data $(\mathrm{B}, \mathrm{T}, \theta)$ determines a integral model of G from a good basis and the apartment $\mathcal{A}(\mathrm{G})$ with an assigned origin and hyperplane structure on it and the generic character $\theta : \mathrm{U} \to \mathrm{S}^1$ is trivial on $\mathrm{U}(\mathfrak{o}) = \mathrm{U} \cap \mathrm{G}(\mathfrak{o})$.

From a generic data, a compatible good basis of a standard representation of $\mathrm{SO}_{2n+1}(k)$ can be assigned. In part 2 of this thesis, we will use this good basis to define a family of open compact subgroup of $\mathrm{SO}_{2n+1}(k)$ whose fixed space in the generic representation will encode important invariants such as the conductor and the local factors.

From now on, we shall fix a good basis

$$\{e_1, e_2, ..., e_n, v_0, f_n, ..., f_2, f_1\},$$

of V, up to scaling in $\mathfrak{o}^\times$, or equivalently a generic data $(\mathrm{B}, \mathrm{T}, \theta)$ of G.

2.3. The groups SO_{2n+1}, SO_{2n}, and GL_n

Recall that V is a split quadratic space over k of dimension $2n + 1$ with an associated bilinear form $\langle\ ,\ \rangle$ whose Gram matrix under given fixed good basis is

$$\begin{bmatrix} & & & & & & 1 \\ & & & & & \iddots & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & \iddots & & & & & \\ 1 & & & & & & \end{bmatrix}$$

and $\mathbb{L}$ is the $\mathfrak{o}$-lattice generated by the good basis of V. The n-plane $X = \oplus_{i=1}^{n} ke_i$ is a maximal isotropic subspace of V and v_0 is an anisotropic vector of V with $\langle v_0, v_0 \rangle = 2$.

2.3. The groups SO_{2n+1}, SO_{2n}, and GL_n

The isotropic subspace $X^\vee = \oplus_{i=1}^n k f_i$ is isomorphic to the dual space of X under the perfect pairing $\langle\ ,\ \rangle : X \times X^\vee \to k$. Let

$$W = X \oplus X^\vee$$

be the split quadratic space of dimension $2n$ over k which is the orthogonal compliment of the anisotropic vector v_0 in V.

We have $\mathrm{G} = \mathrm{SO}(V) \simeq \mathrm{SO}_{2n+1}$ with integral model $\mathrm{SO}(\mathbb{L})$. Define $\mathrm{H} = \mathrm{SO}(W) \simeq \mathrm{SO}_{2n}$ to be the subgroup of G fixing v_0 and $\mathrm{M} = \mathrm{GL}(X) \simeq \mathrm{GL}_n$ to be the subgroup stabilizing X and $X^\vee$ fixing v_0, embedded in H (and hence G) with action on $X^\vee$ by the adjoint operator * via $\langle\ ,\ \rangle$. Denote by $\det : \mathrm{M} \to k^\times$ the determinant map on $\mathrm{GL}(X)$.

The subgroups H and M are split reductive groups with Borel subgroups $\mathrm{B_H} = \mathrm{H} \cap \mathrm{B}$ and $\mathrm{B_M} = \mathrm{M} \cap \mathrm{B}$ defined over $\mathfrak{o}$ both containing T as a maximal split torus. Let us denote by V and N_n the subgroups $\mathrm{H} \cap \mathrm{U}$ and $\mathrm{M} \cap \mathrm{U}$ of G which are maximal unipotent subgroups of H and M respectively.

The bases of the root systems Φ_M, Φ_H and Φ_G of M, H and G respectively are

$$\begin{aligned}
\Delta_\mathrm{M} &= \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{n-1} - \epsilon_n\} \quad (\text{highest root } \epsilon_1 - \epsilon_n), \\
\Delta_\mathrm{H} &= \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\} \quad (\text{highest root } 2\epsilon_1), \\
\Delta_\mathrm{G} &= \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{n-1} - \epsilon_n, \epsilon_n\} \quad (\text{highest root } \beta_\mathrm{G} = \epsilon_1 + \epsilon_2).
\end{aligned}$$

The corresponding bases of the co-roots of G and H are

$$\Delta_\mathrm{G}^\vee = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$$

$$\Delta_\mathrm{H}^\vee = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}.$$

2.3. The groups SO_{2n+1}, SO_{2n}, and GL_n

The sets of the fundamental co-weights of G and H are

$$\Delta^*_{\mathrm{G}} = \{\epsilon_1, \epsilon_1+\epsilon_2, ..., \epsilon_1+\epsilon_2+...+\epsilon_n\}$$

$$\Delta^*_{\mathrm{H}} = \{\epsilon_1, \epsilon_1+\epsilon_2, ..., \epsilon_1+...+\epsilon_{n-1}, \frac{\epsilon_1+...+\epsilon_{n-1}-\epsilon_n}{2}, \frac{\epsilon_1+\epsilon_2+...+\epsilon_n}{2}\}.$$

We have $\Lambda(\mathrm{G}) = X_\bullet(\mathrm{T}) \supset \Lambda_r(\mathrm{G})$ and the co-root lattice $\Lambda_r(\mathrm{G})$ is contained in the co-weight lattice $\Lambda(\mathrm{G})$ with index 2. Similarly, we have $\Lambda(\mathrm{H}) \supset X_\bullet(\mathrm{T}) \supset \Lambda_r(\mathrm{H})$ and the co-root lattice $\Lambda_r(\mathrm{H})$ is contained in $\Lambda(\mathrm{H})$ with index 4.

The apartments $\mathcal{A}(\mathrm{M})$, $\mathcal{A}(\mathrm{H})$ and $\mathcal{A}(\mathrm{G})$ of the maximal torus T have the same underlying affine space E, but different hyperplane structures. Set the following points

$$x_0 = 0 \quad \text{and} \quad x_m = m\frac{\epsilon_1+\epsilon_2+...+\epsilon_n}{2}$$

on E for $m \in \mathbb{Z}$. The corresponding building points of x_i's are vertices (0-facets) of $\mathcal{A}(\mathrm{G})$ and are special vertices of $\mathcal{A}(\mathrm{H})$. These points play a crucial role in the rest of the thesis to express our target family of open compact subgroups. We shall denote by x_i's the building points in both $\mathcal{A}(\mathrm{G})$ and $\mathcal{A}(\mathrm{H})$ when the content is clear. The reductive group M is not semisimple and has center generated by the image of $\lambda^{\mathrm{M}} = \epsilon_1+\epsilon_2+...+\epsilon_n \in \mathcal{A}(\mathrm{M})$. We focus on $\mathcal{A}(\mathrm{M})/\langle\lambda^{\mathrm{M}}\rangle$ instead.

The Weyl groups W_{M}, W_{H} and W_{G} acts on E preserving the hyperplane structure of the affine apartment $\mathcal{A}(\mathrm{M})$, $\mathcal{A}(\mathrm{H})$ and $\mathcal{A}(\mathrm{G})$ respectively. The Weyl group W_{M} is isomorphic to the permutation group S_n on n letters. The Weyl group W_{H} is isomorphic to the semi-direct product of W_{M} and the group generated by composition of even number of reflections s_{ϵ_i}'s. Let us call the simple reflections s_{ϵ_i} the *sign changes* in the later context. The Weyl group W_{G} is isomorphic to the semi-direct product of W_{M} and the group generated by composition of all sign changes s_{ϵ_i}'s. We have $W_{\mathrm{M}} \simeq S_n$, $W_{\mathrm{H}} \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ and $W_{\mathrm{G}} \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$.

2.4. Parabolic subgroups Q, P

Among all parabolic subgroups of H and G, the ones that stabilizes the isotropic flag $0 \subset X$ are of special importance, for it serves as a good first stab when one wants to investigate the parahoric subgroups H_{x_i} and G_{x_i}. These have a close relation with the open compact subgroups $\mathrm{K}(\mathfrak{p}^m)$ which will be defined in Part 2 of this thesis.

Let Q (resp. P) denote the parabolic subgroup of G (resp. H) that stabilizes the isotropic flag $0 \subset X$. Then the subgroup M is a Levi factor of both Q and P. Denote by Y (resp. Z) the unipotent radical of Q (resp. P) which M acts by conjugation. We have Levi decompositions

$$\mathrm{Q} = \mathrm{M} \ltimes \mathrm{Y} \quad \mathrm{P} = \mathrm{M} \ltimes \mathrm{Z}\,.$$

The subgroup Y is a two-step unipotent group which fits into the exact sequence of M-modules

$$0 \to \wedge^2 X \to \mathrm{Y} \xrightarrow{\alpha} X \to 0$$

where the map α is given by $y \mapsto y(v_0) - v_0$. The subgroup Z is a commutative unipotent group isomorphic to $\wedge^2 X$ and is normal in Y. We have the isomorphism $\mathrm{Y} \,/\, \mathrm{Z} \simeq \oplus_{i=1}^{n} \mathrm{U}_{\epsilon_i}$. The roots in Lie(Z) under action of T are $\epsilon_i + \epsilon_j$, $1 \le i < j \le n$.

We write down these groups in the case when $n = 2$ as 5 by 5 matrices under the fixed good basis in the following example.

Example 2.4.1. When $\mathrm{G} = \mathrm{SO}_5$, the subgroups H, M, T, Q, P, Y and Z are as follows.

$$\mathrm{H} = \begin{bmatrix} * & * & & * & * \\ * & * & & * & * \\ & & 1 & & \\ * & * & & * & * \\ * & * & & * & * \end{bmatrix} \cap \mathrm{G}, \quad \mathrm{M} = \begin{bmatrix} * & * & & & \\ * & * & & & \\ & & 1 & & \\ & & & * & * \\ & & & * & * \end{bmatrix} \cap \mathrm{G}, \quad \mathrm{T} = \begin{bmatrix} * & & & & \\ & * & & & \\ & & 1 & & \\ & & & * & \\ & & & & * \end{bmatrix} \cap \mathrm{G}$$

$$\mathrm{Q} = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & * & * \end{bmatrix} \cap \mathrm{G}, \quad \mathrm{P} = \begin{bmatrix} * & * & & * & * \\ * & * & & * & * \\ & & 1 & & \\ & & & * & * \\ & & & * & * \end{bmatrix} \cap \mathrm{G}, \quad \mathrm{Y} = \begin{bmatrix} 1 & & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \cap \mathrm{G}, \quad \mathrm{Z} = \begin{bmatrix} 1 & & & * & * \\ & 1 & & * & * \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \cap \mathrm{G}\,.$$

2.4. Parabolic subgroups Q, P

The Bruhat decompostion shows that the double coset representatives of $\mathrm{B} \backslash \mathrm{G} / \mathrm{B}$ can be chosen from $\mathrm{B}\, N_{\mathrm{G}}(\mathrm{T})\, \mathrm{B}$ and hence

$$\mathrm{G} = \sqcup_s \mathrm{B}\, w_s\, \mathrm{B} = \sqcup_s \mathrm{U}\, w_s\, \mathrm{B}$$

where w_s is a lift of s to $N_{\mathrm{G}}(\mathrm{T})$ and the above union is taken over the Weyl group element in W_{G}. If Q_1 and Q_2 are two parabolic subgroups containing B with Levi factor M_1 (resp. M_2) containing T, then we have the following commutative diagram

(2.4.1)
$$\begin{array}{ccc} \mathrm{B} \backslash \mathrm{G} / \mathrm{B} & \longrightarrow & W_{\mathrm{G}} \\ \downarrow & & \downarrow \\ Q_1 \backslash \mathrm{G} / Q_2 & \longrightarrow & W_{M_1} \backslash W_{\mathrm{G}} / W_{M_2} \end{array}$$

where the horizontal maps are bijections and the vertical maps are quotient maps. One would argue this by looking at $(Q_1 \cap N_{\mathrm{G}}(\mathrm{T})) \backslash N_{\mathrm{G}}(\mathrm{T}) / (Q_2 \cap N_{\mathrm{G}}(\mathrm{T}))$ and it follows from the definition of W_{M_1} and W_{M_2}. This diagram holds after taking $\mathfrak{o}$ points and reduction modulo $\mathfrak{p}$ while the group W_{G} is lifted to the hyperspecial subgroup $\mathrm{G}(\mathfrak{o})$. Similarly we can argue with H.

Let us apply it to our parabolic groups Q and P of G and H respectively. We notice that $W_{\mathrm{M}} = S_n$. Denote by $I \subset W_{\mathrm{G}}$ the set of all sign changes and by $I_0 \subset W_{\mathrm{H}}$ the set of all even sign changes. Then we have Bruhat decompositions (over k and over $\mathfrak{o}/\mathfrak{p}$)

$$\mathrm{G} = \sqcup_{s \in I} \mathrm{B}\, w_s\, \mathrm{Q} = \sqcup_{s \in I} \mathrm{U}\, w_s\, \mathrm{Q}\ , \qquad \mathrm{H} = \sqcup_{s \in I_0} \mathrm{B_H}\, w_s\, \mathrm{P} = \sqcup_{s \in I_0} \mathrm{V}\, w_s\, \mathrm{P}\, .$$

Here again w_s represents any lift of the Weyl group element s to $N_{\mathrm{G}}(\mathrm{T})$.

2.5. Parahoric subgroups G_{x_i}, H_{x_i}

Recall that $x_t = t\dfrac{\epsilon_1 + \epsilon_2 + ... + \epsilon_n}{2}$ is a building point in $\mathcal{A}(\mathrm{G})$ (and $\mathcal{A}(\mathrm{H})$ by abuse of notation) defined for $t \in \mathbb{Z}$. We define x_t by the above formula for $t \in \mathbb{R}$. For any $i \in \mathbb{Z}$, let $x_{i_+} = x_{(i+1)_-}$ be any point in the edge (1-facet) $\{x_t \mid i < t < i+1\}$ whose closure contains the vertices x_i and x_{i+1}.

The open compact subgroups defined by

$$\mathrm{G}_{x_t} = \langle \mathrm{T}(\mathfrak{o}), \mathrm{U}_\alpha(\mathfrak{p}^n) \mid (\alpha + n)(x_t) \geq 0, \alpha \in \Phi_{\mathrm{G}}, n \in \mathbb{Z} \rangle,$$

$$\mathrm{H}_{x_t} = \langle \mathrm{T}(\mathfrak{o}), \mathrm{U}_\alpha(\mathfrak{p}^n) \mid (\alpha + n)(x_t) \geq 0, \alpha \in \Phi_{\mathrm{H}}, n \in \mathbb{Z} \rangle$$

are parahoric subgroups of G and H respectively. The groups G_{x_t} and H_{x_t} have pro-unipotent subgroups, namely, the open compact subgroups

$$\mathrm{G}^+_{x_t} = \langle \mathrm{T}(1+\mathfrak{p}), \mathrm{U}_\alpha(\mathfrak{p}^n) \mid (\alpha + n)(x_t) > 0, \alpha \in \Phi_{\mathrm{G}}, n \in \mathbb{Z} \rangle, \text{ and}$$

$$\mathrm{H}^+_{x_t} = \langle \mathrm{T}(1+\mathfrak{p}), \mathrm{U}_\alpha(\mathfrak{p}^n) \mid (\alpha + n)(x_t) > 0, \alpha \in \Phi_{\mathrm{H}}, n \in \mathbb{Z} \rangle,$$

which are normal in G_{x_t} and H_{x_t} respectively.

Suppose i is an integer. The parahoric subgroups G_{x_i} and H_{x_i} are maximal and admit reductive quotients

$$\mathrm{G}_{x_i} \,/\, \mathrm{G}^+_{x_i} \simeq \begin{cases} \mathrm{G}(\mathfrak{f}), & i: \textit{even} \\ \mathrm{H}(\mathfrak{f}), & i: \textit{odd} \end{cases}, \quad \mathrm{H}_{x_i} \,/\, \mathrm{H}^+_{x_i} \simeq \mathrm{H}(\mathfrak{f})$$

and moreover,

$$\mathrm{G}_{x_{i_+}} \,/\, \mathrm{G}^+_{x_{i_+}} \simeq \mathrm{H}_{x_{i_+}} \,/\, \mathrm{H}^+_{x_{i_+}} \simeq \mathrm{M}(\mathfrak{f}).$$

The non-maximal parahoric subgroup $\mathrm{H}_{x_{i_+}}$ and $\mathrm{H}_{x_{i_-}}$ are contained in H_{x_i}. Their images in the reductive quotient $\mathrm{H}(\mathfrak{f})$ of H_{x_i} equal to the parabolic subgroup $\mathrm{P}(\mathfrak{f})$

and $\overline{P}(\mathfrak{f})$, respectively. The *Iwahori factorization* of $H_{x_{i_+}}$ gives

$$H_{x_{i_+}} = \overline{Z}(\mathfrak{p}^{i+1})\, M(\mathfrak{o})\, Z(\mathfrak{p}^{-i}) = Z(\mathfrak{p}^{-i})\, M(\mathfrak{o})\overline{Z}(\mathfrak{p}^{i+1}).$$

The *Bruhat decomposition* of $H(\mathfrak{f})$ can be lifted to the parahoric subgroup H_{x_i} of H and give a decomposition

$$H_{x_i} = \cup_{s\in I_0}(V \cap H_{x_i}) w_{s,i}\, H_{x_{i_+}},$$

where $w_{s,i}$ represents any lift of the Weyl element s to H_{x_i}.

Consider the maximal parahoric subgroups G_{x_0} and G_{x_1} of G. Denote by

$$K_x = N_G(G_x)$$

the normalizer of G_x in G for any building point x. Then $K_{x_0} = G_{x_0}$ is a hyperspecial maximal open compact subgroup and K_{x_1} is a maximal open compact subgroup contains G_{x_1} with index 2. The intersection of the groups G_{x_0} and G_{x_1} is the parahoric subgroup $G_{x_{0_+}}$, whose image in the reductive quotient $G(\mathfrak{f})$ of G_{x_0} is the parabolic subgroup $Q(\mathfrak{f})$. We have a Iwahori factorization

$$G_{x_{0_+}} = Y(\mathfrak{o})\, M(\mathfrak{o})\overline{Y}(\mathfrak{p}) = \overline{Y}(\mathfrak{p})\, M(\mathfrak{o})\, Y(\mathfrak{o}).$$

The Bruhat decomposition for $G(\mathfrak{f})$ can be lifted to G_{x_0} and give a decomposition

$$G_{x_0} = \cup_{s\in I}(U \cap G_{x_0}) w_{s,0}\, G_{x_{i_+}},$$

where $w_{s,i}$ represents any lift of Weyl element s to K_{x_i}.

The smooth map $G \to \mathfrak{B}$ to the flag variety $\mathfrak{B} = G\,/\,B$ of the split group G is separable and is thus a quotient map. We have $G(k)/\,B(k) = G(\mathfrak{o})/\,B(\mathfrak{o})$ and hence we also have the Iwasawa decomposition

$$G = B\,G(\mathfrak{o}) = U\,T\,G(\mathfrak{o}).$$

Since G_{x_0} is hyperspecial and any lift $w_{s,0}$ of a Weyl element s is contained in $T\,K_{x_1}$, $s \in I$, we also have the decompositions $G = B\,K_{x_0} = B\,K_{x_1}$. A similar argument can be applied to conclude that the decomposition

$$G = B\,K_{x_i}$$

holds for any integer i.

Before we end this chapter and move on to discussion on representations of p-adic groups, we fix the following convention. For any subgroup C of G, we will write $C_{(m)}$ for the pullback of $C(\mathfrak{o}/\mathfrak{p}^m)$ in $G(\mathfrak{o}/\mathfrak{p}^m)$ under the reduction modulo $\mathfrak{p}^m$ map on $G(\mathfrak{o})$. For example,

$$Q_{(m)} = \overline{Y}(\mathfrak{p}^m)\,M(\mathfrak{o})\,Y(\mathfrak{o}) = \overline{Y}(\mathfrak{p}^m)\,Q(\mathfrak{o})$$

is a subgroup of $G(\mathfrak{o})$ contained in $G_{x_{0_+}}$. Let I denote the identity element in G, then the set of subgroups $\{I_{(m)}\}_{m\geq 0}$ forms a system of open compact neighborhood of identity I in the locally pro-finite group G.

CHAPTER 3

Generic representations

We begin with a general theory of smooth representations. In this chapter, G is a general reductive group over k for most of the sections.

3.1. Admissible representations

Let G be a locally compact and totally disconnected topological group. A *representation* of G is a homomorphism π from G to the linear automorphism group of a complex vector space V_π. The dimension of complex vector space V_π is called the dimension of the representation π. We will sometimes denote a representation as a pair (π, V_π) indicating G acts on V_π by π. A representation is said to be *smooth* if every vector in V_π is invariant under elements of an open compact subgroup. For any compact subgroup K of G, we write

$$V_\pi^K = \{v \in V_\pi \mid \pi(k)v = v \ \forall k \in K\}.$$

Then π is smooth if and only if $V_\pi = \cup_K V_\pi^K$ where K runs over all open compact subgroup of G. A representation π is *admissible* if the fixed subspace of any open compact subgroup K is finite dimensional, i.e. $\dim V_\pi^K < \infty$. A character of G is a one dimensional smooth representation, which is clearly admissible.

Let π be any representation of G on a vector space V_π, define the smooth part V_π^∞ of π as the subspace $\cup_K V_\pi^K$, where K runs through all open compact subgroups of G. Then V_π^∞ is an invariant subspace and the action π of G on V_π^∞ is a smooth representation. For a smooth representation π of G on the space V_π, the *contragradient* $\tilde{\pi}$ is defined as the dual action π^* on the smooth part of the dual representation of G on V_π^* given by $\langle \pi^*(g)v_1^*, v_2\rangle = \langle v_1^*, \pi(g^{-1})v_2\rangle$, $\forall v_1^* \in V_\pi^*, v_2 \in V_\pi, g \in G$ with $\langle\ ,\ \rangle$ the perfect duality on $V_\pi^* \times V_\pi$.

In general we have an action of G on the space of complex-valued functions f by right translation R_g, $(R_g f)(x) = f(xg)$ $\forall g, x \in G$. This action again preserves the subspace of locally constant functions, denoted $C^\infty(G)$, and the subspace of locally constant functions of compact support, denoted $C_c^\infty(G)$. $C_c^\infty(G)$ is analogous to the regular representation of G when G is a finite group. Any G-invariant space is naturally a $\mathbb{C}[G]$-module.

Let dg be a left Haar measure on G, which is unique up to scalar. We have a distribution $C_c^\infty(G) \to \mathbb{C}$ of G by $f \mapsto \int_G f(g)dg$. The *modulus character* $\delta_G : G \mapsto \mathbb{R}^+$ of G is defined as the character of G satisfying $d(gx^{-1}) = \delta_G(x)dg$. When G is compact or reductive, this character is trivial and the Haar measure is bi-invariant. Let $P = MN$ be a parabolic subgroup of a reductive group G with Levi factor M and unipotent radical N. Since M normalizes P, the character δ_P is determined by the adjoint action of M on the Lie algebra of N. To be more precise, $\delta_P(m) = |\det \mathrm{Ad}(m)|_{\mathrm{Lie}(N)}|$, $\forall m \in M$. In particular, let B be the Borel subgroup of a reductive group G containing a maximal torus T of G.

For any closed subgroup H of G and any smooth representation σ of H on the vector space W_σ, G acts on the vector space

$$\mathrm{Ind}_H^G W_\sigma = \{f : G \to W_\sigma \text{ locally constant} \mid f(hg) = \sigma(h)f(g),\ \forall h \in H\}$$

by right translation R_g, $R_g f(x) = f(xg)$. This representation is smooth and is called the *inducted representation*, denoted $\mathrm{Ind}_H^G \sigma$. The space $\mathrm{Ind}_H^G W_\sigma$ has an invariant subspace $\mathrm{ind}_H^G W_\sigma$ of functions compactly supported modulo H. This representation of G is called the *compact induction*, denoted by $\operatorname{ind}_H^G \sigma$. When H is an open subgroup, the compact induction $\operatorname{ind}_H^G W_\sigma$ is can be identified with $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W_\sigma$ as a $\mathbb{C}[G]$-module. In particular, $\operatorname{Ind}_{\mathrm{I}}^G \mathbb{C} = C^\infty(G)$ and $\operatorname{ind}_{\mathrm{I}}^G \mathbb{C} = C_c^\infty(G)$.

Let $\operatorname{Rep}(G)$ denote the category of smooth representations of G. The inductions define functors from $\operatorname{Rep}(H)$ to $\operatorname{Rep}(G)$. We list some properties of the inductions.

Proposition 3.1.1. *Let H be closed subgroup of G, and (σ, W_σ) a smooth representation of H.*
(i) The functors $\operatorname{Ind}_H^G -$ and $\operatorname{ind}_H^G -$ are exact.
(ii) Assume $J \supset H$ be a closed subgroup of G, then $\operatorname{Ind}_H^G \sigma = \operatorname{Ind}_J^G(\operatorname{Ind}_H^J \sigma)$.
(iii) Assume G is reductive. Then $\widetilde{\operatorname{ind}_H^G \sigma} \simeq \operatorname{Ind}_H^G \tilde{\sigma}\delta_H$.
(iv) If (π, V_π) is a smooth representation of G, then $\operatorname{ind}_H^G \pi|_H \otimes \sigma \simeq \pi \otimes \operatorname{ind}_H^G \sigma$.
(v) If σ is unitary, then $\operatorname{Ind}_H^G \sigma\delta_H^{1/2}$ is unitarizable.

We will prove the following reciprocity which will be used very often later.

Proposition 3.1.2 (Frobenius reciprocity)**.** *Let H be a closed subgroup of G. Let (π, V_π) be a smooth representation of G and (σ, W_σ) be a smooth representation of H. Then there are canonical isomorphisms:*
(i) $\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma) \simeq \operatorname{Hom}_H(\pi|_H, \sigma)$.
(ii) $\operatorname{Hom}_G(\operatorname{ind}_H^G \sigma, \tilde{\pi}) \simeq \operatorname{Hom}_H(\sigma\delta_H^{-1}, \widetilde{\pi|_H})$.
(iii) Assume H is open. $\operatorname{Hom}_G(\operatorname{ind}_H^G \sigma, \pi) \simeq \operatorname{Hom}_H(\sigma, \pi|_H)$.

Proof. On the induced representation $\operatorname{Ind}_H^G \sigma$, we have a H-invariant map

$$\alpha_\sigma : \operatorname{Ind}_H^G W_\sigma \to W_\sigma,\ f \mapsto f(\mathrm{I}). \tag{3.1.1}$$

This map induces a homomorphism from $\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma)$ to $\operatorname{Hom}_H(\pi|_H, \sigma)$ by composition. Given such a H-invariant map $T : V_\pi \to W_\sigma$, we can recover f by the function $T(\pi(g)v)$. This gives an inverse of the homomorphism, which is hence an isomorphism. This proves (i). Applying Proposition 3.1.1 (iii) and part (i) we get

$$\mathrm{Hom}_G(\mathrm{ind}_H^G\,\sigma,\tilde{\pi}) \simeq \mathrm{Hom}_G(\pi,\mathrm{Ind}_H^G\,\tilde{\sigma}\delta_H) \simeq \mathrm{Hom}_H(\pi|_H,\tilde{\sigma}\delta_H) \simeq \mathrm{Hom}_H(\sigma\delta_H^{-1},\widetilde{\pi|_H})$$

and hence prove (ii). If H is open, then $\mathrm{ind}_H^G\,W_\sigma \simeq \mathbb{C}[G]\otimes_{\mathbb{C}H} W_\sigma$. There is a natural map $W_\sigma \to \mathrm{ind}_H^G\,W_\sigma$ which is H-invariant and induces a homomorphism from $\mathrm{Hom}_G(\mathrm{ind}_H^G\,\sigma,\pi)$ to $\mathrm{Hom}_H(\sigma,\pi|_H)$ by pullback. Since any H-invariant map from W_σ to V_π can be extended to a G-invariant map from $\mathbb{C}[G]\otimes_{\mathbb{C}[H]} W_\sigma$ to V_π,

$$W_\pi \to V_\pi \;\rightsquigarrow\; \mathrm{ind}_H^G\,W_\sigma \simeq \mathbb{C}[G]\otimes_{\mathbb{C}[H]} W_\sigma \to V_\pi.$$

It defines an inverse of the homomorphism. (iii) is thus proved. □

On the other hand, we also have an analog of the restriction map as in the representation theory of finite groups.

Let H be a closed subgroup of G and ξ be a character on H. The normalizer $\mathrm{Norm}_G(H,\xi)$ is the set of elements g in G such that $g\in N_G(H)$ and $\xi(ghg^{-1})=\xi(h)$ for $h\in H$. For any representation (π,V_π) of G, set

$$V_\pi(H,\xi)=\langle \pi(h)v-\xi(h)v; v\in V_\pi, h\in H\rangle,$$

which is an invariant space of $\mathrm{Norm}_G(\xi)$. The ξ-*localization* of π is the quotient space

$$(V_\pi)_{H,\xi}=V_\pi/V_\pi(H,\xi)$$

on which $\mathrm{Norm}_G(H,\xi)$ acts by restricting π on the cosets. This is the maximal quotient of V_π such that H acts by ξ. The ξ-localization defines a functor, called a (modified) *Jacquet functor*, denoted

$$J_{H,\xi}:\ \ \mathrm{Rep}(G)\to\mathrm{Rep}(\mathrm{Norm}_G(H,\xi))$$

$$(\pi,V_\pi)\mapsto(\pi_{H,\xi},(V_\pi)_{H,\xi}).$$

We omit the subscript ξ when it is trivial. $J_H : \mathrm{Rep}(G) \to \mathrm{Rep}(N_G(H))$ is the ordinary Jacquet functor, and $J_H(\pi)$ is called the *Jacquet module* of π at H, which is exactly the H-covariants π_H of π.

We list some of its properties and omit the proofs.

Proposition 3.1.3. *Let H be a closed subgroup of G exhausted by its compact subgroups, and (π, V_π) a smooth representation of G.*
(i) The functors J_H- is exact.
(ii) Assume $H = H_1H_2$ and H_2 normalizes H_1, then $((V_\pi)_{H_1,\xi|_{H_1}})_{H_2,\xi|_{H_2}} = (V_\pi)_{H,\xi}$.
(iii) $V_\pi(H,\xi) = V_{\xi^{-1}\pi}(H)$ and $(V_\pi)_{H,\xi} = (V_{\xi^{-1}\pi})_H$.
(iv) $v \in V_\pi(H,\xi)$ if and only if there exists a compact subgroup $\mathcal{U} \subset H$ such that

$$\int_{\mathcal{U}} \xi^{-1}(h)\pi(h)v \; dv = 0. \tag{3.1.2}$$

Let M, N be closed subgroups, M normalizes N and $P = MN$ is closed. (For example, $P = MN$ is a parabolic subgroup of a reductive group G with Levi factor M and unipotent radical N.) Let ξ be a character of N and $M \subset \mathrm{Norm}_G(N,\xi)$. For any smooth representation (τ, W_τ) of M, define

$$\mathrm{I}_{N,\xi}(\tau) = \mathrm{Ind}_P^G(\tau \otimes \xi)\delta_P^{1/2}, \quad i_{N,\xi}(\tau) = \mathrm{ind}_P^G(\tau \otimes \xi)\delta_P^{1/2};$$

for any smooth representation (π, V_π) of G, define

$$r_{N,\xi}(\pi) = \pi_{N,\xi}\delta_P^{-1/2}.$$

We obtained functors

$$\mathrm{I}_{N,\xi}, i_{N,\xi} : \mathrm{Rep}(M) \to \mathrm{Rep}(G), \quad r_{N,\xi} : \mathrm{Rep}(G) \to \mathrm{Rep}(M).$$

When $\xi = 1$, $\mathrm{I}_{G,M} = \mathrm{I}_{N,1}$ (resp. $i_{G,M} = i_{N,\xi}$) is called a *normalized induction* (resp. *normalized compact induction*) and $r_{M,G} = r_{N,1}$ is called the *normalized Jacquet* *functor at* N. When G/P is compact, these functors preserve admissibility and the property of being unitary and $\mathrm{I}_{N,\xi}$ coincides with $i_{N,\xi}$.

Using the properties of the induction and the ξ-localization (see Proposition 3.1.1, 3.1.3), it is clear that the functors $\mathrm{I}_{N,\xi}$, $i_{N,\xi}$ and $r_{N,\xi}$ are exact. Since for $\pi \in \mathrm{Rep}(G), \tau \in \mathrm{Rep}(M)$, the Frobenius reciprocity implies that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \tau \otimes \xi) \simeq \mathrm{Hom}_P(\pi|_P, \tau \otimes \xi) \simeq \mathrm{Hom}_M(\pi_{N,\xi}, \tau \otimes \xi)$ for any character ξ of N normalized by M. The functor $r_{N,\xi}$ is left adjoint to $\mathrm{I}_{N,\xi}$. We have another form of the Frobenius reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{I}_{N,\xi}(\tau)) \simeq \mathrm{Hom}_M(r_{N,\xi}(\pi), \tau).$$

When G is a reductive group. Suppose $P = MN$ is a proper parabolic subgroup of G with Levi factor M. A parabolically induced representation, called a *parabolic induction*, of G is of the form $\mathrm{Ind}_P^G \tau$ where τ is a smooth representation of M inflated to P by assuming trivial on N. An irreducible representation is said to be *supercuspidal* if it can not be realized as any subrepresentation of a parabolically induced representation of G. The Frobenius reciprocity now shows an irreducible representation π of G is supercuspidal if and only if $r_{N,1}(\pi) = 0$ for any unipotent radical N of a proper parabolic subgroup of G. Conversely, if a nontrivial irreducible representation τ of M occurs in $r_{N,\xi}(\pi)$ for some $P = MN$ and ξ, then π can be embedded into a parabolic induction $\mathrm{I}_{N,\xi}(\tau)$.

Most of the result in this section can be found in [**1**], [**2**].

3.2. Whittaker linear forms

Let G be a connected split reductive group over k and let (B, T, θ) be a generic data of G. Recall that this means that $B = TU$ is a k-rational Borel subgroup, T is a k-split torus contained in B and $\theta : U \to \mathrm{S}^1$ is a generic character of the unipotent radical U of B such that the stabilizer of θ under action of T is in the center of G.

3.2. Whittaker linear forms

Denote by $\mathbb{C}_\theta$ the one dimensional space on which U acts by θ. Then we can consider the induced representation $\mathrm{Ind}_U^G \theta$, acting on the space $\mathrm{Ind}_U^G \mathbb{C}_\theta$ of locally constant functions f on G such that

$$f(ug) = \theta(u)f(g), \forall u \in U, g \in G,$$

on which G acts by right translation R_g.

Theorem 3.2.1 (Gelfand-Kazhdan [**10**], Rodier [**24**], Shalika [**27**])**.** *The representation $\mathrm{Ind}_U^G \theta$ is multiplicity free. That is, for any irreducible smooth representation π of G, the complex vector space $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \theta)$ is of dimension at most 1.*

We say an irreducible smooth representation (π, V_π) of G is θ-*generic* if

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \theta) = \mathbb{C}.$$

A *Whittaker model* of π with respect to the generic character θ is an invariant subspace $\mathscr{W}(\pi, \theta)$ of $\mathrm{Ind}_U^G \mathbb{C}_\theta$ on which the action of G is isomorphic to π. A θ-generic representation π admits a Whittaker model and Theorem 3.2.1 shows such model is unique when exists. By the Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \theta) \simeq \mathrm{Hom}_U(\pi|_U, \theta).$$

Therefore, when π is θ-generic, there is also a nontrivial linear functional ℓ_θ on V_π, unique up to scalar, such that $\ell_\theta(\pi(u)v) = \theta(u)\ell_\theta(v)$. Such a linear form ℓ_θ is called a *Whittaker functional* on V_π. Given a Whittaker functional $\ell_\theta \in \mathrm{Hom}_U(V_\pi, \mathbb{C}_\theta)$, the Whittaker model of (π, V_π) with respect to θ is the space

$$\mathscr{W}(\pi, \theta) = \{W_v : G \to \mathbb{C} \mid W_v(g) = \ell_\theta(\pi(g)v), \forall v \in V_\pi\}, \tag{3.2.1}$$

with G acting by right translation R_g.

The following lemma reduces the question of the uniqueness of the Whittaker model $\mathscr{W}(\pi,\theta)$ to the case when π is a supercuspidal representation of G.

Lemma 3.2.2 (Casselman-Shalika [**6**], Shahidi [**25**])**.** *Let w_G be any lift of the longest Weyl element of G, meaning $B \cap w_G B w_G^{-1} = T$, then $U'_M = M \cap w_G U w_G^{-1}$ is a maximal unipotent subgroup of M and $\theta'_M = \theta \circ Ad(w_G)$ is a generic character on U'_M. Assume (τ, W_τ) is a θ'_M-generic representation of M. Then*

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G \tau, \mathrm{Ind}_U^G \theta) \simeq \mathrm{Hom}_M(\tau, \mathrm{Ind}_{U'_M}^M \theta'_M).$$

In particular, if the parabolic induction $\mathrm{Ind}_P^G \tau$ is irreducible, then it is θ-generic.

Remark 3.2.3. Following the notation as in Lemma 3.2.2, assume τ is θ'_M-generic, and $\ell_\theta \in \mathrm{Hom}_G(\mathrm{Ind}_P^G \tau, \mathrm{Ind}_U^G \theta)$. If π is a θ-generic subrepresentation of $\mathrm{Ind}_P^G \tau$ then the space of the Whittaker model $\mathscr{W}(\pi,\theta)$ is as defined in equation (3.2.1). Indeed, assuming τ is supercuspidal, such θ-generic subquotient is unique. This can be done by analyzing the Jordan composite series of $(\mathrm{Ind}_P^G \tau)|_M$. (See [**2**] Section 2.)

3.3. Modules of the mirabolic group P_{n+1}

We review theory of Bernstein and Zelevinsky on the modules of mirabolic groups.

Assume $n \geq 0$ is an integer. Let X_{n+1} be an $n+1$-dimensional k-vector space. Set $\mathrm{M}_{n+1} = \mathrm{GL}(X_{n+1})$. Fix a complete flag $0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1}$ and hence a Borel subgroup B_{n+1} and a maximal unipotent subgroup N_{n+1} of M_{n+1}. For $1 \leq i \leq j \leq n$, let $Q_{i,j+1}$ be the parabolic subgroup of M_{j+1} stabilizing the flag $0 \subset X_i \subset X_{i+1} \subset \cdots \subset X_{j+1}$ and $U_{i,j+1}$ be its unipotent radical. Then $U_{i,j} \simeq U_{i,j-1} \ltimes X_j$, and $N_{j+1} = N_j U_{j,j+1}$. Let $\xi = \xi^{n+1}$ be a generic character on N_{n+1}. Set $\xi^j = \xi|_{N_j}$ and $\xi_j = \xi^{j+1}|_{U_{j,j+1}}$. Then $\xi^{j+1} = \xi^j \xi_j$ and $\xi = \xi_1 \xi_2 \cdots \xi_n$.

3.3. Modules of the mirabolic group P_{n+1}

The *mirabolic subgroup* of M_{j+1} is defined as the subgroup

$$\mathrm{P}_{j+1} = \mathrm{M}_j \, U_{j,j+1}.$$

It satisfies the inductive properties that

$$\mathrm{P}_j = \mathrm{Norm}_{\mathrm{M}_j}(U_{j,j+1}, \xi_j), \quad \mathrm{Norm}_{\mathrm{P}_{j+1}}(U_{j,j+1}, \xi_j) = \mathrm{P}_j \, U_{j,j+1}.$$

There are only two orbits of characters of $U_{j,j+1}$ under action of P_j, one is the closed orbit consists of the trivial character, the other is an open orbit containing ξ_j. Notice that $\mathrm{Norm}_{P_{j+1}}(U_{j,j+1}, 1) = \mathrm{P}_{j+1} = \mathrm{M}_j \, U_{j,j+1}$. We have exact functors

$$\Phi^- = r_{U_{j,j+1},\xi_j} : \mathrm{Rep}(\mathrm{P}_{j+1}) \to \mathrm{Rep}(\mathrm{P}_j), \quad \Phi^+ = i_{U_{j,j+1},\xi_j} : \mathrm{Rep}(\mathrm{P}_j) \to \mathrm{Rep}(\mathrm{P}_{j+1}),$$

$$\Psi^- = r_{U_{j,j+1},1} : \mathrm{Rep}(\mathrm{P}_{j+1}) \to \mathrm{Rep}(\mathrm{M}_j), \quad \Psi^+ = i_{U_{j,j+1},1} : \mathrm{Rep}(\mathrm{M}_j) \to \mathrm{Rep}(\mathrm{P}_{j+1}).$$

It is immediate that $\Phi^-\Psi^+ = 0$, $\Psi^-\Phi^+ = 0$ and Ψ^- is left adjoint to Φ^+.

The representations of these mirabolic groups have been well-studied by Bernstein and Zelevinsky in late 70s. (See [**1**].) By arguing about the l-sheaves on l-groups ([**1**] §5), they proved that $\Phi^-\Phi^+ \simeq \mathrm{id}$, $\Phi^-\Phi^+ \simeq \mathrm{id}$, and

$$0 \to \Phi^+\Phi^- \to \mathrm{id} \to \Psi^-\Psi^+ \to 0 \tag{3.3.1}$$

forms a short exact sequence. Indeed, it is not hard to check that for $(\sigma, W_\sigma) \in \mathrm{Rep}(\mathrm{P}_{j+1})$, $\Phi^+\Phi^-(W_\sigma) = W_\sigma(U_{j,j+1})$ and $\Psi^+\Psi^-(W_\sigma) \simeq (W_\sigma)_{U_{j,j+1}}$ as P_{j+1}-modules. As a quick result, Φ^- is left adjoint to Φ^+ and Φ^+, Ψ^+ preserve irreducibility.

The exact sequence 3.3.1 shows an irreducible representation σ is either from an irreducible representation of M_n (ie. of the form $\Psi^+\Psi^-(\sigma)$) or is from a smaller mirabolic subgroup P_n (ie. of the form $\Phi^+\Phi^-(\sigma)$). Applying induction on n we conclude the following lemma.

Lemma 3.3.1. *Assume $\sigma \in \mathrm{Rep}(\mathrm{P}_{n+1})$ is irreducible. There exists a unique $k \in \mathbb{N}$ such that the representation $\sigma^{(k)} = \Psi^-(\Phi^-)^{k-1}(\sigma) \in \mathrm{Rep}(\mathrm{M}_{n+1-k})$, called the k^{th}* derivative *of σ, is nonzero. For such an integer k, $\sigma^{(k)}$ is irreducible and*

$$\sigma \simeq (\Phi^+)^{k-1}\Psi^+(\sigma^{(k)}).$$

The $(n+1)^{th}$ derivative $\sigma^{(n+1)}$ of $\sigma \in \mathrm{Rep}(\mathrm{P}_{n+1})$ is a representation of $\mathrm{M}_0 = \mathrm{I}$ and hence a vector space. Since $N_{n+1} = \prod_{j=1}^n U_{j,j+1}$ and $\xi = \prod_{j=1}^n \xi_j$, the $(n+1)^{th}$ derivative is

$$\sigma^{(n+1)} = \Psi^-(\Phi^-)^n(\sigma) = \sigma_{N_{n+1},\xi}.$$

It is either 0 or one dimensional if σ is irreducible. When it is the latter, σ is isomorphic to the induced representation $\mathrm{ind}_{N_{n+1}}^{\mathrm{P}_{n+1}} \xi$, called the (irreducible) *standard representation* of Gelfand-Graev. In general,

$$(\Phi^+)^n\Psi^+(\sigma^{(n+1)}) = \mathrm{ind}_{N_{n+1}}^{\mathrm{P}_{n+1}} \xi \otimes \sigma_{N_{n+1},\xi} = \mathrm{ind}_{N_{n+1}}^{\mathrm{P}_{n+1}} \xi^{\oplus \dim \sigma^{(n+1)}}$$

is called the *nondegenerate part* of σ, denoted $\sigma^{(nd)}$. If $\sigma^{(nd)} = 0$, we say σ is degenerate, otherwise σ is nondegenerate. It is clear that σ is nondegenerate if and only if $\sigma_{N_{n+1},\xi} \neq 0$, hence $\sigma/\sigma^{(nd)}$ is always degenerate.

Further examining the exact sequence (3.3.1) and applying it inductively leads to the the following structure theorem of P_{n+1}-modules.

Theorem 3.3.2 (Bernstein-Zelevinsky [**1**])**.** *Suppose $\sigma \in \mathrm{Rep}(\mathrm{P}_{n+1})$, then σ is glue from $(\Phi^+)^{k-1}\Psi^+(\sigma^{(k)})$. More precisely, there is a natural filtration $0 \subset \sigma_{n+1} \subset \cdots \subset \sigma_2 \subset \sigma_1 = \sigma$ such that $\sigma_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\sigma)$, and the successive quotients are*

$$\sigma_k/\sigma_{k+1} = (\Phi^+)^{k-1}\Psi^+(\sigma^{(k)})$$

In particular, $\sigma_{n+1} = \sigma^{(nd)}$ and $\sigma/\sigma^{(nd)}$ is degenerate.

3.3. Modules of the mirabolic group P_{n+1}

Let $(\tau, W_\tau) \in \mathrm{Rep}(\mathrm{M}_j)$, and denote the restriction of ξ to N_j also be ξ. Define the k^{th} *derivative* $\tau^{(k)}$ of τ as the k^{th} derivative of $\tau|_{\mathrm{P}_j}$, i.e. $(\tau|_{\mathrm{P}_j})^{(k)}$, and $\tau^{(0)} = \tau|_{\mathrm{P}_j}$. Then by uniqueness of the Whittaker functional, $\tau^{(nd)} = \tau_{N_n,\xi}$ is either 0 or of dimension 1. When it is the latter, the representation τ is ξ-generic and admits a unique realization in the space $\mathrm{Ind}_{N_j}^{\mathrm{M}_j} \xi$. Bernstein and Zelevinsky shows in this case, if τ is irreducible admissible then the map from $\mathrm{Ind}_{N_j}^{\mathrm{M}_j} \xi$ to $\mathrm{Ind}_{N_j}^{\mathrm{P}_j} \xi$ by restricting the function to P_n is injective on the realization of τ. Clearly, the kernel in τ is degenerate. When τ is supercuspidal, then $\tau^{(k)} = 0$ for $1 \le k < j$ and hence $\tau = \tau^{(nd)}$ as a P_j-module. Hence the restricting map is an injection on the Whittaker model of τ. This turns it into a P_j-module and is called a Kirillov model.

We can do this similarly for a representation of $\mathrm{SO}_{2n+1}(k)$.

From now on, the notations are as in Chapter 2. Let X_{n+1} be the k-vector space $X \oplus kv_0$, then $0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1}$ forms a complete flag in X_{n+1}. Define as above the unipotent subgroups $U_{i,j+1}$ and maximal unipotent subgroup N_{j+1} of M_{j+1}, for $1 \le i \le j \le n$, corresponding to this flag. Then $Y/Z \simeq U_{n,n+1}$ and we have an exact sequence

$$1 \to \mathrm{Z} \to \mathrm{Q} \to \mathrm{P}_{n+1} \to 1.$$

The generic character θ of U is trivial on Z and factors through a generic character on N_{n+1}, denoted also by θ. Assume (π, V_π) is a smooth representation of G. The representation π_{Z} of Q is naturally a P_{n+1}-module. We can thus talk about the derivative and nondegenerate part of π_{Z} as we defined and discussed above. Then if π is supercuspidal, then $\pi_{\mathrm{Z}} = \pi_{\mathrm{Z}}^{(nd)}$ is a multiple of $\mathrm{ind}_{N_{n+1}}^{\mathrm{P}_{n+1}} \theta \simeq \mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}} \theta$. When π is θ-generic, the natural map from the realization of π in $\mathrm{Ind}_{\mathrm{U}}^{\mathrm{G}} \theta$ to $\mathrm{Ind}_{\mathrm{U}}^{\mathrm{Q}} \theta$ by restricting to Q is never injective. It has at least a kernel containing $\pi(\mathrm{Z})$. When π is supercuspidal, the kernel is exactly $\pi(\mathrm{Z})$. We land at the following useful proposition.

