Entropy, Dimension and Combinatorial Moduli for One-Dimensional Dynamical Systems

Entropy, dimension and combinatorial moduli for one-dimensional dynamical systems

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Abstract

The goal of this thesis is to provide a unified framework in which to analyze the dynamics of two seemingly unrelated families of one-dimensional dynamical systems, namely the family of quadratic polynomials and continued fractions. We develop a combinatorial calculus to describe the bifurcation set of both families and prove they are isomorphic. As a corollary, we establish a series of results describing the behavior of entropy as a function of the parameter. One of the most important applications is the relation between the topological entropy of quadratic polynomials and the Hausdorff dimension of sets of external rays landing on principal veins of the Mandelbrot set.

Contents

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1. Introduction

Families of dynamical systems often exhibit a rich topological structure. Qualitative aspects of the dynamics can change abruptly as parameters change, and typically, in parameter space there are infinitely many islands of stability, which are intertwined with chaotic regimes in an intricate way.

In this work we shall study the bifurcation sets of two families of one-dimensional dynamical systems and establish an isomorphism between them. We will also use the combinatorial description of these sets to study their Hausdorff dimension and the entropy of their associated dynamics; in particular, we will establish a relation between the entropy of Hubbard trees and the geometry of the Mandelbrot set.

The quadratic family. First, we consider the quadratic family $f_c(z) := z^2 + c$, whose bifurcations are captured by the *Mandelbrot set* $\mathcal{M}$

$$\mathcal{M} := \{c \in \mathbb{C} \ : \ f_c^n(0) \text{ does not tend to } \infty \text{ as } n \to \infty\}.$$

Following Douady-Hubbard and Thurston, a combinatorial model for $\mathcal{M}$ comes from uniformizing its exterior via the Riemann mapping $\Phi_M : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$ (see section 2). For each $\theta \in \mathbb{R}/\mathbb{Z}$, the image of the radial arc at angle θ is called the *external ray* $R_M(\theta)$. Conjecturally, the Mandelbrot set is locally connected [DH]; if this is the case, then the Riemann mapping extends to the boundary of the unit disk, so $\mathcal{M}$ is homeomorphic to the quotient of the unit disk by an equivalence relation, which can be represented by Thurston's *quadratic minor lamination* [Th1] (Figure 1).

The bifurcation set of the real quadratic family is the real slice $\partial\mathcal{M} \cap \mathbb{R}$ of the boundary of the Mandelbrot set. A combinatorial model for it is the set $\mathcal{R}$ of external angles of rays whose impression intersects the real axis:

$$\mathcal{R} := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \text{the impression of } R_M(\theta) \text{ intersects } \partial\mathcal{M} \cap \mathbb{R}\}.$$

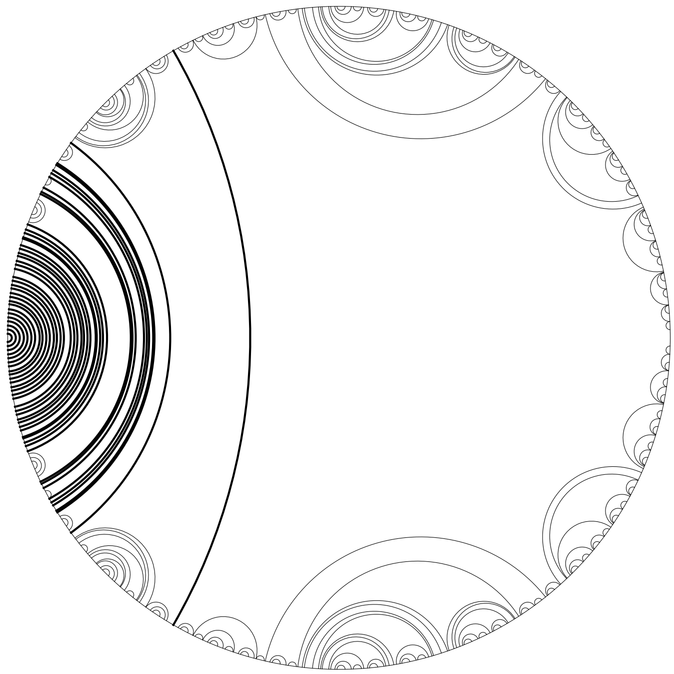


FIGURE 1. Thurston's *quadratic minor lamination*. Two points on the unit circle are joined by a leaf if and only if the two corresponding external rays land on the same point (see section 3.1 for a precise statement); the quotient of the unit disk by the equivalence relation given by the lamination is a topological model for $\mathcal{M}$. Symmetric leaves, which correspond to rays landing on the real axis, are displayed thicker.

Independently of the MLC conjecture, the set $\mathcal{R}$ can be described combinatorially as the set of endpoints of symmetric leaves of the quadratic minor lamination, and it conjecturally coincides with the set of external angles of rays landing on the real slice of $\mathcal{M}$. The set $\mathcal{R}$ will be called the *combinatorial bifurcation set* for the real quadratic family.

Continued fractions. We now turn to the construction of another bifurcation set, related to the dynamics of continued fractions. Let $r \in \mathbb{Q} \cap (0, 1)$ be a rational number; then r admits precisely two continued fraction expansions,

$$r = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{(a_n - 1) + \cfrac{1}{1}}}}}$$

with $a_n \geq 2$, which we will denote as $r = [a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1]$. Given a rational number $r \in \mathbb{Q} \cap (0, 1)$, let us define the *quadratic interval* I_r to be the interval whose endpoints are the quadratic irrationals with periodic continued fraction expansions

$$I_r := ([\overline{a_1, a_2, \ldots, a_n}], [\overline{a_1, a_2, \ldots, a_n - 1, 1}]).$$

Moreover, we define $I_1 := ([\overline{1}], 1]$. Let us define the *exceptional set* $\mathcal{E}$ as the complement of all quadratic intervals:

$$\mathcal{E} := [0, 1] \setminus \bigcup_{r \in \mathbb{Q} \cap (0,1]} I_r.$$

We shall see (section 13) that the set $\mathcal{E}$ appears in a few different dynamical contexts: namely, as the bifurcation set for numbers of generalized bounded type, for α-continued fraction transformations, and as the recurrence spectrum of Sturmian sequences.

Results. The first remarkable fact is that the sets $\mathcal{R}$ and $\mathcal{E}$ are essentially homeomorphic:

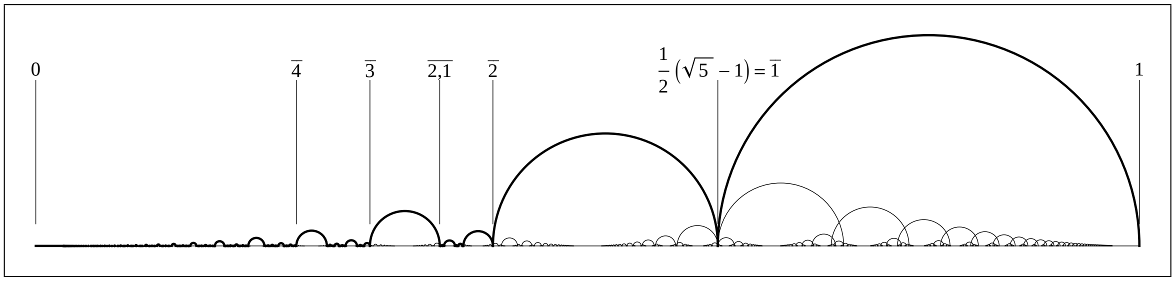


FIGURE 2. The exceptional set $\mathcal{E}$: each half-circle has a quadratic interval I_r as its diameter, so $\mathcal{E}$ is the intersection of the real line with the complement of all half-circles.

Theorem 1.1. *The homeomorphism* $\varphi : [0,1] \to [\frac{1}{4}, \frac{1}{2}]$ *given by*

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \mapsto \varphi(x) = 0.0\underbrace{11\ldots1}_{a_1}\underbrace{00\ldots0}_{a_2}\underbrace{11\ldots1}_{a_3}\ldots$$

maps $\mathcal{E}$ *bijectively onto* $\mathcal{R} \cap (0, \frac{1}{2}]$.

The map φ is a variant of Minkowski's *question mark* function (see section 14). The proof of Theorem 1.1 will be given in section 14; in the following, we shall elaborate on various features of the correspondence.

1.1. **Topology of the bifurcation sets.** The sets $\mathcal{R}$ and $\mathcal{E}$ are both compact and totally disconnected, and their homeomorphism type is easily described. Indeed, let C be the usual Cantor set in the unit interval, and let us add to each connected component U of the complement of C a countable sequence of isolated points which accumulate on the left endpoint of U: the resulting space is homeomorphic, as an embedded subset of the interval, to the sets $\mathcal{R} \cap \left(0, \frac{1}{2}\right]$ and $\mathcal{E}$.

Moreover, the combinatorial bifurcation sets can be generated via a bisection algorithm. Given an interval $I \subseteq \mathbb{R}$ of length smaller than 1, we shall call the *(rational) pseudocenter* of I the (unique!) rational number in I with least denominator.

Proposition 1.2. *Given $x, y \in \mathcal{E}$, let r be the pseudocenter of the interval (x, y). Then the quadratic interval I_r is a connected component of the complement of $\mathcal{E}$.*

As a corollary, the endpoints of $I_r = (\alpha^-, \alpha^+)$ lie in $\mathcal{E}$, so we can proceed inductively taking the pseudocenters of (x, α^-), (α^+, y) and generate new maximal quadratic intervals. The iteration of this procedure generates all of $[0, 1] \setminus \mathcal{E}$.

Similarly, the *dyadic pseudocenter* of an interval J of length smaller than 1 is defined as the unique dyadic number $\theta^* = \frac{p}{2^q}$ with shortest binary expansion (i.e. with the smallest q) among all dyadics in J. Recall moreover that a *hyperbolic component* $W \subseteq \mathcal{M}$ is a connected component of the open set of parameters c for which the critical point of f_c is attracted to a periodic cycle. If W intersects the real axis, we define the *hyperbolic window* associated to W to be the interval $(\theta_2, \theta_1) \subseteq [0, 1/2]$, where the rays $R_M(\theta_1)$ and $R_M(\theta_2)$ land on $\partial W \cap \mathbb{R}$.

By translating Proposition 1.2 to the world of kneading sequences, we get the following bisection algorithm (section 14.2) to generate all real hyperbolic windows (Figure 3).

Theorem 1.3. *The set of all real hyperbolic windows in the Mandelbrot set can be generated as follows. Let $c_1 < c_2$ be two real parameters on the boundary of $\mathcal{M}$, with external angles $0 \leq \theta_2 < \theta_1 \leq \frac{1}{2}$. Let θ^* be the dyadic pseudocenter of the interval (θ_2, θ_1), and let*

$$\theta^* = 0.s_1 s_2 \ldots s_{n-1} s_n$$

be its binary expansion, with $s_n = 1$. *Then the hyperbolic window of smallest period in the interval* (θ_2, θ_1) *is the interval of external angles* (α_2, α_1) *with*

$$\begin{aligned} \alpha_2 &:= 0.\overline{s_1 s_2 \dots s_{n-1}} \\ \alpha_1 &:= 0.\overline{s_1 s_2 \dots s_{n-1} \check{s}_1 \check{s}_2 \dots \check{s}_{n-1}} \end{aligned}$$

(where $\check{s}_i := 1 - s_i$*). All hyperbolic windows are obtained by iteration of this algorithm, starting with* $\theta_2 = 0$, $\theta_1 = 1/2$.

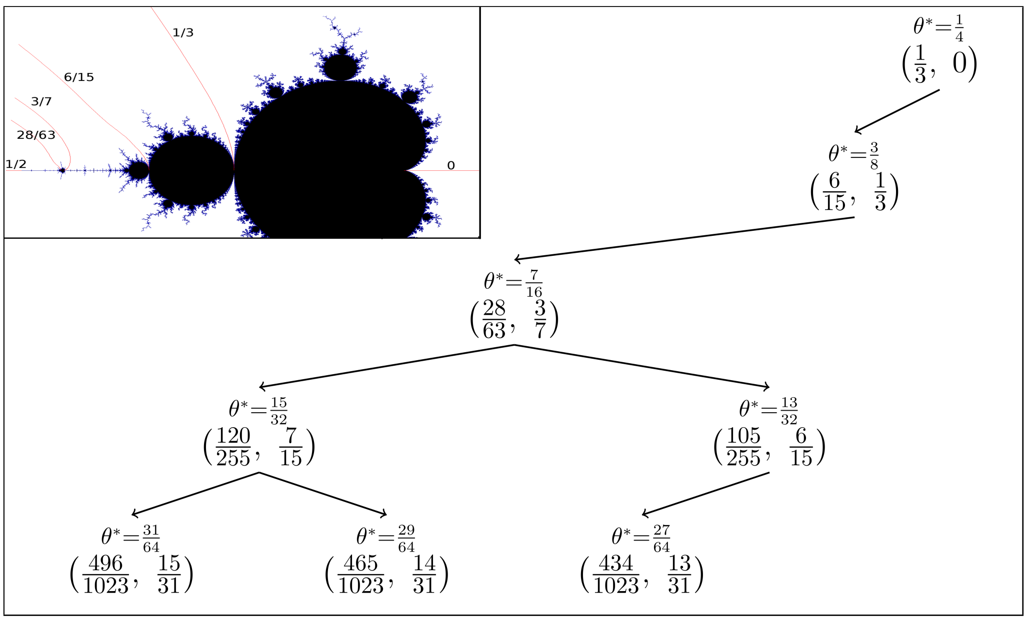


FIGURE 3. The first few generations of the bisection algorithm which produces all real hyperbolic windows between external angles 0 and $\frac{1}{2}$. Every interval represents a hyperbolic component, and we display the angles of rays landing at the endpoints as well as the pseudocenter θ^*. The root of the tree ($\theta^* = \frac{1}{4}$) corresponds to the real slice of the main cardioid, its child is the "basilica" component of period 2 ($\theta^* = \frac{3}{8}$), then $\theta^* = \frac{7}{16}$ corresponds to the "airplane" component of period 3 etc. Some branches of the tree do not appear because some pairs of components have an endpoint in common (due to period doubling).

1.2. **Real Julia sets and numbers of bounded type.** We shall now see that the correspondence between parameter spaces of Theorem 1.1 has an analogue in the dynamical plane. Recall that for each quadratic polynomial f_c, the exterior the Julia set $J(f_c)$ is uniformized by the Riemann mapping Φ_c, and each θ gives rise to a ray $R_c(\theta)$ which lands on the boundary of the Julia set. Let S_c be the set of external rays which land on the real slice of the Julia set:

$$S_c := \{\theta \in \mathbb{R}/\mathbb{Z} : R_c(\theta) \text{ lands on } J(f_c) \cap \mathbb{R}\}.$$

Inside the real slice of the Julia set lies the *Hubbard tree* T_c, which in the real case is just the segment $[c, f_c(c)]$. Let H_c be the set of external rays which land on T_c.

On the other hand, for each positive integer N, the set of *numbers of bounded type* $\mathcal{B}_N$ is the set of numbers in the unit interval with all continued fraction coefficients bounded by N:

$$\mathcal{B}_N := \{x = [a_1, a_2, \ldots] \ : \ 1 \leq a_i \leq N \quad \forall i \geq 1\}.$$

We can interpolate between the discrete family $\mathcal{B}_N$ by defining, for each $t > 0$, the set of *numbers of type bounded by* t as the set

$$\mathcal{B}(t) := \{x \in [0, 1] \ : G^n(x) \geq t \quad \forall n \geq 0\}$$

where $G(x) := \frac{1}{x} - \left[\frac{1}{x}\right]$ is the Gauss map. Note that $\mathcal{B}_N = \mathcal{B}(\frac{1}{N+1})$. We prove the following correspondence (section 14.4):

Theorem 1.4. *Let t belong to the set $\mathcal{E}$, and c be the parameter on $\partial\mathcal{M} \cap \mathbb{R}$ where the external ray of angle $\varphi(t)$ lands. Then the set of numbers of type bounded by t is sent via the homeomorphism φ to the set of external angles of rays landing on $J(f_c) \cap [c, f_c(c)]$. More precisely, we have the identity*

$$2\varphi(\mathcal{B}(t)) = H_c \cap [1/2, 1].$$

As an example, the set of numbers with all partial quotients bounded by 2 is mapped via φ to the set of external angles landing on the Hubbard tree in the Julia set of the "airplane" (the real polynomial f_c with a superattracting cycle of period 3).

1.3. **Metric properties.** In terms of measure theory, we shall see that the bifurcation sets have zero measure, but their dimension is large (section 14.3).

Theorem 1.5. *The combinatorial bifurcation sets $\mathcal{R}$ and $\mathcal{E}$ have zero Lebesgue measure, but full Hausdorff dimension:*

$$\text{Leb } \mathcal{E} = \text{Leb } \mathcal{R} = 0$$

but

$$\text{H.dim } \mathcal{E} = \text{H.dim } \mathcal{R} = 1.$$

We shall also see that the dimension of the set $\mathcal{R}$ is very unevenly distributed, since most points are concentrated near the tip of $\mathcal{M}$; to make this precise, we shall now compare the local dimension of $\mathcal{R}$ near a given parameter to the dimension of the corresponding object in the dynamical plane for that parameter.

Given a parameter $c \in [-2, 1/4]$, one can consider the set of external angles which land on the real slice of the Mandelbrot set, to the right of the chosen parameter c. A combinatorial model for it is the set

$$P_c := \{\theta \in S^1 : \text{the impression of } R_M(\theta) \text{ intersects } \partial\mathcal{M} \cap [c, 1/4]\}.$$

Note that $P_c = \mathcal{R} \cap [-\theta_c, \theta_c]$, where $\theta_c \in [0, 1/2]$ is a characteristic angle of c (see section 3). The Hausdorff dimension of P_c is a decreasing function of c, and takes values between 0 (e.g. at the cusp $c = 1/4$) and 1 (at the tip $c = -2$).

On the other hand, each real quadratic polynomial f_c has a well-defined *topological entropy* $h_{top}(f_c \mid_{\mathbb{R}})$ as a map of a real interval. The function $h_{top}(f_c \mid_{\mathbb{R}})$ is continuous and decreasing in c [MT]. We shall prove the following identity (section 8):

Theorem 1.6. *Let* $c \in [-2, 1/4]$. *Then we have*

$$\frac{h_{top}(f_c\mid_{\mathbb{R}})}{\log 2} = \text{H.dim } S_c = \text{H.dim } P_c.$$

The first equality establishes a relation between entropy, Hausdorff dimension and the Lyapunov exponent of the doubling map (in the spirit of the "entropy formulas" [Ma], [Yo], [LeYo]), while the second one quantifies the fact that the local geometry of the Mandelbrot set near the parameter c reflects the geometry of the corresponding Julia set $J(f_c)$. The analogous result for continued fractions is the following:

Theorem 1.7 ([CT2], Theorem 2)**.** *For each* $t > 0$*, we have the identity*

$$\text{H.dim } \mathcal{B}(t) = \text{H.dim } \mathcal{E} \cap [t, 1].$$

It is worth noting that the dimension of the set S_c also equals the dimension of the set H_c of angles of rays landing on the Hubbard tree, as well as the dimension of the set B_c of *biaccessible angles* (see section 6).

1.4. **Entropy along veins of the Mandelbrot set.** If the parameter c is not real, then the real axis is not preserved by f_c, but the *Hubbard tree* T_c is naturally forward-invariant (see section 4), so one can look at the topological entropy $h_{top}(f_c\mid_{T_c})$ of the restriction of the map to the Hubbard tree.

On the other hand, the appropriate generalization of the real axis is a *vein*, that is an arc v embedded in $\mathcal{M}$. Given a parameter $c \in \partial\mathcal{M}$ which can be connected by a vein v to the center of the main cardioid, we define the set P_c to be the set of external angles of rays which land on the vein v closer than c to the main cardioid.

In the $\frac{p}{q}$-limb, there is a unique parameter $c_{p/q}$ such that the critical point lands on the β fixed point after q iterates (i.e. $f^q(0) = \beta$). The vein $v_{p/q}$ joining $c_{p/q}$ to the center of the main cardioid will be called the *principal vein* of angle p/q. For all parameters c along the vein $v_{p/q}$, the Hubbard tree of f_c is a star with q prongs (Proposition 15.3).

It is easy to check that $v_{1/2}$ is the real axis. Existence of principal veins has been shown by Branner-Douady [BD] via quasiconformal surgery. We shall extend (by using a combinatorial version of the Branner-Douady surgery) the previous equality to principal veins (section 15):

Theorem 1.8. *Let $v = v_{p/q}$ be a principal vein in the Mandelbrot set, and $c \in \partial\mathcal{M}$ a parameter on the vein v. Then we have the formula*

$$\frac{h_{top}(f_c\mid_{T_c})}{\log 2} = \text{H.dim } H_c = \text{H.dim } P_c.$$

The result is a first step towards understanding the relationship between the entropy of Hubbard trees and the combinatorics and geometry of parameter spaces, following the program recently laid out by W. Thurston (see section 18.1). We conjecture that the same equality holds along every vein.

1.5. **Application to α-continued fractions.** Let $\alpha \in [0,1]$. Then the α-continued fraction transformation T_α is a discontinuous map defined on the interval $I_\alpha := [\alpha - 1, \alpha]$ as

$$T_\alpha(x) := \frac{1}{|x|} - c_{\alpha,x} \qquad c_{\alpha,x} := \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor.$$

The family T_α interpolates between well-known maps which generate different types of continued fraction expansion. Indeed $T_1(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ is the usual Gauss map, while T_0 generates the *by-excess continued fraction* expansion, and $T_{1/2}$ generates the *nearest integer continued fraction.*

The parameter α is said to satisfy a *matching condition* when the orbits of the two endpoints of I_α collide after a finite number of steps, i.e. if there exist integers M, N such that $T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 1)$. It turns out that such a condition is stable under perturbation of the parameter, and infinitely many combinatorial types (N, M) appear.

Even more remarkably, the complement of all stable regions is precisely the exceptional set $\mathcal{E}$ [CT]. As a consequence, intervals in the parameter space of α-continued fractions where a matching between the orbits of the endpoints occurs are in one-to-one correspondence with real hyperbolic components of the Mandelbrot set (section 20.2):

Theorem 1.9. *Let $\alpha \in [0, 1]$, and $\theta := \varphi(\alpha)$. Then the α-continued fraction transformation T_α satisfies the matching condition if and only if the external ray $R_M(\theta)$ lies inside a real hyperbolic window.*

The result also has consequences in terms of entropy: indeed, one can consider for each parameter α the measure-theoretic entropy $h(\alpha)$ of the invariant measure for T_α in the Lebesgue class, and study the function $\alpha \mapsto h(\alpha)$. It follows from Theorem 1.9 that intervals over which the entropy of α-continued fractions is monotone are mapped to parameter intervals in the space of quadratic polynomials where the topological entropy is constant (see Figure 4). For instance, the matching interval $([0; \overline{3}], [0; \overline{2, 1}])$, identified in [LM] and [NN], corresponds to the "airplane component" of period 3 in the Mandelbrot set.

1.6. **Tuning.** In the family of quadratic polynomials, Douady and Hubbard [DH] described the small copies of the Mandelbrot set which appear inside $\mathcal{M}$ as images of *tuning operators*. Pushing further the correspondence between quadratic polynomials and continued fractions, we define tuning operators acting on the parameter space of α-continued fractions; we then use them to explain in further detail the fractal structure of the entropy function and characterize its plateaux, i.e. the intervals where it is constant.

