



# Introduction to Quantum Algorithms for Physics and Chemistry

## Citation

Yung, M.-H., Whitfield, J. D., Boixo, S., Tempel, D. G. and Aspuru-Guzik, A. (2014) Introduction to Quantum Algorithms for Physics and Chemistry, in Quantum Information and Computation for Chemistry: Advances in Chemical Physics Volume 154 (ed S. Kais), John Wiley & Sons, Inc., Hoboken, New Jersey. doi: 10.1002/9781118742631.ch03

## Published Version

10.1002/9781118742631.ch03

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# Introduction to Quantum Algorithms for Physics and Chemistry

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March 8, 2012

## Abstract

An enormous number of model chemistries are used in computational chemistry to solve or approximately solve the Schrödinger equation; each with their own drawbacks. One key limitation is that the hardware used in computational chemistry is based on classical physics, and is often not well suited for simulating models in quantum physics. In this review, we focus on applications of quantum computation to chemical physics problems. We describe the algorithms that have been proposed for the electronic-structure problem, the simulation of chemical dynamics, thermal state preparation, density functional theory and adiabatic quantum simulation.

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## 1 Introduction

Controllable quantum systems provide unique opportunities for solving problems in quantum chemistry and many-body physics that are intractable by classical computers. This approach is called “quantum simulation”<sup>1</sup>, and was pioneered by Feynman (1982). There are two different approaches for quantum simulation: analog or digital. In analog quantum simulation, dedicated physical systems are engineered to emulate the behavior of other quantum systems. A classic example is the use of atoms trapped in optical lattices to simulate the (Bose-)Hubbard model. Analog simulators are therefore special-purposed machines. On the other hand, digital simulation uses a universal quantum computer. Interestingly, a universal quantum computer is also capable, in principle, of factoring arbitrary long numbers (Shor 1997), whereas a classical computer is not known to be able to perform the same task. For a recent review, see e.g. Ladd, Jelezko, Laflamme, Nakamura, Monroe & O’Brien (2010) and Hauke, Cucchietti, Tagliacozzo, Deutsch & Lewenstein (2011).

One key advantage of simulations with quantum computers over classical computers is the huge Hilbert space available to faithfully represent quantum systems. Moreover, quantum simulation avoids many problems encountered in classical simulation. For example, many classical algorithms relying on Monte Carlo methods exhibit the so-called fermion sign problem that severely damages the performance of the algorithm. In quantum simulation, this problem can be avoided by either encoding the fully anti-symmetrized wavefunction in the qubit

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<sup>1</sup>Unfortunately, the term “quantum simulation” in the community of computational physics refers to numerical simulation of quantum systems using classical computers.

states, or by performing Jordan-Wigner transformations in the second quantized Hamiltonian and turn it into a spin Hamiltonian as first suggested by Ortiz, Gubernatis, Knill & Laflamme (2001). The latter case will often result in non-local interaction terms, but it can still be simulated efficiently in a quantum computer.

The purpose of this article is to introduce the basic concepts of digital quantum simulation and several recent developments achieved in our group. We note that this is by no means a comprehensive review of the whole literature in quantum simulation. We will selectively cover materials that we find most useful to convey an overall idea about the current status of quantum digital simulation. Several review articles (Zalka 1998b, Buluta & Nori 2009, Brown, Munro & Kendon 2010, Kassal, Whitfield, Perdomo-Ortiz, Yung & Aspuru-Guzik 2011) and book chapters (Kitaev, Shen & Vyalı 2002, Stolze & Suter 2008, Nielsen & Chuang 2011, Williams 2010) already present a different emphasis. This review also contains some new material. Sections 2.2.2 and 2.2.3 present new descriptions of the simulation in the first and second quantized representations, respectively. Section 2.3.3 lays out a new point of view for the perturbative update of thermal states from smaller to bigger quantum systems, and a new bound for the change of a thermal state due to a perturbation.

## 1.1 Quantum computational complexity and chemistry

### 1.1.1 An exponential wall for many-body problems

The theory of computational complexity studies the scaling of the resources necessary to solve a given problem as a function of input size. Problems are considered to be “easy,” or efficiently solvable, if the time (or number of steps) for solving the problem scales as a polynomial of the input size  $n$ . For example, sorting a list of  $n$  items will take at most  $O(n^2)$  steps. On the other hand, problems are considered “hard” if the scaling is exponential in  $n$ . This exponential scaling is essentially true in the worst case for almost all many-body problems in physics and chemistry (Pople 1999). A concise discussion of this point is given by Kohn (1999), where the exponential scaling of the Hilbert space of many-electron problems is referred to as the “Van Vleck catastrophe.” The argument presented is as follows: if for each molecule, the accuracy to which one can approximate the state is  $(1 - \epsilon)$  (under a suitable metric), then for  $n$  non-overlapping (and non-identical) molecules, the approximation worsens exponentially as  $(1 - \epsilon)^n$ . In the next subsection, we discuss the connection of many-body problems with computational complexity further.

### 1.1.2 Computational complexity of quantum simulation

The study of the computational complexity of problems in quantum simulation helps us better understand how quantum computers can surpass classical computers. It has also spurred new developments in computational complexity. For simplicity, computational complexity is often formulated using decision problems. A decision problem resolves if some condition is true or false e.g. is the

ground-state energy of the system below a certain critical value? Although the answer to decision problems is either “yes” or “no,” one can keep asking questions in a binary search fashion. For instance, one could attempt to determine in this way the ground-state energy to an arbitrarily high accuracy.

A complexity class contains a set of computational problems that share some common properties about the computational resources required for solving them. We briefly summarize below a few important examples of complexity classes of decision problems.

### P and NP problems

The complexity class P contains all decision problems that are solvable in a polynomial time with a classical computer (more precisely, a deterministic Turing machine). Roughly speaking, solving a problem in a polynomial time refers to the cases where the number of steps for solving the problem scale as a polynomial power instead of exponentially. This is considered “efficient” but, of course, there could be exceptions. For example, problems that scale as  $O(n^{10000})$  may take very long time to finish, compared with ones that scale exponentially as  $O(1.0001^n)$ .

Nevertheless, from a theoretical perspective, this division allows for considerable progress to be made without considering the minutiae of the specific system or implementation. However from a practical standpoint, the order of the polynomial may be very important; especially in chemistry where an algorithm is applied to many molecules and many geometries. That said, the notion of polynomial versus exponential makes sense when considering Moore’s “law:” the density of transistors in classical computers doubles every two years<sup>2</sup>. If the algorithm runs in exponential-time, one may be forced to wait several lifetimes in order for an instance to become soluble due to better classical hardware.

Practically, common hard problems typically fall into the complexity class NP, which contains decision problems whose “yes” instances can be efficiently verified to be true with a classical computer given an appropriate “solution” or witness. There is no doubt that P is a subclass of NP, i.e.,

$$P \subset NP \quad . \quad (1)$$

As an example, finding the prime factors of an integer belongs to an NP problem; once the factors are given, then it is easy to check the answer by performing a multiplication. Interestingly, finding the ground state energy of the Ising model

$$\sum_{(i,j) \in E} \sigma_z^i \sigma_z^j + \sum_{i \in V} \sigma_z^i, \quad (2)$$

where  $(V, E)$  is a planar graph, is an NP-complete (Barahona 1982). This implies that if a polynomial algorithm for finding the ground state energy is found, then

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<sup>2</sup>The exponential growth in the computational density is expected to cease sometime this century highlighting the importance of new methods of computation such a quantum computation. The growth in CPU clock speed has already ceased.

all of the problems in NP could be solved in polynomial time. In other words, it will imply  $P = NP$ , a result considered highly unlikely. A rigorous proof or disproof of this statement would constitute a significant breakthrough<sup>3</sup>.

It is believed, but not known with certainty, that quantum computers are not capable of solving all NP problems efficiently. Nevertheless, as mentioned above, they can solve the integer-factoring problem efficiently. It is believed that the complexity of integer-factoring is intermediate between P and NP (Shor 1997).

### BQP and QMA problems

The quantum analog of P and NP problems are, respectively, the BQP (bounded-error quantum polynomial time) and QMA (quantum Merlin Arthur) problems<sup>4</sup>. BQP is the class of (decision) problems that are solvable by a quantum computer in polynomial time. QMA is the class of (decision) problems that can be verified by a quantum computer in polynomial time. Like NP-problems, the QMA class covers many problems that are important to physics and chemistry (Liu, Christandl & Verstraete 2007b, Schuch & Verstraete 2009a, Wei, Mosca & Nayak 2010). For example, the ground-state problem of Hamiltonians involving local interaction terms is known to be QMA-complete (Kitaev et al. 2002, Kempe, Kitaev & Regev 2006a). For more discussion on topics of computational complexity and quantum simulation, readers may find the following references useful: Aharonov & Naveh (2002), Rassolov & Garashchuk (2008), Aaronson (2009), Osborne (2011) and Aaronson (2011).

The key point here is that so far it is not known whether quantum computers can solve NP and QMA problems efficiently. In fact, many attempts (see e.g. Young, Knysh & Smelyanskiy (2008), Poulin & Wocjan (2009a), Poulin & Wocjan (2009b), Young & Smelyanskiy (2010), Bilgin & Boixo (2010), Yung, Naga, Whitfield & Aspuru-Guzik (2010), and Hen & Young (2011)) show that exponential resources are required to solve problems in these classes. Nevertheless, many problems in physics and chemistry do exhibit symmetries and structures that we could exploit to construct efficient quantum simulation algorithms. This is the main theme of the discussion in the rest of the paper.

## 1.2 Basic quantum algorithms for digital quantum simulation

Digital quantum simulation cannot be easily understood without a detour into the basics of quantum algorithms. Quantum algorithms are procedures for applying elementary quantum logic gates to complete certain unitary transformations of the input state. The quantum computer state is usually written in terms of qubits (two-level systems). In the two-dimensional Hilbert space of a

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<sup>3</sup>P vs NP is one of the Millennium Problems of the Clay Mathematics Institute [[http://www.claymath.org/millennium/P\\_vs\\_NP/](http://www.claymath.org/millennium/P_vs_NP/)]

<sup>4</sup>More precisely, BQP is analogous to the classical complexity class BPP, which refers to problems that can be solved with randomized algorithms in a classical computer in polynomial time, subject to a bounded error probability.

single qubit, we label the upper and lower eigenstates of  $\sigma^z$  as  $|0\rangle$  and  $|1\rangle$ . Note that the choice of  $\sigma^z$  as the computational basis is arbitrary. This is called the computational basis, and the matrix representation of operators and states are written in this basis unless otherwise stated. The unitary transformations of the qubits may be visualized using quantum circuit diagrams introduced later to explain some of the more complex quantum algorithms.

It is known that any unitary gate can be decomposed into some sets of universal quantum logic gates that contains single- and two-qubit operations (Nielsen & Chuang 2011). The first gate of interest is the single-qubit Hadamard transformation defined (in the computational basis) as

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard gate transforms between the  $\sigma^z$  basis and the  $\sigma^x$  basis ( $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ ) and will be used throughout the article. A second gate of interest is the CNOT (controlled not) gate which is a non-trivial two-qubit gate. It leaves one input qubit unchanged and acts with  $\sigma^x = |0\rangle\langle 1| + |1\rangle\langle 0|$  on the second qubit when the first qubit is in the state  $|1\rangle$ . The first qubit is called the control and the NOT operation is applied coherently when the control qubit is in a superposition of computational basis states. Symbolically, the gate is written as  $\text{CNOT} = |1\rangle\langle 1| \otimes \sigma^x + |0\rangle\langle 0| \otimes I$ . The Hadamard and CNOT gates are not universal for quantum computation, and in fact quantum algorithms with only these gates can be simulated efficiently classically as shown by the Knill-Gottesman theorem (Nielsen & Chuang 2011). Therefore, this gate set must be augmented by other single qubit gates which can always be expressed by single-qubit rotations,  $R_x$ ,  $R_y$ , and  $R_z$  where  $R_x$  is defined as  $\exp[-i\sigma^x\theta/2]$  for real angle  $\theta$ .

There are two elementary algorithms, namely quantum Fourier transform (QFT) and phase estimation algorithm (PEA), that play important roles in many applications in quantum simulation. We turn our attention to them now.

### 1.2.1 Quantum Fourier transform (QFT)

Given a vector with  $N$  elements  $(x_0, x_1, \dots, x_{N-1})$ , in classical computation, the discrete Fourier transform outputs another vector of  $N$  numbers  $(y_0, y_1, \dots, y_{N-1})$  through the following relation:

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i j k / N} \quad . \quad (3)$$

In quantum computation, for any given quantum state,  $|\phi\rangle = \sum_{x=0}^{N-1} \phi(x) |x\rangle$ , the goal of the quantum Fourier transform  $U_{\text{QFT}}$  is to perform the following unitary transformation:

$$U_{\text{QFT}} |\phi\rangle = \sum_{k=0}^{N-1} \tilde{\phi}(k) |k\rangle \quad , \quad (4)$$

where  $\tilde{\phi}(k) = (1/\sqrt{N}) \sum_{x=0}^{N-1} \phi(x) e^{2\pi i x k/N}$  is the Fourier-transform of the function  $\phi(x)$  (compare with Eq. (3)). Due to the linearity of  $U_{\text{QFT}}$ , it is sufficient to consider the transformation of the basis vectors such that

$$U_{\text{QFT}} |x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x k/N} |k\rangle \quad . \quad (5)$$

For a system of  $n$  qubits, the number of gates required for such a transformation is  $O(n^2)$  (Nielsen & Chuang 2011). For the classical case (see Eq. (3)), one will require  $O(n2^n)$  gates to complete the same transformation, e.g. with Fast Fourier transform (FFT). This may seem to suggest that quantum computers are exponentially more efficient in performing the task of discrete Fourier transformation. However, the caveat is that one cannot directly compare QFT with the classical FFT. The reason is that if we want to obtain a particular Fourier-transform coefficient, say  $\tilde{\phi}(k)$ , from the quantum state in Eq. (4), it would still require exponentially many steps to extract the information (phase and amplitude), e.g. through quantum state tomography where many measurements are used to analyze the state (Nielsen & Chuang 2011).

Nevertheless, QFT is essential in many applications in digital quantum simulation. As we shall see in section 2.2.2, it allows us to simulate the time dynamics of particles efficiently by moving between the position and momentum representations. Another important application of the QFT is phase estimation, discussed next.

### 1.2.2 Phase estimation algorithm (PEA)

The phase estimation algorithm  $U_{\text{PEA}}$  is an essential component for many quantum algorithms for quantum simulation, as well as the celebrated factoring algorithm (Shor 1997). Loosely speaking, the PEA can be considered as a realization of the von Neumann measurement scheme (without the last projective measurement) in the eigenvalue basis  $|a_k\rangle$  of any Hermitian observable  $A$  (e.g. Hamiltonian  $H$ ). More precisely, if we prepare a register of  $m$  ancilla qubits initialized in the state  $|000\dots 0\rangle$ , then for any given state  $|\phi\rangle = \sum_k c_k |a_k\rangle$ , we have

$$U_{\text{PEA}} |\phi\rangle |000\dots 0\rangle \approx \sum_k c_k |a_k\rangle |A_k\rangle \quad , \quad (6)$$

where, for the moment, we assume that the  $A_k$ 's are the  $m$ -integer-digit representation (i.e.,  $A_k \in \{0, 1, 2, \dots, 2^m - 1\}$ ) of the eigenvalue of  $A$ . The projective measurement cannot be implemented perfectly in general (hence the  $\approx$  symbol). We will see where the errors come from as we go through the details of the algorithm below.