Proposition 3.3.3. *Assume* (π, V_π) *is an irreducible* θ*-generic and supercuspidal representation of* G. *Then* $\pi_{\mathrm{Z}} \simeq \mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$ *and if* $v \in V_\pi$ *is realized as the Whittaker function* $W_v \in \mathscr{W}(\pi,\theta)$ *in* $\mathrm{Ind}_{\mathrm{U}}^{\mathrm{G}}\,\theta$ *and* $W_v \equiv 0$ *on* Q, *then* $v \in V_\pi(\mathrm{Z})$, *or equivalently,* $J_{\mathrm{Z}}(v) = 0$. *If* π *is supercuspidal but not generic, then* $\pi_{\mathrm{Z}} = 0$.

We note that the proposition says assuming supercuspidality, $\pi_{\mathrm{Z}}^{(nd)}$ is only one copy of $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$ as a P_{n+1}-module if it is generic, and is zero if it is not. This is because that the multiplicity of $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$ in π_{Z} is the same as the multiplicity of θ in $\pi|_{\mathrm{U}}$ by Frobenius reciprocity, which is 1 when π is irreducible θ-generic and 0 when π is not generic. This result was used by Gelbart and Piatetski-Shapiro to prove the existence and uniqueness of a Rankin-Selberg L-function for $\mathrm{G} \times \mathrm{M}$ when the representations on both factors are generic. (See [**9**] §8, §9.)

3.4. A Lemma

We have seen when a representation (π, V_π) of G is irreducible θ-generic and supercuspidal, then its Jacquet module π_{Z} is isomorphic to the irreducible Q-module $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$. Before we end this chapter, we introduce a lemma of Moy and Prasad. Together with Proposition 3.3.3 it will play a crucial role in understanding the fixed vectors of $\mathrm{K}(\mathfrak{p}^m)$, which is at the heart of the study of newforms and will be introduced in Part 2. We shall see later that such vectors are always fixed by H_{x_m} for some m.

Lemma 3.4.1 (Moy-Prasad [**21**]). *Assume* $m \geq 0$ *is an integer. Suppose that* (ρ, W) *is a smooth representation of* H. *Then the natural projection map under the Jacquet functor* J_{Z}

$$J_{\mathrm{Z}} : W^{\mathrm{H}_{x_m+}} \to W_{\mathrm{Z}}^{\mathrm{M}(\mathfrak{o})}$$

is an injection.

Proof. Let i be an integer. Recall that we have a Iwahori factorization on non-maximal parahoric subgroup $\mathrm{H}_{x_{i_+}}$ of H

$$\mathrm{H}_{x_{i_+}} = \mathrm{Z}(\mathfrak{p}^{-i})\,\mathrm{M}(\mathfrak{o})\overline{\mathrm{Z}}(\mathfrak{p}^{i+1}).$$

Suppose $u_0 \in W^{\mathrm{H}_{x_{m+}}}$ is nonzero and $J_{\mathrm{Z}}(u_0) = 0$. By Proposition 3.1.3 (iv) there exists a minimal integer $i \geq m$ such that

$$\int_{\mathrm{Z}(\mathfrak{p}^{-j})} \rho(n)u_0 \, dn = 0, \ \forall j \geq i, \quad \text{and} \quad \int_{\mathrm{Z}(\mathfrak{p}^{-(i-1)})} \rho(n)u_0 \, dn \neq 0.$$

If $i = m$, then $u_0 = 0$, a contradiction. Assume $i \geq m+1$. Then u_0 is invariant under $\mathrm{M}(\mathfrak{o})$ and $\overline{\mathrm{Z}}(\mathfrak{p}^i)$. The vector

$$w_1 = \int_{\mathrm{H}_{x_{i_+-1}}} \rho(n)u_0 \, dn \neq 0$$

is invariant under the $\mathrm{H}_{x_{i_+-1}} = \mathrm{Z}(\mathfrak{p}^{-(i-1)})\,\mathrm{M}(\mathfrak{o})\overline{\mathrm{Z}}(\mathfrak{p}^i)$. The image of $\mathrm{H}_{x_{i_+-1}}$ in the reductive quotient $\mathrm{H}(\mathfrak{f})$ of H_{x_i} by the pro-unipotent radical $\mathrm{H}_{x_i}^+$ is the opposite parabolic subgroup $\overline{P}(\mathfrak{f})$.

Consider the representation (τ, W) of the finite reductive group $\mathrm{H}(\mathfrak{f})$ by restricting π to H_{x_i} on the space $W^{\mathrm{H}_{x_i}^+}$. Then $w_1 \in \mathsf{W}^{\overline{\mathrm{P}}(\mathfrak{f})}$. The theory of representations of finite group of Lie type shows (c.f. [**21**, Proposition 6.1]) summing over $\mathrm{Z}(\mathfrak{f})$ forms an isomorphism from $\mathsf{W}^{\overline{\mathsf{Z}}(\mathfrak{f})}$ to $\mathsf{W}^{\mathsf{Z}(\mathfrak{f})}$ for any W of finite dimension. Since any representation of $\mathrm{H}(\mathfrak{f})$ is a direct sum of irreducible (and hence finite dimensional) representations of $\mathrm{H}(\mathfrak{f})$ by Zorn's Lemma, it is an isomorphism for any representation of $\mathrm{H}(\mathfrak{f})$. We get a nonzero vector

$$w_1' = \int_{\mathsf{Z}(\mathfrak{f})} \tau(n)w_1 \, dn.$$

in $\mathsf{W}^{\mathrm{P}(\mathfrak{f})}$.

We construct another nonzero vector w_2 in W by

$$\begin{aligned}
0 \neq w_2 &= \int_{\mathrm{M}(\mathfrak{f})} \tau(m) w_1' \, dm = \int_{\mathrm{P}(\mathfrak{f})} \tau(p) w_1 \, dp = \int_{\mathrm{H}_{x_{i+}}} \rho(h) w_1 \, dh \\
&= \int_{\mathrm{Z}(\mathfrak{p}^{-i})\,\mathrm{M}(\mathfrak{o})\overline{\mathrm{Z}}(\mathfrak{p}^{i+1})} \rho(h) w_1 \, dh \\
&= (\mathrm{const}) \int_{\mathrm{Z}(\mathfrak{p}^{-i})} \rho(h) w_1 \, dh \\
&= (\mathrm{const}) \int_{\mathrm{Z}(\mathfrak{p}^{-(i)})} \int_{\mathrm{Z}(\mathfrak{p}^{-(i-1)})} \rho(h_2 h_1) u_0 \, dh_1 dh_2 \\
&= (\mathrm{const}) \int_{\mathrm{Z}(\mathfrak{p}^{-i})} \rho(h) u_0 \, dh = 0, \qquad \text{a contradiction.}
\end{aligned}$$

The last equality is by changing the order of the integration and fact that Z is commutative. Therefore, u_0 must be 0. The map is injective. □

The original proof in [**21**] deals with irreducible admissible representations of H in which case the map is an isomorphism. The surjectivity fails when removing the admissible condition because of the use of Jacquet's Lemma, while injectivity stays valid by passing through the Zorn's Lemma. I thank Jiu-Kang Yu for his discussion with me on removing the admissibility condition.

Corollary 3.4.2. *Assume* $(\pi, V_\pi) \in \mathrm{Rep}(\mathrm{G})$ *is irreducible and supercuspidal. If* π *is* θ*-generic and* $v \in V_\pi^{\mathrm{H}_{x_m}}$ *for some integer* $m \geq 0$*, then the associated Whittaker function* W_v *in* $\mathscr{W}(\pi, \theta)$ *is determined by its restriction to* Q *which lies in* $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\, \theta$*. If* π *is non-generic, then* $V_\pi^{\mathrm{H}_{x_m}} = 0$ *for all* $m \in \mathbb{Z}$*.*

Proof. $\pi|_{\mathrm{H}}$ is a smooth representation of H. If π is θ-generic, then by Proposition 3.3.3 and Lemma 3.4.1 $W_v(\mathrm{Q}) = 0 \Rightarrow J_{\mathrm{Z}}(v) = 0 \Rightarrow v = 0$. If π is not generic, then Proposition 3.3.3 implies $(V_\pi)_{\mathrm{Z}} = 0$ and Lemma 3.4.1 implies $V_\pi^{\mathrm{H}_{x_m}} = 0$ for all $m \geq 0$. □

3.5. Hecke algebras

Let G be a connected k-split reductive group over $\mathfrak{o}$ and fix a generic data $(B, T, \theta : U \to \mathrm{S}^1)$ of G. The Hecke algebra $\mathcal{H}(G)$ is the algebra of smooth compactly supported functions on G with multiplication given by convolution $*$. Suppose K is an open compact subgroup of G. Denote by $\mathcal{H}(G, K)$ the subalgebra of bi-K-invariant functions in $\mathcal{H}(G)$. The algebra $\mathcal{H}(G)$ is generated by characteristic functions ch_K on each open compact subset K of G. Denote by e_K the function $\mathrm{vol}(K)^{-1}\,\mathrm{ch}_K$ in $\mathcal{H}(G)$ for K an open compact subgroup of G. Then e_K is an idempotent of $\mathcal{H}(G)$ and $\mathcal{H}(G, K) = e_K * \mathcal{H}(G) * e_K$, which contains e_K as a unit. Since $f \in \mathcal{H}(G)$ is smooth and has compact support, there exists an open compact subgroup K such that $f \in \mathcal{H}(G, K)$. Hence $\mathcal{H}(G) = \cup_K \mathcal{H}(G, K)$ with K running through open compact subgroups of G. We say a $\mathcal{H}(G)$-module $\mathcal{V}$ is *smooth* if for all $v \in \mathcal{V}$, $v \in \mathcal{H}(G, K)\mathcal{V}$ for some K, or, equivalently, $\mathcal{H}(G)\mathcal{V} = \mathcal{V}$.

Fix $(\pi, V_\pi) \in \mathrm{Rep}(G)$ and fix a Haar measure dg on G. Any function f in the Hecke algebra induces an operator $\pi(f)$ on the space of the representation. We have $\mathcal{H}(G) \to \mathrm{End}_{\mathbb{C}}(V_\pi)$ and $\mathcal{H}(G, K) \to \mathrm{End}_{\mathbb{C}}(V_\pi^K) = \mathrm{End}_{\mathbb{C}}(V_\pi)^K$ given by

$$f \mapsto \pi(f) = \int_G f(g)\pi(g)\ dg.$$

Since naturally the operator $\pi(f_2) \circ \pi(f_1)$ is given by the convolution $\pi(f_2 * f_1)$ for $f_1, f_2 \in \mathcal{H}(G)$. The space V_π is endowed the structure of a smooth $\mathcal{H}(G)$-module. Here the smoothness is given by the facts $V_\pi^K = \pi(e_K)V_\pi$ and $V_\pi = \cup_K V_\pi^K$. Suppose (π_1, V_1) and (π_2, V_2) are two smooth representations of G and $T : V_1 \to V_2$ is a G-homomorphism, then it is also a $\mathcal{H}(G)$-module map. On the other hand, any smooth $\mathcal{H}(G)$-module endows a smooth action of G on it as follows.

Proposition 3.5.1. *Suppose $\mathcal{V}$ is a smooth $\mathcal{H}(G)$-module, then there is a unique smooth representation $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{V})$ such that $\pi(f)v = fv$ for $f \in \mathcal{H}(G)$, $v \in \mathcal{V}$.*

Proof. Let us claim that we have canonical isomorphism $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} \mathcal{V} \simeq \mathcal{V}$ by multiplication which hence induces a canonical action of G on $\mathcal{V}$ via the action of left translation on the first factor. The multiplication is surjective by smoothness, and injective since it is injective on $\mathcal{H}(G, K) \otimes_{\mathcal{H}(G,K)} e_K\mathcal{V} = e_K\mathcal{V}$. The action can be given explicitly by $\pi(g)v = \mathrm{vol}(K)^{-1}\,\mathrm{ch}_{gK}\, v$ for open compact subgroup K such that $v \in e_K\mathcal{V}$. □

As a result, the category of smooth $\mathcal{H}(G)$-module is equivalent to the category of smooth representation of G. In particular, a representation (π, V_π) is irreducible if and only if it is a simple smooth $\mathcal{H}(G)$-module.

Proposition 3.5.2. *Assume* $(\pi_i, V_i) \in \mathrm{Rep}(G)$ *are irreducible for* $i = 1, 2$. *Suppose* $T : V_1^K \to V_2^K$ *is a* $\mathcal{H}(G, K)$*-module map. Then it extends to a* $\mathcal{H}(G)$*-module map* $\tilde{T} : V_1 \to V_2$ *uniquely.*

Proof. We have seen that $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} V_i = V_i$ and $V_i^K = \pi(e_K)V_i$. Let us claim that $\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} V_i^K \simeq V_i$. Clearly $\mathcal{H}(G)V_i^K$ is a smooth $\mathcal{H}(G)$-submodule of the simple smooth $\mathcal{H}(G)$-module V_i. To show injectivity, assume that $\sum_{j=1}^d \pi(f_j)v_j = 0$. Let K' be an open compact subgroup contained in K as a normal subgroup such that $f_j \in \mathcal{H}(G, K')$ for all j. Then since $\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} V_i^K \simeq \mathcal{H}(G) \otimes_{\mathcal{H}(G,K')} (\mathcal{H}(K', K) \otimes_{\mathcal{H}(K',K)} V_i^K) \simeq \mathcal{H}(G) \otimes_{\mathcal{H}(G,K')} V_i^{K'}$, the element $\sum_{j=1}^d f_j \otimes v_j = e_{K'} \otimes \sum_{j=1}^d \pi(f_j)v_j \in \mathcal{H}(G) \otimes_{\mathcal{H}(G,K')} V_i^{K'}$ is 0. Hence the kernel is trivial. By tensoring $\mathcal{H}(G)$ the $\mathcal{H}(G, K)$-module map T thus extend canonically to a $\mathcal{H}(G)$-module map, hence a G-homomorphism, $\tilde{T} : V_1 \simeq \mathcal{H}(G) \otimes_{\mathcal{H}(G)} V_1 \to V_2 \simeq \mathcal{H}(G) \otimes_{\mathcal{H}(G)} V_2$. The uniqueness is by the Schur Lemma which says that G-homomorphism between irreducible representations is unique up to scaling. □

Corollary 3.5.3. *For* $(\pi, V_\pi) \in \mathrm{Rep}(G)$, *assuming* $V_\pi^K \neq 0$ *then* π *is irreducible if and only if* V_π^K *is a simple* $\mathcal{H}(G, K)$*-module.*

In particular, we get the following:

Corollary 3.5.4. *If* $(\pi, V_\pi) \in \mathrm{Rep}(G)$ *is irreducible and the Hecke algebra* $\mathcal{H}(G,K)$ *is commutative on* V_π^K. *Then* $\dim V_\pi^K \leq 1$.

We will show that this applies to hyperspecial open compact subgroup K of G by proving $\mathcal{H}(G,K)$ commutative using the Satake isomorphism.

Let $\rho \in \frac{1}{2}\,\mathrm{X}^\bullet(T)$ be half of the sum of the positive roots of G, i.e. $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_G^+}\alpha$. One note that if $\lambda \geq \mu$ then $\langle \lambda-\mu, \rho\rangle \geq 0$. Notice that $\delta_B|_T = 2\rho$. The following proof is based on [**12**].

Definition 3.5.5. Assume K is an open compact subgroup of G such that $G = BK$ and $T(\mathfrak{o}) = T \cap K$. The *Satake transform* $\mathcal{S} : \mathcal{H}(G,K) \to \mathcal{H}(T, T(\mathfrak{o}))$ is defined by

$$f \mapsto \mathcal{S}f(t) = \delta_B^{1/2}(t)\int_U f(tu)\ du.$$

Let us show that the Satake transform is well-defined. Since $T(\mathfrak{o}) = T\cap K$, so

$$\mathcal{S}f(t) = \delta_B^{1/2}(t)\int_U f(tu)\ du = \delta_B^{-1/2}(t)\int_U f(ut)\ du$$

is a bi-$T(\mathfrak{o})$-invariant function on T. For $f_1, f_2 \in \mathcal{H}(G,K)$ and $t\in T$,

$$\begin{aligned}
\mathcal{S}(f_1 * f_2)(t) &= \delta_B(t)^{-1/2}\int_U\int_G f_1(g)f_2(g^{-1}u_2t)\ dg\,du_2\\
&= \delta_B(t)^{-1/2}\int_U\int_{BG(\mathfrak{o})} f_1(g)f_2(g^{-1}u_2t)\ dg\,du_2\\
&= \delta_B(t)^{-1/2}\int_U\int_B f_1(b)f_2(b^{-1}u_2t)\ db\,du_2\\
&= \delta_B(t)^{-1/2}\int_U\int_T\int_U f_1(t'u_1)f_2(u_1^{-1}t'^{-1}u_2t)\ du_1\,dt'\,du_2\\
&= \delta_B(t)^{-1/2}\int_T\int_U\int_U f_1(t'u_1)f_2(t'^{-1}u_2t)\ du_2\,du_1\,dt'\\
&= \int_T \mathcal{S}f_1(t')\mathcal{S}f_2(t'^{-1}t)\ dt' = (\mathcal{S}f_1 * \mathcal{S}f_2)(t).
\end{aligned}$$

Proposition 3.5.6. *The Satake transform is an algebra homomorphism.*

Assume K is the hyperspecial maximal compact subgroup $G(\mathfrak{o})$. It satisfies the properties $G = BK$ and $T \cap K = T(\mathfrak{o})$ with $T/(T \cap K) \simeq \mathrm{X}_\bullet(T)$. Furthermore, $G = \sqcup_{\lambda \in P^+} K\varpi^\lambda K$. Let $b \in \mathcal{A}(G)$ be the barycenter of the fundamental alcove C, then the parahoric subgroup G_b is a Iwahori subgroup and

$$K = \sqcup_{s \in W_G} G_b w_s G_b, \quad (w_s\text{: any lift of } s \text{ in } K).$$

Recall there is a partial order $\geq$ on $\mathrm{X}_\bullet(T) \subset (W_G)_{\mathrm{aff}}$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a sum of positive co-roots, $\check{\alpha} \in \check{\Phi}_G^+$. Then $\lambda \geq s(\lambda)$ for all $s \in W_G$ given $\lambda \in P^+$. We have the following property for $\lambda, \mu \in P^+$

$$K\varpi^\lambda K \cap U\varpi^\mu K \neq \emptyset \Rightarrow \mu \leq \lambda \tag{3.5.1}$$

which will be prove at the end of the section.

Using these property we can show the following famous result.

Theorem 3.5.7 (Cartier [**3**])**.** *Assume K is the hyperspecial maximal compact subgroup of* G*. Then the Satake transform $\mathcal{S}$ induces an algebra isomorphism onto its image $\mathcal{H}(T, T(\mathfrak{o}))^{W_G}$.*

Proof. Let K be $G(\mathfrak{o})$. The Weyl group $W_G \simeq N_G(T)/T$ acts on T by conjugation and induces an action on $\mathcal{H}(T, T(\mathfrak{o}))$. The hyperspecial subgroup $G(\mathfrak{o})$ contains a lift of W_G. Hence the image of $\mathcal{S}$ is bi-W_G-invariant and sits in the W_G-invariants. Let us further show that $\mathcal{S}$ is indeed an isomorphism onto $\mathcal{H}(T, T(\mathfrak{o}))^{W_G}$. Let

$$\mathrm{ch}'_{T(\mathfrak{o})\varpi^\lambda T(\mathfrak{o})} = \frac{1}{|W_G|} \sum_{s \in W_G} \mathrm{ch}_{T(\mathfrak{o})\varpi^{w(\lambda)} T(\mathfrak{o})} .$$

Then $\{\mathrm{ch}_{K\varpi^\lambda K}\}_{\lambda \in P^+}$ forms a basis of the $\mathbb{C}$-vector space $\mathcal{H}(G, K)$ and $\{\mathrm{ch}'_{T(\mathfrak{o})\varpi^\lambda T(\mathfrak{o})}\}_{\lambda \in P^+}$ is a basis of the $\mathbb{C}$-vector space $\mathcal{H}(T, T(\mathfrak{o}))^{W_G}$. For $\lambda \in P^+$, there are constants

$c_\lambda(\mu) \in \mathbb{C}$, for $\mu \in P^+$, such that

$$\mathcal{S}(\mathrm{ch}_{K\varpi^\lambda K}) = \sum_{\mu \in P^+} c_\lambda(\mu)\, \mathrm{ch}'_{T(\mathfrak{o})\varpi^\mu T(\mathfrak{o})}\,.$$

By direct computation, for $\lambda, \mu \in P^+$ the coefficient $c_\lambda(\mu)$ is equal to

$$\begin{aligned}\mathcal{S}(\mathrm{ch}_{K\varpi^\lambda K})(\varpi^\mu) &= \delta_B^{-1/2}(\varpi^\mu)\int_U \mathrm{ch}_{K\varpi^\lambda K}(u\varpi^\mu)\, du \\ &= q^{\langle\mu,\rho\rangle}\,\mathrm{vol}(U\varpi^\mu K \cap K\varpi^\lambda K)\end{aligned}$$

which is nonzero only if $\lambda \geq \mu$. In particular, $c_\lambda(\lambda) = q^{\langle\lambda,\rho\rangle}\,\mathrm{vol}(\varpi^\lambda K) = q^{\langle\lambda,\rho\rangle}$ is nonzero. Hence

$$\mathcal{S}(\mathrm{ch}_{K\varpi^\lambda K}) = q^{\langle\lambda,\rho\rangle}\,\mathrm{ch}'_{T(\mathfrak{o})\varpi^\lambda T(\mathfrak{o})} + \sum_{\mu \in P^+, \lambda > \mu} c_\lambda(\mu)\, \mathrm{ch}'_{T(\mathfrak{o})\varpi^\mu T(\mathfrak{o})}\,. \tag{3.5.2}$$

Since $\geq$ is a partial order on P^+, this implies $\mathcal{S}$ is bijective onto $\mathcal{H}(T, T(\mathfrak{o}))^{W_G}$. □

Corollary 3.5.8. *The spherical Hecke algebra $\mathcal{H}(G, G(\mathfrak{o}))$ is commutative and isomorphic to the coordinate ring $\mathbb{C}[\hat{T}]^{W_G}$ of $\hat{T}/W_G$.*

Proof. Since T is commutative, it is clear that $\mathcal{H}(T, T(\mathfrak{o}))$ is commutative. Moreover, the algebra structure of $\mathcal{H}(T, T(\mathfrak{o}))$ is isomorphic to $\mathrm{X}_\bullet(T) \otimes_{\mathbb{Z}} \mathbb{C}$. By duality, this is $\mathrm{X}^\bullet(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{C}$ which is the $\mathbb{C}$-algebra of the coordinate ring of the variety $\hat{T}$. This is compatible with the actions of W_G on T and $\hat{T}$. □

Let us now give a proof for the property (3.5.1). We shall apply the following facts regarding an Iwahori subgroup G_b compatible with Φ_G^+.

(i) G_b admits a Iwahori decomposition $G_b = (G_b \cap \overline{U})(G_b \cap T)(G_b \cap U)$.

(ii) Assume K an open compact subgroup containing G_b, then $K = \cup_{w \in I_K} G_b w G_b$ for some subset I_K of $\tilde{W}_G$.

(iii) For $w, w' \in \tilde{W}_G$, $G_b w G_b w' G_b \subset \bigsqcup_{w'' \leq w'} G_b w w'' G_b$.

The following proof is due to Haines and Rostami [**13**].

Proof of (3.5.1). Assume $K \supset G_b$ is an open compact subgroup such that $G = \cup_\mu U\varpi^\mu K$ and $K = \sqcup_{s\in W_G} G_b w_s G_b$ with w_s a lift of s in K. Assume $\lambda \in P^+$, then since $G_b\varpi^\lambda G_b w_s G_b = G_b\varpi^\lambda w_{s_2} G_b$, we get

$$K\varpi^\lambda K = \bigcup_{s_1,s_2\in W_G} G_b w_{s_1} G_b \varpi^\lambda G_b w_{s_2} G_b = \bigcup_{s_1,s_2\in W_G} G_b w_{s_1} G_b \varpi^\lambda w_{s_2} G_b$$

Assume $U\varpi^\mu K \cap G_b w_{s_1} G_b \varpi^\lambda w_{s_2} G_b \neq \emptyset$. Since $U\varpi^\mu K = \bigcup_{s\in W_G} U\varpi^\mu w_s G_b$, there exist $u \in U$, $s, s_1, s_2 \in W_G$ such that

$$u\varpi^\mu w_s \in G_b w_{s_1} w_{s_2'} G_b$$

for some $s_2' \in \tilde{W}_G$, $s_2' \leq \lambda s_2$. Take a co-character γ such that $u = \varpi^{-\gamma} u' \varpi^\gamma$ for some $u \in G_b$. Then

$$G_b \varpi^\gamma \varpi^\mu w_s G_b \subset G_b \varpi^\gamma G_b w_{s_1} w_{s_2'} G_b.$$

This implies $\mu s \leq s_1 s_2' \leq s_1 \lambda s_2$ and $\leq \lambda$ since $\lambda \in P^+$ and $s_1, s_2 \in W_G$. Hence we can find a minimal μ' such that $\varpi^{\mu'} K = \varpi^\mu K$ and $\mu' \leq \lambda$. □

In the case $G = \mathrm{SO}(V)$, other than the hyperspecial open compact subgroups K_{x_i}, i: even, the rest of the family K_{x_i} for i odd are also subgroups that satisfy the properties used to prove (3.5.1) for the Satake isomorphism. Consider the Iwahori subgroup G_{x_i+b} where b is the barycenter of the alcove C. Then $\mathrm{K}_{x_i} = \bigsqcup_{w\in N_{\mathrm{K}_{x_i}}(T)/T(\mathfrak{o})} \mathrm{G}_{x_i+b}\, w\, \mathrm{G}_{x_i+b}$ and $N_{\mathrm{K}_{x_i}}(T)/T(\mathfrak{o}) \simeq W_G$. We have $\mathrm{G} = \mathrm{B}\,\mathrm{K}_{x_i}$ and $\mathrm{K}_{x_i} \cap \mathrm{T} = \mathrm{T}(\mathfrak{o})$. The open compact groups K_{x_i} admit the property (3.5.1) and

$$\mathrm{K}_{x_i}\,\varpi^\lambda\,\mathrm{K}_{x_i} \subset \bigcup_{\mu\leq\lambda} U\varpi^\mu\,\mathrm{K}_{x_i}, \quad \forall\lambda \in P^+. \tag{3.5.3}$$

Following the same line as the proof of Theorem 3.5.7 we can also get:

Proposition 3.5.9. *The Satake transform* $\mathcal{S} : \mathcal{H}(\mathrm{G}, \mathrm{K}_{x_i}) \to \mathcal{H}(\mathrm{T}, \mathrm{T}(\mathfrak{o}))$ *is an isomorphism onto* $\mathcal{H}(\mathrm{T}, \mathrm{T}(\mathfrak{o}))^{W_{\mathrm{G}}}$*. Hence the Hecke algebra* $\mathcal{H}(\mathrm{G}, \mathrm{K}_{x_i})$ *is commutative.*

As a result, we obtain the following Corollary.

Corollary 3.5.10. *Let* π *be any irreducible smooth representation of* G*. Then the* K_{x_i}*-invariants* $\pi^{\mathrm{K}_{x_i}}$ *in* π *has dimension at most* 1*.*

CHAPTER 4

Local factors of generic representations

For a generic representation of $\mathrm{SO}_{2n+1}(k)$, the Langlands functorial lifting to $\mathrm{GL}_{2n}(k)$ has been established by Soudry and Jiang and hence the local Langlands correspondence from generic representations π of $\mathrm{SO}_{2n+1}(k)$ to $2n$-dimensional symplectic Weil-Deligne representations $(\rho, \mathrm{Sp}(\mathbb{M}), N)$ of the Weil group of k, called the Langlands parameter $\mathbb{M}$ of π, is valid. The standard L-functions $L(\pi, std, s)$ of the Langlands parameters have then an integral representation, the zeta integrals, which by Soudry is the Rankin-Selberg L-functions $L(\pi, s)$ for $\mathrm{SO}_{2n+1}(k) \times \mathrm{GL}_1(k)$. The so defined ε-factors $\varepsilon(\pi, s, \psi)$, conductors a_π and root numbers ε_π of the representations are equal to the ones defined for the Langlands parameters. We shall introduce the construction of these local factors in this chapter. The notation follow Chapter 2 and 3 as before and $\mathrm{G} = \mathrm{SO}_{2n+1}$. A generic data $(\mathrm{B}, \mathrm{T}, \theta)$ of G is fixed.

In this chapter, $(\pi, V_\pi) \in \mathrm{Rep}(\mathrm{G})$ is always an irreducible θ-generic supercuspidal representation of G. Fix a Whittaker functional ℓ_θ on V_π with respect to θ and hence a realization of π to the Whittaker model $\mathscr{W}(\pi, \theta)$ by $v \mapsto W_v(g) = \ell_\theta(\pi(g).v)$ for $v \in V_\pi$. Recall that by Corollary 3.4.2, W_v is uniquely determined by its restriction to Q which is a function in $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\, \theta$ of compact support modulo U. The restriction of W_v to T is slowly increasing by smoothness of π.

4.1. Standard L-function for $\mathrm{SO}_{2n+1}(k)$

In this section, we will construct a zeta integral by Rankin-Selberg convolution for $\mathrm{G} \times \mathrm{GL}_1(k)$ which interpolate the standard L-function. It was first constructed by

Novodvorsky and studied systematically by Ginzburg [**11**] (global case) and Soudry [**28**] (local case) for general $\mathrm{SO}_{2n+1}(k) \times \mathrm{GL}_r(k)$. These Rankin-Selberg L-functions are known to agree with the tensor product L-functions, up to a normalization. We review the general idea of this construction before we introduce the special cases $r = 1$. We will also treat the case $r = n$ in later sections.

Assume $1 \le i \le j \le n$ and $1 \le r \le n$ are integers. Let M_{j+1}, N_{j+1} and $U_{i,j+1}$ be as defined in Section 3.3. Define the subgroup $\mathrm{Y}'_{(r,n)}$ as the unipotent radical of the parabolic subgroup preserving the isotropic flag

$$0 \subset ke_{r+1} \subset ke_{r+1} \oplus ke_{r+2} \subset \cdots \subset ke_{r+1} \oplus ke_{r+2} \oplus \cdots \oplus ke_n.$$

$\mathrm{Y}'_{(r,n)}$ normalizes the intersection $\mathrm{U} \cap \mathrm{Y}'_{(r,n)}$ and the character $\theta_{(r)} = \theta|_{\mathrm{U}\cap \mathrm{Y}'_{(r,n)}}$. Then $\theta_{(r)}$ is a character of $\mathrm{U} \cap \mathrm{Y}'_{(r,n)}$. Let $\mathrm{X}'_{(r,n)}$ be the subgroup such that $\mathrm{Y}'_{(r,n)} = (\mathrm{U} \cap \mathrm{Y}'_{(r,n)}) \rtimes \mathrm{X}'_{(r,n)}$. Then

$$\mathrm{X}'_{(1,n)} = \prod_{i=2}^{n} \mathrm{U}_{\epsilon_i - \epsilon_1}$$

is abelian and isomorphic to k^{n-1}.

Definition 4.1.1. For $v \in V_\pi$, define the *zeta integral* attached to v as

$$I(v,s) = \int_{k^\times} \int_{\mathrm{X}'_{(1,n)}} W_v(\vec{x}\, \epsilon_1(a)) |a|^{s-\frac{1}{2}} \, d\vec{x}\, da, \; s \in \mathbb{C}. \tag{4.1.1}$$

By a change of variables, the zeta integral $I(v,s)$ can also be written as

$$I(v,s) = \int_{k^\times} \int_{\mathrm{X}'_{(1,n)}} W_v(\epsilon_1(a)\, \vec{x}) |a|^{s-(n-\frac{1}{2})} \, d\vec{x}\, da. \tag{4.1.2}$$

Since π is smooth, every vector is fixed by some open compact subgroup. The zeta integral $I(v,s)$ is a finite sum of functions of the form

$$\int_{a\in k^\times} W_{v'}(\epsilon_1(a)) |a|^{s-(n-\frac{1}{2})} \, da = \sum_{m\in\mathbb{Z}} q^{m(n-\frac{1}{2})} \left(\int_{\mathfrak{p}^m - \mathfrak{p}^{m+1}} W_{v'}(\epsilon_1(a)) \, da \right) q^{-ms}.$$

Since $W_v|_{\mathrm{T}}$ is a slowly increasing function on $\mathrm{T} \simeq \mathbb{G}_m^n$. For $\Re(s) \gg 0$, the function $I(v,s)$ converges absolutely to a rational function in $X = q^{-s}$ and therefore has a meromorphic continuation to all $s \in \mathbb{C}$.

Proposition 4.1.2. *For $v \in V_\pi$, the zeta integral $I(v,s)$ converges absolutely on a right half plane to a rational function in $X = q^{-s}$ and has a meromorphic continuation to the whole complex plane.*

For vectors in V_π that is invariant under elements in $\mathrm{Q}(\mathfrak{o})$, the zeta integral attached to them can be rewritten into a simpler form.

Lemma 4.1.3 (Simpler formula for $I(v,s)$)**.** *If v is fixed by $\mathrm{Q}(\mathfrak{o})$, then*

$$I(v,s) = \int_{k^\times} W_v(\epsilon_1(a))|a|^{s-(n-\frac{1}{2})}\, da.$$

Proof. For $\alpha = \epsilon_i - \epsilon_1$, $i = 1, 2, ..., n$, $\mathrm{G}_\alpha \simeq \mathrm{SL}_2(k)$. For $c_i \neq 0$, $i = 2, 3, ..., n$,

$$x_{\epsilon_i-\epsilon_1}(c_i)x_{\epsilon_1-\epsilon_{i+1}}(y_i) = x_{\alpha_i}(-c_iy_i)x_{\epsilon_1-\epsilon_{i+1}}(y_i)x_{\epsilon_i-\epsilon_1}(c_i).$$

Assume $\vec{x} = \prod_{i=2}^n x_{\epsilon_i-\epsilon_1}(c_i)$ with $c_j \in \mathfrak{o}$ for $j > i$ and $c_i \notin \mathfrak{o}$. Suppose v is invariant under elements in $\mathrm{Q}(\mathfrak{o})$. For all $y_2, y_3, ..., y_i \in \mathfrak{o}$,

$$\begin{aligned} \vec{x}\, v &= \prod_{j=2}^{n} x_{\epsilon_j-\epsilon_1}(c_j)\ v = \prod_{j=2}^{i} x_{\epsilon_j-\epsilon_1}(c_j)\ v \\ &= \left(\prod_{j=2}^{i-1} x_{\epsilon_j-\epsilon_1}(c_j)\right) x_{\epsilon_i-\epsilon_1}(c_i)x_{\epsilon_1-\epsilon_{i+1}}(y_i)\ v \\ &= \left(\prod_{j=2}^{i-1} x_{\epsilon_j-\epsilon_1}(c_j)\right) x_{\alpha_i}(-c_iy_i)\ v \\ &= x_{\alpha_i}(-c_iy_i)\left(\prod_{j=2}^{i-1} x_{\epsilon_j-\epsilon_1}(c_j)\right)\ v \\ &= x_{\alpha_i}(-c_iy_i)\cdots x_{\alpha_3}(-c_3y_3)x_{\alpha_2}(-c_2y_2)\ v. \end{aligned}$$

Choose y_i with close to 0 enough such that $c_j y_j \in \mathfrak{o}$. Using the other expression (4.1.2) of $I(v,s)$, one can get

$$\begin{aligned} I(v,s) &= \int_{k^\times}\int_{\mathrm{X}'_{(1,n)}} W_v(\epsilon_1(a)\,\vec{x})|a|^{s-(n-\frac{1}{2})}\,d\vec{x}\,da \\ &= \int_{k^\times}\int_{\mathfrak{o}}\cdots\int_{\mathfrak{o}} \ell_\theta(\epsilon_1(a)\prod_{i=2}^{n} x_{\epsilon_i-\epsilon_1}(c_i)\,v)|a|^{s-(n-\frac{1}{2})}\,dc_n\cdots dc_2\,da \\ &= \int_{k^\times} \ell_\theta(\epsilon_1(a)\,v)|a|^{s-(n-\frac{1}{2})}\,da, \end{aligned}$$

which proves the assertion. □

Remark 4.1.4. In the proof of the simpler formula, we see that to obtain the simpler formula, it is enough to require v to be invariant under elements in $\mathrm{X}'_{(1,n)}(\mathfrak{o})$, $\mathrm{U}_{\epsilon_1}(\mathfrak{o})$, $\mathrm{U}_{\epsilon_1-\epsilon_i}(\mathfrak{o})$ for $i=3,...,n$ and $\mathrm{U}_{\alpha_i}(\mathfrak{p})$ for $i=1,...,n-1$.

Using this simpler formula, we can argue that the complex valued function $I(v,s)$ can achieve any constant function for some $v\in V_\pi$. This is done by the fact that the linear form $I(v,s)$ on V_π passes through a linear form on $(V_\pi)_{\mathrm{Z}}$, which contains the whole space $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\theta$ by genericity assumption. We look at the function W_0 in $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\theta$ which is $\mathrm{Q}(\mathfrak{o})$-invariant on the right, supported on $\mathrm{U}\,\mathrm{Q}(\mathfrak{o})$ and takes 1 on the identity. Then W_0 is well-defined since θ is trivial on $\mathrm{U}\cap\mathrm{Q}(\mathfrak{o})$. Any preimage of W_0 in V_π under J_{Z} is fixed by $\mathrm{Q}(\mathfrak{o})$ since J_{Z} is a Q-homorphism. Applying the simpler formula, it is clear that the zeta integral attached to such a preimage is a constant function. By rescaling we get any constant function.

Let the set

$$I(\pi)=\{I(v,s)\mid v\in V_\pi\}$$

be the vector space of zeta integrals attached to the representation space V_π. We have seen that $\mathbb{C}\subset I(\pi)$. Since $I(v,s)$ has meromorphic continuation to a rational function in $X=q^{-s}$, we can view $I(\pi)\subset\mathbb{C}(q^{-s})$. Since $I(\epsilon_1(\varpi^m)v,s)=q^{-m(s-\frac{1}{2})}I(v,s)$, so multiplying by q^{-ms} for any $m \in \mathbb{Z}$ preserves the space. It is indeed a sub-$\mathbb{C}[q^{-s}, q^s]$-module of $\mathbb{C}(q^{-s})$ hence a fractional ideal. Since the polynomial ring $\mathbb{C}[X, X^{-1}]$ is a principal ideal domain. The fractional ideal $I(\pi)$ is hence principal and admits a generator. This is how the L-function of π is defined.

Proposition 4.1.5. *For an irreducible generic representation π of* G*, the set*

$$I(\pi) = \{I(v, s) \mid v \in V_\pi\} \subset \mathbb{C}(q^{-s})$$

*is a fractional ideal of the principal ideal domain $\mathbb{C}[q^{-s}, q^s]$. The L-*function *of π is defined as the generator of the fractional ideal which is of the form*

$$L(\pi, s) = \frac{1}{P_\pi(q^{-s})}, \quad P_\pi(X) \in \mathbb{C}[X],\ P_\pi(0) = 1.$$

In particular, if π is supercuspidal, then $L(\pi, s) = 1$, or equivalently, $P_\pi(X) = 1$.

Proof. We have seen that $I(\pi)$ is a fraction ideal. Suppose $1/P_\pi(q^{-s}) \in \mathbb{C}(q^{-s})$ is a generator. Since $\mathbb{C}[X, X^{-1}]^\times = \langle cX^m\ ;\ c \in \mathbb{C}, m \in \mathbb{Z}\rangle$. The generator $1/P_\pi(X)$ can be chosen to be of the form $A(X)/B(X)$ for some polynomials $A, B \in \mathbb{C}[X]$ relatively prime in $\mathbb{C}[X, X^{-1}]$. Since $1 \in I(\pi)$, there exist a polynomial $R(X) \in \mathbb{C}[X]$ such that $A(X)R(X)$ equals $B(X)$ up to a unit in $\mathbb{C}[X, X^{-1}]$. Since A, B are coprime, $A = 1$ and $P_\pi(X) \in \mathbb{C}[X]$.

To show last assertion in the proposition, we need to show that $I(\pi) = \mathbb{C}[q^{-s}, q^s]$. The inclusion is clear. To show $I(\pi) \supset \mathbb{C}[q^{-s}, q^s]$ we use the fact that $\pi_{\mathrm{Z}} = \mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$. Then for every $v \in V_\pi$, the function $W_v|_{\mathrm{T}}$ is compactly supported. Since the zeta integral is a finite sum of functions of the form

$$\begin{aligned} &\sum_{m \in \mathbb{Z}} q^{m(n-\frac{1}{2})}\Big(\int_{\mathfrak{p}^m - \mathfrak{p}^{m+1}} W_v(\epsilon_1(a))\ da\Big)\, q^{-ms} \\ = &\sum_{M \le m \le N} q^{m(n-\frac{1}{2})}\Big(\int_{\mathfrak{p}^m - \mathfrak{p}^{m+1}} W_v(\epsilon_1(a))\ da\Big)\, q^{-ms} \in \mathbb{C}[q^{-s}, q^s] \end{aligned}$$

for some $M, N \in \mathbb{Z}$, it must be in $\mathbb{C}[q^{-s}, q^{s}]$.

Remark 4.1.6. If the representation $\pi \in \mathrm{Rep}(\mathrm{G})$ is not supercuspidal, it is a subrepresentation of an prarabolically induced representation by an irreducible supercuspidal generic representation of a Levi. By Lemma 3.2.2, we can still work with the Whittaker functional on the parabolic induction. The discussion in this section works as well and the local factors can be defined for any generic representation in the same way. (See [**28**] for detail of the general case.)

Let us do an example with the unramified representations of G.

Example 4.1.7. Let $\chi = \prod_{i=1}^{n} |\cdot|^{s_i}$ be a character of $\mathrm{T} \simeq \mathbb{G}_m^n$ with $s_i \in \mathbb{C}$. Let π_χ be the unique irreducible generic subrepresentation of $\mathrm{I}_{\mathrm{U}}(\chi) = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi \delta_{\mathrm{B}}^{1/2}$. Assume π is the whole space $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi \delta_{\mathrm{B}}^{1/2}$. Let $y_i = q^{-s_i}$ and let $(y_1, y_2, ..., y_n) \in (\mathbb{C}^\times)^n$ be the Satake parameter of π in the Langlands dual group $\hat{\mathrm{T}}$ of T which gives a semisimple elment

$$t_\chi = \mathrm{diag}(y_1, y_2, ..., y_n, y_n^{-1}, ..., y_2^{-1}, y_1^{-1})$$

in the Langlands dual group $\mathrm{Sp}_{2n}(\mathbb{C})$ of G. Then it is expected that the standard L-function $L(\pi_\chi, std, s)$ of π is

$$\det(\mathrm{I} - t_\chi q^{-s})^{-1} = \sum_{m \geq 0} \mathrm{tr}\,\mathrm{Sym}^m(t_\chi) q^{-ms}.$$

Let χ_λ be the character of $\hat{\mathrm{T}}$ on the irreducible finite dimensional representation of $\mathrm{Sp}_{2n}(\mathbb{C})$ of highest weight $\lambda \in P^+ \subset \mathrm{X}^\bullet(\hat{\mathrm{T}})$. Denote by $W_\chi \in \mathscr{W}(\pi_\chi, \theta)$ the normalized (spherical) Whittaker function that is invariant under elements in $\mathrm{G}(\mathfrak{o})$ attached to a (spherical) function f_χ in $\mathrm{I}_{\mathrm{U}}(\chi)^{\mathrm{G}(\mathfrak{o})}$. This Whittaker function W_χ is determined by its value on T because of the Iwasawa decomposition $\mathrm{G} = \mathrm{U}\,\mathrm{T}\,\mathrm{G}(\mathfrak{o})$.

The Casselman-Shalika formula [**6**] shows that on T the function W_χ satisfies

$$W_\chi(\varpi^\lambda) = \delta_{\mathrm{B}}^{1/2}(\varpi^\lambda)\chi_\lambda(t_\chi)$$

for any co-character $\lambda \in P^+$ and 0 otherwise. Since $\mathrm{G}(\mathfrak{o})$ contains $\mathrm{Q}(\mathfrak{o})$, we can apply the simpler formula for $I(f_\chi, s)$. We get

$$\begin{aligned} I(f_\chi, s) &= \int_{k^\times} W_\chi(\epsilon_1(a))|a|^{s-(n-\frac{1}{2})}\, da = \mathrm{vol}(\mathfrak{o}^\times)\sum_{m\geq 0} q^{m(n-\frac{1}{2})} W_\chi(\varpi^{m\epsilon_1})q^{-ms} \\ &= \mathrm{vol}(\mathfrak{o}^\times)\sum_{m\geq 0}\chi_{m\epsilon_1}(t_\chi)q^{-ms}. \end{aligned}$$

The irreducible representation of $\mathrm{Sp}_{2n}(\mathbb{C})$ with highest weight ϵ_1 is the $2n$ dimensional standard representation and the one with highest weight $m\epsilon_1$ is its m-th symmetric power. The zeta integral $I(f_\chi, s)$ becomes $\mathrm{vol}(\mathfrak{o}^\times)\sum_{m\geq 0}\mathrm{tr}\,\mathrm{Sym}^m(t_\chi)q^{-ms}$ which is a scalar multiple of the standard L-function $L(\pi, std, s)$.

4.2. ε-factor and conductor

In this section we develop a functional equation for the zeta integrals.

It is clear that the linear form $I(v, s)$ depends only on $W_v|_{\mathrm{Q}}$ and hence only on $J_{\mathrm{Z}}(v)$. We are allowed to focus on the P_{n+1}-module $(V_\pi)_{\mathrm{Z}}$. Indeed, the linear form factor through the Jacquet module $(V_\pi)_{\mathrm{Y}'_{(1,n)},\theta_{(1)}}$. That is, it satisfies

$$I(\pi(y)v, s) = \theta_{(1)}(y)\, I(v, s), \quad \forall y \in \mathrm{Y}'_{(1,n)}\,. \tag{4.2.1}$$

The space of linear forms satisfying (4.2.1) turns out to be one dimensional for $s \in \mathbb{C}$ where it is defined. It leads to a functional equation for the zeta integrals $I(v, s)$.

Let ω_s denote the character $|\cdot|^s$ of $\mathrm{M}_1 \simeq k^\times$. The subgroup $\mathrm{M}_1\,\mathrm{X}'_{(1,n)}$ normalizes the character $\theta_{(1)}$ of $\mathrm{Y}_{(1,n)}$.

Lemma 4.2.1. $\mathrm{Hom}_{\mathrm{M}_1}(J_{\mathrm{Y}'_{(1,n)},\theta_{(1)}}(\pi)\otimes\omega_{s'}, \mathbb{C}) \simeq \mathbb{C}$.