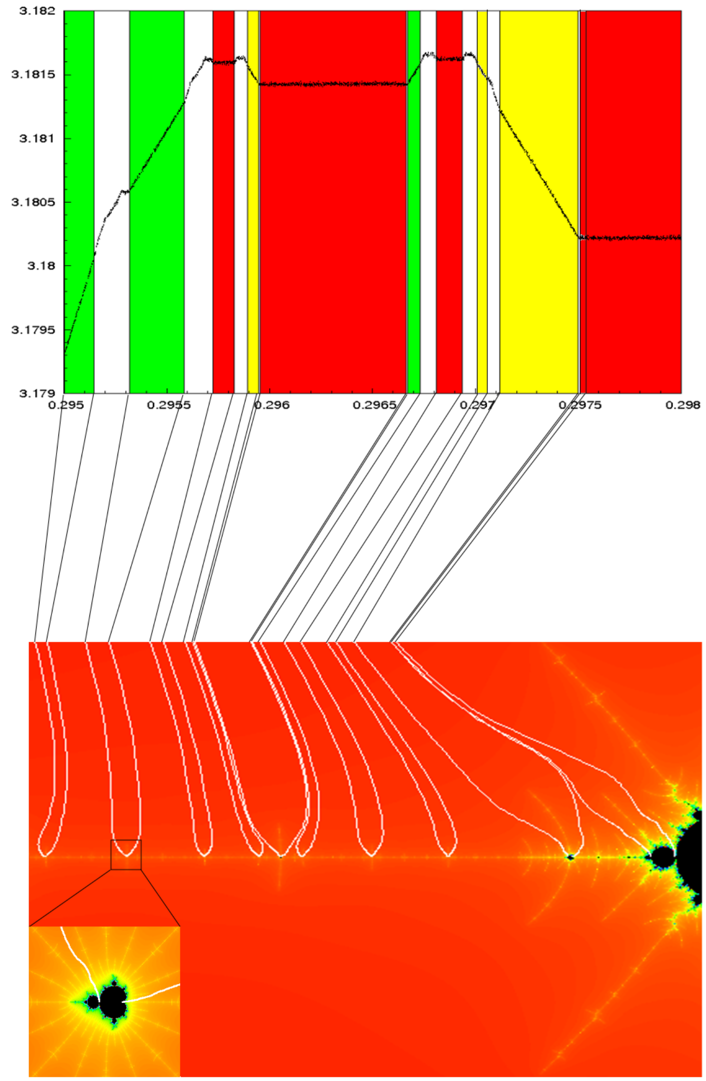


FIGURE 4. Correspondence between the parameter space of α-continued fraction transformations and the Mandelbrot set. On the top: the entropy of α-continued fraction as a function of α, from [LM]; the strips correspond to *matching intervals*. At the bottom: a section of the Mandelbrot set along the real line, with external rays landing on the real axis. The hyperbolic component at the very right hand side of the picture has period 6 and is the doubling of the airplane component. Matching intervals on the top figure correspond to hyperbolic components on the bottom via the map φ. For instance, the maximal quadratic interval $I_r = ([\overline{3,2,1,1,2}, \overline{3,2,1,1,1,1}])$ for $r = \frac{13}{44}$ maps to the cardioid of the small Mandelbrot set in the zoomed box, which has period 9; the angle of the displayed ray which lands on its cusp has binary expansion $\theta = 0.\overline{011100101}$.

Our construction is the following: we associate, to each rational number r indexing a maximal interval, a *tuning map* τ_r from the whole parameter space of α-continued fraction transformations to a subset W_r, called *tuning window*. Note that τ_r also maps the bifurcation set $\mathcal{E}$ into itself. A tuning window W_r is called *neutral* if the alternating sum of the partial quotients of r is zero.

Theorem 1.10. *The function h is constant on every neutral tuning window W_r, and every plateau of h is the interior of some neutral tuning window W_r.*

Even more precisely, we will characterize the set of rational numbers r such that the interior of W_r is a plateau (section 23.4). On non-neutral tuning windows, instead, the entropy function is non-constant and h reproduces, on a smaller scale, its behavior on the whole parameter space $[0, 1]$. To make this precise, we define the *monotonicity* $M(f, I)$ of the monotone function f on the interval I to be $+1, -1$ or 0 according to whether f is increasing, decreasing, or constant on I. We can now formulate the following product formula (section 22):

Theorem 1.11. *Let $h : [0, 1] \to \mathbb{R}$ be the measure-theoretic entropy of α-continued fractions, and I_r, I_p two maximal quadratic intervals. Then the monotonicity of h on the tuned interval $I_{\tau_r(p)}$ is given by*

$$M(h, I_{\tau_r(p)}) = -M(h, I_r) \cdot M(h, I_p).$$

As a consequence, we can also completely classify the local monotonic behavior of the entropy function $\alpha \mapsto h(\alpha)$. A corollary is that the entropy function is non-monotone in a very strong sense: indeed, the set of parameters α such that the entropy is not locally monotone at α has Hausdorff dimension 1.

Structure of the thesis. The work is organized as follows. We first provide (sections 2-6) background material on external rays and laminations and discuss the topology of Hubbard trees, in order to to prepare for the proof of the first equality of Theorems 1.6 and 1.8, namely Theorem 7.1 in section 7. In section 8 we introduce the discussion of entropy in the real quadratic family in more detail, and present the strategy of proof of Theorem 1.6. Then (section 10) we introduce the combinatorial coding and prove the second part of Theorem 1.6 in section 12. In section 13 we construct the exceptional set for continued fractions, and we discuss the main correspondence in section 14, thus proving Theorems 1.1, 1.3, 1.4 and 1.5. Then we turn to the complex case, analyzing the principal vein and the combinatorial surgery map (sections 15-17), giving the proof of the second part of Theorem 1.8 at the end of section 17. Section 18 presents open questions and pictures on the entropy of Hubbard trees. The last part (sections 19-24) introduces the theory of α-continued fractions, leading up to the construction of tuning operators and the classification of the plateaux for the entropy, hence to the proofs of Theorems 1.9, 1.10 and 1.11.

Some topics in the thesis have appeared in articles and preprints of the author, to which we sometimes refer for further details. In particular, part of sections 13 and 14 appear in [CT], [BCIT] and [CT2], while sections 19-24 are part of [CT3].

2. External rays

Let $f(z)$ be a monic polynomial of degree d. Recall that the *filled Julia set* $K(f)$ is the set of points which do not escape to infinity under iteration:

$$K(f) := \{z \in \mathbb{C} \ : \ f^n(z) \text{ does not tend to } \infty \text{ as } n \to \infty\}.$$

The *Julia set* $J(f)$ is the boundary of $K(f)$. If $K(f)$ is connected, then the complement of $K(f)$ in the Riemann sphere is simply connected, so it can be uniformized by the Riemann mapping $\Phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K(f)$ which maps the exterior of the closed unit disk $\overline{\mathbb{D}}$ to the exterior of $K(f)$. The Riemann mapping is unique once we impose $\Phi(\infty) = \infty$ and $\Phi'(\infty) = 1$. With this choice, Φ conjugates the action of f on the exterior of the filled Julia set to the map $z \mapsto z^d$, i.e.

$$f(\Phi(z)) = \Phi(z^d). \tag{1}$$

By Carathéodory's theorem (see e.g. [Po]), the Riemann mapping extends to a continuous map $\overline{\Phi}$ on the boundary $\overline{\Phi} : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus \text{int } K(f)$ if and only if the Julia set is locally connected. If this is the case, the restriction of $\overline{\Phi}$ to the boundary is sometimes called the *Carathéodory loop* and it will be denoted as

$$\gamma : \mathbb{R}/\mathbb{Z} \to J(f).$$

As a consequence of the eq. (1), the action of f on the set of angles is semiconjugate to multiplication by d (mod 1):

$$\gamma(d \cdot \theta) = f(\gamma(\theta)) \qquad \text{for each } \theta \in \mathbb{R}/\mathbb{Z}. \tag{2}$$

In the following we will only deal with the case of quadratic polynomials of the form $f_c(z) := z^2 + c$, so $d = 2$ and we will denote as

$$D(\theta) := 2 \cdot \theta \mod 1$$

the doubling map of the circle. Moreover, we will add the subscript c when we need to make the dependence on the polynomial f_c more explicit. Given $\theta \in \mathbb{R}/\mathbb{Z}$, the *external ray* $R_c(\theta)$ is the image of the radial arc at angle $2\pi\theta$ via the Riemann mapping $\Phi_c : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K(f_c)$:

$$R_c(\theta) := \{\Phi_c(\rho e^{2\pi i\theta})\}_{\rho>1}.$$

The ray $R_c(\theta)$ is said to *land* at x if

$$\lim_{\rho\to 1^+} \Phi_c(\rho e^{2\pi i\theta}) = x.$$

If the Julia set is locally connected, then all rays land; in general, by Fatou's theorem, the set of angles for which $R_c(\theta)$ does not land has zero Lebesgue measure, and indeed it also has zero capacity and hence zero Hausdorff dimension (see e.g. [Po]). It is however known that there exist non-locally connected Julia sets for polynomials [Mi2]. The ray $R_c(0)$ always lands on a fixed point of f_c which is traditionally called the β fixed point and denoted as β. The other fixed point of f_c is called the α fixed point. Note that in the case $c = \frac{1}{4}$ one has $\alpha = \beta$. Finally, the critical point of f_c will be denoted by 0, and the critical value by c.

Analogously to the Julia sets, the exterior of the Mandelbrot set can be uniformized by the Riemann mapping

$$\Phi_M : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

with $\Phi_M(\infty) = \infty$, and $\Phi'(\infty) = 1$, and images of radial arcs are called *external rays*. Every angle $\theta \in \mathbb{R}/\mathbb{Z}$ determines an external ray

$$R_M(\theta) := \Phi_M(\{\rho e^{2\pi i\theta} : \rho > 1\})$$

which is said to *land* at x if the limit $\lim_{\rho\to 1^+} \Phi_M(\rho e^{2\pi i\theta})$ exists. According to the *MLC conjecture* [DH], the Mandelbrot set is locally connected, and therefore all rays land on some point of the boundary of $\mathcal{M}$.

2.1. **Biaccessibility and regulated arcs.** A point $z \in J(f_c)$ is called *accessible* if it is the landing point of at least one external ray. It is called *biaccessible* if it is the landing point of at least two rays, i.e. there exist θ_1, θ_2 two distinct angles such that $R_c(\theta_1)$ and $R_c(\theta_2)$ both land at z. This is equivalent to say that $J(f_c) \setminus \{z\}$ is disconnected.

Let $K = K(f_c)$ be the filled Julia set of f_c. Assume K is connected and locally connected. Then it is also path-connected (see e.g. [Wi], Chapter 8), so given any two points x, y in K, there exists an arc in K with endpoints x, y.

If K has no interior, then the arc is uniquely determined by its endpoints x, y. Let us now describe how to choose a canonical representative inside the Fatou components in the case K has interior. In this case, each bounded Fatou component eventually maps to a periodic Fatou component, which either contains an attracting cycle, or it contains a parabolic cycle on its boundary, or it is a periodic Siegel disk.

Since we will not deal with the Siegel disk case in the rest of the thesis, let us assume we are in one of the first two cases. Then there exists a Fatou component U_0 which contains the critical point, and a biholomorphism $\phi_0 : U_0 \to \mathbb{D}$ to the unit disk mapping the critical point to 0. The preimages $\phi_0^{-1}(\{\rho e^{2\pi i\theta} \ : \ 0 \leq \rho < 1\})$ of radial arcs in the unit disk are called *radial arcs* in U_0. Any other bounded Fatou component U is eventually mapped to U_0; let $k \geq 0$ be the smallest integer such that $f_c^k(U) = U_0$. Then the map $\phi := \phi_0 \circ f_c^k$ is a biholomorphism of U onto the unit disk, and we define radial arcs to be preimages under ϕ of radial arcs in the unit disk.

An embedded arc I in K is called *regulated* (or *legal* in Douady's terminology [Do2]) if the intersection between I and the closure of any bounded Fatou component is contained in the union of at most two radial arcs. With this choice, given any two points x, y in K, there exists a unique regulated arc in K with endpoints x, y ([Za1], Lemma 1). Such an arc will be denoted by $[x, y]$, and the corresponding open arc by $(x, y) := [x, y] \setminus \{x, y\}$. A *regulated tree* inside K is a finite tree whose edges are regulated arcs. Note that, in the case K has non-empty interior, regulated trees as defined need not be invariant for the dynamics, because f_c need not map radial arcs to radial arcs. However, by construction, radial arcs in any bounded Fatou component U different from U_0 map to radial arcs in $f_c(U)$. In order to deal with U_0, we need one further hypothesis. Namely, we will assume that f_c has an attracting or parabolic cycle of period p with real multiplier. Then we can find a parametrization $\phi_0 : U_0 \to \mathbb{D}$ such that the interval $I := \phi_0^{-1}((-1, 1))$ is preserved by the p-th iterate of f_c, i.e. $f_c^p(I) \subseteq I$. The interval I will be called the *bisector* of U_0. Now note that, if the regulated arc $[x, y]$ does not contain 0 in its interior and it only intersects the critical Fatou component U_0 in its bisector, then we have

$$f_c([x, y]) = [f_c(x), f_c(y)].$$

The *spine* of f_c is the regulated arc $[-\beta, \beta]$ joining the β fixed point to its preimage $-\beta$. The biaccessible points are related to the points which lie on the spine by the following lemma.

Lemma 2.1. *Let $f_c(z) = z^2 + c$ be a quadratic polynomial whose Julia set is connected and locally connected. Then the set of biaccessible points is*

$$\mathcal{B} = J(f_c) \cap \bigcup_{n \geq 0} f_c^{-n}((-\beta, \beta)).$$

Proof. Let $f = f_c$, and $x \in J(f) \cap (-\beta, \beta)$. The set $V := R_c(0) \cup [-\beta, \beta] \cup R_c(1/2)$ disconnects the plane in two parts, $\mathbb{C} \setminus V = A_1 \cup A_2$. We claim that x is the limit of points in the basin of infinity U_∞ on both sides of V, i.e. for each $i = 1, 2$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq A_i \cap U_\infty$ with $x_n \to x$; since the Riemann mapping Φ

extends continuously to the boundary, this is enough to prove that there exist two external angles $\theta_1 \in (0, 1/2)$ and $\theta_2 \in (1/2, 1)$ such that $R_c(\theta_1)$ and $R_c(\theta_2)$ both land on x. Let us now prove the claim; if it is not true, then there exists an open neighborhood Ω of x and an index $i \in \{1, 2\}$ such that $\Omega \cap A_i$ is connected and contained in the interior of the filled Julia set $K(f)$, hence $\Omega \cap A_i$ is contained in some bounded Fatou component. This implies that $\Omega \cap V$ lies in the closure of a bounded Fatou component, and x on its boundary. However, this contradicts the definition of regulated arc, because if U is a bounded Fatou component intersecting a regulated arc I, then $\partial U \cap I$ does not disconnect $\overline{U} \cap I$. Suppose now that $x \in J(f)$ is such that $f^n(x)$ belongs to $(-\beta, \beta)$ for some n. Then by the previous argument $f^n(x)$ is biaccessible, and since f is a local homeomorphism outside the spine, x is also biaccessible.

Conversely, suppose x is biaccessible, and the two rays at angles θ_1 and θ_2 land on x, with $0 < \theta_1 < \theta_2 < 1$. Then there exists some n for which $1/2 \leq D^n(\theta_2) - D^n(\theta_1) < 1$, hence $R_c(D^n(\theta_1))$ and $R_c(D^n(\theta_2))$ must lie on opposite sides with respect to the spine, and since they both land on $f^n(x)$, then $f^n(x)$ belongs to the spine. Since the point β is not biaccessible ([Mc], Theorem 6.10), $f^n(x)$ must belong to $(-\beta, \beta)$. □

Lemma 2.2. *We have that* $\alpha \in [0, c]$.

Proof. Indeed, since $\alpha \in (-\beta, 0)$ ([Za1], Lemma 5), we have $-\alpha \in (\beta, 0)$ and $\alpha = f(-\alpha) \in (\beta, c)$. Thus, since $0 \in (\alpha, \beta)$ we have $\alpha \in (0, c)$. □

Lemma 2.3. *For* $x \in [0, \beta)$, *we have* $x \in (f(x), \beta)$.

Proof. Let us consider the set $S = \{x \in [0, \beta] \ : \ x \in (\beta, f(x))\}$. The set is open by continuity of f. Since the β fixed point is repelling, the set S contains points in a neighborhood of β, so it is not empty. Suppose $S \neq [0, \beta)$ and let $x \in \partial S$, $x \neq \beta$. By continuity of f, x must be a fixed point of f, but the only fixed point of f in the arc is β. □

For more general properties of biaccessibility we refer to [Za1].

3. Laminations

A powerful tool to construct topological models of Julia sets and the Mandelbrot set is given by laminations, following Thurston's approach. As we will see, laminations represent equivalence relations on the boundary of the disk arising from external rays which land on the same point. We now give the basic definitions, and refer to [Th1] for further details.

A *geodesic lamination* λ is a set of hyperbolic geodesics in the closed unit disk $\overline{\mathbb{D}}$, called the *leaves* of λ, such that no two leaves intersect in $\mathbb{D}$, and the union of all leaves is closed.

A *gap* of a lamination λ is the closure of a component of the complement of the union of all leaves. In order to represent Julia sets of quadratic polynomials, we need to restrict ourselves to *invariant laminations*.

Let $d \geq 2$. The map $g(z) := z^d$ acts on the boundary of the unit disk, hence it induces a dynamics on the set of leaves. Namely, the image of a leaf $\overline{pq}$ is defined as the leaf joining the images of the endpoint: $g(\overline{pq}) = \overline{g(p)g(q)}$. A lamination λ is *forward invariant* if the image of any leaf L of λ still belongs to λ. Note that the image leaf may be *degenerate*, i.e. consist of a single point on the boundary of the disk.

A lamination is *invariant* if in addition to being forward invariant it satisfies the additional conditions:

- *Backward invariance*: if $\overline{pq}$ is in λ, then there exists a collection of d disjoint leaves in λ, each joining a preimage of p to a preimage of q.
- *Gap invariance*: for any gap G, the hyperbolic convex hull of the image of $G_0 = \overline{G} \cap S^1$ is either a gap, a leaf, or a single point.

In this thesis we will only deal with quadratic polynomials, so $d = 2$ and the invariant laminations for the map $g(z) = z^2$ will be called *invariant quadratic laminations*. A leaf of maximal length in a lamination is called a *major leaf*, and its image a *minor leaf*. Typically, a quadratic invariant lamination has 2 major leaves, but the minor leaf is always unique.

If $J(f_c)$ is a Julia set of a quadratic polynomial, one can define the equivalence relation $\sim_c$ on the unit circle $\partial\mathbb{D}$ by saying that $\theta_1 \sim_c \theta_2$ if the rays $R_c(\theta_1)$ and $R_c(\theta_2)$ land on the same point.

From the equivalence relation $\sim_c$ one can construct a quadratic invariant lamination in the following way. Let E be an equivalence class for $\sim_c$. If $E = \{\theta_1, \theta_2\}$ contains two elements, then we define the leaf L_E as $L_E := (\theta_1, \theta_2)$. If $E = \{\theta\}$ is a singleton, then we define L_E to be the degenerate leaf $L_E := \{\theta\}$. Finally, if $E = \{\theta_1, \dots, \theta_k\}$ contains more than two elements, with $0 \le \theta_1 < \theta_2 < \dots < \theta_k < 1$, then we define L_E to be the union of the leaves $L_E := (\theta_1, \theta_2) \cup (\theta_2, \theta_3) \cup \dots \cup (\theta_k, \theta_1)$. Finally, we let the associated lamination λ_c be

$$\lambda_c := \bigcup_{E \text{ equiv. class of } \sim_c} L_E.$$

The lamination λ_c is an invariant quadratic lamination. The equivalence relation $\sim_c$ can be extended to a relation $\cong_c$ on the closed disk $\overline{\mathbb{D}}$ by taking convex hulls, and the quotient of the disk by $\cong_c$ is a model for the Julia set:

Theorem 3.1 ([Do2]). *If the Julia set $J(f_c)$ is connected and locally connected, then it is homeomorphic to the quotient of $\overline{\mathbb{D}}$ by the equivalence relation $\cong_c$.*

We define the the *characteristic leaf* of a quadratic polynomial f_c with Julia set connected and locally connected to be the minor leaf of the invariant lamination λ_c. The endpoints of the characteristic leaf are called *characteristic angles*.

3.1. **The abstract Mandelbrot set.** In order to construct a model for the Mandelbrot set, Thurston [Th1] defined the *quadratic minor lamination* QML as the union of the minor leaves of all quadratic invariant laminations (see Figure 1).

As in the Julia set case, the lamination determines an equivalence relation $\cong_M$ on $\overline{\mathbb{D}}$ by identifying points on the same leaf, and also points in the interior of finite ideal polygons whose sides are leaves. The quotient

$$\mathcal{M}_{abs} := \overline{\mathbb{D}} / \cong_M$$

is called *abstract Mandelbrot set.* It is a compact, connected and locally connected space. Douady [Do2] constructed a continuous surjection

$$\pi_M : \mathcal{M} \to \mathcal{M}_{abs}$$

which is injective if and only if $\mathcal{M}$ is locally connected.

The idea behind the construction is that leaves of QML connect external angles whose corresponding rays in parameter space land on the same point. However, since we do not know whether $\mathcal{M}$ is locally connected, additional care is required. Indeed, let $\sim_M$ denote the equivalence relation on $\partial\mathbb{D}$ induced by the lamination QML, and $\theta_1 \asymp_M \theta_2$ denote that the external rays $R_M(\theta_1)$ and $R_M(\theta_2)$ land on the same point. The following theorem summarizes a few key results comparing the analytic and combinatorial models of the Mandelbrot set:

Theorem 3.2. *Let $\theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}$ be two angles. Then the following are true:*

(1) if $\theta_1 \asymp_M \theta_2$, then $\theta_1 \sim_M \theta_2$;

(2) if $\theta_1 \sim_M \theta_2$ and θ_1, θ_2 are rational, then $\theta_1 \asymp_M \theta_2$;

(3) if $\theta_1 \sim_M \theta_2$ and θ_1, θ_2 are not infinitely renormalizable, then $\theta_1 \asymp_M \theta_2$.

Proof. (1) and (2) are contained in ([Th1], Theorem A.3). (3) follows from Yoccoz's theorem on landing of rays at finitely renormalizable parameters (see [Hu] for the proof). Indeed, Yoccoz proves that external rays $R_M(\theta)$ with non-infinitely renormalizable combinatorics land, and moreover that the intersections of nested parapuzzle pieces contain a single point. Along the boundary of each puzzle piece lie pairs of external rays with rational angles (see also [Hu], sections 5 and 12) which land on the same point, and since the intersection of the nested sequence of puzzle pieces is a single point $c \in \partial\mathcal{M}$, the rays θ_1 and θ_2 land on the same point c. $\square$

The following criterion makes it possible to check whether a leaf belongs to the quadratic minor lamination by looking at its dynamics under the doubling map:

Proposition 3.3 ([Th1]). *A leaf m is the minor leaf of some invariant quadratic lamination (i.e. it belongs to QML) if and only if the following three conditions are met:*

(a) all forward images of m have disjoint interiors;

(b) the length of any forward image of m is never less than the length of m;

(c) if m is a non-degenerate leaf, then m and all leaves on the forward orbit of m are disjoint from the interiors of the two preimage leaves of m of length at least $1/3$.

For the rest of the thesis we shall work with the abstract, locally connected model of $\mathcal{M}$ and study its dimension using combinatorial techniques; only at the very end (Proposition 17.12) we shall compare the analytical and combinatorial models and prove that our results hold for the actual Mandelbrot set even without assuming the MLC conjecture.

4. Hubbard trees

Assume now that the polynomial $f = f_c(z) = z^2 + c$ has connected Julia set (i.e. $c \in \mathcal{M}$), and no attracting fixed point (i.e. c lies outside the main cardioid). The *critical orbit* of f is the set $Crit(f) := \{f^k(0)\}_{k\geq 0}$. Let us now give the fundamental

Definition 4.1. *The* Hubbard tree T *for* f *is the smallest regulated tree which contains the critical orbit, i.e.*

$$T := \bigcup_{i,j\geq 0} [f^i(0), f^j(0)].$$

Note that, according to this definition, the set T need not be closed in general. We shall establish a few fundamental properties of Hubbard trees.

Lemma 4.2. *The following properties hold:*

(1) T *is the smallest forward-invariant set which contains the regulated arc* $[\alpha, 0]$*;*

(2) $T = \bigcup_{n\geq 0}[\alpha, f^n(0)]$.