Suppose that we are given an eigenstate  $|a_k\rangle$  of the Hermitian observable  $A$ . The goal of PEA is to determine  $A_k$ , given that we are able to simulate a unitary operator  $W$  where

$$W |a_k\rangle = e^{2\pi i \phi_k} |a_k\rangle \quad , \quad (7)$$



and  $\phi_k \equiv A_k/2^m$ . The first step of the PEA is to apply Hadamard gates to each of the ancilla qubits. This results in an equal superposition of states

$$|S\rangle \equiv \frac{1}{\sqrt{2^m}} \sum_{x=0}^{2^m-1} |x\rangle \quad (8)$$

where  $x$  is a  $m$ -digit binary number. Then, taking each ancilla qubit  $j$  as a control qubit, we apply the controlled- $W^{2^j-1}$  gate to the state  $|S\rangle|a_k\rangle$ ; this effectively performs the following operation:

$$|x\rangle|a_k\rangle \rightarrow |x\rangle W^x |a_k\rangle \quad . \quad (9)$$

Of course, from Eq. (7), the right-hand side gives only a phase factor, namely  $\exp(2\pi i x \phi_k)$ . The resulting state is

$$\left( \frac{1}{\sqrt{2^m}} \sum_{x=0}^{2^m-1} e^{2\pi i x \phi_k} |x\rangle \right) |a_k\rangle \quad . \quad (10)$$

Comparing this state with that in Eq. (4), and assuming the special cases where the phase angle  $\phi_x$  can be expressed exactly by  $m$  binary digits, the application of the inverse of the quantum Fourier transform  $U_{\text{QFT}}$  will convert the state in Eq. (10) into the following state

$$|A_k\rangle|a_k\rangle \quad . \quad (11)$$

Since the unitary operator  $U_{\text{PEA}}$  is linear, the procedure applies to any initial state. For this particular case, where  $A_k$ 's are integers, we have shown that PEA is effectively a projective measurement as advertised in Eq. (6).

For the general case, where the  $A_k$ 's are real numbers, the corresponding  $\phi_k$ 's will have precision beyond  $1/2^m$ ; this is the source of the errors in the expression of Eq. (6). The overall error decreases when we increase the number of ancilla qubits and perform several QFTs in parallel (we refer to Kaye, Laflamme & Mosca (2007) for a detailed error analysis). More precisely, if we want to achieve a  $p$ -bit precision of  $\phi_k$  with an error less than  $\epsilon$ , one will need more than  $m = p + \log(2 + 1/2\epsilon)$  ancilla qubits. In general, implementing the operator  $W^k$  requires  $k$  times as many resources as those needed for simulating  $W$ . Therefore, the scaling of the quantum gates of PEA grows exponentially when we increase the precision  $p$  of the phase measurement. This result is consistent with that of the general sampling theory in classical signal processing, where the precision of the Fourier spectrum  $\delta\omega$  goes as the inverse of the total time  $T$  sampled, i.e.,  $\delta\omega \sim O(1/T)$ . This is because the cost of the quantum simulation is proportional to  $T$ , and  $T$  grows exponentially with the number of bits of precision.

## 2 Digital quantum simulation

### 2.1 Overview

Broadly speaking, the steps involved in carrying out a digital quantum simulation consist of three parts: state preparation, time evolution, and measure-

ment of observables. Measurement of Hermitian observables can be achieved via the phase estimation method (Abrams & Lloyd 1999, Jaksch & Papageorgiou 2003, Knill, Ortiz & Somma 2007) described before. Other applications (Lidar & Wang 1999, Wu, Byrd & Lidar 2002, Somma, Ortiz, Gubernatis, Knill & Laflamme 2002, Somma, Ortiz, Knill & Gubernatis 2003, Byrnes & Yamamoto 2006, Kassal & Aspuru-Guzik 2009) or quantities of physical interest such as the partition function (Master, Yamaguchi & Yamamoto 2003, Wocjan, Chiang, Nagaj & Abeyesinghe 2009), can be obtained through variants of the methods employed in state preparation and time evolution, and we will skip them in this review. Below we give an overview of state preparation and simulation of time evolution. It turns out that many methods of state preparation also depend on the time evolution itself. Therefore, we will first cover the methods of time evolution before state preparation.

## 2.2 Simulation of time evolution

The simulation of the time evolution of quantum state  $|\psi\rangle$  under Hamiltonian  $H$  according to the Schrödinger's equation ( $\hbar = 1$ ),

$$i\frac{\partial}{\partial t}|\psi\rangle = H(t)|\psi\rangle \quad , \quad (12)$$

is one of the key applications of quantum computation. If, for example, the time-evolution operator

$$U(t) = \exp(-iHt) \quad (13)$$

can be simulated efficiently, then the eigenvalues of  $H$  can be obtained through the phase estimation algorithm<sup>5</sup>. As mentioned in the introduction, Feynman (1982) investigated the possibility of simulating quantum systems using another quantum system, and conjectured that there existed a class of universal quantum simulators that evolved under a Hamiltonian with local interactions. This conjecture was justified by Lloyd (1996), who argued that any Hamiltonian

$$H = \sum_{i=1}^m H_i \quad (14)$$

which can be decomposed into  $m$  local terms  $\{H_i\}$  can be simulated efficiently by a universal quantum computer. Each  $H_i$  term acts on at most  $k$  qubits (or quantum subsystems). The key idea is based on the Trotter splitting or “trotterization” of all non-commuting operators,

$$e^{-iHt} \approx \left( e^{-iH_1 t/n} e^{-iH_2 t/n} \dots e^{-iH_m t/n} \right)^n \quad , \quad (15)$$

where the approximation can be made arbitrarily tight by refining the time-slicing, i.e., increasing  $n$ .

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<sup>5</sup>Moreover, it can also be exploited for quantum cooling (see section 2.4).

There exist higher order approximations (Suzuki-Trotter formulas) which reduce the error even further. For instance, the second-order approximation is given by

$$e^{-iHt} \approx \left( \left( e^{-ih_1 \frac{\Delta t}{2}} \dots e^{-ih_{N-1} \frac{\Delta t}{2}} \right) e^{-ih_N \Delta t} \left( e^{-ih_{N-1} \frac{\Delta t}{2}} \dots e^{-ih_1 \frac{\Delta t}{2}} \right) \right)^{\frac{t}{\Delta t}} + O(t(\Delta t)^2) \quad (16)$$

A quantum circuit on  $n$  qubits which approximates  $U(\tau)$ , with error at most  $\epsilon$ , is efficient if the number of one- and two-qubit gates involved is polynomial in the scaling of the problem, i.e.,  $\text{poly}(n, \tau, 1/\epsilon)$  with  $\tau = t/\|H\|$ .

### 2.2.1 Suzuki-Trotter formulas

We now briefly review the use of Suzuki-Trotter formulas in quantum simulation for time-independent sparse Hamiltonians, providing an introduction to the quantum simulation literature. Continuing the work of Lloyd (1996), works by Aharonov & Ta-Shma (2003) and Childs (2004) show that black-box sparse Hamiltonians are too efficiently simulatable. Sparsity here means that the number of elements per row is bounded by some polynomial of  $n$ , while the dimension of the Hilbert space is  $D = 2^n$ . It is also required that each matrix element can be retrieved efficiently. Ref. (Aharonov & Ta-Shma 2003) used a coloring scheme to decompose the Hamiltonian into a sum of  $2 \times 2$  block diagonal matrices. This coloring scheme has been updated in several references (Berry, Ahokas, Cleve & Sanders 2006, Berry & Childs 2009, Childs & Kothari 2011). The coloring scheme and blackbox simulation will not be discussed further.

Berry et al. (2006) were the first to approach the general problem of simulating non-commuting Hamiltonians by using higher order Suzuki-Trotter formulas. Papageorgiou & Zhang (2010) returned to this issue and their contributions will be discussed later. The important results of Berry et al. (2006) are

1. the use of higher order Trotter-Suzuki decompositions to bound the number of non-commuting exponentials,  $N_{exp}$ , necessary to carry out a simulation for some amount of time  $t$ ,
2. a proof of a no-go theorem for sub-linear black-box simulation and
3. improvements upon the coloring scheme of (Aharonov & Ta-Shma 2003) and (Childs 2004) for black box simulation of sparse Hamiltonians.

The simulations in this review are concerned with the first two results and they will be explained in more detail after describing the Suzuki-Trotter formulas.

M. Suzuki has studied and extended the Trotter formula essentially continuously since 1990 and this work was reviewed in Hatano & Suzuki (2005). The recursive formulas introduced by Suzuki define a fractal pattern where a combination of forward and backward propagation leads to an improved approximation of the desired exponential. Suzuki defines higher order Trotter formulas

in a recursive way. Beginning with the split operator formula,  $e^{Ax/2}e^{Bx}e^{Ax/2}$ , for  $m$  operators, the following series of equations were derived:

$$S_2(x) = \left( \prod_{k=1}^m e^{h_k x} \right) \left( \prod_{k=m}^1 e^{h_k x} \right) \quad (17)$$

$$S_4(x) = S_2(z_2 x)^2 S_2((1 - 4z_2)x) S_2(z_2 x)^2 \quad (18)$$

$$\vdots \quad \vdots$$

$$S_{2k}(x) = S_k(z_k x)^2 S_k((1 - 4z_k)x) S_k(z_k x)^2 \quad (19)$$

The values of the constants  $\{z_j\}$  are selected so that  $S_{2j}$  is correct through  $2j^{th}$  order and it can be shown (Hatano & Suzuki 2005) that  $z_i = (4 - 4^{1/(2i-1)})^{-1}$ . If there are  $m$  non-commuting Hamiltonians, then the first order approximation takes  $m = N_{exp}$  and for the split operator formula,  $S_2$ , the number of exponentials is  $2m - 1$ . In general,  $2(m - 1)5^{k-1} + 1$  exponentials are used for the  $S_{2k}$  approximant.

For the  $k^{th}$  order Suzuki-Trotter, with  $m$  Hamiltonians in the sum, and error tolerance given by  $\varepsilon$ , Berry et al. (2006) gives a bound on the number of exponentials by bounding each order of the Suzuki-Trotter formula. Papageorgiou & Zhang (2010) presented an improvement by noting that the relative ratio of Hamiltonian norms is also important. The main idea is that if some of the Hamiltonians have very small weight, then their exponentials can be effectively ignored.

The optimal order of Trotter decomposition,  $k^*$ , is determined by selecting the best compromise between time-step length and a decomposition using more exponentials. In Berry et al. (2006) this was worked out for unstructured sparse Hamiltonians  $N_{exp} \geq \|H\|t$ . The lower bound on the generic cost of simulating an evolution was by contradiction, and relied on the lower bounds to quantum mechanical problems based on the polynomial method (Beals, Buhrman, Cleve, Mosca & de Wolf 1998). This bound could be violated given sub-linear simulation time. In a departure from the methods discussed so far, Childs (2010) used a quantum walk based approach to push the scaling closer to linear in the reweighed time and Raesi, Wiebe & Sanders (2011) looked at designing quantum circuits for quantum simulation.

For problems in chemistry, it is more natural to represent Hamiltonians in terms of first- and second-quantized forms. In the following, we will describe how to exploit the special structure of molecular Hamiltonians to simulate the time dynamics in quantum computers.

### 2.2.2 First-quantized representation

In the first-quantized form, the non-relativistic molecular Hamiltonians  $H$  decomposes in a kinetic  $T$  and potential  $V$  terms, i.e.,

$$H = T + V \quad . \quad (20)$$

The kinetic term includes the contribution from the nuclei and electrons separately,

$$T = - \sum_i \frac{\hbar^2}{2M_i} \nabla_i^2 - \sum_j \frac{\hbar^2}{2m_e} \nabla_j^2 \quad , \quad (21)$$

where  $M_i$  is the mass of the nucleus  $i$ , and  $m_e$  is the electron mass. The potential energy term comprises of the Coulomb interaction among the nuclei, among the electrons, and between the nuclei and electrons. Explicitly:

$$V(\mathbf{r}, \mathbf{R}) = \frac{e^2}{4\pi\epsilon_0} \sum_{i<j} \frac{Z_i Z_j}{|\mathbf{R}_i - \mathbf{R}_j|} + \frac{e^2}{4\pi\epsilon_0} \sum_{i<j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} - \frac{e^2}{4\pi\epsilon_0} \sum_{i,j} \frac{Z_i}{|\mathbf{R}_i - \mathbf{r}_j|} \quad , \quad (22)$$

where  $e$  is the electric charge, and  $Z_i$  is the charge of nuclei  $i$ . The coordinates of nuclei  $i$  and electron  $j$  are denoted by  $\mathbf{R}_i$  and  $\mathbf{r}_j$ . We will use the notation  $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots)$  (and similarly for  $\mathbf{R}$ ). We also ignore the spin degrees of freedom, which can be incorporated easily.

The general wavefunction can be represented in the position basis as

$$|\Psi\rangle = \sum_{\mathbf{r}, \mathbf{R}} \Psi(\mathbf{r}, \mathbf{R}) |\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \dots\rangle |\mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3 \dots\rangle \quad , \quad (23)$$

where each electronic or nuclear coordinate is represented on its own grid over  $m$  qubits resulting in a total of  $Bm$  qubits to represent the state of  $B$  particles. Note that the grid encoded in  $m$  qubits has  $2^m$  points. The complex wavefunction  $\Psi(\mathbf{r}, \mathbf{R})$  in addition to being properly normalized must also be anti-symmetrized (or symmetrized for Bosons). Abrams & Lloyd (1997) and Ward, Kassal & Aspuru-Guzik (2009) consider the necessary anti-symmetrization process for fermions in first quantization.

To simulate the dynamics (Zalka 1998b, Wiesner 1996, Kassal, Jordan, Love, Mohseni & Aspuru-Guzik 2008), we note that although the kinetic and potential terms do not commute with each other, both can be represented as diagonal operators in momentum and position basis respectively. By using the quantum Fourier transform  $U_{\text{QFT}}$ , it is natural to decompose the time evolution as

$$e^{-iHt} \approx \left( U_{\text{QFT}}^\dagger e^{-iTt/n} U_{\text{QFT}} e^{-iVt/n} \right)^n \quad . \quad (24)$$

In fact, this method is known as the split-operator method (Feit, Fleck Jr & Steiger 1982, Kosloff 1988). Higher-order Suzuki-Trotter formulas can also be applied, as described before. This method was applied to quantum computing in a number of works (Wiesner 1996, Zalka 1998a, Zalka 1998b, Strini 2002, Benenti & Strini 2008, Kassal et al. 2008).