4.2. ε-factor and conductor

This is a special case of [**28**, Theorem 8.2], which shows in general for $1 \leq r \leq n$ the space $\mathrm{Hom}_{\mathrm{M}_r}(J_{\mathrm{Y}'_{(r,n)},\theta_{(r)}}(\pi) \otimes \tau|\det|^{s'}, \mathbb{C})$ is one dimensional, where $\tau \in \mathrm{Rep}(\mathrm{M}_r)$ is irreducible generic and supercuspidal. If π is non supercuspidal, the lemma is still valid also arguing a case by case argument using the Schur's lemma and the Mackey's formula using the fact that π_{Z} is glued from the induced representations $(\Phi^+)^{k-1}\Psi^+(\pi^{(k)})$, $1 \leq k \leq n+1$.

Proof. Since π is generic supercuspidal, the P_{n+1}-module π_{Z} is $\mathrm{ind}_{\mathrm{N}_{n+1}}^{\mathrm{P}_{n+1}}\theta$. Note that $\mathrm{Norm}_{\mathrm{G}}(\mathrm{Y}'_{(1,n)}, \theta_{(1)}) = \mathrm{M}_1\,\mathrm{Y}'_{(1,n)}$ and $\delta_{\mathrm{M}_1\,\mathrm{Y}_{(1,n)}}|_{\mathrm{M}_1} = \omega_{n-1}^{-1}$ on M_1. By Frobenius reciprocity and Proposition 3.1.1 (iii), ($s' = s - \frac{1}{2}$)

$$\mathrm{Hom}_{\mathrm{M}_1}(J_{\mathrm{Y}'_{(1,n)},\theta_{(1)}}(\pi) \otimes \omega_{s'}, \mathbb{C}) \simeq \mathrm{Hom}_{\mathrm{P}_{n+1}}(\pi_{\mathrm{Z}}, \mathrm{I}_{\mathrm{Y}'_{(1,n)},\theta_{(1)}^{-1}}(\omega_{-s'}\omega_{n-1}^{1/2}))$$

$$\simeq \mathrm{Hom}_{\mathrm{P}_{n+1}}(\mathrm{ind}_{\mathrm{N}_{n+1}}^{\mathrm{P}_{n+1}}\theta \otimes i_{\mathrm{Y}'_{(1,n)},\theta_{(1)}}(\omega_{s'}\omega_{n-1}^{-1/2}), \mathbb{C}).$$

Applying Proposition 3.1.1 (iii) again, this space is equal to

$$\mathrm{Hom}_{\mathrm{P}_{n+1}}(i_{\mathrm{Y}'_{(1,n)},\theta_{(1)}}(\omega_{s'}\omega_{n-1}^{-1/2}), \delta_{\mathrm{P}_{n+1}}^{-1}\,\mathrm{Ind}_{\mathrm{N}_{n+1}}^{\mathrm{P}_{n+1}}\theta^{-1}) \simeq \mathrm{Hom}_{\mathrm{M}_1}(\omega_{s'}\omega_{n-1}^{1/2}, \omega_{-1}\,\mathrm{Ind}_{\mathrm{N}_1}^{\mathrm{M}_1}\theta|_{\mathrm{N}_1}^{-1})$$

$$= \mathrm{Hom}_{k^\times}(|\cdot|^{s+\frac{n}{2}}, C^\infty(k^\times)) = \mathbb{C}. \quad \square$$

Set an element

$$u_0 = \begin{bmatrix} & & & 1 \\ & -1 & & \\ & & \ddots & \\ & & & -1 & \\ 1 & & & \end{bmatrix} \in \mathrm{G}$$

which lifts an odd sign change Weyl element s_{ϵ_1} of G.

The element u_0 is in G, but not in Q. It stabilizes the group $\mathrm{Y}'_{(1,n)}$ and the character $\theta_{(1)}$. Let $\tilde{I}(v,s)$ be the linear form $I(u_0 v, 1-s)$ for $s \in \mathbb{C}$, $v \in V_\pi$. Then

$$\tilde{I}(\epsilon_1(a)v, s) = |a|^{-(s-\frac{1}{2})}\tilde{I}(v,s)$$

and factors through a linear form in $\mathrm{Hom}_{\mathrm{M}_1}(J_{\mathrm{Y}'_{(1,n)},\theta_{(1)}}(\pi)\otimes\omega_{s'},\mathbb{C})$ for all $s\in\mathbb{C}$ where it is defined. It has been shown in Lemma 4.2.1 that this vector space is one dimensional. Therefore for all $s\in\mathbb{C}$ with finite exceptions of values of q^{-s}, there exists a complex number $\gamma(\pi,s,\psi)$ independent of v such that the functional equation

$$I(u_0v,1-s)=\gamma(\pi,\psi,s)I(v,s)$$

holds. Since $I(v,s)\in\mathbb{C}(q^{-s})$ for all v, the function $\gamma(\pi,s,\psi)$ in s lies in $\mathbb{C}(q^{-s})$ and is called the *γ-factor* of π.

Notice that $I(v,s)/L(\pi,s)\in\mathbb{C}[q^{-s},q^{s}]$. Let us define the local invariants, the *conductor* a_π, and the *root number* ε_π, of the representation π.

Knowing $L(\pi,s)$ agrees with the standard L-function $L(\pi,std,s)$, these invariants agree with the Artin conductor and the root number of the corresponding Langlands parameter $(\rho,\mathrm{Sp}(\mathbb{M}),N)$ of π.

Theorem/Definition 4.2.2. *The* ε-factor *of π is the rational function $\varepsilon(\pi,s,\psi)$ in $X=q^{-s}$ satisfies the functional equation*

$$\frac{I(u_0v,1-s)}{L(\pi,1-s)}=\varepsilon(\pi,s,\psi)\frac{I(v,s)}{L(\pi,s)}. \tag{4.2.2}$$

It is a unit in $\mathbb{C}[q^{-s},q^{s}]$ and has the form

$$\varepsilon(\pi,s,\psi)=\varepsilon_\pi q^{-a_\pi(s-\frac{1}{2})}$$

for some number $\varepsilon_\pi\in\{\pm1\}$, the root number *of π, and some integer a_π, the* conductor *of π.*

Proof. Since the definition of $\varepsilon(\pi,s,\psi)$ does not depend on the choice of v in Equation (4.2.2), choose v_* such that $I(v_*,s)=L(\pi,s)$. Then

$$\varepsilon(\pi,s,\psi)=\frac{I(u_0v_*,1-s)}{L(\pi,1-s)}\in\mathbb{C}[q^{-s},q^{s}].$$

By applying the functional equation (4.2.2) twice, one sees $\varepsilon(\pi, 1-s, \psi)\varepsilon(\pi, s, \psi) = 1$. We conclude that the ε-factor $\varepsilon(\pi, s, \psi) \in \mathbb{C}[q^{-s}, q^{s}]^{\times} = \langle cq^{-ms} \,;\, c \in \mathbb{C}, m \in \mathbb{Z}\rangle$. Take $\varepsilon_\pi = \varepsilon(\pi, \frac{1}{2}, \psi)$, then for some integer $a_\pi \in \mathbb{Z}$, $\varepsilon(\pi, s, \psi) = \varepsilon_\pi q^{-a_\pi(s-\frac{1}{2})}$. Since $\varepsilon(\pi, \frac{1}{2}, \psi)^2 = 1$, the number $\varepsilon_\pi = \pm 1$. $\square$

We will show in Part 2 that the conductor a_π defined above must be nonnegative.

4.3. Rankin-Selberg convolutions for $\mathrm{SO}_{2n+1}(k) \times \mathrm{GL}_n(k)$

In this section, we review the construction of the Rankin-Selberg convolutions for $\mathrm{G} \times \mathrm{M}_r$ with $r = n$. Notations are as in Section 4.1. The group M_n equals to the Levi subgroup M of Q and P. We land at the simplest case with $\mathrm{Y}'_{(n,n)} = \mathrm{X}'_{(n,n)} = \mathrm{I}$, and $\theta_{(n)} = 1$. Write $s' = s - \frac{1}{2}$ for $s \in \mathbb{C}$. Any unramified character of M_n is of the form $\omega_{s'} \circ \det$ for some $s \in \mathbb{C}$.

For $\tau \in \mathrm{Rep}(\mathrm{M})$, set $\tau_s = \tau|\det|^{s-\frac{1}{2}}$ as an unramified twist of τ for $s \in \mathbb{C}$. Consider the normalized induction

$$\rho_{\tau,s} = \mathrm{I}_{\mathrm{H},\mathrm{M}}(\tau_s) \in \mathrm{Rep}(\mathrm{H})$$

for $s \in \mathbb{C}$. Then $(\rho_{\tau,s}, V_{\rho_{\tau,s}})$ is irreducible for all but a finite set of values of q^{-s}. Assume $\rho_{\tau,s}$ is irreducible. Note that $\delta_{\mathrm{P}_{n+1}}|_{\mathrm{M}_n} = |\det|$ and $\delta_{\mathrm{P}}|_{\mathrm{M}_n} = |\det|^{n-1}$. Using the theory of mirabolic group P_{n+1} in §2.3, the space of H-invariant bilinear forms $\mathrm{Hom}_{\mathrm{H}}(\pi|_{\mathrm{H}} \otimes \rho_{\tau,s}, \mathbb{C})$ is canonically isomorphic to

$$\mathrm{Hom}_{\mathrm{M}_n}(\pi_{\mathrm{Z}} \otimes \tau_s|\det|^{-\frac{n-1}{2}}, \mathbb{C}) \simeq \mathrm{Hom}_{\mathrm{P}_{n+1}}(\mathrm{ind}_{\mathrm{N}_{n+1}}^{\mathrm{P}_{n+1}} \theta \otimes \Psi^{+}(\tau_{s-\frac{n}{2}}), \mathbb{C})$$

$$\simeq \mathrm{Hom}_{\mathrm{P}_{n+1}}(\Psi^{+}(\tau_{s-\frac{n}{2}}), \delta_{\mathrm{P}_{n+1}}^{-1} \mathrm{Ind}_{\mathrm{N}_{n+1}}^{\mathrm{P}_{n+1}} \theta^{-1}) \simeq \mathrm{Hom}_{\mathrm{M}_n}(\tau_s|\det|^{-\frac{n-1}{2}}, \mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}_n} \theta|_{\mathrm{N}_n}^{-1})$$

$$\simeq \mathrm{Hom}_{\mathrm{N}_n}(\tau|_{\mathrm{N}_n}, \theta|_{\mathrm{N}_n}^{-1}),$$

which is one dimensional if τ is $\theta|_{\mathrm{N}_n}^{-1}$-generic, and is zero otherwise.

On the other hand, by Frobenius reciprocity this space of H-invariant bilinear forms is canonically isomorphic to the space of H-embeddings

$$\rho_{\tau,s} \hookrightarrow \mathrm{I}_{\mathrm{H},\mathrm{M}} \operatorname{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta|_{\mathrm{N}_n}^{-1} = \operatorname{Ind}_{\mathrm{V}}^{\mathrm{H}} \theta^{-1}$$

which by dimension one gives a unique realization of $\rho_{\tau,s}$ in the space of functions $f(h,s) \in \operatorname{Ind}_{\mathrm{V}}^{\mathrm{H}} \theta^{-1}$ such that $f(nzh,s) = \theta(n)^{-1} f(h,s)$ for $n \in \mathrm{N}_n, z \in \mathrm{Z}, h \in \mathrm{H}$. One should be aware that $\theta|_{\mathrm{V}}$ is not a generic character of the maximal unipotent subgroup V of H.

By abusing the notation, let us also denote by θ the character $\theta|_{\mathrm{N}_n}$ of N_n when the content is clear. Assume $(\tau, W_\tau) \in \mathrm{Rep}(\mathrm{M})$ is irreducible θ^{-1}-generic. Let $\ell_{\overline{\theta}}$ be a θ^{-1}-Whittaker functional on the space W_τ of τ, and $\mathscr{W}(\tau_s, \theta^{-1})$ be the Whittaker model of τ_s. The map

$$V_{\rho_{\tau,s}} \hookrightarrow \operatorname{Ind}_{\mathrm{V}}^{\mathrm{H}} \theta^{-1}, \quad \xi(h,s) \mapsto f_\xi(h,s) = \ell_{\overline{\theta}}(\xi(h,s))$$

gives the unique realization of $\rho_{\tau,s}$ in the space $\operatorname{Ind}_{\mathrm{V}}^{\mathrm{H}} \theta^{-1}$ into $\mathrm{I}_{\mathrm{H},\mathrm{M}} \mathscr{W}(\tau_s, \theta^{-1})$. For $\xi \in V_{\rho_{\tau,s}}$, the function f_ξ satisfies

$$f_\xi(nmzh, s) = \theta(n)^{-1} \ell_{\overline{\theta}}(\tau_s(m)\xi(h,s)),$$

for $n \in \mathrm{N}_n, m \in \mathrm{M}, z \in \mathrm{Z}, h \in \mathrm{H}$. We warm that f_ξ is not a Whittaker function attached to ξ since $\theta^{-1}|_{\mathrm{V}}$ is a not a generic character of V.

Theorem/Definition 4.3.1. *For $v \in V_\pi$, $\xi \in V_{\rho_{\tau,s}}$, the* zeta integral *attached to $v \otimes \xi$ is a complex-valued function*

$$\zeta(v \otimes \xi, s) = \int_{\mathrm{V}\backslash\mathrm{H}} W_v(h) f_\xi(h,s)\, dh.$$

It defines a H*-invariant bilinear form in* $\mathrm{Hom}_{\mathrm{H}}(\pi \otimes \rho_{\tau,s}, \mathbb{C})$ *for all but a finite set of values of q^{-s}, which is unique up to a scaling.*

Again since the representations $\pi|_{\mathrm{H}}$ and $\rho_{\tau,s}$ of H are smooth, and by Cartan decomposition $\mathrm{V}\,\mathrm{T}\setminus\mathrm{H}$ is compact, we get for $v \otimes \xi \in V_\pi \otimes V_{\rho_{\tau,s}}$ the zeta integral $\zeta(v \otimes \xi, s)$ is a finite sum of functions in q^{-s} of the form

$$\int_{\mathrm{T}} W_{v'}(t) f_{\xi'}(t,s)\ dt$$

for some v', ξ'. $W_v|_{\mathrm{T}}$ has compact support on T and the function $f_{\xi'}$ on T agrees with the Whittaker function $W_{\xi'(1,s)}$ attached to $\xi'(1,s) \in W_\tau$ restricted to T. Since the Whittaker functions on T is slowly increasing in q^{-s}. We again conclude

Proposition 4.3.2. *The zeta integrals converge absolutely to a rational function in q^{-s} for $\Re(s) \gg 0$ and admit meromorphic continuations to the whole complex plane.*

Let w_{H}, and w_{M} be lifts of the longest Weyl elements of H and M in H_{x_0} respectively, such that ${}^{w_{\mathrm{H}}}\mathrm{V} \cap \mathrm{V} = \mathrm{I}$ and ${}^{w_{\mathrm{M}}}\mathrm{N}_n \cap \mathrm{N}_n = 1$. (Notice that ${}^{w_{\mathrm{G}}}g = {}^*g = g^{-1}$, ${}^{w_{\mathrm{M}}}m = {}^t m$.)

Set $w_{\mathrm{P}} = w_{\mathrm{M}} w_{\mathrm{H}}$ and $\omega = w_{\mathrm{G}}^{-1} w_{\mathrm{H}}$. Then the parabolic subgroup $\mathrm{P}^\omega = \mathrm{M}^\omega \ltimes \mathrm{Z}^\omega$ is associated to $\mathrm{P} = \mathrm{M} \ltimes \mathrm{Z}$ in H by w_{P} such that ${}^{w_{\mathrm{P}}}(\mathrm{M}^\omega \cap \mathrm{V}) = {}^{w_{\mathrm{M}}}\overline{\mathrm{N}_n} \subset \mathrm{V}$ and $(\mathrm{M} \cap \mathrm{V})^{w_{\mathrm{P}}} = \overline{\mathrm{N}_n}^{w_{\mathrm{H}}} \subset \mathrm{V}$. Conjugating by the element ω defines an outer automorphism of H which preserving V. Set $\omega_0 = w_{\mathrm{P}} \omega^{-1} = w_{\mathrm{M}} w_{\mathrm{G}}$, which lifts the Weyl element $s_{\epsilon_1} \cdots s_{\epsilon_n}$ of G in G_{x_0} and set $\omega_m = \varpi^{-m(\epsilon_1 + \cdots \epsilon_n)} \omega_m$ which lifts it in K_{x_m}.

For $\xi \in \mathrm{I}_{\mathrm{M},\mathrm{H}}\, \tau_s$, the function

$$(A(w_{\mathrm{P}}^{-1}, s)\xi)(h) = \int_{\mathrm{Z}} \xi(\omega_0 z h \omega)\ dz$$

satisfies the property that

$$(A(w_{\mathrm{P}}^{-1}, s)\xi)(mzh) = \delta_{\mathrm{P}}(m) |\det(m)|^{-1/2} \tau_s({}^t m^{-1}) (A(w_{\mathrm{P}}^{-1}, s)\xi)(h)$$

$$= \delta_{\mathrm{P}}^{1/2}(m) \tilde{\tau}_{1-s}(m) (A(w_{\mathrm{P}}^{-1}, s)\xi)(h), \quad m \in \mathrm{M}, z \in \mathrm{Z}, h \in \mathrm{H}$$

and defines an intertwining operator $A(w_{\mathrm{P}}^{-1}, s) : \mathrm{I}_{\mathrm{H,M}}(\tau_s) \to \mathrm{I}_{\mathrm{H,M}}(\tilde{\tau}_{1-s})$. It induces an operator, also denoted by $A(w_{\mathrm{P}}, s)$,

$$A(w_{\mathrm{P}}^{-1}, s) : \mathrm{I}_{\mathrm{H,M}}\,\mathscr{W}(\tau_s, \theta^{-1}) \to \mathrm{I}_{\mathrm{H,M}}\,\mathscr{W}(\tilde{\tau}_{1-s}, \theta^{-1})$$

on subspaces of functions in $\mathrm{Ind}_{\mathrm{V}}^{\mathrm{H}}\,\theta^{-1}$ by

$$f_\xi \mapsto (A(w_{\mathrm{P}}^{-1}, s) f_\xi)(h) = \int_{\mathrm{Z}} f_\xi(d_{\mathrm{M}} \omega_0 z h \omega)\; dz, \quad d_{\mathrm{M}} \in \mathrm{T}, s.t.\ \alpha(d_{\mathrm{M}}) = -1\ \forall \alpha \in \Delta_{\mathrm{M}}.$$

(Note that d_{M} is to ensure ${}^{d_{\mathrm{M}}}\theta^{-1}(n^{-1}) = \theta^{-1}(n)$, for $n \in \mathrm{N}_n$.) The *normalized intertwining operator* is the operator

$$A^*(w_{\mathrm{P}}^{-1}, s) = \gamma(\tau, \wedge^2, 2s-1, \psi) A(w_{\mathrm{P}}^{-1}, s)$$

where $\gamma(\tau, \wedge^2, s, \psi) = \varepsilon(\tau, \wedge^2, s, \psi)\dfrac{L(\tilde{\tau}, \wedge^2, 1-s)}{L(\tau, \wedge^2, s)}$, the γ-factor associated to the exterior square L-function of τ, is the local coefficient of Shahidi such that $A^*(w_{\mathrm{P}}^{-1}, s)$ has no zero. ([**28**] [**29**])

Let us similarly consider the zeta integrals on $V_\pi \otimes V_{\rho_{\tilde{\tau},1-s}}$ for $\pi \times \tilde{\tau}$. Then for all but a finite set of values of q^{-s}, the bilinear form

$$\zeta(\omega v \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi, 1-s) = \int_{\mathrm{V}\backslash\mathrm{H}} W_v(h\omega)(A^*(w_{\mathrm{P}}^{-1}, s) f_\xi)(h, 1-s)\; dh,$$

for $v \in V_\pi, \xi \in V_{\rho_{\tau,s}}$, is again H-invariant and defines an element in the one dimensional vector space $\mathrm{Hom}_{\mathrm{H}}(\pi \otimes \rho_{\tau,s}, \mathbb{C})$. By uniqueness, it is a scalar multiple of $\zeta(v \otimes \xi, s)$ on which s it is defined.

Theorem/Definition 4.3.3. *For all but a finite set of values of q^{-s}, there is a number $\gamma(\pi \times \tau, s, \psi)$, independent of v and ξ, such that for $v \in V_\pi$, $\xi \in V_{\rho_{\tau,s}}$, the functional equation*

$$\zeta(\omega v \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi, 1-s) = \gamma(\pi \times \tau, s, \psi)\zeta(v \otimes \xi, s)$$

holds whenever it is defined. This is called the γ-factor associate with π and τ.

Remark 4.3.4. Soudry [**28**] [**29**] further showed this γ-factor is multiplicative in π and τ. It agrees with the gamma factor defined by the Langlands-Shahidi method [**26**]. The thus defined L-factor shall agree with the L-factor defined by the Langlands-Shahidi method and agree with the tensor product L-function of the Langlands parameter of $\pi \times \tau$ on the Galois side. We define this L-factor in the next section.

4.4. Rankin-Selberg L-function of $\pi \times \tau$

Consider $\rho_{\tau,s} = \mathrm{I}_{\mathrm{H,M}}\, \tau_s$ as space of sections with s a parameter, taking values on W_τ-valued functions $\xi(\cdot, s)$ on H, such that

$$\xi(mzh, s) = |\det m|^{s+\frac{n-2}{2}} \tau(m)\xi(h, s)$$

for $m \in \mathrm{M}, z \in \mathrm{Z}, h \in \mathrm{H}$. We say a section $\xi(h, s) \in \mathrm{I}_{\mathrm{H,M}}\, \tau_s$ is *standard* if it satisfies one of the following condition

i τ is unramified and $f_\xi(k, s) = L(\tau, \wedge^2, 2s)$ for all k in the hyperspecial open compact subgroup H_{x_m} of H for some m.

ii. The restriction of ξ to H_{x_m} is independent of s for some m.

iii. $f_\xi = A^*(w_\mathrm{P}, 1-s) f_{\xi'}$ for some ξ' satisfied condition (ii) in $\mathrm{I}_{\mathrm{H,M}}\, \widetilde{\tau}_{1-s}$.

Lemma 4.4.1. *The set of poles and zeros of the zeta integral* $\zeta(v\otimes\xi, s)$ *is independent of the choice of the generic character* θ *of* U.

Proof. Let θ' be another generic character of U. Since the orbit of generic character of U under adjoint action of T is unique, $\theta' = \theta^t$ for some $t \in \mathrm{T}$. The complex-valued function constructed via θ_t is

$$\begin{aligned}
\zeta_t(v \otimes \xi, s) &= \int_{\mathrm{V}\setminus\mathrm{H}} \ell_{\theta^t}(\pi(h)v)\ell_{\overline{\theta^t}}(\xi(h,s))\, dh \\
&= \int_{\mathrm{V}\setminus\mathrm{H}} \ell_{\theta}(\pi(t)\pi(h)v)\ell_{\overline{\theta}}(\tau_s(t)\xi(h,s))\, dh \\
&= \int_{\mathrm{V}\setminus\mathrm{H}} \ell_{\theta}(\pi(th)v)\ell_{\overline{\theta}}(\xi(th,s)|\det t|^{-(s+\frac{n-2}{2})})\, dh \\
&= |\det t|^{-(s+\frac{n-2}{2})} \int_{\mathrm{V}\setminus\mathrm{H}} \ell_{\theta}(\pi(h)v)\ell_{\overline{\theta}}(\xi(h,s))\, dh \\
&= |\det t|^{-(s+\frac{n-2}{2})}\, \zeta(v \otimes \xi, s).
\end{aligned}$$

Since $|\det t|^{-(s+\frac{n-2}{2})}$ is an entire function, the new zeta integral $\zeta_t(v \otimes \xi, s)$ has the same set of poles and zeros as original zeta integral $\zeta(v \otimes \xi, s)$. ☐

Proposition 4.4.2. *Define $I(\pi \times \tau) \subset \mathbb{C}(q^{-s})$ as the set*

$$I(\pi \times \tau) = \left\{ \zeta_t(v \otimes \xi, s) \mid v \in \pi,\ \xi\text{: standard section in } \mathrm{Ind}_{\mathrm{P}}^{\mathrm{H}} \tau_s,\ t \in \mathrm{T} \right\}.$$

Then $I(\pi \times \tau)$ contains $\mathbb{C}$, the constant function, and is a fractional ideal of $\mathbb{C}[q^{-s}, q^s]$.

Proof. Since $\pi_{\mathrm{Z}}|_{\mathrm{Q}} \simeq \mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$, there exists $v_* \in V_\pi^{\mathrm{Q}_{(m)}}$ for some $m \geq 0$ such that $W_{v_*}|_{\mathrm{Q}} \in \mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$ is supported on $\mathrm{V}\,\mathrm{Q}(\mathfrak{o})$ and $W_{v_*}(\mathrm{I}) = 1$. Choose $\xi_*(h,s) \in V_{\rho_{\tau,s}}$ such that it supports on $\mathrm{P}\,K \cap \mathrm{H}$ with $K \subset \mathrm{Q}_{(m)}$ an open compact subgroup of G small enough such that ξ_* is fixed by K and $f_{\xi_*}(1,s) = 1$. The choice of K can be chosen to be independent of s since τ_s is a twist of τ by a unramified character for all s. Therefore $\xi_*(h,s)$ is a standard section and $\zeta(v_* \otimes \xi_*, s) \in I(\pi \times \tau)$. The zeta integral becomes

$$\begin{aligned}
\zeta(v_* \otimes \xi_*, s) &= \int_{\mathrm{V}\setminus\mathrm{P}\,K\cap\mathrm{H}} W_{v_*}(h) f_{\xi_*}(h,s)\, dh \\
&= \int_{\mathrm{V}\setminus\mathrm{P}} W_{v_*}(p) f_{\xi_*}(p,s)\, dp \;=\; W_{v_*}(\mathrm{I}) f_{\xi_*}(\mathrm{I},s) \;=\; 1
\end{aligned}$$

a constant function in q^{-s} with suitable choices of Haar measures on H and P.

We have seen from the proof of Lemma 4.4.1 that

$$\zeta_t(v \otimes \xi, s) = |\det t|^{-(s+\frac{n-2}{2})}\, \zeta(v \otimes \xi, s).$$

Take $t = \varpi^{\epsilon_1}$, then $q^{\pm s} I(\pi \times \tau) \subset I(\pi \times \tau)$. The set is then a $\mathbb{C}$-algebra contained in the fraction field of $\mathbb{C}[q^{-s}, q^s]$ containing $\mathbb{C}$ and closed under multiplying by $q^{\pm s}$. The assertion follows. □

We do an example with π being supercuspidal. Recall that such π has the property that $\pi_{\mathrm{Z}}|_{\mathrm{Q}} \simeq \mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\, \theta$ as a representation of Q.

Example 4.4.3. Assume π is irreducible, generic and supercuspidal. Recall that the zeta integrals are in the space of bilinear forms $\mathrm{Hom}_{\mathrm{H}}(\pi|_{\mathrm{H}} \otimes \rho_{\tau,s}, \mathbb{C})$, which is isomorphic to $\mathrm{Hom}_{\mathrm{N}_n}(\tau|_{\mathrm{N}_n}, \theta^{-1})$. The space $\mathrm{Hom}_{\mathrm{N}_n}(\tau|_{\mathrm{N}_n}, \theta^{-1})$ is nonzero for all s. Hence the zeta integrals are indeed well-defined for all $s \in \mathbb{C}$ and hence are entire functions. In particular, for $v \otimes \xi \in V_\pi \otimes V_{\rho_{\tau,s}}$, the Laurent series $\zeta(v \otimes \xi, s)$ in determinant $X = q^{-s}$ is in $\mathbb{C}[X, X^{-1}]$.

Another way to look at this is that the zeta integral is a finite sum of functions of the form $\int_{\mathrm{T}} W_{v'}(t) W_{\xi'(\mathrm{I})}(t) |\det t|^{s-\frac{n}{2}}\, dt$ while $W_{v'}|_{\mathrm{T}}$ is of compact support. Hence such function is a finite sum of the form $c_i W_{v_i}(\varpi^{a_i}) W_{\xi_i(\mathrm{I})}(\varpi^{a_i}) q^{b_i s}$ for some $v_i \otimes \xi_i \in V_\pi \otimes V_{\rho_{\tau,s}}$, $a_i, b_i \in \mathbb{Z}$ and $c_i \in \mathbb{C}$. Therefore, the zeta integral must sits in $\mathbb{C}[q^{-s}, q^s]$.

We are ready to define the L-factor of $\pi \times \tau$ for $\mathrm{G} \times \mathrm{M}$ as the g.c.d of the set $I(\pi \times \tau)$, which can be normalized to be $1/P(q^{-s})$ for some polynomial $P(X) \in \mathbb{C}[X]$.

Definition 4.4.4. The *L-factor $L(\pi \times \tau, s)$ associate with π and τ* is defined as the generator of the fractional ideal $I(\pi \times \tau)$ of $\mathbb{C}[q^{-s}, q^s]$ such that

$$L(\pi \times \tau, s) = \frac{1}{P_{\pi \times \tau}(q^{-s})}, \quad P_{\pi \times \tau}(X) \in \mathbb{C}[X], \text{ and } P_{\pi \times \tau}(0) = 1.$$

In particular, when π is irreducible generic and supercuspidal, $L(\pi \times \tau, s) = 1$.

By the definition of the $L(\pi\times\tau, s)$ and Lemma 4.4.1, there exist $v_i \otimes \xi_i \in V_\pi \otimes V_{\rho_{\tau,s}}$ and some $a_i \in \mathbb{Z}$, $i = 1, 2, ..., d$ such that

$$L(\pi \times \tau, s) = \sum_{i=1}^{d} q^{a_i s} \zeta(v_i \otimes \xi_i, s) \in \mathbb{C}(q^{-s}).$$

Moreover, since $L(\pi \times \tau, s)$ is the generator of the fractional ideal $I(\pi \times \tau)$, for all $\zeta(v \otimes \xi, s)$ we shall have $\dfrac{\zeta(v \otimes \xi, s)}{L(\pi \times \tau, s)} \in \mathbb{C}[q^{-s}, q^s]$. Then we have following.

Theorem/Definition 4.4.5. *The* ε-factor $\varepsilon(\pi \times \tau, s)$ associate with π and τ *is an entire function*

$$\varepsilon(\pi \times \tau, s, \psi) = \gamma(\pi \times \tau, s, \psi) \frac{L(\pi \times \tau, s)}{L(\pi \times \tilde{\tau}, 1 - s)}$$

satisfies $\varepsilon(\pi \times \tilde{\tau}, 1 - s, \psi)\varepsilon(\pi \times \tau, s, \psi) = 1$. *We have the functional equation*

$$\frac{\zeta(\omega v \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi, 1 - s)}{L(\pi \times \tilde{\tau}, 1 - s)} = \varepsilon(\pi \times \tau, s, \psi) \frac{\zeta(v \otimes \xi, s)}{L(\pi \times \tau, s)}$$

and the ε*-factor* $\varepsilon(\pi \times \tau, s)$ *is a unit in* $\mathbb{C}[q^{-s}, q^s]^\times$.

Proof. For simplicity, we will only prove the case when π is irreducible generic and supercuspidal. Notice that it implies $I(\pi \times \tau, s) = I(\pi \times \tilde{\tau}) = \mathbb{C}[q^{-s}, q^s]$ and $L(\pi \times \tau, s) = L(\pi \times \tilde{\tau}, s) = 1$. For $\xi \in V_{\rho_{\tau,s}}$, there is an open compact subset K' of H such that ξ is supported on $\mathrm{P}\,K'$ and $\overline{\mathrm{Z}} \cap \mathrm{P}\,K' \subset K'$. Since $\mathrm{H} = (\overline{\mathrm{Z}}\,\mathrm{M}\,\mathrm{Z})K'$ and commutators of $\overline{\mathrm{Z}}$ and Z are in M, the function

$$(A(w_{\mathrm{P}}, 1 - s)A(w_{\mathrm{P}}^{-1}, s)\xi)(h) = \int_{\mathrm{Z}} \int_{\mathrm{Z}} \xi(\omega_0 z_1 \omega_0 z_2 h)\ dz_1\, dz_2 = \int_{\mathrm{Z}} \int_{\overline{\mathrm{Z}}} \xi(\overline{z_1} z_2 h)\ d\overline{z_1}\, dz_2$$

gotten by applying intertwining operator twice is supported on $\mathrm{P}\,K'$ as well. Take $v_* \otimes \xi_*$ as defined in the proof of Proposition 4.4.2, then

$$\zeta(v_* \otimes \xi_*, s) = \zeta(v_* \otimes A(w_{\mathrm{P}}, 1 - s)A(w_{\mathrm{P}}^{-1}, s)\xi_*, s) = 1,$$

up to a normalization of the Haar measure on Z. Since $A^*(w_{\mathrm{P}}^{-1}, s)\xi_*$ is a standard section and $\gamma(\tilde{\tau}, \wedge^2, 2-2s, \psi)\gamma(\tau, \wedge^2, 2s-1, \psi) = 1$. We show that

$$\varepsilon(\pi \times \tau, s, \psi) = \zeta(\omega v_* \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi_*, 1-s) \in \mathbb{C}[q^{-s}, q^s]$$

and by applying the functional equation twice that

$$\zeta(v_* \otimes A(w_{\mathrm{P}}, 1-s)A(w_{\mathrm{P}}^{-1}, s)\xi_*, s) = \varepsilon(\pi \times \tilde{\tau}, 1-s, \psi)\varepsilon(\pi \times \tau, s, \psi)\zeta(v_* \otimes \xi_*, s).$$

It follows that $\varepsilon(\pi \times \tilde{\tau}, 1-s, \psi)\varepsilon(\pi \times \tau, s, \psi) = 1$ and $\varepsilon(\pi \times \tau, s, \psi) \in \mathbb{C}[q^{-s}, q^s]^\times$. $\square$

When π is supercuspidal, the L-function is trivial and the ε-factor equals to the γ-factor. We quote the main theorem [**29**, Theorem 3] of Soudry in his work, *Full multiplicativity of gamma factors for* $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$, to end this chapter.

Theorem 4.4.6 (Soudry [**29**])**.** *The γ-factor $\gamma(\pi \times \tau, s, \psi)$ attached to π and τ is multiplicative in both the first and the second factor.*

CHAPTER 5

The Fourier transform $\Psi(v, X; X_1, X_2, ..., X_n)$

The notation of this chapter follows those in Chapter 4 as well as in Chapter 2 and Chapter 3. A generic data $(\mathrm{B}, \mathrm{T}, \theta)$ of G is fixed, and (π, V_π) shall be an irreducible θ-generic supercuspidal representation of G. The restriction of θ to the maximal unipotent subgroups V and N_n of H and M respectively is still denoted by θ. Notice that $\theta|_{\mathrm{V}}$ is not a generic character of V but $\theta|_{\mathrm{N}_n}$ is a generic character of N_n. Fix a Whittaker functional ℓ_θ on V_π and hence an embedding, $v \mapsto W_v$, of V_π to the realization, the Whittaker model $\mathscr{W}(\pi, \theta)$, of π in the space $\mathrm{Ind}_{\mathrm{V}}^{\mathrm{G}} \theta$ of Whittaker functions.

The k-split torus $\mathrm{T} \simeq \mathbb{G}_m^n$ has complex dual group $\hat{\mathrm{T}}$ a complex torus of rank n contained in the complex dual group $\hat{\mathrm{G}} \simeq \mathrm{Sp}_{2n}(\mathbb{C})$ of G. The action of the Weyl group W_{M} (resp. W_{H}, W_{G}) on $\hat{\mathrm{T}}$ is induced from its action on $\mathrm{X}_\bullet(\mathrm{T}) = \mathrm{X}^\bullet(\hat{\mathrm{T}})$. Its coordinate ring $\mathbb{C}[\hat{\mathrm{T}}]$ is the $\mathbb{C}$-algebra of the group $\mathrm{X}^\bullet(\hat{\mathrm{T}}) = \mathrm{X}_\bullet(\mathrm{T})$ which is identified to $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, ..., X_n, X_n^{-1}]$ by $\epsilon_i \mapsto X_i$ and $W_{\mathrm{M}} \simeq S_n$ acts on by permuting the subindices of X_i's. Notice that the group algebra $\mathbb{C}[\mathrm{X}_\bullet(\mathrm{T})]$ are the complex-valued functions on $\mathrm{T} / \mathrm{T}(\mathfrak{o})$ with finite support which is the set $\mathcal{H}(\mathrm{T}, \mathrm{T}(\mathfrak{o}))$. Let $i \geq 0$ be an integer. We recall we have Satake transforms from spherical Hecke algebras $\mathcal{H}(\mathrm{M}, \mathrm{M}(\mathfrak{o}))$, $\mathcal{H}(\mathrm{H}, \mathrm{H}_{x_i})$ and $\mathcal{H}(\mathrm{G}, \mathrm{G}(\mathfrak{o}))$ to $\mathbb{C}[\hat{T}]$ onto the invariants of the Weyl groups of M, H, and G respectively. Denote by ς_{M}, $\varsigma_{\mathrm{H},i}$ and ς_{G} the inverse of the Satake isomorphisms of M, H and G respectively.

Notation 5.0.7. The coordinate of a complex dual torus element $\underline{x}$ is the n-tuple $(x_1, x_2, ..., x_n)$ with $x_i = \epsilon_i(\underline{x})$. Under this notation, $\underline{x}$ is the diagonal element

$\mathrm{diag}(x_1, x_2, ..., x_n, x_1^{-1}, x_2^{-1}, ..., x_n^{-1}) \in \mathrm{Sp}_{2n}(\mathbb{C})$. However, in this thesis $q^{-s}\underline{x}$ represents scalar multiplication by q^{-s} in M, that is, multiplying q^{-s} on each of the coordinates of $\underline{x}$. This convention does matter when one wants to deal with the trace of $\underline{x}$ acting on a finite dimensional representation of each of the dual groups $\hat{\mathrm{T}} \subset \hat{\mathrm{M}} \subset \hat{\mathrm{G}}$. Let us denote by p the map $\underline{x} \mapsto p(\underline{x}) = \mathrm{diag}(x_1, x_2, ..., x_n) \in \mathrm{GL}_n(\mathbb{C})$.

5.1. Spherical Whittaker functions on $\mathrm{GL}_n(k)$

Assume (τ, W_τ) is an irreducible generic unramified representation of M. Let K_n be the hyperspecial maximal open compact subgroup $\mathrm{M}(\mathfrak{o})$ of M. Then τ admits a nonzero vector fixed by K_n, a *spherical vector*, and a nonzero Whittaker functional $\ell_{\mathrm{M}} \in \mathrm{Hom}_{\mathrm{N}_n}(\tau|_{\mathrm{N}_n}, \theta^{-1})$ with a unique Whitaker model $\mathscr{W}(\tau, \theta^{-1})$ in $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta^{-1}$. On the other hand, the spherical vectors, meaning K_n-invariants, in $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}_n} \theta^{-1}$ collects spherical Whittaker functions with respect to θ^{-1} of all irreducible generic unramified representations of M.

Let us consider the space $(\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta^{-1})^{\mathrm{K}_n}$ as a $\mathcal{H}(\mathrm{M}, \mathrm{K}_n)$-module. Since $\mathcal{H}(\mathrm{M}, \mathrm{K}_n) \simeq \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$ is commutative, it decomposes any $\mathcal{H}(\mathrm{M}, \mathrm{K}_n)$-module into eigenspaces. Each eigenvalue is a linear form on $\mathbb{C}[\hat{\mathrm{T}}//W_{\mathrm{M}}]$ respecting the ring structures. An eigenvalue is hence the the evaluation map at a point $\underline{x}$, called the *Satake parameter*, on the complex variety $\hat{\mathrm{T}}//W_{\mathrm{M}}$ composing the Satake isomorphism. To be more explicit, suppose $\mathcal{W}_{\underline{x}} \in (\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta^{-1})^{\mathrm{K}_n}$ is the an eigenvector of $\mathcal{H}(\mathrm{M}, \mathrm{K}_n)$ with Satake parameter $\underline{x} \in \hat{\mathrm{T}}$ normalized such that $\mathcal{W}_{\underline{x}}(\mathrm{I}) = 1$, then for $P \in \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$ one has

$$\varsigma_{\mathrm{M}}(P)\mathcal{W}_{\underline{x}}(m) = \int_{\mathrm{M}} \varsigma_{\mathrm{M}}(P)(m')\mathcal{W}_{\underline{x}}(mm')\, dm' = P(\underline{x})\mathcal{W}_{\underline{x}}(m).$$

The smooth $\mathcal{H}(\mathrm{M})$-module generated by $\mathcal{W}_{\underline{x}}$ is simple and gives an irreducible unramified smooth representation $\tau_{\underline{x}}$ of M with Satake parameter $\underline{x}$. By uniqueness of the Whittaker model, $\mathcal{W}_{\underline{x}}$ is uniquely determined.

5.1. Spherical Whittaker functions on $\mathrm{GL}_n(k)$

Casselman and Shalika [**5**, Proposition 2.6] showed any irreducible unramified representation can be embedded into the unramified principal series $\mathrm{I}_{\mathrm{M,T}}\,\chi$ for some unramified character χ of T. In particular, $\tau_{\underline{x}}$ is isomorphic to the principal series $\mathrm{I}_{\mathrm{M,T}}\,\chi_{\underline{x}}$ with $\chi_{\underline{x}}$ the unramified character such that $\chi_{\underline{x}}(\varpi^\lambda) = \lambda(\underline{x})$ for all $\lambda \in \mathrm{X}_\bullet(\mathrm{T})$. This can be check easily since $\mathrm{Hom}_{\mathrm{M}}(\tau_{\underline{x}}, \mathrm{I}_{\mathrm{M,T}}\,\chi) \neq 0$ by Frobenius reciprocity if and only if the space $\mathrm{Hom}_{\mathrm{T}}(\tau_{\underline{x}}|_{\mathrm{T}}, \delta_{\mathrm{B_M}}^{1/2}\chi)$ is nonzero. Hence we may take $\chi = \chi_{\underline{x}}$ or any of its W_{M}-orbits. Conversely, Jacquet and Shalika [**15**] showed that for any $\underline{x} \in \hat{\mathrm{T}}$ the representation $\mathrm{I}_{\mathrm{M,T}}\,\chi_{\underline{x}}$ can be embedded into the space of Whittaker functions $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta^{-1}$. Hence all $\underline{x}$ can appear as an eigenvalue.

By Casselman-Shalika's formula [**6**], for each $\underline{x} \in \hat{\mathrm{T}}$ the unique eigenvector $\mathcal{W}_{\underline{x}} \in (\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta^{-1})^{\mathrm{K}_n}$ has the formula: if $m = n\varpi^\lambda k$, $n \in \mathrm{N}_n$, $\lambda \in \mathrm{X}_\bullet(\mathrm{T})$, $k \in \mathrm{K}_n$,

$$\mathcal{W}_{\underline{x}}(m) = \theta^{-1}(n) q^{-\langle\lambda,\rho_{\mathrm{M}}\rangle}\chi_\lambda^{\mathrm{M}}(\underline{x}),\ \text{if } \lambda \in \mathrm{P}_{\mathrm{M}}^+; = 0, \text{ if otherwise.} \tag{5.1.1}$$

Here $\chi_\lambda^{\mathrm{M}}$ is the Weyl character which equals to the trace of the irreducible representation of the complex dual group $\hat{\mathrm{M}}$ with highest weight λ, P_{M}^+ is the fundamental Weyl chamber of M and $\rho_{\mathrm{M}} \in \mathrm{X}^\bullet(\mathrm{T})$ is half of the sum of positive roots in Φ_{M}^+.

It is known that $\chi_\lambda^{\mathrm{M}}$ agrees with the degree n Schur polynomial with indetermininat $\epsilon_1, \epsilon_2, ..., \epsilon_n$. Then for each given $m \in \mathrm{M}$, there exists $\mathcal{W}(m) \in \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$ such that $\mathcal{W}_{\underline{x}}(m)$ is a specialization.

Proposition 5.1.1. *Define* $\mathcal{W}$ *as a function on* $\mathrm{M}\times\hat{\mathrm{T}}$ *satisfying* $\forall n \in \mathrm{N}_n, k \in \mathrm{K}_n$*,*

$$\mathcal{W}(n\varpi^\lambda k;\cdot) = \theta^{-1}(n)q^{-\langle\lambda,\rho_{\mathrm{M}}\rangle}\chi_\lambda^{\mathrm{M}} \quad \text{in } \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}, \quad \forall\lambda \in P_{\mathrm{M}}^+,$$

with the first factor supported on $\bigsqcup_{\lambda\in P_{\mathrm{M}}^+} \mathrm{N}_n\,\varpi^\lambda\,\mathrm{K}_n$*. It has the properties*

$$\mathcal{W}(d_{\mathrm{M}}w_{\mathrm{M}}\,{}^t m^{-1};\underline{x}) = \mathcal{W}(m;\underline{x}^{-1}), \quad \mathcal{W}(m;q^{-s}\underline{x}) = \mathcal{W}(m;\underline{x})|\det m|^s, \quad \forall m \in \mathrm{M}\,.$$

Proof. Let us show the first property. We note that w_{M} is a lift of the longest Weyl element in K_n whose action on the root system Φ^+_{M} reverses the polarization Φ^+_{M}, *outer* : $m \mapsto {}^t m^{-1}$ is an outer automorphism whose induced action on Φ_{M} switches Φ^+_{M} and Φ^-_{M} and acts as (-1) on $\mathrm{X}_\bullet(\mathrm{T}) = \mathrm{X}^\bullet(\hat{\mathrm{T}})$, and $d_{\mathrm{M}} \in \mathrm{T} \cap \mathrm{K}_n$ is a torus element such that $\overline{{}^{d_{\mathrm{M}}}\theta|^{-1}_{\mathrm{N}_n}} = \theta|^{-1}_{\mathrm{N}_n}$. The operator $\mathrm{Ad}(w_{\mathrm{M}}) \circ \mathit{outer}$ then preserves N and P^+_{M}. We get for $m = n\varpi^\lambda k$, $n \in \mathrm{N}_n, \lambda \in P^+_{\mathrm{M}}, k \in \mathrm{K}_n$,

$$\begin{aligned} \mathcal{W}(d_{\mathrm{M}} w_{\mathrm{M}} {}^t m^{-1}; \underline{x}) &= \theta^{-1}(n)\mathcal{W}(\varpi^{w_{\mathrm{M}}(-\lambda)}; \underline{x}) \\ &= \theta^{-1}(n) q^{-\langle -\lambda, -\rho_{\mathrm{M}}\rangle} \chi^{\mathrm{M}}_\lambda({}^{w_{\mathrm{M}}}\underline{x}) = \mathcal{W}(m; \underline{x}^{-1}). \end{aligned}$$

The second equality is because $w_{\mathrm{M}}(\rho_{\mathrm{M}}) = -\rho_{\mathrm{M}}$ and $\langle\ ,\ \rangle$ and Weyl character are invariant under action of Weyl elements.