Proof. Let now T_1 be the smallest forward-invariant set which contains the regulated arc $[\alpha, 0]$. By definition, T is forward-invariant and contains $[\alpha, 0]$ since $\alpha \in [0, c]$, so $T_1 \subseteq T$. Let now

$$T_2 := \bigcup_{n\geq 0}[\alpha, f^n(0)].$$

Since $[f^i(0), f^j(0)] \subseteq [\alpha, f^i(0)] \cup [\alpha, f^j(0)]$, then $T \subseteq T_2$. By definition,

$$T_1 = \bigcup_{n\geq 0} f^n([\alpha, 0]).$$

Since $f^i([\alpha, 0]) \supseteq [\alpha, f^i(0)]$, then $T_2 \subseteq T_1$, hence $T = T_1 = T_2$. □

The tree thus defined need not have finitely many edges. However, in the following we will restrict ourself to the case when T is a finite tree. Let us introduce the definition:

Definition 4.3. *A polynomial* f *is* topologically finite *if the Julia set is locally connected and the Hubbard tree* T *is homeomorphic to a tree with finitely many edges.*

Recall that a polynomial is called *postcritically finite* if the critical orbit is finite. Postcritically finite polynomials are also topologically finite, but it turns out that

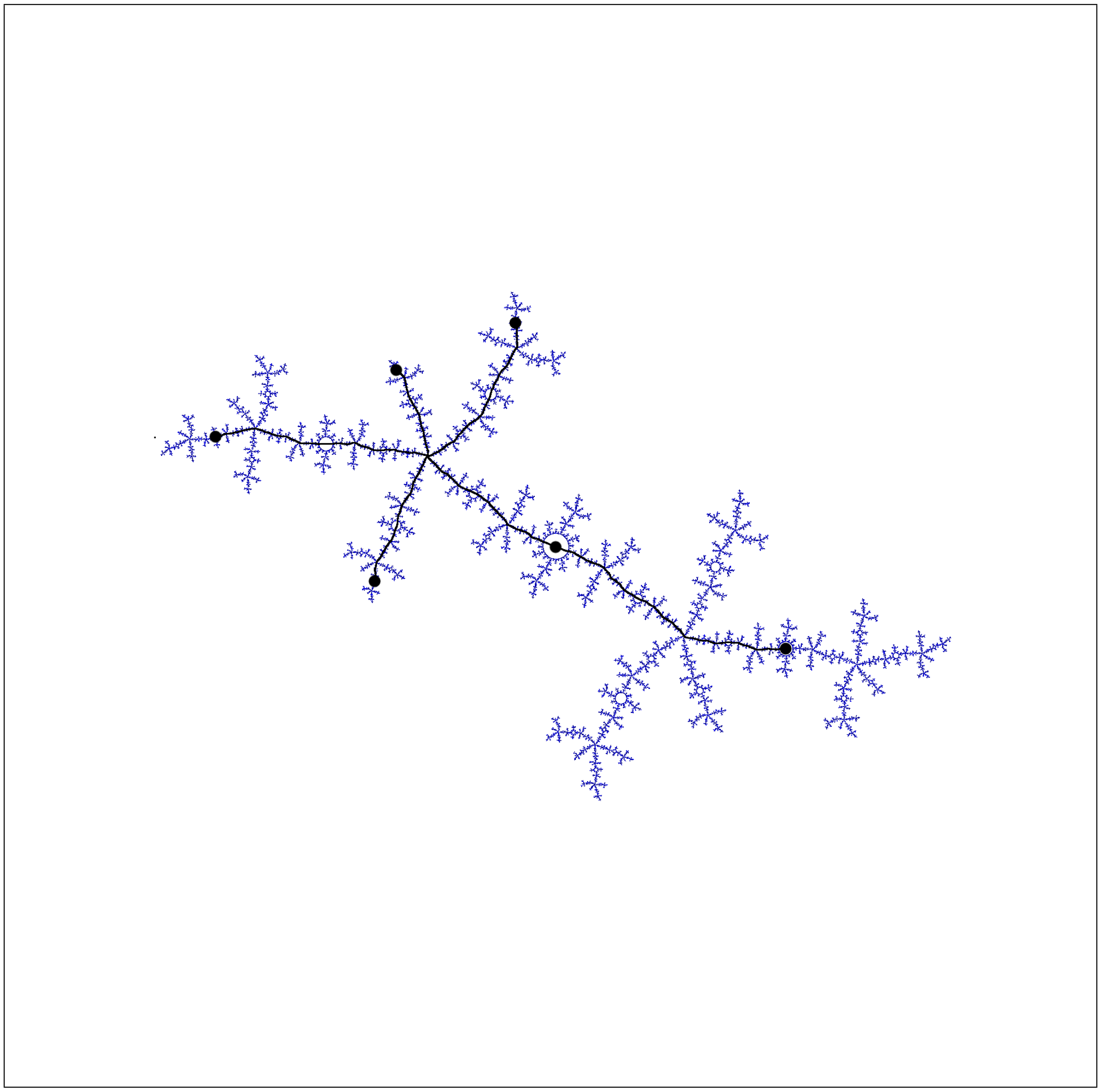


FIGURE 5. The Hubbard tree of the quadratic polynomial with characteristic leaf $(19/63, 20/63)$. The map f_c is postcritically finite, and the critical point belongs to a cycle of period 6. The parameter c belongs to the principal vein in the 2/5-limb.

the class of topologically finite polynomials is much bigger and indeed it contains all polynomials along the veins of the Mandelbrot set (see also section 15.1).

Proposition 4.4. *Let f have locally connected Julia set. Suppose there is an integer $n \geq 1$ such that $f^n(0)$ lies on the regulated arc $[\alpha, \beta]$, and let N be the smallest such* *integer. Then f is topologically finite, and the Hubbard tree T of f is given by*

$$T = \bigcup_{i=0}^{N} [\alpha, f^i(0)].$$

Proof. Let $T_N := \bigcup_{i=0}^{N} [\alpha, f^i(0)]$. By Lemma 4.2 (2), $T_N \subseteq T$. Note now that for each i we have

$$f([\alpha, f^i(0)]) \subseteq [\alpha, c] \cup [\alpha, f^{i+1}(0)]$$

thus

$$f(T_N) \subseteq T_N \cup [\alpha, f^{N+1}(0)].$$

Now, either $f^N(0)$ lies in $[\alpha, -\alpha]$, or by Lemma 2.3, $f^N(0)$ lies between β and $f^{N+1}(0)$. In the first case, $[\alpha, f^{N+1}(0)] \subseteq [\alpha, c]$ and in the second case $[\alpha, f^{N+1}(0)] \subseteq [\alpha, f^N(0)]$; in both cases, $[\alpha, f^{N+1}(0)] \subseteq T_N$, so T_N is forward-invariant and it contains $[\alpha, 0]$, so it contains T by Lemma 4.2 (1). $\square$

Proposition 4.5. *If the Julia set of f is locally connected and the critical value c is biaccessible, then f is topologically finite.*

Proof. Since c is biaccessible, by Lemma 2.1 there exists $n \geq 0$ such that $f^n(c)$ belongs to the spine $[-\beta, \beta]$ of the Julia set. Then either $f^n(c)$ or $f^{n+1}(c)$ lie on $[\alpha, \beta]$, so f is topologically finite by Proposition 4.4. $\square$

Let us define the *extended Hubbard tree* $\widetilde{T}$ to be the union of the Hubbard tree and the spine:

$$\widetilde{T} := T \cup [-\beta, \beta].$$

Note the extended tree is also forward invariant, i.e. $f(\widetilde{T}) \subseteq \widetilde{T}$. Moreover, it is related to the usual Hubbard tree in the following way:

Lemma 4.6. *The extended Hubbard tree eventually maps to the Hubbard tree:*

$$\widetilde{T} \setminus \{\beta, -\beta\} \subseteq \bigcup_{n \geq 0} f^{-n}(T).$$

Proof. Since $f([\alpha, -\beta)) = [\alpha, \beta)$, we just need to check that every element $z \in [\alpha, \beta)$ eventually maps to the Hubbard tree. Indeed, either there exists $n \geq 0$ such that $f^n(z) \in [\alpha, c] \subseteq T$, or, by Lemma 2.3, the sequence $\{f^n(z)\}_{z \geq 0}$ all lies on $[0, \beta)$ and it is ordered along the segment, i.e. for each n, $f^{n+1}(z)$ lies in between 0 and $f^n(z)$. Then the sequence must have a limit point, and such limit point would be a fixed point of f. However, f has no fixed points on $[0, \beta)$, contradiction. □

4.1. **Valence.** If T is a finite tree, then the *degree* of a point $x \in T$ is the number of connected components of $T \setminus \{x\}$, and is denoted by $deg(x)$. Moreover, let us denote by $deg(T)$ denote the largest degree of a point on the tree:

$$deg(T) := \max\{deg(x) \ : \ x \in T\}.$$

On the other hand, for each $z \in J(f)$, we call *valence* of z the number of external rays which land on z and denote it as

$$val(z) := \#\{\theta \in \mathbb{R}/\mathbb{Z} \ : \ R_c(\theta) \text{ lands on } z\}.$$

The valence of z also equals the number of connected components of $J(f) \setminus \{z\}$ ([Mc], Theorem 6.6), also known as the *Urysohn-Menger index* of $J(f)$ at z.

Proposition 4.7. *Let T be the extended Hubbard tree for a topologically finite quadratic polynomial f. Then the number of rays N landing on $x \in T$ is bounded above by*

$$N \leq 2 \cdot deg(T).$$

The proposition follows easily from the

Lemma 4.8. *Let T be the extended Hubbard tree for f, and $x \in T$ a point on the tree which never maps to the critical point. Then the number of rays N landing on x is bounded above by*

$$N \leq \max\{deg(f^n(x)) \ : \ n \geq 0\}.$$

Proof. Note that, since the forward orbit of x does not contain the critical point, f^n is a local homeomorphism in a neighborhood of x; thus, for each $n \geq 0$, $val(f^n(x)) = val(x)$ and $deg(f^n(x)) \geq deg(x)$. Suppose now the claim is false: let N be such that $deg(f^N(x)) = \max\{deg(f^n(x)) \ : \ n \geq 0\} < val(x)$, and denote $y = f^N(x)$. Then there are two angles θ_1, θ_2 such that the rays $R_c(\theta_1)$ and $R_c(\theta_2)$ both land at y, and the sector between $R_c(\theta_1)$ and $R_c(\theta_2)$ does not intersect the tree. Then, there exists $M \geq 0$ such that the rays $R_c(D^M(\theta_1))$ and $R_c(D^M(\theta_2))$ lie on opposite sides of the spine, thus their common landing point $z := f^M(y)$ must lie on the spine. Moreover, since $val(z) = val(x) \geq 2$ while only one ray lands on the β fixed point, z must lie in the interior of the spine. This means that the sector between the rays $R_c(D^M(\theta_1))$ and $R_c(D^M(\theta_2))$ intersects the spine, so $deg(f^M(y)) > deg(y)$, contradicting the maximality of N. □

Proof of Proposition 4.7. If $val(x) > 0$, then x lies in the Julia set $J(f_c)$. Now, if the forward orbit of x does not contain the critical point, the claim follows immediately from the Lemma. Otherwise, let $n \geq 0$ be such that $f^n(x) = 0$ is the critical point. Note that this n is unique, because otherwise the critical point would be periodic, so it would not lie in the Julia set. Hence, by applying the Lemma to the critical value $f^{n+1}(x)$, we have

$$val(f^{n+1}(x)) \leq deg(T).$$

Finally, since the map f_c is locally a double cover at the critical point,

$$val(x) = val(f^n(x)) = 2 \cdot val(f^{n+1}(x)) \leq 2 \cdot deg(T).$$

$\square$

5. Topological entropy

Let $f : X \to X$ be a continuous map of a compact metric space (X, d). A measure of the complexity of the orbits of the map is given by its *topological entropy*. Let us now recall its definition. Useful references are [dMvS] and [CFS].

Given $x \in X$, $\epsilon > 0$ and n an integer, we define the ball $B_f(x, \epsilon, n)$ as the set of points whose orbit remains close to the orbit of x for the first n iterates:

$$B_f(x, \epsilon, n) := \{y \in X \ : \ d(f^i(x), f^i(y)) < \epsilon \ \forall 0 \leq i \leq n\}.$$

A set $E \subseteq X$ is called (n, ϵ)-*spanning* if every point of X remains close to some point of E for the first n iterates, i.e. if $X = \bigcup_{x \in E} B_f(x, \epsilon, n)$. Let $N(n, \epsilon)$ be the minimal cardinality of a (n, ϵ)-spanning set. The topological entropy is the growth rate of $N(n, \epsilon)$ as a function of n:

Definition 5.1. *The* topological entropy *of the map $f : X \to X$ is defined as*

$$h_{top}(f) := \lim_{\epsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log N(n, \epsilon).$$

When f is a piecewise monotone map of a real interval, it is easier to compute the entropy by looking at the number of laps. Recall the *lap number* $L(g)$ of a piecewise monotone interval map $g : I \to I$ is the smallest cardinality of a partition of I in intervals such that the restriction of g to any such interval is monotone. The following result of Misiurewicz and Szlenk relates the topological entropy to the growth rate of the lap number of the iterates of f:

Theorem 5.2 ([MS]). *Let $f : I \to I$ be a piecewise monotone map of a close bounded interval I, and let $L(f^n)$ be the lap number of the iterate f^n. Then the following* *equality holds:*

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log L(f^n).$$

Another useful property of topological entropy is that it is invariant under dynamical extensions of bounded degree:

Proposition 5.3 ([Bo])**.** *Let $f : X \to X$ and $g : Y \to Y$ be two continuous maps of compact metric spaces, and let $\pi : X \to Y$ a continuous, surjective map such that $g \circ \pi = \pi \circ f$. Then*

$$h_{top}(g) \leq h_{top}(f).$$

Moreover, if there exists a finite number d such that for each $y \in Y$ the fiber $\pi^{-1}(y)$ has cardinality always smaller than d, then

$$h_{top}(g) = h_{top}(f).$$

In order to resolve the ambiguities arising from considering different restrictions of the same map, if K is an f-invariant set we shall use the notation $h_{top}(f, K)$ to denote the topological entropy of the restriction of f to K.

Proposition 5.4 ([Do3], Proposition 3)**.** *Let $f : X \to X$ a continuous map of a compact metric space, and let Y be a closed subset of X such that $f(Y) \subseteq Y$. Suppose that, for each $x \in X$, the distance $d(f^n(x), Y)$ tends to zero, uniformly on any compact subset of $X \setminus Y$. Then $h_{top}(f, Y) = h_{top}(f, X)$.*

The following proposition is the fundamental step to relate entropy and Hausdorff dimension of invariant subsets of the circle ([Fu], Proposition III.1; see also [Bi]):

Proposition 5.5. *Let $d \geq 1$, and $\Omega \subset \mathbb{R}/\mathbb{Z}$ be a closed, invariant set for the map $Q(x) := dx \mod 1$. Then the topological entropy of the restriction of Q to Ω is* *related to the Hausdorff dimension of Ω in the following way:*

$$\text{H.dim } \Omega = \frac{h_{top}(Q, \Omega)}{\log d}.$$

6. Invariant sets of external angles

Let f_c be a topologically finite quadratic polynomial, and T_c its Hubbard tree. One of the main players in the rest of the thesis is the set H_c of angles of external rays landing on the Hubbard tree:

$$H_c := \{\theta \in \mathbb{R}/\mathbb{Z} \ : R_c(\theta) \text{ lands on } T_c\}.$$

Note that, since T_c is compact and the Carathéodory loop is continuous by local connectivity, H_c is a closed subset of the circle. Moreover, since $T_c \cap J(f_c)$ is f_c-invariant, then H_c is invariant for the doubling map, i.e. $D(H_c) \subseteq H_c$.

Similarly, we will denote by S_c the set of angles of rays landing on the spine $[-\beta, \beta]$, and B_c the set of angles of rays landing on the set of biaccessible points.

Proposition 6.1. *Let f_c be a topologically finite quadratic polynomial. Then*

$$\text{H.dim } H_c = \text{H.dim } S_c = \text{H.dim } B_c.$$

Proof. Lemma 2.1 implies the inclusion

$$S_c \setminus \{0, 1/2\} \subseteq B_c \subseteq \bigcup_{n\geq 0}^{\infty} D^{-n}(S_c)$$

hence

$$\text{H.dim } S_c \leq \text{H.dim } B_c \leq \sup_{n\geq 0} \text{H.dim } D^{-n}(S_c) = \text{H.dim } S_c.$$

Moreover, it is clear that $H_c \subseteq B_c$, and by Lemma 4.6 one also has

$$S_c \setminus \{0, 1/2\} \subseteq \bigcup_{n\geq 0}^{\infty} D^{-n}(H_c)$$

hence H.dim $S_c \leq$ H.dim $H_c \leq$ H.dim B_c. $\square$

We will now characterize the set H_c and other similar sets of angles purely in terms of the dynamics of the doubling map on the circle, as the set of points whose orbit never hits certain open intervals.

In order to do so, we will make use of the following lemma:

Lemma 6.2. *Let $X \subseteq S^1$ be a closed, forward invariant set for the doubling map D, so that $D(X) \subseteq X$, and let $U \subseteq S^1$ be an open set, disjoint from X. Suppose moreover that*

(1) $D^{-1}(X) \setminus X \subseteq U$;

(2) $\partial U \subseteq X$.

Then X equals the set of points whose orbit never hits U:

$$X = \{\theta \in S^1 \ : \ D^n(\theta) \notin U \quad \forall n \geq 0\}.$$

Proof. Let θ belong to X. By forward invariance, $D^n(\theta) \in X$ for each $n \geq 0$, and since X and U are disjoint, then $D^n(\theta) \notin U$ for all n. Conversely, let us suppose that θ does not belong to X, and let V be the connected component of the complement of X containing θ; since the doubling map is uniformly expanding, there exists some n such that $f^n(V)$ is the whole circle, hence there exists an integer $k \geq 1$ such that $D^k(V) \cap X \neq \emptyset$, but $D^{k-1}(V) \cap X = \emptyset$; then, $D^{k-1}(V)$ intersects $D^{-1}(X) \setminus X$, so by (1) it intersects U. Moreover, since $\partial U \subseteq X$ we have $D^{k-1}(V) \cap \partial U = \emptyset$, so $D^{k-1}(V)$ is an open set which intersects U but does not intersect its boundary, hence $D^{k-1}(V) \subseteq U$ and, since $\theta \in V$, we have $D^{k-1}(\theta) \in U$. $\square$

Let us now describe combinatorially the set of angles of rays landing on the Hubbard tree. Let T_c be the Hubbard tree of f_c; since T_c is a compact set, then $H_c = \gamma^{-1}(T_c)$ is a closed subset of the circle. Among all connected components of the complement of H_c, there are finitely many $U_1, U_2, \dots, U_r$ which contain rays which land on the preimage $f_c^{-1}(T_c)$. The angles of rays landing on the Hubbard tree are precisely the angles whose future trajectory for the doubling map never hits the U_i:

Proposition 6.3 ([TaoL])**.** *Let T_c be the Hubbard tree of f_c, and $U_1, U_2, \dots, U_r$ be the connected components of the complement of H_c which contain rays landing on $f_c^{-1}(T_c)$. Then the set H_c of angles of rays landing on T_c equals*

$$H_c = \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ D^n(\theta) \notin U_i \quad \forall n \geq 0 \ \forall i = 1, \dots, r\}.$$

Proof. It follows from Lemma 6.2 applied to $X = H_c$ and $U = U_1 \cup \dots \cup U_r$. Indeed, $D(H_c) \subseteq H_c$ since $T_c \cap J(f_c)$ is forward-invariant under f_c. The set U is disjoint from H_c by definition of the U_i. Moreover, if θ belongs to $D^{-1}(H_c) \setminus H_c$, then $R_c(\theta)$ lands on $f_c^{-1}(T_c)$, so θ belongs to some U_i. Finally, let us check that for each i we have the inclusion $\partial U_i \subseteq H_c$. Indeed, if U is non-empty then H_c has no interior (since it is invariant for the doubling map and does not coincide with the whole circle), so angles on the boundary of U_i are limits of angles in H_c, so their corresponding rays land on the Hubbard tree by continuity of the Riemann mapping on the boundary. □

7. Entropy of Hubbard trees

We are now ready to prove the relationship between the topological entropy of a topologically finite quadratic polynomial f_c and the Hausdorff dimension of the set of rays which land on the Hubbard tree T_c:

Theorem 7.1. *Let $f_c(z) = z^2 + c$ be a topologically finite quadratic polynomial, let T_c be its Hubbard tree and H_c the set of external angles of rays which land on the Hubbard tree. Then we have the identity*

$$\frac{h_{top}(f_c\mid_{T_c})}{\log 2} = \text{H.dim } H_c.$$

Proof. Let $\gamma : \mathbb{R}/\mathbb{Z} \to J(f_c)$ the Carathéodory loop. We know that

$$\gamma(D(\theta)) = f_c(\gamma(\theta)).$$

By Proposition 4.7, the cardinality of the preimage of any point is bounded; hence, by Theorem 5.3, we have

$$h_{top}(f_c, J(f_c) \cap T_c) = h_{top}(D, \gamma^{-1}(J(f_c) \cap T_c)) = h_{top}(D, H_c).$$

Moreover, Proposition 5.4 implies

$$h_{top}(f_c, J(f_c) \cap T_c) = h_{top}(f_c, T_c).$$

Then we conclude, by the dimension formula of Proposition 5.5, that

$$\text{H.dim } H_c = \frac{h_{top}(D, H_c)}{\log 2}.$$

□

The exact same argument applies to any compact, forward invariant set X in the Julia set:

Theorem 7.2. *Let f_c be a topologically finite quadratic polynomial, and $X \subseteq J(f_c)$ compact and invariant (i.e. $f_c(X) \subseteq X$). Let define the set*

$$\Theta_c(X) := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ R_c(\theta) \text{ lands on } X\};$$

then we have the equality

$$\frac{h_{top}(f_c \mid_X)}{\log 2} = \text{H.dim } \Theta_c(X).$$

8. Entropy formula for real quadratic polynomials

Let us now turn to the main result of the first part of the thesis, namely a formula relating the topological entropy of real quadratic polynomials to the Hausdorff dimension of a certain section of the combinatorial model for the real slice of the Mandelbrot set (Theorem 1.6 in the introduction).

A fundamental theme in the study of parameter spaces in holomorphic dynamics is that the local geometry of the Mandelbrot set near a parameter c reflects the geometry of the Julia set $J(f_c)$, hence it is related to dynamical properties of f_c. In this section we will establish an instance of this principle, by looking at the Hausdorff dimension of certain sets of external rays.

Recall that a measure of the complexity of a continuous map is its *topological entropy*, which is essentially defined as the growth rate of the number of itineraries under iteration (see section 5).

In our case, the map $f_c(z) = z^2 + c$ is a degree-two ramified cover of the Riemann sphere $\hat{\mathbb{C}}$, hence a generic point has exactly 2 preimages, and the topological entropy of f_c always equals $\log 2$, independently of the parameter [Ly]. If c is real, however, then f_c can also be seen as a real interval map, and its restriction to the invariant interval $[-\beta, \beta]$ also has a well-defined topological entropy, which we will denote by $h_{top}(f_c, [-\beta, \beta])$. The dependence of $h_{top}(f_c, [-\beta, \beta])$ on c is much more interesting: indeed, it is a continuous, decreasing function of c [MT], and it is constant on each baby Mandelbrot set [Do3].