In the context of quantum computing, it remains to find a method to induce a coordinate-dependent phase factor such that

$$|\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \dots\rangle |\mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3 \dots\rangle \rightarrow e^{-iV(\mathbf{r}, \mathbf{R})\delta t} |\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \dots\rangle |\mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3 \dots\rangle \quad , \quad (25)$$

where  $\delta t \equiv t/n$ , and similarly for the kinetic term in the Fourier basis. An efficient method<sup>6</sup> is implicitly described in the book Kitaev et al. (2002) (pages 131-135), which was further developed and adapted to the chemistry context by Kassal et al. (2008). We sketch the idea here for completeness. First, we will assume that the potential energy term is rescaled to become dimensionless, and projected into a range of integer values such that  $0 \leq V(\mathbf{r}, \mathbf{R}) \leq 2^m - 1$ , where  $m$  should be sufficiently large to allow appropriate resolution of  $V(\mathbf{r}, \mathbf{R})$  in the integer representation. Next, we define a more compact notation  $|\mathbf{r}, \mathbf{R}\rangle \equiv |\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \dots\rangle |\mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3 \dots\rangle$ , and an algorithmic operation  $\mathcal{A}$  to be performed in the position basis:

$$\mathcal{A}|\mathbf{r}, \mathbf{R}\rangle |\mathbf{s}\rangle \rightarrow |\mathbf{r}, \mathbf{R}\rangle |\mathbf{s} \oplus V(\mathbf{r}, \mathbf{R})\rangle \quad , \quad (26)$$

where  $|\mathbf{s}\rangle$ ,  $\mathbf{s} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots$ , is a quantum state of  $m$  ancilla qubits, and  $\oplus$  is addition modulo  $2^m$ . Suppose now that the ancilla qubits are initialized in the following state:

$$|\mathbf{q}\rangle \equiv \frac{1}{\sqrt{M}} \sum_{\mathbf{s}=0}^{M-1} e^{2\pi i \mathbf{s}/M} |\mathbf{s}\rangle \quad , \quad (27)$$

where  $M \equiv 2^m$ . This state is the Fourier transform of  $|1\rangle$ . Then the desired phase generating operation described in Eq. (25) can be achieved using controlled  $\sigma^z$  rotations after applying  $\mathcal{A}$  to the state  $|\mathbf{r}, \mathbf{R}\rangle |\mathbf{q}\rangle$ . A similar procedure is applied to the kinetic term to complete the Trotter cycle.

An alternative approach to implement the controlled-phase operation described in Eq. (25) is the following: first include a register of qubits initialized as  $|\mathbf{0}\rangle$ . Then in a similar (but not identical) way as that described in Eq. (26), we define the operation

$$\tilde{\mathcal{A}}|\mathbf{r}, \mathbf{R}\rangle |\mathbf{0}\rangle \rightarrow |\mathbf{r}, \mathbf{R}\rangle |V(\mathbf{r}, \mathbf{R}) \delta t\rangle \quad , \quad (28)$$

where we used  $\mathbf{0} \oplus V(\mathbf{r}, \mathbf{R}) \delta t = V(\mathbf{r}, \mathbf{R}) \delta t$ . The state  $|V(\mathbf{r}, \mathbf{R}) \delta t\rangle$  is the binary representation  $\{x_1 x_2 x_3 \dots x_m\}$  defined through the following equality,

$$V(\mathbf{r}, \mathbf{R}) \delta t \equiv 2\pi \times 0.x_1 x_2 x_3 \dots x_m = 2\pi \sum_{k=1}^m \frac{x_k}{2^k} \quad . \quad (29)$$

Now, we can decompose the overall phase as follows,

$$e^{-iV(\mathbf{r}, \mathbf{R}) \delta t} = e^{-i2\pi x_1/2} e^{-i2\pi x_2/2^2} \dots e^{-i2\pi x_m/2^m} \quad (30)$$

This decomposition can be achieved through the application of  $m$  local phase gates  $R_k \equiv |0\rangle\langle 0| + \exp(-2\pi i/2^k) |1\rangle\langle 1|$  for each ancilla qubit. This approach requires the ancilla to be un-computed (i.e. the inverse of the operation in Eq. (28)) in the last step.

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<sup>6</sup>An alternative method was proposed by Benenti & Strini (2008), but it scales exponentially with the number qubits.

### 2.2.3 Second-quantized representation

The first-quantization method is universally applicable to any molecule. The shortcoming is that it does not take into account the physical symmetrization properties of the underlying quantum system. When a suitable set of basis functions is employed, the size of the problem can be significantly reduced. This is known as the second-quantization approach in quantum chemistry, which can be extended for quantum simulation.

Most studies on quantum simulation based on first quantization methods use grids to represent wave functions, while works employing second quantization methods generally use atomic or molecular orbitals as a basis set for the wave functions. We will take the later approach here. Nevertheless, the choice of basis is not the key difference between the first and second quantization. Indeed, a basis set of delta functions (or approximations to delta functions) could be used to represent a grid within second quantization. On the other hand, the storage of the same wave function is very different in second and first quantization. For example, a two-particle wave function with the first particle at site  $i$  and the second at site  $j$ , is represented as  $|\text{coord}_i\rangle|\text{coord}_j\rangle$  in first quantization, and as  $|0 \cdots 1_i \cdots 1_j \cdots 00\rangle$  in second quantization.

The starting point of the second-quantization approach (Aspuru-Guzik, Dutoi, Love & Head-Gordon 2005, Wang, Kais, Aspuru-Guzik & Hoffmann 2008, Whitfield, Biamonte & Aspuru-Guzik 2011) is the Born-Oppenheimer approximation, where the nuclear coordinates  $\mathbf{R}$  are taken to be classical variables. This allows us to focus on the electronic structure problem. Ignoring the nuclear kinetic and the nuclear-nuclear interaction terms, the molecular Hamiltonian in Eq. (20) can be expressed as

$$H = \sum_{pq} h_{pq} a_p^\dagger a_q + \frac{1}{2} \sum_{pqrs} h_{pqrs} a_p^\dagger a_q^\dagger a_r a_s \quad . \quad (31)$$

where the fermionic creation operator  $a_p^\dagger$  creates an electron in the  $p$  mode from the vacuum, i.e.,  $a_p^\dagger |\text{vac}\rangle = |p\rangle$ . Denote  $\chi_p(\mathbf{r})$  as the single-particle wavefunction corresponding to mode  $p$ .<sup>7</sup> Then, the explicit form for the single-electron integrals is given by

$$h_{pq} \equiv - \int d\mathbf{r} \chi_p^*(\mathbf{r}) \left( \frac{\hbar^2}{2m_e} \nabla^2 + \frac{e^2}{4\pi\epsilon_0} \sum_i \frac{Z_i}{|\mathbf{R}_i - \mathbf{r}|} \right) \chi_q(\mathbf{r}) \quad , \quad (32)$$

and the electron-electron Coulomb interaction term is,

$$h_{pqrs} \equiv \frac{e^2}{4\pi\epsilon_0} \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{\chi_p^*(\mathbf{r}_1) \chi_q^*(\mathbf{r}_2) \chi_r(\mathbf{r}_2) \chi_s(\mathbf{r}_1)}{|\mathbf{r}_1 - \mathbf{r}_2|} \quad . \quad (33)$$

These integrals have to be pre-calculated with classical computers before encoding them into the quantum algorithms. If we keep  $k$  single-particle orbitals, then

<sup>7</sup>Here  $\mathbf{r}$  refers to the coordinates of one particular electron.

there are  $O(k^4)$  terms. More details of the formalism of second-quantized electronic structure theory in the Born-Oppenheimer approximation can be found in Helgaker, Jorgensen & Olsen (2000).

To simulate time dynamics in a quantum computer, we can apply the same Trotterization idea described above (see Eq. (15)), and simulate separately the terms

$$\exp(-ih_{pq}a_p^\dagger a_q \delta t) \quad \text{and} \quad \exp(-ih_{pqrs}a_p^\dagger a_q^\dagger a_r a_s \delta t) \quad . \quad (34)$$

Since the simulation of every single exponential term in a quantum computer is costly, due to error-correction overheads as discussed in Clark, Metodi, Gasster & Brown (2009), one simplification we can make is to group the terms of single-particle terms into two-particle terms. This is possible for electronic problems with a fixed number  $N$  of electrons. Consider any  $N$ -fermionic state, then the identity operator  $I_N$  is equivalent to a summation of the following single-body number operators,

$$(1/N) \sum_s a_s^\dagger a_s \quad \Leftrightarrow \quad I_N \quad , \quad (35)$$

which means that we can write

$$a_p^\dagger a_q = \frac{1}{N-1} \sum_s a_p^\dagger a_s^\dagger a_s a_q \quad . \quad (36)$$

The last equation is a sum of two-electron terms, and can be absorbed into the pre-computed values of  $h_{pqrs}$ . Now, denoting the new values as  $\tilde{h}_{pqrs}$ , the Hamiltonian  $H$  reduces to

$$H = \frac{1}{2} \sum_{pqrs} \tilde{h}_{pqrs} a_p^\dagger a_q^\dagger a_r a_s \quad . \quad (37)$$

Therefore, we are left only with simulating the two-body term in Eq. (34).

One challenge we need to overcome is the fermionic nature of the operators  $a_p^\dagger$  and  $a_q$ , which comes from the anti-symmetrization requirement of fermionic wavefunctions. A first step to overcome this challenge is to map the occupation representation to the qubit configuration. Explicitly, for each fermionic mode  $j$ , we represent the qubit state  $|0\rangle_j \equiv |\downarrow\rangle_j$  as an unoccupied state, and similarly  $|1\rangle_j \equiv |\uparrow\rangle_j$  as an occupied state. To enforce the exchange symmetry, we apply the Jordan-Wigner transformation (Ortiz et al. 2001, Whitfield et al. 2011):

$$a_j^\dagger = \left( \prod_{m<j} \sigma_m^z \right) \sigma_j^- \quad \text{and} \quad a_j = \left( \prod_{m<j} \sigma_m^z \right) \sigma_j^+ \quad , \quad (38)$$

where

$$\sigma^\pm \equiv (\sigma^x \pm i\sigma^y)/2 \quad . \quad (39)$$

By using Eq. (38) and (39), we can now write the fermionic Hamiltonian in Eq. (37) as a spin Hamiltonian involving products of Pauli matrices  $\{\sigma^x, \sigma^y, \sigma^z\}$ :

$$H_{spin} = \sum_{pqrs} \sum_{abcd} g_{pqrs}^{abcd} \theta_{pqrs} \sigma_p^a \sigma_q^b \sigma_r^c \sigma_s^d \quad , \quad (40)$$



where the set of indices  $\{p, q, r, s\}$  is summed over the fermionic modes, and  $\{a, b, c, d\}$  is either  $x$  or  $y$ . The operator  $\theta_{pqrs}$  keeps track of the  $\sigma_z$ 's; for example, if  $p > q > r > s$ , we then have

$$\theta_{pqrs} = \left( \prod_{p>i>q} \sigma_i^z \right) \times \left( \prod_{r>j>s} \sigma_j^z \right) . \quad (41)$$

The punchline here is that the Hamiltonian becomes a polynomial sum of products of spin operators, and each operator is locally equivalent to  $\sigma_z$ . Therefore, the non-trivial part of simulating the time dynamics of the fermionic Hamiltonian is to simulate the non-local interaction terms of the following form:

$$\exp(-ig\sigma^z\sigma^z\sigma^z\dots\sigma^z\delta t) \quad , \quad (42)$$

where  $g$  is some constant. This can be achieved by a series of controlled-NOT together with a local operation (see e.g. Figure 4.19 of Nielsen & Chuang (2011)), or the phase generating method similar to the one described in the previous section (cf. Eq. (25)). The explicit circuits for simulating the time evolution operators can be found in Whitfield et al. (2011).

## 2.2.4 Open-system dynamics

In quantum mechanics, the time evolution dynamics of a closed system is always described by a unitary transformation of states,  $U(t)\rho U^\dagger(t)$ . However, non-unitary dynamics occurs when the dynamics of the system of interest  $S$  is coupled to the environment  $B$ , as in,

$$\rho_S(t) \equiv \text{Tr}_B [U(t)\rho_{SB}U^\dagger(t)] . \quad (43)$$

After some approximations this evolution can often be described by a (Markovian) quantum master equation in Lindblad form (Breuer & Petruccione 2002, Lindblad 1975, Gorini, Kossakowski & Sudarshan 1976),

$$\frac{d}{dt}\rho_s(t) = -i[H_s, \rho_s] + \sum_{\alpha, \beta} m_{\alpha\beta} \left( [\Lambda_\alpha \rho_s, \Lambda_\beta^\dagger] + [\Lambda_\alpha, \rho_s \Lambda_\beta^\dagger] \right) \quad , \quad (44)$$

where  $H_s$  is the system Hamiltonian,  $m_{\alpha\beta}$  is a positive matrix, and  $\Lambda_\alpha$  is a linear basis of traceless operators. This quantum master equation is relevant in many physical, chemical, and biological processes at finite temperature (Mohseni, Rebentrost, Lloyd & Aspuru-Guzik 2008, Rebentrost, Mohseni, Kassar, Lloyd & Aspuru-Guzik 2009). Further, this equation has many applications in quantum information processing, including preparing entangled states (from arbitrary initial states) (Kraus, Büchler, Diehl, Kantian, Micheli & Zoller 2008, Krauter, Muschik, Jensen, Wasilewski, Petersen, Cirac & Polzik 2011, Muschik, Polzik & Cirac 2011, Cho, Bose & Kim 2011, Müller, Hammerer, Zhou, Roos & Zoller 2011), quantum memories (Pastawski, Clemente & Cirac 2011), and dissipative quantum computation (Verstraete, Wolf & Ignacio Cirac 2009). It has been

shown that the quantum master equation can be simulated by a unitary quantum circuit with polynomial resource scaling (Bacon, Childs, Chuang, Kempe, Leung & Zhou 2001, Kliesch, Barthel, Gogolin, Kastoryano & Eisert 2011). The basic idea is as follows: we first re-write the master equation (Eq. (44)) in the form,

$$\frac{d}{dt}\rho_s(t) = \mathcal{L}(\rho_s) \quad , \quad (45)$$

where  $\mathcal{L}$  is a super-operator. Similar to the unitary dynamics, we can define the super-operator version of the propagator  $\mathcal{K}(t_1, t_0)$  through the relation,

$$\rho_s(t_1) = \mathcal{K}(t_1, t_0)(\rho_s(t_0)) \quad (46)$$

for all values of time  $t_1 \geq t_0$ . Suppose we consider a finite time interval  $T$ , which can be divided into  $m$  small time intervals  $\Delta t$ , i.e.,  $T = m\Delta t$ . Then similar arguments (Kliesch et al. 2011) based on Trotterization show that the following approximation,

$$\mathcal{K}(T) \approx \underbrace{\mathcal{K}(\Delta t)\mathcal{K}(\Delta t)\mathcal{K}(\Delta t)\dots\mathcal{K}(\Delta t)}_{m \text{ times}} \quad , \quad (47)$$

indeed converges when the division size goes to zero, i.e.,  $\Delta t \rightarrow 0$ . The remaining part of the argument is to show that each of the small-time propagator terms  $\mathcal{K}(\Delta t)$  can be simulated efficiently with a quantum circuit. This is generally true if the superoperator  $\mathcal{L}$  is a finite (polynomial) sum of local terms (Bacon et al. 2001).