To see the second property, we use the Weyl character formula: for a regular semisimple element $t \in \mathrm{T}$,

$$\chi^{\mathrm{M}}_\lambda(t) = \frac{\sum_{s \in W_{\mathrm{M}}} \mathrm{sign}(s) t^{s(\lambda+\rho_{\mathrm{M}})}}{\sum_{s \in W_{\mathrm{M}}} \mathrm{sign}(s) t^{s(\rho_{\mathrm{M}})}}$$

where $\rho_{\mathrm{M}} = \frac{1}{2}\sum_{\lambda \in \check{\Phi}^+_{\mathrm{M}}} \lambda$ and $t^\lambda = \lambda(t)$ for $t \in \hat{\mathrm{T}}$. Denote by $deg(\lambda)$ the degree map on the free $\mathbb{Z}$-module $\mathrm{X}_\bullet(\mathrm{T})$ with respect to the basis $\epsilon_1, \epsilon_2, ..., \epsilon_n$. One sees $\det \varpi^\lambda = \varpi^{deg\lambda}$. Since W_{M} acts by permuting ϵ_i's, it preserves the degree map on $\mathrm{X}_\bullet(\mathrm{T})$. We then get

$$\chi^{\mathrm{M}}_\lambda(q^{-s}\underline{x}) = q^{(deg\lambda)s}\chi^{\mathrm{M}}_\lambda(\underline{x}) = |\det \varpi^\lambda|^s \chi^{\mathrm{M}}_\lambda(\underline{x}).$$

The assertion follows easily by applying the formula. □

Corollary 5.1.2. *Let $\tau_{\underline{x}}$ denote the unique irreducible unramified subrepresentation of $\mathrm{Ind}^{\mathrm{M}}_{\mathrm{N}_n} \theta^{-1}$ with Satake parameter $\underline{x} \in \hat{\mathrm{T}}$. Then $L_{d_{\mathrm{M}}} \widetilde{\tau_{\underline{x}}} = \tau_{\underline{x}^{-1}}$ and $(\tau_{\underline{x}})_s = \tau_{q^{-s}\underline{x}}$.*

Here $L_{d_{\mathrm{M}}}$ denotes the left translation by d_{M} which intertwines $\mathrm{Ind}^{\mathrm{M}}_{\mathrm{N}_n} \theta$ and $\mathrm{Ind}^{\mathrm{M}}_{\mathrm{N}_n} \theta^{-1}$.

There is one more interesting property of the function $\mathcal{W}$. One notices that the Weyl invariants $\mathrm{X}_\bullet(\mathrm{T})^{W_\mathrm{M}}$ in the co-character lattice is generated by

$$\lambda^\mathrm{M} = \epsilon_1 + \epsilon_2 + \cdots \epsilon_n$$

and $\langle \lambda^\mathrm{M}, \gamma \rangle = deg\gamma$ for any character $\gamma \in \mathrm{X}^\bullet(\mathrm{T})$. (Again, the deg is the degree map on $\mathrm{X}^\bullet(\mathrm{T})$ with respect to the basis $\epsilon_1, \epsilon_2, ..., \epsilon_n$.) Notice that all roots in Φ_M has degree zero, so $\langle \lambda^\mathrm{M}, \rho_\mathrm{M} \rangle = 0$ and $\varpi^{\lambda^\mathrm{M}}$ centralizes M. By using the Weyl character formula, we have

$$\mathcal{W}(\varpi^{\lambda^\mathrm{M}} m; \underline{x}) = \lambda^\mathrm{M}(\underline{x}) \mathcal{W}(m; \underline{x}), \quad \forall m \in \mathrm{M}\,. \tag{5.1.2}$$

A consequence of (5.1.2) is the support $\bigsqcup_{\lambda \in I_{\underline{x}}} \mathrm{N}_n \, \varpi^\lambda \, \mathrm{K}_n$, $I_{\underline{x}} \subset P^+_\mathrm{M}$, of an eigenvector $\mathcal{W}_{\underline{x}}$ is invariant under shifting the set $I_{\underline{x}}$ by λ^M. In particular, these are not in the subspace $\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}_n} \theta^{-1}$ of functions of compact support modulo N_n.

If we write the complex dual torus point $\underline{x}$ in the coordinate $(x_1, x_2, ..., x_n)$, $x_i = \epsilon_i(\underline{x})$, then (5.1.2) reads

$$\mathcal{W}(\varpi^{\lambda^\mathrm{M}} m; X_1, X_2, ..., X_n) = (\prod_{i=1}^n X_i) \, \mathcal{W}(m; X_1, X_2, ..., X_n), \quad \forall m \in \mathrm{M}\,.$$

The two properties can also be rewritten in terms of the coordinates by

$$\mathcal{W}(d_\mathrm{M} w_\mathrm{M} {}^t m^{-1}; X_1, X_2, ..., X_n) = \mathcal{W}(m; X_1^{-1}, X_2^{-1}, ..., X_n^{-1}) \quad \forall m \in \mathrm{M},$$

$$\mathcal{W}(m; q^{-s} X_1, q^{-s} X_2, ..., q^{-s} X_n) = \mathcal{W}(m; X_1, X_2, ..., X_n) \, |\det m|^s \quad \forall m \in \mathrm{M}\,.$$

and the property of being an eigenvector becomes

$$\varsigma_\mathrm{M}(P) \mathcal{W}(m; X_1, X_2, ..., X_n) = P(X_1, X_2, ..., X_n) \mathcal{W}(m; X_1, X_2, ..., X_n)$$

for all $P \in \mathbb{C}[X_1^\pm, X_2^\pm, ..., X_n^\pm]^{S_n}$.

5.2. Fourier transforms of Whittaker functions

Suppose a function f on M lies in $(\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta)^{\mathrm{K}_n}$. We have $f(nmk) = \theta(n)f(m)$ for $n \in \mathrm{N}_n, m \in \mathrm{M}, k \in \mathrm{K}_n$ and

$$f(m) \neq 0 \text{ only if } C_1 < |\det m| < C_2 \text{ for some positive numbers } C_1, C_2.$$

Under action of $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$, the space $(\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta)^{\mathrm{K}_n}$ decomposes into direct sum of lines indexed by the Satake parameters appearing in it. We then have a Fourier expansion of f as the well-defined function with a complex variable q^{-s} introduced

$$\Psi_f(q^{-s}) = \int_{\mathrm{N}_n \backslash \mathrm{M}} f(m)\mathcal{W}(m; q^{-s}\underline{x})\ dm \in (\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}})[q^{-s}, q^{s}],$$

which is an expansion into $\sum_{r\in\mathbb{Z}} a_r(\underline{x})q^{-rs}$ with coefficient

$$a_r(\underline{x}) = \int_{\mathrm{N}_n \backslash \mathrm{M}} f(m)\mathcal{W}(m; \underline{x})\, \mathrm{ch}_{\varpi^r\mathfrak{o}^\times}(\det m)\ dm$$

$\neq 0$ for $c_1 \leq r \leq c_2$, and c_1, c_2 are some integers depending on C_1, C_2. We shall call this the *Fourier transform* of f.

In their work on conductors for the GL_n case Jacquet, Piatetski-Shapiro, and Shalika proved that this Fourier transform $\Psi_f(q^{-s})$ uniquely determines f.

The idea goes as follows. We are focusing on the representation $\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta$, whose contragradient is $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta^{-1}$. The pairing

$$(W, f) = \int_{\mathrm{N}_n \backslash \mathrm{M}} f(m)W(m)\ dm$$

on $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta^{-1} \otimes \mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta$ defines the M-equivarient perfect duality. All continuous linear forms on $\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta$ can be realized by taking $(W, \cdot)$ on $\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta$ for some $W \in \mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta^{-1}$. For $f \in (\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta)^{\mathrm{K}_n}$, its dual W in $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}}\,\theta^{-1}$ must also be K_n-invariant which has $\mathcal{W}_{\underline{x}}$ as a basis. Hence $\Phi_f(q^{-s}) \equiv 0$ forces $f = 0$.

Proposition 5.2.1 ([**14**] Lemma 3.5)**.** *Assume* $f \in (\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta)^{\mathrm{K}_n}$*. If the Fourier transform* $\Psi_f(q^{-s}) = 0$*, then* $f = 0$*.*

Proof. Consider the regular representation $(\Sigma, C^\infty(\mathrm{M}))$ of M, which decomposes continuously to irreducible representations σ_x: $\Sigma = \int_x \sigma_x \; d\mu(x)$. ($\mu$ a distribution of M.) The representation $\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta$ is an invariant subspace of Σ with countable dimension. We thus has for almost all σ_x, there is an intertwining operator T_x that maps $\mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta$ to σ_x such that the unitary structure is compatible, namely

$$\langle f_1, f_2 \rangle = \int_x \langle A_x f_1, A_x f_2 \rangle \; d\mu(x), \quad f_1, f_2 \in \mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta,$$

and $f = 0$ if $A_x f = 0$ for all x. When f is K_n-invariant, $T_x f \neq 0$ only if σ_x is unramified. On the other hand, since $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta^{-1}$ is its contragradient, for every x, there exists some W_x in the K_n-invariants of $\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta^{-1}$ such that $\langle f', W_x \rangle = \langle A_x f', A_x f \rangle$ for all $f' \in \mathrm{ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta$. Take $f' = f$. Since W_x is a linear combination of $\mathcal{W}_{\underline{x}}$, by assumption $\langle A_x f, A_x f \rangle = \int_{\mathrm{M}} f(m) W_x(m) \; dm = 0$. Hence $A_x f = 0$ for all x, which implies $f = 0$. □

This proof can be weaken and works on $f \in (\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta)^{\mathrm{K}_n}$ with the weaker property that $f(m) \neq 0$ only if $C_1 < |\det m|$ for some $C_1 > 0$. Then the Fourier transform $\Psi_f(q^{-s})$ is a Laurent series in q^{-s} with coefficients in $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$.

The idea introduced by Jacquet, Piatetski-Shapiro, and Shalika in 1979 is to consider the restriction of functions in $(\mathrm{Ind}_{\mathrm{N}_{n+1}}^{\mathrm{P}_{n+1}} \theta)^{\mathrm{M}(\mathfrak{o})}$ to M, which hence lies in $(\mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{M}} \theta)^{\mathrm{K}_n}$, as source of f to show properties of *new vectors* for GL_{n+1}. We will define *new vectors* for SO_{2n+1} in the Part 2. To prepare our discussion in Part 2, we will make the Fourier transforms with the restriction of functions in $(\mathrm{Ind}_{\mathrm{U}}^{\mathrm{Q}} \theta)^{\mathrm{M}(\mathfrak{o})}$ to M as a source of f. Let us define it below.

Assume π is an irreducible generic and supercuspidal representation of G. Recall $\pi \to \pi_{\mathrm{Z}}|_{\mathrm{Q}} \simeq \mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}} \theta$ by $v \mapsto W_v|_{\mathrm{Q}}$. Define $\Psi(v, q^{-s}; \underline{x}) \in \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$ as the Fourier transform $\Psi_{W_v\delta_{\mathrm{P}}^{-1/2}}(q^{-s'})$, $s' = s - \frac{1}{2}$, of $W_v\delta_{\mathrm{P}}^{-1/2}|_{\mathrm{M}}$ for $v \in V_\pi^{\mathrm{M}(\mathfrak{o})}$. Namely,

$$\Psi(v, q^{-s}; \underline{x}) = \int_{\mathrm{N}_n \backslash \mathrm{M}} \delta_{\mathrm{P}}^{-1/2}(m) W_v(m) \mathcal{W}(m; q^{-s'}\underline{x})\ dm. \tag{5.2.1}$$

Suppose $\tau_{\underline{x}}$ is the Whittaker model of a generic unramified representation of M with Satake parameter $\underline{x}$. The contragradient of $\tau_{\underline{x}}$ has Satake parameter $\underline{x}^{-1}$ and has Whittaker model $\tau_{\underline{x}^{-1}}$. Not so surprisingly, the zeta integrals on spherical vectors can be unwound to the Fourier transforms of the Whittaker functions. Let us give this computation below.

Take $\xi_m^0(h, s) \in \rho_{\tau_{\underline{x}},s}$ to be the unique H_{x_m}-spherical standard section such that

$$f_{\xi_m^0}(m, s) = L(\tau_{\underline{x}}, \wedge^2, 2s)\mathcal{W}(m; q^{-s'}\underline{x})\delta_{\mathrm{P}}^{1/2}(m),$$

where as always $s' = s - \frac{1}{2}$. Recall that $\widetilde{\tau_{\underline{x}}} = \tau_{\underline{x}^{-1}}$. As well take $\tilde{\xi}_m^0(h, 1-s) \in \rho_{\tau_{\underline{x}^{-1}},1-s}$ to be the unique H_{x_m}-spherical standard section such that

$$f_{\tilde{\xi}_m^0}(m, 1-s) = L(\tau_{\underline{x}^{-1}}, \wedge^2, 2(1-s))\mathcal{W}(m; q^{s'}\underline{x}^{-1})\delta_{\mathrm{P}}^{1/2}(m).$$

Note that this is $L(\tau_{\underline{x}^{-1}}, \wedge^2, 2(1-s))\mathcal{W}(d_{\mathrm{M}}w_{\mathrm{M}}{}^tm^{-1}; q^{-s'}\underline{x})\delta_{\mathrm{P}}^{1/2}(m)$.

Then

$$\begin{aligned}
& (A(w_{\mathrm{P}}^{-1}, s)f_{\xi_m^0})(\mathrm{I}, 1-s) \\
= {} & \int_{\mathrm{Z}} f_{\xi_m^0}(d_{\mathrm{M}}\omega_0 z\omega)\ dz = \int_{\mathrm{Z}} f_{\xi_m^0}(d_{\mathrm{M}}\varpi^{m\lambda^{\mathrm{M}}}\omega_m z\omega)\ dz \\
= {} & \int_{\mathrm{Z}} (m\lambda^{\mathrm{M}})(\underline{x}) f_{\xi_m^0}(d_{\mathrm{M}}{}^{\omega_m}z)\ dz \\
= {} & \lambda^{\mathrm{M}}(\underline{x})^m L(\tau_{\underline{x}}, \wedge^2, 2s)\ \frac{L(L_{d_{\mathrm{M}}}\tau_{\omega_m(\underline{x})}, \wedge^2, 1-(1-2s))}{L(\tau_{\underline{x}}, \wedge^2, 2s)} \\
= {} & \lambda^{\mathrm{M}}(\underline{x})^m L(\tau_{\underline{x}}, \wedge^2, 2s-1) \\
= {} & \lambda^{\mathrm{M}}(\underline{x})^m \gamma(\tau_{\underline{x}}, \wedge^2, 2s-1)^{-1} L(\tau_{\underline{x}^{-1}}, \wedge^2, 2(1-s)).
\end{aligned}$$

5.2. Fourier transforms of Whittaker functions

We observe that

$$(A^*(w_{\mathrm{P}}^{-1}, s)f_{\xi_m^0})(\mathrm{I}, 1-s) = \lambda^{\mathrm{M}}(\underline{x})^m f_{\tilde{\xi}_m^0}(\mathrm{I}, 1-s).$$

Since we know that the image of $f_{\xi_m^0}$ must be ${}^{\omega}\,\mathrm{H}_{x_m}$-spherical and hence a multiple of $f_{(\omega\omega_m^{-1})\tilde{\xi}_m^0}$, we get the multiple is $\lambda^{\mathrm{M}}(\underline{x})^m$, and

$$A^*(w_{\mathrm{P}}^{-1}, s)f_{\xi_m^0} = \lambda^{\mathrm{M}}(\underline{x})^m f_{(\omega\omega_m^{-1})\tilde{\xi}_m^0}.$$

(One note the element ω_m normalizes H_{x_m} and $\omega\omega_m^{-1} \in \mathrm{H}$.)

Now for any given Satake parameter $\underline{x} \in \hat{\mathrm{T}}$ of M, consider the Rankin-Selberg zeta integral for $\pi \times \tau_{\underline{x}}$ on $v_m \otimes \xi_m^0 \in V_\pi^{\mathrm{H}_{x_m}} \otimes \rho_{\tau_{\underline{x}},s}^{\mathrm{H}_{x_m}}$.

$$\begin{aligned}
\zeta(v_m \otimes \xi_m^0, s) &= \int_{\mathrm{V}\setminus\mathrm{H}} W_{v_m}(h) f_{\xi_m^0}(h,s)\, dh = \int_{\mathrm{V}\setminus\mathrm{P}} W_{v_m}(p) f_{\xi_m^0}(p,s)\, dp \\
&= \int_{\mathrm{N}_n\setminus\mathrm{M}} \delta_{\mathrm{P}}^{-1}(m) W_{v_m}(m) f_{\xi_m^0}(m,s)\, dm \\
&= L(\tau_{\underline{x}}, \wedge^2, 2s) \int_{\mathrm{N}_n\setminus\mathrm{M}} \delta_{\mathrm{P}}^{-1/2}(m) W_{v_m}(m) \mathcal{W}(m; q^{-s'}\underline{x})\, dm.
\end{aligned}$$

This is equal to the Fourier transform multiplying a factor $L(\tau_{\underline{x}}, \wedge^2, 2s)$.

We obtain the following new interpretation for Rankin-Selberg zeta integral at H_{x_m}-fixed vectors $v_m \otimes \xi_m^0$ interms of the Fourier transform.

Lemma 5.2.2. $\forall v_m \in V_\pi^{\mathrm{H}_{x_m}}$, $\zeta(v_m \otimes \xi_m^0, s) = L(\tau_{\underline{x}}, \wedge^2, 2s)\Psi(v_m, q^{-s}; \underline{x}) \in \mathbb{C}[q^{-s}, q^s]$.

Here the equation lives in $\mathbb{C}[q^{-s}, q^s]$ under the assumption that π is supercuspidal and $L(\pi \times \tau_{\underline{x}}, s) = 1$ with all zeta integrals live in the principal ideal ring.

Similarly we get the following new interpretation for Rankin-Selberg zeta integral on the other side of the functional equation at ${}^{\omega}\,\mathrm{H}_{x_m}$-fixed vectors $\omega v_m \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi_m^0$ interms of the Fourier transform.

Lemma 5.2.3. $\forall v_m \in V_\pi^{\mathrm{H}_{x_m}}$, $\zeta(\omega v_m \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi_m^0, s) =$

$$\lambda^{\mathrm{M}}(\underline{x})^m L(\tau_{\underline{x}^{-1}}, \wedge^2, 2(1-s))\Psi(\omega_m v_m, q^{-(1-s)}; \underline{x}^{-1}) \in \mathbb{C}[q^{-s}, q^s].$$

Let us compute this below.

$$\begin{aligned}
& \zeta(\omega v_m \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi_m^0, s) \\
= \; & \lambda^{\mathrm{M}}(\underline{x})^m \int_{\mathrm{V}\setminus\mathrm{H}} W_{v_m}(h\omega) f_{\tilde{\xi}_m^0}(h\omega\omega_m^{-1}, s)\; dh \\
= \; & \lambda^{\mathrm{M}}(\underline{x})^m \int_{\mathrm{V}\setminus\mathrm{H}} W_{v_m}(h\omega_m) f_{\tilde{\xi}_m^0}(h, s)\; dh \\
= \; & \lambda^{\mathrm{M}}(\underline{x})^m \int_{\mathrm{N}_n\setminus\mathrm{M}} \delta_{\mathrm{P}}^{-1}(m) W_{\omega_m v_m}(m) f_{\tilde{\xi}_m^0}(m, s)\; dm \\
= \; & \lambda^{\mathrm{M}}(\underline{x})^m L(\tau_{\underline{x}^{-1}}, \wedge^2, 2(1-s)) \int_{\mathrm{N}_n\setminus\mathrm{M}} \delta_{\mathrm{P}}^{-1/2}(m) W_{\omega_m v_m}(m) \mathcal{W}(m; q^{s'}\underline{x}^{-1})\; dm \\
= \; & \lambda^{\mathrm{M}}(\underline{x})^m L(\tau_{\underline{x}^{-1}}, \wedge^2, 2(1-s))\Psi(\omega_m v_m, q^{-(1-s)}; \underline{x}^{-1}) \in \mathbb{C}[q^{-s}, q^s].
\end{aligned}$$

Since $\varepsilon(\pi \times \tau_{\underline{x}}, s, \psi) = \gamma(\pi \times \tau_{\underline{x}}, s, \psi)$ and it is known in [**29**] that $\gamma(\pi \times \tau_{\underline{x}}, s, \psi)$ is multiplicative. By the fact that $\tau_{\underline{x}} \simeq \mathrm{I}_{\mathrm{M},\mathrm{T}}\, \chi_{\underline{x}}$ with $\chi_{\underline{x}}(\varpi^\lambda) = \lambda(\underline{x})$. One has

$$\varepsilon(\pi \times \tau_{\underline{x}}, s, \psi) = \prod_{i=1}^n \varepsilon(\pi \otimes (\chi_{\underline{x}} \circ \epsilon_i), s, \psi) = \lambda^{\mathrm{M}}(\underline{x})^{a_\pi} \varepsilon_\pi^n q^{-na_\pi s'}.$$

The functional equation for $\pi \times \tau_{\underline{x}}$

$$\zeta(\omega v_m \otimes A^*(w_{\mathrm{P}}^{-1}, s)\xi_m^0, s) = \varepsilon(\pi \times \tau_{\underline{x}}, s, \psi)\zeta(v_m \otimes \xi_m^0, s)$$

hence can be translated into relations of the Fourier transforms and local invariants:

Proposition 5.2.4. $\forall v_m \in V_\pi^{\mathrm{H}_{x_m}}$, $\forall \underline{x} \in \hat{\mathrm{T}}$,

$$L(\tau_{\underline{x}^{-1}}, \wedge^2, 2(1-s))\Psi(\omega_m v_m, q^{-(1-s)}; \underline{x}^{-1})$$

$$= \lambda^{\mathrm{M}}(\underline{x})^{a_\pi - m} \varepsilon_\pi^n q^{-na_\pi s'} L(\tau_{\underline{x}}, \wedge^2, 2s)\Psi(v_m, q^{-s}; \underline{x}) \in \mathbb{C}[q^{-s}, q^s].$$

5.3. Actions of Hecke operators

Let us first show the existence of vectors fixed by H_{x_m} for each $m \in \mathbb{Z}$.

Lemma 5.3.1. *For any given $\underline{x} \in \hat{\mathrm{T}}$, there exists a vector $v_m \in V_\pi^{\mathrm{H}_{x_m}}$ for each $m \in \mathbb{Z}$ such that the complex variable function $\Psi(v_m, q^{-s}; \underline{x})$ is not identically zero.*

Proof. Since $L(\pi \times \tau_{\underline{x}}) \neq 0$, there exists $v_{(i)} \in V_\pi$, $\xi_{(i)} \in V_{\rho_{\tau_{\underline{x}},s}}$, and $a_{(i)} \in \mathbb{Z}$ for $i = 1, 2, ..., r$ such that $L(\pi \times \tau_{\underline{x}}, s) = \sum_{i=1}^r q^{a_{(i)}s}\zeta(v_{(i)} \otimes \xi_{(i)}, s)$. Since $\xi_m^0 \in V_{\rho_{\tau_{\underline{x}},s}}^{\mathrm{H}_{x_m}} \neq 0$, the spherical standard section defined in the previous section, $V_{\rho_{\tau_{\underline{x}},s}} = \mathcal{H}(\mathrm{H})\xi_m^0$. Since the zeta integral is a H-invariant bilinear form, one can to take $\xi_{(i)} = \xi_m^0$. However, by the same fact, one can replace $v_{(i)}$ by its average over H_{x_m}, i.e. its image under $e_{\mathrm{H}_{x_m}} \in \mathcal{H}(\mathrm{H})$. Since $\sum_{i=1}^r q^{a_{(i)}s}\zeta(v_{(i)} \otimes \xi_m^0, s)$ is nonzero, there exists an i such that $v^i \in V_\pi^{\mathrm{H}_{x_m}}$ is nonzero with $\zeta(v_{(i)} \otimes \xi_m^0, s) \neq 0$. By Lemma 5.2.2, $\zeta(v_{(i)} \otimes \xi_m^0, s) = L(\tau_{\underline{x}}, \wedge^2, 2s)\Psi(v_{(i)}, q^{-s}; \underline{x}) \neq 0$, which implies $\Psi(v_{(i)}, q^{-s}; \underline{x}) \neq 0$. $\square$

Recall by definition, for $v \in V_\pi^{\mathrm{K}_n}$ the function $\Psi(v, q^{-s}; \underline{x})$ in $\mathbb{C}[\hat{\mathrm{T}}]^{W_\mathrm{M}}[q^{-s}, q^s]$ is defined as the Fourier transform

$$\int_{\mathrm{N}_n \backslash \mathrm{M}_n} \delta_\mathrm{P}^{-1/2}(m)W_v(m)\mathcal{W}(m; q^{-s'}\underline{x})\, dm, \quad s' = s - \frac{1}{2}.$$

Suppose $P \in \mathbb{C}[\hat{\mathrm{T}}]^{W_\mathrm{M}}$. Since $P(q^{-s'}\underline{x})\mathcal{W}(m; q^{-s}\underline{x}) = \varsigma_\mathrm{M}(P)\mathcal{W}(m; q^{-s}\underline{x})$, we have

$$\begin{aligned}
& P(q^{-s'}\underline{x})\Psi(v, q^{-s}; \underline{x}) \\
= & \int_{\mathrm{N}_n \backslash \mathrm{M}_n} \delta_\mathrm{P}^{-1/2}(m)W_v(m)(\varsigma_\mathrm{M}(P)\mathcal{W})(m; q^{-s'}\underline{x})\, dm \\
= & \int_{\mathrm{N}_n \backslash \mathrm{M}_n} \delta_\mathrm{P}^{-1/2}(m)W_v(m)\left(\int_\mathrm{M} \varsigma_\mathrm{M}(P)(m')\mathcal{W}(mm'; q^{-s'}\underline{x})\, dm'\right) dm \\
= & \int_\mathrm{M}\int_{\mathrm{N}_n \backslash \mathrm{M}_n} \delta_\mathrm{P}^{-1/2}(mm'^{-1})\varsigma_\mathrm{M}(P)(m')W_v(mm'^{-1})\mathcal{W}(m; q^{-s'}\underline{x})\, dm'\, dm \\
= & \int_{\mathrm{N}_n \backslash \mathrm{M}_n} \delta_\mathrm{P}^{-1/2}(m)\left(\int_\mathrm{M} \delta_\mathrm{P}^{1/2}(m')\varsigma_\mathrm{M}(P)(m')W_v(mm'^{-1})\, dm'\right)\mathcal{W}(m; q^{-s'}\underline{x})\, dm
\end{aligned}$$

(We note that Fubini's Theorem applies since $\varsigma_{\mathrm{M}}(P)$ is compactly supported on M.)

Following the ideas in [**14**] and [**22**], we define an action of $\mathcal{H}(\mathrm{M}, \mathrm{K}_n)$ on V_π by

$$f * v = \int_{\mathrm{M}} \delta_{\mathrm{P}}^{1/2}(m') f(m') \pi(m'^{-1}) v \; dm', \quad \forall f \in \mathcal{H}(\mathrm{M}, \mathrm{K}_n). \tag{5.3.1}$$

It is clear that this action preserves the subspace $V_\pi^{\mathrm{K}_n}$. Then from above we obtain

$$P(q^{-s'}\underline{x})\Psi(v, q^{-s}; \underline{x}) = \Psi(\varsigma_{\mathrm{M}}(P) * v, q^{-s}; \underline{x}) \tag{5.3.2}$$

for all $P \in \mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$ and $v \in V_\pi^{\mathrm{K}_n}$.

Since $\Psi(v, q^{-s}; \underline{x})$ lies in $\mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}[q^{-s}, q^s]$. Evaluating at $s = 1/2$ (or equivalently, $s' = 0$) defines a $\mathbb{C}$-linear map $\Xi : V_\pi^{\mathrm{K}_n} \to \mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$ which by (5.3.2) satisfies the identity

$$P \cdot \Xi(v) = \Xi(\varsigma_{\mathrm{M}}(P) * v), \quad \forall P \in \mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}. \tag{5.3.3}$$

Lemma 5.3.2. *$\Xi : V_\pi^{\mathrm{K}_n} \to \mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$ is a $\mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$-module homomorphism, with $\mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$ acting on $V_\pi^{\mathrm{K}_n}$ by the action of $\mathcal{H}(\mathrm{M}, \mathrm{K}_n)$ defined above composing the Satake transform ς_{M} and on $\mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$ by multiplication. It is surjective and has kernel*

$$\ker \Xi = \{v \in V_\pi^{\mathrm{K}_n} \mid W_v|_{\mathrm{T}} = 0\}.$$

Proof. We have seen it commutes with the action of $\mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$. To show the kernel, for π irreducible generic supercuspidal the map $v \mapsto W_v|_{\mathrm{Q}}$ induces a surjective Q-homomorphism from V_π to $\mathrm{Ind}_{\mathrm{U}}^{\mathrm{Q}} \theta$. There exists $v \in V_\pi^{\mathrm{K}_n}$ such that $W_v|_{\mathrm{M}}$ supports on $\mathrm{N}_n \mathrm{K}_n$. Then $\Xi(v) = \Psi(v, q^{-1/2}; \underline{x}) = \mathrm{vol}(\mathfrak{o}^\times)^n$ is a unit in $\mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$. Hence the $\mathbb{C}[\hat{\mathrm{M}}]^{W_{\mathrm{M}}}$-module homomorphism is surjective.

By Iwasawa decomposition of M, the kernel is contained in the given set. To prove the other inclusion, we recall $\Psi(v, q^{-s}; \underline{x}) = \Psi(v, q^{-1/2}; q^{-s}\underline{x})$. Hence $\Xi(v) \equiv 0$

implies that $W_v|_{\mathrm{M}}$ has trivial Fourier transform. By Proposition 5.2.1, $W_v|_{\mathrm{M}} = 0$ and in particular $W_v|_{\mathrm{T}} = 0$.

Recall that Lemma 3.4.1 and Corollary 3.4.2 show that $\{v \in V_\pi^{\mathrm{H}_{x_m}} \mid W_v|_{\mathrm{Q}} = 0\} = 0$ for each integer m. We would like to focus on the subspaces $V_\pi^{\mathrm{H}_{x_m}}$, $m \in \mathbb{Z}$, on which many good properties are valid.

In order to preserve the subspace $V_\pi^{\mathrm{H}_{x_m}}$, we consider the intermediate Satake transform. The map $\jmath_m : \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m}) \to \mathcal{H}(\mathrm{M}, \mathrm{K}_n)$ for $m \in \mathbb{Z}$ defined by

$$\phi \mapsto \jmath_m(\phi)(m) = \delta_{\mathrm{P}}^{1/2}(m) \int_{\mathrm{Z}} \phi(mz)\, dz \tag{5.3.4}$$

fits into the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}[\widehat{\mathrm{T}}]^{W_{\mathrm{H}}} & \xrightarrow[\varsigma_{\mathrm{H},m}]{\sim} & \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m}) \\
{\scriptstyle incl}\downarrow & & \downarrow{\scriptstyle \jmath_m} \\
\mathbb{C}[\widehat{\mathrm{T}}]^{W_{\mathrm{M}}} & \xrightarrow[\varsigma_{\mathrm{M}}]{\sim} & \mathcal{H}(\mathrm{M}, \mathrm{K}_n)
\end{array}$$

and is an injective algebra homomorphism. Therefore

$$\jmath_m(\phi) * v = \int_{\mathrm{M}} \delta_{\mathrm{P}}(m') \left(\int_{\mathrm{Z}} \phi(m'z')\, dz' \right) \pi(m'^{-1}) v\, dm', \quad \forall \phi \in \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m}).$$

Let us similarly define the action of $\mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m})$ on V_π by

$$\phi * v = \int_{\mathrm{H}} \phi(h') \pi(h'^{-1}) v\, dh', \quad \forall \phi \in \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m}), \tag{5.3.5}$$

which preserves $V_\pi^{\mathrm{H}_{x_m}}$. By taking an inverse, the Iwasawa decomposition $\mathrm{H} = \mathrm{P}\,\mathrm{H}_{x_m}$ can also be written as $\mathrm{H} = \mathrm{H}_{x_m}\mathrm{P}$. For $v_m \in V_\pi^{\mathrm{H}_{x_m}}$ and $\phi \in \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m})$, the vector $\phi * v$ becomes

$$\int_{\mathrm{P}} \delta_{\mathrm{P}}(p') \phi(p') \pi(p'^{-1}) v\, dp'.$$

One observes that the Whittaker function associated to $\phi * v$ restricted to M is

$$\begin{aligned} W_{\phi * v}(m) &= \int_{\mathrm{P}} \delta_{\mathrm{P}}(p')\phi(p')W_v(mp'^{-1})\ dp' \\ &= \int_{\mathrm{M}}\int_{\mathrm{Z}} \delta_{\mathrm{P}}(m')\phi(m'z')W_v(mz'^{-1}m'^{-1})\ dz'\, dm' \\ &= \int_{\mathrm{M}} \delta_{\mathrm{P}}(m')\left(\int_{\mathrm{Z}} \phi(m'z')\ dz'\right) W_v(mm'^{-1})\ dm', \quad \forall m \in \mathrm{M}, \end{aligned}$$

which equals to the Whittaker function associated to $\jmath_m(\phi) * v$ restricted to M.

Since the Fourier transform depends only on the restriction of the Whittaker function to M, we conclude

$$P(q^{-s'}\underline{x})\Psi(v_m, q^{-s}; \underline{x}) = \Psi(\varsigma_{\mathrm{H},m}(P) * v_m, q^{-s}; \underline{x}) \tag{5.3.6}$$

or equivalently,

$$P \cdot \Xi(v_m) = \Xi(\varsigma_{\mathrm{H},m}(P) * v_m) \text{ in } \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$$

for all $P \in \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$ and $v_m \in V_\pi^{\mathrm{H}_m}$.

We obtain the following modified version of Lemma 5.3.2.

Lemma 5.3.3. *For integer $m \geq 0$, the $\mathbb{C}$-linear map $\Xi : V_\pi^{\mathrm{H}_{x_m}} \to \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$ gotten from restriction is an injective $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$-module homomorphism, with $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$ acting on $V_\pi^{\mathrm{K}_n}$ by the action of $\mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m})$ defined above composing the Satake transform $\varsigma_{\mathrm{H},m}$ and on $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$ by multiplication.*

The following Corollary is immediate from the injectivity of Ξ on $V_\pi^{\mathrm{H}_{x_m}}$.

Corollary 5.3.4. *Assume $m \in \mathbb{Z}$. For any nonzero vector $v_m \in V_\pi^{\mathrm{H}_{x_m}}$, the H_{x_m}-fixed vectors $\varsigma_{\mathrm{H},m}(P) * v_m$ for all $P \in \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$ are distint and nonzero.*

This result will be used in computing the dimension of subspaces of fixed vectors in Part 2.

5.4. Fourier transform Ψ and Jacquet's polynomial Ω

Let us write the results into coordinates $X_1 = \epsilon_1$, $X_2 = \epsilon_2$,...,$X_n = \epsilon_n$ and discuss them in the ring $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, ..., X_n, X_n^{-1}]^{S_n}$, which we shall denote by $\mathscr{S}_n$. Under Satake isomorphism for each $1 \leq i \leq n$ the generator $[\mathrm{M}(\mathfrak{o})\varpi^\lambda \mathrm{M}(\mathfrak{o})]$, the characteristic function of the double coset $\mathrm{M}(\mathfrak{o})\varpi^\lambda \mathrm{M}(\mathfrak{o})$, for $\lambda = \epsilon_1 + \epsilon_2 + ... + \epsilon_i$, in the Hecke algebra $\mathcal{H}(\mathrm{M}, \mathrm{M}(\mathfrak{o}))$ maps to the sum of characteristic functions $\sum_{s\in W_{\mathrm{M}}} \mathrm{ch}_{\varpi^{s(\lambda)}\,\mathrm{T}(\mathfrak{o})}$ and has corresponding element in the ring $\mathscr{S}_n$ as

$$T_i := \sum_{s\in S_n} X_{s(1)}X_{s(2)}\cdots X_{s(i)}$$

which is the elementary symmetric polynomial. Hence $\mathscr{S}_n = \mathbb{C}[T_1, T_2, ..., T_n, T_n^{-1}]$ and T_n gives a $\mathbb{Z}$-grading on the ring $\mathscr{S}_n = \oplus_{d\in\mathbb{Z}}\mathscr{S}_{n,d}$ by the degree of T_n.

Recall that we have the $\mathscr{S}_n$-module map $\Xi : V_\pi^{\mathrm{K}_n} \to \mathscr{S}_n$ defined by $\Xi(v) = \Psi(v, q^{-1/2}; \underline{x})$ whose restriction to the subset $V_\pi^{\mathrm{H}_{xm}}$ is injective. (See Lemma 5.3.3.)

Lemma 5.4.1. *For $v \in V_\pi^{\mathrm{K}_n}$, if v is invariant under $x_{\epsilon_n}(\mathfrak{p}^k)$ then $\deg_{T_n} \Xi(v) \geq -k$. As a result image of $V_\pi^{\mathrm{Q}(\mathfrak{o})}$ under Ξ is contained in $\oplus_{d\geq 0}\mathscr{S}_{n,d} = \mathbb{C}[T_1, T_2, ..., T_n]$.*

Proof. Since if v is also invariant under $x_{\epsilon_n}(\mathfrak{p}^k)$ then the Whittaker function $W_v|_{\mathrm{M}}$ has support contained in $\cup_{\langle\mu,\epsilon_n\rangle\geq -k} \mathrm{M}(\mathfrak{o})\varpi^\mu \mathrm{M}(\mathfrak{o})$ on which $\deg_{T_n} \mathcal{W}(\cdot; \underline{x}) \geq -k$. □

Note that for $v \in V_\pi^{\mathrm{H}_{xm}}$, $m \geq 0$ integer, we have $L(\tau_{\underline{x}}, \wedge^2, 2s)\Psi(v, q^{-s}; \underline{x})$ in $\mathscr{S}_n[q^{-s}, q^s]$ and hence is entire in s. We take $s = \frac{1}{2}$ and obtain that

$$\Omega(v; X_1, X_2, ..., X_n) := \Xi(v)/ \prod_{1\leq i<j\leq n} (1 - q^{-1}X_iX_j) \in \mathscr{S}_n \tag{5.4.1}$$

which gives a factorization in $\mathscr{S}_n$ as

$$\Xi(v) = \left(\prod_{1\leq i<j\leq n} (1 - q^{-1}X_iX_j)\right) \Omega(v; X_1, X_2, ..., X_n).$$

We note that again $\Omega(v) = 0$ implies $v = 0$ provided that $v \in V_\pi^{\mathrm{H}_{x_m}}$.

By Proposition 5.2.4 the functional equation for $v \in V_\pi^{\mathrm{H}_{x_m}}$ gives the following important relation.

Proposition 5.4.2. *For $v \in V_\pi^{\mathrm{H}_{x_m}}$, we have the following identity in $\mathscr{S}_n$.*

$$\Omega(\omega_m v; X_1^{-1}, X_2^{-1}, ..., X_n^{-1}) = \varepsilon_\pi^n T_n^{a_\pi - m}\, \Omega(v; X_1, X_2, ..., X_n). \tag{5.4.2}$$

Note that the factor $\left(\prod_{1\le i<j\le n}(1 - q^{-1}X_iX_j)\right)$ is a prime in $\mathscr{S}_n$ and lives in the zeroth graded piece $\mathscr{S}_{n,0}$. Now combining Lemma 5.4.1 and Proposition 5.4.2 we obtain the following observation.

Proposition 5.4.3. *For $v \in V_\pi^{\mathrm{H}_{x_m}}$ nonzero, if v is invariant under $x_{\epsilon_n}(\mathfrak{p}^k)$ and $x_{-\epsilon_1}(\mathfrak{p}^l)$ then*

$$\Omega(v; X_1, X_2, ..., X_n) \in \oplus_{-k\le d\le l-a_\pi}\mathscr{S}_{n,d}.$$

Proof. In Lemma 5.4.1 we have seen that $\Omega(v; X_1, X_2, ..., X_n) \in \oplus_{-k\le d}\mathscr{S}_{n,d}$. However, since $\omega_m v$ is invariant under $x_{\epsilon_n}(\mathfrak{p}^{l-m})$, we also have $\Omega(\omega_m v; X_1, X_2, ..., X_n) \in \oplus_{m-l\le d}\mathscr{S}_{n,d}$ and hence $\Omega(\omega_m v; X_1^{-1}, X_2^{-1}, ..., X_n^{-1}) \in \mathbb{C}[T_{n-1}T_n^{-1}, ..., T_1T_n^{-1}, T_n^{-1}, T_n]$ so has degree in T_n less than or equal to $m - l$. Then apply the identity (5.4.2). $\square$

Remark 5.4.4. The results in this section hold for general irreducible generic representations as well in which case the Fourier transform $\Psi(v, q^{-s}; \underline{x})$ is a Laurent series in $X = q^{-s}$ by smoothness of v and converges for $\Re(s)$ large enough by the slowly increasing property of the Whittaker function W_v, and the definition of $\Omega(v)$ is multiplied by an extra factor $\prod_{i=1}^n P_\pi(q^{-1/2}X_i) \in \mathscr{S}_n$ which was 1 in the supercuspidal case. Since $\prod_{i=1}^n P_\pi(q^{-1/2}X_i)$ contains a constant term the result regarding the degree is still valid.

Part 2

Test vectors

CHAPTER 6

Review for cases of lower rank

In this chapter, we summarize the known results for the lower rank case. When $n = 1$, this is the classical theory for PGL_2 proved by Casselman [**4**]. When $n = 2$, this is studied by the recent work of Roberts and Schmidt on PGSp_4 [**23**].

6.1. Rank 1: $\mathrm{SO}_3(k) \simeq \mathrm{PGL}_2(k)$

Let V_1 be the set of traceless 2 by 2 matrices over k which is the Lie algebra $\mathfrak{sl}_2$. The group GL_2 acts on V_1 by taking conjugate on every matrix in V_1. The center of GL_2 acts trivially and V_1 becomes the 3 dimensional adjoint representation of PGL_2. This action preserves a volume form

$$\varphi : A = \begin{bmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{bmatrix} \mapsto -2 \det A = 2a_1^2 + 2a_2a_3$$

on V_1. $\varphi : V_1 \to k$ is a quadratic form on V_1 of discriminant -2 and it makes V_1 a split quadratic space with a good basis

$$\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

We thus obtain an isomorphism from PGL_2 to $\mathrm{SO}(V_1)$. Or more explicitly,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ad-bc)^{-1} \begin{bmatrix} a^2 & -2ab & -b^2 \\ -ac & ad+bc & bd \\ -c^2 & 2cd & d^2 \end{bmatrix}.$$

6.1. Rank 1: $\mathrm{SO}_3(k) \simeq \mathrm{PGL}_2(k)$

Set $\mathrm{G} = \mathrm{SO}(V_1)$. Set $\mathrm{G} = \mathrm{SO}(V_1)$ and let $(\mathrm{B}, \mathrm{T}, \theta)$ be a generic data compatible with the good basis.

Assume $m \geq 0$ is an integer. The congruence subgroup $\Gamma_0(\mathfrak{p}^m)$ of $\mathrm{GL}_2(k)$ is defined as the open compact subgroup

$$\Gamma_0(\mathfrak{p}^m) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}) \mid c \equiv 0 (\mathrm{mod}\, \mathfrak{p}^m), a, d \in \mathfrak{o}^\times \right\}.$$

The normalizer of $\Gamma_0(\mathfrak{p}^m)$ in $\mathrm{PGL}_2(k)$ is generated by $\Gamma_0(\mathfrak{p}^m)$ and $\begin{bmatrix} & 1 \\ \varpi^m & \end{bmatrix}$, called the Atkin-Lehner element of level $\mathfrak{p}^m$. The Atkin-Lehner element has order 2 in the adjoint group $\mathrm{PGL}_2(k)$ and its image in $\mathrm{SO}(V_1)$ is

$$u_m = \begin{bmatrix} & & \varpi^{-m} \\ & -1 & \\ \varpi^m & & \end{bmatrix}.$$

The normalizer of $\Gamma_0(\mathfrak{p}^m)$ contains it with index 2 for $m \geq 1$ and equals to itself for $m = 0$. Let $\mathrm{K}(\mathfrak{p}^m)$ denotes the image of $\Gamma_0(\mathfrak{p}^m)$ in $\mathrm{SO}(V_1)$. The subgroups

$$\mathrm{T}(\mathfrak{o}), \mathrm{U}_{\epsilon_1}(\mathfrak{o}), \mathrm{U}_{-\epsilon_1}(\mathfrak{p}^m)$$

are contained in $\mathrm{K}(\mathfrak{p}^m)$. Together with u_m these subgroups generate the stabilizer of the lattices

$$\mathbb{L}_m = \mathfrak{o}e \oplus \mathfrak{p}^m v_0 \oplus \mathfrak{p}^m f \text{ and } \mathbb{L}_m^\vee = \mathfrak{p}^{-m} e \oplus \mathfrak{p}^{-m} v_0 \oplus \mathfrak{o} f$$

in $\mathrm{SO}(V_1)$. Therefore, $\mathrm{K}(\mathfrak{p}^m)$ is equal to $\mathrm{Stab}(\mathbb{L}_m)$ for $m = 0$ and is a normal subgroup of index two in $\mathrm{Stab}_{\mathrm{G}}(\mathbb{L}_m)$ for $m \geq 1$.

6.1. Rank 1: $\mathrm{SO}_3(k) \simeq \mathrm{PGL}_2(k)$

Let π be a generic irreducible representation of $\mathrm{G} = \mathrm{SO}(V_1)$. Then there exists some vector $v_* \in \pi$ such that $I(v,s) = L(\pi,s)$. Recall that

$$I(v,s) = \int_{k^\times} W_v(\epsilon_1(a))|a|^{s-\frac{1}{2}}\ da,\ \forall v \in \pi.$$

We are allowed to assume that v_* is fixed by $\mathrm{T}(\mathfrak{o})$ and $\mathrm{U}_{\epsilon_1}(\mathfrak{o})$ by taking an average. Let a_π denote the conductor of π. We recall that we have a functional equation

$$\frac{I(u_0 v_*, 1-s)}{L(\pi, 1-s)} = \varepsilon(\pi, s, \psi)\frac{I(v_*, s)}{L(\pi, s)}$$

whose right hand side simply equals to $\varepsilon_\pi q^{-a_\pi(s-\frac{1}{2})}$. Using the property that

$$I(u_0\epsilon_1(\varpi^{a_\pi})v_*, 1-s) = q^{a_\pi(s-\frac{1}{2})}I(u_0 v_*, 1-s),$$

the equation becomes

$$\frac{I(u_{a_\pi} v_*, 1-s)}{L(\pi, 1-s)} = \varepsilon_\pi \Rightarrow I(\varepsilon_\pi^{-1}u_{a_\pi}v_*, s) = L(\pi, s).$$

Therefore the Whittaker functions W_{v_*} and $W_{\varepsilon_\pi^{-1}u_{a_\pi}v_*}$ agree on $\mathrm{Q} = \mathrm{U}_\epsilon\, \epsilon_1(k)$ and are fixed by $\mathrm{H}_{x_{a_\pi}} = \mathrm{T}(\mathfrak{o})$. We get v_* and $\varepsilon_\pi^{-1}u_{a_\pi}v_*$ have the same image under the Jacquet functor J_Z, which is the identity map since $\mathrm{Z} = \mathrm{I}$, and hence are the same. We get $v_* = \varepsilon_\pi^{-1}u_{a_\pi}v_*$ is fixed by the subgroups

$$\mathrm{T}(\mathfrak{o}), \mathrm{U}_{\epsilon_1}(\mathfrak{o}),\ \text{and}\ \mathrm{U}_{-\epsilon_1}(\mathfrak{p}^{a_\pi}) = u_{a_\pi}\, \mathrm{U}_{\epsilon_1}(\mathfrak{o})u_{a_\pi}^{-1}$$

and is hence fixed by the subgroup $\mathrm{K}(\mathfrak{p}^{a_\pi})$.