Given a subset A of $\partial\mathcal{M}$, one can define the *harmonic measure* ν_M as the probability that a random ray from infinity lands on A:

$$\nu_M(A) := \mathrm{Leb}(\{\theta \in S^1 \ : \ R_M(\theta) \text{ lands on } A\}).$$

If one takes $A := \partial\mathcal{M} \cap \mathbb{R}$ to be the real slice of the boundary of $\mathcal{M}$, then the harmonic measure of A is zero. However, the set of rays which land on the real axis

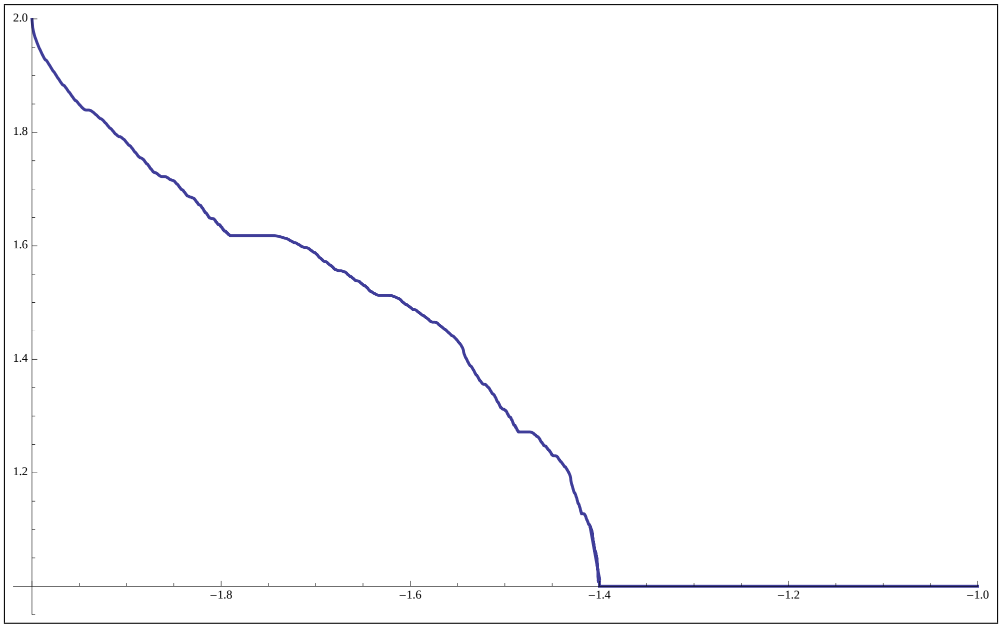


FIGURE 6. Topological entropy of the real quadratic family $f_c(z) := z^2 + c$, as a function of c. For each value of $c \in [-2, -1]$, we plot the growth number $e^{h_{top}(f_c)}$.

has full Hausdorff dimension [Za2]. (By comparison, the set of rays which land on the main cardioid has zero Hausdorff dimension.) As a consequence, it is more useful to look at Hausdorff dimension than Lebesgue measure; for each c, let us consider the section

$$P_c := \{\theta \in S^1 : \text{the impression of } R_M(\theta) \text{ intersects } \partial\mathcal{M} \cap [c, 1/4]\}$$

of all parameter rays which (at least combinatorially) land on the real axis, to the right of c. The function

$$c \mapsto \text{H.dim } P_c$$

decreases with c, taking values from 1 to 0. In the dynamical plane, one can consider the set of rays which land on the real slice of $J(f_c)$, and let S_c be the set of external angles of rays landing on $J(f_c) \cap \mathbb{R}$. This way, we construct the the function $c \mapsto$ H.dim S_c, which we want to compare to the Hausdorff dimension of P_c.

The main result is an identity relating entropy and dimension:

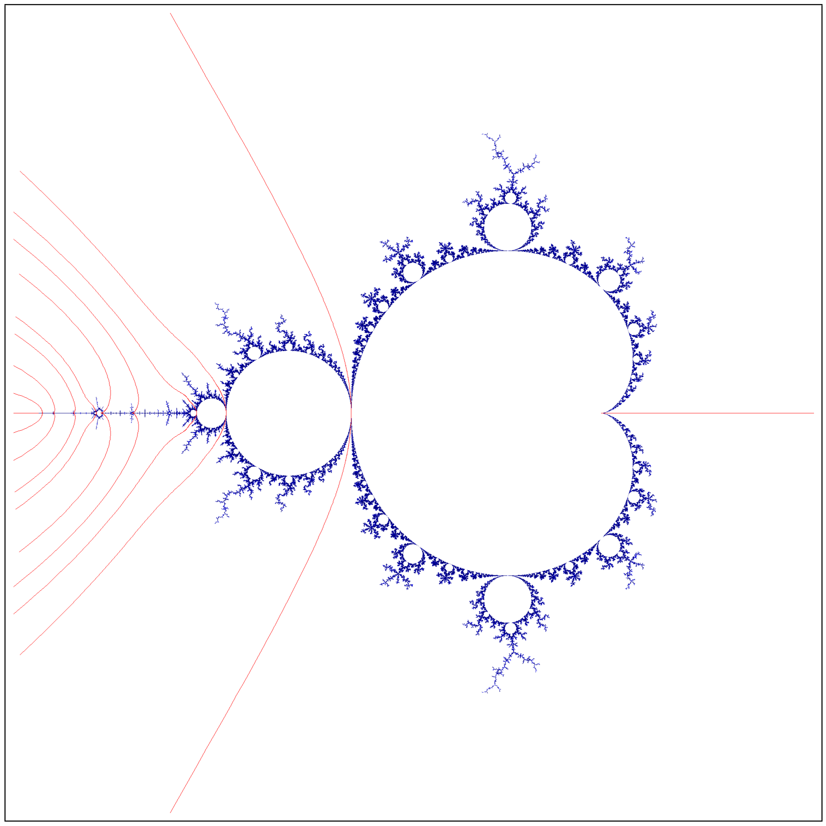


FIGURE 7. A few rays which land on the real slice of the Mandelbrot set.

Theorem 8.1. *Let $c \in [-2, 1/4]$. Then we have*

$$\frac{h_{top}(f_c, [-\beta, \beta])}{\log 2} = \text{H.dim } S_c = \text{H.dim } P_c.$$

The first equality is in line with the classical "entropy formula" relating Hausdorff dimension, entropy and Lyapunov exponents, while the second equality can be seen as an instance of Douady's principle relating the local geometry of the Mandelbrot set to the geometry of the corresponding Julia set. Indeed, we can replace P_c with the set of angles of rays landing on $[c, c+\epsilon]$ in parameter space, as long as $[c, c+\epsilon]$ does not lie in a tuned copy of the Mandelbrot set. Note that the set of rays which possibly do not land has zero capacity, hence the result is independent of the MLC conjecture.

A first study of the dimension of the set of angles of rays landing on the real axis has been done in [Za2], where it is proven that the set of angles of parameter rays landing on the real slice of $\mathcal{M}$ has dimension 1. Zakeri also provides estimates on the dimension along the real axis, and specifically asks for dimension bounds for parameters near the Feigenbaum point ($-1.75 \leq c \leq c_{Feig}$, see [Za2], Remark 6.9). Our result gives an identity rather than an estimate, and the dimension of S_c can be exactly computed in the case c is postcritically finite (see following examples).

Recall the dimension of S_c also equals the dimension of the set B_c of angles landing at biaccessible points (Proposition 6.1). Smirnov [Sm] first showed that such set has positive Hausdorff dimension for Collet-Eckmann maps. More recent work on biaccessible points is due to Zakeri [Za3] and Zdunik [Zd]. The first equality in Theorem 8.1 has also been established independently by Bruin-Schleicher [BS].

A precise statement of the asymptotic similarity between $\mathcal{M}$ and Julia sets near Misiurewicz points is proven in [TanL].

Examples

(1) If $c = 0$, then $f_c^n(z) = z^{2^n}$ has only one lap for each n, hence the entropy is zero. Moreover, the characteristic ray is $\theta = 0$, hence P_c consists of only one element and it has zero dimension. Moreover, the Julia set is a circle and the set of rays landing on the real axis $S_c = \{0, \frac{1}{2}\}$ consists of two elements, hence the dimension is 0.

(2) If $c = -2$, then f_c is a 2-1 surjective map from $[-2, 2]$ to itself, hence the entropy is $\log 2$. The Julia set is a real segment, hence all rays land on the real axis and the Hausdorff dimension of S_c is 1. Similarly, the set of rays P_c is the set of all parameter rays which land on the real axis, which has Hausdorff dimension 1.

(3) The *basilica map* $f_c(z) = z^2 - 1$ has a superattracting cycle of period 2, and for each n, f_c^n has $2n+1$ critical points, hence the entropy is $\lim_{n\to\infty} \frac{\log(2n+1)}{n} = 0$. The rays which land on the Hubbard tree are just $\frac{1}{3}, \frac{2}{3}$, and the set of rays which land on the real Julia set is countable, hence it has dimension 0. In parameter space, the only rays which land on the real axis to the right of $c = -1$ are $\theta = 0, 1/3, 2/3$, hence their dimension is still zero.

(4) The *airplane map* has a superattracting cycle of period 3, and its characteristic angle is $\theta_c = \frac{3}{7}$. The set of angles whose rays land on the Hubbard tree is the set of binary numbers with expansion which does not contain any sequence of three consecutive equal symbols. It is a Cantor set which can be generated by the automaton in Figure 8, and its Hausdorff dimension is $\log_2 \frac{\sqrt{5}+1}{2}$.

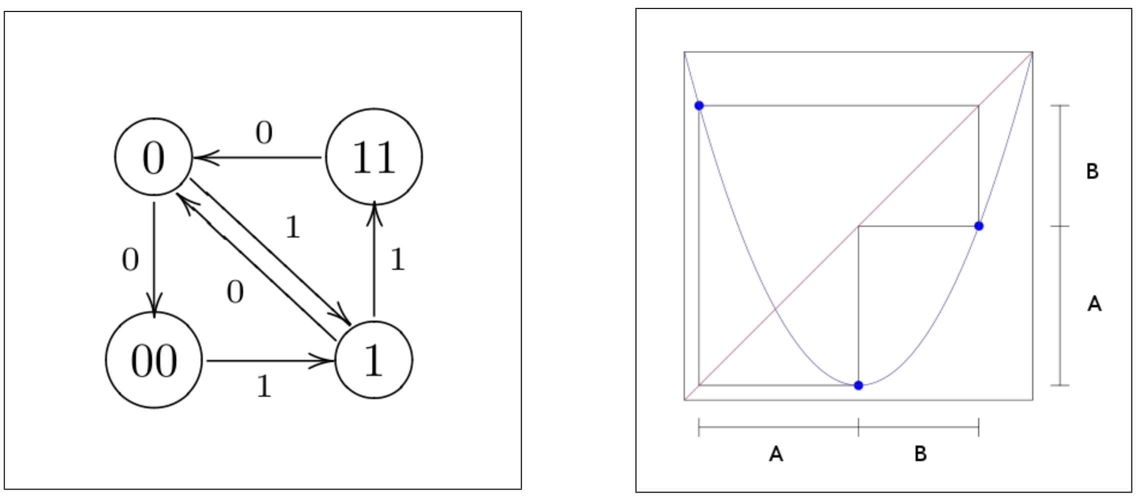


FIGURE 8. To the right: the combinatorics of the airplane map of period 3. To the left: the automaton which produces all symbolic orbits of points on the real slice of the Julia set.

On the other hand, the topological dynamics of the real map is encoded by the right-hand side diagram: the interval A is mapped onto $A \cup B$, and B is mapped onto A. Then the number of laps of f_c^n is given by the Fibonacci numbers, hence the topological entropy is the logarithm of the golden mean. It is harder to characterize explicitly the set of parameter rays which land on the boundary of $\mathcal{M}$ to the right of the characteristic ray: however, as a consequence of Theorem 8.1, the dimension of such set is also $\log_2 \frac{\sqrt{5}+1}{2}$.

A more complicated example is the *Feigenbaum point* c_{Feig}, the accumulation point of the period doubling cascades. As a corollary of Theorem 8.1, we are able to answer a question of Zakeri ([Za2], Remark 6.9):

Corollary 8.2. *The set of biaccessible angles for the Feigenbaum parameter c_{Feig} has dimension zero:*

$$\text{H.dim } B_{c_{Feig}} = 0.$$

8.1. **Sketch of the argument.** The proof of Theorem 8.1 is carried in two steps. We already proved (Theorem 7.1 in section 7) the relationship between topological entropy $h_{top}(f_c \mid_{T_c})$ of the map restricted to the Hubbard tree and the Hausdorff dimension of the set H_c of angles landing on the tree, for all topologically finite polynomials f_c. The bulk of the argument is then proving the identity of Hausdorff dimensions between the real Julia set and the slices of $\mathcal{M}$:

Theorem 8.3. *For any $c \in [-2, \frac{1}{4}]$, we have the equality*

$$\text{H.dim } S_c = \text{H.dim } P_c.$$

It is not hard to show that $P_c \subseteq H_c \subseteq S_c$ for any real parameter c (Corollary 9.7); it is much harder to give a lower bound for the dimension of P_c in terms of the dimension of H_c; indeed, it seems impossible to include a copy of H_c in P_c when c belongs to some *tuning window*, i.e. to some baby Mandelbrot set. However, for non-renormalizable parameters we can prove the following:

Proposition 8.4. *Given a non-renormalizable, real parameter c and another real parameter $c' > c$, there exists a piecewise linear map $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that*

$$F(H_{c'}) \subseteq P_c.$$

The proposition immediately implies equality of dimension for all non-renormalizable parameters. By applying tuning operators, we then get equality for all finitely-renormalizable parameters, which are dense hence the result follows from continuity.

Proposition 8.4 will be proved in section 11. Its proof relies on the definition of a class of parameters, which we call *dominant*, which are a subset of the set of non-renormalizable parameters. We will show that for these parameters (which can be defined purely combinatorially) it is easier to construct an inclusion of the Hubbard tree into parameter space; finally, the most technical part (section 12.3) will be proving that such parameters are dense in the set of non-renormalizable angles.

9. Combinatorial description: the real case

Suppose $c \in \partial\mathcal{M} \cap \mathbb{R}$. By definition, the *dynamic root* r_c of f_c is the critical value c if c belongs to the Julia set, otherwise it is the smallest value of $J(f_c) \cap \mathbb{R}$ larger than c. This means that r_c lies on the boundary of the bounded Fatou component containing c.

Recall that the *impression* of a parameter ray $R_M(\theta)$ is the set of all $c \in \partial\mathcal{M}$ for which there is a sequence $\{w_n\}$ such that $|w_n| > 1$, $w_n \to e^{2\pi i\theta}$, and $\Phi_M^{-1}(w_n) \to c$. We denote the impression of $R_M(\theta)$ by $\hat{R}_M(\theta)$. It is a non-empty, compact, connected subset of $\partial\mathcal{M}$. Every point of $\partial\mathcal{M}$ belongs to the impression of at least one parameter ray. Conjecturally, every parameter ray $R_M(\theta)$ lands at a well-defined point $c(\theta) \in \partial\mathcal{M}$ and $\hat{R}_M(\theta) = c(\theta)$.

In the real case, much more is known to be true. First of all, every real Julia set is locally connected [LvS]. The following result summarizes the situation for real maps.

Theorem 9.1 ([Za2], Theorem 3.3)**.** *Let $c \in \partial\mathcal{M} \cap \mathbb{R}$. Then there exists a unique angle $\theta_c \in [0, 1/2]$ such that the rays $R_c(\pm\theta_c)$ land at the dynamic root r_c of f_c. In the parameter plane, the two rays $R_M(\pm\theta_c)$, and only these rays, contain c in their impression.*

The theorem builds on the previous results of Douady-Hubbard [DH] and Tan Lei [TanL] for the case of periodic and preperiodic critical points and uses density of hyperbolicity in the real quadratic family to get the claim for all real maps.

To each angle $\theta \in S^1$ we can associate a *length* $\ell(\theta)$ as the length (along the circle) of the chord delimited by the leaf joining θ to $1-\theta$ and containing the angle $\theta = 0$. In formulas, it is easy to check that

$$\ell(\theta) := \begin{cases} 2\theta & \text{if } 0 \leq \theta < \frac{1}{2} \\ 2 - 2\theta & \text{if } \frac{1}{2} \leq \theta < 1. \end{cases}$$

For a real parameter c, we will denote as ℓ_c the length of the characteristic leaf

$$\ell_c := \ell(\theta_c).$$

The key to analyzing the symbolic dynamics of f_c is the following interpretation in terms of the dynamics of the tent map. Since all real Julia sets are locally connected, for c real all dynamical rays $R_c(\theta)$ have a well-defined limit $\gamma_c(\theta)$, which belongs to $J(f_c)$. Let us moreover denote by T the full tent map on the interval $[0,1]$, defined as $T(x) := \min\{2x, 2-2x\}$. The following diagram is commutative:

$$\begin{tikzcd} S^1 \arrow[loop above, "D"] \arrow[r, "\gamma_c"] \arrow[d, "\ell"] & J(f_c) \arrow[loop above, "f_c"] \\ {[0,1]} \arrow[loop right, "T"] & \end{tikzcd}$$

This means that we can understand the dynamics of f_c on the Julia set in terms of the dynamics of the tent map on the space of lengths. First of all, the set of external angles corresponding to rays which land on the real slice of the Julia set can be given the following characterization:

Proposition 9.2. *Let $c \in [-2, \frac{1}{4}]$. Then the set S_c of external angles of rays which land on the real slice $J(f_c) \cap \mathbb{R}$ of the Julia set is*

$$S_c = \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ T^n(\ell(\theta)) \leq \ell_c \quad \forall n \geq 1\}.$$

Proof. Let X be the set of angles of rays landing on the segment $[c, \beta]$. Since $f_c^{-1}([c, \beta]) = [-\beta, \beta]$, then $D^{-1}(X)$ is the set of angles landing on the spine. Thus, if we set $U := (\theta_c, 1 - \theta_c)$ then the hypotheses of Lemma 6.2 hold, hence we get the following description:

$$S_c = \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ D^n(\theta) \notin (\theta_c, 1 - \theta_c) \ \ \forall n \geq 1\}$$

hence by taking the length on both sides

$$\theta \in S_c \quad \Leftrightarrow \quad \ell(D^n(\theta)) \leq \ell(\theta_c) \ \ \forall n \geq 1$$

and by the commutative diagram we have $\ell(D^n(\theta)) = T^n(\ell(\theta))$, which, when substituted into the previous equation, yields the claim. □

Recall that for a real polynomial f_c the Hubbard tree is the segment $[c, f_c(c)]$. Let us denote as $L_c := \ell(D(\theta_c))$ the length of the leaf which corresponds to $f_c(c) = c^2 + c$. The set of angles which land on the Hubbard tree can be characterized as:

Proposition 9.3. *The set H_c of angles of external rays which land on the Hubbard tree for f_c is:*

$$H_c := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ T^n(\ell(\theta)) \geq L_c \quad \forall n \geq 0\}.$$

Proof. Since the Hubbard tree is $[c, f_c(c)]$ and its preimage is $[0, c]$, one can take $U = (D(\theta_c), 1 - D(\theta_c))$ (where we mean the interval containing zero) and $X = H_c$, and we get by Lemma 6.2

$$H_c = \{\theta \in S^1 \ : \ D^n(\theta) \notin U \ \ \forall n \geq 0\}$$

hence in terms of length

$$H_c = \{\theta \in S^1 \ : \ \ell(D^n(\theta)) \geq \ell(D(\theta_c)) \ \ \forall n \geq 0\}$$

which yields the result when you substitute $\ell(D^n(\theta)) = T^n(\ell(\theta))$ and $L_c = \ell(D(\theta_c))$.

□

9.1. **The real slice of the Mandelbrot set.** Let us now turn to parameter space. We are looking for a combinatorial description of the set of rays which land on the real axis. However, in order to account for the fact that some rays might not land, let us define the set $\mathcal{R}$ of *real parameter angles* as the set of angles of rays whose prime-end impression intersects the real axis:

$$\mathcal{R} := \{\theta \in S^1 \ : \ \hat{R}_M(\theta) \cap \mathbb{R} \neq \emptyset\}.$$

The set $\mathcal{R}$ is also the closure (in S^1) of the union of the angles of rays landing on the boundaries of all real hyperbolic components. Combinatorially, elements of $\mathcal{R}$ correspond to leaves which are maximal in their orbit under the dynamics of the tent map:

Proposition 9.4. *The set $\mathcal{R}$ of real parameter angles can be characterized as*

$$\mathcal{R} = \{\theta \in S^1 \ : \ T^n(\ell(\theta)) \leq \ell(\theta) \ \ \forall n \geq 0\}.$$

Proof. Let θ_c be the characteristic angle of a real quadratic polynomial. Since the corresponding dynamical ray $R_c(\theta)$ lands on the spine, by Proposition 9.2 applied to $\ell(\theta_c) = \ell_c$ we have for each $n \geq 0$

$$T^n(\ell(\theta_c)) \leq \ell(\theta_c).$$

Conversely, if θ does not belong to $\mathcal{R}$ then it belongs to the opening of some real hyperbolic component W. By symmetry, we can assume θ belongs to $[0, 1/2]$: then

θ must belong to the interval (α, ω), whose endpoints have binary expansion

$$\alpha = 0.\overline{s_1 \dots s_n}$$
$$\omega = 0.\overline{\check{s}_1 \dots \check{s}_n s_1 \dots s_n}$$

where n is the period of W, and $s_1 = 0$ (recall the notation $\check{s}_i := 1 - s_i$); in this case it is easy to check that both $\ell(\alpha) = 2\alpha$ and $\ell(\omega) = 2\omega$ are fixed points of T^n, and $T^n(x) > x$ if $x \in (2\alpha, 2\omega)$. The description is equivalent to the one given in ([Za2], Theorem 3.7). □

Note moreover that the image of characteristic leaves are the shortest leaves in the orbit:

Proposition 9.5. *The set $\mathcal{R} \setminus \{0\}$ of non-zero real parameter angles can be characterized as*

$$\mathcal{R} \setminus \{0\} = \{\theta \in [1/4, 3/4] \ : \ T^n(\ell(D(\theta))) \geq \ell(D(\theta)) \ \ \forall n \geq 0\}.$$

Proof. Since $\theta \in \mathcal{R} \setminus \{0\}$, then $\ell(\theta) \geq 2/3$, so $\ell(D(\theta)) \leq 1/3$. The claim follows then from the previous proposition by noting that T maps $[1/2, 1]$ homeomorphically to $[0, 1]$ and reversing the orientation. □

In the following it will be useful to introduce the following slice of $\mathcal{R}$, by taking for each $c \in [-2, 1/4]$ the set of angles of rays whose impression intersects the real axis to the right of c.

Definition 9.6. *Let $c \in [-2, 1/4]$. Then we define the set*

$$P_c := \mathcal{R} \cap [1 - \theta_c, \theta_c]$$

where $\theta_c \in [0, 1/2]$ is the characteristic ray of f_c, and $[1 - \theta_c, \theta_c]$ is the interval containing 0.

A corollary of the previous description is that parameter rays landing on $\partial\mathcal{M} \cap \mathbb{R}$ to the right of c also land on the Hubbard tree of c:

Corollary 9.7. *Let $c \in [-2, 1/4]$. Then the inclusion*

$$P_c \setminus \{0\} \subseteq H_c$$

holds.

Proof. Let $\theta \neq 0$ belong to P_c. Then $\ell(\theta) \leq \ell(\theta_c)$, hence also $\ell(D(\theta)) \geq \ell(D(\theta_c))$. Now, by Proposition 9.4,

$$T^n(\ell(D(\theta))) \geq \ell(D(\theta)) \geq \ell(D(\theta_c))$$

for each $n \geq 0$, hence θ belongs to H_c by Proposition 9.3. $\square$

10. Compact coding of kneading sequences

In order to describe the combinatorics of the real slice, we will now associate to each real external ray an infinite sequence of positive integers. The notation is inspired by the correspondence with continued fractions established in Theorem 1.1. Indeed, because of the isomorphism, the set of integer sequences which arise from parameters on the real slice of $\mathcal{M}$ is exactly the same as the set of sequences of partial quotients of elements of the bifurcation set $\mathcal{E}$ for continued fractions.

Let $\Sigma := (\mathbb{N}^+)^{\mathbb{N}}$ be the space of infinite sequences of positive integers, and $\sigma : \Sigma \to \Sigma$ be the shift operator. Sequences of positive integers will also be called *strings*.

Let us now associate a sequence of integers to each angle. Indeed, let $\theta \in \mathbb{R}/\mathbb{Z}$, and write θ as a binary sequence: if $0 \leq \theta < 1/2$, we have

$$\theta = 0.\underbrace{0\ldots0}_{a_1}\underbrace{1\ldots1}_{a_2}\underbrace{0\ldots0}_{a_3}\ldots \qquad a_i \geq 1$$

while if $1/2 \leq \theta < 1$ we have

$$\theta = 0.\underbrace{1\dots1}_{a_1}\underbrace{0\dots0}_{a_2}\underbrace{1\dots1}_{a_3}\dots \qquad a_i \geq 1.$$

In both cases, let us define the sequence w_θ by counting the number of repetitions of the same symbol:

$$w_\theta := (a_1, a_2, a_3, \dots).$$

Note moreover that w_θ only depends on $\ell(\theta)$, which in both cases is given by

$$\ell(\theta) = 0.\underbrace{0\dots0}_{a_1-1}\underbrace{1\dots1}_{a_2}\underbrace{0\dots0}_{a_3}\dots \qquad a_i \geq 1.$$

Note that we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[r, "\ell"] \arrow[loop above, "D"] & {[0,1]} \arrow[r] \arrow[loop above, "T"] & \Sigma \arrow[loop above, "F"] \end{tikzcd}$$

where $F((a_1, a_2, \dots)) = (a_1 - 1, a_2, \dots)$ if $a_1 > 1$, and $F((1, a_2, \dots)) = (a_2, \dots)$.