### 2.3 State preparation

We have discussed how quantum dynamics can be simulated efficiently with a quantum computer, but we have not yet discussed how quantum states of physical or chemical interest can be initialized on the quantum computer. In fact, both thermal and ground states of physical Hamiltonians can be prepared by incorporating the methods of simulating the time dynamics, as we shall explain later in this section.

We first consider a strategy to prepare quantum states that can be efficiently described by some integrable general function, e.g., a Gaussian wave packet. Before we provide a general description, it may be instructive to consider the case of creating a general (normalized) two-qubit state,

$$f_{00}|00\rangle + f_{01}|01\rangle + f_{10}|10\rangle + f_{11}|11\rangle \quad (48)$$

from the initial state  $|00\rangle$ . First of all, we will assume that all the coefficients  $f_{ij}$ 's are real numbers, as the phases can be generated by the method described in Eq. (25). Now, we can write the state in Eq. (48) as

$$g_0|0\rangle \otimes \left( \frac{f_{00}}{g_0}|0\rangle + \frac{f_{01}}{g_0}|1\rangle \right) + g_1|1\rangle \otimes \left( \frac{f_{10}}{g_1}|0\rangle + \frac{f_{11}}{g_1}|1\rangle \right) \quad , \quad (49)$$

where  $g_0 \equiv \sqrt{f_{00}^2 + f_{01}^2}$  is the probability to find the first qubit in the state  $|0\rangle$ , and similarly for  $g_1 \equiv \sqrt{f_{10}^2 + f_{11}^2}$ . The form in Eq. (49) suggests that we can use the following method to generate the general state of Eq. (48) from  $|00\rangle$ .

1. Apply a rotation, such that  $|0\rangle \rightarrow g_0 |0\rangle + g_1 |1\rangle$ , to the first qubit. The resulting state becomes,

$$(g_0 |0\rangle + g_1 |1\rangle) |0\rangle. \quad (50)$$

2. Perform the following controlled operation:

$$|x\rangle |0\rangle \rightarrow |x\rangle \left( \frac{f_{x0}}{g_x} |0\rangle + \frac{f_{x1}}{g_x} |1\rangle \right), \quad (51)$$

where  $x = \{0, 1\}$ .

The final state is exactly the same as that in Eq. (49) or Eq. (48).

Consider, more generally, the preparation of the following  $n$ -qubit quantum state (Zalka 1998b, Grover & Rudolph 2002, Kaye & Mosca 2004, Ward et al. 2009):

$$\sum_{x=0}^{2^n-1} f(x) |x\rangle. \quad (52)$$

Here again we will assume that  $f(x)$  is real. We can image that this is the wavefunction of a particle in 1D. The first qubit describes whether the particle is located in the left half  $|0\rangle$  or right half  $|1\rangle$  of the line divided by  $L \equiv 2^n$  divisions. The first step is therefore to rotate the first qubit as  $\cos \theta_0 |0\rangle + \sin \theta_1 |1\rangle$ , where

$$\cos^2 \theta_0 = \sum_{0 \leq x < L/2} f(x)^2 \quad (53)$$

represents the probability of locating the particle at the left side, i.e.  $0 \leq x < L/2$ . The next step is to apply the following controlled rotation:

$$|x\rangle |0\rangle \rightarrow |x\rangle \left( \frac{\cos \theta_{x0}}{\cos \theta_x} |0\rangle + \frac{\cos \theta_{x1}}{\cos \theta_x} |1\rangle \right), \quad (54)$$

where

$$\cos^2 \theta_{00} = \sum_{0 \leq x < L/4} f(x)^2 \quad \text{and} \quad \cos^2 \theta_{01} = \sum_{L/4 \leq x < L/2} f(x)^2 \quad (55)$$

represents the probability for finding the particle in the ‘00’ division ( $0 \leq x < L/4$ ) and the ‘01’ division ( $L/4 \leq x < L/2$ ) respectively; an analogous arguments apply for the ‘10’ and ‘11’ divisions. In the remaining steps, similar controlled operations described in Eq. (54) are applied, which depend on the division of the controlling qubits. The  $\theta$  rotation angles have to be calculated explicitly. It is therefore necessary that the function  $f(x)$  is efficiently integrable (Grover & Rudolph 2002). This is expected, as otherwise such a simple algorithm would be able to solve the random-field Ising spin models and other NP-complete problems. We will cover the creation of thermal states later. In the next section, we will consider methods for preparing ground states.

<b>0</b>				<b>1</b>			
<b>00</b>		<b>01</b>		<b>10</b>		<b>11</b>	
<b>000</b>	<b>001</b>	<b>010</b>	<b>011</b>	<b>100</b>	<b>101</b>	<b>110</b>	<b>111</b>
x = 0	1	2	3	4	5	6	7

Figure 1: Example for the state preparation method. The space is divided in  $L = 8$  divisions. the ‘0’ division refers to the left half of the space, ( $0 \leq x < L/2$ ), and similarly for the ‘1’ division. Finer resolution is achieved by increasing the number of labeling digits.

### 2.3.1 Preparing ground states

#### Phase-estimation based methods

Finding ground states of classical Hamiltonians, e.g. a random-field Ising model, is known to be NP-hard. Therefore, it is not expected that a quantum computer would be able to solve it efficiently in general. Furthermore, preparing the ground-state of a general quantum Hamiltonian  $H$  is even more challenging as both eigenvalues and eigenvectors are required to be obtained, and this problem belongs to the QMA complexity class, the quantum analog of NP. Fortunately, many problems in physics and chemistry exhibit structures and symmetries that allow us to arrive at solutions that are approximation of the exact solutions; for example, the BCS wavefunction related to superconductivity and superfluidity, and the Laughlin wavefunction related to the fractional quantum Hall effect (FQHE), both provide good predictions for the corresponding many-body problems. The quality of other approximated solutions, such as the mean-field or Hartree-Fock approximation, may vary from problem to problem.

The quality of the approximated solution (or trial solution)  $|\psi_T\rangle$  can be quantified by the fidelity  $F$  defined by

$$F \equiv |\langle e_0 | \psi_T \rangle|^2 \quad , \quad (56)$$

where  $|e_0\rangle$  is the target ground state (assumed unique) of the Hamiltonian  $H$  of interest. The physical meaning of  $F$  is that if one can implement a projective measurement  $\{|e_k\rangle\langle e_k|\}$  in the eigenvector basis  $\{|e_k\rangle\}$  of  $H$  to the trial state  $|\psi_T\rangle$ , then the probability of getting the ground state  $|e_0\rangle$  is exactly equal to  $F$ , and can be implemented with the phase estimation algorithm (Abrams & Lloyd 1999). A similar procedure can produce low energy eigenstates even if there is no gap (Poulin & Wocjan 2009a).

With the methods of the previous paragraph, if the projection on the ground state fails, the initial approximation must be reconstructed again. Because the projection fails with probability  $1 - F$ , the approximate preparation must be

done  $1/(1 - F)$  times in average. This can be improved using phase amplification (a trick similar to Grover’s search) to  $\sqrt{1/(1 - F)}$  “coherent” initial state preparations. A different method is possible if, as is often the case, we can evolve with a Hamiltonian  $\tilde{H}$  for which the state approximation is a ground state. Assume that the approximated ground state has an energy gap bounded by  $\Delta$  for  $\tilde{H}$  and the exact ground state has a similar gap for  $H$ . Then we can transform a single preparation of the approximated state into the exact ground state using around  $1/(1 - F)$  phase estimations, each implemented with a time evolution for a time of  $1/\Delta$  (Boixo, Knill & Somma 2010).

Therefore, a quantum computer, even if it can not solve all ground-state problems efficiently, is capable to leverage classical trial states, and solve a border class of problems than those efficiently solvable by classical computers.

### Adiabatic state preparation

The adiabatic method is an alternative way to prepare ground states (Farhi, Goldstone, Gutmann & Sipser 2000, Aharonov & Ta-Shma 2003, Perdomo, Truncik, Tubert-Brohman, Rose & Aspuru-Guzik 2008, Boixo, Knill & Somma 2009, Biamonte, Bergholm, Whitfield, Fitzsimons & Aspuru-Guzik 2010, Boixo et al. 2010). The original idea is due to Farhi et al. (2000). We first must be able to efficiently prepare the ground state  $|\psi(0)\rangle$  of a certain initial Hamiltonian  $H(0) = H_i$ . Then we change the Hamiltonian  $H(t)$  slowly, e.g.,

$$H(t) = (1 - t/T) H_i + (t/T) H_f . \quad (57)$$

Notice that for many reasonable choices of  $H_i$  and most physical Hamiltonians  $H_f$  the Hamiltonian  $H(t)$  can be simulated using the methods of Sec. 2.2. Nevertheless, common two-body Hamiltonians could be simulated directly.

If the change from  $H_i$  (when  $t = 0$ ) to the target Hamiltonian  $H_f$  (when  $t = T$ ) is slow enough, then the state  $|\psi(t)\rangle$ , satisfying the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle \quad , \quad (58)$$

follows the corresponding eigenstate of  $H(t)$  adiabatically. This means that  $|\psi(T)\rangle$  is close to the ground state of the target Hamiltonian  $H_f$ . A sufficient condition for the total time  $T$  to ensure the adiabaticity for a linear interpolation between two Hamiltonians is

$$T \gg \frac{\|\partial_s H(s)\|}{\Delta_{\min}^2} \quad , \quad (59)$$

where  $s \equiv t/T$ . Here

$$\Delta_{\min} \equiv \min_{0 \leq s \leq 1} (E_1(s) - E_0(s)) \quad (60)$$

is the minimum gap between the instantaneous eigen-energies  $E_1(s)$  and  $E_0(s)$  of the first excited state and the ground state. The following bound has a

better dependence on the minimum gap and it also holds for general (non-linear) interpolations if the rate of change of the instantaneous eigenstate  $|\partial_s\psi(s)\rangle$  is known (Boixo et al. 2009)

$$T > \frac{\mathcal{L}^2}{\Delta_{\min}} . \quad (61)$$

Here  $\mathcal{L}$  is the path length given by the equation<sup>8</sup>

$$\mathcal{L} = \int \|\partial_s\psi(s)\| ds . \quad (62)$$

Using the methods of Sec. 2.2 adiabatic evolutions can be simulated efficiently on a quantum circuit. That is, for cases where one may not be able to physically implement  $H(t)$ , it is still possible to turn the adiabatic state preparation into a quantum algorithm and simulate the adiabatic process in a digital quantum computer. Furthermore, in this case the total time of the adiabatic evolution can be improved to<sup>9</sup> (Boixo et al. 2010):

$$T > \frac{\mathcal{L}}{\Delta_{\min}} . \quad (63)$$

The remaining question is, in terms of finding ground states, ‘how good are adiabatic algorithms?’. As we have seen, the performance, or computational complexity, of adiabatic algorithms generically depends on the scaling of the minimal gap  $\Delta_{\min}$ . Even for classical target Hamiltonians  $H_f$ , whether adiabatic algorithms success in solving NP-problems is still a controversial issue (Altshuler, Krovi & Roland 2010, Knysch & Smelyanskiy 2010). Numerical results suggest that for the classical satisfiability (SAT) problems, the scaling of the gap would be exponential (Young et al. 2008, Young & Smelyanskiy 2010, Hen & Young 2011). If the target Hamiltonian is quantum, the problem is QMA-complete. Nevertheless, we can in principle apply the adiabatic algorithm to the trial states to improve the ground-state fidelity (Oh 2008), which gives us higher probability to project into the exact ground state by the phase estimation algorithm discussed in the previous section.

### 2.3.2 Preparing thermal states using quantum Metropolis

We now consider the preparation of density matrices for thermal states

$$\rho_{th} = e^{-\beta H} / \text{Tr} (e^{-\beta H}) , \quad (64)$$

where  $H$  can be a quantum or classical Hamiltonian, and  $\beta = 1/T$  is the inverse temperature. We simplify the notation by choosing our units so that the Boltzmann constant  $k_B$  is 1. Zalka (1998b) and Terhal & DiVincenzo (2000) proposed to simulate the Markovian dynamics of the system by modeling the interaction

<sup>8</sup>More precisely, for this equation we must make a choice of phases such that  $\langle \partial_s g(s) | g(s) \rangle$ .

<sup>9</sup>Boixo & Somma (2010) have shown that this expression for the total evolution time is also optimal.

with a heat-bath by some ancilla qubits. A similar idea has been recently investigated by Wang, Ashhab & Nori (2011). Terhal & DiVincenzo (2000) also attempted to prepare thermal states by generalizing classical Metropolis-type sampling (Gould, Tobochnik & Christian 2007). This first quantum Metropolis algorithm was limited by the fact that it was not possible to control the update rule for the Metropolis algorithm, which would generally lead to a slow convergence rate of the underlying Markov chain. A significant improvement upon this work has been presented recently in Temme, Osborne, Vollbrecht, Poulin & Verstraete (2011) with the “quantum Metropolis sampling” algorithm. This algorithm also constructs a Markov-chain whose fixed point is a thermal state Eq. (64), but the transitions between states can be engineer to achieve faster convergence. The underlying time cost of this algorithm scales as  $O(1/\Delta)$  (Aldous 1982), where  $\Delta$  is the eigenvalue gap of the Markov matrix associated with the Metropolis algorithm.

Szegedy (2004) introduced a quantum algorithm to speedup classical Markov chains. Richter (2007) extended this methods to some quantum walks with decoherence. Szegedy’s method has also been applied to achieve a quadratic speedup of classical simulated annealing algorithms (Somma, Boixo, Barnum & Knill 2008, Wocjan & Abeyesinghe 2008). Further, Yung & Aspuru-Guzik (2012) achieved a similar speedup of the quantum Metropolis sampling algorithm for quantum Hamiltonians. This algorithm outputs a coherent encoding of the thermal state (CETS):

$$|\psi_{th}\rangle = \sum_k \sqrt{e^{-\beta E_k} / \mathcal{Z}} |e_k\rangle \quad , \quad (65)$$

where  $\mathcal{Z}$  is the partition function, and  $E_k$  is the eigenvalue associated with the eigenvector  $|e_k\rangle$  of the Hamiltonian  $H$ .

Markov chain approaches are practical for many applications. However, for systems like spin glasses, the eigenvalue gap  $\Delta$  of the corresponding Markov matrices typically become exponential small, making it inefficient. Several alternative approaches have been already introduced in the literature. Among them: exploiting the transfer-matrix structure of spin systems (Lidar & Biham 1997, Yung et al. 2010); mapping the CETS as the ground state of certain Hamiltonian (Somma, Batista & Ortiz 2007, Ohzeki & Nishimori 2011); and methods based on quantum phase estimation (Poulin & Wocjan 2009b, Bilgin & Boixo 2010, Riera, Gogolin & Eisert 2011). In the next section we modified one of these algorithms to prepare thermal states building up from small to bigger subsystems.