For each vector $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$, $u_{a_\pi}v$ is fixed by $\mathrm{K}(\mathfrak{p}^{a_\pi})$ as well. Hence we have

$$(*) \quad \frac{I(u_{a_\pi}v, 1-s)}{L(\pi, 1-s)} = \varepsilon_\pi \frac{I(v,s)}{L(\pi,s)}$$

Since v and $u_{a_\pi}v$ are $\mathrm{U}_{\epsilon_1}(\mathfrak{o})$-fixed, the right hand side of $(*)$ is in $\mathbb{C}[q^{-s}] = \mathbb{C}[q^{-s}, q^s] \cap \mathbb{C}[[q^{-s}]]$. Similarly, the left hand side of $(*)$ is in $\mathbb{C}[q^{1-s}]$ and hence in $\mathbb{C}$. Therefore every vector $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ has $I(v,s) = cL(\pi,s)$ for some $c \in \mathbb{C}$. Again v and cv_* are fixed by $\mathrm{H}_{x_{a_\pi}} = \mathrm{T}(\mathfrak{o})$ and have the same image under the Jacquet functor J_{Z}, which is the identity map. Therefore, $v = cv_*$. We can conclude the following.

Theorem 6.1.1 (Casselman)**.** *The fixed subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ is one dimensional. There is a unique vector v_* on this line such that $I(v_*,s) = L(\pi,s)$ and hence $W_{v_*}(I) = 1$. Moreover, v_* is an eigenvector of u_{a_π} with eigenvalue ε_π.*

The line $V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ encodes all of the local invariants of the generic representation π of $\mathrm{SO}_3(k)$. The vector v_* can be used as a test vector of π. Since $V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ is one dimensional, the Hecke operators in $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p}^{a_\pi}))$ acts on it by a character. v_* is thus a Hecke eigenform. Casselman in his paper [**4**] showed that a_π is the lowest exponent one can/will get to obtain a nontrivial fixed subspace. Such vector is called a *new form* of the representation.

6.2. Rank 2: $\mathrm{SO}_5(k) \simeq \mathrm{PGSp}_4(k)$

There is a analogous theory of new forms for $\mathrm{GSp}_4(k)$ studied by Roberts and Schmidt [**23**] in 2006 which works for generic representations with trivial central character.

Let D be a 4 dimensional vector space equipped with a skew-symmetric bilinear form. Fix a basis $\{d_1, d_2, d_3, d_4\}$ of D such that the skew-symmetric bilinear form has Gram matrix

$$J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

The symplectic similitude group $\mathrm{GSp}(D)$ is the subgroup of the automorphism group $\mathrm{GL}(D)$ of D conformal with respect to the bilinear form. The vector space D is a standard representation of $\mathrm{GSp}(D)$.

6.2. Rank 2: $\mathrm{SO}_5(k) \simeq \mathrm{PGSp}_4(k)$

Consider the exterior square representation $W_1 = (\wedge^2 D)^* \simeq \wedge^2 D$ of $\mathrm{GSp}(D)$. The skew-symmetric form induces a linear functional on $\wedge^2 D$ and hence a vector

$$w = d_1 \wedge d_4 + d_2 \wedge d_3$$

on W_1. The similitude group $\mathrm{GSp}(D)$ preserves the line $\ell = kw$ and acts on the 5 dimensional vector space $V = W_1/\ell$. The Grassmannian $\mathbf{G}(2,4) = \{planes \subset D\}$ is embedded as a quadratic hypersurface (an isotropic space of a quadratic form) in W_1 and is stable under action of $\mathrm{GSp}(D)$. Therefore the action of $\mathrm{GSp}(D)$ on W_1/ℓ preserves a quadratic form φ which is nondegenerate of discriminant 2. This induces a map

$$j : \mathrm{PGSp}(D) \to \mathrm{SO}(V).$$

The set

$$\{e_1 = d_1 \wedge d_2, e_2 = d_1 \wedge d_3, v_0 = d_2 \wedge d_3, f_2 = -d_2 \wedge d_4, f_1 = d_3 \wedge d_4\}$$

forms a good basis of V and the Gram matrix of φ is

$$\begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 2 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

Let $\mathrm{G} = \mathrm{SO}(V)$ and notations such as H, Q and Z are as in Part 1.

Denote by $\mathrm{GSp}(D)_0$ the set of elements in $\mathrm{GSp}(D)$ with determinant in $\mathfrak{o}^\times$. Assume $m \geq 0$ is a nonnegative integer. Roberts and Schmidt in [**23**] consider the open compact subgroup of $\mathrm{GSp}(D)$, called the *paramodular subgroup of level* $\mathfrak{p}^m$, which is the intersection of the stabilizer of the lattice

$$\mathbb{M}_m = \mathfrak{p}^{-m} d_1 \oplus \mathfrak{o} d_2 \oplus \mathfrak{o} d_3 \oplus \mathfrak{o} d_4$$

and the subgroup $\mathrm{GSp}(D)_0$. Explicitly, it consists of matrices in the set

$$\begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-m} \\ \mathfrak{p}^m & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^m & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^m & \mathfrak{p}^m & \mathfrak{p}^m & \mathfrak{o} \end{bmatrix} \cap \mathrm{GSp}(D)_0.$$

The element $\begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^m & & & \\ & -\varpi^m & & \end{bmatrix}$ in $\mathrm{GSp}(D)$ normalizes the paramodular subgroup of level $\mathfrak{p}^m$ whose square lies in the center. It is an analog of the Atkin-Lehner element of level $\mathfrak{p}^m$ for $\mathrm{GL}_2(k)$.

Denote by $\mathrm{K}(\mathfrak{p}^m)$ the image of the paramodular subgroup of level $\mathfrak{p}^m$ under j. Then $\mathrm{K}(\mathfrak{p}^m)$ is an open compact subgroup of $\mathrm{SO}(V)$ stabilizing the lattice

$$\mathbb{L}_m = \mathfrak{o}e_1 \oplus \mathfrak{o}e_2 \oplus \mathfrak{p}^m v_0 \oplus \mathfrak{p}^m f_2 \oplus \mathfrak{p}^m f_1 = (\wedge^2 \mathbb{M}_m)^\vee.$$

The group $\mathrm{K}(\mathfrak{p}^m)$ contains the subgroup $\mathrm{Q}_{(m)}$ and the affine Weyl element $w_{s,m}$ for $s \in I_0$. Let us denote $w_{s_{\epsilon_1+\epsilon_2},m}$ by t_m. We note that in this case the set of even number of sign changes I_0 consists only one element $s_{\epsilon_1+\epsilon_2}$ which lifts to t_m in $\mathrm{K}(\mathfrak{p}^m)$. The Atkin-Lehner element $\begin{bmatrix} & & 1 & \\ & & & -1 \\ \varpi^m & & & \\ & -\varpi^m & & \end{bmatrix}$ maps to

$$u_m = \begin{bmatrix} & & & & \varpi^{-m} \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ \varpi^m & & & & \end{bmatrix}$$

in G under j and also stabilizes $\mathbb{L}_m$. u_m is a lift of the odd sign change s_{ϵ_1} and $I = \{s_{\epsilon_1}, s_{\epsilon_1+\epsilon_2}\}$. One can then check the following properties: u_m normalizes $\mathrm{K}(\mathfrak{p}^m)$; $\mathrm{K}(\mathfrak{p}^m)$ is generated by $\mathrm{Q}_{(m)}$ and $u_m \mathrm{Q}_{(m)} u_m^{-1}$; $\mathrm{Stab}_{\mathrm{G}}(\mathbb{L}_m)$ is generated by $\mathrm{K}(\mathfrak{p}^m)$ and u_m and contains $\mathrm{K}(\mathfrak{p}^m)$ with index 2. Let $t'_m = t_m w_{\epsilon_1-\epsilon_2,0}$. We will use a decomposition

$$\mathrm{K}(\mathfrak{p}^m) = \mathrm{Z}(\mathfrak{p}^{-m})\, \mathrm{Q}_{(m)} \cup \mathrm{Z}(\mathfrak{p}^{-m}) t'_m\, \mathrm{Z}(\mathfrak{p}^{-m+1})\, \mathrm{Q}_{(m)}\,. \tag{6.2.1}$$

6.2. Rank 2: $\mathrm{SO}_5(k) \simeq \mathrm{PGSp}_4(k)$

Our goal is to obtain a theory of test vectors for generic representations of G.

Let (π, V_π) be an irreducible generic representation of G. Assume a_π is the conductor of π and $\varepsilon(\pi, s, \psi) = \varepsilon_\pi q^{-a_\pi(s-\frac{1}{2})}$.

Theorem 6.2.1 (Roberts-Schmidt)**.** *The fixed subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ is one dimensional. There is a unique vector v_* on this line such that $I(v_*, s) = L(\pi, s)$ and hence $W_{v_*}(\mathrm{I}) = 1$. Moreover, v_* is an eigenvector of u_{a_π} with eigenvalue ε_π.*

We will summarize the proof in [**23**] of this theorem in the case when π is generic and supercuspidal. We note that in this case, the L-function $L(\pi, s) = 1$ and the Jacquet module π_{Z} is an irreducible P_3-module and is isomorphic to $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$ via the restriction of the Whittaker functions $v \mapsto W_v|_{\mathrm{Q}}$ to Q, which factors through the Jacquet functor J_{Z}.

Let us denote by $[K_2 h K_1]$ the characteristic function of the double coset $K_2 h K_1$ on G which lies in the Hecke algebra $\mathcal{H}(\mathrm{G})$ and induces an operator $V_\pi^{K_1} \to V_\pi^{K_2}$. The Hecke algebra $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p}^m))$ is generated by $[\mathrm{K}(\mathfrak{p}^m) h\, \mathrm{K}(\mathfrak{p}^m)]$ and induces operators on the $\mathrm{K}(\mathfrak{p}^m)$-fixed subspace of V_π. The operators $[\mathrm{K}(\mathfrak{p}^m) h\, \mathrm{K}(\mathfrak{p}^m)]$ and $[\mathrm{K}(\mathfrak{p}^m) h^{-1}\, \mathrm{K}(\mathfrak{p}^m)]$ on $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ are adjoint to each other. For a fixed level $\mathfrak{p}^m$, set

$$T_\lambda = [\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\, \mathrm{K}(\mathfrak{p}^m)] \in \mathrm{End}(V_\pi^{\mathrm{K}(\mathfrak{p}^m)})$$

for $\lambda \in \mathrm{X}_\bullet(\mathrm{T})$. Since $w_{s_{\epsilon_1+\epsilon_2},m}$ lies in $\mathrm{K}(\mathfrak{p}^m)$, one can easily see the Hecke operators

$$T_{\epsilon_1}(= T_{-\epsilon_1}), \quad T_{\epsilon_1+\epsilon_2}(= T_{-(\epsilon_1+\epsilon_2)})$$

at level $\mathfrak{p}^m$ are self-adjoint and hence diagonalizable. Here we note that $\varpi^{\epsilon_1} = u_{m-1}u_m$ and $\varpi^{\epsilon_1+\epsilon_2} = t_{m-1}t_m$.

Define operators θ_λ, δ_λ between the fixed subspaces $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ and $V_\pi^{\mathrm{K}(\mathfrak{p}^{m-1})}$ for $\lambda \in \mathrm{X}_\bullet(\mathrm{T})$ as

$$\theta_\lambda = [\mathrm{K}(\mathfrak{p}^m)\varpi^{-\lambda}\,\mathrm{K}(\mathfrak{p}^{m-1})] : V_\pi^{\mathrm{K}(\mathfrak{p}^{m-1})} \to V_\pi^{\mathrm{K}(\mathfrak{p}^m)},$$

$$\delta_\lambda = [\mathrm{K}(\mathfrak{p}^{m-1})\varpi^{\lambda}\,\mathrm{K}(\mathfrak{p}^{m})] : V_\pi^{\mathrm{K}(\mathfrak{p}^{m})} \to V_\pi^{\mathrm{K}(\mathfrak{p}^{m-1})}.$$

We have the following observation

$$\theta_{\epsilon_1} = u_m\theta_0 u_{m-1}, \quad \theta_{\epsilon_1+\epsilon_2} = \theta_0, \quad \delta_{\epsilon_1} = u_{m-1}\delta_0 u_m, \quad \delta_{\epsilon_1+\epsilon_2} = \delta_0.$$

Roberts and Schmidt proves the following relation.

Lemma 6.2.2 ([**23**], Proposition 6.1)**.** *For $m \geq 2$, on $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ the operators satisfy*

$$T_{\epsilon_1} \circ T_{\epsilon_1+\epsilon_2} - T_{\epsilon_1+\epsilon_2} \circ T_{\epsilon_1} = \theta_{\epsilon_1} \circ \delta_{\epsilon_1+\epsilon_2} - \theta_{\epsilon_1+\epsilon_2} \circ \delta_{\epsilon_1}$$

$$= (u_m\theta_0 u_{m-1}) \circ \delta_0 - \theta_0 \circ (u_{m-1}\delta_0 u_m)$$

Denote by $\mathfrak{c}(\pi)$ the maximal ideal such that $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ is nonzero. In particular, the operator δ_λ is the zero map on $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ for any $\lambda \in \mathrm{X}_\bullet(\mathrm{T})$. One can immediately get:

Lemma 6.2.3. *Assume $\mathfrak{c}(\pi) \subset \mathfrak{p}^2$. The Hecke operators T_{ϵ_1} and $T_{\epsilon_1+\epsilon_2}$ at level $\mathfrak{c}(\pi)$ commute and can be simultaneously diagonalized on $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$.*

Just like for the classical modular forms, we study the eigenvectors of the Hecke operators T_{ϵ_1} and $T_{\epsilon_1+\epsilon_2}$ on the subspace $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi)}$ and called them the Hecke eigenforms of π. These Hecke eigenforms form a basis of $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$. It has been shown that the zeta integral

$$I(v,s) = \int_{a\in k^\times} \ell_\theta(\epsilon_1(a)\,v)|a|^{s-\frac{1}{2}}\;da$$

of a Hecke eigenform v can be expressed by its Hecke eigenvalues.

Proposition 6.2.4 ([**23**], Lemma 7.4.4)**.** *Assume $\mathfrak{c}(\pi) \subset \mathfrak{p}^2$. Let $v \in V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ be a Hecke eigenform and for $\lambda = \epsilon_1, \epsilon_1 + \epsilon_2$ let $\mu_\lambda \in \mathbb{C}$ be the constant such that*

$$T_\lambda v = \mu_\lambda v.$$

Assume

$$c_{(a,b)} = \ell_\theta(\varpi^{a\epsilon_1 + b\epsilon_2}\, v)$$

for $a, b \in \mathbb{Z}$, then

$$\mu_{\epsilon_1} c_{(a,0)} = q^3 c_{(a+1,0)} + q^2 c_{(a,1)} + c_{(a-1,0)},\ a \geq 0$$

$$\mu_{\epsilon_1+\epsilon_2} c_{(a,0)} = q^4 c_{(a+1,1)},\ a \in \mathbb{Z}$$

which combine together to the recurrence relation

$$q^3 c_{(a+1,0)} - \mu_{\epsilon_1} c_{(a,0)} + (1 + q^{-2}\mu_{\epsilon_1+\epsilon_2}) c_{(a-1,0)} = 0,\ a \geq 0.$$

Proof. Using the decomposition (6.2.1), we can write $\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)$ into left cosets. Assume $m \geq 2$. We have

$$\begin{aligned} & \mathrm{K}(\mathfrak{p}^m)\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \\ = \;& \cup_{s\in I_0} \mathrm{Z}(\mathfrak{p}^{-m}) w_{s,m}\, \mathrm{Z}(\mathfrak{p}^{-m+1})\, \mathrm{Q}_{(m)}\, \varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \\ = \;& \cup_{s\in I_0} \mathrm{Z}(\mathfrak{p}^{-m}) w_{s,m}\, \mathrm{M}(\mathfrak{o}) x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \\ = \;& \mathrm{Z}(\mathfrak{p}^{-m})\, \mathrm{M}(\mathfrak{o}) x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \cup \mathrm{Z}(\mathfrak{p}^{-m}) t'_m\, \mathrm{M}(\mathfrak{o}) x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m). \end{aligned}$$

Since the Bruhat decomposition of M over $\mathfrak{f}$ implies

$$\mathrm{M}(\mathfrak{o}) = \overline{\mathrm{B}_\mathrm{M}}(\mathfrak{o})\, \mathrm{N}_n(\mathfrak{p}) \cup \overline{\mathrm{B}_\mathrm{M}}(\mathfrak{o}) w_{\epsilon_1-\epsilon_2,0} \overline{\mathrm{B}_\mathrm{M}}(\mathfrak{o})\, \mathrm{N}_n(\mathfrak{p}),$$

the decomposition becomes

$$\begin{aligned} & \mathrm{K}(\mathfrak{p}^m)\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \\ = \; & \mathrm{Z}(\mathfrak{p}^{-m})x_{\epsilon_1-\epsilon_2}(\mathfrak{o})x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \cup \mathrm{Z}(\mathfrak{p}^{-m})w_{\epsilon_1-\epsilon_2,0}x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \cup \\ & \mathrm{Z}(\mathfrak{p}^{-m})t'_m x_{\epsilon_1-\epsilon_2}(\mathfrak{o})x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \cup \mathrm{Z}(\mathfrak{p}^{-m})t'_m w_{\epsilon_1-\epsilon_2,0}x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \\ = \; & \mathrm{Z}(\mathfrak{p}^{-m})x_{\epsilon_1-\epsilon_2}(\mathfrak{o})x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \cup \mathrm{Z}(\mathfrak{p}^{-m})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m) \cup \\ & \mathrm{Z}(\mathfrak{p}^{-m})t'_m x_{\epsilon_1-\epsilon_2}(\mathfrak{o})x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m) \cup t'_m x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m). \end{aligned}$$

Since v is fixed by $x_{-\epsilon_1}(\mathfrak{p}^m)$, whose commuter with $x_{\epsilon_1}(\mathfrak{p}^{-1})$ lies in $\mathrm{K}(\mathfrak{p}^m)$, we get

$$\ell_\theta(\varpi^{a\epsilon_1} g\, v) = \ell_\theta(\varpi^{a\epsilon_1} x_{\epsilon_2}(c) g\, v) = \psi(c)\ell_\theta(\varpi^{a\epsilon_1} g\, v),\ \forall c \in \mathfrak{p}^{-1}$$

and hence $\ell_\theta(\varpi^{a\epsilon_1} g\, v) = 0$ for $g \in \mathrm{Z}(\mathfrak{p}^{-m})t'_m x_{\epsilon_1-\epsilon_2}(\mathfrak{o})x_{\epsilon_1}(\mathfrak{o})\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m)$. The definition $T_{\epsilon_1} v = \int_{\mathrm{K}(\mathfrak{p}^m)\varpi^{\epsilon_1}\,\mathrm{K}(\mathfrak{p}^m)} g\, v\; dg$ results in for integer $a \geq 0$

$$\mu_{\epsilon_1} c_{(a,0)} = q^3 c_{(a+1,0)} + q^2 c_{(a,1)} + \ell_\theta((a-1)\varpi^{\epsilon_1} \int_{\mathfrak{o}} x_{-\epsilon_1}(y\varpi^{m-1})\, v\; dy).$$

We use the following trick

$$\begin{aligned} I(\int_{\mathfrak{o}} x_{-\epsilon_1}(y\varpi^{m-1})\, v\; dy, s) &= \gamma(\pi, s, \psi)^{-1} q^{m(s-\frac{1}{2})} I(u_m \int_{\mathfrak{o}} x_{-\epsilon_1}(y\varpi^{m-1})\, v\; dy, 1-s) \\ &= \gamma(\pi, s, \psi)^{-1} q^{m(s-\frac{1}{2})}\, \mathrm{vol}(\mathfrak{o}) I(u_m v, 1-s) = I(v, s) \end{aligned}$$

gotten by applying the functional equation twice. Here we used the simpler formula by the fact that the vector $u_m \int_{\mathfrak{o}} x_{-\epsilon_1}(y\varpi^{m-1})\, v$ is fixed by $\mathrm{Q}(\mathfrak{o})$. Then comparing the coefficients of q^{-s} on this equation, one can get

$$\ell_\theta(\varpi^{(a-1)\epsilon_1} \int_{\mathfrak{o}} x_{-\epsilon_1}(y\varpi^{m-1})\, v\; dy) = c_{(a-1,0)}.$$

Let us do the other Hecke operator $T_{\epsilon_1+\epsilon_2}$. Similarly we can get

$$\begin{aligned}
&\mathrm{K}(\mathfrak{p}^m)\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)\\
=\;& \cup_{s\in I_0}\mathrm{Z}(\mathfrak{p}^{-m})w_{s,m}\,\mathrm{Z}(\mathfrak{p}^{-m+1})\,\mathrm{Q}_{(m)}\,\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)\\
=\;& \cup_{s\in I_0}\mathrm{Z}(\mathfrak{p}^{-m})w_{s,m}\,\mathrm{Z}(\mathfrak{p}^{-m+1})x_{\epsilon_1}(\mathfrak{o})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)\\
=\;& \mathrm{Z}(\mathfrak{p}^{-m})x_{\epsilon_1}(\mathfrak{o})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)\cup t'_m\,\mathrm{Z}(\mathfrak{p}^{-m+1})x_{\epsilon_1}(\mathfrak{o})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)
\end{aligned}$$

It follows v is fixed by $x_{-\epsilon_1}(\mathfrak{p}^m)$ that

$$\ell_\theta(\varpi^{a\epsilon_1}g\,v)=0$$

for $g \in \mathrm{Z}(\mathfrak{p}^{-m})t'_m\,\mathrm{Z}(\mathfrak{p}^{-m+1})x_{\epsilon_1}(\mathfrak{o})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)$. (We again omit some detail of computation here.) Since $T_{\epsilon_1+\epsilon_2}v = \int_{\mathrm{K}(\mathfrak{p}^m)\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)} g\,v\,dg$, this results in

$$\mu_{\epsilon_1+\epsilon_2}c_{(a,0)} = q^4 c_{(a+1,1)}.$$

The last assertion follows by some easy algebra. □

Remark 6.2.5. There are two parts which we omitted in the proof for showing that on some cosets $g\,\mathrm{K}(\mathfrak{p}^m)$. The result $\ell_\theta(\epsilon_1(a)g\,v)=0$ uses highly the fact that $x_{\epsilon_n}(\mathfrak{p}^{-1})^{w_{s,m}}$ sits in $\mathrm{Q}_{(m-1)}$ for $s\in I_0$, $s\neq 1$. However, this can not be achieved for $n>2$. The recurrence relation currently can not be obtained for $n>2$.

Since the zeta integral $I(v,s)$ on any fixed vector is a generating function of $\mathrm{vol}(\mathfrak{o}^\times)c_{(a,0)}q^{3a/2}$, $a\geq 0$, the recurrence relation on eigenforms shows the following:

Lemma 6.2.6 ([**23**], Proposition 7.4.5)**.** *Assume $\mathfrak{c}(\pi)\subset\mathfrak{p}^2$. Then if $v\in V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ is an eigenform and $T_\lambda v=\mu_\lambda v$, then*

$$(1-q^{-3/2}\mu_{\epsilon_1}q^{-s}+(1+q^{-2}\mu_{\epsilon_1+\epsilon_2})q^{-2s})I(v,s)=(1-q^{-1})c_{(0,0)}.$$

Equivalently,

$$I(v,s) = \frac{(1-q^{-1})\ell_\theta(v)}{1 - q^{-3/2}\mu_{\epsilon_1} q^{-s} + (1+q^{-2}\mu_{\epsilon_1+\epsilon_2})q^{-2s}}.$$

We recall that $\mathrm{K}(\mathfrak{p}^m)\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)$ has a decomposition

$$\mathrm{Z}(\mathfrak{p}^{-m})x_{\epsilon_1}(\mathfrak{o})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m) \cup t'_m\,\mathrm{Z}(\mathfrak{p}^{-m+1})x_{\epsilon_1}(\mathfrak{o})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m)$$

and hence equals to

$$\mathrm{Z}(\mathfrak{p}^{-m})x_{\epsilon_1}(\mathfrak{o})x_{\epsilon_2}(\mathfrak{o})\varpi^{\epsilon_1+\epsilon_2}\,\mathrm{K}(\mathfrak{p}^m) \cup (\epsilon_1+\epsilon_2)(\varpi^{-1})\,\mathrm{Q}_{(m-1)}\,\mathrm{K}(\mathfrak{p}^m).$$

On the other hand, $\mathrm{K}(\mathfrak{p}^{m-1}) = (\mathrm{Z}(\mathfrak{p}^{-m+1}) \cup t'_{m-1}\,\mathrm{Z}(\mathfrak{p}^{-m+2}))\,\mathrm{Q}_{(m-1)}$. The next step is to make sure at the level $\mathfrak{c}(\pi)$, a Hecke eigenform v satisfies $I(v,s) \neq 0$ which is equivalent to the condition $\ell_\theta(v) \neq 0$.

Suppose $\mathfrak{c}(\pi) = \mathfrak{p}^m$ and $m \geq 2$. For $v \in \mathrm{K}(\mathfrak{p}^m)$, $\delta_0 v \in \pi^{\mathrm{K}(\mathfrak{p}^{m-1})} = 0$ implies for all $\lambda \in \mathrm{X}_\bullet(\mathrm{T})$ and all integers $a, b \geq 0$

$$q^{m-1}\int_{\mathrm{Q}_{(m-1)}} \ell_\theta(\varpi^\lambda g\, v)\; dg + \ell_\theta(\int_{\mathrm{Z}(\mathfrak{p}^{-m+2})\,\mathrm{Q}_{(m-1)}} \varpi^\lambda t_{m-1} g\, v\; dg) = 0.$$

Then

$$\begin{aligned} & \int_{\mathrm{Q}_{(m-1)}} \ell_\theta(\varpi^{(a-1)\epsilon_1+(b-1)\epsilon_2} g\, v)\; dg \\ = \; & -q^{-1}\ell_\theta(\int_{\mathrm{Q}_{(m-1)}} \varpi^{a\epsilon_1+b\epsilon_2} t'_m g\, v\; dg) \\ = \; & -q^{-1}q^3\ell_\theta(\varpi^{a\epsilon_1+b\epsilon_2}\, v) \\ = \; & -q^2 c_{(a,b)} \end{aligned}$$

Using this result, the Hecke operator $T_{\epsilon_1+\epsilon_2}$ acts on v as

$$\mu_{\epsilon_1+\epsilon_2} c_{(a,b)} = q^4 c_{(a+1,b+1)} - q^2 c_{(a,b)}. \tag{6.2.2}$$

We hence get a relation $(\mu_{\epsilon_1+\epsilon_2} + q^2)c_{(a,b)} = q^4 c_{(a+1,b+1)}$ for integers $a, b \geq 0$. Using this relation and the relations from Proposition 6.2.4, since by the fact that v is fixed by $x_{\epsilon_2}(\mathfrak{o})$ we have $c_{(a,b)} = 0$ for $b < 0$, we get $c_{(0,0)} = 0 \Rightarrow c_{(a,b)} = 0$ for $a, b \in \mathbb{Z} \Rightarrow \ell_\theta(\mathrm{T}\ v) = 0$.

Let us assume π is generic and supercuspidal. In this case, the Jacquet module π_{Z} is non-degenerate and isomorphic to $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$ as a Q-module. We shall prove the assertion that $\ell_\theta(v) \neq 0$ and Theorem 6.2.1.

Knowing that $\mathrm{Q} = \mathrm{U}\,\mathrm{T}\,\mathrm{Q}(\mathfrak{o})$, $\ell_\theta(\mathrm{T}\ v) = 0$ results in $\ell_\theta(\mathrm{Q}\ v) = 0$ and hence $J_{\mathrm{Z}}(v) = 0$. Since v is fixed by H_{x_m}, thus $J_{\mathrm{Z}}(v) = 0$ implies $v = 0$. We conclude the following for a generic supercuspidal representation π.

Lemma 6.2.7 ($n = 2$)**.** *Assume $\mathfrak{c}(\pi) \subset \mathfrak{p}^2$. For any eigenform $v \in V_\pi^{K(\mathfrak{c}(\pi))}$, $\ell_\theta(v) = 0$ if and only if $v = 0$.*

Since we assume π is supercuspidal, we have $L(\pi, s) = 1$ and $a_\pi \geq 2$. For any paramodular vector v of level $\mathfrak{p}^m$, v is fixed by $\mathrm{Q}(\mathfrak{o})$ and H_m. This gives $\Omega(v; X_1, X_2) \in \mathbb{C}[X_1, X_2]$ and

$$\Omega(\omega_m v; X_1^{-1}, X_2^{-1}) = \varepsilon_\pi^2 (X_1 X_2)^{a_\pi - m} \Omega(v; X_1, X_2).$$

Recall that $v \neq 0$ if and only if $\Omega(v; X_1, X_2) \neq 0$. This forces $v \neq 0 \Rightarrow m \geq a_\pi$. In particular, $\mathfrak{c}(\pi) \subset \mathfrak{p}^{a_\pi} \subset \mathfrak{p}^2$. Suppose v is a Hecke eigenform at level $\mathfrak{c}(\pi)$. Then by Lemma 6.2.6 $I(v, s) \in \mathbb{C}[q^{-s}, q^s]$ implies $\mu_{\epsilon_1} = 0$ and $\mu_{\epsilon_1+\epsilon_2} = -q^2$. In particular, every eigenform has same set of eigenvalues and the values $c_{(a,b)}$ for $a, b \in \mathbb{Z}$ are uniquely determines by $c_{(0,0)}$ by the recurrence relations in Proposition 6.2.4 and (6.2.2). The Whittaker functions of all Hecke eigenforms with fixed $c_{(0,0)}$ agrees on Q. These Hecke eigenforms hence have the same image in $\mathrm{ind}_{\mathrm{U}}^{\mathrm{Q}}\,\theta$ under J_{Z}, which is injective on $\mathrm{K}(\mathfrak{c}(\pi))$, and are thus the same. We then conclude that $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ is one dimensional.

Let v_* be the unique Hecke eigenform at level $\mathfrak{c}(\pi) = \mathfrak{p}^m$ with $\ell_\theta(v_*) = c_{(0,0)} = (1-q^{-1})^{-1}$ and $I(v_*, s) = 1$. Then $u_m v_* = \varepsilon v_*$ for some $\varepsilon \in \mathbb{C}$. The functional equation

$$\mathrm{vol}(\mathfrak{o}) I(u_m v_*, 1-s) = \varepsilon_\pi q^{(a_\pi - m)(s-\frac{1}{2})} I(v_*, s)$$

then can be written as $\varepsilon = \varepsilon_\pi q^{(a_\pi - m)(s-\frac{1}{2})}$ which implies

$$\varepsilon = \varepsilon_\pi, \quad a_\pi = m.$$

Moreover, computing $c_{(a,b)}$ we get $c_{(a,b)} = 0$ unless $(a,b) = (0,0), (1,1)$ and

$$\Psi(v_*, X; X_1, X_2) = 1 - X_1 X_2 X^2, \ \Omega(v_*; X_1, X_2) = 1.$$

□

CHAPTER 7

Open compact subgroups and their fixed vectors

In this chapter we define the open compact subgroups $\mathrm{K}(\mathfrak{p}^m)$ of our group $\mathrm{G} = \mathrm{SO}(V)$. These subgroups play the central role in our study of generic representations of G. We recall that V is a split quadratic space of dimension $2n+1$ over k, $(\mathrm{B}, \mathrm{T}, \theta)$ is a generic data of G and $\{e_1, ..., e_n, v_0, f_n, ..., f_1\}$ is a compatible good basis of V.

7.1. Definition of $\mathrm{K}(\mathfrak{p}^m)$, $m \geq 0$

To define the family of subgroups of G, we first define a family of quadratic lattices over $\mathfrak{o}$ in the quadratic space V that defines G.

Definition 7.1.1. For integer $m \geq 0$, let $\mathbb{L}_m$ be the quadratic lattice

$$(\bigoplus_{i=1}^{n} \mathfrak{o} e_i \oplus \mathfrak{p}^m f_i) \oplus \mathfrak{p}^m v_0$$

with associated bilinear form $\langle\ ,\ \rangle_m := \varpi^{-m} \langle\ ,\ \rangle : \mathbb{L}_m \times \mathbb{L}_m \to \mathfrak{o}$.

The Gram matrix for the quadratic lattice $\mathbb{L}_m$ is

$$\begin{bmatrix} & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & 2\varpi^m & & & \\ & 1 & & & & \\ \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & \end{bmatrix}, \quad m \geq 0.$$

The quadratic lattice $\mathbb{L}_m / \mathfrak{p}\mathbb{L}_m$ over the residue field $\mathfrak{f}$ is nondegenerate for $m = 0$ and degenerate for $m \geq 1$. The special fiber of the smoothen of the group scheme $\mathrm{SO}(\mathbb{L}_m)$ is SO_{2n+1} for $m = 0$ and O_{2n} for $m \geq 1$. The smoothen process is as defined in [**8**] by Gan and Yu.

7.1. Definition of $\mathrm{K}(\mathfrak{p}^m)$, $m \geq 0$

For $m \geq 0$, let $\mathrm{J}(\mathfrak{p}^m)$ denote the subgroup $\mathrm{SO}(\mathbb{L}_m)$ of $\mathrm{G}(k)$. Namely,

$$\mathrm{J}(\mathfrak{p}^m) = \{g \in \mathrm{G} \mid g\mathbb{L}_m \subset \mathbb{L}_m\},$$

while the condition that g preserves $\langle\ ,\ \rangle_m$ on $\mathbb{L}_m$ is automatic by $g \in \mathrm{G}$. In particular, $\mathbb{L}_0 = \mathbb{L}$ and $\mathrm{J}(\mathfrak{o}) = \mathrm{G}(\mathfrak{o})$ is the hyperspecial maximal subgroup G_{x_0} of G. Furthermore, $\mathrm{J}(\mathfrak{p})$ is the normalizer K_{x_1} of the parahoric subgroup G_{x_1}.

We shall now define the open compact subgroup $\mathrm{K}(\mathfrak{p}^m)$. It is a normal subgroup of $\mathrm{J}(\mathfrak{p}^m)$ and admits a smooth integral model. The definitions of $\mathrm{J}(\mathfrak{p}^m)$ and $\mathrm{K}(\mathfrak{p}^m)$ depend only on the generic data $(\mathrm{B}, \mathrm{T}, \theta)$ and are independent of the choice of compatible basis.

Definition 7.1.2. Define $\mathrm{K}(\mathfrak{o}) = \mathrm{J}(\mathfrak{o})$. For $m \geq 1$, define the open compact subgroup $\mathrm{K}(\mathfrak{p}^m)$ as the kernel of the composite map

$$\mathrm{SO}(\mathbb{L}_m) \xrightarrow{\text{mod } \mathfrak{p}} \mathrm{SO}(\mathbb{L}_m/\varpi\mathbb{L}_m) \to \mathrm{O}_{2n}(\mathfrak{f}) \xrightarrow{det} \{\pm 1\}.$$

By definition, $\mathrm{K}(\mathfrak{p}^m)$ is a normal subgroup of $\mathrm{J}(\mathfrak{p}^m)$ with index 2 for $m \geq 1$. Let us follow the convention for $n = 2$ in [**23**] and denote by

$$u_m = \begin{bmatrix} & & & \varpi^{-m} \\ & -1 & & \\ & & \ddots & \\ & & & -1 & \\ \varpi^m & & & \end{bmatrix} \in \mathrm{J}(\mathfrak{p}^m) - \mathrm{K}(\mathfrak{p}^m)$$

a lift of the Weyl group element s_{ϵ_1} to $N_{\mathrm{G}}(\mathrm{T})$ in $\mathrm{J}(\mathfrak{p}^m)$ that represents the nontrivial coset in $\mathrm{J}(\mathfrak{p}^m)/\mathrm{K}(\mathfrak{p}^m)$. The element u_m normalizes $\mathrm{K}(\mathfrak{p}^m)$ and is an analog of the Atkin-Lehner element $\left[\begin{smallmatrix} & 1 \\ -\varpi^m & \end{smallmatrix}\right]$ of PGL_2. The element ω_m also normalizes $\mathrm{K}(\mathfrak{p}^m)$.

One should further notice that the hyperspecial maximal open compact subgroup

$$\mathrm{H}_{x_m} = \mathrm{SO}\Big(\bigoplus_{i=1}^{n} \mathfrak{o}e_i \oplus \mathfrak{p}^m f_i\Big)$$

of H is contained in $\mathrm{K}(\mathfrak{p}^m)$. The following is a useful way to decompose $\mathrm{K}(\mathfrak{p}^m)$.

Proposition 7.1.3. *Assume $m \geq 1$ is an integer.*

$$\mathrm{K}(\mathfrak{p}^m) = \left(\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)\right)\left(\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\right)\mathrm{H}_{x_m} \tag{7.1.1}$$

$$= \left(\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\right)\left(\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)\right)\mathrm{H}_{x_m}. \tag{7.1.2}$$

Proof. The subgroup H is the fixer of the anisotropic vector v_0 in G and hence is the fixer of $v_0 \in \varpi^{-m}\mathbb{L}_m$. By definition $\mathrm{K}(\mathfrak{p}^m)$ is the stabilizer of $\mathbb{L}_m$ (resp. its dual $\mathbb{L}_m^{*_m}$ under $\langle\ ,\ \rangle_m$) which fixes $\varpi^m v_0$ (resp. v_0) modulo $\mathfrak{p}\mathbb{L}_m$ (resp. $\mathfrak{p}\mathbb{L}_m^{*_m}$). Therefore, we can identify the orbit space $\mathrm{K}(\mathfrak{p}^m)v_0$, which equals $\mathbb{L}_m^{*_m}$, with the left coset space $\mathrm{K}(\mathfrak{p}^m)/(\mathrm{K}(\mathfrak{p}^m)\cap\mathrm{H})$, which equals $\mathrm{K}(\mathfrak{p}^m)/\,\mathrm{H}_{x_m}$. We claim we can use some operation $x_{-\epsilon_i}(\mathfrak{p}^m)$'s and then some operations $x_{\epsilon_i}(\mathfrak{o})$'s to bring any vector in $\mathbb{L}_m^{*_m}$ back to v_0. This is a tedious routine work. Assume $v = \sum_{i=1}^n a_i e_i + cv_0 + \sum_{j=1}^n b_i\varpi^m f_i$ for some $a_i, b_i \in \mathfrak{o}$, $i = 1, 2, ..., n$ and $c \in 1+\mathfrak{p} \subset \mathfrak{o}^\times$. Then by Hensel's lemma there exists $c_n \in \mathfrak{o}$ such that $x_{\epsilon_1}(-c_1)v = v-(cc_1+c_1^2b_1\varpi^m)e_1+c_1b_1\varpi^m v_0$ and $cc_1+c_1^2b_1\varpi^m = a_1$. Then continuing this process there exists $c_1, c_2, ..., c_n \in \mathfrak{o}$ such that one sees $v' = \prod_{i=1}^n x_{\epsilon_{n+1-i}}(-c_{n+1-i})v$ is a vector v' of the form $v' = (c+c'\varpi^m)v_0 + \sum_{j=1}^n b_i\varpi^m f_i$ for some $c' \in \mathfrak{o}$. Write $c'' = c+c'\varpi^m \in 1+\mathfrak{p} \subset \mathfrak{o}^\times$. Then this orbit of v_0 under $\mathrm{K}(\mathfrak{p}^m)$ becomes $v'' = \prod_{i=1}^n x_{-\epsilon_{n+1-i}}(-b_{n+1-i}c''^{-1})v' \in (1+\mathfrak{p})v_0$. Since G preserve a quadratic form, and v_0 is anisotropic, this scalar in $1+\mathfrak{p}$ must be 1. Hence $v'' = v_0$ and the claim follows. This shows the containment $\subset$ side of (7.1.1) while the containing $\supset$ side is clear. A similar argument with the lattice $\mathbb{L}_m$ shows (7.1.2). □

As well we have:

Corollary 7.1.4. *Assume $m \geq 1$ is an integer. The subgroup $\mathrm{K}(\mathfrak{p}^m)$ is equal to*

$$\mathrm{H}_{x_m}\left(\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)\right)\left(\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\right) \text{ and } \mathrm{H}_{x_m}\left(\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\right)\left(\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)\right).$$

Proof. This is gotten by taking an inverse of (7.1.1) and (7.1.2). □

7.1. Definition of $\mathrm{K}(\mathfrak{p}^m)$, $m \geq 0$

The open compact subgroups is really only defined up to conjugacy. The ones we defined form two descending filtrations each with the same parity on m in the sense that for each m there is one member in the conjugacy class of $\mathrm{K}(\mathfrak{p}^m)$ in G such that we have the descending chains of subgroups with the same parity on m. Let us describe them in a more explicit way below.

Let C be the fundamental alcove in the affine apartment $\mathcal{A}(\mathrm{G})$ of T with respect to the polarization Φ_{G}^{+}. The closure $\overline{C}$ of C is a fundamental domain under the action of the affine Weyl group. For $m \in \mathbb{Z}$, the building points x_m are congruent to either x_0 or x_1, depending on the parity of m. $\mathrm{J}(\mathfrak{p}^m) = \mathrm{SO}(\mathbb{L}_m)$ is an open compact subgroup of G and is contained in the (unique) maximal open compact subgroup K_{x_m} of G.

Definition 7.1.5. For integer $m \geq 0$, the congruence subgroup $\mathrm{K}_0(\mathfrak{p}^m)$ is the unique open compact subgroup contained in either K_{x_0} or K_{x_1} that is conjugate to $\mathrm{K}(\mathfrak{p}^m)$.

More precisely, if $m = 2m' + e$, $e \in \{0, 1\}$, then $\mathrm{K}_0(\mathfrak{p}^m)$ is a subgroup of $\mathrm{SO}(\mathbb{L}'_m)$, which is the kernel of the composite map

$$\mathrm{SO}(\mathbb{L}'_m) \xrightarrow{\text{mod } \mathfrak{p}} \mathrm{SO}(\mathbb{L}'_m/\varpi\mathbb{L}'_m) \to \mathrm{O}_{2n}(\mathfrak{f}) \xrightarrow{det} \{\pm 1\},$$

where

$$\mathbb{L}'_m = (\bigoplus_{i=1}^{n} \mathfrak{o}e_i \oplus \mathfrak{p}^e f_i) \oplus \mathfrak{p}^{m'+e} v_0$$

is the quadratic lattice in V. The quadratic lattices $(\mathbb{L}'_m, \langle\ ,\ \rangle)$ and $(\mathbb{L}_m, \langle\ ,\ \rangle_m)$ are isomorphic. The open compact subgroups $\mathrm{K}_0(\mathfrak{p}^m)$ and $\mathrm{K}(\mathfrak{p}^m)$ are conjugate by $\varpi^{m'(\epsilon_1+\epsilon_2+\ldots+\epsilon_n)}$ in T and H_{x_e} is contained in $\mathrm{K}_0(\mathfrak{p}^m)$.

This family forms two descending chains by the parity of m. One sees

$$\mathrm{K}(\mathfrak{o}) = \mathrm{K}_0(\mathfrak{o}) \supset \mathrm{K}_0(\mathfrak{p}^2) \supset \mathrm{K}_0(\mathfrak{p}^4) \supset \cdots \supset \mathrm{H}_{x_0},$$

$$\mathrm{K}(\mathfrak{p}) = \mathrm{K}_0(\mathfrak{p}) \supset \mathrm{K}_0(\mathfrak{p}^3) \supset \mathrm{K}_0(\mathfrak{p}^5) \supset \cdots \supset \mathrm{H}_{x_1}.$$

Moreover, any open compact subgroup K of G containing either H_{x_0} or H_{x_1} contains $\mathrm{K}_0(\mathfrak{p}^m)$ for some $m \geq 0$. Namely, we have

$$\mathrm{H}_{x_0} = \cap_{m:even} \mathrm{K}(\mathfrak{p}^m), \text{ and } \mathrm{H}_{x_1} = \cap_{m:odd} \mathrm{K}(\mathfrak{p}^m). \tag{7.1.3}$$

7.2. $\mathrm{K}(\mathfrak{p}^m)$ with $m = 0, 1$

Recall that $\mathrm{J}(\mathfrak{o})$ is the special orthogonal group of the quadratic lattice $\mathbb{L}$ and is hence equal to $\mathrm{G}(\mathfrak{o})$. We thus have

$$\mathrm{K}(\mathfrak{o}) = \mathrm{J}(\mathfrak{o}) = \mathrm{G}(\mathfrak{o}) = \mathrm{K}_{x_0} = \mathrm{G}_{x_0}.$$

On the other hand, one can check that the parahoric subgroup G_{x_1} stabilizes the quadratic lattice $\mathbb{L}_1$ and is hence contained in $\mathrm{J}(\mathfrak{p}) = \mathrm{SO}(\mathbb{L}_1)$. Since $\mathrm{J}(\mathfrak{p})$ is its normalizer, and K_{x_1} is a maximal open compact subgroup of G. We obtain

$$\mathrm{J}(\mathfrak{p}) = \mathrm{K}_{x_1} \text{ and } \mathrm{G}_{x_1} = \mathrm{K}(\mathfrak{p})$$

while the second equality is gotten from the fact that the group in the first equality contains $\mathrm{G}_{x_1} \subset \mathrm{K}(\mathfrak{p})$ with same index.

We conclude that when $m = 0$, the open compact group $\mathrm{K}(\mathfrak{o})$ is the hyperspecial maximal open compact subgroup G_{x_0} of G; when $m = 1$, $\mathrm{K}(\mathfrak{p})$ is equal to the maximal parahoric subgroup G_{x_1} of G.

Recall that in Chapter 2 of Part 1 we have many good property with these two maximal open compact subgroups K_{x_i}, for $i \geq 0$ integers.

The Iwasawa factorization $\mathrm{G} = \mathrm{B}\,\mathrm{K}_{x_i}$ can then be rewritten as

$$\mathrm{G} = \mathrm{B}\,\mathrm{J}(\mathfrak{o}) = \mathrm{B}\,\mathrm{J}(\mathfrak{p})$$

and by the Cartan decomposition the double cosets of $\mathrm{J}(\mathfrak{o})\backslash \mathrm{G} / \mathrm{J}(\mathfrak{o})$ have representatives $\{\varpi^\lambda\}_{\lambda \in P^+}$, or equivalently,

$$\mathrm{G} = \sqcup_{\lambda \in P^+} \mathrm{K}(\mathfrak{o})\varpi^\lambda \mathrm{K}(\mathfrak{o}).$$

More generally, for any parahoric subgroup G_x, denote by W_x by the subgroup $N_{\mathrm{G}_x}(\mathrm{T})/\mathrm{T}(\mathfrak{o})$ of the extended affine Weyl group $\tilde{W}_{\mathrm{G}}$. Assume x, x' lie in the closed fundamental alcove C. Then $\mathrm{G} = \sqcup_\sigma \mathrm{G}_x \, \sigma \, \mathrm{G}_{x'}$ where σ runs through a set of representatives for the double cosets $W_x \backslash \tilde{W}_{\mathrm{G}} / W_{x'}$. (See [**17**] Proposition 3.1). In particular, $W_{x_1} \simeq W_{\mathrm{H}}$ and let u_1 be a representative of $\mathrm{K}(\mathfrak{p})\backslash \mathrm{J}(\mathfrak{p})$ then

$$\mathrm{G} = \left(\sqcup_{\lambda \in P_{\mathrm{H}}^+} \mathrm{K}(\mathfrak{p})\varpi^\lambda \mathrm{K}(\mathfrak{p})\right) \sqcup \left(\sqcup_{\lambda \in P_{\mathrm{H}}^+} \mathrm{K}(\mathfrak{p}) u_1 \varpi^\lambda \mathrm{K}(\mathfrak{p})\right),$$

where P_{H}^+ denotes the closure of the fundamental Weyl chamber for H. It is then clear that since $P^+ \sqcup u_1 P^+ = P_{\mathrm{H}}^+$ so

$$\mathrm{G} = \sqcup_{\lambda \in P_{\mathrm{G}}^+} \mathrm{J}(\mathfrak{p})\varpi^\lambda \mathrm{J}(\mathfrak{p}). \tag{7.2.1}$$

Another way to view this is to see that K_{x_i} contains a Iwahori subgroup for all integers i. In particular, $\mathrm{K}_{x_i} = \cup_{s \in W'_{x_i}} \mathrm{G}_{b+x_i} \, w_{s,i} \, \mathrm{G}_{b+x_i}$ with $W'_{x_i} = W_{\mathrm{G}}$ for all integers $i \geq 0$. The result (7.2.1) follows $W'_{x_i} \backslash \tilde{W}_{\mathrm{G}} / W'_{x_i} = (\mathrm{T} / \mathrm{T}(\mathfrak{o}))^{W_{\mathrm{G}}}$.