If θ_c is the characteristic angle of a real hyperbolic component, we denote by w_c the string associated to the *postcharacteristic leaf* $L_c = (D(\theta_c), 1 - D(\theta_c))$. For instance, the airplane component has root $\theta_c = 3/7 = 0.\overline{011}$, so $D(\theta_c) = 1/7 = 0.\overline{001}$ and $w_c = \overline{(2,1)}$.

10.1. **Extremal strings.** Let us now define the *alternate lexicographic order* on the set of strings of positive integers. Let $S = (a_1, \dots, a_n)$ and $T = (b_1, \dots, b_n)$ be two finite strings of positive integers of equal length, and let $k := \min\{i \geq 1 \ : \ a_i \neq b_i\}$ the first different digit. We will say that $S < T$ if $k \leq n$ and either

$$k \text{ is odd and } a_k > b_k$$

or

$$k \text{ is even and } a_k < b_k.$$

For instance, in this order $(2,1) < (1,2)$, and $(2,1) < (2,3)$. The order can be extended to an order on the set $\Sigma := (\mathbb{N}^+)^{\mathbb{N}}$ of infinite strings of positive integers. Namely, if $S = (a_1, a_2, \dots)$ and $T = (b_1, b_2, \dots)$ are two infinite strings, then $S < T$ if there exists some $n \geq 1$ for which $(a_1, a_2, \dots, a_n) < (b_1, b_2, \dots, b_n)$. We will denote as $\overline{S}$ the infinite periodic string $(S, S, \dots)$.

Note that as a consequence of our ordering we have, for two angles θ and θ',

$$w_\theta < w_{\theta'} \Leftrightarrow \ell(\theta) > \ell(\theta')$$

and on the other hand, for two real $c, c' \in \partial\mathcal{M} \cap \mathbb{R}$,

$$w_c < w_{c'} \Leftrightarrow \ell(\theta_c) < \ell(\theta'_c).$$

The following is a convenient criterion to compare periodic strings:

Lemma 10.1 ([CT], Lemma 2.12)**.** *Let S, T be finite strings of positive integers. Then*

$$ST < TS \Leftrightarrow \overline{S} < \overline{T}. \tag{3}$$

In order to describe the real kneading sequences, we need the

Definition 10.2. *A finite string of positive integers S is called* extremal *if*

$$XY < YX$$

for every splitting $S = XY$ where X, Y are nonempty strings.

For instance, the string $(2,1,2)$ is extremal because $(2,1,2) < (2,2,1) < (1,2,2)$. Note that a string whose first digit is strictly larger than the others is always extremal.

Extremal strings are very useful because they parametrize purely periodic (i.e. rational with odd denominator) parameter angles on the real axis:

Lemma 10.3. *A purely periodic angle $\theta \in [1/4, 3/4]$ belongs to the set $\mathcal{R}$ if and only if there exists an extremal string S for which*

$$w_{D(\theta)} = \overline{S}.$$

Proof. Let $\theta \in [1/4, 1/2]$ be purely periodic for the doubling map. Then we can write its expansion as

$$\theta = 0.\overline{01^{a_1}0^{a_2}\dots 0^{a_n-1}}$$

with $a_i \geq 1$, and n even. Then $x := \ell(D(\theta)) = 0.\overline{0^{a_1-1}1^{a_2}\dots 1^{a_n}0}$, and by Proposition 9.5 the angle θ belongs to $\mathcal{R}$ if and only if

$$T^n(x) \geq x \qquad \text{for all } n \geq 0.$$

By writing out the binary expansion one finds out that this is equivalent to the statement

$$0.\overline{0^{a_k-1}1^{a_{k+1}}\dots 1^{a_{k-1}}0} \geq 0.\overline{0^{a_1-1}1^{a_2}\dots 1^{a_n}0} \qquad \text{for all } 1 \leq k \leq n$$

which in terms of strings reads

$$(\overline{a_k, \dots, a_n, a_1, \dots, a_{k-1}}) \geq (\overline{a_1, \dots, a_n}) \qquad \text{for all } 1 \leq k \leq n.$$

The condition is clearly satisfied if $S = (a_1, \dots, a_n)$ is extremal. Conversely, if the condition is satisfied then S must be of the form $S = P^k$ with P an extremal string. $\square$

10.2. **Dominant strings.** The order $<$ is a total order on the strings of positive integers of fixed given length; in order to be able to compare strings of different lengths we define the partial order

$$S << T \quad \text{if } \exists i \leq \min\{|S|, |T|\} \text{ s.t. } S_1^i < T_1^i$$

where $S_1^i := (a_1, \dots, a_i)$ denotes the truncation of S to the first i characters. Let us note that:

(1) if $|S| = |T|$, then $S < T$ if and only if $S << T$;

(2) if S, T, U are any strings, $S << T \Rightarrow SU << T, S << TU$;

(3) If $S << T$, then $S \cdot z < T \cdot w$ for any $z, w \in (\mathbb{N}^+)^{\mathbb{N}}$.

Definition 10.4. *A finite string S of positive integers is called* dominant *if it has even length and*

$$XY << Y$$

for every splitting $S = XY$ where X, Y are finite, nonempty strings.

Let us remark that every dominant string is extremal, while the converse is not true. For instance, the strings $(5, 2, 4, 3)$ and $(5, 2, 4, 5)$ are both extremal, but the first is dominant while the second is not. On the other hand, a string whose first digit is strictly large than the others is always dominant (as a corollary, there exist dominant strings of arbitrary length).

Definition 10.5. *A real parameter c is* dominant *if there exists a dominant string S such that*

$$w_c = \overline{S}.$$

The airplane parameter $\theta_c = 0.\overline{011}$ is dominant because $w_c = \overline{(2, 1)}$, and $(2, 1)$ is dominant. On the other hand, the period-doubling of the airplane ($\theta_c = 0.\overline{011100}$) is not dominant because its associated sequence is $\overline{(3)}$, and dominant strings must be of even length. In general, we will see that tuning always produces non-dominant parameters.

However, the key result is that dominant parameters are dense in the set of non-renormalizable angles:

Proposition 10.6. *Let $\theta_c \in [0, 1/2]$ be the characteristic angle of a real, non-renormalizable parameter c, with $c \neq -1$. Then θ_c is limit point from below of characteristic angles of dominant parameters.*

Since the proof of the proposition is quite technical, it will be postponed to section 12.3.

11. A copy of the Hubbard tree inside parameter space

We saw that the set of rays which land on the real axis in parameter space also land in the dynamical plane. In order to establish equality of dimensions, we would like to prove the other inclusion. Unfortunately, in general there is no copy of H_c inside P_c (for instance, is c is the basilica tuned with itself, then the Hubbard tree is a countable set, while only two pairs of rays land in parameter space to the right of c). However, outside of the baby Mandelbrot sets, one can indeed map the combinatorial model for the Hubbard tree into the combinatorial model of parameter space:

Proposition 11.1. *Given a non-renormalizable, real parameter c and another real parameter $c' > c$, there exists a piecewise linear map $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that*

$$F(H_{c'}) \subseteq P_c.$$

Proof. Let us denote $\ell := \ell(c)$ and $\ell' := \ell(c')$ the lengths of the characteristic leaves. Let us now choose a dominant parameter c'' in between c and c' and such that its corresponding string $w_{c''} = \overline{S}$ with S dominant, in such a way that S is a prefix of w_c and not a prefix of $w_{c'}$. Let us denote by $\ell'' := \ell(c'')$ the length of the characteristic leaf of c''.

If $S = (s_1, s_2, \dots, s_n)$ (recall n must be even), let us define the dyadic number

$$s := 0.01^{s_1}0^{s_2} \dots 1^{s_{n-1}}0^{s_n}$$

and the "length" of S to be $N := s_1 + s_2 + \cdots + s_n$. Then, let us construct the map

$$F(\theta) := \begin{cases} s + \frac{1-\theta}{2^{N+1}} & \text{if } 0 \leq \theta < \frac{1}{2} \\ (1-s) + \frac{\theta}{2^{N+1}} & \text{if } \frac{1}{2} \leq \theta < 1 \end{cases} \tag{4}$$

Let us now check that F maps $[0, \frac{1}{2}) \cap H_{c'}$ into $P_{c''} \subseteq P_c$ (then the other half follows by symmetry). In order to verify the claim, let us pick $\theta \in H_{c'}$, $0 < \theta < \frac{1}{2}$. We need to check that $\phi := F(\theta)$ satisfies:

(1) $\ell(\phi) \leq \ell''$;
(2) $T^n(\ell(\phi)) \leq \ell(\phi) \qquad \forall n \geq 0$.

(1) Since θ belongs to $H_{c'}$, by Proposition 9.3 we have

$$\ell(\theta) \geq L_{c'} \geq L_{c''}.$$

Moreover, equation (4) implies

$$\ell(\phi) = 2s + 2^{-N}(1 - \ell(\theta)/2)$$

while by the definition of s one has

$$\ell'' = 2s + 2^{-N}(1 - L_{c''}/2)$$

hence combining with the previous inequality we get $\ell(\phi) \leq \ell''$.

(2) If $1 \leq n < N$, then either $T^n(\ell(\phi)) \leq \frac{1}{2} < \ell(\phi)$, or $T^n(\ell(\phi))$ is of the form

$$0.1^{s_k}0^{s_{k+1}} \ldots 0^{s_n} \ldots$$

which is less than $0.1^{s_1}0^{s_2} \ldots 1^{s_n}$ because of dominance. If instead $n > N$, $T^n(\ell(\phi)) = T^{n-N-1}(\ell(\theta)) \leq \ell'$, and $\ell' < \ell(\phi)$ because $\ell(\phi)$ begins with $0.1^{s_1}0^{s_2} \ldots 0^{s_n}$, and S is not a prefix of $w_{\theta'}$. Finally, let $\hat{\theta} := \max\{\theta, 1-\theta\}$ and analyze the N^{th} iterate: we have

$$T^N(\ell(\phi)) = \hat{\theta} \leq 2s + \frac{\hat{\theta}}{2^N} = \ell(\phi)$$

because $\hat{\theta}$ belongs to $H_{c'} \subseteq H_{c''}$, and $\max\{\theta \in [0,1] \ : \theta \in H_{c''}\} = 2s/(1-2^{-N})$. $\square$

12. Renormalization and tuning

The Mandelbrot set has the remarkable property that near every point of its boundary there are infinitely many copies of the whole $\mathcal{M}$, called *baby Mandelbrot sets*. A *hyperbolic component* W of the Mandelbrot set is a connected component of the interior of $\mathcal{M}$ such that all $c \in W$, the orbit of the critical point is attracted to a periodic cycle under iteration of f_c.

Douady and Hubbard [DH] related the presence of baby copies of $\mathcal{M}$ to renormalization in the family of quadratic polynomials. More precisely, they associated to any hyperbolic component W a *tuning map* $\iota_W : \mathcal{M} \to \mathcal{M}$ which maps the main cardioid of $\mathcal{M}$ to W, and such that the image of the whole $\mathcal{M}$ under ι_W is a baby copy of $\mathcal{M}$.

The tuning map can be described in terms of external angles in the following terms [Do1]. Let W be a hyperbolic component, and η_0, η_1 the angles of the two external rays which land on the root of W. Let $\eta_0 = 0.\overline{\Sigma_0}$ and $\eta_1 = 0.\overline{\Sigma_1}$ be the (purely periodic) binary expansions of the two angles which land at the root of W. Let us define the map $\tau_W : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ in the following way:

$$\theta = 0.\theta_1\theta_2\theta_3 \ldots \mapsto \tau_W(\theta) = 0.\Sigma_{\theta_1}\Sigma_{\theta_2}\Sigma_{\theta_3} \ldots$$

where $\theta = 0.\theta_1\theta_2 \ldots$ is the binary expansion of θ, and its image is given by substituting the binary string Σ_0 to every occurrence of 0 and Σ_1 to every occurrence of 1.

Proposition 12.1 ([Do3], Proposition 7)**.** *The map τ_W has the property that, if θ is a characteristic angle of the parameter $c \in \partial\mathcal{M}$, then $\tau_W(\theta)$ is a characteristic angle of the parameter $\iota_W(c)$.*

If W is a real hyperbolic component, then ι_W preserves the real axis. The image of the tuning operator is the *tuning window* $\Omega(W)$ with

$$\Omega(W) := [\omega(W), \alpha(W)]$$

where

$$\begin{aligned} \alpha(W) &:= 0.\overline{\Sigma_0} \\ \omega(W) &:= 0.\Sigma_0\overline{\Sigma_1}. \end{aligned}$$

The point $\alpha(W)$ will be called the *root* of the tuning window. Overlapping tuning windows are nested, and we call *maximal tuning window* a tuning window which is not contained in any other tuning window.

Let us describe the behavior of Hausdorff dimension with respect to the tuning operator:

Proposition 12.2. *Let W be a hyperbolic component of period p with root $r(W)$, and let $c \in \mathcal{M}$. Then we have the equalities*

$$\text{H.dim } H_{\tau_W(c)} = \max\left\{\text{H.dim } H_{r(W)}, \text{H.dim } \tau_W(H_c)\right\}$$

$$\text{H.dim } P_{\tau_W(c)} = \max\left\{\text{H.dim } P_{r(W)}, \text{H.dim } \tau_W(P_c)\right\}.$$

Moreover,

$$\text{H.dim } \tau_W(H_c) = \frac{1}{p}\text{H.dim } H_c.$$

Proof. Let $c' := \tau_W(c)$. The Julia set of $f_{c'}$ is constructed by taking the Julia set of $f_{r(W)}$ and inserting a copy of the Julia set of f_c inside every bounded Fatou component. Hence in particular, the extended Hubbard tree of $J(f_{c'})$ contains a topological copy T_1 of the extended Hubbard tree of $f_{r(W)}$ which contains the critical value c'. The set of angles which land on T_1 are precisely the image $\tau_W(H_c^{ext})$ via tuning of the set H_c^{ext} of angles which land on the extended Hubbard tree of H_c. Let $\theta \in H_{c'}$ be an angle whose ray lands on the Hubbard tree of $f_{c'}$. Then either θ also belongs to

$H_{r(W)}$ or it lands on a small copy of the extended Hubbard tree of $f_{r(W)}$, hence it eventually maps to T_1. Hence we have the inclusions

$$H_{r(W)} \cup \tau_W(H_c) \subseteq H_{c'} \subseteq H_{r(W)} \cup \bigcup_{n \geq 0} D^{-n}(\tau_W(H_c^{ext}))$$

from which the claim follows, recalling that $H_c^{ext} \setminus \{-\beta, \beta\} \subseteq \bigcup_{n\geq 0} D^{-n}(H_c)$.

In parameter space, one notices that the set of rays landing on the vein v for c' either land between 0 and $r(W)$, or between $r(W)$ and c'. In the latter case, they land on the small copy of the Mandelbrot set with root $r(W)$, so they are in the image of τ_W. Hence

$$P_{c'} = P_{r(W)} \cup \tau_W(P_c)$$

and the claim follows. The last claim follows by looking at the commutative diagram

$$\begin{tikzcd} H_c \arrow[loop above, "D"] \arrow[r, "\tau_W"] & \tau_W(H_c). \arrow[loop above, "D^p"] \end{tikzcd}$$

Since τ_W is injective and continuous restricted to H_c (because H_c does not contain dyadic rationals) we have by Proposition 5.3

$$h_{top}(D, H_c) = h_{top}(D^p, \tau_W(H_c))$$

and, since H_c is forward invariant we can apply Proposition 5.5 and get

$$\text{H.dim } \tau_W(H_c) = \frac{h_{top}(D^p, \tau_W(H_c))}{p \log 2} = \frac{1}{p}\frac{h_{top}(D, H_c)}{\log 2} = \frac{1}{p}\text{H.dim } H_c$$

from which the claim follows.

□

12.1. **Scaling and continuity at the Feigenbaum point.** Among all tuning operators is the operator τ_W where W is the basilica component of period 2 (the associated strings are $\Sigma_0 = 01$, $\Sigma_1 = 10$). We will denote this particular operator simply with τ. The fixed point of τ is the external angle of the *Feigenbaum point* c_{Feig}.

Let us explicitly compute the dimension at the Feigenbaum parameter. Indeed, let c_0 be the airplane parameter of angle $\theta_0 = 3/7$, and consider the sequence of parameters of angles $\theta_n := \tau^n(\theta_0)$ given by successive tuning.

The set H_{c_0} is given by all angles with binary sequences which do not contain 3 consecutive equal symbols, hence the Hausdorff dimension is easily computable (see example 4 in the introduction):

$$\text{H.dim } H_{\theta_0} = \log_2 \frac{\sqrt{5}+1}{2}.$$

Now, by repeated application of Proposition 12.2 we have

$$\text{H.dim } H_{\theta_n} = \frac{\text{H.dim } H_{\theta_0}}{2^n}.$$

Note that the angles θ_n converge from above to the Feigenbaum angle θ_F, also $\text{H.dim } H_{c_{Feig}} = 0$; moreover, since θ_n is periodic of period 2^n,

$$\theta_n - \theta_F \asymp 2^{-2^n}$$

and together with

$$\text{H.dim } H_{\theta_n} - \text{H.dim } H_{\theta_F} = \frac{\text{H.dim } H_{\theta_0}}{2^n} \tag{5}$$

we have proved the

Proposition 12.3. *For the Feigenbaum parameter c_{Feig} we have*

$$\text{H.dim } S_{c_{Feig}} = 0$$

and moreover, the entropy function $\theta \mapsto h(\theta)$ is not Hölder-continuous at the Feigenbaum point. Similarly, the dimension of the set of biaccessible angles for the Feigenbaum parameter is 0.

Note that it also follows that the entropy $h(c) := h_{top}(f_c, [-\beta, \beta])$ as a function of the parameter c has vertical tangent at $c = c_{Feig}$, as shown in Figure 6. Indeed, if $c_n \to c_{Feig}$ is the sequence of period doubling parameters converging to the Feigenbaum point, it is a deep result [Ly2] that $|c_n - c_{Feig}| \asymp \lambda^{-n}$, where $\lambda \cong 4.6692\ldots$ is the Feigenbaum constant; hence, by equation (5), we have

$$\frac{h(c_n) - h(c_{Feig})}{|c_n - c_{Feig}|} \asymp \left(\frac{\lambda}{2}\right)^n \to \infty.$$

12.2. **Proof of Theorem 8.3.** Let us now turn to the proof of equality of dimensions between H_c and P_c. Recall we already established $P_c \subseteq H_c$, hence we are left with proving that for all real parameters $c \in \partial\mathcal{M} \cap \mathbb{R}$,

$$\text{H.dim } H_c \leq \text{H.dim } P_c.$$

By Proposition 12.3, the inequality holds for the Feigenbaum point and for all $c > c_{Feig}$. Moreover, by Proposition 11.1 and continuity of entropy ([MT], see also section 16), we have the inequality for any $c \in \partial\mathcal{M} \cap \mathbb{R}$ which is non-renormalizable. Let now τ be the tuning operator whose fixed point is the Feigenbaum point: since the root of its tuning window is the basilica map which has zero entropy, by Proposition 12.2 we have, for each $n \geq 0$ and each $c \in \mathcal{M}$,

$$\text{(6)} \qquad \text{H.dim } H_{\tau^n(c)} = \text{H.dim } \tau^n(H_c) \qquad \text{H.dim } P_{\tau^n(c)} = \text{H.dim } \tau^n(P_c).$$

Now, each renormalizable parameter $c \in \mathcal{M} \cap (-2, c_{Feig})$ is either of the form $c = \tau^n(c_0)$ with c_0 non-renormalizable, or $c = \tau^n(\tau_W(c_0))$ with W a real hyperbolic component such that its root $r(W)$ is outside the baby Mandelbrot set determined by the image of τ.

(1) In the first case we note that (since tuning operators behave well under the operation of concatenation of binary strings), by applying the operator τ^n to both sides of the inclusion of Proposition 11.1 we get for each $c' > c_0$ a piecewise linear map F_0 such that

$$F_0(\tau^n(H_{c'})) \subseteq \tau^n(P_{c_0})$$

hence, by continuity of entropy and of tuning operators,

$$\text{H.dim } H_c = \sup_{c'>c_0} \text{H.dim } H_{\tau^n(c')} = \text{H.dim } \tau^n(H_{c'}) \leq \text{H.dim } \tau^n(P_{c_0}) = \text{H.dim } P_c.$$

(2) In the latter case $c = \tau^n(\tau_W(c_0))$, by Proposition 12.2 we get

$$\tau^n(P_{\tau_W(c_0)}) = \tau^n(P_{r(W)}) \cup \tau^n(\tau_W(P_{c_0}))$$

and since the period of W is larger than 2 we have the inequality

$$\text{H.dim } \tau^n(\tau_W(P_{c_0})) \leq \text{H.dim } \tau^{n+1}(P_{c_0}) \leq \text{H.dim } \tau^{n+1}(\mathcal{R}) \leq \tau^n(P_{r(W)})$$

where in the last inequality we used the fact that the set of rays $\tau(\mathcal{R})$ land to the right of the root $r(W)$. Thus we proved that

$$\text{H.dim } \tau^n(P_{\tau_W(c_0)}) = \text{H.dim } \tau^n(P_{r(W)})$$

and the same reasoning for H_c yields

$$\text{H.dim } \tau^n(H_{\tau_W(c_0)}) = \text{H.dim } \tau^n(H_{r(W)}).$$

Finally, putting together the previous equalities with eq.(6) and applying the case (1) to $\tau^n(r(W))$ (recall $r(W)$ is non-renormalizable), we have the equalities

$$\text{H.dim } P_c = \text{H.dim } \tau^n(P_{\tau_W(c_0)}) = \text{H.dim } \tau^n(P_{r(W)}) = \text{H.dim } P_{\tau^n(r(W))} =$$

$$= \text{H.dim } H_{\tau^n(r(W))} = \text{H.dim } \tau^n(H_{r(W)}) = \text{H.dim } \tau^n(H_{\tau_W(c_0)}) = \text{H.dim } H_c.$$

12.3. **Density of dominant parameters.** In order to prove Proposition 10.6, we will need the following definitions: given a string S, the set of its *prefixes-suffixes* is

$$\begin{aligned} PS(S) &:= \{Y : Y \text{ is both a prefix and a suffix of } S\} = \\ &= \{Y : Y \neq \emptyset, \exists\ X, Z \text{ s.t. } S = XY = YZ\}. \end{aligned}$$

Note that an extremal string S of even length is dominant if and only if $PS(S)$ is empty. Moreover, let us define the set of *residual suffixes* as

$$RS(S) := \{Z\ :\ S = YZ,\ Y \in PS(S)\}.$$

Proof of Proposition 10.6. By density of the roots of the maximal tuning windows in the set of non-renormalizable angles, it is enough to prove that every $\theta \in (0, \frac{1}{2})$ which is root of a maximal tuning window, $\theta \neq 1/3$, can be approximated from the right by dominant points. Hence we can assume $w_\theta = \overline{S}$, S an extremal string of even length, and 1 is not a prefix of S. If S is dominant, a sequence of approximating dominant parameters is given by the strings

$$S^n 11, \qquad n \geq 1.$$

The rest of the proof is by induction on $|S|$. If $|S| = 2$, then S itself is dominant and we are in the previous case. If $|S| > 2$, either S is dominant and we are done, or $PS(S) \neq \emptyset$ and also $RS(S) \neq \emptyset$. Let us choose $Z_\star \in RS(S)$ such that

$$\overline{Z_\star} := \min\{\overline{Z}\ :\ Z \in RS(S)\}$$

and $Y_\star \in PS(S)$ such that $S = Y_\star Z_\star$. Let $\alpha(Y_\star)$ be the root of the maximal tuning window $\overline{Y_\star}$ belongs to. Then by Lemma 12.7, $\overline{Z_\star} > \alpha(Y_\star)$, and by minimality

$$\alpha(Y_\star) < \overline{Z} \quad \forall Z \in RS(S).$$

Now, since $Y_\star$ has odd length and belongs to the window of root $\alpha(Y_\star)$, then one can write $\alpha(Y_\star) = \overline{P}$ with $Y_\star << P$, hence also $S << P$. Moreover,

$$|P| \leq |Y_\star| + 1 \leq |S|$$

and actually $|Y_\star| + 1 < |S|$ because otherwise the first digit of $Y_\star$ would appear twice at the beginning of S, contradicting the fact that S is extremal. Suppose now $\alpha(Y_\star) \neq \overline{1}$. Then $|P| < |S|$ and by induction there exists $\gamma = \overline{T}$ such that T is dominant,

$$\alpha(Y_\star) < \overline{T} < \overline{Z} \quad \forall Z \in RS(S)$$

and γ can also be chosen close enough to $\alpha(Y_\star)$ so that P is prefix of T, which implies

$$S << T.$$

By Lemma 12.5, $S^n T^m$ is a dominant string for m large enough, of even length if m is even, and arbitrarily close to $\overline{S}$ as n tends to infinity. If $\alpha(Y_\star) = \overline{1}$, the string $S^n 1^{2m}$ is also dominant for n, m large enough. $\square$

Lemma 12.4. *If S is an extremal string and $Y \in PS(S)$, then Y is an extremal string of odd length.*

Proof. Suppose $S = XY = YZ$. Then by extremality $XY < YX$, hence $XYY < YXY$ and, by substituting YZ for XY, $YZY < YYZ$. If $|Y|$ were even, it would follow that $ZY < YZ$, which contradicts the extremality of $S = YZ < ZY$. Hence $|Y|$ is odd. Suppose now $Y = AB$, with A and B non-empty strings. Then $S = XAB < BXA$. By considering the first $k := |Y|$ characters on both sides of this equation, $Y = AB = S_1^k \leq (BXA)_1^k = BA$. If $Y = AB = BA$, then $Y = P^k$ for some string P, hence by Lemma 10.1 we have $PZP^{k-1} < P^k Z = S$, which contradicts the extremality of S, hence $AB < BA$ and Y is extremal. □

Lemma 12.5. *Let S be an extremal string of even length, and T be a dominant string. Suppose moreover that*

(1) $S << T$;

(2) $\overline{T} < \overline{Z} \quad \forall Z \in RS(S)$.