### 2.3.3 Preparing thermal states with perturbative updates

The quantum Metropolis algorithms of the previous subsection extend the advantages of Markov chain Monte Carlo methods to the quantum case, even if we do not know how to diagonalize the quantum Hamiltonian. It is expected that, as in the classical case, they will exhibit good performance for most Hamiltonians. Nevertheless, for very complex systems, such as strongly correlated

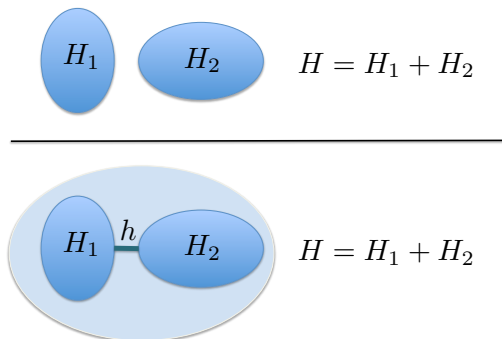


Figure 2: Pictorial representation of the perturbative update method. The top figure depicts two quantum systems with Hamiltonians  $H_1$  and  $H_2$  whose thermal states we can prepare (maybe through prior perturbative updates). The bottom figure depicts a single system where the two quantum systems of the top figure have been linked with Hamiltonian  $h$ . The perturbative update technique is a method to prepare the thermal state of the linked system from thermal states of the smaller subsystems.

molecules, it might be difficult to design rules to choose appropriate Markov chain update candidate states, or the convergence rate to the thermal state might still be too slow. In this subsection we will show how, even in the worst case, quantum algorithms for preparing thermal states will exhibit a substantial speedup over classical algorithms, elaborating upon the method of Bilgin & Boixo (2010).

The core of this algorithm is a perturbative update subroutine that builds the thermal state  $\rho^{(\epsilon)} \propto e^{-\beta(H+\epsilon h)}$  from the state  $\rho^{(0)} \propto e^{-\beta H}$ . We can use this subroute to build thermal states of complex systems out of thermal states of their components (see Fig. 2). For this we start with the thermal states of subsystems with Hamiltonians  $H_1$  and  $H_2$ , and use them to prepare the thermal state of the linked system with Hamiltonian  $H_1 + H_2 + h$ . The coupling Hamiltonian  $h$  is introduced perturbatively with a sequence of updates:

$$\rho^{(0)} \rightarrow \rho^{(\epsilon)} \rightarrow \rho^{(2\epsilon)} \rightarrow \dots \rightarrow \rho^{(1)}. \quad (66)$$

Quite possibly the thermal states of the smaller subsystems have themselves been prepared with a perturbative update over still smaller pieces. As in the quantum Metropolis case, it is not necessary to know how to diagonalize the corresponding Hamiltonian.

The perturbative update subroutine is probabilistic, and succeeds with probability  $1 - \epsilon\beta\text{Tr}\rho^{(0)}h$ , which gives the dominant cost of the algorithm. If the perturbative update fails, we must reconstruct the state  $\rho^{(0)}$ . The probability of failure is given by the maximum change of a thermal state  $\rho^{(0)} \propto e^{-\beta H}$  introduced by a perturbation  $\epsilon h$ , which we now bound. We denote with  $Z = \text{Tr}\rho^{(0)}$  the partition function of  $\rho^{(0)}$ . Using the Dyson series expansion in imaginary



time we write

$$\frac{e^{-\beta(H+\epsilon h)\lambda}}{Z} = \frac{e^{-\beta H\lambda}}{Z} - \frac{\epsilon\beta}{Z} \int_0^\lambda d\lambda_1 e^{-\beta H(\lambda-\lambda_1)} h e^{-\beta H\lambda_1} + \dots \quad (67)$$

The appropriate measure of the difference between two density matrices is the trace norm  $\|\cdot\|_{\text{Tr}}$ . The reason is that this norm bounds the difference of arbitrary measurement results for those two states. The trace norm for an arbitrary operator  $A$  is given by the sum of the eigenvalues of  $\sqrt{A^\dagger A}$ , and often scales with the dimension of the operator. We want to do better for the trace norm of the difference between a thermal state and its perturbation, because their dimension grows exponentially (in the number of subsystems). We give such a bound next.

We will use the following inequality which applies to all unitarily invariant matrix norms<sup>10</sup>

$$\left\| \int_0^1 A^t X B^{1-t} dt \right\| \leq 1/2 \|AX + XB\| \quad (68)$$

Applying this inequality to the trace norm of the Dyson series of a perturbed thermal state we obtain the bound

$$(1/Z) \left\| \epsilon\beta \int_0^1 d\lambda_1 e^{-\beta H(1-\lambda_1)} h e^{-\beta H\lambda_1} \right\|_{\text{Tr}} \leq \epsilon\beta \|h\| \quad (69)$$

where  $\|h\|$  is the operator norm of  $h$ . Notice that the operator norm  $\|h\|$  is the highest eigenvalue of  $h$ , and does not scale with the dimension of  $H$  (or even  $h$ )<sup>11</sup>.

The perturbative update subroutine is composed of two operations. The first operation implements the quantum map

$$(\rho^{(0)}) \rightarrow (1 - \epsilon\beta h/2)\rho^{(0)}(1 - \epsilon\beta h/2)\mathcal{N}, \quad (70)$$

where  $\mathcal{N}$  is just a normalization factor. Similar to the algorithms of the previous section, this map is implemented with phase estimation and a conditional rotation on an ancillary system. The ancillary system is then measured. This measurement can fail, which corresponds to implementing the wrong transformation in the thermal state. The success rate is  $1 - \epsilon\beta \text{Tr} \rho^{(0)} h$ . When the measurement of the ancilla system fails, the thermal state can not be recovered, and we must start again from the beginning. The cost of the phase estimation is  $\epsilon^{-1}\beta^{-1}\|h\|^{-2}$ . This operation can be understood as an update of the Gibbs probabilities of  $\rho^{(0)}$  to those of  $\rho^{(\epsilon)}$ . The second operation of the perturbative

<sup>10</sup>See, for instance, Theorem 5.4.7 in (Bhatia 2007).

<sup>11</sup>Although strictly speaking we have derived the bound on the change of the thermal state here only to second order, it can be shown to be valid to all orders. For that, we use the exact formula for the perturbation  $\epsilon\beta \int_0^1 d\lambda_1 e^{-\beta H(1-\lambda_1)} h e^{-\beta(H+h)\lambda_1}$ , and the same matrix norm inequality. In addition, we need to use the bound for the change on the partition from Poulin & Wocjan (2009b).

update is a transformation to the eigenbasis of  $\rho^{(\epsilon)}$ . This is implemented by “dephasing” in that basis, which is achieved by evolving for a random amount of time (with expectation time  $\epsilon\|h\|$ ) using the Hamiltonian  $H + \epsilon h$ . This completes the perturbative update subroutine.

## 2.4 Algorithmic quantum cooling

Yung, Boixo & Aspuru-Guzik (2011) presented an algorithmic quantum cooling approach that transforms any input state  $\rho_{in}$  into an output state  $\rho_{out}$  which is guaranteed to have lower energy with respect to a given Hamiltonian  $H$ . Explicitly,

$$\text{Tr}(H\rho_{out}) < \text{Tr}(H\rho_{in}) \quad . \quad (71)$$

In principle, this algorithm can cool the resulting quantum state to a state arbitrarily close to the ground state of  $H$ . Nevertheless, like the ground-state algorithms of Sec. 2.3.1, the efficiency is related to the energy gap  $\Delta$  between the ground state and the excited state(s). Depending on how the algorithm is implemented, this dependence can scale like  $O(1/\Delta^2)$  or  $O(1/\Delta)$ .

Algorithmic quantum cooling first entangles an ancilla qubit with the system state. When the ancilla qubit is measured, a result of  $|0\rangle$  correlates with a cooler system state. On average, however, there is no gain or loss of energy. This measurement is used to gain information, just like a Maxwell’s demon. The measurement outcome of the ancilla qubit in algorithmic quantum cooling can be mapped into a 1D random walk. The walker starts at  $x = 0$ . For the cooling outcome, the walker makes a step towards the positive side  $x > 0$ , and towards the negative side  $x < 0$  for the heating outcome. If the walker moves too far to the negative side, the procedure is restarted. For some range of parameters, whenever the walker goes to the negative side  $x < 0$ , it is guaranteed that the quantum state is hotter than the original state. Therefore, removing these hot walkers will reduce the average energy over an ensemble of walkers, just like in evaporative (or “coffee”) cooling of gas molecules. The procedure stops once the walker has moved sufficiently to the positive side.

### 2.4.1 Basic idea of the quantum cooling method

We now sketch the basic working mechanism of algorithmic quantum cooling. The core component of this cooling algorithm consists of four quantum gates (see Fig. 3)<sup>12</sup>. The first gate is a so-called Hadamard gate:

$$H \equiv \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \langle 0| + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \langle 1| \quad . \quad (72)$$

It is followed by a local phase gate,

$$R_z(\gamma) \equiv |0\rangle \langle 0| - ie^{i\gamma} |1\rangle \langle 1| \quad , \quad (73)$$

where the parameter  $\gamma$  plays a role in determining the overall efficiency of the cooling performance of the algorithm. The interaction with the Hamiltonian

<sup>12</sup>Similar quantum circuits are used in DQC1 and phase estimation, for instance.

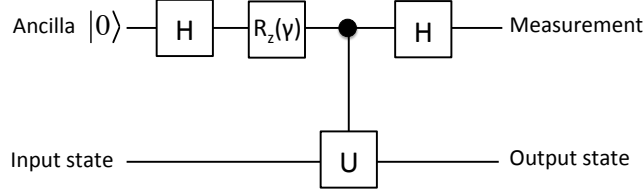


Figure 3: Quantum circuit diagram for the quantum cooling algorithm. Here  $H = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \langle 0| + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \langle 1|$  is the Hadamard gate,  $R_z(\gamma) = |0\rangle \langle 0| - ie^{i\gamma} |1\rangle \langle 1|$  is a local phase gate, and  $U(t) = e^{-iH_s t}$ , is the time  $t$  evolution operator simulating the dynamics of the system under the Hamiltonian  $H_s$ .

$H$ , which can be either quantum or classical, is encoded in the time evolution operator

$$U(t) = e^{-iH_s t} . \quad (74)$$

As explained above, time evolution can be implemented efficiently in a quantum computer.

The operation of the circuit in Fig. 3 on input state  $|\psi_{in}\rangle$  is as follows.

**Step 1** State initialization,

$$|\psi_{in}\rangle |0\rangle , \quad (75)$$

with the ancilla state in  $|0\rangle$ .

**Step 2** Apply the Hadamard gate,  $H = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \langle 0| + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \langle 1|$ , and the local phase gate  $R_z(\gamma) = |0\rangle \langle 0| - ie^{i\gamma} |1\rangle \langle 1|$  to the ancilla qubit,

$$|\psi_{in}\rangle (|0\rangle - ie^{i\gamma} |1\rangle) / \sqrt{2} . \quad (76)$$

**Step 3** Apply the controlled- $U(t)$  to the system state,

$$(|\psi_{in}\rangle |0\rangle - ie^{i\gamma} U(t) |\psi_{in}\rangle |1\rangle) / \sqrt{2} . \quad (77)$$

**Step 4** Apply the Hadamard to the ancilla qubit again. This produces the following output state:

$$\Lambda_0 |\psi_{in}\rangle |0\rangle + \Lambda_1 |\psi_{in}\rangle |1\rangle , \quad (78)$$

where  $\Lambda_j \equiv (I + (-1)^{j+1} ie^{i\gamma} U) / 2$  for  $j = \{0, 1\}$ .

A projective measurement on the ancilla qubit in the computational basis  $\{|0\rangle, |1\rangle\}$  yields one of the two (unnormalized) states,

$$(I \pm ie^{i\gamma} U) |\psi_{in}\rangle \quad (79)$$

Their mean energy is either higher (for outcome  $|1\rangle$ ,  $x$  is decreased by 1) or lower (for outcome  $|0\rangle$ ,  $x$  is increased by 1) than that of the initial state  $|\psi_{in}\rangle$ .

To justify this assertion, let us expand the input state,

$$|\psi_{in}\rangle = \sum_k c_k |e_k\rangle \quad , \quad (80)$$

in the eigenvector basis  $\{|e_k\rangle\}$  of the Hamiltonian  $H$ . Note that

$$|(1 \pm ie^{i\gamma}U)|e_k\rangle|^2 = |c_k|^2 (1 \pm \sin \phi_k) \quad , \quad (81)$$

where  $\phi_k \equiv E_k t - \gamma$  depends on the eigen-energy  $E_k$  of  $H$ . For simplicity, we will assume that one can always adjust the two parameters,  $\gamma$  and  $t$ , such that

$$-\frac{\pi}{2} \leq \phi_k < \frac{\pi}{2} \quad (82)$$

for all non-negative integers  $k$ . Then, the factors  $(1 - \sin \phi_k)$  are in descending order of the eigen-energies, and the opposite is true for the factors  $(1 + \sin \phi_k)$ . Therefore, apart from an overall normalization constant, the action of the operator  $(I \pm ie^{i\gamma}U)$  is to scale each of the probability weights  $|c_k|^2$  by an eigen-energy dependent factor  $(1 \pm \sin \phi_k)$ , i.e.,

$$|c_k|^2 \rightarrow |c_k|^2 (1 \pm \sin \phi_k) \quad . \quad (83)$$

The probability weights scale to larger values, i.e.,

$$(1 - \sin \phi_k) / (1 - \sin \phi_j) > 1 \quad (84)$$

for the eigen-energy  $E_k < E_j$  in the cooling case (i.e., for outcome  $|0\rangle$ ), and vice versa for the heating case (i.e., for outcome  $|1\rangle$ ). Further cooling can be achieved by applying the quantum circuit repeatedly and reject/recycle the random walker when  $x < 0$ .

#### 2.4.2 Connection with heat-bath algorithmic cooling

The algorithmic quantum cooling approach is related to the well-known heat-bath algorithmic cooling (HBAC) (Boykin, Mor, Roychowdhury, Vatan & Vrijen 2002, Baugh, Moussa, Ryan, Nayak & Laflamme 2005, Schulman, Mor & Weinstein 2005). HBAC aims to polarize groups of spins as much as possible, i.e. to prepare the state

$$|\uparrow\uparrow\uparrow \dots \uparrow\rangle \quad . \quad (85)$$

This state is important for providing fresh ancilla qubits for quantum error correction as well as for NMR quantum computation. In HBAC, some reversible operations are first performed to redistribute the entropy among a group of spins. Some of the spins will become more polarized. For a closed system, there is a so-called Shannon bound (Schulman et al. 2005) which limits the compression of the entropy. In order to decrease the entropy of the whole system, the depolarized spins interact with a physical heat bath that acts as an entropy sink. We note that from an algorithm point of view, the existence of a physical heat bath can be replaced by the (imperfect) preparation of polarized spins by other methods. The method of algorithmic quantum cooling from Yung et al. (2011) may be considered as a generalization of the HBAC, as it is applicable to cool *any* physical system that is simulable by a quantum computer, not just non-interacting spins.