As we have discussed in Section 3.5, these properties leads to the following facts regarding the Satake transform on the Hecke algebras.

Lemma 7.2.1. *The Satake transform* $\mathcal{S} : \mathcal{H}(\mathrm{G}, \mathrm{K}_{x_i}) \to \mathcal{H}(T, T(\mathfrak{o}))$, $f \mapsto \mathcal{S}f(t) = \delta_B^{1/2}(t) \int_U f(tu)\, du$ *induces an isomorphism to* $\mathcal{H}(\mathrm{T}, \mathrm{T}(\mathfrak{o}))^{W'_{x_i}}$ *and is hence a commutative algebra. Any simple* $\mathcal{H}(\mathrm{G}, \mathrm{K}_{x_i})$*-module is of dimension at most* 1.

Proof. This is a recall of Theorem 3.5.7 and Proposition 3.5.9. The last statement uses Proposition 3.5.2 to prove dimension at most one. ☐

Lemma 7.2.2. *Any simple $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{o}))$-module is of dimension at most 1 and simple module of $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p})$ is of dimension at most 2.*

Proof. Since $\mathrm{K}(\mathfrak{o}) = \mathrm{J}(\mathfrak{o})$ so the first assertion is just a repetition of the previous lemma. Since $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p})) = \mathcal{H}(\mathrm{G}, \mathrm{J}(\mathfrak{p})) + R_{u_1}\mathcal{H}(\mathrm{G}, \mathrm{J}(\mathfrak{p}))$ as a subalgebra of $\mathcal{H}(\mathrm{G})$, where $R_{u_1}f(g) = f(gu_1)$, and any $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p}))$-module map $T : \mathcal{V}_1 \to \mathcal{V}_2$ extends uniquely to a $\mathcal{H}(\mathrm{G})$-module map between $\mathcal{H}(\mathrm{G})\mathcal{V}_1$ and $\mathcal{H}(\mathrm{G})\mathcal{V}_2$. Hence we cannot have a simple $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p})$-module of dimension more than 2 which is against the unique extension property since $\mathcal{H}(\mathrm{G}, \mathrm{J}(\mathfrak{p}))$ is commutative by the previous lemma. □

In general, we have:

Lemma 7.2.3. *The commutative algebra $\mathcal{H}(\mathrm{G}, \mathrm{K}_{x_m})$ is a subalgebra of $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p}^m))$ and there is a $\mathbb{C}$-linear map from $\mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m})$ to $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p}^m))$.*

7.3. Existence of Fix vectors

Assume (π, V_π) is an irreducible admissible generic representation of G. Let G^c denote the group generated by the root subgroups U_α, $\alpha \in \Phi_\mathrm{G}$. Assume π has no subspace fixed by G^c. We are interested in the fixed subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$, or equivalently $V_\pi^{\mathrm{K}_0(\mathfrak{p}^m)}$, of the open compact subgroups $\mathrm{K}(\mathfrak{p}^m)$, or equivalently $\mathrm{K}_0(\mathfrak{p}^m)$, defined in the previous sections.

The two families $\mathrm{K}(\mathfrak{p}^m)$ and $\mathrm{K}_0(\mathfrak{p}^m)$ both have their advantages so we will switch them back and forth. For example, (7.1.3) implies that

$$V_\pi^{\mathrm{H}_{x_0}} = \cup_{m:even} V_\pi^{\mathrm{K}(\mathfrak{p}^m)}, \text{ and } V_\pi^{\mathrm{H}_{x_1}} = \cup_{m:odd} V_\pi^{\mathrm{K}(\mathfrak{p}^m)}, \tag{7.3.1}$$

while the containment between the subgroups $\mathrm{K}_0(\mathfrak{p}^m)$ with the same parity implies the contained between the fixed subspaces, namely

$$V_\pi^{\mathrm{K}_0(\mathfrak{o})} \subset V_\pi^{\mathrm{K}_0(\mathfrak{p}^2)} \subset V_\pi^{\mathrm{K}_0(\mathfrak{p}^4)} \subset \cdots \subset V_\pi^{\mathrm{H}_{x_0}},$$

$$V_\pi^{\mathrm{K}_0(\mathfrak{p})} \subset V_\pi^{\mathrm{K}_0(\mathfrak{p}^3)} \subset V_\pi^{\mathrm{K}_0(\mathfrak{p}^5)} \subset \cdots \subset V_\pi^{\mathrm{H}_{x_1}}.$$

As a result of these properties, showing existence of fixed vectors of H_{x_i} shall implies fixed vectors of $\mathrm{K}(\mathfrak{p}^N)$ for certain $N \geq 0$ and hence existence of fixed vectors of $\mathrm{K}(\mathfrak{p}^m)$ for all $N \geq m$ with same parity as m.

This leads to a existence and non-exsitence theorem of the fixed vectors.

Theorem 7.3.1 (Existence 1)**.** *Assume π is irreducible generic and supercuspidal, then there exists a nonzero fixed vector of $\mathrm{K}(\mathfrak{p}^m)$ for some m with both parities and hence for all $\mathrm{K}(\mathfrak{p}^m)$ with m sufficiently large integers. On the other hand, any irreducible supercuspidal representation of G that is not generic contains no fixed vector of $\mathrm{K}(\mathfrak{p}^m)$ for any integer m.*

Proof. By Lemma 5.3.1, $V_\pi^{\mathrm{H}_{x_i}}$ for both $i = 0, 1$ is nonzero when π is irreducible generic and supercuspidal. By Corollary 3.4.2, $V_\pi^{\mathrm{H}_{x_i}}$ is zero for $i = 0, 1$ when π is irreducible supercuspidal but non-generic. □

On the other hand, we have nice properties with the fixed vectors of $\mathrm{K}(\mathfrak{p}^m)$ which separates vector of different "level" m, and the term *level* is hence well-defined.

Proposition 7.3.2. $n > 2$. *Let $v_1, v_2, ..., v_r$ be nonzero vectors in V_π and v_i is invariant under $\mathrm{K}(\mathfrak{p}^{m_i})$ for $1 \leq i \leq r$ with distinct $m_i \geq 0$, then they are linearly independent.*

Proof. Without lost of generality suppose $m_1 > m_2 > ... > m_r \geq 0$, and $v_1 + v_2 + ... + v_r = 0$. Let Σ be the group generated by $\mathrm{K}(\mathfrak{p}^{m_1})$ and $\mathrm{K}(\mathfrak{p}^{m_2}) \cap ... \cap K(\mathfrak{p}^{m_r})$ and fixes the vector

$$v_1 = -(v_2 + ... + v_r).$$

We claim that Σ contains G^c and hence v_1 must be zero which leads to a contradiction.

For $\gamma \in \Phi_{\mathrm{G}}$ with root subgroup U_γ contained in Z, one sees

$$x_\gamma(\mathfrak{p}^{-m_1}), x_{-\gamma}(\mathfrak{p}^{m_1}) \subset \mathrm{K}(\mathfrak{p}^{m_1}) \subset \Sigma, \text{ and}$$

$$x_{-\gamma}(\mathfrak{p}^{m_2}), x_\gamma(\mathfrak{p}^{-m_r}) \subset \mathrm{K}(\mathfrak{p}^{m_2}) \cap ... \cap \mathrm{K}(\mathfrak{p}^{m_r}) \subset \Sigma.$$

The group Σ therefore contains $x_\gamma(\mathfrak{p}^{m_2-2m_1}) = {}^{w_{s_\gamma,m_1}}x_{-\gamma}(\mathfrak{p}^{m_2})$. On the other hand, $x_{-\gamma}(\mathfrak{p}^{2m_1-m_2}) \subset x_{-\gamma}(\mathfrak{p}^{m_2}) \subset \Sigma \Rightarrow w_{s_\gamma,2m_1-m_2} \in \Sigma$. Then the element $\check{\gamma}(\varpi^{m_1-m_2}) = w_{s_\gamma,m_1}^{-1} w_{s_\gamma,2m_1-m_2}$ is also contained in Σ. Conjugating $x_\gamma(\mathfrak{p}^{-m_r})$ and $x_{-\gamma}(\mathfrak{p}^{m_1})$ by arbitrary power of $\check{\gamma}(\varpi^{m_1-m_2})$ we get $\mathrm{U}_{\pm\gamma}(k) \subset \Sigma$ for $\gamma \in \Phi_{\mathrm{G}}$.

One can conjugate $x_{\alpha_i}(\mathfrak{o}) \subset \mathrm{K}(\mathfrak{p}^{m_1}) \subset \Sigma$ by arbitrary power of $\check{\gamma}(\varpi^{m_1-m_2})$ for all such γ. Then one sees all simple root subgroups are contained in the group Σ and so are all positive root subgroups. By a similar method all negative root subgroups are in Σ as well. $\mathrm{T}(\mathfrak{o})$ is contained in $\mathrm{K}(\mathfrak{p}^{m_1})$ and hence in Σ. The group Σ therefore contains the Chevalley group G^c.

By assumption, there is no nonzero vector invariant under G^c hence under Σ. This is contradict to $v_1 \neq 0$. □

Definition 7.3.3. Every nonzero vector in $\pi^{K(\mathfrak{p}^m)}$ is called a *fixed vector of level m.*

Proposition 7.3.4. $\dim V_\pi^{\mathrm{K}(\mathfrak{o})} \leq 1$ *and* $\dim V_\pi^{\mathrm{K}(\mathfrak{p})} \leq 2$.

Proof. Since $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ is a simple $\mathcal{H}(\mathrm{G}, \mathrm{K}(\mathfrak{p}^m))$-module so it follows by Lemma 7.2.2. □

Proposition 7.3.5. *If π has conductor $a_\pi = 0$, then* $\dim V_\pi^{\mathrm{K}(\mathfrak{o})} = 1$.

Proof. If $a_\pi = 0$ then the representation is unramified and $V_\pi^{\mathrm{G}(\mathfrak{o})} \neq 0$. Since $\mathrm{K}(\mathfrak{o}) = \mathrm{G}(\mathfrak{o})$ and by Proposition 7.3.4 $\dim V_\pi^{\mathrm{K}(fo)} \leq 1$, so the dimension must be 1. □

In general, we have the following theorem regarding the fixed subspace at level smaller or equal to the conductor a_π of the representation.

Theorem 7.3.6 (Existence 2)**.** *Assume π is irreducible and supercuspidal, then* $\dim V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})} \leq 1$ *and* $\dim V_\pi^{\mathrm{K}(\mathfrak{p}^m)} = 0$ *for m less than the conductor a_π. Moreover, $\Omega(v)$ is a constant for $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$.*

Since $V_\pi^{\mathrm{K}(\mathfrak{p}^m)} \subset V_\pi^{\mathrm{H}_{xm}}$ for each integer $m \geq 0$, we shall prove Main Theorem 7.3.6 by the $\mathbb{C}$-linear map $\Omega : V_\pi^{\mathrm{H}_{xm}} \to \mathscr{S}_n$ constructed in Section 5.4. Recall that $\mathscr{S}_n = \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$ with a grading $\oplus_{d\in\mathbb{Z}}\mathscr{S}_{n,d}$. Let us first prove a lemma on the image of $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ under Ω.

Lemma 7.3.7. *Assume $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ is a fixed vector of level $m \geq 0$. Then*

$$\Omega(v; X_1, X_2, ..., X_n) \in \oplus_{0\leq d\leq m-a_\pi}\mathscr{S}_{n,d}.$$

Proof. This is by the facts that $x_{\epsilon_n}(\mathfrak{o}), x_{-\epsilon_1}(\mathfrak{p}^m) \subset \mathrm{K}(\mathfrak{p}^m)$ and Proposition 5.4.3. □

Let us prove the second Existence Theorem.

Proof of Theorem 7.3.6. If π is not generic, then the Existence Theorem has shown $\dim V_\pi^{\mathrm{K}(\mathfrak{p}^m)} = 0$. Assume π is generic and assume there exists a nonzero fixed vector $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ of some level m. Then by Lemma 7.3.7 and the degree of $\Omega(v)$, we get $m - a_\pi \geq 0$ and if $m = a_\pi$ then we claim $\Omega(v)$ lies in $\mathbb{C}$. If $\Omega(v) \notin \mathbb{C}$ for some $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$, then the image of $\Omega(v; X_1^{-1}, X_2^{-1}, ..., X_n^{-1})$ in $\oplus_{d<0}\mathscr{S}_{n,d}$ is nonzero. However, the functional equation (5.4.2)

$$\Omega(v; X_1^{-1}, X_2^{-1}, ..., X_n^{-1}) = \varepsilon_\pi^n \Omega(\omega_{a_\pi} v; X_1, X_2, ..., X_n)$$

, the vector $\omega_{a_\pi} v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ is nonzero but $\Omega(\omega_{a_\pi} v; X_1, X_2, ..., X_n) \notin \mathscr{S}_{n,0}$, which is a contradiction. The claim follows. Since Ω is injective on each $V_\pi^{\mathrm{H}_{xm}}$ so the dimension of $V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ is less than or equal to 1. □

Remark 7.3.8. We have remark at the end of Chapter 5 in Remark 5.4.4 that the results we have used to prove the second Existence Theorem still hold after relaxing the condition that π is supercuspidal. Hence the second Existence Theorem is also true for non-supercuspidal representations.

7.4. Fixed vectors at the level equal to the conductor

By the Existence Theorem 2, the conductor is the minimal possible level of a nonzero fixed vector. We have seen the uniqueness of such vector. In this section, we shall investigate more property of vectors at this level.

For simplicity we shall still assume π is irreducible generic and supercuspidal, which implies the L-factors are trivial.

Recall that the conductor is defined by the ε-factor, or equivalently the functional equation. We have two useful functional equations:

$$I(u_m v, 1-s) = \varepsilon_\pi q^{(m-a_\pi)(s-\frac{1}{2})} I(v,s), \quad \forall v \in V_\pi,$$

$$\Omega(\omega_m v; X_1^{-1}, X_2^{-1}, ..., X_n^{-1}) = \varepsilon_\pi^n T_n^{a_\pi - m}\, \Omega(v; X_1, X_2, ..., X_n), \quad \forall v \in V_\pi^{\mathrm{H}_{x_m}}.$$

In particular, for $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ we have

$$I(u_{a_\pi} v, 1-s) = \varepsilon_\pi I(v,s) \text{ and } \Omega(\omega_{a_\pi} v) = \varepsilon_\pi^n \Omega(v)$$

and both are equal to some constant functions.

Assume there exists v_* which is a nonzero vector in $V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$. We obtain the following properties.

Lemma 7.4.1. *$I(v_*, s) = \mathrm{vol}(\mathfrak{o}^\times)\ell_\theta(v_*) \neq 0$ and $u_{a_\pi} v_* = \varepsilon_\pi v_*$.*

Proof. Since v_* is nonzero so $\Omega(v_*)$ is a nonzero constant by Theorem 7.3.6, which let us normalize to 1. Therefore $\Xi(v_*) = \prod_{1\le i<j\le n}(1-q^{-1}X_iX_j)$, and hence has nonzero constant term 1 in $\mathscr{S}_n$. On the other hand, the constant term of $\Xi(v_*)$ equals $\mathrm{vol}(\mathrm{T}(\mathfrak{o}))W_v(\mathrm{I}) = \mathrm{vol}(\mathfrak{o}^\times)^n \ell_\theta(v_*)$. Hence $\ell_\theta(v_*)$ is nonzero. By using this, since

$u_{a_\pi} v_* = \varepsilon v_*$ for some $\varepsilon \in \mathbb{C}$ by dimension one and $\varepsilon I(v_*, 1-s) = I(u_{a_\pi} v_*, 1-s) = \varepsilon_\pi I(v_*, s) \neq 0$ is independent of s, so we get $\varepsilon = \varepsilon_\pi$. □

Proposition 7.4.2. *The Whittaker functional ℓ_θ is nonzero on v_* and the order two group $\mathrm{J}(\mathfrak{p}^{a_\pi}) / \mathrm{K}(\mathfrak{p}^{a_\pi})$ acts on the subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ by a quadratic character which equals to the root number ε_π.*

Proof. Since u_{a_π} represents the nontrivial element of $\mathrm{J}(\mathfrak{p}^{a_\pi}) / \mathrm{K}(\mathfrak{p}^{a_\pi})$, so the assertion follows the previous lemma. □

Proposition 7.4.3. *The $\mathbb{C}[\hat{\mathrm{T}}]^{W_\mathrm{H}}$-submodule $\Omega(V_\pi^{\mathrm{H}_{a_\pi}})$ of $\mathscr{S}_n$ contains $\mathbb{C}[\hat{\mathrm{T}}]^{W_\mathrm{H}}$.*

Proof. Ξ is a $\mathbb{C}[\hat{\mathrm{T}}]^{W_\mathrm{H}}$-module map on $V_\pi^{\mathrm{H}_{a_\pi}}$ hence so is Ω. Since the image contains a unit 1 because $\Omega(v_*) = 1$ for a vector v_* in $V_\pi^{\mathrm{H}_{xm}}$, so the assertion follows. □

To end this discussion, let us give some examples of supercuspidal representations with a nonzero fixed vector at the level equal to the conductor.

Example 7.4.4. Let τ be an inflation of an irreducible cuspidal representation τ of $\mathrm{G}(\mathfrak{f}) \simeq \mathrm{G}_{x_0} / \mathrm{G}_{x_0}^+$ to G_{x_0}. Assume τ is generic in the sense that the $\mathrm{Z}(\mathfrak{f})$-covariants $\tau_{\mathrm{Z}(\mathfrak{f})}$ is the standard representation $\mathrm{ind}_{\mathrm{N}_{n+1}}^{\mathrm{P}_{n+1}} \psi$ of Gelfand and Kazhdan of the mirabolic group P_{n+1}. The compactly induced representation

$$\pi = \mathrm{ind}_{\mathrm{G}_{x_0}}^{\mathrm{G}} \tau$$

of G has a nonzero subspace of $\mathrm{G}_{x_0}^+$-invariants which is isomorphic to τ as a G_{x_0}-space. Hence π is a generic depth zero supercuspidal representation of conductor $a_\pi = 2n$.[1] By Mackey's restriction formula we have

$$\pi|_{\mathrm{K}_0(\mathfrak{p}^{2n})} = \mathrm{ind}_{\mathrm{G}_{x_0}}^{\mathrm{G}} \tau|_{\mathrm{K}_0(\mathfrak{p}^{2n})} = \sum_{g \in \mathrm{G}_{x_0} \backslash \mathrm{G} / \mathrm{K}_0(\mathfrak{p}^{2n})} \mathrm{ind}_{\mathrm{G}_{x_0}^g \cap \mathrm{K}_0(\mathfrak{p}^{2n})}^{\mathrm{K}_0(\mathfrak{p}^{2n})} \tau^g|_{\mathrm{G}_{x_0}^g \cap \mathrm{K}(\mathfrak{p}^{2n})}.$$

[1]In [**7**], DeBaker and Reeder conjecture that all generic depth zero supercuspidal representations of G are arisen in this way.

There exists $g = \varpi^{-\sum_{i=1}^{n}(i-1)\epsilon_i} \in \mathrm{T}$ such that the intersection $\mathscr{C}$ of G_{x_0} with the group $\mathrm{K}_0(\mathfrak{p}^{2n})^{g^{-1}}$ has image ${}^{w_\mathrm{P}}\,\mathrm{B}_\mathrm{H}(\mathfrak{f})$ in the reductive quotient $\mathrm{G}(\mathfrak{f})$. Then since $\tau|_{\mathrm{B}_\mathrm{H}(\mathfrak{f})^{w_\mathrm{M}}} = \mathrm{ind}_\mathrm{I}^{\mathrm{B}_\mathrm{H}(\mathfrak{f})^{w_\mathrm{M}}} 1$ contains a trivial representation of $\mathrm{B}_\mathrm{H}(\mathfrak{f})^{w_\mathrm{M}}$, so the representation $\pi|_{\mathrm{K}_0(\mathfrak{p}^{2n})}$ contains a trivial representation of $\mathrm{K}_0(\mathfrak{p}^{2n})$. Hence the fixed subspace $\pi^{\mathrm{K}_0(\mathfrak{p}^{2n})}$ is nonzero.

The example above was modeled by Mark Reeder and is the supercuspidal representation of G with the smallest conductor. The next smallest conductor is $2n+1$ and occurs as the conductor of the simple supercuspidal representations of minimal positive depth $1/2n$.

Example 7.4.5. Let G_b^{++} be the prop-p-Sylow subgroup of G_b^{+}, the pro-unipotent radical of the Iwahori subgroup G_b, occurs as the next Moy-Prasad subgroup of G_b in the filtration $\mathrm{G}_b \supset \mathrm{G}^+ \supset \mathrm{G}_b^{++} \supset \ldots$ and we have $\mathrm{G}_b^{+} \,/\, \mathrm{G}_b^{++} \simeq \oplus_{i=0}^{n} \mathrm{U}_{\psi_i}(\mathfrak{f})$. Set $\mathrm{K}_b^{+} = \mathrm{K}_b \cap \mathrm{G}_b^{+}$. Let

$$\pi = \mathrm{ind}_{\mathrm{K}_b^+}^{\mathrm{G}} \chi$$

be a simple supercuspidal representation for some affine generic character χ, which is the inflation of a character on $\mathrm{G}_b^{+} \,/\, \mathrm{G}_b^{++}$ to K_b^{+} and is generic in the sense that it is nontrivial on $\mathrm{U}_{\psi_i}(\mathfrak{f})$ for $0 \leq i \leq n$. By Mackey's restriction formula we have

$$\pi|_{\mathrm{K}_0(\mathfrak{p}^{2n+1})} = \mathrm{ind}_{\mathrm{K}_b^+}^{\mathrm{G}} \tau|_{\mathrm{K}_0(\mathfrak{p}^{2n+1})} = \sum_{g \in \mathrm{K}_b^+ \backslash \mathrm{G} \,/\, \mathrm{K}_0(\mathfrak{p}^{2n+1})} \mathrm{ind}_{(\mathrm{K}_b^+)^g \cap \mathrm{K}_0(\mathfrak{p}^{2n+1})}^{\mathrm{K}_0(\mathfrak{p}^{2n+1})} \chi^g|_{(\mathrm{K}_b^+)^g \cap \mathrm{K}(\mathfrak{p}^{2n})}.$$

There exists $g = \varpi^{-\sum_{i=1}^{n}(i-1)\epsilon_i} \in \mathrm{T}$ such that the intersection $\mathscr{C}$ of K_b^{+} with the group $\mathrm{K}_0(\mathfrak{p}^{2n+1})^{g^{-1}}$ has trivial image in the quotient $\mathrm{G}_b^{+} \,/\, \mathrm{G}_b^{++} \simeq \oplus_{i=0}^{n} \mathrm{U}_{\psi_i}(\mathfrak{f})$. Then since $\chi|_{\mathrm{K}_b^+ \cap \mathrm{K}(\mathfrak{p}^{2n})^{g^{-1}}}$ is trivial, so the representation $\pi|_{\mathrm{K}_0(\mathfrak{p}^{2n+1})}$ contains a trivial representation of $\mathrm{K}_0(\mathfrak{p}^{2n+1})$ and the fixed subspace $\pi^{\mathrm{K}_0(\mathfrak{p}^{2n+1})}$ is nonzero.

CHAPTER 8

Action of the Hecke operators

In the previous chapter, we defined the open compact subgroups $\mathrm{K}(\mathfrak{p}^m)$ for $\mathrm{G}(k)$ and have discussed many properties for the groups and the subspaces fixed by them. It is natural for us to look at the action of the Hecke operators given by bi-$\mathrm{K}(\mathfrak{p}^m)$-invariant functions on the fixed subspaces $\mathrm{K}(\mathfrak{p}^m)$. Since the subgroups contains the hyperspecial open compact subgroups H_{x_m} of the smaller orthogonal group $\mathrm{H}(k)$, the action will be very close to how the spherical Hecke algebra act. We hence will be able to see many nice properties carried by such operators.

In this chapter, we will define the level raising operators, which sends fixed vectors of smaller level to the larger ones, by using the spherical Hecke algebra for $\mathrm{H}(k)$. Then we put our attention on the Hecke actions of $\mathrm{K}(\mathfrak{p}^m)$-double cosets. Some of these, which we shall call T_1, T_2,..., T_n' can be simultaneously diagonalized and make the fixed subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ decompose into common eigenspaces. From this observation, we then argue about the vectors at the minimal level and shall prove that this subspace must be of dimension one.

We will fixed the notation as defined in Part 1 and denote by b the barycenter of the fundamental alcove C. The alcoves containing $x_m \pm b$ contains both x_m and $x_{m\pm1}$. The parahoric subgroup $\mathrm{H}_{x_m\pm b}$ is a Iwahori subgroup of H with Iwahori factorizations

$$\begin{aligned}\mathrm{H}_{x_m\pm b} &= (\mathrm{H}_{x_m\pm b}\cap \mathrm{V})(\mathrm{H}_{x_m\pm b}\cap \mathrm{T})(\mathrm{H}_{x_m\pm b}\cap\overline{\mathrm{V}})\\ &= (\mathrm{H}_{x_m\pm b}\cap^{\omega_0}\mathrm{V})(\mathrm{H}_{x_m\pm b}\cap \mathrm{T})(\mathrm{H}_{x_m\pm b}\cap^{\omega_0}\overline{\mathrm{V}})\end{aligned}$$

and contained in the parahoric subgroups H_{x_m} and $\mathrm{H}_{x_{m\pm1}}$ with $\mathrm{H}_{x_m+b} \,/\, \mathrm{H}^+_{x_m} \simeq \mathrm{B}_{\mathrm{H}}(\mathfrak{f})$, $\mathrm{H}_{x_m+b} \,/\, \mathrm{H}^+_{x_{m+1}} \simeq {}^{\omega_{m+1}}\,\mathrm{B}_{\mathrm{H}}(\mathfrak{f})$ (or $\mathrm{H}_{x_m-b} \,/\, \mathrm{H}^+_{x_m} \simeq \overline{\mathrm{B}_{\mathrm{H}}}(\mathfrak{f})$, $\mathrm{H}_{x_m-b} \,/\, \mathrm{H}^+_{x_{m-1}} \simeq {}^{\omega_{m-1}}\overline{\mathrm{B}_{\mathrm{H}}}(\mathfrak{f})$). We have decompositions

$$\mathrm{H}_{x_m} = \bigcup_{s\in W_{\mathrm{H}}} \mathrm{H}_{x_m+b}\, w_{s,m}\, \mathrm{H}_{x_m+b}, \text{ and } \mathrm{H}_{x_{m\pm1}} = \bigcup_{s\in W_{\mathrm{H}}} \mathrm{H}_{x_m\pm b}\, w_{s,m\pm1}\, \mathrm{H}_{x_m\pm b}$$

where again $w_{s,m}$ (resp. $w_{s,m\pm1}$) denotes any lift of s to H_{x_m} (resp. $\mathrm{H}_{x_{m\pm1}}$).

8.1. Level raising operators

Since the union of the fixed vectors under $\mathrm{K}_0(\mathfrak{p}^m)$ is equal to the union of the fixed subspaces of H_{x_0} and H_{x_1}. That is, we have

$$\cup_{m\geq 0} V_\pi^{\mathrm{K}_0(\mathfrak{p}^m)} = V_\pi^{\mathrm{H}_{x_0}} \cup V_\pi^{\mathrm{H}_{x_1}}.$$

To produce a fixed vector from another, we consider the action

$$\phi * v = \int_{\mathrm{H}} \phi(h')\pi(h'^{-1})v\; dh', \quad \forall \phi \in \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_m}), \quad \forall v \in V_\pi^{\mathrm{H}_{x_m}} \tag{8.1.1}$$

for integers m defined in Section 5.3.

Recall that we have an injective $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$-module homomorphism

$$\Xi : V_\pi^{\mathrm{H}_{x_i}} \to \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$$

satisfying that for $P \in \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$, $v \in V_\pi^{\mathrm{H}_{x_i}}$

$$P \cdot \Xi(v) = \Xi(\varsigma_{\mathrm{H},i}(P) * v) \text{ in } \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}. \tag{8.1.2}$$

For $v \in \mathrm{K}_0(\mathfrak{p}^m)$ and nonzero $\phi \in \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_i})$, the vector $\phi * v$ is then a nonzero fixed vector in $V_\pi^{\mathrm{K}_0(\mathfrak{p}^l)}$ for some level l with same parity as m. Note that the vector space $\mathcal{H}(\mathrm{H}, \mathrm{H}_{x_i})$ is generated by the characteristic functions of the double cosets $\mathrm{H}_{x_i}\, \varpi^\lambda\, \mathrm{H}_{x_i}$,

$\lambda \in P_{\mathrm{H}}^{+}$. We introduce the following notion[1]: the *norm* of a co-character λ is the map $\|\cdot\| : \mathrm{X}_{\bullet}(\mathrm{T}) \to \mathbb{Z}$ such that

$$\|\lambda\| = \max_{1\le i\le n} |m_i|, \quad \text{if } \lambda = \sum_{i=1}^{n} m_i\epsilon_i. \tag{8.1.3}$$

This integer-valued function satisfies the triangle inequality and $\|\lambda\| = 0$ if and only if $\lambda = 0$. Moreover, it is preserved under action of the Weyl group.

Proposition 8.1.1. *Define $\phi_\lambda \in \mathcal{H}(\mathrm{H}, \mathrm{H}_{x_i})$ as the characteristic function of the double coset $\mathrm{H}_{x_i}\,\varpi^{\lambda}\,\mathrm{H}_{x_i}$. Then $\phi_\lambda : V_\pi^{\mathrm{K}_0(\mathfrak{p}^m)} \to V_\pi^{\mathrm{K}_0(\mathfrak{p}^{m+2l})}$ for $m \equiv i \pmod 2$ and $\|\lambda\| \le l$.*

Proof. This is implied by the fact that

$$\varpi^{\lambda}\left(\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{p}^{m'+l})\right)\left(\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^{m'+l+i})\right)\varpi^{-\lambda} \subset \left(\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{p}^{m'})\right)\left(\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^{m'+i})\right).$$

□

There is an isomorphism between the fixed subspace of $\mathrm{K}(\mathfrak{p}^m)$ and the one of $\mathrm{K}_0(\mathfrak{p}^m)$ by translating by $\varpi^{-m'\lambda^{\mathrm{M}}} \in \mathrm{G}$:

$$V_\pi^{\mathrm{K}_0(\mathfrak{p}^{2m'+i})} \to V_\pi^{\mathrm{K}(\mathfrak{p}^{2m'+i})}, \quad v' \mapsto v = \pi(\varpi^{-m'\lambda^{\mathrm{M}}})v',$$

for integer $m' \ge 0$ and $i \in \{0, 1\}$. (Recall $\lambda^{\mathrm{M}} = \epsilon_1 + \epsilon_2 + ... + \epsilon_n \in \mathrm{X}_{\bullet}(\mathrm{T})$.) We define for $\lambda \in P_{\mathrm{H}}^{+}$ the level raising operators η_λ as follows:

$$\eta_\lambda(v) = \int_{\mathrm{H}_{x_i}\,\varpi^{\lambda}\,\mathrm{H}_{x_i}} \pi(\varpi^{-(m'+\|\lambda\|)\lambda^{\mathrm{M}}} h^{-1} \varpi^{m'\lambda^{\mathrm{M}}})v \, dh \tag{8.1.4}$$

which is equal to

$$\eta_\lambda(v) = \int_{\mathrm{H}_{x_{2(m'+\|\lambda\|)+i}}\,\varpi^{-(\lambda+\|\lambda\|\lambda^{\mathrm{M}})}\,\mathrm{H}_{x_{2m'+i}}} \pi(h)v \, dh. \tag{8.1.5}$$

[1] The definition is credit to Cheng-Chiang Tsai and the action is inspired by [**22**] in the PGL(n) case.

Then this operator induces an injective map from fixed space of lower level $2m'+i$ to fixed space of higher level $2(m'+\|\lambda\|)+i$.

In addition to η_λ, to go to level with different parity, we define θ_λ to be the operator

$$\theta_\lambda(v) = \int_{\mathrm{K}(\mathfrak{p}^{m+1})\varpi^{-\lambda}\,\mathrm{K}(\mathfrak{p}^m)} \pi(g)v\;dg, \quad v \in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$$

by the Hecke action. This gives us an operator which raises the level by one. Similarly the Hecke action gives operators

$$\delta_\lambda(v) = \int_{\mathrm{K}(\mathfrak{p}^{m-1})\varpi^{\lambda}\,\mathrm{K}(\mathfrak{p}^m)} \pi(g)v\;dg, \quad v \in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$$

which lowers the level by one. That is,

$$\theta_\lambda = [\mathrm{K}(\mathfrak{p}^{m+1})\varpi^{-\lambda}\,\mathrm{K}(\mathfrak{p}^m)] : V_\pi^{\mathrm{K}(\mathfrak{p}^m)} \to V_\pi^{\mathrm{K}(\mathfrak{p}^{m+1})}$$

$$\delta_\lambda = [\mathrm{K}(\mathfrak{p}^{m-1})\varpi^{\lambda}\,\mathrm{K}(\mathfrak{p}^m)] : V_\pi^{\mathrm{K}(\mathfrak{p}^m)} \to V_\pi^{\mathrm{K}(\mathfrak{p}^{m-1})}.$$

We also define the companion operators

$$\tilde{\theta}_\lambda = \omega_{m+1}\circ\theta_\lambda\circ\omega_m \text{ and } \tilde{\delta}_\lambda = \omega_{m-1}\circ\delta_\lambda\circ\omega_m.$$

We remark that when λ is minuscule, the level raising (resp. level lowering) operators θ_λ (resp. δ_λ) only give two distinct operators. This is because for each $s \in W_{\mathrm{H}}$, one has

$$\mathrm{K}(\mathfrak{p}^{m\pm1})\varpi^{\mp\lambda}\,\mathrm{K}(\mathfrak{p}^m) = \mathrm{K}(\mathfrak{p}^{m\pm1})w_{s,m\pm1}\varpi^{\mp\lambda}w_{s,m}\,\mathrm{K}(\mathfrak{p}^m),$$

while $w_{s,m\pm1}w_{s,m}$ exhaust $\varpi^{\mp\lambda}$ for λ minuscule co-characters listed above. We follow Roberts and Schmidt [**23**] and define the dual operators

$$\theta_\lambda^* = u_{m+1}\circ\theta_\lambda\circ u_m \text{ and } \delta_\lambda^* = u_{m-1}\circ\delta_\lambda\circ u_m.$$

Then these level raising operators θ_λ (resp. level lowering operators δ_λ) for minuscule λ are equal to either of the operators θ_0 and θ_0^* (resp. the operators δ_0 and δ_0^*).

Definition 8.1.2. The *level raising operators* on the fixed subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ of level m are the injective linear maps η_λ, $\lambda \in P_{\mathrm{H}}^+$, and the operators θ_0 and θ_0^*.

Let us now show that θ_0 is also an injective linear map. This implies that $\theta_{\epsilon_1} = \theta_0^*$ is injective as well. (However, in general the level lowering operators are not injective unlike θ_0 and θ_0^*.)

Proposition 8.1.3. *$\theta_0(v) = \int_{\mathrm{K}(\mathfrak{p}^{m+1})\,\mathrm{K}(\mathfrak{p}^m)} \pi(k)v\; dk \neq 0$ for nonzero $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$.*

Proof. Using the decomposition $\mathrm{K}(\mathfrak{p}^m) = \mathrm{H}_{x_m}\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)$ we can get $\mathrm{K}(\mathfrak{p}^{m+1})\,\mathrm{K}(\mathfrak{p}^m) = \mathrm{H}_{x_{m+1}}\,\mathrm{K}(\mathfrak{p}^m) = \bigcup_{s\in W_{\mathrm{H}}} \mathrm{H}_{b+x_m}\, w_{s,m+1}\,\mathrm{K}(\mathfrak{p}^m)$. One also sees $\mathrm{H}_{b+x_m}\, w_{s,m+1}\,\mathrm{H}_{x_m} = (\mathrm{H}_{b+x_m} \cap^{\omega_0} \mathrm{V})w_{s,m+1}w_{s^{-1},m}\,\mathrm{H}_{x_m}$ which implies

$$\mathrm{K}(\mathfrak{p}^{m+1})\,\mathrm{K}(\mathfrak{p}^m) = \bigcup_{s\in W_{\mathrm{H}}} \overline{\mathrm{Z}}(\mathfrak{p}^{m+1})\,\mathrm{N}_n(\mathfrak{o})w_{s,m+1}w_{s^{-1},m}\,\mathrm{K}(\mathfrak{p}^m).$$

Here $w_{s,m+1}w_{s^{-1},m}$ lies in T and equals to ϖ^μ for some $\mu \in P_{\mathrm{H}}^+$ such that $\langle \mu, \epsilon_i\rangle \in \{0,-1\}$ for $1 \le i \le n$ and $\deg \mu$ is even.

Assume $\theta_0(v) = 0$ then it implies $W_{\omega_m\theta_0(v)}(\varpi^\lambda) = 0$ for all $\lambda \in P^+$. Notice that ${}^{\omega_m}(\overline{\mathrm{Z}}(\mathfrak{p}^{m+1})\,\mathrm{N}_n(\mathfrak{o})) = \mathrm{N}(\mathfrak{o})\,\mathrm{Z}(\mathfrak{p}^{-m+1})$. Since $\omega_m\theta_0(v)$ is a positive sum of

$$\sum_{z\in \mathrm{Z}(\mathfrak{p}^{-m+1}/\mathfrak{p}^{-m+2})}\ \sum_{n\in \mathrm{N}_n(\mathfrak{o}/\mathfrak{p})} \pi(zn\varpi^{\mu'})(\omega_m v)$$

with $\langle \mu', \epsilon_i\rangle \in \{0,1\}$ for $1 \le i \le n$ and $\deg \mu$ even. For $\lambda \in P^+$, we get $W_{\omega_m\theta_0(v)}(\varpi^\lambda)$ is a positive sum of $W_{\omega_m v}(\varpi^{\lambda+\mu'})$.

Since $\omega_m v \in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ is nonzero, so $W_{\omega_m v}|_{\mathrm{T}} \neq 0$. Take λ to be the maximal element in P^+ under the Bruhat order $\ge$ such that $W_{\omega_m v}(\varpi^\lambda) \neq 0$. (This is feasible since $\Xi(\omega_m v)$ is in $\mathbb{C}[\hat{\mathrm{T}}]$.) Then since for $\mu' \neq 0$, $\lambda + \mu' \ge 0$, we get that $0 = W_{\omega_m\theta_0(v)}$ is a multiple of $W_{\omega_m v}(\varpi^\lambda) \neq 0$, a contradiction. Hence $\theta_0(v)$ must be nonzero. □

Corollary 8.1.4. *If $V_\pi^{\mathrm{K}(\mathfrak{p}^c)} \neq 0$, then $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ is nonzero for all $m \ge c$.*

Proof. This is immediate by injectivity of the level raising operators η_λ and θ_0. $\square$

Before we end this section, we give an early version of the dimension count.

Proposition 8.1.5. *Suppose $\mathfrak{c}(\pi) = \mathfrak{p}^{c(\pi)}$ is the maximal ideal such that the fixed space $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ is nonzero, then*

$$\dim V_\pi^{\mathrm{K}(\mathfrak{p}^m)} = \dim V_\pi^{\mathrm{K}_0(\mathfrak{p}^m)} \geq \#\left\{\lambda \in P_{\mathrm{H}}^+ \mid \|\lambda\| \leq \left\lfloor \frac{m - c(\pi)}{2} \right\rfloor \right\}$$

where P_{H}^+ denotes the fundamental Weyl chamber of H*.*

Proof. We recall that ϕ_λ, $\lambda \in P_{\mathrm{H}}^+$, forms a basis of $\mathcal{H}(\mathrm{H}, \mathrm{H}_{x_i})$ and the linear map Ξ on $\pi^{\mathrm{H}_{x_i}}$ is a $\mathcal{H}(\mathrm{H}, \mathrm{H}_{x_i})$-module homomorphism. Take any nonzero vector v in $V_\pi^{\mathrm{K}_0(\mathfrak{p}^{c(\pi)})}$ (or $\theta_0(v) \in V_\pi^{\mathrm{K}_0(\mathfrak{p}^{c(\pi)+1})}$ if the parity does not match) then $v \in V_\pi^{\mathrm{H}_{x_i}}$ and $\Xi(v) \neq 0$, we get $\Xi(\phi_\lambda * v)$, $\lambda \in P_{\mathrm{H}}^+$ are linearly independent. The statement follows the fact that $\phi_\lambda * v$ sits in $V_\pi^{\mathrm{K}_0(\mathfrak{p}^{c(\pi)+2\|\lambda\|})} \subset V_\pi^{\mathrm{K}_0(\mathfrak{p}^m)}$ for $c(\pi) + 2\|\lambda\| \leq m$. $\square$

8.2. Hecke operators

The decomposition

$$\mathrm{K}(\mathfrak{p}^m) = \left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\mathrm{H}_{x_m} = \mathrm{H}_{x_m}\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)$$

provides some advantages in working with the double coset of $\mathrm{K}(\mathfrak{p}^m)$, especially those ones whose double coset representatives are in the maximal torus. The computation can then be reduced to computing the double cosets in $\mathrm{H}_{x_m}\,\mathrm{T}\,\mathrm{H}_{x_m}$ on which one has the Cartan decomposition and where the Iwasawa decomposition can also be useful. In this section, we will first work on the composition of two Hecke actions of double cosets. In the next section, we will look at how the values of the Whittaker functions varies after applying the Hecke action.

From now on, we assume that the rank $n \geq 2$ and the level $m \geq 2$.

8.2. Hecke operators

We consider λ to be the minuscule co-weights

$$\lambda_1 = \epsilon_1,\ \lambda_2 = \epsilon_1 + \epsilon_2, ...,\ \lambda_{n-1} = \epsilon_1 + \epsilon_2 + ... + \epsilon_{n-1},\ \text{and}$$

$$\lambda_n = \epsilon_1 + \epsilon_2 + ... + \epsilon_{n-1} + \epsilon_n,\ \lambda_n^* = \epsilon_1 + \epsilon_2 + ... + \epsilon_{n-1} - \epsilon_n$$

in P_{H}^+. Denote by T_i the Hecke operator $\mathrm{ch}_{\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_i}\,\mathrm{K}(\mathfrak{p}^m)}$ and T_i^* its dual $u_m \circ T_i \circ u_m$. Then T_n^* is equal to the operator $\mathrm{ch}_{\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^*}\,\mathrm{K}(\mathfrak{p}^m)}$. Assume π is a supercuspidal representation of G. Then π is unitary and has a G-invariant Hermitian form on the space V_π.

For open compact subgroup K and $h \in \mathrm{G}$, let us denote the Hecke operator on the V_π^K given the characteristic function of the double coset KhK by T_h and write T_λ for T_{ϖ^λ}. Then one has $\langle T_h v, w\rangle = \langle v, T_{h^{-1}} w\rangle$ for $v, w \in V_\pi^K$. That is, T_h and $T_{h^{-1}}$ are adjoint. Then on the fixed subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$, one sees $T_1, T_2, ..., T_{n-1}$ and $T_n' = T_n + T_n^*$ are self-adjoint. A self-adjoint operator on a finite dimensional vector space is diagonalizable. We shall show that the operators $T_1, T_2, ..., T_{n-1}$ and T_n' commute and hence can be diagonalized simultaneously.