Then, for any $n \geq 1$ and for m sufficiently large, $S^n T^m$ is a dominant string.

Proof. Let us check that $S^n T^m$ by checking all its splittings. We have four cases:

(1) From (1), we have

$$S^n T^m << T^a, \quad a \geq 1$$

$$S^n T^m << S^b T^m, \quad b < n.$$

(2) If $S = XY$, $XY << YX$ by extremality, hence

$$S^n T^m << Y S^b T^m \quad \forall b \geq 1.$$

(3) Since T is dominant, $T << U$ whenever $T = QU$, thus

$$S^n T^m << T << U.$$

(4) One is left to prove that $S^n T^m << Y T^m$ whenever $S = XY$. If $Y \notin PS(S)$, then $XY << Y$ and the proof is complete. Otherwise, $S = XY = YZ$, $|Y| \equiv 1 \mod 2$ by Lemma 12.4. Moreover, since $YZ < ZY$, by a few repeated applications of Lemma 10.1, we have $\overline{ZS^{n-1}} > \overline{Z}$, hence (2) implies $\overline{T} < \overline{ZS^{n-1}}$, and by Lemma 12.6 we have $ZS^{n-1}\overline{T} > \overline{T}$, hence for m large enough $ZS^{n-1}T^m >> T^m$ and then

$$S^n T^m << Y T^m.$$

$\square$

Lemma 12.6. *Let Y, Z be finite strings of positive integers such that $\overline{Y} < \overline{Z}$. Then*

$$Z\overline{Y} > \overline{Y}.$$

Proof. By Lemma 10.1, for any $k \geq 0$ we have

$$\overline{Y^k} < \overline{Z} \Rightarrow Y^k Z < Z Y^k$$

hence, by taking the limit as $k \to \infty$, $Z\overline{Y} \geq \overline{Y}$. Equality cannot hold because otherwise Y and Z have to be multiple of the same string, which contradicts the strict inequality $\overline{Y} < \overline{Z}$. $\square$

Lemma 12.7. *Let θ be a non-renormalizable, real parameter angle such that $w_\theta = \overline{S}$ and S is an extremal string of even length, and let $Y \in PS(S)$, $S = YZ$. Let ϕ the parameter angle such that $w_\phi = \overline{Y}$, and let $\Omega = [\omega, \alpha]$ be the maximal tuning window which contains ϕ. Then if $w_\alpha = \overline{S_0}$, we have*

$$\overline{Z} > \overline{S_0}.$$

Proof. Since ϕ lies in the tuning window Ω, Y is a concatenation of the strings S_0 and S_1. As a consequence, $Y\overline{S_0}$ is also a concatenation of strings S_0 and S_1, so $Y\overline{S_0} \geq S_1\overline{S_0}$. Moreover, by Lemma 10.1, $\overline{S} < \overline{Y} < \overline{S_0}$. We now claim that

$$\beta := \overline{ZY} > \overline{S_0}.$$

Indeed, suppose $\beta \leq \overline{S_0}$; then, $\overline{S} = Y\beta \geq Y\overline{S_0} \geq S_1\overline{S_0}$, which combined with the fact that $\overline{S} < \overline{S_0}$ implies θ lies in the tuning window Ω, contradicting the fact that θ is non-renormalizable.

Now, suppose $\overline{Z} \leq \overline{S_0}$; then $\overline{Z} \leq \overline{S_0} \leq \overline{ZY}$, which implies Z has to be prefix of $\overline{S_0}$, hence $Z = S_0^k V$ with V prefix of S_0, $V \neq \emptyset$ since $|Z|$ is odd. If $S_0 \neq (1,1)$, then S_0 is extremal and, by Lemma 10.1, $\overline{Z} = \overline{S_0^k V} > \overline{S_0}$, contradiction. In the case $S_0 = (1, 1)$, then Z must be just a sequence of 1's of odd length, which forces $\overline{S} = \overline{1}$, hence S cannot be extremal. $\square$

13. The exceptional set for continued fractions

We will now start by constructing the exceptional set $\mathcal{E}$ for continued fractions. Let $S = (s_1, \ldots, s_n)$ be a finite string of positive integers: we will use the notation

$$[S] := [s_1, \ldots, s_n] = \cfrac{1}{s_1 + \cfrac{1}{\ddots + \frac{1}{s_n}}}.$$

Moreover, $\overline{S}$ will be the periodic infinite string SSS... and $[\overline{S}]$ the quadratic irrational with purely periodic continued fraction $[\overline{s_1, \ldots, s_n}]$. The symbol $|S|$ will denote the length of the string S. We will denote the denominator of the rational number r as $\mathrm{den}(r)$.

13.1. **Pseudocenters.** Let us start out by defining a useful tool in our analysis of intervals defined by continued fractions.

Lemma 13.1. *Let $J = (\alpha, \beta)$, $\alpha, \beta \in \mathbb{R}$, $|\alpha - \beta| < 1$. Then there exists a unique rational $p/q \in J$ such that $q = \min\{q' \geq 1 \ : \ p'/q' \in J\}$.*

Proof. Let $d := \min\{q \geq 1 \ : \ p/q \in J\}$. If $d = 1$ we are done. Let $d > 1$ and assume by contradiction that $\frac{c}{d}$ and $\frac{c+1}{d}$, both belong to J. Then there exists $k \in \mathbb{Z}$ such that $\frac{k}{d-1} < \frac{c}{d} < \frac{c+1}{d} < \frac{k+1}{d-1}$, hence $cd - c - 1 < kd < cd - c$, which is a contradiction since kd is an integer. $\square$

Definition 13.2. *The number $\frac{p}{q}$ which satisfies the properties of the previous lemma will be called the* pseudocenter *of J.*

Lemma 13.3. *Let $\alpha, \beta \in (0,1)$ be two irrational numbers with continued fraction expansions $\beta = [S, b_0, b_1, b_2, \ldots]$ and $\alpha = [S, a_0, a_1, a_2, \ldots]$, where S stands for a finite string of positive integers. Assume $b_0 > a_0$. Then the pseudocenter of the interval J with endpoints α and β is*

$$r = [S, a_0 + 1] (= [S, a_0, 1]).$$

Proof. Suppose there exists $s \in \mathbb{Q} \cap J$ with $\mathrm{den}(s) < \mathrm{den}\ (r)$. Since $s \in J$, then $s = [0; S, s_0, s_1, \ldots, s_k]$ with $a_0 \le s_0 \le b_0$ and $k \ge 0$. The choice $s_0 \ge a_0 + 1$ gives rise to $\mathrm{den}(s) \ge \mathrm{den}(r)$, so $s_0 = a_0$. On the other hand, $[S, a_0]$ does not belong to the interval, so $k \ge 1$ and $s_1 \ge 1$, still implying $\mathrm{den}(s) \ge \mathrm{den}(r)$. $\square$

13.2. **Quadratic intervals.**

Definition 13.4. *Let $0 < a < 1$ be a rational number with continued fraction expansion*

$$a = [a_1, \ldots, a_N] = [a_1, \ldots, a_N - 1, 1], \ a_N \ge 2.$$

We define the quadratic interval I_a *associated to a to be the open interval with endpoints*

$$[\overline{a_1, \ldots, a_{N-1}, a_N}] \quad \text{and} \quad [\overline{a_1, \ldots, a_{N-1}, a_N - 1, 1}]. \tag{7}$$

Moreover, we define $I_1 := (\frac{\sqrt{5}-1}{2}, 1]$ (recall that $\frac{\sqrt{5}-1}{2} = [\overline{1}]$).

The exceptional set $\mathcal{E}$ *is defined as the complement of all quadratic intervals:*

$$\mathcal{E} := [0,1] \setminus \bigcup_{r \in \mathbb{Q} \cap (0,1]} I_r.$$

Note that the ordering of the endpoints in (7) depends on the parity of N: given $a \in \mathbb{Q}$, we will denote by A^+ and A^- the two strings of positive integers which represent a as a continued fraction, with the convention that A^+ is the string of *even* length and A^- the string of *odd* length, so that

$$I_a = ([\overline{A^-}], [\overline{A^+}]), \qquad a = [A^+] = [A^-].$$

Example

If $a = \frac{1}{3} = [3] = [2, 1]$, $[\overline{A^+}] = [\overline{2, 1}]$, $[\overline{A^-}] = [\overline{3}]$, $I_a = (\frac{\sqrt{13}-3}{2}, \frac{\sqrt{3}-1}{2})$.

Note that a is the pseudocenter of I_a, hence by uniqueness $I_a = I_{a'} \Leftrightarrow a = a'$.

Lemma 13.5. *Quadratic intervals satisfy the following properties:*

(1) *If $\xi \in \overline{I}_a$, then a is a convergent to ξ.*

(2) *If $I_a \cap I_b \neq \emptyset$, then either a is a convergent to b or b is a convergent to a.*

(3) *If $I_a \subsetneq I_b$ then b is convergent to a, hence $den(a) < den(b)$.*

Proof. (1) Since $\xi \in I_a$, either $\xi = [a_1, \dots, a_N, \dots]$ or $\xi = [a_1, \dots, a_N - 1, \dots]$. In the first case the claim holds; in the second case one has to notice that neither $[a_1, \dots, a_N - 1]$ nor all elements of the form $[a_1, \dots, a_N - 1, k, \dots]$ with $k \geq 2$ belong to I_a, so $k = 1$ and a is a convergent of ξ.

(2) Fix $\xi \in I_a \cap I_b$. By the previous point, both a and b are convergents of ξ, hence the rational with the shortest expansion is a convergent of the other.

(3) From (1) since $a \in I_a \subseteq I_b$. □

Definition 13.6. *A quadratic interval I_a is* maximal *if it is not properly contained in any I_b with $b \in \mathbb{Q} \cap (0, 1]$.*

The interest in maximal quadratic intervals lies in the

Proposition 13.7. *Every quadratic interval I_a is contained in a unique maximal quadratic interval.*

A good way to visualize the family of quadratic intervals is to plot, for any rational a, the geodesic γ_a on the hyperbolic upper half plane with the same endpoints as I_a, as in Figure 2 in the introduction: one can see the maximal intervals corresponding to the "highest" geodesics, in such a way that every γ_a has some maximal geodesic (possibly itself) above it and no two maximal γ_a intersect.

The proof of Proposition 13.7 will be given in two lemmas:

Lemma 13.8. *Every quadratic interval I_a is contained in some maximal quadratic interval.*

Proof. If I_a were not contained in any maximal interval, then there would exist an infinite chain $I_a \subsetneq I_{a_1} \subsetneq I_{a_2} \subsetneq \dots$ of proper inclusions, hence by the lemma every a_i is a convergent of a, but rational numbers can only have a finite number of convergents. □

Lemma 13.9. *If I_a is maximal then for all $a' \in \mathbb{Q} \cap (0,1)$*

$$I_a \cap I_{a'} \neq \emptyset \ \Rightarrow I_{a'} \subset I_a,$$

and equality holds iff $a = a'$. In particular, distinct maximal intervals do not intersect.

Proof. We need the following lemma, for the proof of which we refer to ([CT], section 4):

Lemma 13.10. *If $I_a \cap I_b \neq \emptyset$, $I_a \setminus I_b \neq \emptyset$ and $I_b \setminus I_a \neq \emptyset$, then either I_a or I_b is not maximal.*

Let now I_{a_0} be the maximal interval which contains $I_{a'}$. Since $I_a \cap I_{a_0} \neq \emptyset$, by Lemma 13.10 either $I_a \subseteq I_{a_0}$ or $I_{a_0} \subseteq I_a$, hence by maximality $I_a = I_{a_0}$ and $I_{a'} \subseteq I_a$. Since a is the pseudocenter of I_a, $I_a = I_{a'} \Rightarrow a = a'$. □

13.3. **The bisection algorithm.** We will now describe an algorithmic way to produce all maximal intervals. This will also provide an alternative proof of the fact the $\mathcal{E}$ has zero measure.

Let $\mathcal{F}$ be a family of disjoint open intervals which *accumulate only at* 0, i.e. such that for every $\epsilon > 0$ the set $\{J \in \mathcal{F} : J \cap [\epsilon, 1] \neq \emptyset\}$ is finite, and denote $F = \bigcup_{J \in \mathcal{F}} J$. The complement $(0,1] \setminus F$ will then be a countable union of closed disjoint intervals C_j, which we refer to as *gaps*. Note that some C_j may well be a single point. To any gap which is not a single point we can associate its pseudocenter $c \in \mathbb{Q}$ as defined in the previous sections, and moreover consider the interval I_c associated to this rational value. The following proposition applies.

Proposition 13.11. *Let I_a and I_b be two maximal intervals such that the gap between them is not a single point, and let c be the pseudocenter of the gap. Then I_c is a maximal interval and it is disjoint from both I_a and I_b.*

Proof. Pick I_{c_0} maximal such that $I_c \subseteq I_{c_0}$, so by Lemma 13.5 $den(c_0) \leq den(c)$. On the other hand, since maximal intervals do not intersect, then I_{c_0} is contained in the gap and since c is pseudocenter, then $den(c) \leq den(c_0)$ and equality holds only if $c = c_0$. □

The proposition implies that if we add to the family of maximal intervals $\mathcal{F}$ all intervals which arise as gaps between adjacent intervals then we will get another family of maximal (hence disjoint) intervals, and we can iterate the procedure.

For instance, let us start with the collection $\mathcal{F}_1 := \{I_{1/n}, n \geq 1\}$. All these intervals are maximal, since the continued fraction of their pseudocenters has only one digit (apply Lemma 13.5).

Let us now construct the families of intervals $\mathcal{F}_n$ recursively as follows:

$$\mathcal{G}_n := \{C \text{ connected component of } (0,1] \setminus F_n\}$$

$$\mathcal{F}_{n+1} := \mathcal{F}_n \cup \{I_r \;:\; r \text{ pseudocenter of } C, C \in \mathcal{G}_n, C \text{ not a single point}\}$$

(where F_n denotes the union of all intervals belonging to $\mathcal{F}_n$).

It is thus clear that the union $\mathcal{F}_\infty := \bigcup \mathcal{F}_n$ will be a countable family of maximal intervals. The union of all elements of $\mathcal{F}_\infty$ will be denoted by F_∞; its complement (the set of numbers which do not belong to any of the intervals produced by the algorithm) has the following property:

Lemma 13.12. *The set $(0,1) \setminus F_\infty$ consists of irrational numbers of bounded type; more precisely, the elements of $(\frac{1}{n+1}, \frac{1}{n}] \setminus F_\infty$ have partial quotients bounded by n.*

Proof. Let $\gamma = [c_1, c_2, ..., c_n, ...] \notin F_\infty$; we claim that $c_k \leq c_1$ for all $k \in \mathbb{N}$. Since $\gamma \notin F_\infty$, $\forall n \geq 1$ we can choose $J_n \in \mathcal{G}_n$ such that $\gamma \in J_n$. Clearly, $J_{n+1} \subseteq J_n$.

Furthermore, γ cannot be contained in either $I_{\frac{1}{c_1}}$ nor $I_{\frac{1}{c_1+1}}$, so all J_n are produced by successive bisection of the gap $([\overline{c_1, 1}], [\overline{c_1}])$, hence by Lemma 13.3 for every n, the endpoints of J_n are quadratic irrationals with continued fraction expansion bounded by c_1. It may happen that there exists n_0 such that $J_n = \{\gamma\}\ \forall n \geq n_0$, so γ is an endpoint of J_{n_0}, hence it is irrational and c_1-bounded. Otherwise, let p_n/q_n be the pseudocenter of J_n; by uniqueness of the pseudocenter, diam $J_n \leq 2/q_n$, and $q_{n+1} > q_n$ since $J_{n+1} \subseteq J_n$. This implies γ cannot be rational, since the minimum denominator of a rational sitting in J_n is $q_n \to +\infty$. Moreover, diam $J_n \to 0$, so γ is limit point of endpoints of the J_n, which are c_1-bounded, hence γ is also c_1-bounded. □

Proposition 13.13. *The family $\mathcal{F}_\infty$ is precisely the family of all maximal intervals; hence $F_\infty = [0,1] \setminus \mathcal{E}$.*

Proof. If I_c a maximal interval does not belong to $\mathcal{F}_\infty$, then its pseudocenter belongs to the complement of F_∞, but the previous lemma asserts that this set does not contain any rational. □

Note that Proposition 13.13 and Lemma 13.12 imply that the exceptional set $\mathcal{E}$ consists of numbers of bounded type, hence it has zero measure.

13.4. **Maximal intervals and strings.** In order to deal with strings representing continued fractions, recall the total ordering on the space of finite strings of given length (section 10): given two distinct finite strings S and T of equal length, let $l := \min\{i : S_i \neq T_i\}$. We will set

$$S < T := \begin{cases} S_l < T_l \text{ if } l \equiv 0 \mod 2 \\ S_l > T_l \text{ if } l \equiv 1 \mod 2. \end{cases}$$

The exact same definition also gives a total ordering on the space of infinite strings. Note that if S and T have equal length $L \in \mathbb{N} \cup \{\infty\}$,

$$S < T \Leftrightarrow [S] < [T]$$

i.e. this ordering can be obtained by pulling back the order structure on $\mathbb{R}$, via identification of a string with the value of the corresponding continued fraction.

The set of pseudocenters of maximal quadratic intervals is a canonically defined subset of $\mathbb{Q} \cap (0,1)$ and will be denoted by

$$\mathbb{Q}_E := \{r \in (0,1) \ : \ I_r \text{ is maximal}\}.$$

Using the order on the set of strings, we can give an explicit characterization of the continued fraction expansion of those rationals which are pseudocenters of maximal intervals:

Proposition 13.14 ([CT], Proposition 4.5)**.** *A rational number $r = [S]$ belongs to $\mathbb{Q}_E$ if and only if, for any splitting $S = AB$ of S into two strings A, B of positive length, either*

$$AB < BA$$

or $A = B$ with $|A|$ odd.

We shall sometimes refer to $\mathbb{Q}_E$ as the set of *extremal rational values.* Using the criterion, for instance, one can check that $[3,2]$ belongs to $\mathbb{Q}_E$ (because $(3,2) < (2,3)$), and so does $[3,3]$, while $[2,2,1,1]$ does not (indeed, $(2,1,1,2) < (2,2,1,1)$).

13.5. **An alternative description.** A striking feature of the exceptional set $\mathcal{E}$ is that, even though it was defined in terms of quadratic intervals and bisection algorithm, it has an equivalent characterization in terms of the dynamics of the Gauss map G. Indeed, it coincides with the set of points which are closer to zero than all their forward iterates:

Proposition 13.15. *The set $\mathcal{E}$ can be described as*

$$\mathcal{E} = \{x \in [0,1] \ : G^k(x) \geq x \ \ \forall k \in \mathbb{N}\}.$$

The proof is a simple consequence of Proposition 13.14: see [BCIT], Lemma 3.3.

We shall see that the set $\mathcal{E}$ arises in several contexts related to the dynamics of continued fractions. In particular, it is the bifurcation set for the family of α-continued fractions (see section 19) and for the set of numbers of generalized bounded type (see section 14.4). We shall now see a third interpretation, related to Sturmian sequences (and hence to geodesics on the flat torus).

13.6. **The recurrence spectrum of Sturmian sequences.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. A *Sturmian sequence* of slope α is a binary sequence of the type

$$S_{\alpha,\beta} = \lfloor \alpha(n+1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor \quad \text{or} \quad S_{\alpha,\beta} = \lceil \alpha(n+1) + \beta \rceil - \lceil \alpha n + \beta \rceil \tag{8}$$

where β is some other real value. Sturmian sequences have also a geometric interpretation: they can be viewed as cutting sequences of half-lines on the plane with respect to the integral lattice $\mathbb{Z}^2$.

Given a sequence X (finite or infinite) and a positive integer m, the set of m-factors of X is the set of substrings of X of length m:

$$\mathcal{F}_m(X) := \{S = (x_{n+1}, \ldots, x_{n+m}) \ : 0 \leq n < |X| - m + 1\}.$$

The *recurrence function* of a binary sequence $X \in \{0,1\}^{\mathbb{N}}$ is the function $R_X : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by[1]

$$R_X(n) := \inf\{m \in \mathbb{N} \ : \ \forall S \in \mathcal{F}_m(X), \ \ \mathcal{F}_n(S) = \mathcal{F}_n(X)\}$$

[1]We follow the usual convention $\inf \emptyset = +\infty$.

while the *recurrence quotient* of X is the maximal linear growth rate of $R_X(n)$:

$$R_X := \limsup_{n\to+\infty} \frac{R_X(n)}{n}.$$

It is well-known that the recurrence quotient of a Sturmian sequence $S_{\alpha,\beta}$ depends only on the continued fraction expansion of its slope $\alpha = [a_0; a_1, a_2, ...]$. In fact, the following formula holds ([Cas], Corollary 1):

$$R_{S_{\alpha,\beta}} = \rho(\alpha) := 2 + \limsup_{k\to+\infty} [a_k; a_{k-1}, a_{k-2}, ..., a_1].$$

So, if $\limsup a_k = N$, then $\rho(\alpha) \in (N+2, N+3)$; if otherwise α has unbounded partial quotients, then $\rho(\alpha) = +\infty$. The *recurrence spectrum of Sturmian sequences* is defined by

$$\mathcal{RS} := \{\rho(\alpha),\ \alpha \in \mathbb{R} \setminus \mathbb{Q}\};$$

it follows immediately from ([Cas], Theorem 1) and Proposition 13.15 that we have the following characterization of $\mathcal{RS}$ in terms of $\mathcal{E}$:

$$\mathcal{RS} = \left\{ 2 + \frac{1}{x} \ : \ x \in \mathcal{E} \right\}.$$

14. The correspondence

In this section we will establish the isomorphism between the combinatorial bifurcation set $\mathcal{R}$ of the real quadratic family and the exceptional set $\mathcal{E}$ for continued fractions, thus proving Theorem 1.1, and then draw consequences on the structure of these sets.