## 3 Special topics

### 3.1 Adiabatic non-destructive measurements

In Section. 2.3 we reviewed several methods to prepare ground states and thermal states of quantum systems of interest in physics and chemistry. In particular, in subsection 2.3.1 we gave an overview of the adiabatic method for preparing ground states. The adiabatic model may be naturally more robust against noise, offering a method to perform small to medium size simulations without using sophisticated error correction schemes. Because of this and other reasons, adiabatic based quantum computation is possibly easier to realize physically than quantum computation based on the circuit model. In this section we review a method to effect non-destructive measurements of constants of the motion within the adiabatic model.

As explained in subsection 2.3.1, it is in principle possible to adiabatically prepare the ground state of a physical or chemical system with Hamiltonian  $H_f$ . There we said that this can be done by interpolating slowly enough between a simple initial Hamiltonian  $H_i$  and the final Hamiltonian  $H_f$ . Following Biamonte et al. (2010), we now add an ancillary qubit subsystem with orthonormal basis  $\{|p_0\rangle, |p_1\rangle\}$ . This auxiliary system will be use for the adiabatic non-destructive measurements. During the adiabatic ground state preparation, this subsystem is acted upon with Hamiltonian  $\delta|p_1\rangle\langle p_1|$ , and therefore it remains in the state  $|p_0\rangle$ . The choice of  $\delta > 0$  will be explained shortly.

The measurement procedure begins by bringing the ancillary qubit and the system being simulated into interaction, adiabatically<sup>13</sup>. We choose the interaction Hamiltonian  $H_{int} = A \otimes |p_1\rangle\langle p_1|$ . Here  $A$  is any observable corresponding to a constant of the motion, that is  $[A, H] = 0$ . In particular, the Hamiltonian  $H_f$  itself can be used to obtain the ground state energy. The total Hamiltonian becomes

$$H_f + \delta|p_1\rangle\langle p_1| + \underbrace{A \otimes |p_1\rangle\langle p_1|}_{H_{SP}} . \quad (86)$$

If the energy bias  $\delta$  is bigger than the expectation value of the observable  $A$ , the state does not change during this initial interaction (Biamonte et al. 2010).

After the initial interaction, we apply a Hadamard gate to the ancillary qubit. We denote the time at which we apply this gate as  $t = 0$ . Let  $|s_0\rangle$  be the ground state of  $H_f$ . After a further time  $t$  the system plus ancilla qubit evolves to

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|s_0\rangle \otimes (|p_0\rangle + e^{-i\omega t}|p_1\rangle) \quad (87)$$

where  $\omega = (a_0 + \delta)/\hbar$ , and  $a_0 = \langle s_0|A|s_0\rangle$  is the expectation value we wish to measure. Finally, we again apply a Hadamard gate to the probe. The resulting

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<sup>13</sup>The interaction Hamiltonian is typically a three-body Hamiltonian, which makes direct simulations more difficult. This difficulty can be overcome using gadgets (Kempe, Kitaev & Regev 2006b, Oliveira & Terhal 2005, Biamonte & Love 2008, Biamonte et al. 2010) or the average Hamiltonian method (Waugh, Huber & Haeberlen 1968)

state is

$$|\psi(t)\rangle = |s_0\rangle \otimes (\cos(\omega t/2) |p_0\rangle + i \sin(\omega t/2) |p_1\rangle), \quad (88)$$

yielding probability,

$$P_0(t) = \frac{1}{2} (1 + \cos(\omega t)) = \cos^2(\omega t/2). \quad (89)$$

Measuring the probe does not disturb the state of the simulator which can be reused for another measurement. This measurement can be repeated until sufficient statistics have been accumulated to reconstruct  $\omega$ . We refer to Biamente et al. (2010) for details on numerical simulations and considerations of the influence of noise.

### 3.2 TDDFT and quantum simulation

Density Functional Theory (DFT) and its time-dependent extension (TDDFT) have become arguably the most widely used methods in computational chemistry and physics. In DFT and TDDFT, the properties of a many-body system can be obtained as functionals of the simple one-electron density rather than the correlated many-electron wavefunction. This represents a great conceptual leap from usual wavefunction-based methods such as Hartree-Fock, configuration interaction and coupled cluster methods and therefore the connections between DFT/TDDFT and quantum computation have just begun to be explored. Since TDDFT is a time-dependent theory, it is more readily applicable to quantum simulation than DFT which is strictly a ground state theory. For recent developments in the connections between DFT and quantum complexity<sup>14</sup> see Schuch & Verstraete (2009b), while for applications of DFT to adiabatic quantum computation see Gaitan & Nori (2009). In this section, we provide a brief overview of the fundamental theorems of TDDFT, which establish its use as a tool for simulating quantum many-electron atomic, molecular and solid-state systems and we mention recent extensions of TDDFT to quantum computation (Tempel & Aspuru-Guzik 2011).

In its usual formulation, TDDFT is applied to a system of  $N$ -electrons described by the Hamiltonian

$$\hat{H}(t) = \sum_{i=1}^N \frac{\hat{p}_i^2}{2m} + \sum_{i<j}^N w(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|) + \int v(\mathbf{r}, t) \hat{n}(\mathbf{r}) d^3\mathbf{r}, \quad (90)$$

where  $\hat{\mathbf{p}}_i$  and  $\hat{\mathbf{r}}_i$  are respectively the position and momentum operators of the  $i$ th electron,  $w(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|)$  is the electron-electron repulsion and  $v(\mathbf{r}, t)$  is a time-dependent one-body scalar potential which includes the potential due to nuclear charges as well as any external fields. The electron-electron repulsion,  $w(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|)$ , leads to an exponential scaling of the Hilbert space with system-size and makes simulation of the many-electron Schrödinger equation

<sup>14</sup>It turns out that finding a universal functional for DFT is QMA hard. Liu, Christandl & Verstraete (2007a) proved a related results:  $N$ -representability is also QMA hard.

on a classical computer intractable.  $\hat{n}(\mathbf{r}) = \sum_i^N \delta(\mathbf{r} - \hat{\mathbf{r}}_i)$  is the electron density operator, whose expectation value yields the one-electron probability density,  $\langle \hat{n}(\mathbf{r}) \rangle \equiv n(\mathbf{r}, t)$ . The basic theorems of TDDFT prove that, in principle, one can simulate the evolution of the Hamiltonian in Eq. 90 using  $n(\mathbf{r}, t)$  *directly* and thereby avoid calculating and storing the exponential amount of information in the many-electron wavefunction.

The first basic theorem of TDDFT, known as the ‘‘Runge-Gross’’ (RG) theorem’’ (Runge & Gross 1984), establishes the existence of a one-to-one mapping between the expectation value of  $\hat{n}(\mathbf{r})$  and the scalar potential  $v(\mathbf{r}, t)$ . i.e

$$n(\mathbf{r}, t) \leftrightarrow v(\mathbf{r}, t). \quad (91)$$

However,  $v(\mathbf{r}, t)$  is the only part of the Hamiltonian in Eq. 90 that is non-universal. i.e.  $\sum_{i=1}^N \frac{\hat{p}_i^2}{2m} + \sum_{i<j}^N w(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|)$  is the same operator for each electronic system. Therefore, due to the uniqueness of the solution to the time-dependent Schrödinger equation, the RG theorem establishes a one-to-one mapping between the density and the wavefunction. This implies that the wavefunction is in fact a unique functional of the density,

$$\psi(\mathbf{r}_1, \dots, \mathbf{r}_N; t) \equiv \psi[n](\mathbf{r}_1, \dots, \mathbf{r}_N; t), \quad (92)$$

as is any observable of the system. The RG theorem implies the remarkable fact that the one-electron density contains the same quantum information as the many-electron wavefunction. This means that in principle, if one had a means of directly simulating  $n(\mathbf{r}, t)$ , one could extract all observables of the system without ever needing to simulate the many-body wavefunction.

The second basic TDDFT theorem is known as the ‘‘van Leeuwen (VL) theorem’’ (van Leeuwen 1999). It gives an analytic expression for a time-dependent one-body scalar potential that applied to another system with a different, and possibly simpler, electron-electron repulsion  $w'(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|)$ , gives the same density evolution as the original Hamiltonian of Eq. 90. When  $w'(|\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j|) = 0$ , this auxiliary system is referred to as the ‘‘Kohn-Sham system’’ (Kohn & Sham 1965). Due to its simplicity and accuracy, the Kohn-Sham system is in practice used in most DFT and TDDFT calculations. Since the Kohn-Sham system is non-interacting, its wavefunction is simply described by a Slater determinant of single-electron orbitals, which satisfy the time-dependent Kohn-Sham equations,

$$i \frac{\partial}{\partial t} \phi_i(\mathbf{r}, t) = \left[ -\frac{1}{2} \nabla^2 + v_{ks}[n](\mathbf{r}, t) \right] \phi_i(\mathbf{r}, t). \quad (93)$$

The true interacting density is obtained from the orbitals by square-summing; that is,  $n(\mathbf{r}, t) = \sum_i |\phi_i(\mathbf{r}, t)|^2$ . Naturally, the set of single-particle equations in Eq. 93 are far easier to solve than evolution under the Hamiltonian in Eq. 90. In practice, the Kohn-Sham potential,  $v_{ks}[n](\mathbf{r}, t)$ , must be approximated as a density functional, but the VL theorem rigorously guarantees its existence and uniqueness.

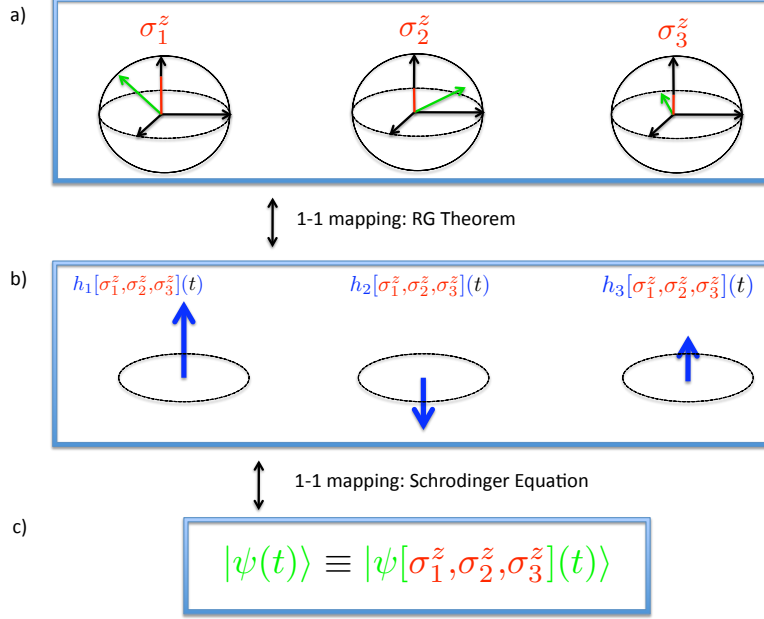


Figure 4: **Runge-Gross theorem for a 3 qubit example** - The set of expectation values  $\{\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z\}$ , defined by the Bloch vector components of each qubit along the z-axis in (a), is uniquely mapped onto the set of local fields  $\{h_1, h_2, \dots, h_N\}$  in (b) through the RG theorem. Then, through the Schrödinger equation, the set of fields is uniquely mapped onto the wavefunction. These two mappings together imply that the N-qubit wavefunction in (c) is in fact a unique functional of the set of expectation values  $\{\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z\}$ .

Tempel & Aspuru-Guzik (2011) recently extended the RG and VL theorems to systems of interacting qubits described by the class of universal 2-local Hamiltonians

$$\hat{H}(t) = \sum_{i=1}^{N-1} J_{i,i+1}^{\perp} (\hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y) + \sum_{i=1}^{N-1} J_{i,i+1}^{\parallel} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \sum_{i=1}^N h_i(t) \hat{\sigma}_i^z. \quad (94)$$

This Hamiltonians apply to a variety of different systems, particularly in solid-state quantum computing. Benjamin & Bose (2003) and Benjamin & Bose (2004) have shown that any set of universal two-qubit and single-qubit gates can be implemented with the Hamiltonian of Eq. 94, and therefore it can be used to perform universal quantum computation. The RG theorem applied to such universal Hamiltonians establishes a one-to-one mapping between the set



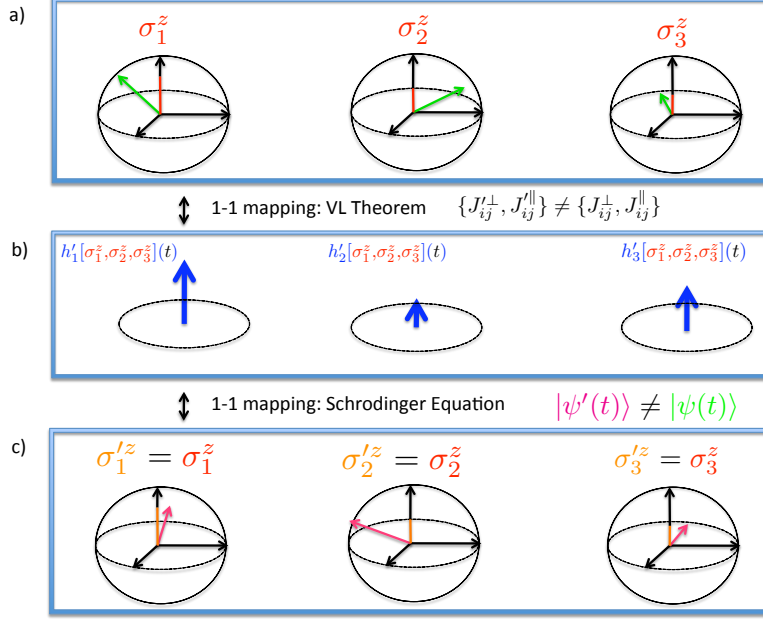


Figure 5: **Van Leeuwen theorem for a 3 qubit example** - The set  $\{\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z\}$  (a) obtained from evolution under Eq. 94, is uniquely mapped to a new set of fields  $\{h'_1, h'_2, \dots, h'_N\}$  (b) for a Hamiltonian with different two-qubit interactions. Evolution under this new Hamiltonian returns the same expectation values  $\{\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z\}$ , although the wavefunction is different and hence projections of the Bloch vectors along other axes are in general different (c).

of local fields  $\{h_1, h_2, \dots, h_N\}$  used to implement a given computation, and the set of single-qubit expectation values  $\{\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z\}$  (see Fig. 4). This implies that one can use single-qubit expectation values as the basic variables in quantum computations rather than wavefunctions and *directly* extract all observables of interest with only knowledge of the spin densities. Naturally, certain properties such as entanglement will be difficult to extract as functionals of the set of spin densities  $\{\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z\}$ . Nevertheless, Tempel & Aspuru-Guzik (2011) give an explicit entanglement functional.