Lemma 8.2.1. $\mathrm{H}_{x_m}\,\varpi^\lambda\,\mathrm{H}_{x_m} = \cup_{s\in W_{\mathrm{H}}} \mathrm{H}_{x_m+b}\,\varpi^{s(\lambda)}\,\mathrm{H}_{x_m}$, *if* $\lambda \in P_{\mathrm{H}}^+$ *minuscule.*

Proof. For $1 \le i \le n-1$, one has $w_{\mathrm{H}}(\lambda_i) = -\lambda$. On one hand, $\lambda \in P_{\mathrm{H}}^+$ implies $\varpi^\lambda(\mathrm{H}_{x_m+b}\cap \mathrm{V})\varpi^{-\lambda} \subset (\mathrm{H}_{x_m+b}\cap\mathrm{V})$. On the other hand, $(\mathrm{H}_{x_m+b}\cap\overline{\mathrm{V}})^{w_{s,m}} \subset \mathrm{H}_{x_m+b}$. Therefore since w_{H} can be lifted to H_{x_m}, by using the Bruhat decomposition we get

$$\begin{aligned}
\mathrm{H}_{x_m}\,\varpi^\lambda\,\mathrm{H}_{x_m} &= \mathrm{H}_{x_m}\,\varpi^{-\lambda}\,\mathrm{H}_{x_m} \\
&= \cup_{s\in W_{\mathrm{H}}} \mathrm{H}_{x_m+b}\,w_{s,m}\,\mathrm{H}_{x_m+b}\,\varpi^{-\lambda}\,\mathrm{H}_{x_m} \\
&= \cup_{s\in W_{\mathrm{H}}} \mathrm{H}_{x_m+b}\,w_{s,m}(\mathrm{H}_{x_m+b}\cap\overline{\mathrm{V}})\,\mathrm{T}(\mathfrak{o})(\mathrm{H}_{x_m+b}\cap\mathrm{V})\varpi^{-\lambda}\,\mathrm{H}_{x_m} \\
&= \cup_{s\in W_{\mathrm{H}}} \mathrm{H}_{x_m+b}\,w_{s,m}\varpi^{-\lambda}\,\mathrm{H}_{x_m} \\
&= \cup_{s'\in W_{\mathrm{H}}} \mathrm{H}_{x_m+b}\,\varpi^{s'(\lambda)}\,\mathrm{H}_{x_m}\,.
\end{aligned}$$

For $\lambda = \lambda_n$ or λ_n^*, one can check $\varpi^{-w_{\mathrm{H}}(\lambda)}(\mathrm{H}_{x_{m+b}} \cap \mathrm{V})\varpi^{w_{\mathrm{H}}(\lambda)} \subset (\mathrm{H}_{x_{m+b}} \cap \mathrm{V})$ so a simliar computation as above leads to same conclusion. □

Using the decomposition of $\mathrm{K}(\mathfrak{p}^m)$ we have

$$\begin{aligned}
&\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda}\,\mathrm{K}(\mathfrak{p}^m)\varpi^{\mu}\,\mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{s\in W_{\mathrm{H}}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda}\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)\,\mathrm{H}_{x_{m+b}}\,\varpi^{s(\mu)}\,\mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{s\in W_{\mathrm{H}}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda}\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)(\mathrm{H}_{b+x_m}\cap \mathrm{P})(\mathrm{H}_{x_{m+b}}\cap\overline{\mathrm{Z}})\,\varpi^{s(\mu)}\,\mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{s\in W_{\mathrm{H}}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda}\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o}/\mathfrak{p})\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m/\mathfrak{p}^{m+1})\overline{\mathrm{Z}}(\mathfrak{p}^{m+1}/\mathfrak{p}^{m+2})\,\varpi^{s(\mu)}\,\mathrm{K}(\mathfrak{p}^m)
\end{aligned}$$

In the computation we use the fact that ${}^{\varpi^{\lambda}}(\mathrm{H}_{x_{m+b}}\cap \mathrm{P}) \subset \mathrm{K}(\mathfrak{p}^m)$, and

$$\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)(\mathrm{H}_{x_{m+b}}\cap \mathrm{P})\overline{\mathrm{Z}}(\mathfrak{p}^{m+1}) \subset (\mathrm{H}_{x_{m+b}}\cap \mathrm{P})\prod_{i=1}^{n} x_{\epsilon_i}(\mathfrak{o})\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m)\overline{\mathrm{Z}}(\mathfrak{p}^{m+1}).$$

For $\lambda = \lambda_j$, $1 \le j \le n$, or $\lambda = \lambda_n^*$, the decomposition is equal to

$$\begin{aligned}
&\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda}\,\mathrm{K}(\mathfrak{p}^m)\varpi^{\mu}\,\mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{s\in W_{\mathrm{H}}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda} x_{\epsilon_n}(\mathfrak{o})\prod_{i=1}^{n} x_{-\epsilon_i}(\mathfrak{p}^m/\mathfrak{p}^{m+1})\overline{\mathrm{Z}}(\mathfrak{p}^{m+1}/\mathfrak{p}^{m+2})\,\varpi^{s(\mu)}\,\mathrm{K}(\mathfrak{p}^m).
\end{aligned}$$

For each root α of Lie(Z), if $\langle \lambda, -\alpha\rangle = \langle s(\mu), \alpha\rangle = -2$, then for $b_\alpha \in \mathfrak{o}^{\times}$,

$$\varpi^{\lambda} x_{-\alpha}(b_\alpha \varpi^{m+1})\varpi^{s(\mu)} = x_\alpha(b_\alpha^{-1}\varpi^{-m+1})\varpi^{\lambda} w_{s_\alpha,m+1}\varpi^{s(\mu)} x_\alpha(b_\alpha^{-1}\varpi^{-m+1}).$$

Since $w_{s_\alpha,m} x_\alpha(b_\alpha^{-1}\varpi^{-m+1}) \in \mathrm{K}(\mathfrak{p}^m)$, $w_{s_\alpha,m+1} w_{s_\alpha^{-1},m} = \varpi^{-\alpha}$ and $s_\alpha(s(\mu)) - \alpha = s(\mu) + \alpha$, we obtain that

$$\varpi^{\lambda} x_{-\alpha}(b_\alpha \varpi^{m+1})\varpi^{s(\mu)}\,\mathrm{K}(\mathfrak{p}^m) = x_\alpha(b_\alpha^{-1}\varpi^{-m+1})\varpi^{\lambda}\varpi^{s(\mu)+\alpha}\,\mathrm{K}(\mathfrak{p}^m).$$

Otherwise, we get

$$\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda x_{-\alpha}(b_\alpha\varpi^{m+1})\varpi^{s(\mu)}\,\mathrm{K}(\mathfrak{p}^m) = \mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\varpi^{s(\mu)}\,\mathrm{K}(\mathfrak{p}^m)$$

and the factor $x_\alpha(b_\alpha\varpi^{m+1})$ can be eliminated from the representative of the double coset since $x_{-\alpha}(\mathfrak{p}^{m+1})$ commutes with $\overline{\mathrm{Z}}$ and $\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)$. We obtain

$$\begin{aligned}&\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)\varpi^\mu\,\mathrm{K}(\mathfrak{p}^m)\\ =& \bigcup_{\substack{s\in W_{\mathrm{H}},\nu\geq_{\mathrm{Z}}0\\ \langle s(\mu)+\nu,\nu\rangle=0\\ \langle\lambda-\nu,\nu\rangle=0}} \mathrm{K}(\mathfrak{p}^m)\varpi^\lambda x_{\epsilon_n}(\mathfrak{o})\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m/\mathfrak{p}^{m+1})\varpi^{\nu+s(\mu)}\,\mathrm{K}(\mathfrak{p}^m).\end{aligned}$$

Here $\geq_{\mathrm{Z}}$ represents the Bruhat order on $\mathrm{X}_\bullet(\mathrm{T})$ with respect to roots in $\mathrm{Lie}(\mathrm{Z})$.

Then we have for $(c_i)_{1\leq i\leq n}\in(\mathfrak{o}/\mathfrak{p})^n$, and $c_i, c_j\in(\mathfrak{o}/\mathfrak{p})^\times$,

$$x_{-\epsilon_i}(c_i\varpi^m)x_{-\epsilon_j}(c_j\varpi^m) \quad=\quad x_{\epsilon_i-\epsilon_j}(c_i^{-1}c_j)x_{-\epsilon_i}(c_i\varpi^m)x_{\epsilon_i-\epsilon_j}(-c_i^{-1}c_j).$$

Then if $\langle\lambda,-\epsilon_i\rangle=\langle\lambda,-\epsilon_j\rangle=\langle\nu+s(\mu),\epsilon_i\rangle=\langle\nu+s(\mu),\epsilon_j\rangle=-1$, we get

$$\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda x_{-\epsilon_i}(c_i\varpi^m)x_{-\epsilon_j}(c_j\varpi^m)\,\mathrm{K}(\mathfrak{p}^m) = \mathrm{K}(\mathfrak{p}^m)\varpi^\lambda x_{-\epsilon_i}(c_i\varpi^m)\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m).$$

Since $\mathrm{U}_{\epsilon_i-\epsilon_j}$ commutes with $\mathrm{U}_{-\epsilon_{i'}}$ for $i'\neq i,j$, we conclude that

$$\begin{aligned}&\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)\varpi^\mu\,\mathrm{K}(\mathfrak{p}^m)\\ =& \bigcup_{\substack{s\in W_{\mathrm{H}},\nu\geq_{\mathrm{Z}}0\\ \langle s(\mu)+\nu,\nu\rangle=\langle\lambda-\nu,\nu\rangle=0}} \mathrm{K}(\mathfrak{p}^m)\varpi^\lambda x_{\epsilon_{j_{s,\nu}}}(\mathfrak{o}/\mathfrak{p})x_{-\epsilon_{i_{s,\nu}}}(\mathfrak{p}^m/\mathfrak{p}^{m+1})\varpi^{\nu+s(\mu)}\,\mathrm{K}(\mathfrak{p}^m)\\ =& \bigcup_{\substack{s\in W_{\mathrm{H}},\nu\geq_{\mathrm{Z}}0\\ \langle s(\mu)+\nu,\nu\rangle=\langle\lambda-\nu,\nu\rangle=0}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda+\nu+s(\mu)}x_{\epsilon_{j_{s,\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o})x_{-\epsilon_{i_{s,\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m)\end{aligned}$$

where $i_{s,\nu}$ is any index i such that $\langle\lambda,-\epsilon_i\rangle=\langle\nu+s(\mu),\epsilon_i\rangle=-1$ and the factor $x_{-\epsilon_{i_{s,\nu}}}(\mathfrak{p}^m/\mathfrak{p}^{m+1})$ is eliminated if there is no such i; while $j_{s,\nu}$ is any index j such that

$\langle \lambda, \epsilon_j \rangle = \langle \nu + s(\mu), -\epsilon_j \rangle = -1$ and the factor $x_{\epsilon_{j_{s,\nu}}}(\mathfrak{p}^m/\mathfrak{p}^{m+1})$ is eliminated if there is no such j. We note that $\langle s(\lambda) + \gamma_s - \mu - \nu, i_{s,\nu} \rangle = \langle s(\lambda) + \gamma_s - \mu - \nu, j_{s,\nu} \rangle = 0$.

Proposition 8.2.2. *($m \geq 2$) The operators $T_1, T_2, ..., T_{n-1}$ and T'_n commute with each other on the subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$.*

Proof. We note that $T_\lambda \circ T_\mu(v) = \int_{\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda \mathrm{K}(\mathfrak{p}^m)\varpi^\mu \mathrm{K}(\mathfrak{p}^m)} \pi(g)v \, dg$ for some suitable choice of Haar measure dg. This statement is trivial for $n = 1$. We assume $n \geq 2$.

Assume λ and μ are minuscule co-weights in P_{H}^+. Recall that we have shown

$$\begin{aligned}
&\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda \mathrm{K}(\mathfrak{p}^m)\varpi^\mu \mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{\substack{s \in W_{\mathrm{H}}, \nu \geq_{\mathrm{Z}} 0 \\ \langle s(\mu)+\nu,\nu\rangle = \langle \lambda-\nu,\nu\rangle = 0}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda+\nu+s(\mu)} x_{\epsilon_{j_{s,\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o}) x_{-\epsilon_{i_{s,\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m) \mathrm{K}(\mathfrak{p}^m)
\end{aligned}$$

where $i_{s,\nu}$ is any index i such that $\langle \lambda, -\epsilon_i \rangle = \langle \nu + s(\mu), \epsilon_i \rangle = -1$ and $j_{s,\nu}$ is any index j such that $\langle \lambda, \epsilon_j \rangle = \langle \nu + s(\mu), -\epsilon_j \rangle = -1$.

On the other hand,

$$\begin{aligned}
&\mathrm{K}(\mathfrak{p}^m)\varpi^\mu \mathrm{K}(\mathfrak{p}^m)\varpi^\lambda \mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{\substack{s \in W_{\mathrm{H}}, \nu \geq_{\mathrm{Z}} 0 \\ \langle s(\lambda)+\nu,\nu\rangle = \langle \mu-\nu,\nu\rangle = 0}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\mu+\nu+s(\lambda)} x_{\epsilon_{j'_{s,\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o}) x_{-\epsilon_{i'_{s,\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m) \mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{\substack{s' \in W_{\mathrm{H}}, \nu \geq_{\mathrm{Z}} 0 \\ \langle \lambda-\nu,\nu\rangle = \langle s'(\mu)+\nu,\nu\rangle = 0}} \mathrm{K}(\mathfrak{p}^m)\varpi^{s'(\mu)-\nu+\lambda} x_{-\epsilon_{j_{s',\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m) x_{\epsilon_{i_{s',\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o}) \mathrm{K}(\mathfrak{p}^m) \\
= &\bigcup_{\substack{s' \in W_{\mathrm{H}}, \nu \geq_{\mathrm{Z}} 0 \\ \langle \lambda-\nu,\nu\rangle = \langle s'(\mu)+\nu,\nu\rangle = 0}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda+\nu+s'(\mu)} x_{-\epsilon_{j_{s',\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m) x_{\epsilon_{i_{s',\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o}) \mathrm{K}(\mathfrak{p}^m)
\end{aligned}$$

where $i'_{s,\nu}$ is any index i such that $\langle \mu, -\epsilon_i \rangle = \langle \nu + s(\lambda), \epsilon_i \rangle = -1$ which implies $\langle s'(\mu), \epsilon_{s'(i)} \rangle = \langle -\nu + \lambda, -\epsilon_{s'(i)} \rangle = -1$ and hence equivalent to $\langle s'(\mu) + \nu, \epsilon_{s'(i)} \rangle = \langle \lambda, -\epsilon_{s'(i)} \rangle = -1$, similar for $j'_{s,\nu}$. The second equality is by conjugating by $w_{s',m} \in$

$\mathrm{K}(\mathfrak{p}^m)$ such that $s'(s(\lambda)) = \lambda$ and the third equality is by conjugating by $w_{s,m}$ such that $\gamma_s = \varpi^\nu$ provided that $\langle \lambda + s'(\mu), \nu\rangle = 0$.

Note that $\langle \lambda+\nu+s(\mu), \epsilon_{j_{s,\nu}}\rangle = \langle \lambda+\nu+s(\mu), \epsilon_{i_{s,\nu}}\rangle = 0$. If either $i_{s,\nu}$ and $j_{s,\nu}$ both exist or $\langle \lambda+\nu+s(\mu), \epsilon_i\rangle = 0$ for some k not equal to either $i_{s,\nu}, j_{s,\nu}$, then since

$$w \in \{(w_{s_{\epsilon_k},m} w_{s_{\epsilon_{i_{s,\nu}}},m}), (w_{s_{\epsilon_k},m} w_{s_{\epsilon_{j_{s,\nu}}},m}), (w_{s_{\epsilon_{i_{s,\nu}}},m} w_{s_{\epsilon_{j_{s,\nu}}},m})\} \subset \mathrm{K}(\mathfrak{p}^m),$$

one has (w chosen depending on existence of $i_{s,\nu}, j_{s,\nu}$ and k)

$$\begin{aligned} & \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda+\nu+s(\mu)} x_{\epsilon_{j_{s,\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o}) x_{-\epsilon_{i_{s,\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m) \\ = \; & \mathrm{K}(\mathfrak{p}^m) w \left(\varpi^{\lambda+\nu+s(\mu)} x_{\epsilon_{j_{s,\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o}) x_{-\epsilon_{i_{s,\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\right) w^{-1}\,\mathrm{K}(\mathfrak{p}^m) \\ = \; & \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda+\nu+s(\mu)} x_{-\epsilon_{j_{s,\nu}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m) x_{\epsilon_{i_{s,\nu}}}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m). \end{aligned}$$

In particular, any k such that $\langle \nu, \epsilon_k\rangle \neq 0$ satisfies $\langle \lambda+\nu+s(\mu), \epsilon_i\rangle = 0$ and $k \neq i_{s,\nu}, j_{s,\nu}$. Therefore to compare $T_\lambda \circ T_\mu$ and $T_\mu \circ T_\lambda$, we only need to compare the set

$$\bigcup_{\substack{s\in W_{\mathrm{H}},\, \langle \lambda+s(\mu),\epsilon_i\rangle \neq 0, \forall i\neq i_{s,0}, j_{s,0} \\ \lambda+s(\mu)+\epsilon_{i_{s,0}}+\epsilon_{j_{s,0}} \in P_{\mathrm{H}}^+}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda+s(\mu)} x_{\epsilon_{j_{s,0}}}(\mathfrak{p}^{-1}/\mathfrak{o}) x_{-\epsilon_{i_{s,0}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m)$$

with the set

$$\bigcup_{\substack{s\in W_{\mathrm{H}},\, \langle \lambda+s(\mu),\epsilon_i\rangle \neq 0, \forall i\neq i_{s,0}, j_{s,0} \\ \lambda+s(\mu)+\epsilon_{i_{s,0}}+\epsilon_{j_{s,0}} \in P_{\mathrm{H}}^+}} \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda+s(\mu)} x_{-\epsilon_{j_{s,0}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m) x_{\epsilon_{i_{s,0}}}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m),$$

with ν taken to be 0 and $\langle \lambda + s(\mu), \epsilon_i\rangle \neq 0, \forall i \neq i_{s,0}, j_{s,0}$, while s is taken to satisfies $\lambda + s(\mu) + \epsilon_{i_{s,0}} + \epsilon_{j_{s,0}} \in P_{\mathrm{H}}^+$ since $\mathrm{K}(\mathfrak{p}^m)$ contains lift of the Weyl group W_{H} of H.

We first note that if $i + j \leq n$, then for $\lambda = \lambda_i$ and $\mu = \lambda_j$, these two sets are empty by looking at the degree of $\lambda + s(\mu)$. Hence $T_i \circ T_j = T_j \circ T_i$ if $i + j \leq n$.

Assume $\lambda = \lambda_i$ and $\mu = \lambda_j$ and $i, j \neq n$, $i + j > n$. Then $i_{s,0} \leq i < n$ and there exists no $j_{s,0}$. In the subindex set $\{s \in W_{\mathrm{H}} \mid \langle \lambda + s(\mu), \epsilon_i \rangle \neq 0, \forall i \neq i_{s,0}, \lambda + s(\mu) + \epsilon_{i_{s,0}} \in P_{\mathrm{H}}^{+}\}$, the co-characters $\lambda + s(\mu) + \epsilon_{i_{s,0}}$ take

$$\lambda_n + \lambda_{(i+j-n)} \text{ and } \lambda_n^* + \lambda_{(i+j-n)} \text{ with } i + j - n \leq i_{s,0} \leq i.$$

For each $i + j - n \leq i_{s,0} \leq i$, since $\langle \lambda_n + \lambda_{(i+j-n-1)} - \epsilon_{i_{s,0}}, \epsilon_{i_{s,0}} \rangle = 0$, the set

$$\begin{aligned} &\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n + \lambda_{(i+j-n-1)} - \epsilon_{i_{s,0}}} x_{-\epsilon_{i_{s,0}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m) \\ \cup\ &\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^* + \lambda_{(i+j-n-1)} - \epsilon_{i_{s,0}}} x_{-\epsilon_{i_{s,0}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m) \end{aligned}$$

by conjugating by $w_{s_{\epsilon_n},m} w_{s_{\epsilon_{i_{s,0}}},m} \in \mathrm{K}(\mathfrak{p}^m)$ is equal to the set

$$\begin{aligned} &\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n + \lambda_{(i+j-n)} - \epsilon_{i_{s,0}}} x_{\epsilon_{i_{s,0}}}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m) \\ \cup\ &\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^* + \lambda_{(i+j-n)} - \epsilon_{i_{s,0}}} x_{\epsilon_{i_{s,0}}}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m). \end{aligned}$$

Therefore comparing $\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_i}\,\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_j}\,\mathrm{K}(\mathfrak{p}^m)$ and $\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_j}\,\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_i}\,\mathrm{K}(\mathfrak{p}^m)$ we again obtain $T_i \circ T_j = T_j \circ T_i$.

We claim that $(T_n + T_n^*) \circ T_j = T_j \circ (T_n + T_n^*)$ also holds for all $1 \leq j < n$. Recall that we only care about either $i_{s,0}$ or $j_{s,0}$ exists. Note that in $\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^*}\,\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_j}\,\mathrm{K}(\mathfrak{p}^m)$ and $\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_{nj}}\,\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^*}\,\mathrm{K}(\mathfrak{p}^m)$ we have for $j \leq i_{s,0} < n$

$$\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_{n-1} + \lambda_{j-1}} x_{\epsilon_n}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m) = \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n + \lambda_{j-1} - \epsilon_{i_{s,0}}} x_{-\epsilon_{i_{s,0}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m),$$

$$\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_{n-1} + \lambda_{j-1}} x_{-\epsilon_n}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m) = \mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^* + \lambda_{j-1} - \epsilon_{i_{s,0}}} x_{\epsilon_{i_{s,0}}}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m)$$

by conjugating by $(w_{s_{\epsilon_{i_{s,0}} + \epsilon_n},m}) \in \mathrm{K}(\mathfrak{p}^m)$. We only have to compare the sets

$$\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n + \lambda_{j-1} - \epsilon_{i_{s,0}}} x_{-\epsilon_{i_{s,0}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m),$$

$$\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^* + \lambda_{j-1} - \epsilon_{i_{s,0}}} x_{-\epsilon_{i_{s,0}}}(\mathfrak{p}^{m-1}/\mathfrak{p}^m)\,\mathrm{K}(\mathfrak{p}^m)$$

with the sets

$$\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n+\lambda_{j-1}-\epsilon_{i_{s,0}}}x_{\epsilon_{i_{s,0}}}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m),$$

$$\mathrm{K}(\mathfrak{p}^m)\varpi^{\lambda_n^*+\lambda_{j-1}-\epsilon_{i_{s,0}}}x_{\epsilon_{i_{s,0}}}(\mathfrak{p}^{-1}/\mathfrak{o})\,\mathrm{K}(\mathfrak{p}^m)$$

for $s \in W_{\mathrm{H}}$, $j \leq i_{s,0} < n$. We see they are the same by conjugating by $w_{s_{\epsilon_{i_{s,0}}+\epsilon_n},m}$. □

As a result, we see that for $m \geq 2$ the subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ decomposes into orthogonal direct sum of common eigenspaces of the Hecke operators T_1, T_2, ..., T_{n-1}, T'_n.

8.3. Hecke eigenvectors

We call a vector in $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ a *Hecke eigenvector* if it is a common eigenvector of $T_1, T_2, ..., T_{n-1}$ and T'_n. Let $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ be such a Hecke eigenvector. Denote by μ_i the Hecke eigenvalue of T_{λ_i}, $1 \leq i < n$ and by μ_n the Hecke eigenvalue of T'_n of v. Let $c_\nu(v)$ the value of its Whittaker function at ϖ^ν, namely $c_\nu(v) = W_v(\varpi^\nu)$ In this section, we obtain a relationship among these numbers attached to v for all $\nu \in P^+$.

8.3. Hecke eigenvectors

We begin with the computation of the double coset $\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)$ for λ minuscule co-characters in P_{H}^+.

$$\begin{aligned}
&\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)\\
=\;&\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)\mathrm{H}_{x_m}\,\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)\\
=\;&\bigcup_{s\in W_{\mathrm{H}}}\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)\mathrm{H}_{b+x_m}\,\varpi^{s(\lambda)}\,\mathrm{K}(\mathfrak{p}^m)\\
=\;&\bigcup_{s\in W_{\mathrm{H}}}\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)(\mathrm{N}_n(\mathfrak{o})\,\mathrm{Z}(\mathfrak{p}^{-m}))(\mathrm{H}_{b+x_m}\cap\overline{\mathrm{V}})\,\varpi^{s(\lambda)}\,\mathrm{K}(\mathfrak{p}^m)\\
=\;&\bigcup_{s\in W_{\mathrm{H}}}\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\mathrm{N}_n(\mathfrak{o})\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)\mathrm{Z}(\mathfrak{p}^{-m})(\mathrm{H}_{b+x_m}\cap\overline{\mathrm{V}})\,\varpi^{s(\lambda)}\,\mathrm{K}(\mathfrak{p}^m)\\
=\;&\bigcup_{s\in W_{\mathrm{H}}}\mathrm{N}_n(\mathfrak{o})\left(\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o})\right)\mathrm{Z}(\mathfrak{p}^{-m})\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)(\mathrm{H}_{b+x_m}\cap\overline{\mathrm{V}})\,\varpi^{s(\lambda)}\,\mathrm{K}(\mathfrak{p}^m)\\
=\;&\bigcup_{s\in W_{\mathrm{H}}}(\mathrm{K}(\mathfrak{p}^m)\cap\mathrm{U})\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)(\mathrm{H}_{b+x_m}\cap\overline{\mathrm{V}})\,\varpi^{s(\lambda)}\,\mathrm{K}(\mathfrak{p}^m).
\end{aligned}$$

We shall do some algorithms to best replace negative roots by positive roots. Since $\mathrm{H}_{x_m}\,\varpi^\lambda\,\mathrm{H}_{x_m}\subset\cup_{\mu\le_{\mathrm{H}}\lambda}\mathrm{V}\,\varpi^\mu\,\mathrm{H}_{x_m}$ where $\le_{\mathrm{H}}$ is the Bruhat order with respect to Φ_{H}^+, it is expected to be contained in $\cup_{\mu\le_{\mathrm{H}}\lambda}\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)\mathrm{U}\,\varpi^\mu\,\mathrm{K}(\mathfrak{p}^m)$. For notation convenience, we will also denote by $\le_{\mathrm{Z}}$ the Bruhat order on $\mathrm{X}_\bullet(\mathrm{T})$ with respect to roots in $\operatorname{Lie}\mathrm{Z}$ and $\le_{\mathrm{M}}$ to be the Bruhat order on $\mathrm{X}_\bullet(\mathrm{T})$ with respect to Φ_{M}^+.

We use the following nice tricks to do the job.

Lemma 8.3.1. *Assume for some $1\le i<j\le n$, $\langle\gamma,\epsilon_i+\epsilon_j\rangle=-2$. Then*

$$x_{-\epsilon_i-\epsilon_j}(\mathfrak{p}^{m+1})\varpi^\gamma\,\mathrm{K}(\mathfrak{p}^m)=\varpi^\gamma\,\mathrm{K}(\mathfrak{p}^m)\cup x_{\epsilon_i+\epsilon_j}(\varpi^{-m-1}\mathfrak{o}^\times)\varpi^{\gamma+\epsilon_i+\epsilon_j}\,\mathrm{K}(\mathfrak{p}^m).$$

Proof. Let $c \in \mathfrak{o}^\times$. Then $x_{-\epsilon_i-\epsilon_j}(c\varpi^{m+1})\varpi^\gamma \mathrm{K}(\mathfrak{p}^m)$ equals

$$\begin{aligned}
& \varpi^\gamma x_{-\epsilon_i-\epsilon_j}(c\varpi^{m-1})\,\mathrm{K}(\mathfrak{p}^m) \\
= \; & \varpi^\gamma x_{\epsilon_i+\epsilon_j}(c^{-1}\varpi^{-m+1}) w_{s_{\epsilon_i+\epsilon_j},m-1} x_{\epsilon_i+\epsilon_j}(c^{-1}\varpi^{-m+1})\,\mathrm{K}(\mathfrak{p}^m) \\
= \; & \varpi^\gamma x_{\epsilon_i+\epsilon_j}(c^{-1}\varpi^{-m+1})\varpi^{\epsilon_i+\epsilon_j}\,\mathrm{K}(\mathfrak{p}^m) \\
= \; & x_{\epsilon_i+\epsilon_j}(c^{-1}\varpi^{-m-1})\varpi^{\gamma+\epsilon_i+\epsilon_j}\,\mathrm{K}(\mathfrak{p}^m).
\end{aligned}$$

On the other hand, $x_{-\epsilon_i-\epsilon_j}(\mathfrak{p}^{m+2})\varpi^\gamma \mathrm{K}(\mathfrak{p}^m) = \varpi^\gamma \mathrm{K}(\mathfrak{p}^m)$ by the assumption. □

Lemma 8.3.2. *Assume* $1 \le i < j \le n$.

(i) If $\langle \gamma, \epsilon_i - \epsilon_j \rangle = -2$, *then*

$$x_{\epsilon_j-\epsilon_i}(\mathfrak{p})\varpi^\gamma \mathrm{K}_n(\mathfrak{p}^m) = \varpi^\gamma \mathrm{K}(\mathfrak{p}^m) \cup x_{\epsilon_i-\epsilon_j}(\varpi^{-1}\mathfrak{o}^\times) w^{\gamma+\epsilon_i-\epsilon_j}\,\mathrm{K}(\mathfrak{p}^m).$$

(ii) If $\langle \gamma, \epsilon_i - \epsilon_j \rangle = -1$, *then*

$$x_{\epsilon_j-\epsilon_i}(\mathfrak{o})\varpi^\gamma \mathrm{K}_n(\mathfrak{p}^m) = \varpi^\gamma \mathrm{K}(\mathfrak{p}^m) \cup x_{\epsilon_i-\epsilon_j}(\varpi^{-1}\mathfrak{o}^\times) w^{\gamma+\epsilon_i-\epsilon_j}\,\mathrm{K}(\mathfrak{p}^m).$$

Proof. This is a very similar argument as the previous lemma. We omit it here. □

Let us write each $s(\lambda)$ as a sum

$$s(\lambda) = s(\lambda)_+ - s(\lambda)_- \tag{8.3.1}$$

such that $\|s(\lambda)_\pm\| \le 1$ and $\langle s(\lambda)_+, \epsilon_i \rangle \ge 0$ for $1 \le i \le n$ and define $\widetilde{s(\lambda)}$ as

$$\widetilde{s(\lambda)} = \begin{cases} s(\lambda)_+ & \text{if } \deg s(\lambda)_- \text{ is even} \\ s(\lambda)_+ - \epsilon_{i_s} & \text{if } \deg s(\lambda)_- \text{ is odd and } i_s = \max_i \langle s(\lambda), \epsilon_i \rangle < 0 \end{cases} \tag{8.3.2}$$

and the numbers $1 \le i_1 < i_2 < ... < i_{d_s^+} \le n$ and $1 \le j_1 < j_2 < ... < j_{d_s^-} \le n$ as

$$s(\lambda) = (\epsilon_{j_1} + \epsilon_{j_2} + ... + \epsilon_{j_{d_s^+}}) - (\epsilon_{i_1} + \epsilon_{i_2} + ... + \epsilon_{i_{d_s^-}}),$$

where $d_s^{\pm}$ are the degrees of $s(\lambda)_{\pm}$.

Algorithm 1. Assume we can take $i < j$ to be the smallest two indices such that $\langle s(\lambda), \epsilon_i + \epsilon_j \rangle = -2$. One notice that

$$\prod_{1 \le l < k \le n} x_{-\epsilon_l - \epsilon_k}(\mathfrak{p}^{m+1}) \varpi^{s(\lambda)} \mathrm{K}(\mathfrak{p}^m) = \prod_{1 \le k, l \le d_s^+,\, j_k, j_l > i,j} x_{-\epsilon_{j_k} - \epsilon_{j_l}}(\mathfrak{p}^{m+1}) \varpi^{s(\lambda)} \mathrm{K}(\mathfrak{p}^m).$$

The commutator of $x_{\epsilon_i+\epsilon_j}(\mathfrak{p}^{-m-1})$ with $\prod_{j_k, j_l > i,j} x_{-\epsilon_{j_k}-\epsilon_{j_l}}(\mathfrak{p}^{m+1})$ is $\prod_{j_l > i,j} x_{\epsilon_i - \epsilon_{j_l}}(\mathfrak{o}) x_{\epsilon_j - \epsilon_{j_l}}(\mathfrak{o})$. Hence by Lemma and $\langle s(\lambda) + \epsilon_i + \epsilon_j, \epsilon_i + \epsilon_j \rangle = 0$ we have

$$\prod_{1 \le l < k \le n} x_{-\epsilon_l - \epsilon_k}(\mathfrak{p}^{m+1})\, \varpi^{s(\lambda)} \mathrm{K}(\mathfrak{p}^m) = \prod_{(l,k) \neq (i,j)} x_{-\epsilon_l - \epsilon_k}(\mathfrak{p}^{m+1})\, \varpi^{s(\lambda)} \mathrm{K}(\mathfrak{p}^m)$$

$$\cup \prod_{j_l > i,j} x_{\epsilon_i - \epsilon_{j_l}}(\mathfrak{o}) x_{\epsilon_j - \epsilon_{j_l}}(\mathfrak{o}) x_{\epsilon_i+\epsilon_j}(\varpi^{-m-1}\mathfrak{o}^\times) \prod_{l,k \neq i,j} x_{-\epsilon_l - \epsilon_k}(\mathfrak{p}^{m+1})\, \varpi^{s(\lambda)+\epsilon_i+\epsilon_j} \mathrm{K}(\mathfrak{p}^m).$$

Repeating **Algorithm 1** we obtain that

$$\prod_{1 \le l < k \le n} x_{-\epsilon_l - \epsilon_k}(\mathfrak{p}^{m+1})\, \varpi^{s(\lambda)} \mathrm{K}(\mathfrak{p}^m) =$$

$$\bigcup_{\mu \ge_{\mathrm{Z}} s(\lambda),\, \langle \mu, \mu - s(\lambda) \rangle = 0} \prod_{\beta \in J_{\mu - s(\lambda)}} \prod_{\epsilon_i + \epsilon_j = \beta,\, j_l > i,j} x_{\epsilon_i - \epsilon_{j_l}}(\mathfrak{o}) x_\beta(\varpi^{-m-1}\mathfrak{o}^\times)\, \varpi^\mu \mathrm{K}(\mathfrak{p}^m)$$

where the notation $\mu \ge_{\mathrm{Z}} s(\lambda)$ means $\mu - s(\lambda)$ is a sum $\beta_1 + \beta_2 + ... + \beta_k$ of roots β_i in Lie(Z) uniquely determined such that $\beta_i - \beta_j \ge 0$ for $i < j$ and $J_{\mu - s(\lambda)} = \{\beta_i\}_{i=1}^k$.

The commutators of $x_{\epsilon_i+\epsilon_j}(\mathfrak{p}^{-m-1})$ with $\overline{\mathrm{N}_n}(\mathfrak{p})$ lie in $\mathrm{Z}(\mathfrak{p}^{-m})$. Hence we see that for $\vec{c} = (c_1, c_2, ..., c_n) \in (\mathfrak{o}\mathfrak{p})^n$, $(b_\beta)_\beta \in (\mathfrak{o}/\mathfrak{p})_\beta$,

$$(\mathrm{K}(\mathfrak{p}^m) \cap \mathrm{U}) \left(\prod_{i=1}^n x_{-\epsilon_i}(c_i \varpi^m) \right) \overline{\mathrm{N}_n}(\mathfrak{p})\, \mathrm{N}_n(\mathfrak{o}) \prod_{\beta \in J_{\mu - s(\lambda)}} x_\beta(b_\beta \varpi^{-m-1})\, \mathrm{T}(\mathfrak{o}) =$$

$$(\mathrm{K}(\mathfrak{p}^m) \cap \mathrm{U}) \prod_{\substack{\beta \in J_{\mu - s(\lambda)} \\ \epsilon_i + \epsilon_j = \beta}} x_{\epsilon_i}(b_\beta c_i \varpi^{-1}) x_{\epsilon_j}(b_\beta c_j \varpi^{-1}) x_\beta(b_\beta \varpi^{-m-1}) \left(\prod_{i=1}^n x_{-\epsilon_i}(c_i \varpi^m) \right) \overline{\mathrm{N}_n}(\mathfrak{p})\, \mathrm{T}(\mathfrak{o}).$$

Algoritm 2. Assume we take $i < j$ so that i is the smallest number and j is the largest number such that $\langle \mu, \epsilon_i - \epsilon_j \rangle = -2$, $\mu \geq_{\mathrm{Z}} s(\lambda)$ and $\langle \mu, \mu - s(\lambda) \rangle = 0$. By a similar argument as in **Algorithm 1** since

$$\prod_{1 \leq k < l \leq n} x_{\epsilon_l - \epsilon_k}(\mathfrak{p})\, \varpi^{\mu}\, \mathrm{K}(\mathfrak{p}^m) = \left(\prod_{1 \leq k \leq d_s^-} \prod_{1 \leq l \leq d_s^+,\, j_l > i_k} x_{\epsilon_{j_l} - \epsilon_{i_k}}(\mathfrak{p}) \right) \varpi^{\mu}\, \mathrm{K}(\mathfrak{p}^m)$$

by Lemma we have

$$\prod_{1 \leq k < l \leq n} x_{\epsilon_l - \epsilon_k}(\mathfrak{p}) \varpi^{\mu}\, \mathrm{K}(\mathfrak{p}^m) = \prod_{1 \leq k < l \leq n,\, (k,l) \neq (i,j)} x_{\epsilon_l - \epsilon_k}(\mathfrak{p})\, \varpi^{\mu}\, \mathrm{K}(\mathfrak{p}^m) \cup$$

$$\left(\prod_{i < j_l,\, i_k < j} x_{\epsilon_{j_l} - \epsilon_j}(\mathfrak{o}) x_{\epsilon_i - \epsilon_{i_k}}(\mathfrak{o}) \right) x_{\epsilon_i - \epsilon_j}(\varpi^{-1} \mathfrak{o}^{\times}) \prod_{1 \leq k < l \leq n,\, k, l \neq i, j} x_{\epsilon_l - \epsilon_k}(\mathfrak{p})\, \varpi^{\mu + \epsilon_i - \epsilon_j}\, \mathrm{K}(\mathfrak{p}^m).$$

Repeating Algorithm 2 we obtain

$$\begin{aligned} &\prod_{1 \leq k < l \leq n} x_{\epsilon_l - \epsilon_k}(\mathfrak{p}) \varpi^{\mu}\, \mathrm{K}(\mathfrak{p}^m) \\ = &\bigcup_{\substack{\nu \geq_{\mathrm{M}} \mu \\ \langle \nu, \nu - \mu \rangle = 0}} \prod_{\beta \in I_{\nu - \mu}} \prod_{\substack{i < j_l,\, i_k < j \\ \epsilon_i - \epsilon_j = \beta}} x_{\epsilon_{j_l} - \epsilon_j}(\mathfrak{o}) x_{\epsilon_i - \epsilon_{i_k}}(\mathfrak{o}) x_{\beta}(\varpi^{-1} \mathfrak{o}^{\times})\, \varpi^{\nu}\, \mathrm{K}(\mathfrak{p}^m) \end{aligned}$$

where the notation $\nu \geq_{\mathrm{M}} \mu$ means $\nu - \mu$ is a sum $\beta_1 + \gamma_2 + ... + \gamma_k$ of roots in $\mathrm{Lie}(\mathrm{N}_n)$ such that $\beta_i - \beta_j \geq 0$ for $i < j$ and $I_{\nu - \mu} = \{\beta_i\}_{i=1}^{k}$.

Hence together with a computation of the communicator with $x_{-\epsilon_i}(\mathfrak{p}^m)$ we can conclude the following. For $\vec{c}=(c_1,c_2,...,c_n)\in(\mathfrak{o}/\mathfrak{p})^n$, $\vec{b}=(b_\beta)_\beta\in(\mathfrak{o}/\mathfrak{p})_\beta$

$$
\begin{aligned}
&(\mathrm{K}(\mathfrak{p}^m)\cap \mathrm{U})\left(\prod_{i=1}^n x_{-\epsilon_i}(\mathfrak{p}^m)\right)(\mathrm{H}_{b+x_m}\cap\overline{\mathrm{V}})\,\varpi^{s(\lambda)}\,\mathrm{K}(\mathfrak{p}^m)\\
=&\bigcup_{\substack{\mu\geq_{\mathrm{Z}}s(\lambda),\,\nu\geq_{\mathrm{M}}\mu\\ \langle\mu,\mu-s(\lambda)\rangle=\langle\nu,\nu-\mu\rangle=0}}(\mathrm{K}(\mathfrak{p}^m)\cap\mathrm{U})\prod_{\substack{\beta\in J_{\mu-s(\lambda)}\\ \epsilon_i+\epsilon_j=\beta}}x_{\epsilon_i}(b_\beta c_i\varpi^{-1})x_{\epsilon_j}(b_\beta c_j\varpi^{-1})x_\beta(b_\beta\varpi^{-m-1})\\
&\left(\prod_{i=1}^n x_{-\epsilon_i}(c_i\varpi^m)\right)\prod_{\beta\in I_{\nu-\mu}}x_\beta(b_\beta\varpi^{-1})\,\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m)\\
=&\bigcup_{\substack{\mu\geq_{\mathrm{Z}}s(\lambda),\,\nu\geq_{\mathrm{M}}\mu\\ \langle\mu,\mu-s(\lambda)\rangle=\langle\nu,\nu-\mu\rangle=0}}(\mathrm{K}(\mathfrak{p}^m)\cap\mathrm{U})\prod_{\substack{\beta\in J_{\mu-s(\lambda)}\\ \epsilon_i+\epsilon_j=\beta}}x_{\epsilon_i}(b_\beta c_i\varpi^{-1})x_{\epsilon_j}(b_\beta c_j\varpi^{-1})x_\beta(b_\beta\varpi^{-m-1})\\
&\prod_{\beta\in I_{\nu-\mu}}\prod_{\epsilon_{i'}-\epsilon_j=\beta}x_{-\epsilon_j}(b_\beta c_{i'}\varpi^{m-1})x_\beta(b_\beta\varpi^{-1})\left(\prod_{i=1}^n x_{-\epsilon_i}(c_i\varpi^m)\right)\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m)
\end{aligned}
$$

For μ,ν in the index set above, let us denote by $\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})$ the set

$$
\begin{aligned}
\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b}):=(\mathrm{K}(\mathfrak{p}^m)\cap\mathrm{U})&\prod_{\substack{\beta\in J_{\mu-s(\lambda)}\\ \epsilon_i+\epsilon_j=\beta}}x_{\epsilon_i}(b_\beta c_i\varpi^{-1})x_{\epsilon_j}(b_\beta c_j\varpi^{-1})x_\beta(b_\beta\varpi^{-m-1})\\
&\prod_{\beta\in I_{\nu-\mu}}\prod_{\epsilon_{i'}-\epsilon_j=\beta}x_{-\epsilon_j}(b_\beta c_{i'}\varpi^{m-1})x_\beta(b_\beta\varpi^{-1})\prod_{i=1}^n x_{-\epsilon_i}(c_i\varpi^m)\,\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m).
\end{aligned}
$$

Then

$$
\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)=\cup_{s\in W_{\mathrm{H}}}\cup_{\substack{\mu\geq_{\mathrm{Z}}s(\lambda),\,\nu\geq_{\mathrm{M}}\mu\\ \langle\mu,\mu-s(\lambda)\rangle=\langle\nu,\nu-\mu\rangle=0}}\cup_{\vec{c}\in(\mathfrak{o}/\mathfrak{p})^n,\vec{b}\in(\mathfrak{o}/\mathfrak{p})_{\beta\in\Phi_{\mathrm{H}}^+}}\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b}).
$$

We note that if $g\in\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})$, for all $t\in\mathrm{T}$ and $v\in\pi^{\mathrm{K}(\mathfrak{p}^m)}$ the value $W_{\pi(g)v}(t)$ is a multiple of

$$
W_v(t\prod_{\beta\in I_{\nu-\mu}}\prod_{\epsilon_{i'}-\epsilon_j=\beta}x_{-\epsilon_j}(b_\beta c_{i'}\varpi^{m-1})\prod_{\langle\nu,\epsilon_i\rangle=-1}x_{-\epsilon_i}(c_i\varpi^m)\,\varpi^\nu),
$$

by the property of the Whittaker function that U acts on the left by character θ.

One observes that for $c, d \in \mathfrak{o}^\times$ and $a, b \in \mathbb{Z}$,

$$
\begin{aligned}
& x_{-\epsilon_i}(c\varpi^{m-a})x_{-\epsilon_k}(d\varpi^{m-b}) \\
= \;& x_{\epsilon_k-\epsilon_i}(cd^{-1}\varpi^{b-a})x_{-\epsilon_k}(d\varpi^{m-b})x_{\epsilon_k-\epsilon_i}(-cd^{-1}\varpi^{b-a}) \\
= \;& x_{\epsilon_k-\epsilon_i}(cd^{-1}\varpi^{b-a})x_{-\epsilon_k}(d\varpi^{m-b})x_{-\epsilon_k+\epsilon_i}(-dc^{-1}\varpi^{a-b})\varpi^{(a-b)(\epsilon_i-\epsilon_k)}w_{s_{\epsilon_k-\epsilon_i},m}x_{-\epsilon_k+\epsilon_i}(-dc^{-1}\varpi^{a-b})
\end{aligned}
$$

for some lift $w_{s_{\epsilon_k-\epsilon_i},m}$ of the Weyl element $s_{\epsilon_k-\epsilon_i}$ to $\mathrm{K}(\mathfrak{p}^m)$.

We have

$$
\begin{aligned}
& \prod_{\beta\in I_{\nu-\mu}}\prod_{\epsilon_{i'}-\epsilon_j=\beta} x_{-\epsilon_j}(b_\beta c_{i'}\varpi^{m-1}) \prod_{\langle\nu,\epsilon_i\rangle=-1} x_{-\epsilon_i}(c_i\varpi^m)\,\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m) \\
= \;& \prod_{\beta\in I_{\nu-\mu}}\prod_{\epsilon_{i'}-\epsilon_j=\beta,\, j\neq j_0} x_{\epsilon_{j_0}-\epsilon_j}(b^{-1}_{\epsilon_{i'_0}-\epsilon_{j_0}}c^{-1}_{i'_0}c_{i'}b_\beta) \prod_{\langle\nu,\epsilon_i\rangle=-1,\, i\neq i_0} x_{\epsilon_{i_0}-\epsilon_i}(c^{-1}_{i_0}c_i) \\
& x_{-\epsilon_{j_0}}(b_{\epsilon_{i'_0}-\epsilon_{j_0}}c_{i'_0}\varpi^{m-1})x_{-\epsilon_{i_0}}(c_{i_0}\varpi^m)\,\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m)
\end{aligned}
$$

where j_0 is the smallest j such that $\beta = \epsilon_{i'_0} - \epsilon_{j_0}$ and $b_\beta \neq 0$ for some $\beta \in I_{\nu-\mu}$ (note $j > 1$), and i_0 is the smallest i such that $\langle\nu, \epsilon_i\rangle = -1$ and $c_i \neq 0$.

If $i_0 < j_0$, then

$$
\begin{aligned}
& x_{-\epsilon_{j_0}}(b_\beta c_{i'_0}\varpi^{m-1})x_{-\epsilon_{i_0}}(c_{i_0}\varpi^m)\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m) \\
= \;& x_{\epsilon_{i_0}-\epsilon_{j_0}}(b_\beta c_{i'_0}c^{-1}_{i_0}\varpi^{-1})x_{-\epsilon_{i_0}}(c_{i_0}\varpi^m)x_{\epsilon_{i_0}-\epsilon_{j_0}}(-b_\beta c_{i'_0}c^{-1}_{i_0}\varpi^{-1})\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m) \\
= \;& x_{\epsilon_{i_0}-\epsilon_{j_0}}(b_\beta c_{i'_0}c^{-1}_{i_0}\varpi^{-1})x_{-\epsilon_{i_0}}(c_{i_0}\varpi^m)\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m)
\end{aligned}
$$

This is nice if $i_0 = 1$ and $\langle\nu, \epsilon_1\rangle = -1$. We continue if $i_0 > 1$ and $\langle\nu, \epsilon_1\rangle \geq 0$.

$$
\begin{aligned}
= \;& x_{\epsilon_{i_0}-\epsilon_{j_0}}(b_\beta c_{i'_0}c^{-1}_{i_0}\varpi^{-1})x_{-\epsilon_{i_0}}(c_{i_0}\varpi^m)x_{-\epsilon_1}(\varpi^{m-\langle\nu,\epsilon_1\rangle})\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m) \\
= \;& x_{\epsilon_{i_0}-\epsilon_{j_0}}(b_\beta c_{i'_0}c^{-1}_{i_0}\varpi^{-1})x_{\epsilon_1-\epsilon_{i_0}}(c_{i_0}\varpi^{\langle\nu,\epsilon_1\rangle})x_{-\epsilon_1}(\varpi^{m-\langle\nu,\epsilon_1\rangle})x_{-\epsilon_1+\epsilon_{i_0}}(-c^{-1}_{i_0}\varpi^{-\langle\nu,\epsilon_1\rangle}) \\
& \varpi^{s_{\epsilon_1-\epsilon_{i_0}}(\nu)+\langle\nu,\epsilon_1\rangle(\epsilon_1-\epsilon_{i_0})}\,\mathrm{K}(\mathfrak{p}^m) \quad (\text{while } s_{\epsilon_1-\epsilon_{i_0}}(\nu)+\langle\nu,\epsilon_1\rangle(\epsilon_1-\epsilon_{i_0}) = \nu-\epsilon_1+\epsilon_{i_0}).
\end{aligned}
$$

Hence if $g \in \mathscr{E}_{s,\mu,\nu}(\vec{c}, \vec{b})$, for $t \in \mathrm{T}, v \in \pi^{\mathrm{K}(\mathfrak{p}^m)}$ the value $W_{\pi(g)v}(t)$ is a multiple of $W_v(t\,x_{-\epsilon_1}(\varpi^{m-\langle\nu,\epsilon_1\rangle})x_{-\epsilon_1+\epsilon_{i_0}}(-c^{-1}_{i_0}\varpi^{-\langle\nu,\epsilon_1\rangle})\,\varpi^{\nu-\epsilon_1+\epsilon_{i_0}})$. Note $\nu \geq_{\mathrm{H}} \nu - \epsilon_1 + \epsilon_{i_0}$.