Let T, F, G denote respectively the *tent map*, the *Farey map* and the *Gauss map*, given by[2]

$$T(x) := \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1 \end{cases} \qquad F(x) := \begin{cases} \dfrac{x}{1-x} & \text{if } 0 \le x < \frac{1}{2} \\ \dfrac{1-x}{x} & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

and $G(0) := 0$, $G(x) := \left\{ \frac{1}{x} \right\}, \quad x \neq 0$.

The action of F and T can be nicely illustrated with different symbolic codings of numbers. Given $x \in [0,1]$ we can expand it in (at least) two ways: using a *continued fraction expansion*, i.e.

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \equiv [a_1, a_2, a_3, \dots] \quad , \quad a_i \in \mathbb{N}$$

and a *binary expansion*, i.e.

$$x = \sum_{i \ge 1} b_i \, 2^{-i} \equiv 0.b_1 \, b_2 \dots \quad , \quad b_i \in \{0,1\}$$

The action of T on binary expansions is as follows: for $\omega \in \{0,1\}^{\mathbb{N}}$,

$$T(0.\,0\,\omega) = 0.\,\omega \qquad T(0.\,1\,\omega) = 0.\,\hat{\omega} \tag{9}$$

where $\hat{\omega} = \hat{\omega}_1 \hat{\omega}_2 \dots$ and $\hat{0} = 1$, $\hat{1} = 0$. The actions of F and G are given by $F([a_1, a_2, a_3, \dots]) = [a_1 - 1, a_2, a_3, \dots]$ if $a_1 > 1$, while $F([1, a_2, a_3, ...]) = [a_2, a_3, ...]$, and $G([a_1, a_2, a_3, \dots]) = [a_2, a_3, \dots]$. As a matter of fact, the map G is obtained by

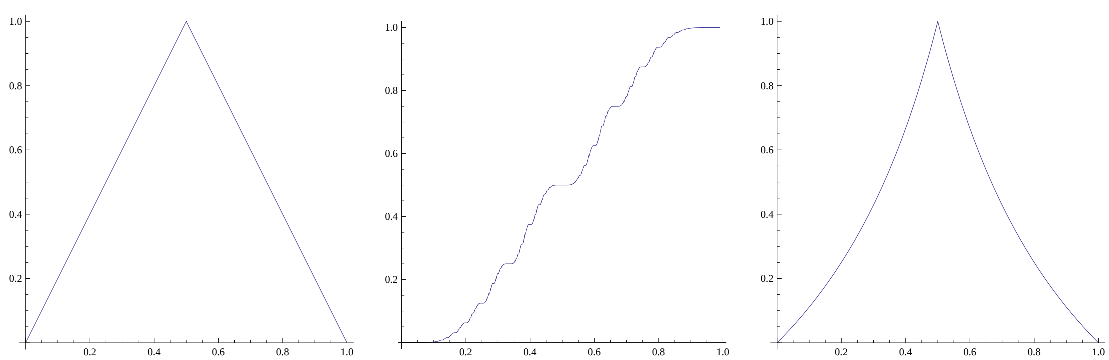
[2]Here $\lfloor x \rfloor$ and $\{x\}$ denote the integer and the fractional part of x, respectively, so that $x = \lfloor x \rfloor + \{x\}$.

accelerating the dynamics of F:

$$G(x) = F^{\lfloor 1/x \rfloor}(x) \qquad \text{if } x \neq 0. \tag{10}$$

Now, given $x = [a_1, a_2, a_3, \dots]$, one may ask what is the number obtained by interpreting the partial quotients a_i as the lengths of successive blocks in the dyadic expansion of a real number in $[0, 1]$; this defines *Minkowski's question mark* function $? : [0, 1] \to [0, 1]$

$$?(x) = \sum_{k \geq 1} (-1)^{k-1}\, 2^{-(a_1 + \cdots + a_k - 1)} = 0.\underbrace{00\dots0}_{a_1-1}\underbrace{11\dots1}_{a_2}\underbrace{00\dots0}_{a_3}\cdots \tag{11}$$



(A) Tent map (B) Minkowski map (C) Farey map

FIGURE 9. The tent, Minkowski and Farey maps.

The question mark function $?(x)$ has the following properties (see [Sa]):

- it is strictly increasing from 0 to 1 and Hölder continuous of exponent $\beta = \frac{\log 2}{2 \log \frac{\sqrt{5}+1}{2}}$;
- x is rational iff $?(x)$ is of the form $k/2^s$, with k and s integers;
- x is a quadratic irrational iff $?(x)$ is a (non-dyadic) rational;
- $?(x)$ is a singular function: its derivative vanishes Lebesgue-almost everywhere;
- it satisfies the functional equation $?(x) + ?(1 - x) = 1$.

14.1. **From continued fractions to kneading sequences.** We are now ready to prove Theorem 1.1, namely that the map $\varphi : [0,1] \to [\frac{1}{4}, \frac{1}{2}]$ given by

$$x = [a_1, a_2, a_3, ...] \mapsto \varphi(x) = 0.0\underbrace{11\dots1}_{a_1}\underbrace{00\dots0}_{a_2}\underbrace{11\dots1}_{a_3}\dots$$

is an orientation-reversing homeomorphism which takes $\mathcal{E}$ onto $\mathcal{R} \cap \left(0, \frac{1}{2}\right]$.

Proof of Theorem 1.1. The key step is that Minkowski's question mark function conjugates the Farey and tent maps, i.e.

$$?(F(x)) = T(?(x)) \qquad \forall x \in [0,1]. \tag{12}$$

Recall that by Proposition 9.5 the bifurcation set for the real quadratic family is characterized as

$$\mathcal{R} \setminus \{0\} = \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ T^n(\ell(D(\theta))) \geq \ell(D(\theta)) \ \ \forall n \geq 0\}$$

while by Proposition 13.15

$$\mathcal{E} = \{x \in [0,1] \ : \ F^n(x) \geq x \ \ \forall n \geq 0\}.$$

Hence, since the Minkowski map is a conjugacy, each $\theta \in \mathbb{R}/\mathbb{Z}$ belongs to $\mathcal{R}$ if and only if

$$\ell(D(\theta)) = ?(x)$$

for some $x \in \mathcal{E}$. The claim follows by writing out explicitly the question mark function as in eq. (11). $\square$

In the following subsections we will investigate a few consequences of such a correspondence.

14.2. **Binary pseudocenters and real hyperbolic components.** In section 13, we described an algorithm which produces all connected components of the complement of the exceptional set $\mathcal{E}$ by taking successive pseudocenters of nested gaps. Using the correspondence of Theorem 1.1 we shall now describe an algorithm to produce all real hyperbolic components by successive bisections. The equivalent concept to the pseudocenter of section 13.1 is the following:

Definition 14.1. *The* binary pseudocenter *of a real interval* $[a, b]$ *with* $|a - b| < 1$ *is the unique dyadic rational number* $\theta^* = p/2^q$ *with shortest binary expansion (i.e. with smallest* q*) among all numbers in* (a, b)*.*

E.g., the pseudocenter of the interval $[\frac{13}{15}, \frac{14}{15}]$ is $\frac{7}{8} = 0.111$, since $\frac{13}{15} = 0.\overline{1101}$ and $\frac{14}{15} = 0.\overline{1110}$. As a consequence, the set of all real hyperbolic components can be generated by a bisection algorithm:

Theorem 14.2. *Let* $c_1 < c_2$ *be two real parameters on the boundary of* $\mathcal{M}$*, with external angles* $0 \leq \theta_2 < \theta_1 \leq \frac{1}{2}$*. Let* θ^* *be the pseudocenter of the interval* $[\theta_2, \theta_1]$*, and let*

$$\theta^* = 0.s_1 s_2 \dots s_{n-1} s_n$$

be its binary expansion, with $s_n = 1$*. Then the hyperbolic component of smallest period in the interval* $[\theta_2, \theta_1]$ *is the interval of external angles* $[\alpha_2, \alpha_1]$ *with*

$$\begin{aligned} \alpha_1 &:= 0.\overline{s_1 s_2 \dots s_{n-1}} \\ \alpha_2 &:= 0.\overline{s_1 s_2 \dots s_{n-1} \check{s}_1 \check{s}_2 \dots \check{s}_{n-1}} \end{aligned}$$

where $\check{s}_i := 1 - s_i$*.*

Example

Suppose we want to find all hyperbolic components between the airplane parameter (of period 3) and the basilica parameter (of period 2). The ray landing on the root of the airplane component has angle $\theta_1 = \frac{3}{7}$, while the ray landing immediately to the left of the basilica has angle $\theta_2 = \frac{2}{5}$. Let us apply the algorithm:

$$\begin{array}{rcl} \theta_2 = \frac{2}{5} & = & 0.011001100110\ldots \\ \theta_1 = \frac{3}{7} & = & 0.011011011011\ldots \\ \hline \\ \theta^* & = & 0.01101 \end{array}$$

hence $\alpha_1 = 0.\overline{0110} = \frac{2}{5}$ and $\alpha_2 = 0.\overline{01101001} = \frac{7}{17}$ and we get the component of period 4 which is the doubling of the basilica. Note we do not always get the doubling of the previous component; indeed, the next step is

$$\begin{array}{rcl} \theta_2 = \frac{7}{17} & = & 0.011010010110\ldots \\ \theta_1 = \frac{3}{7} & = & 0.011011011011\ldots \\ \hline \\ \theta^* & = & 0.011011 \end{array}$$

hence $\alpha_1 = 0.\overline{01101}$ and we get a component of period 5. Iteration of the algorithm eventually produces all hyperbolic components. We conjecture that a similar algorithm holds in every vein.

Proof of Theorem 14.2. The algorithm is a translation, via the correspondence φ, of the bisection algorithm of Proposition 13.11, and it produces all connected components of the complement of $\mathcal{R}$, i.e. all real hyperbolic components by Proposition 13.13. $\square$

14.3. **Measure and dimension.** We now prove that the bifurcation sets $\mathcal{E}$ and $\mathcal{R}$ both have zero Lebesgue measure and their Hausdorff dimension is equal to 1, establishing Theorem 1.5.

Proof of Theorem 1.5. Let us recall the dynamical characterization of $\mathcal{E}$ (Proposition 13.15):

$$\mathcal{E} = \{x \in [0,1] \ : \ G^n(x) \geq x \ \ \forall n \geq 0\}.$$

Let us now fix $t > 0$. If $x \in \mathcal{E} \cap [t,1]$, we have by the above characterization that

$$G^n(x) \geq x \geq t$$

for all $n \geq 0$, hence the orbit of any element of $\mathcal{E} \cap [t,1]$ never enters the interval $[0,t)$, so by ergodicity of the Gauss map the set $\mathcal{E} \cap [t,1]$ has measure zero. The exact same argument, replacing the Gauss map with the tent map, which is also ergodic, gives the proof of the fact that $\mathcal{R}$ has measure zero.

In order to discuss the Hausdorff dimension of $\mathcal{E}$, let us denote by $\mathcal{B}_N$ the set

$$\mathcal{B}_N := \{x = [a_1, a_2, \dots] \ : \ 1 \leq a_k \leq N \ \ \forall N \geq 1\}$$

of numbers with continued fraction bounded by N. Now fix N and note that, if

$$x = [N+1, a_1, a_2, \dots]$$

with all $a_k \leq N$, then we have the inequality

$$G^n(x) = [a_n, a_{n+1}, \dots] \geq \min \mathcal{B}_N = [\overline{N,1}] > [N+1, a_1, \dots] = x$$

so x belongs to $\mathcal{E}$. This means the map $F_N(x) := 1/(N+1+x)$ is a bi-Lipschitz map which maps $\mathcal{B}_N$ into $\mathcal{E}$, so

$$F_N(\mathcal{B}_N) \subseteq \mathcal{E} \tag{13}$$

and

$$\text{H.dim } \mathcal{E} \geq \text{H.dim } \mathcal{B}_N$$

for each N; the fact that $\lim_{N\to\infty} \text{H.dim}\, \mathcal{B}_N = 1$ yields the claim that the Hausdorff dimension of $\mathcal{E}$ equals 1.

We can use our dictionary to obtain the analogous result for $\mathcal{R}$; indeed, the equivalent of $\mathcal{B}_N$ is the set

$$C_K := \{x \in [0, 1/2] : \underline{x} \text{ does not contain } K+1 \text{ consecutive equal digits}\}$$

where $\underline{x}$ is the binary expansion of x; if we then define the map

$$\Phi_K(x) := \frac{1}{2} - \frac{x}{2^{K+2}}$$

by using the correspondence of Theorem 1.1, the inclusion of eq. (13) becomes

$$\Phi_K(C_K) \subseteq \mathcal{R}$$

so $\mathcal{R}$ contains a Lipschitz copy of C_K for every K. Now, the set C_K is self-similar, therefore its Hausdorff dimension can be computed by standard techniques (see [Fa], Theorem 9.3). More precisely, if $a_K(n)$ is the number of binary sequences of n digits whose first digit is 1 and do not contain $K+1$ consecutive equal digits, one has the following linear recurrence: [3]

$$a_K(n+K) = a_K(n+K-1) + ... + a_K(n+1) + a_K(n) \tag{14}$$

which implies that for any fixed integer $K \geq 2$ the Hausdorff dimension of C_K is $\log_2(\lambda_K)$, where λ_K is the only positive real root of the characteristic polynomial

$$P_K(t) := t^K - (t^{K-1} + ... + t + 1).$$

[3]Sequences satisfying this relation are known as *multinacci* sequences, being a generalization of the usual Fibonacci sequence; the positive roots of their characteristic polynomials are Pisot numbers.

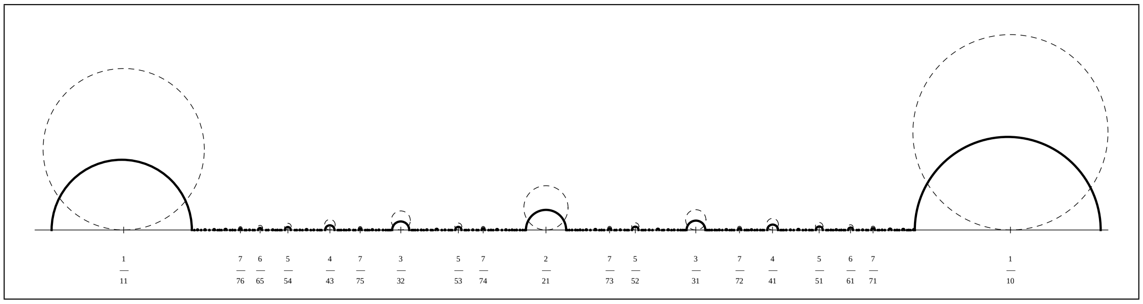


FIGURE 10. Comparison between the set $\mathcal{E}$ and a horoball packing.

A simple estimate on the unique positive root of P_K yields

$$\text{H.dim } \mathcal{R} = \lim_{K \to +\infty} \text{H.dim } C_K = 1.$$

$\square$

Remark 14.3. *The inclusion* (13) *can be interpreted geometrically by saying that for every* $\frac{p}{q} \in \mathbb{Q}_E \cap \left(\frac{1}{N+1}, \frac{1}{N}\right)$

$$B\left(\frac{p}{q}, \frac{1}{(N+2)q^2}\right) \subseteq I_{p/q} \subseteq B\left(\frac{p}{q}, \frac{1}{(N-1)q^2}\right)$$

where $B(x, r)$ *denotes the euclidean ball of center* x *and radius* r. *This means that in any fixed subinterval* $(\frac{1}{N+1}, \frac{1}{N})$ *the size of the geodesic over* $I_{p/q}$ *is comparable to the diameter of the horocycles* $\partial B(\frac{p}{q} + \frac{\imath}{Nq^2}, \frac{1}{Nq^2})$ *(which, for any fixed* N*, all lie in the same* $SL_2(\mathbb{Z})$*-orbit). The picture shows this comparison for* $N = 10$.

14.4. **Numbers of generalized bounded type and real Julia sets.** We will now see how the correspondence of Theorem 1.1 does not only induce an isomorphism between bifurcation sets in parameter space, but it also induces a correspondence between the combinatorial model of any real Julia set and certain sets of numbers with bounded continued fraction. This will prove Theorem 1.4 stated in the introduction.

Definition 14.4. *For each $t > 0$, the set $\mathcal{B}(t)$ of* numbers of type bounded by t *is the set*

$$\mathcal{B}(t) := \{x \in [0,1] \ : \ G^n(x) \geq t \ \ \forall n \geq 0\}.$$

Note that if $t = \frac{1}{N+1}$, then $\mathcal{B}(\frac{1}{N+1})$ is exactly the set $\mathcal{B}_N$ of numbers whose continued fraction expansion has all partial quotients a_i bounded by N. Thus, the family $\{\mathcal{B}(t)\}_{t>0}$ interpolates between the usual countable family of numbers of bounded type.

Let us start by proving some elementary properties of the family $\mathcal{B}(t)$.

Lemma 14.5. *The sets $\mathcal{B}(t)$ have the following properties:*

(i) $\mathcal{B}(0) = [0,1]$; $\mathcal{B}(t) = \emptyset$ if $t > g = \frac{\sqrt{5}-1}{2}$, in fact $t \mapsto \mathcal{B}(t)$ is monotone decreasing;

(ii) $\mathcal{B}(t)$ is forward-invariant for the Gauss map G;

(iii) $\mathcal{B}(t)$ is closed and, if $t > 0$, with no interior and of zero Lebesgue measure;

(iv) the union $\bigcup_{t>0} \mathcal{B}(t)$ is the set of bounded type numbers;

(v) $\bigcap_{t'<t} \mathcal{B}(t') = \mathcal{B}(t)$;

(vi) $\mathcal{E} = \{t \in [0,1] \ : \ t \in \mathcal{B}(t)\}$.

Proof. Points (i), (ii), (iv), (v), (vi) are immediate by definition.
(iii) Let us consider the Farey map $F : [0,1] \to [0,1]$

$$F(x) := \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

One can easily check that if $x := [a_1, a_2, a_3, ...]$ then $F(x) = [a_1 - 1, a_2, a_3, ...]$ if $a_1 > 1$ (while $F(x) = [a_2, a_3, ...]$ in the case $x := [1, a_2, a_3, ...]$) and so it is clear that for each $x \in [0,1]$

$$\inf_{k \geq 1} G^k(x) = \inf_{k \geq 1} F^k(x).$$

Therefore one can write

$$\mathcal{B}(t) = \{x \in [0,1] \ : \ F^k(x) \geq t \ \forall k \in \mathbb{N}\} \tag{15}$$

which is closed by continuity of F. For $t > 0$, $\mathcal{B}(t)$ has no interior because it does not contain any rational number, and it has measure zero by ergodicity of the Gauss map. □

Given a family $K(t)$ of compact sets, we define the *bifurcation locus* of $K(t)$ to be the set ot t for which the function

$$t \mapsto K(t)$$

is not locally constant at t. We have the following

Proposition 14.6. *The set $\mathcal{E}$ is the bifurcation locus of the family $\{\mathcal{B}(t)\}_{0\leq t<1}$.*

In order to prove the proposition and establish a few basic relations between $\mathcal{B}(t)$ and $\mathcal{E}$, let us define for each $0 \leq t \leq g$ the function

$$m(t) := \min \mathcal{B}(t).$$

We shall list some elementary properties of m.

Lemma 14.7. *The function $t \mapsto m(t)$ is monotone increasing and*

(i) For any $0 \leq t \leq g$, $m(t) \in \mathcal{E}$;

(ii) $t \leq m(t)$ $\forall t \in [0, g]$;

(iii) $t = m(t) \iff t \in \mathcal{E}$;

(iv) the function m is left-continuous: $m(t) = \sup_{t'<t} m(t') = \lim_{t'\to t^-} m(t')$;

(v) if (α, β) is a connected component of $[0, g] \setminus \mathcal{E}$ then

$$\begin{matrix} m(t) = \beta \\ \mathcal{B}(t) = \mathcal{B}(\beta) \end{matrix} \qquad \forall t \in (\alpha, \beta].$$

Proof. (i): since $m(t) \in \mathcal{B}(t)$, $G^n(m(t)) \geq m(t)$, hence $m(t) \in \mathcal{E}$. (ii): $x \in \mathcal{B}(t) \Rightarrow x \geq t$, hence $m(t) \geq t$. (iii) is a consequence of Lemma 14.5-(vi):

$$t = m(t) \iff t \in \mathcal{B}(t) \iff t \in \mathcal{E}.$$

(iv) follows from Lemma 14.5-(v). (v): let us pick t such that $\alpha < t < \beta$. Since (α, β) is a connected component of $[0, g] \setminus \mathcal{E}$ we have $\beta \in \mathcal{E}$ and so $\beta \in \mathcal{B}(\beta) \subset \mathcal{B}(t)$, and

$$\beta \geq \min \mathcal{B}(t) = m(t).$$

On the other hand, since $(\alpha, \beta) \cap \mathcal{E} = \emptyset$ and $m(t) \in \mathcal{E} \cap [t, 1]$ it follows that

$$m(t) \geq \beta.$$

We have thus proved that $m(t) = \beta$. Now, from (ii) and monotonicity, $\mathcal{B}(m(t)) \subseteq \mathcal{B}(t)$. Moreover, if $x \in \mathcal{B}(t)$, by G-invariance $G^n(x) \in \mathcal{B}(t)$, hence $G^n(x) \geq m(t)$ and $x \in \mathcal{B}(m(t))$, hence $\mathcal{B}(t) = \mathcal{B}(m(t)) = \mathcal{B}(\beta)$. □

Note that from Lemma 14.7 it also follows that

$$m(t) = \min(\mathcal{E} \cap [t, 1]).$$

Proof of Proposition 14.6. By Lemma 14.7-(v), the function $t \mapsto \mathcal{B}(t)$ is locally constant outside $\mathcal{E}$. On the other hand, if $t \in \mathcal{E}$, then $t \in \mathcal{B}(t)$ by definition, but $t \notin \mathcal{B}(t')$ for any $t' > t$, so t must belong to the bifurcation set. □

We finally turn to the isomorphism between set of numbers of bounded type and the Hubbard trees of real Julia sets. Let us recall the statement of Theorem 1.4, namely that there is an explicit isomorphism between the set H_c of angles of rays landing on the Hubbard tree of f_c (for c real), and the set $\mathcal{B}(t)$ of numbers of type bounded by t when the characteristic angle of f_c equals $\varphi(t)$ (and φ is the isomorphism of Theorem 1.1).

Proof of Theorem 1.4. Let $c \in [-2, 1/4]$ be a real parameter on the boundary of the Mandelbrot set with characteristic angle $\theta_c \in [0, 1/2]$. By Proposition 9.3, the set of angles landing on the Hubbard tree is given by

$$H_c = \{\theta \in S^1 \ : \ T^n(\ell(\theta)) \geq L_c \ \ \forall n \geq 0\}$$

with $L_c := \ell(D(\theta_c))$. If we let $x :=?^{-1}(\ell(\theta))$ and $t :=?^{-1}(\ell(D(\theta_c)))$, then we have, by the conjugacy of eq. (12) and the characterization of $\mathcal{B}(t)$ in terms of the Farey map (eq. (15)),

$$\theta \in H_c \Leftrightarrow x \in \mathcal{B}(t)$$

hence by following through the definition of φ we get

$$H_c \cap [1/2, 1] = 2\varphi(\mathcal{B}(t)).$$

Note that the "upper part" of H_c can be obtained by symmetry:

$$H_c \cap [0, 1/2] = 1 - 2\varphi(\mathcal{B}(t)).$$

$\square$

15. The complex case

The result of Theorem 1.6 lends itself to a natural generalization for complex quadratic polynomials, which we will now describe and then prove in the following sections.

In the real case, we related the entropy of the restriction of f_c on an invariant interval to the Hausdorff dimension of a certain set of angles of external rays landing on the real slice of the Mandelbrot set.

In the case of complex quadratic polynomials, the real axis is no longer invariant, but we can replace it with the Hubbard tree (section 4) T_c. In particular, recall that we defined the polynomial f_c to be topologically finite if the Julia set is connected and locally connected and the Hubbard tree is homeomorphic to a finite tree (see Figure 11, left). We thus define the entropy $h_{top}(f_c\mid_{T_c})$ of the restriction of f_c to the Hubbard tree, and we want to compare it to the Hausdorff dimension of some subset of combinatorial parameter space.