In addition to the RG theorem, one can derive a VL theorem for qubits. The VL theorem for qubits provides an exact prescription for simulating universal Hamiltonians with other universal Hamiltonians that have different, and possibly easier-to-realize, two-qubit interactions. In analogy to the Kohn-Sham

system in electronic TDDFT, one can consider an auxiliary Hamiltonian,

$$\begin{aligned} \hat{H}'(t) = & \sum_{i=1}^{N-1} J'_{i,i+1}{}^{\perp} (\hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y) + \sum_{i=1}^{N-1} J'_{i,i+1}{}^{\parallel} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \\ & + \sum_{i=1}^N h'_i [\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z](t) \hat{\sigma}_i^z, \end{aligned} \quad (95)$$

with simpler two-qubit couplings  $\{J'^{\perp}, J'^{\parallel}\}$ . The VL theorem guarantees the existence of the auxiliary fields  $\{h'_1, h'_2, \dots, h'_N\}$  as functionals of the spin densities which reproduce any given set  $\{\sigma_1^z, \sigma_2^z, \dots, \sigma_N^z\}$  that one might wish to simulate. In this way, one can construct an entire class of density functionals for quantum computing that map between different universal Hamiltonians as illustrated in figure 5.

TDDFT applied to universal qubit Hamiltonians provides a potentially powerful tool for simulating quantum computations on classical computers, similar to how it has been applied in computational chemistry for simulating electronic systems. By choosing the auxiliary "Kohn-Sham" system to be less entangled than the original system, one can hope to simplify simulations of quantum algorithms by finding simple approximations to the auxiliary local fields  $\{h'_1, h'_2, \dots, h'_N\}$  as functionals of the spin density. As in electronic TDDFT, the development of approximate density functionals for qubit systems will be a necessary next step, which is discussed in Tempel & Aspuru-Guzik (2011).

## 4 Conclusion and outlook

To the best of our knowledge, the first quantum simulation experiment was performed by Somaroo, Tseng, Havel, Laflamme & Cory (1999) in a two-qubit NMR system in 1999, where a truncated quantum harmonic or anharmonic oscillator (4 levels) was simulated. Strictly speaking, this belongs to an "analog" simulation; because the Hamiltonian of the quantum oscillator was directly simulated by the Hamiltonian of the spins, instead of applying quantum algorithms. The progress of quantum simulation is still gaining momentum. In Table 1, we list several recent experiments on digital quantum simulation. Earlier references may be found from them and Buluta & Nori (2009), Kassal et al. (2011), and Jones (2011).

So far, none of the simulations implement active error correction. An important aspect of digital quantum simulation is the resource estimation when fault-tolerant structures are considered. We refer to Brown, Clark & Chuang (2006) and Clark et al. (2009) for further explore this area. In short, to achieve large-scale quantum simulation, there are still many technological challenges to overcome. For example, in many currently available setups, the performances of the two-qubit gates are still too noisy for fault-tolerant simulation. However, by experimenting with small-scale quantum simulation experiments, we believe that valuable lessons can be learnt to optimize the performance of the currently available technology.

Table 1: A brief survey of recent experiments on digital quantum simulation

Physical implementations	What is simulated?	Scale
Nuclear Magnetic Resonance (NMR)	• Thermal states of a frustrated magnet (Zhang, Yung, Laflamme, Aspuru-Guzik & Baugh 2011)	4 qubits
	• Ground state of a pair of interacting Heisenberg spins subject to simulated external fields (Li, Yung, Chen, Lu, Whitfield, Peng, Aspuru-Guzik & Du 2011)	3 qubits
	• Isomerization reaction dynamics (Lu, Xu, Xu, Chen, Gong, Peng & Du 2011)	3 qubits
	• Ground state of hydrogen molecule $H_2$ (Du, Xu, Peng, Wang, Wu & Lu 2010)	2 qubits
Trapped Ions	• Time dynamics of spin systems (Lanyon, Hempel, Nigg, Muller, Gerritsma, Zahring, Schindler, Barreiro, Rambach, Kirchmair, Hennrich, Zoller, Blatt & Roos 2011)	6 qubits
	• Dissipative open-system dynamics (Barreiro, Müller, Schindler, Nigg, Monz, Chwalla, Hennrich, Roos, Zoller & Blatt 2011)	5 qubits
Quantum Optics	• 1D quantum walk of a topological system (Kitagawa, Broome, Fedrizzi, Rudner, Berg, Kassa, Aspuru-Guzik, Demler & White 2011)	4 steps
	• 1D quantum walk with tunable decoherence (Broome, Fedrizzi, Lanyon, Kassa, Aspuru-Guzik & White 2010)	6 steps
	• Ground state of hydrogen molecule $H_2$ (Lanyon, Whitfield, Gillett, Goggin, Almeida, Kassa, Biante, Mohseni, Powell, Barbieri, Aspuru-Guzik & White 2010)	2 qubits

A related question is “what is the best way to implement quantum simulation?”. For classical computers, there is no doubt that silicon-based semiconductors work successfully. For quantum computers, a general feature of the currently proposed technologies, such as quantum dots, quantum optics, trapped ions, nuclear and electron spins, impurity, superconducting devices etc., is that there is a trade-off between controllability and reliability. Usually, systems that can be controlled easily suffer more from decoherence from the environment. There are two approaches to tackle this problem: one may either look for new systems that are good for both control and can be isolated from the environment, or develop hybrid structures that combine the advantages from both sides. For example, there has been progress in coupling superconducting devices with spin ensembles (Duty 2010). The former provides the controllability and the latter provides reliability. In short, the future of quantum computation and quantum simulation is still full of challenges and opportunities. We hope this article can stimulate more ideas that can help move this field forward.

## 5 Acknowledgements

The authors would like to acknowledge NSF CCI grant number: CHE-1037992.

## References

- Aaronson, S. (2009). Computational complexity: Why quantum chemistry is hard, *Nature Physics* **5**(10): 707–708.
- Aaronson, S. (2011). Why Philosophers Should Care About Computational Complexity, *arXiv:1108.1791* p. 58.
- Abrams, D. & Lloyd, S. (1997). Simulation of many-body Fermi systems on a universal quantum computer, *Phys. Rev. Lett.* **79**: 2586.
- Abrams, D. & Lloyd, S. (1999). Quantum Algorithm Providing Exponential Speed Increase for Finding Eigenvalues and Eigenvectors, *Physical Review Letters* **83**(24): 5162–5165.
- Aharonov, D. & Naveh, T. (2002). Quantum NP - A Survey, *arXiv:quant-ph/0210077* p. 23.
- Aharonov, D. & Ta-Shma, A. (2003). Adiabatic quantum state generation and statistical zero knowledge, *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, ACM, pp. 20–29.
- Aldous, D. J. (1982). Some Inequalities for Reversible Markov Chains, *Journal of the London Mathematical Society* **s2-25**(3): 564–576.
- Altshuler, B., Krovi, H. & Roland, J. (2010). Anderson localization makes adiabatic quantum optimization fail, *Proc Natl Acad Sci U S A* **107**(28): 12446–50.
- Aspuru-Guzik, A., Dutoi, A. D., Love, P. J. & Head-Gordon, M. (2005). Simulated quantum computation of molecular energies., *Science (New York, N. Y.)* **309**(5741): 1704–7.
- Bacon, D., Childs, A., Chuang, I., Kempe, J., Leung, D. & Zhou, X. (2001). Universal simulation of Markovian quantum dynamics, *Physical Review A* **64**(6).
- Barahona, F. (1982). On the computational complexity of ising spin glass models, *J. Phys. A: Math. Gen* **15**(10): 3241–3253.
- Barreiro, J. T., Müller, M., Schindler, P., Nigg, D., Monz, T., Chwalla, M., Hennrich, M., Roos, C. F., Zoller, P. & Blatt, R. (2011). An open-system quantum simulator with trapped ions, *Nature* **470**(7335): 486–491.

- Baugh, J., Moussa, O., Ryan, C. a., Nayak, A. & Laflamme, R. (2005). Experimental implementation of heat-bath algorithmic cooling using solid-state nuclear magnetic resonance., *Nature* **438**(7067): 470–3.
- Beals, R., Buhrman, H., Cleve, R., Mosca, M. & de Wolf, R. (1998). Quantum lower bounds by polynomials, *Proceedings of FOCS' 98*, pp. 352–361.
- Benenti, G. & Strini, G. (2008). Quantum simulation of the single-particle Schrödinger equation, *American Journal of Physics* **76**(7): 657.
- Benjamin, S. C. & Bose, S. (2003). Quantum computing with an always-on heisenberg interaction, *Phys. Rev. Lett.* **90**: 247901.
- Benjamin, S. C. & Bose, S. (2004). Quantum computing in arrays coupled by “always-on” interactions, *Phys. Rev. A* **70**: 032314.
- Berry, D. W., Ahokas, G., Cleve, R. & Sanders, B. C. (2006). Efficient Quantum Algorithms for Simulating Sparse Hamiltonians, *Communications in Mathematical Physics* **270**(2): 359–371.
- Berry, D. W. & Childs, A. M. (2009). Black-box Hamiltonian simulations and unitary implementation, *arXiv:0910.4157* .
- Bhatia, R. (2007). *Positive definite matrices*, Princeton University Press.
- Biamonte, J. D., Bergholm, V., Whitfield, J. D., Fitzsimons, J. & Aspuru-Guzik, A. (2010). Adiabatic Quantum Simulators, *University Computing* p. 9.
- Biamonte, J. D. & Love, P. J. (2008). Realizable hamiltonians for universal adiabatic quantum computers, *Physical Review A (Atomic, Molecular, and Optical Physics)* **78**(1): 012352–7.
- Bilgin, E. & Boixo, S. (2010). Preparing Thermal States of Quantum Systems by Dimension Reduction, *Physical Review Letters* **105**(17).
- Boixo, S., Knill, E. & Somma, R. D. (2009). Eigenpath traversal by phase randomization, *Quantum Information and Computation* **9**: 833–855.
- Boixo, S., Knill, E. & Somma, R. D. (2010). Fast quantum algorithms for traversing paths of eigenstates, *arXiv:1005.3034* .
- Boixo, S. & Somma, R. D. (2010). Necessary condition for the quantum adiabatic approximation, *Physical Review A* **81**(3): 032308.
- Boykin, P. O., Mor, T., Roychowdhury, V., Vatan, F. & Vrijen, R. (2002). Algorithmic cooling and scalable NMR quantum computers., *Proceedings of the National Academy of Sciences of the United States of America* **99**(6): 3388–93.
- Breuer, H.-P. & Petruccione, F. (2002). *The theory of open quantum systems*, Oxford University Press.

- Broome, M. A., Fedrizzi, A., Lanyon, B. P., Kassal, I., Aspuru-Guzik, A. & White, A. G. (2010). Discrete Single-Photon Quantum Walks with Tunable Decoherence, *Physical Review Letters* **104**(15): 1–4.
- Brown, K. L., Munro, W. J. & Kendon, V. M. (2010). Using Quantum Computers for Quantum Simulation, *Entropy* **12**(11): 2268–2307.
- Brown, K. R., Clark, R. J. & Chuang, I. L. (2006). Limitations of quantum simulation examined by simulating a pairing Hamiltonian using nuclear magnetic resonance., *Physical Review Letters* **97**(5): 050504.
- Buluta, I. & Nori, F. (2009). Quantum simulators., *Science (New York, N.Y.)* **326**(5949): 108–11.
- Byrnes, T. & Yamamoto, Y. (2006). Simulating lattice gauge theories on a quantum computer, *Physical Review A* **73**(2): 1–16.
- Childs, A. M. (2004). *Quantum information processing in continuous time*, Ph.D., MIT, Cambridge, MA.
- Childs, A. M. (2010). On the relationship between continuous- and discrete-time quantum walk, *Comm. Math. Phys.* **294**: 581.
- Childs, A. M. & Kothari, R. (2011). Simulating sparse Hamiltonians with star decompositions, *Theory of Quantum Computation Communication and Cryptography TQC 2010* **6519**: 94–103.
- Cho, J., Bose, S. & Kim, M. (2011). Optical Pumping into Many-Body Entanglement, *Physical Review Letters* **106**(2): 1–4.
- Clark, C., Metodi, T., Gasster, S. & Brown, K. (2009). Resource requirements for fault-tolerant quantum simulation: The ground state of the transverse Ising model, *Physical Review A* **79**(6): 1–9.
- Du, J., Xu, N., Peng, X., Wang, P., Wu, S. & Lu, D. (2010). NMR Implementation of a Molecular Hydrogen Quantum Simulation with Adiabatic State Preparation, *Physical Review Letters* **104**(3): 1–4.
- Duty, T. (2010). Towards superconductor-spin ensemble hybrid quantum systems, *Physics* **3**: 80.
- Farhi, E., Goldstone, J., Gutmann, S. & Sipser, M. (2000). Quantum Computation by Adiabatic Evolution, *arXiv:quant-ph/0001106v1* .
- Feit, M., Fleck Jr, J. & Steiger, A. (1982). Solution of the Schrödinger equation by a spectral method, *Journal of Computational Physics* **47**(3): 412–433.
- Feynman, R. P. (1982). Simulating physics with computers, *International Journal of Theoretical Physics* **21**(6-7): 467–488.

- Gaitan, F. & Nori, F. (2009). Density functional theory and quantum computation, *Phys. Rev. B* **79**: 205117.
- Gorini, V., Kossakowski, A. & Sudarshan, E. C. G. (1976). Completely positive dynamical semigroups of n-level systems, *J. Math. Phys* **17**: 821.
- Gould, H., Tobochnik, J. & Christian, W. (2007). *An introduction to computer simulation methods: applications to physical systems*, Pearson Addison Wesley.
- Grover, L. & Rudolph, T. (2002). Creating superpositions that correspond to efficiently integrable probability distributions, *arXiv:quant-ph/0208112* .
- Hatano, N. & Suzuki, M. (2005). Finding exponential product formulas of higher orders, in A. Das & B. Chakrabarti (eds), *Quantum Annealing and Other Optimization Methods*, Lectures Notes in Physics, Springer, pp. 37–68.
- Hauke, P., Cucchietti, F. M., Tagliacozzo, L., Deutsch, I. & Lewenstein, M. (2011). Can One Trust Quantum Simulators?, *arXiv:1109.6457* .
- Helgaker, T., Jorgensen, P. & Olsen, J. (2000). *Molecular Electronic-Structure Theory*, John Wiley and Sons.
- Hen, I. & Young, A. P. (2011). Exponential complexity of the quantum adiabatic algorithm for certain satisfiability problems, *Physical Review E* **84**(6): 061152.
- Jaksch, P. & Papageorgiou, A. (2003). Eigenvector Approximation Leading to Exponential Speedup of Quantum Eigenvalue Calculation, *Physical Review Letters* **91**(25): 1–4.
- Jones, J. A. (2011). Quantum computing with NMR., *Progress in nuclear magnetic resonance spectroscopy* **59**(2): 91–120.
- Kassal, I. & Aspuru-Guzik, A. (2009). Quantum algorithm for molecular properties and geometry optimization., *The Journal of chemical physics* **131**(22): 224102.
- Kassal, I., Jordan, S. P., Love, P. J., Mohseni, M. & Aspuru-Guzik, A. (2008). Polynomial-time quantum algorithm for the simulation of chemical dynamics., *Proceedings of the National Academy of Sciences of the United States of America* **105**(48): 18681–6.
- Kassal, I., Whitfield, J. D., Perdomo-Ortiz, A., Yung, M.-H. & Aspuru-Guzik, A. (2011). Simulating chemistry using quantum computers., *Annual review of physical chemistry* **62**: 185–207.
- Kaye, P., Laflamme, R. & Mosca, M. (2007). *An Introduction to Quantum Computing*, 1 edn, Oxford University Press, USA.