If $i_0 > j_0 > 1$, then $\langle \nu, \epsilon_1 \rangle = a \geq 0$ and (noting $\langle \nu, \epsilon_{i_0} \rangle = -1$ and $\langle \nu, \epsilon_{j_0} \rangle = 0$)

$$\begin{aligned}
& x_{-\epsilon_{j_0}}(b_\beta c_{i_0'} \varpi^{m-1}) x_{-\epsilon_{i_0}}(c_{i_0} \varpi^m) \varpi^\nu \, \mathrm{K}(\mathfrak{p}^m) \\
= \; & x_{-\epsilon_{j_0}}(b_\beta c_{i_0'} \varpi^{m-1}) x_{-\epsilon_{i_0}}(c_{i_0} \varpi^m) x_{-\epsilon_1}(\varpi^{m-a}) \varpi^\nu \, \mathrm{K}(\mathfrak{p}^m) \\
= \; & x_{-\epsilon_{j_0}}(b_\beta c_{i_0'} \varpi^{m-1}) x_{\epsilon_1 - \epsilon_{i_0}}(c_{i_0} \varpi^a) x_{-\epsilon_1}(\varpi^{m-a}) x_{-\epsilon_1 + \epsilon_{i_0}}(-c_{i_0}^{-1} \varpi^{-a}) \varpi^{s_{\epsilon_1 - \epsilon_{i_0}}(\nu) + a(\epsilon_1 - \epsilon_{i_0})} \, \mathrm{K}(\mathfrak{p}^m) \\
= \; & x_{\epsilon_1 - \epsilon_{i_0}}(c_{i_0} \varpi^a) x_{\epsilon_1 - \epsilon_{j_0}}(b_\beta c_{i_0'} \varpi^{a-1}) x_{-\epsilon_1}(\varpi^{m-a}) x_{\epsilon_1 - \epsilon_{j_0}}(-b_\beta c_{i_0'} \varpi^{a-1}) x_{-\epsilon_1 + \epsilon_{i_0}}(-c_{i_0}^{-1} \varpi^{-a}) \\
& \varpi^{s_{\epsilon_1 - \epsilon_{i_0}}(\nu) + a(\epsilon_1 - \epsilon_{i_0})} \, \mathrm{K}(\mathfrak{p}^m) \quad (\text{while } s_{\epsilon_1 - \epsilon_{i_0}}(\nu) + a(\epsilon_1 - \epsilon_{i_0}) = \nu - \epsilon_1 + \epsilon_{i_0}) \\
= \; & x_{\epsilon_1 - \epsilon_{i_0}}(c_{i_0} \varpi^a) x_{\epsilon_1 - \epsilon_{j_0}}(b_\beta c_{i_0'} \varpi^{a-1}) x_{-\epsilon_1}(\varpi^{m-a}) x_{-\epsilon_1 + \epsilon_{i_0}}(-c_{i_0}^{-1} \varpi^{-a}) x_{\epsilon_1 - \epsilon_{j_0}}(-b_\beta c_{i_0'} \varpi^{a-1}) \\
& x_{-\epsilon_{j_0} + \epsilon_{i_0}}(-b_\beta c_{i_0'} c_{i_0}^{-1} \varpi^{-1}) \varpi^{\nu - \epsilon_1 + \epsilon_{i_0}} \, \mathrm{K}(\mathfrak{p}^m) \quad (\text{noting } \langle \nu - \epsilon_1 + \epsilon_{i_0}, \epsilon_1 - \epsilon_{j_0} \rangle = a - 1) \\
= \; & x_{\epsilon_1 - \epsilon_{i_0}}(c_{i_0}) x_{\epsilon_1 - \epsilon_{j_0}}(b_\beta c_{i_0'} \varpi^{-1}) x_{-\epsilon_1}(\varpi^m) x_{-\epsilon_1 + \epsilon_{i_0}}(-c_{i_0}^{-1}) x_{\epsilon_{j_0} - \epsilon_{i_0}}(-b_\beta^{-1} c_{i_0'}^{-1} c_{i_0} \varpi) \\
& \varpi^{s_{\epsilon_{j_0} - \epsilon_{i_0}}(\nu - \epsilon_1 + \epsilon_{i_0}) + \epsilon_{j_0} - \epsilon_{i_0}} \, \mathrm{K}(\mathfrak{p}^m) \quad (\text{while } s_{\epsilon_{j_0} - \epsilon_{i_0}}(\nu - \epsilon_1 + \epsilon_{i_0}) + \epsilon_{j_0} - \epsilon_{i_0} = \nu - \epsilon_1 + \epsilon_{j_0}) \\
= \; & x_{\epsilon_1 - \epsilon_{i_0}}(c_{i_0}) x_{\epsilon_1 - \epsilon_{j_0}}(b_\beta c_{i_0'} \varpi^{-1}) x_{\epsilon_{j_0} - \epsilon_{i_0}}(-b_\beta^{-1} c_{i_0'}^{-1} c_{i_0} \varpi) \\
& x_{-\epsilon_1}(\varpi^m) x_{-\epsilon_1 + \epsilon_{i_0}}(-c_{i_0}^{-1}) x_{-\epsilon_1 + \epsilon_{j_0}}(b_\beta^{-1} c_{i_0'}^{-1} \varpi) \varpi^{\nu - \epsilon_1 + \epsilon_{j_0}} \, \mathrm{K}(\mathfrak{p}^m).
\end{aligned}$$

Hence if $g \in \mathscr{E}_{s,\mu,\nu}(\vec{c}, \vec{b})$, for $t \in \mathrm{T}, v \in \pi^{\mathrm{K}(\mathfrak{p}^m)}$ the value $W_{\pi(g)v}(t)$ is a multiple of $W_v(t \, x_{-\epsilon_1}(\varpi^m) x_{-\epsilon_1 + \epsilon_{i_0}}(-c_{i_0}^{-1}) x_{-\epsilon_1 + \epsilon_{j_0}}(b_\beta^{-1} c_{i_0'}^{-1} \varpi) \, \varpi^{\nu - \epsilon_1 + \epsilon_{j_0}})$. Note $\nu \geq_{\mathrm{H}} \nu - \epsilon_1 + \epsilon_{j_0}$.

Lemma 8.3.3. *(Assume $n \geq 2$.) For $c, c_i, c_j \in k$ and $\nu \in \mathrm{X}_\bullet(\mathrm{T})$,*

$$W_{\pi(x_{-\epsilon_1}(c) x_{-\epsilon_1 + \epsilon_i}(c_i) x_{-\epsilon_1 + \epsilon_j}(c_j) \varpi^\nu) v}(\varpi^{a\epsilon_1}) = W_{\pi(\varpi^\nu) v}(\varpi^{a\epsilon_1}).$$

Proof. By functional equation

$$\begin{aligned}
& I(\pi(x_{-\epsilon_1}(c) x_{-\epsilon_1 + \epsilon_i}(c_i) x_{-\epsilon_1 + \epsilon_j}(c_j) \varpi^\nu) v, s) \\
= \; & \gamma(\pi, s, \psi)^{-1} I(\pi(u_0 x_{-\epsilon_1}(c) x_{-\epsilon_1 + \epsilon_i}(c_i) x_{-\epsilon_1 + \epsilon_j}(c_j) \varpi^\nu) v, 1 - s) \\
= \; & \gamma(\pi, s, \psi)^{-1} I(\pi(x_{\epsilon_1}(c) x_{\epsilon_1 + \epsilon_i}(c_i) x_{\epsilon_1 + \epsilon_j}(c_j) u_0 \varpi^\nu) v, 1 - s) \\
= \; & \gamma(\pi, s, \psi)^{-1} I(\pi(u_0 \varpi^\nu) v, 1 - s) = I(\pi(\varpi^\nu) v, s).
\end{aligned}$$

Since $I(\pi(x_{-\epsilon_1}(c)x_{-\epsilon_1+\epsilon_i}(c_i)x_{-\epsilon_1+\epsilon_j}(c_j)\varpi^\nu)v,s)$ and $I(\pi(\varpi^\nu)v,s)$ are the generating functions of the two Whittaker values for $a\in\mathbb{Z}$, comparing the coefficients of q^{-as} the assertion follows. □

We have enough information for computing $c_{a\epsilon_1}(T_\lambda(v))$ by summing it over $\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})$.

Let us list the values $c_{a\epsilon_1}(\int_{\cup_{\vec{c},\vec{b}}\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})}\pi(h)v\ dh)$ for some easy cases.

Proposition 8.3.4. *If $\mu=\nu=\widetilde{s(\lambda)}$ and $\langle\nu,\alpha_i\rangle\geq 0$ for $i>1$. Then $\nu=\sum_{i=1}^{d_s^+}\epsilon_i$, $\sum_{i=2}^{d_s^++1}\epsilon_i$ or $-\epsilon_1+\sum_{i=2}^{d_s^++1}\epsilon_i$, and for $a\geq 0$*
(i) $c_{a\epsilon_1}(\int_{\cup_{\vec{c},\vec{b}}\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})}\pi(h)v\ dh)=q^{(2n-d_s^+)d_s^++\frac{3}{2}d_s^-}\ c_{a\epsilon_1+\nu}(v)$, if $\nu=\sum_{i=1}^{d_s^+}\epsilon_i$;
(ii) $c_{a\epsilon_1}(\int_{\cup_{\vec{c},\vec{b}}\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})}\pi(h)v\ dh)=q^{(2n-d_s^+-1)d_s^++\frac{3}{2}d_s^-}\ c_{a\epsilon_1+\nu}(v)$, if $\nu=\sum_{i=2}^{d_s^++1}\epsilon_i$;
(iii) $c_{a\epsilon_1}(\int_{\cup_{\vec{c},\vec{b}}\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})}\pi(h)v\ dh)=q^{(2(n-1)-d_s^+)d_s^++\frac{3}{2}(d_s^--1)}\ c_{a\epsilon_1+\nu}(v)$, if $\nu=-\epsilon_1+\sum_{i=2}^{d_s^++1}\epsilon_i$.

Definition 8.3.5. For each $\nu\in\mathrm{X}_\bullet(\mathrm{T})$ with $\langle\nu,\epsilon_n\rangle=0$ set $\nu^!$ as the shift of the coordinate under basis $(\epsilon_1,\epsilon_2,...,\epsilon_{n-1},\epsilon_n)$ by one to the right. More explicitly, $(\sum_{i=1}^{n-1}a_i\epsilon_i)^!=\sum_{i=1}^{n-1}a_i\epsilon_{i+1}$.

Then we conclude the following proposition.

Proposition 8.3.6. *Assume $\lambda=\lambda_i$ for some $i<n$ and $v\in V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$, then*

$$c_{a\epsilon_1}(T_i(v))=\sum_{\nu\leq_{\mathrm{H}}\lambda_i}a_{a\epsilon_1+\nu}c_{a\epsilon_1+\nu}(v)+\sum_{\nu\leq_{\mathrm{H}}\lambda_i}a_{a\epsilon_1+\nu^!}c_{a\epsilon_1+\nu^!}(v)+\sum_{\nu+2\epsilon_1\leq_{\mathrm{H}}\lambda_i}a_{a\epsilon_1+\nu}c_{a\epsilon_1+\nu}(v)$$

for some $a_\nu\in\mathbb{R}$, $\nu\in P^+$. Moreover, $a_{a\epsilon_i+\lambda_i}$, $a_{a\epsilon_1+\lambda_i^!}$ are positive numbers for $a\geq 0$. If $\lambda=\lambda_n$ or λ_n^*, then

$$c_{a\epsilon_1}(T_i(v))=\sum_{\nu\leq_{\mathrm{H}}\lambda}a_{a\epsilon_1+\nu}c_{a\epsilon_1+\nu}(v)+\sum_{\nu+2\epsilon_1\leq_{\mathrm{H}}\lambda}a_{a\epsilon_1+\nu}c_{a\epsilon_1+\nu}(v)$$

and $a_{a\epsilon_1+\lambda}$ is positive for $a\geq 0$.

For example, one has

$$c_{a\epsilon_1}(T_1(v)) = a_{(a+1)\epsilon_1}c_{(a+1)\epsilon_1}(v) + a_{a\epsilon_1+\epsilon_2}c_{a\epsilon_1+\epsilon_2}(v) + a_{(a-1)\epsilon_1}c_{(a-1)\epsilon_1}(v)$$

with $a_{(a+1)\epsilon_1} = q^{2n-1}$, $a_{a\epsilon_1+\epsilon_2} = 1$, $a_{(a-1)\epsilon_1} = q^{2n-2}$.

We wish to get a relation of $c_{a\epsilon_1}(v)$ for $a \in \mathbb{Z}$. To get rid of $c_{a\epsilon_1+\epsilon_2}(v)$ in the expression, we can go one more step and use

$$\begin{aligned} c_{(a-1)\epsilon_1}(T_2(v)) &= a_{a\epsilon_1+\epsilon_2}c_{a\epsilon_1+\epsilon_2}(v) + a_{(a-1)\epsilon_1+\epsilon_2+\epsilon_3}c_{(a-1)\epsilon_1+\epsilon_2+\epsilon_3}(v) \\ &\quad + a_{(a-1)\epsilon_1}c_{(a-1)\epsilon_1}(v) + a_{(a-2)\epsilon_1+\epsilon_2}c_{(a-2)\epsilon_1+\epsilon_2}(v). \end{aligned}$$

Then by replacing the terms with $a\epsilon_1+\epsilon_2$, $(a-2)\epsilon_1+\epsilon_2$ by the previous relation, there is only one term which is not of the desired form, namely $c_{(a-1)\epsilon_1+\lambda_2^!}$. Since $c_{(a-i+1)+\lambda_i^!}$ always has nonzero coefficient in the expression of $c_{(a-i+1)\epsilon_1}(T_{\lambda_i^!}(v))$, we continue this process till we meet the expression for $c_{(a-n+1)\epsilon_1}((T_n+T_n^*)(v))$, which involves no more shifted terms but only those $\nu \le \lambda_n, \lambda_n^*$ terms. In other words, the relation for $c_{(a-n+1)\epsilon_1}(T_n(v))$ and $c_{(a-n+1)\epsilon_1}(T_n^*(v))$ can be totally reduced to terms with only $c_{(a+1)\epsilon_1}(v)$, $c_{a\epsilon_1}(v)$, $c_{(a-1)\epsilon_1}(v)$,...,$c_{(a-n+1)\epsilon_1}(v)$ and these of $T_\lambda(v)$ involved.

Corollary 8.3.7. *Assume a fixed vector v is a simultaneous eigenvector of the Hecke operators $T_1, T_2, ..., T_{n-1}$ and $T_n+T_n^*$. There exist linearly independent combinations of the eigenvalues $c_0, c_1..., c_{n-1}$ and $c_n = q^{2n-1}$ such that the relation*

$$c_nc_{(a+n)\epsilon_1}(v) + c_{n-1}c_{(a+n-1)\epsilon_1} + ... + c_1c_{(a+1)\epsilon_1}(v) + c_0c_{a\epsilon_1}(v) = 0 \qquad \textit{holds for } a \ge 0.$$

Recall $I(v,s) = \mathrm{vol}(\mathfrak{o}^\times)\sum_{a\ge 0} q^{a(n-\frac{1}{2})}c_{a\epsilon_1}(v)q^{-as}$. The recurrence relation leads to

$$(q^{-n(n-\frac{1}{2})}c_n + ... + q^{-(n-\frac{1}{2})}c_1q^{-(n-1)s} + c_0q^{-ns})I(v,s) = \mathrm{vol}(\mathfrak{o}^\times)c_0(v). \tag{8.3.3}$$

Corollary 8.3.8. *Assume a fixed vector v is a simultaneous eigenvector of the Hecke operators $T_1, T_2, ..., T_{n-1}$ and $T_n + T_n^*$. Then if $\ell_\theta(v) \neq 0$, then $I(v,s)$ is a nonzero constant and the eigenvalues are unique. As a result, the values $c_{a\epsilon_1+\lambda_i}(v)$ for $a \geq 0$, $0 \leq i \leq n$ are uniquely determined by $\ell_\theta(v)$. On the other hand, if $\ell_\theta = 0$, then $c_{a\epsilon_1+\lambda_i}(v) = 0$ for $a \geq 0$, $0 \leq i \leq n$.*

Proof. Since we assume π is generic and supercuspidal so $I(v,s) \in \mathbb{C}[q^{-s}, q^s]$. Hence by the expression of $I(v,s)$ in (8.3.3) it must be a constant and we have $c_i = 0$ for $i = 0, 1, ..., n-1$ which determines a nonsingular system of n linear equations of the n eigenvalues. Therefore the eigenvalues are uniquely determined. Solving back we get all other Whittaker values in the expression of $c_{a\epsilon_1}(T_i(v))$, $\forall a, \forall i$. □

8.4. Minimal level

In this section we investigate the Hecke eigenvectors at the minimal level. Assume $\mathfrak{c}(\pi) = \mathfrak{p}^{c(\pi)}$ is the maximal idea of $\mathfrak{o}$ such that the fixed space $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ is nonzero and thus there exists nonzero fixed vectors of level $c(\pi)$, minimal among all. By definition the fixed space $V_\pi^{\mathrm{K}(\mathfrak{p}^{c(\pi)-1})}$ of level smaller than $c(\pi)$ must be zero. We have discussed the fixed vector of level 0 or 1 in Chapter 7. Let us assume $c(\pi) \geq 2$. Indeed, since by Theorem 7.3.6 $c(\pi) \geq a_\pi$, and $a_\pi \geq 2n \geq 2$ for π generic supercuspidal, this assumption always holds.

Recall that we have seen for $m \geq 2$,

$$\mathrm{K}(\mathfrak{p}^m)\varpi^\lambda \mathrm{K}(\mathfrak{p}^m) = \cup_{\substack{s\in W_{\mathrm{H}},\, \mu\geq_{\mathrm{Z}} s(\lambda),\, \nu\geq_{\mathrm{M}}\mu \\ \langle\mu,\mu-s(\lambda)\rangle=\langle\nu,\nu-\mu\rangle=0}} \cup_{\vec{c}\in(\mathfrak{o}/\mathfrak{p})^n, \vec{b}\in(\mathfrak{o}/\mathfrak{p})_{\beta\in\Phi_{\mathrm{H}}^+}} \mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b})$$

with

$$\mathscr{E}_{s,\mu,\nu}(\vec{c},\vec{b}) = (\mathrm{K}(\mathfrak{p}^m)\cap \mathrm{U}) \prod_{\substack{\beta\in J_{\mu-s(\lambda)} \\ \epsilon_i+\epsilon_j=\beta}} x_{\epsilon_i}(b_\beta c_i \varpi^{-1}) x_{\epsilon_j}(b_\beta c_j \varpi^{-1}) x_\beta(b_\beta \varpi^{-m-1})$$

$$\prod_{\beta\in I_{\nu-\mu}}\prod_{\epsilon_{i'}-\epsilon_j=\beta} x_{-\epsilon_j}(b_\beta c_{i'}\varpi^{m-1})x_\beta(b_\beta\varpi^{-1})\prod_{i=1}^{n} x_{-\epsilon_i}(c_i\varpi^m)\,\varpi^\nu\,\mathrm{K}(\mathfrak{p}^m)$$

which equals to

$$(\mathrm{K}(\mathfrak{p}^m)\cap \mathrm{U})\prod_{\substack{\beta\in J_{\mu-s(\lambda)}\\ \epsilon_i+\epsilon_j=\beta}} x_{\epsilon_i}(b_\beta c_i\varpi^{-1})x_{\epsilon_j}(b_\beta c_j\varpi^{-1})x_\beta(b_\beta\varpi^{-m-1})\prod_{\substack{\beta\in I_{\nu-\mu}\\ \epsilon_{i'}-\epsilon_j=\beta}} x_\beta(b_\beta\varpi^{-1})\cdot$$

$$\prod_{\substack{\beta\in I_{\nu-\mu},\, j\neq j_0\\ \epsilon_{i'}-\epsilon_j=\beta}} x_{\epsilon_{j_0}-\epsilon_j}(b^{-1}_{\epsilon_{i'_0}-\epsilon_{j_0}}c^{-1}_{i'_0}c_{i'}b_\beta)\prod_{\substack{\langle\nu,\epsilon_i\rangle=-1\\ i\neq i_0}} x_{\epsilon_{i_0}-\epsilon_i}(c^{-1}_{i_0}c_i)\left(\varpi^\nu x_{-\epsilon_{j_0}}(b_{\epsilon_{i'_0}-\epsilon_{j_0}}c_{i'_0}\varpi^{m-1})x_{-\epsilon_{i_0}}(c_{i_0}\varpi^{m-1})\right)\mathrm{K}(\mathfrak{p}^m)$$

where j_0 is the smallest j such that $\beta=\epsilon_{i'_0}-\epsilon_{j_0}$ and $b_\beta\neq 0$ for $\beta\in I_{\nu-\mu}$, and i_0 is the smallest i such that $\langle\nu,\epsilon_i\rangle=-1$ and $c_i\neq 0$. Define $a_{\nu,m}$ as the size $|(\mathrm{K}(\mathfrak{p}^m)\cap\mathrm{U})^{\varpi^\nu}/(\mathrm{K}(\mathfrak{p}^m)\cap\mathrm{U})^{\varpi^\nu}\cap\mathrm{K}(\mathfrak{p}^m)|$ for any given $\nu\in\mathrm{X}_\bullet(\mathrm{T})$, $m\in\mathbb{N}$.

We similarly get for $\lambda\in\{0,\epsilon_1\}$,

$$\begin{aligned}&\mathrm{K}(\mathfrak{p}^{m-1})\varpi^\lambda\,\mathrm{K}(\mathfrak{p}^m)\\ =\ &\cup_{\substack{s\in W_{\mathrm{H}},\mu'\geq_{\mathrm{Z}}0,\nu'\geq_{\mathrm{M}}0\\ \langle\mu',s(\lambda)+\gamma_s\rangle=0\\ \langle\nu',2(s(\lambda)+\gamma_s+\mu')+\nu'\rangle=0}}\cup_{\vec{c}\in(\mathfrak{o}/\mathfrak{p})^n,\vec{b}\in(\mathfrak{o}/\mathfrak{p})_\beta}\prod_{i=1}^n x_{\epsilon_i}(\mathfrak{o}/\mathfrak{p})\prod_{\substack{\beta\in J_{\mu'}\\ \epsilon_i+\epsilon_j=\beta}} x_{\epsilon_i}(b_\beta c_i)x_{\epsilon_j}(b_\beta c_j)x_\beta(b_\beta\varpi^{-m+1})\\ &\prod_{\substack{\beta\in I'_{\nu'},\, j\neq\tilde{j}_0\\ \epsilon_{i'}-\epsilon_j=\beta}} x_{\epsilon_{\tilde{j}_0}-\epsilon_j}(b^{-1}_{\epsilon_{i'_0}-\epsilon_{\tilde{j}_0}}c^{-1}_{i'_0}c_{i'}b_\beta)\prod_{\substack{\langle s(\lambda)+\gamma_s+\mu'+\nu',\epsilon_i\rangle=0\\ i\neq\tilde{i}_0}} x_{\epsilon_{\tilde{i}_0}-\epsilon_i}(c^{-1}_{\tilde{i}_0}c_i)\Big(\varpi^{s(\lambda)+\gamma_s+\mu'+\nu'}\\ &x_{-\epsilon_{\tilde{j}_0}}(b_{\epsilon_{i'_0}-\epsilon_{\tilde{j}_0}}c_{i'_0}\varpi^{m-1})x_{-\epsilon_{\tilde{i}_0}}(c_{\tilde{i}_0}\varpi^{m-1})\Big)\,\mathrm{K}(\mathfrak{p}^m)\end{aligned}$$

where $\tilde{j}_0$ is the smallest j such that $\beta=\epsilon_{i'_0}-\epsilon_{j_0}$ and $\tilde{i}_0$ is the smallest i such that $\langle s(\lambda)+\gamma_s+\mu'+\nu',\epsilon_i\rangle=0$ and $c_i\neq 0$ and $\varpi^{\gamma_s}=w_{s,m-1}w_{s^{-1},m}$ as before.

For $\lambda'\in P^+$, $\lambda\in P^+_{\mathrm{H}}$ minuscule and $v\in V_\pi^{\mathrm{K}(\mathfrak{p}^{c(\pi)})}$, we have

$$c_{\lambda'}(T_\lambda(v))=\sum_{s,\mu,\nu}a_{\nu,m}q^{3|J_{\mu-s(\lambda)}|+|I_{\nu-\mu}|}\sum_{c_{i_0},c'_{i'_0}\in\mathfrak{o}/\mathfrak{p}}c_{\lambda'+\nu}(\pi(x_{-\epsilon_{j_0}}(c'_{i'_0}\varpi^{c(\pi)-1})x_{-\epsilon_{i_0}}(c_{i_0}\varpi^{c(\pi)-1}))v).$$

And for $\lambda'' \in P^+$, $\lambda \in \{0, \epsilon_1\}$ and $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{c(\pi)})}$, by $\delta_\lambda(v) = 0$ we have

$$\sum_{s,\mu',\nu'} a'_{\nu,m} q^{3|J_{\mu'}|+|I'_{\nu'}|} \sum_{c_{\tilde{i}_0}, c'_{i'_0} \in \mathfrak{o}/\mathfrak{p}} c_{\lambda''+\nu}(\pi(x_{-\epsilon_{\tilde{j}_0}}(c'_{i'_0}\varpi^{c(\pi)-1}) x_{-\epsilon_{\tilde{i}_0}}(c_{\tilde{i}_0}\varpi^{c(\pi)-1}))v) = 0$$

with $0 \leq_{\mathrm{H}} \nu = s(\lambda) + \gamma_s + \mu' + \nu' \leq_{\mathrm{H}} \lambda$ and $a'_{\nu,m} = q^{\deg(\nu)}$. Then we can solve for $\sum_{c_{\tilde{i}_0}, c'_{i'_0} \in \mathfrak{o}/\mathfrak{p}} c_{\lambda''+\nu}(\pi(x_{-\epsilon_{\tilde{j}_0}}(c'_{i'_0}\varpi^{c(\pi)-1}) x_{-\epsilon_{\tilde{i}_0}}(c_{\tilde{i}_0}\varpi^{c(\pi)-1}))v)$, $0 \leq_{\mathrm{H}} \nu \leq_{\mathrm{H}} \lambda$, by $c_{\lambda''+\nu''}(v)$ by choosing $\lambda'' = \lambda' - (\lambda_n - \lambda_i)$, $i = 0, 1, 2, ..., n$, for all $\lambda' \in \mathrm{P}^+$ in lexicographic order for each λ''. This implies for $1 \leq j < n$ there exists $b_{j,\nu,m}$ such that

$$\mu_j v = c_{\lambda'}(T_j(v)) = \sum_{\substack{s \in W_{\mathrm{H}}, \nu \geq_{\mathrm{H}} s(\lambda_j) \\ \langle \nu, \nu - s(\lambda_j) \rangle = 0}} b_{j,\nu,m} c_{\lambda'+\nu}(v)$$

and there exists $b_{n,\nu,m}, b^*_{n,\nu,m}$ such that

$$\mu_n v = c_{\lambda'}((T_n + T_n^*)(v)) = \sum_{\substack{s \in W_{\mathrm{H}}, \nu \geq_{\mathrm{H}} s(\lambda_n) \\ \langle \nu, \nu - s(\lambda_n) \rangle = 0}} b_{n,\nu,m} c_{\lambda'+\nu}(v) + \sum_{\substack{s \in W_{\mathrm{H}}, \nu \geq_{\mathrm{H}} s(\lambda_n^*) \\ \langle \nu, \nu - s(\lambda_n^*) \rangle = 0}} b^*_{n,\nu,m} c_{\lambda'+\nu}(v)$$

if $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{c(\pi)})}$ is a Hecke eigenvector with Hecke eigenvalues $\mu_1, \mu_2, ..., \mu_n$.

By Corollary 8.3.8, if v is a Hecke eigenvector, then the values $c_{a\epsilon_1+\lambda_i}(v)$ for $a \geq 0$, $0 \leq i \leq n$ are uniquely determined by $\ell_\theta(v) = c_0(v)$. With the relation above, since $\ell_\theta(v)$ determines $c_{a\epsilon_1+\lambda_i}(v)$ so it determines the values $c_{a\epsilon_1+b\epsilon_2+\lambda_i}(v)$ for $a, b \geq 0$, $0 \leq i \leq n$, as well. Continue a similar process we can argue that $\ell_\theta(v)$ determines $c_{\lambda''}(v)$ for all $\lambda \in P^+$. As a result if $\ell_\theta(v) = 0$, then v must be equal to 0, and once $\ell_\theta(v)$ is determined, then $W_v|_{\mathrm{T}/\mathrm{T}(\mathfrak{o})}$ is determined. This implies $\Xi(v)$ is uniquely determined by $\ell_\theta(v)$. However, we know that the $\mathbb{C}$-linear map Ξ is injective on fixed vectors of fixed level by Lemma 5.3.3, so this implies such eigenvector is unique up to scaling. This leads to the following Multiplicity One Theorem.

Theorem 8.4.1 (Multiplicity One)**.** $\dim V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))} = 1$ *and if v is a nonzero fixed vector of minimal level $c(\pi)$, then $\ell_\theta(v)$ must be nonzero.*

Proof. Since the Hecke operators $T_1, T_2, ..., T_{n-1}$ and $T_n + T_n^*$ are self-adjoint and commute with each other. The $\mathrm{K}(\mathfrak{p}^m)$-fixed subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^m)}$ decomposes into common eigenspaces of $T_1, T_2, ..., T_{n-1}$ and $T_n + T_n^*$ for all m. When $m = c(\pi)$, for each set of eigenvalues $\mu_1, \mu_2, ..., \mu_n$, if $\ell_\theta(v) = 0$, then we have seen eigenvectors of this eigenspace must be 0, hence we may assume nonzero common eigenvectors take nonzero value under the Whittaker functional ℓ_θ, and hence by Corollary 8.3.8, are uniquely determined by the value under ℓ_θ. Hence $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))}$ is of dimension 1 unless ℓ_θ is trivial on this subspace. However, this implies that every eigenvector is zero, which leads to $V_\pi^{\mathrm{K}(\mathfrak{c}(\pi))} = 0$ and contradicts with the existence of fixed vectors. $\square$

The Multiplicity One Theorem implies the following theorem regarding the conductor, which together with our discussion in Section 7.4 gives a result on all local invariants attached to π.

Theorem 8.4.2 (Conductor Theorem)**.** *The minimal level $c(\pi)$ is the conductor $a(\pi)$ and the order two group $\mathrm{J}(\mathfrak{p}^{c(\pi)})/\mathrm{K}(\mathfrak{p}^{c(\pi)})$ acts on the subspace $V_\pi^{\mathrm{K}(\mathfrak{p}^{c(\pi)})}$ by a quadratic character which equals to the root number ε_π.*

Proof. By Corollary 8.3.8 and Theorem 8.4.1, $I(v, s)$ is a nonzero constant for any nonzero $v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{c(\pi)})}$. Since $u_{c(\pi)}v \in V_\pi^{\mathrm{K}(\mathfrak{p}^{c(\pi)})}$ is also nonzero, by Multiplicity One there exists a nonzero constant ε such that $u_{c(\pi)}v = \varepsilon v$. Rescale v and assume $I(v, s) = 1$. Then by the functional equation we have $I(u_{c(\pi)}v, 1-s) = \varepsilon_\pi q^{(c(\pi)-a_\pi)s'} I(v, s)$ which implies $\varepsilon = \varepsilon_\pi q^{(c(\pi)-a_\pi)s'}$. Hence we get $\varepsilon = \varepsilon_\pi$ and $c(\pi) = a_\pi$. $\square$

CHAPTER 9

Main Theorems

We shall finally put all pieces together and get the main results on newforms and oldforms. In this chapter, we give the definition of the *new vector* for a generic representation of $\mathrm{SO}_{2n+1}(k)$ for a non-Archimedean local field k and prove the theory of newforms for the case when the representation is supercuspidal. We give a conjecture on oldforms at the end, which predicts that all fixed vectors are obtained by applying level raising operators on the new vector.

9.1. New vectors and old vectors

Assume (π, V_π) is a smooth irreducible generic representation of G with local invariants conductor a_π and root number ε_π.

Definition 9.1.1. A nonzero vector v of π is a *new vector* of π if v is fixed by $\mathrm{K}(\mathfrak{p}^{a_\pi})$.

Main Theorem 1. *Assume π is supercuspidal. Then the fixed subspace of V_π of the open compact subgroup $\mathrm{K}(\mathfrak{p}^m)$ is nonzero if and only if $m \geq a_\pi$.*

Proof. This is a combination of Theorem 7.3.6 and Corollary 8.1.4.

Main Theorem 2. *The subspace $\pi^{\mathrm{K}(\mathfrak{p}^{a_\pi})}$ is a line generated by the new vectors and the order group group $\mathrm{J}(\mathfrak{p}^{a_\pi})/\mathrm{K}(\mathfrak{p}^{a_\pi})$ acts on this line by quadratic character ε_π. Moreover, the Whittaker functional ℓ_θ is nontrivial on this line.*

Proof. The existence and uniqueness of the new vector is by Theorem 8.4.1 and Theorem 8.4.2. The last assertion is Theorem 8.4.1. □ □

Proposition 9.1.2. *Assume v is a new vector, then $I(v,s)$ is a nonzero constant function and $\Omega(v)$ is a nonzero constant in $\mathscr{S}_n = \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{M}}}$. Moreover, $\omega_{a_\pi} v = \varepsilon_\pi^n v$.*

Proof. By Corollary 8.3.8, since $v \neq 0$ is a Hecke eigenvector so $I(v,s)$ is a nonzero constant. By Lemma 7.3.6 $\Omega(v) \in \mathbb{C}$. Since v is nonzero, so $\Omega(v)$ is a nonzero constant. By the functional equation (5.4.2), we get $\Omega(\omega_{a_\pi} v) = \epsilon_\pi^n \Omega(v)$. Hence by injectivity of the $\mathbb{C}$-linear map Ω, we get $\omega_{a_\pi} v = \epsilon_\pi^n v$. □

Proposition 9.1.3. *Assume v is a new vector. The fixed vectors $\theta_0(v)$ and $\theta_0^*(v)$ of level $a_\pi + 1$ are linearly independent. As a result,* $\dim \pi^{\mathrm{K}(\mathfrak{p}^{a_\pi+1})} \geq 2$.

Proof. Notice $\omega_m \mathrm{K}(\mathfrak{p}^m)$ is $\mathrm{K}(\mathfrak{p}^m)$ if n is even and is $u_m \mathrm{K}(\mathfrak{p}^m)$ if n is odd. Recall that $\mathrm{K}(\mathfrak{p}^{m+1}) \mathrm{K}(\mathfrak{p}^m) = \cup_{s \in W_{\mathrm{H}}} (\mathrm{H}_{x_{m+b}} \cap {}^{\omega_0} \mathrm{V}) w_{s,m+1} w_{s^{-1},m} \mathrm{K}(\mathfrak{p}^m)$. One observes that ${}^{\omega_{m+1}}(\mathrm{H}_{x_{m+b}} \cap {}^{\omega_0} \mathrm{V}) \subset \mathrm{V}$ and $\varpi^{a\epsilon_1} \omega_{m+1} w_{s,m+1} w_{s^{-1},m} \omega_m$ is a torus element and is dominant only if it is $\varpi^{a\epsilon_1}$ or $\varpi^{(a-1)\epsilon_1}$. Hence if n is even, then $W_{\theta_0(v)} = W_{\tilde{\theta}_0(v)}(\varpi^{a\epsilon_1})$ is nonzero scalar times of $W_v(\varpi^{a\epsilon_1})$ since $u_{a_\pi+1}\theta_0 u_{a_\pi}(v)$ is $\mathrm{K}(\mathfrak{p}^{a_\pi+1})$-fixed; if n is odd, then $W_{u_{a_\pi+1}\theta_0 u_{a_\pi}(v)} = W_{\tilde{\theta}_0(v)}(\varpi^{a\epsilon_1})$ is nonzero scalar times of $W_v(\varpi^{(a-1)\epsilon_1})$ since $\theta_0 v$ is $\mathrm{K}(\mathfrak{p}^{a_\pi+1})$-fixed. Hence $I(\tilde{\theta}_0(v), s)$ is a nonzero scalar time of $I(v,s)$ if n is even, and a nonzero scalar times of $q^{-s'} I(v,s)$ if n is odd.

By the functional equation, we have

$$I(u_{a_\pi+1}\tilde{\theta}_0 u_{a_\pi}(v), 1-s) = \varepsilon_\pi q^{s'} I(\tilde{\theta}_0(\varepsilon_\pi v), s).$$

We get $I(\theta_0^*(v), 1-s) = q^{s'} I(\theta_0(v), s)$ is a nonzero scalar times of $q^{s'} I(v,s)$ if n is even and $I(\theta_0(v), 1-s) = q^{s'} I(\theta_0^*(v), s)$ is a nonzero scalar times of $I(v,s)$ if n is odd. Sine 1 and $q^{s'}$ are linearly independent so $\theta_0^*(v)$ and $\theta_0(v)$ must be linearly independent. □

From the proof above we also obtain the following.

Corollary 9.1.4. *Assume v is a new vector, then $I(\theta_0(v), s)$ is a scalar times of $I(v, s)$ and $I(\theta_0^*(v), s)$ is a scalar times of $q^{-s'} I(v, s)$.*

Lemma 9.1.5. *Assume v is a fixed vector, namely $v \in \pi^{\mathrm{K}(\mathfrak{p}^m)}$ for some m, and $\Omega(v) \in \oplus_{d \geq 0} \mathscr{S}_{n,d}$ then $\mathrm{vol}(\mathfrak{o}^\times)^{n-1} I(v, s) = \Omega(v; q^{-s'}, 0, 0, ..., 0)$.*

Proposition 9.1.6. *If $m \equiv a_\pi (\bmod 2)$, $\dim \pi^{\mathrm{K}(\mathfrak{p}^m)} \geq \begin{pmatrix} n + \frac{m - a_\pi}{2} \\ n \end{pmatrix} + \begin{pmatrix} n + \frac{m - a_\pi}{2} - 1 \\ n \end{pmatrix}$.*

Proof. Note that if $m < a_\pi$, then this lower bound is 0. Assume $m \geq a_\pi$. By Proposition 8.1.5 and $c(\pi) = a_\pi$, this is a matter of counting number of $\lambda \in P_{\mathrm{H}}^+$ such that $\|\lambda\| \leq k$ for $k = \frac{m - a_\pi}{2}$. Since $\lambda = a_1 \epsilon_1 + a_2 \epsilon_2 + ... + a_n \epsilon_n$ is in P_{H}^+ if and only if $a_1 \geq a_2 \geq ... \geq |a_n|$. Then assume $a_n \geq 0$, this is two times the number of the tuple $(a_1 - a_2, a_2 - a_3, ..., a_{n-1} - a_n, a_n)$ with nonnegative integer entries with sum $\leq k$. There are $\begin{pmatrix} n + k \\ n \end{pmatrix}$ of them. Assume $a_n < 0$, then this is the number of the tuple $(a_1 - a_2, a_2 - a_3, ..., a_{n-1} - |a_n|, |a_n| - 1)$ with nonnegative integer entries with sum $\leq k - 1$. There are $\begin{pmatrix} n + k - 1 \\ n \end{pmatrix}$ of them. ☐

Proposition 9.1.7. *If $m \equiv a_\pi + 1 (\bmod 2)$, $\dim \pi^{\mathrm{K}(\mathfrak{p}^m)} \geq 2 \begin{pmatrix} n + \frac{m - 1 - a_\pi}{2} \\ n \end{pmatrix}$.*

Proof. Note that if $m < a_\pi + 1$, then this lower bound is 0. Assume $m \geq a_\pi + 1$ Let $v_1 = \theta_0(v_0) \in \mathrm{K}(\mathfrak{p}^{a_\pi + 1})$ be a nonzero fixed vector of level $a_\pi + 1$ for v_0 a new vector. By Proposition 9.1.3, the vector $v_1' = u_{a_\pi + 1} v_1 = \theta_0^*(u_{a_\pi} v_0) = \varepsilon_\pi \theta_0^*(v)$ is linearly independent to v_1. Set $\mathrm{H}'_{x_{a_\pi + 1}} = \langle \mathrm{H}_{x_{a_\pi} + 1}, u_{a_\pi + 1} \rangle$ whose reductive quotient is isomorphic to $\mathrm{O}_{2n}(\mathfrak{f})$. We get two independent vectors $v_1 + v_1'$ and $v_1 - v_1'$ which are in the $+1$ and -1 space of $\mathrm{J}(\mathfrak{p}^{a_\pi + 1})$ respectively. Then since $\mathrm{H}'_{x_{a_\pi + 1}}$ contains $w_{s, a_\pi + 1}$ for $s \in W_{\mathrm{G}}$ so $\mathrm{H}'_{x_{a_\pi + 1}} \mathrm{T} \mathrm{H}'_{x_{a_\pi + 1}} = \sqcup_{\lambda \in P^+} \mathrm{H}'_{x_{a_\pi + 1}} \varpi^\lambda \mathrm{H}'_{x_{a_\pi + 1}}$ and the characteristic functions $[\mathrm{H}'_{x_{a_\pi + 1}} \varpi^\lambda \mathrm{H}'_{x_{a_\pi + 1}}]$, $\lambda \in P^+$, are independent. Notice that $\Omega(v_1 + v_1')$ and $\Omega(v_1 - v_1')$ are also independent and moreover not in $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$, since they are contained in

$\oplus_{d\geq 0}\mathscr{S}_{n,d}$ by Proposition 5.4.3 but not in $\mathbb{C}$ by Lemma 9.1.5 and Corollary 9.1.4. Hence $\Omega(\eta_\lambda(v_1+v_1'))$, $\lambda \in P^+$, and $\Omega(\eta_{\lambda(v_1-v_1')})$, $\lambda \in P^+$, are linearly independent. Therefore we obtain that the dimension of $\dim \pi^{\mathrm{K}(\mathfrak{p}^m)}$ is two times the number of $\lambda \in P^+$ such that $\|\lambda\| \leq \frac{m-(a_\pi+1)}{2}$. Then since $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + ... + a_n\epsilon_n$ is in P^+ if and only if $a_1 \geq a_2 \geq ... \geq a_n$. Same computation as in the previous lemma gives the assertion. □

Combining the two Propositions above, we can write down the lower bound of the dimension of the two cases in one formula.

Main Theorem 3. $\dim \pi^{\mathrm{K}(\mathfrak{p}^m)} \geq \begin{pmatrix} n + \lfloor \frac{m-a_\pi}{2} \rfloor \\ n \end{pmatrix} + \begin{pmatrix} n + \lfloor \frac{m-a_\pi+1}{2} \rfloor - 1 \\ n \end{pmatrix}.$

Definition 9.1.8. A nonzero fixed vector is an *old vector* if it is obtained by level raising operators θ_λ and η_λ from the new vectors.

We conjecture that all fixed vectors are obtained in this way, that is they are all old vectors. This conjecture is partially implied by $\Omega(\pi^{\mathrm{H}_{a_\pi}}) = \mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$, which we have known $\supset$, or knowing the $\mathbb{C}[\hat{\mathrm{T}}]^{W_{\mathrm{H}}}$-module $\pi^{\mathrm{H}_{a_\pi}}$ is of rank one.

Conjecture 9.1.9. *All nonzero fixed vectors of level greater than a_π are old vectors.*

As a corollary to the old form conjecture:

Conjecture 9.1.10. *The lower bound of the dimension given in Main Theorem 3 is the exact dimension.*

When $n = 2$ this is a theorem by Roberts and Schmidt [**23**].

Remark 9.1.11. It is expected that the theories of newforms and oldforms hold for general generic representations of G including non-supercuspidal representations. This is a work in progress.

Bibliography

[1] I. N. Bernstein and A. V. Zelevinsky. Representation of the group GL(n,F) where F is a non-Archimedean local field. *Russian Math. Surveys*, 31(3):1–68, 1976.

[2] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive p-adic groups. I. *Ann. Sci. Ecole Norm. Sup.* 4^e *serie*, 10(4):441–472, 1977.

[3] P. Cartier. Representations of p-adic groups. In *Automorphic forms, representations, and L-functions*, volume I, Providence, Rhode Island, 1979. American Mathematical Society, Proceedings of Symposia in Pure Mathematics.

[4] W. Casselman. On some results of Atkin and Lehner. *Math. Ann.*, 201:301–314, 1973.

[5] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. I. The spherical function. *Comp. Math.*, 40(3):387–406, 1980.

[6] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Comp. Math.*, 41(2):207–231, 1980.

[7] S. DeBacker and M. Reeder. Depth-zero supercuspidal L-packets and their stability. *Ann. of Math.*, 169:795–901, 2009.

[8] W. T. Gan and J. Yu. Group schemes and local densities. *Duke Math. J.*, 105(3):497–524, 2010.

[9] S. Gelbart and I. I. Piatetski-Shapiro. L-functions for $G \times GL(n)$. *Lecture Notes in Mathematics*, 1987:53–146, 2001.

[10] I. Gelf́and and D. A. Kajdan. Representations of the group $GL(n, K)$ where K is a local field. *Funkcional. Anal. i Prilozen*, 6:13–44, 1972.

[11] D. Ginzburg. L-functions for $SO_n \times GL_k$. *J. Reine angew. math*, 405, 1990.

[12] B. Gross. On the Satake isomorphism. *LMS Lecture Notes*, 254:223–238, 1998.

[13] T. J. Haines and S. Rostami. The Satake isomorphism for special maximal parahoric Hecke algebras. *Represent. Theory*, 14, 2010.

[14] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika. Conducteur des represéntations du groupe linéaire. *Math. Ann.*, 256:199–214, 1981.

[15] H. Jacquet and J. Shalika. The Whittaker models of induced representations. *Pacific J. Math.*, 1:107–120, 1983.

[16] J. Lansky and A. Raghuram. Conductors and newforms for U(1,1). *Proc. Indian Acad. Sci. (Math Sci.)*, 114(4):319–343, 2004.

[17] J. M. Lansky. Decomposition of double cosets in $\mathfrak{p}$-adic groups. *Pacific Journal of Mathematics*, 197(1):97–117, 2001.

[18] M. Miyauchi. Conductors and newforms for non-supercuspidal representations of unramified U(2, 1). *preprint, arXiv:1112.4899*, 2011.

[19] M. Miyauchi. On epsilon factors of supercuspidal representations of unramified U(2,1). *preprint, arXiv:1111.2212*, 2011.

[20] M. Miyauchi. On local newforms for unramified U(2, 1). *preprint, arXiv:1105.6004*, 2011.

[21] A. Moy and G. Prasad. Jacquet functors and unrefined minimal K-types. *Comment. Math. Helv.*, 71:98–121, 1996.

[22] M. Reeder. Old forms on GL_n. *Amer. J. Math.*, 113(5):911–930, 1991.

[23] B. Roberts and R. Schmidt. *Local Newforms for $GSp(4)$*, volume 1981 of *Lecture Notes in Mathematics*. Springer, Berlin Heidelberg, 2007.

[24] F. Rodier. Whittaker models for admissible representations of reductive $\mathfrak{p}$-adic split groups. In *Harmonic Analysis on Homogeneous Spaces*, Providence, Rhode Island, 1973. American Mathematical Society, Proceedings of Symposia in Pure Mathematics.

[25] F. Shahidi. On certain L-functions. *Amer. J. Math.*, 103:297–355, 1981.

[26] F. Shahidi. A proof of Langlands conjecture on Plancherel measures; Complementary series for p-adic groups. *Ann. of Math.*, 132:273–330, 1990.

[27] J. Shalika. The multiplicity one theorem for $GL(n)$. *Ann. of Math.*, 100:171–193, 1974.

[28] D. Soudry. Rankin-Selberg convolutions for $SO_{2l+1} \times GL_n$: Local theory. *Mem. Amer. Math. Soc.*, 105(500), September 1993.

[29] D. Soudry. Full multiplicativity of gamma factors for $SO_{2l+1} \times GL_n$. *Israel J. Math.*, 120(1):511–561, 2000.