- Kaye, P. & Mosca, M. (2004). Quantum networks for generating arbitrary quantum states.
- Kempe, J., Kitaev, A. & Regev, O. (2006a). The complexity of the local hamiltonian problem, *SIAM Journal on Computing* **35**(5): 1070–1097.
- Kempe, J., Kitaev, A. & Regev, O. (2006b). The complexity of the local hamiltonian problem, *SIAM Journal on Computing* **35**(5): 1070.
- Kitaev, A., Shen, A. & Vyalys, M. (2002). *Classical and quantum computation*, Graduate studies in mathematics, American Mathematical Society.
- Kitagawa, T., Broome, M. A., Fedrizzi, A., Rudner, M. S., Berg, E., Kassal, I., Aspuru-Guzik, A., Demler, E. & White, A. G. (2011). Observation of topologically protected bound states in a one dimensional photonic system, *arXiv:1105.5334* p. 5.
- Kliesch, M., Barthel, T., Gogolin, C., Kastoryano, M. & Eisert, J. (2011). Dissipative Quantum Church-Turing Theorem, *Physical Review Letters* **107**(12): 1–5.
- Knill, E., Ortiz, G. & Somma, R. D. (2007). Optimal quantum measurements of expectation values of observables, *Physical Review A (Atomic, Molecular, and Optical Physics)* **75**(1): 012328–13.
- Knysh, S. & Smelyanskiy, V. (2010). On the relevance of avoided crossings away from quantum critical point to the complexity of quantum adiabatic algorithm, *arXiv:1005.3011* p. 8.
- Kohn, W. (1999). Nobel Lecture: Electronic structure of matter—wave functions and density functionals, *Reviews of Modern Physics* **71**(5): 1253–1266.
- Kohn, W. & Sham, L. J. (1965). Self-consistent equations including exchange and correlation effects, *Phys. Rev.* **140**: 1133.
- Kosloff, R. (1988). Time-dependent quantum-mechanical methods for molecular dynamics, *The Journal of Physical Chemistry* **92**(8): 2087–2100.
- Kraus, B., Büchler, H., Diehl, S., Kantian, A., Micheli, A. & Zoller, P. (2008). Preparation of entangled states by quantum Markov processes, *Physical Review A* **78**(4): 1–9.
- Krauter, H., Muschik, C., Jensen, K., Wasilewski, W., Petersen, J., Cirac, J. & Polzik, E. (2011). Entanglement Generated by Dissipation and Steady State Entanglement of Two Macroscopic Objects, *Physical Review Letters* **107**(8): 1–5.
- Ladd, T. D., Jelezko, F., Laflamme, R., Nakamura, Y., Monroe, C. & O’Brien, J. L. (2010). Quantum computers., *Nature* **464**(7285): 45–53.



- Lanyon, B. P., Hempel, C., Nigg, D., Müller, M., Gerritsma, R., Zahring, F., Schindler, P., Barreiro, J. T., Rambach, M., Kirchmair, G., Hennrich, M., Zoller, P., Blatt, R. & Roos, C. F. (2011). Universal Digital Quantum Simulation with Trapped Ions, *Science* p. 13.
- Lanyon, B. P., Whitfield, J. D., Gillett, G. G., Goggin, M. E., Almeida, M. P., Kassal, I., Biamonte, J. D., Mohseni, M., Powell, B. J., Barbieri, M., Aspuru-Guzik, A. & White, A. G. (2010). Towards quantum chemistry on a quantum computer, *Nature Chemistry* **2**(2): 106–111.
- Li, Z., Yung, M.-H., Chen, H., Lu, D., Whitfield, J. D., Peng, X., Aspuru-Guzik, A. & Du, J. (2011). Solving Quantum Ground-State Problems with Nuclear Magnetic Resonance, *Scientific Reports* **1**: 88.
- Lidar, D. & Biham, O. (1997). Simulating Ising spin glasses on a quantum computer, *Physical Review E* **56**(3): 3661–3681.
- Lidar, D. & Wang, H. (1999). Calculating the thermal rate constant with exponential speedup on a quantum computer, *Physical Review E* **59**(2): 2429–2438.
- Lindblad, G. (1975). On the generators of quantum dynamical semigroups, *Commun. Math. Phys* **48**: 119.
- Liu, Y., Christandl, M. & Verstraete, F. (2007a). Quantum computational complexity of the N-Representability problem: QMA complete, *Physical Review Letters* **98**(11): 110503.
- Liu, Y.-K., Christandl, M. & Verstraete, F. (2007b). Quantum Computational Complexity of the N-Representability Problem: QMA Complete, *Physical Review Letters* **98**(11): 110503.
- Lloyd, S. (1996). Universal Quantum Simulators, *Science* **273**(5278): 1073–1078.
- Lu, D., Xu, N., Xu, R., Chen, H., Gong, J., Peng, X. & Du, J. (2011). Simulation of Chemical Isomerization Reaction Dynamics on a NMR Quantum Simulator, *Physical Review Letters* **107**(2): 8–11.
- Master, C., Yamaguchi, F. & Yamamoto, Y. (2003). Efficiency of free-energy calculations of spin lattices by spectral quantum algorithms, *Physical Review A* **67**(3).
- Mohseni, M., Rebentrost, P., Lloyd, S. & Aspuru-Guzik, A. (2008). Environment-assisted quantum walks in energy transfer of photosynthetic complexes, *Journal of Chemical Physics* **129**(174106).
- Müller, M., Hammerer, K., Zhou, Y. L., Roos, C. F. & Zoller, P. (2011). Simulating open quantum systems: from many-body interactions to stabilizer pumping, *New Journal of Physics* **13**(8): 085007.

- Muschik, C., Polzik, E. & Cirac, J. (2011). Dissipatively driven entanglement of two macroscopic atomic ensembles, *Physical Review A* **83**(5): 1–19.
- Nielsen, M. A. & Chuang, I. L. (2011). *Quantum Computation and Quantum Information: 10th Anniversary Edition*, Quantum Computation and Quantum Information, Cambridge University Press.
- Oh, S. (2008). Quantum computational method of finding the ground-state energy and expectation values, *Physical Review A* **77**(1): 012326.
- Ohzeki, M. & Nishimori, H. (2011). Quantum Annealing: An Introduction and New Developments, *Journal of Computational and Theoretical Nanoscience* **8**(6): 963–971.
- Oliveira, R. & Terhal, B. M. (2005). The complexity of quantum spin systems on a two-dimensional square lattice. *Quant. Inf. Comp.* Vol. 8, No. 10, pp. 0900-0924 (2008).
- Ortiz, G., Gubernatis, J., Knill, E. & Laflamme, R. (2001). Quantum algorithms for fermionic simulations, *Physical Review A* **64**(2): 1–14.
- Osborne, T. J. (2011). Hamiltonian complexity, *arXiv:1106.5875* p. 14.
- Papageorgiou, A. & Zhang, C. (2010). On the efficiency of quantum algorithms for hamiltonian simulation, *Quantum Information Processing* pp. 1–21.
- Pastawski, F., Clemente, L. & Cirac, J. (2011). Quantum memories based on engineered dissipation, *Physical Review A* **83**(1): 1–12.
- Perdomo, A., Truncik, C., Tubert-Brohman, I., Rose, G. & Aspuru-Guzik, A. (2008). Construction of model Hamiltonians for adiabatic quantum computation and its application to finding low-energy conformations of lattice protein models, *Physical Review A* **78**(1): 1–15.
- Pople, J. (1999). Nobel Lecture: Quantum chemical models, *Reviews of Modern Physics* **71**(5): 1267–1274.
- Poulin, D. & Wocjan, P. (2009a). Preparing Ground States of Quantum Many-Body Systems on a Quantum Computer, *Physical Review Letters* **102**(13): 1–4.
- Poulin, D. & Wocjan, P. (2009b). Sampling from the Thermal Quantum Gibbs State and Evaluating Partition Functions with a Quantum Computer, *Physical Review Letters* **103**(22).
- Raesi, S., Wiebe, N. & Sanders, B. C. (2011). Designing Quantum Circuits for Efficient Many-Body Quantum Simulation, *Arxiv:1108.4318* .
- Rassolov, V. a. & Garashchuk, S. (2008). Computational complexity in quantum chemistry, *Chemical Physics Letters* **464**(4-6): 262–264.

- Reberntrost, P., Mohseni, M., Kassal, I., Lloyd, S. & Aspuru-Guzik, A. (2009). Environment-assisted quantum transport, *New Journal of Physics* **11**: 033003.
- Richter, P. (2007). Quantum speedup of classical mixing processes, *Physical Review A* **76**(4).
- Riera, A., Gogolin, C. & Eisert, J. (2011). Thermalization in nature and on a quantum computer, *arXiv:1102.2389* p. 12.
- Runge, E. & Gross, E. K. U. (1984). Density-functional theory for time-dependent systems, *Phys. Rev. Lett.* **52**: 997.
- Schuch, N. & Verstraete, F. (2009a). Computational complexity of interacting electrons and fundamental limitations of density functional theory, *Nature Physics* **5**(10): 732–735.
- Schuch, N. & Verstraete, F. (2009b). Computational Complexity of interacting electrons and fundamental limitations of density functional theory, *Nature Phys.* **5**: 732. Also see appendix of arxiv:0712.0483.
- Schulman, L., Mor, T. & Weinstein, Y. (2005). Physical Limits of Heat-Bath Algorithmic Cooling, *Physical Review Letters* **94**(12): 1–4.
- Shor, P. W. (1997). Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer, *SIAM Journal on Computing* **26**(5): 1484.
- Somaroo, S., Tseng, C., Havel, T., Laflamme, R. & Cory, D. (1999). Quantum Simulations on a Quantum Computer, *Physical Review Letters* **82**(26): 5381–5384.
- Somma, R., Batista, C. & Ortiz, G. (2007). Quantum Approach to Classical Statistical Mechanics, *Physical Review Letters* **99**(3): 1–4.
- Somma, R., Boixo, S., Barnum, H. & Knill, E. (2008). Quantum Simulations of Classical Annealing Processes, *Physical Review Letters* **101**(13).
- Somma, R., Ortiz, G., Gubernatis, J. E., Knill, E. & Laflamme, R. (2002). Simulating physical phenomena by quantum networks, *Physical Review A* **65**(4).
- Somma, R., Ortiz, G., Knill, E. & Gubernatis, J. (2003). Quantum Simulations of Physics Problems, *Proceedings of SPIE* **5105**: 12.
- Stolze, J. & Suter, D. (2008). *Quantum computing: a short course from theory to experiment*, Physics textbook, Wiley-VCH.
- Strini, G. (2002). Error Sensitivity of a Quantum Simulator I: a First Example, *Fortschritte der Physik* **50**(2): 171.

- Szegedy, M. (2004). Quantum Speed-Up of Markov Chain Based Algorithms, *45th Annual IEEE Symposium on Foundations of Computer Science*, IEEE, pp. 32–41.
- Temme, K., Osborne, T. J., Vollbrecht, K. G., Poulin, D. & Verstraete, F. (2011). Quantum Metropolis sampling., *Nature* **471**(7336): 87–90.
- Tempel, D. G. & Aspuru-Guzik, A. (2011). Quantum computing without wavefunctions: Time-dependent density functional theory for universal quantum computation, *arXiv:1108.0097* .
- Terhal, B. M. & DiVincenzo, D. P. (2000). Problem of equilibration and the computation of correlation functions on a quantum computer, *Physical Review A* **61**(2).
- van Leeuwen, R. (1999). Mapping from Densities to Potentials in Time-Dependent Density-Functional Theory, *Phys. Rev. Lett.* **82**(19): 3863–3866.
- Verstraete, F., Wolf, M. M. & Ignacio Cirac, J. (2009). Quantum computation and quantum-state engineering driven by dissipation, *Nature Physics* **5**(9): 633–636.
- Wang, H., Ashhab, S. & Nori, F. (2011). Quantum algorithm for simulating the dynamics of an open quantum system, *Physical Review A* **83**(6): 1–11.
- Wang, H., Kais, S., Aspuru-Guzik, A. & Hoffmann, M. R. (2008). Quantum algorithm for obtaining the energy spectrum of molecular systems., *Physical chemistry chemical physics : PCCP* **10**(35): 5388–93.
- Ward, N. J., Kassal, I. & Aspuru-Guzik, A. (2009). Preparation of many-body states for quantum simulation., *The Journal of chemical physics* **130**(19): 194105.
- Waugh, J., Huber, L. & Haeberlen, U. (1968). Approach to high-resolution NMR in solids, *Physics Review Letters* **20**(5): 180–182.
- Wei, T.-C., Mosca, M. & Nayak, A. (2010). Interacting Boson Problems Can Be QMA Hard, *Physical Review Letters* **104**(4): 1–4.
- Whitfield, J. D., Biamonte, J. & Aspuru-Guzik, A. (2011). Simulation of electronic structure Hamiltonians using quantum computers, *Molecular Physics* **109**(5): 735–750.
- Wiesner, S. (1996). Simulations of Many-Body Quantum Systems by a Quantum Computer, *quant-ph/9603028* .
- Williams, C. P. (2010). *Explorations in Quantum Computing*, Texts in Computer Science, Springer.
- Wocjan, P. & Abeyesinghe, A. (2008). Speedup via quantum sampling, *Physical Review A* **78**(4).

- Wocjan, P., Chiang, C.-F., Nagaj, D. & Abeyesinghe, A. (2009). Quantum algorithm for approximating partition functions, *Physical Review A* **80**(2).
- Wu, L.-a., Byrd, M. & Lidar, D. (2002). Polynomial-Time Simulation of Pairing Models on a Quantum Computer, *Physical Review Letters* **89**(5): 1–4.
- Young, a., Knysh, S. & Smelyanskiy, V. (2008). Size Dependence of the Minimum Excitation Gap in the Quantum Adiabatic Algorithm, *Physical Review Letters* **101**(17): 1–4.
- Young, a. P. & Smelyanskiy, V. N. (2010). First-Order Phase Transition in the Quantum Adiabatic Algorithm, *Physical Review Letters* **104**(2): 1–4.
- Yung, M.-H. & Aspuru-Guzik, A. (2012). A Quantum-Quantum Metropolis Algorithm, *Proc. Nat. Acad. Sci.* **In press**: 7.
- Yung, M.-H., Boixo, S. & Aspuru-Guzik, A. (2011). Algorithmic quantum cooling via random walk, *under preparation* .
- Yung, M.-H., Nagaj, D., Whitfield, J. & Aspuru-Guzik, A. (2010). Simulation of classical thermal states on a quantum computer: A transfer-matrix approach, *Physical Review A* **82**(6): 5.
- Zalka, C. (1998a). Efficient Simulation of Quantum Systems by Quantum Computers, *Fortschritte der Physik* **46**(6-8): 877–879.
- Zalka, C. (1998b). Simulating quantum systems on a quantum computer, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **454**(1969): 313–322.
- Zhang, J., Yung, M.-H., Laflamme, R., Aspuru-Guzik, A. & Baugh, J. (2011). Digital Quantum Simulation of the Statistical Mechanics of a Frustrated Magnet, *arXiv:1108.3270* p. 7.