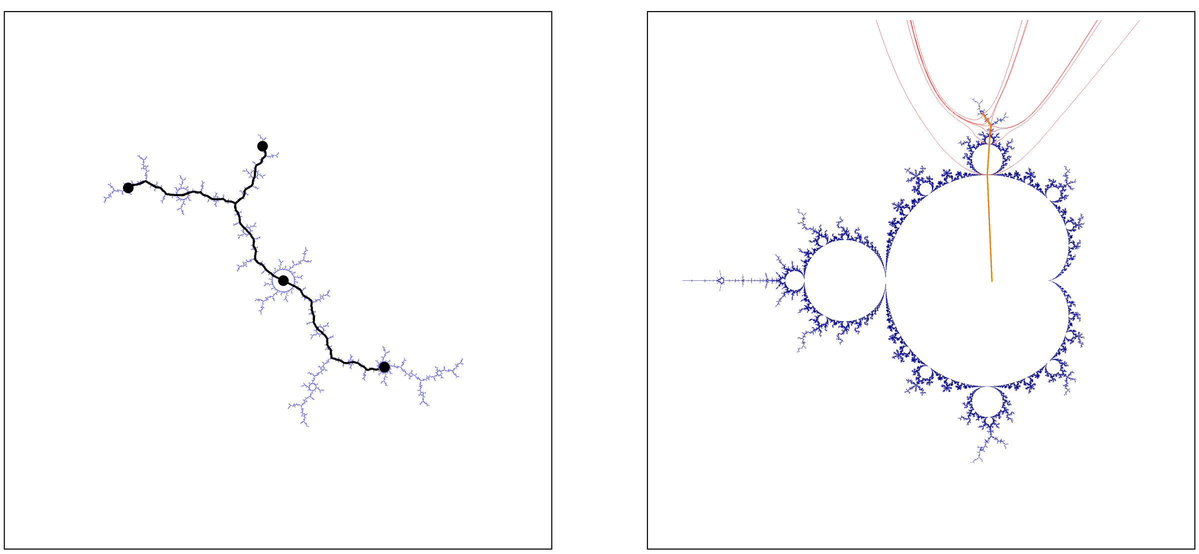


Figure 11. To the left: the Hubbard tree of the complex polynomial of period 4 and characteristic angles $\theta = 3/15, 4/15$. To the right: the vein joining the center of the main cardioid with the main antenna in the 1/3-limb ($\theta = 1/4$), and external rays landing on it.

In parameter space, a generalization of the real slice is a *vein*: a vein v is an embedded arc in $\mathcal{M}$, joining a parameter $c \in \partial\mathcal{M}$ with the center of the main cardioid. Given a vein v and a parameter c on v, we can define the set P_c as the set of external angles of rays which land (at least combinatorially) on v closer than c to the main cardioid:

$$P_c := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ \hat{R}_M(\theta) \text{ intersects } v \cap [0, c]\}$$

where $[0, c]$ is the segment of vein joining c to the center of the main cardioid (see Figure 11, right), and $\hat{R}_M(\theta)$ is the impression of the ray $R_M(\theta)$.

In the $\frac{p}{q}$-limb, there is a unique parameter $c_{p/q}$ such that the critical point lands on the β fixed point after q iterates (i.e. $f^q(0) = \beta$). The vein $v_{p/q}$ joining $c_{p/q}$ to $c = 0$ will be called the *principal vein* of angle p/q. Note that $v_{1/2}$ is the real section of $\mathcal{M}$.

We can thus extend the result of Theorem 1.6 to principal veins:

Theorem 15.1. *Let $v = v_{p/q}$ be principal vein in the Mandelbrot set, and $c \in v \cap \partial\mathcal{M}$ a parameter along the vein. Then we have the equalities*

$$\frac{h_{top}(f_c \mid_{T_c})}{\log 2} = \text{H.dim } H_c = \text{H.dim } P_c.$$

We conjecture that the previous equality holds along any vein v. Note that the statement can be given in more symmetric terms in the following way. If one defines for each $A \subseteq \mathcal{M}$,

$$\Theta_M(A) := \{\theta \in S^1 \ : \ R_M(\theta) \text{ lands on } A\}$$

and similarly, for each $A \subseteq J(f_c)$,

$$\Theta_c(A) := \{\theta \in S^1 \ : \ R_c(\theta) \text{ lands on } A\}$$

then Theorem 15.1 is equivalent to the statement

$$\text{H.dim } \Theta_c([0,c]) = \text{H.dim } \Theta_M([0,c]).$$

In the following sections we will develop in detail the tools needed to prove Theorem 15.1. In particular, in section 16 we prove continuity of entropy along principal veins by developing a generalization of kneading theory to tree maps. Then (section 17) we develop the combinatorial surgery map, which maps the combinatorial model of real Hubbard trees to Hubbard trees along the vein. Finally (section 17.5), we use the surgery to transfer the inclusion of Hubbard tree in parameter space of section 11 from the real vein to the other principal veins.

15.1. **Veins.** A *vein* in the Mandelbrot set is a continuous, injective arc inside $\mathcal{M}$. Branner and Douady [BD] showed that there exists a vein joining the parameter at angle $\theta = 1/4$ to the main cardiod of $\mathcal{M}$. In his thesis, J. Riedl [Ri] showed existence of veins connecting any tip at a dyadic angle $\theta = \frac{p}{2^q}$ to the main cardioid. Another proof of this fact is due to J. Kahn (see [Do2], Section V.4, and [Sch], Theorem 5.6). Riedl also shows that the quasiconformal surgery preserves local connectivity of Julia sets, hence by using the local connectivity of real Julia sets [LvS] one concludes that all Julia sets of maps along the dyadic veins are locally connected ([Ri], Corollary 6.5) .

Let us now see how to define veins combinatorially just in terms of laminations. Recall that the quadratic minor lamination QML is the union of all *minor leaves* of all invariant laminations corresponding to all quadratic polynomials. The degenerate leaf $\{0\}$ is the natural *root* of QML. No other leaf of QML contains the angle 0 as its endpoint. Given a rooted lamination, we define a partial order on the set of leaves by saying that $\ell_1 < \ell_2$ if ℓ_1 separates ℓ_2 from the root.

Definition 15.2. *Let ℓ be a minor leaf. Then the* combinatorial vein *defined by ℓ is the set*

$$P(\ell) := \{\ell' \in QML \ : \{0\} < \ell' \leq \ell\}$$

of leaves which separate ℓ from the root of the lamination.

15.2. **Principal veins.** Let $\frac{p}{q}$ be a rational number, with $0 < p < q$ and p, q coprime. The $\frac{p}{q}$-limb in the Mandelbrot set is the set of parameters which have rotation number $\frac{p}{q}$ around the α fixed point. In each limb, there exists a unique parameter $c = c_{p/q}$ such that the critical point maps to the β fixed point after exactly q steps, i.e. $f_c^q(0) = \beta$. For instance, $c_{1/2} = -2$ is the Chebyshev polynomial. These parameters represent the "highest antennas" in the limbs of the Mandelbrot set. The *principal vein* $v_{p/q}$ is the vein joining $c_{p/q}$ to the main cardioid. We shall denote by $\tau_{p/q}$ the external angle of the ray landing at $c_{p/q}$ in parameter space.

Proposition 15.3. *Each parameter $c \in v_{p/q}$ is topologically finite, and the Hubbard tree T_c is a q-pronged star. Moreover, the valence of any point $x \in T_c$ is at most $2q$.*

Proof. Let τ be the point in the Julia set of f_c where the ray at angle $\tau_{p/q}$ lands. Since $c \in [\alpha, \tau]$, then $f^{q-1}(c) \in [\alpha, \beta]$, hence by Lemma 4.4 the extended Hubbard tree is a q-pronged star. The unique point with degree larger than 1 is the α fixed point, which has degree q, so the second claim follows from Lemma 4.8. $\square$

Note that, by using combinatorial veins, the statement of Theorem 15.1 can be given in purely combinatorial form as follows. Given a set λ of leaves in the unit disk, let us denote by H.dim λ the Hausdorff dimension of the set of endpoints of (non-degenerate) leaves of λ. Moreover, if the leaf ℓ belongs to QML we shall denote as $\lambda(\ell)$ the invariant quadratic lamination which has ℓ as minor leaf. The statement of the theorem then becomes that, for each $\ell \in P(\tau_{p/q})$, the following equality holds:

$$\text{H.dim } P(\ell) = \text{H.dim } \lambda(\ell).$$

We conjecture that the same equality holds for every $\ell \in QML$. In the following sections we will develop the proof of Theorem 15.1.

16. Kneading theory for Hubbard trees

In this section we will analyze the symbolic dynamics of some continuous maps of trees, in order to compute their entropy as zeros of some power series. As a consequence, we will see that the entropy of Hubbard trees varies continuously along principal veins. We will also look at the question of producing piecewise linear, uniformly expanding models of tree maps. Our work is a generalization to tree maps of Milnor and Thurston's *kneading theory* [MT] for interval maps. The general strategy is similar to [BdC], but our view is towards application to Hubbard trees.

16.1. **Counting laps and entropy.** Let $f : T \to T$ be a continuous map of a finite tree T. We will assume f is a local homeomorphism onto its image except at one point, which we call the *critical point*. At the critical point, the map is a branched cover of degree 2. Let us moreover assume T is a *rooted tree*, i.e. it has a distinguished end β. The choice of a root defines a partial ordering on the tree; namely, $x < y$ if x disconnects y from the root.

Let C_f be a finite set of points of T such that $T \setminus C_f$ is a union of disjoint open intervals I_k, and the map f is monotone on each I_k with respect to the above-mentioned ordering. The critical point and the branch points of the tree are included in C_f.

For each subtree $J \subseteq T$, the *number of laps* of the restriction of f^n to J is defined as $\ell(f^n\mid_J) := \#(J \cap \bigcup_{i=0}^{n-1} f^{-i}(C_f)) + \#\text{Ends}(J) - 1$, in analogy with the real case. Denote $\ell(f^n) := \ell(f^n\mid_T)$. The *growth number* s of the map $f : T \to T$ is the exponential growth rate of the number of laps:

$$s := \lim_{n\to\infty} \sqrt[n]{\ell(f^n)}. \tag{16}$$

Lemma 16.1 ([BdC], Lemma 4.1)**.** *The limit in eq.* (16) *exists, and it is related to the topological entropy $h_{top}(f\,|_T)$ in the following way:*

$$s = e^{h_{top}(f|_T)}.$$

The proof is the same as in the analogous result of Misiurewicz and Szlenk for interval maps ([dMvS], Theorem II.7.2). In order to compute the entropy of f, let us define the generating function

$$\mathcal{L}(t) := 1 + \sum_{n=1}^{\infty} \ell(f^n)t^n$$

where $\ell(f^n)$ is the number of laps of f^n on all T. Moreover, for $a, b \in T$, let us denote as $\ell(f^n\,|_{[a,b]})$ the number of laps of the restriction of f^n to the interval $[a, b]$. Thus we can construct for each $x \in T$ the function

$$\mathcal{L}(x, t) := 1 + \sum_{n=1}^{\infty} \ell(f^n\,|_{[\beta,x]})t^n$$

and for each n we shall denote $L_{n,x} := \ell(f^n\,|_{[\beta,x]})$. Let us now relate the generating function $\mathcal{L}$ to the kneading sequence.

Before doing so, let us introduce some notation; for $x \notin C_f$, the sign $\epsilon(x) \in \{\pm 1\}$ is defined according as to whether f preserves or reverses the orientation of a neighbourhood of x. Finally, let us define

$$\eta_k(x) := \epsilon(x) \cdots \epsilon(f^{k-1}(x))$$

for $k \geq 1$, and $\eta_0(x) := 1$. Moreover, let us introduce the notation

$$\chi_k(x) := \begin{cases} 1 & \text{if } f(x) \in I_k \\ 0 & \text{if } f(x) \notin I_k \end{cases}$$

and $\hat{\chi}_k(x) := 1 - \chi_k(x)$.

Let us now focus on the case when T is the Hubbard tree of a quadratic polynomial along the principal vein $v_{p/q}$. Then we can set $C_f := \{\alpha, 0\}$ the union of the α fixed point and the critical point, so that

$$T \setminus C_f = I_0 \cup I_1 \cup \cdots \cup I_q$$

where the critical point separates I_0 and I_1, and the α fixed point separates $I_1, I_2, \ldots, I_q$. The dynamics is the following:

- $f : I_k \mapsto I_{k+1}$ homeomorphically, for $1 \leq k \leq q-1$;
- $f : I_q \mapsto I_0 \cup I_1$ homeomorphically;
- $f(I_0) \subseteq I_0 \cup I_1 \cup I_2$.

We shall now write a formula to compute the entropy of f on the tree as a function of the itinerary of the critical value.

Proposition 16.2. *Suppose the critical point for f is not periodic. Then we have the equality*

$$\mathcal{L}(c,t)\left[1 - 2t\Theta_1(t) + \frac{4t^2}{1+t}\Theta_2(t)\right] = \Theta_3(t)$$

as formal power series, where

$$\Theta_1(t) := \sum_{k=0}^{\infty} \eta_k(c)\hat{\chi}_0(f^k(c))t^k$$

$$\Theta_2(t) := \sum_{k=0}^{\infty} \eta_k(c)\chi_2(f^k(c))t^k$$

depend only on the itinerary of the critical value c, and $\Theta_3(t)$ is some power series with real, non-negative, bounded coefficients. (Note that, in order to deal with the prefixed case, we extend the definitions of ϵ, $\hat{\chi}_0$ and χ_2 by setting $\epsilon(\alpha) = \hat{\chi}_0(\alpha) = \chi_2(\alpha) = 1$.)

Proof. We can compute the number of laps recursively. Let us suppose $x \in T$ such that $f^n(x) \neq 0$ for all $n \geq 0$. Then for $n \geq 2$ we have the following formulas:

$$\ell(f^n \mid_{[\beta,x]}) = \begin{cases} \ell(f^{n-1} \mid_{[\beta,f(x)]}) & \text{if } x \in I_0 \cup \{0\} \\ -\ell(f^{n-1} \mid_{[\beta,f(x)]}) + 2\ell(f^{n-1} \mid_{[\beta,c]}) + 1 & \text{if } x \in I_1 \\ \ell(f^{n-1} \mid_{[\beta,f(x)]}) + 2\ell(f^{n-1} \mid_{[\beta,c]}) - 2\ell(f^{n-1} \mid_{[\beta,\alpha]}) & \text{if } x \in I_2 \cup \dots \cup I_{q-1} \cup \{\alpha\} \\ -\ell(f^{n-1} \mid_{[\beta,f(x)]}) + 2\ell(f^{n-1} \mid_{[\beta,c]}) + 1 & \text{if } x \in I_q \end{cases}$$

Now, recalling the notation $L_{n,x} := \ell(f^n \mid_{[\beta,x]})$, the previous formula can be rewritten as

$$L_{n,x} = \epsilon(x) L_{n-1,f(x)} + 2\hat{\chi}_0(x) L_{n-1,c} - 2\chi_2(x) L_{n-1,\alpha} + \frac{1-\epsilon(x)}{2}.$$

Moreover, for $n = 1$ we have

$$L_{1,x} = \epsilon(x) + 2\hat{\chi}_0(x) + \frac{1-\epsilon(x)}{2} + R(x)$$

where

$$R(x) := \begin{cases} 1 & \text{if } x \in I_q \\ -1 & \text{if } x = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Hence by multiplying every term by t^n and summing up we get

$$\mathcal{L}(x,t) = t\epsilon(x)\mathcal{L}(f(x),t) + 2t\hat{\chi}_0(x)\mathcal{L}(c,t) - 2t\chi_2(x)\tilde{\mathcal{L}}(\alpha,t) + S(x,t)$$

with $S(x,t) := \frac{1-\epsilon(x)}{2}\frac{t}{1-t} + tR(x) + 1$. If we now apply the formula to $f^k(x)$ and multiply everything by $\eta_k(x)t^k$ we have for each $k \geq 0$

$$\eta_k(x)t^k\mathcal{L}(f^k(x),t) - \eta_k(x)\epsilon(f^k(x))t^{k+1}\mathcal{L}(f^{k+1}(x),t) =$$

$$= 2t^{k+1}\eta_k(x)\hat{\chi}_0(f^k(x))\mathcal{L}(c,t) - 2t^{k+1}\eta_k(x)\chi_2(f^k(x))\tilde{\mathcal{L}}(\alpha,t) + \eta_k(x)t^k S(f^k(x),t)$$

so, by summing over all $k \geq 0$, the left hand side is a telescopic series and we are left with

$$\mathcal{L}(x,t) = 2t\Theta_1(x,t)\mathcal{L}(c,t) - 2t\Theta_2(x,t)\tilde{\mathcal{L}}(\alpha,t) + \Theta_3(x,t) \tag{17}$$

where we used the notation $\tilde{\mathcal{L}}(x,t) := \sum_{n=1}^{\infty} \ell(f^n |_{[\beta,x]})t^n$ and

$$\Theta_3(x,t) := \sum_{k=0}^{\infty} \eta_k(x)S(f^k(x),t)t^k = 1 + \sum_{k=1}^{\infty} \frac{1 + \eta_{k-1}(x)(\epsilon(f^{k-1}(x)) + 2R(f^{k-1}(x)))}{2} t^k$$

is a power series whose coefficients are all real and lie between 0 and 1. The claim now follows by plugging in the value $x = c$ in eq. (17), and using Lemma 16.3 to write $\tilde{\mathcal{L}}(\alpha,t)$ in terms of $\mathcal{L}(c,t)$.

□

Lemma 16.3. *We have the following equalities of formal power series:*

(1)
$$\tilde{\mathcal{L}}(\alpha,t) = \frac{2t\mathcal{L}(c,t)}{1+t}$$

(2)
$$\mathcal{L}(t)t^{q-1} = \frac{(1-t^q)\mathcal{L}(c,t)}{1+t} + P(t)$$

where $P(t)$ is a polynomial.

Proof. (1) We can compute $\ell(f^n |_{[\beta,\alpha]})$ recursively, since we have for $n \geq 2$

$$\ell(f^n |_{[\beta,\alpha]}) = 2\ell(f^{n-1} |_{[\beta,c]}) - \ell(f^{n-1} |_{[\beta,\alpha]})$$

while $\ell(f |_{[\beta,\alpha]}) = 2$, hence by multiplying each side by t^n and summing over n we get

$$\tilde{\mathcal{L}}(\alpha,t) = 2t\mathcal{L}(c,t) - t\tilde{\mathcal{L}}(\alpha,t)$$

and the claim holds.

(2) If we let $\mathcal{L}_{[\alpha,c]}(t) := 1 + \sum_{n=1}^{\infty} \ell(f^n \mid_{[\alpha,c]}) t^n$, we have by (1) that

$$\mathcal{L}_{[\alpha,c]}(t) = \frac{(1-t)\mathcal{L}(c,t)}{1+t}.$$

Now, since the Hubbard tree can be written as the union $T = \bigcup_{i=0}^{q-1} [\alpha, f^i(c)]$, for each $n \geq 1$ we have

$$\ell(f^n \mid_T) = \sum_{i=0}^{q-1} \ell(f^n \mid_{[\alpha, f^i(c)]}) = \sum_{i=0}^{q-1} \ell(f^{n+i} \mid_{[\alpha,c]})$$

hence multiplying both sides by t^{n+q-1} and summing over n we get

$$\mathcal{L}(t) t^{q-1} = (1 + t + \cdots + t^{q-1}) \mathcal{L}_{[\alpha,c]}(t) + P(t)$$

for some polynomial $P(t)$. The claim follows by substituting $\mathcal{L}_{[\alpha,c]}(t)$ using (1). $\square$

Proposition 16.4. *Let s be the growth number of the tree map $f : T \to T$. If $s > 1$, then the smallest positive, real zero of the function*

$$\Delta(t) := 1 + t - 2t(1+t)\Theta_1(t) + 4t^2 \Theta_2(t)$$

lies at $t = \frac{1}{s}$. If $s = 1$, then $\Delta(t)$ has no zeros inside the interval $(0, 1)$.

Proof. Recall $s := \lim_{n\to\infty} \sqrt[n]{\ell(f^n)}$, so the convergence radius of the series $\mathcal{L}(t)$ is precisely $r = \frac{1}{s}$. By Proposition 16.2,

$$\mathcal{L}(c,t) = \frac{\Theta_3(t)(1+t)}{\Delta(t)}$$

can be continued to a meromorphic function in the unit disk, and by Lemma 16.3, also $\mathcal{L}(t)$ can be continued to a meromorphic function in the unit disk, and the set of poles of the two functions inside the unit disk coincide (note both power series expansions begin with 1, hence they do not vanish at 0).

Let us now assume $s > 1$. Then $\mathcal{L}(c,t)$ must have a pole on the circle $|t| = \frac{1}{s}$, and since the coefficients of its power series are all positive, it must have a pole on the positive real axis. This implies $\Delta(1/s) = 0$. Moreover, since $\Theta_3(t)$ has real non-negative coefficients, it cannot vanish on the positive real axis, hence $\Delta(t) \neq 0$ for $0 < t < 1/s$.

If instead $s = 1$, $\mathcal{L}(c,t)$ is holomorphic on the disk, so for the same reason $\Delta(t)$ cannot vanish inside the interval $(0,1)$. □

16.2. **Continuity of entropy along veins.**

Theorem 16.5. *Let $v = v_{p/q}$ be the principal vein in the p/q-limb of the Mandelbrot set. Then the entropy $h_{top}(f_c\,|_{T_c})$ of f_c restricted to its Hubbard tree depends continuously, as c moves along the vein, on the angle of the external ray landing at c.*

Proof. Let $\ell \in P(\tau_{p/q})$ be the minor leaf associated to the parameter $c \in \partial\mathcal{M}$, $\ell = (\theta^-, \theta^+)$. Since the entropy does not change under period doubling, we may assume that c is not the period doubling of some other parameter along the vein; thus, there exist $\{\ell_n\}_{n\geq 1} \subseteq P(\tau_{p/q})$ a sequence of leaves of QML which tends to ℓ. Since $c \in \partial\mathcal{M}$, the orbit $f_c^n(0)$ never goes back to 0, so we can apply Propositions 16.2 and 16.4. Thus we can write

$$\mathcal{L}(c,t) = \frac{F(t)}{\Delta(t)} \tag{18}$$

and the entropy $h_{top}(f_c\,|_{T_c})$ is then $\log s$, where $1/s$ is the smallest real positive root of $\Delta(t)$. Finally note that both $F(t)$ and $\mathcal{L}(c,t)$ have real non-negative coefficients, and do not vanish at $t = 0$. The coefficients of $\Delta(t)$ and $F(t)$ depend on the coefficients of $\Theta_1(t)$, $\Theta_2(t)$ and $\Theta_3(t)$, which in turn depend only on the itinerary of the angle θ^- with respect to the doubling map D and the partition given by the complement, in the unit circle, of the set

$$\{\theta_1, \ldots, \theta_q, \tau_{p/q}, \tau_{p/q} + 1/2\}$$

where $\theta_1, \ldots, \theta_q$ are the angles of rays landing on the α fixed point. Let $\Delta_n(t), F_n(t)$ denote the functions $\Delta(t), F(t)$ of equation 18 relative to the parameter corresponding to the leaf ℓ_n. If $f_c^n(0) \neq \alpha$ for all $n \geq 0$, then $D^n(\theta^-)$ always lies in the interior of the partition, so if θ_n^- is sufficiently close to θ^-, its itinerary will share a large initial part with the itinerary of θ^-, hence the power series for $\Delta(t)$ and $\Delta_n(t)$ share arbitrarily many initial coefficients and their coefficients are uniformly bounded, so $\Delta_n(t)$ converges uniformly on compact subsets of the disk to $\Delta(t)$, and similarly $F_n(t) \to F(t)$. Let us now suppose, possibly after passing to a subsequence, that $s_n^{-1} \to s_*^{-1}$. Then by uniform convergeence on compact subsets of $\mathbb{D}$, s_*^{-1} is either 1 or a real, non-negative root of $\Delta(t)$, so in either case

$$\liminf_{n\to\infty} s_n^{-1} \geq s^{-1}.$$

Now, if we have $s_*^{-1} < s^{-1}$, then by Rouché's theorem Δ_n must have a non-real zero z_n inside the disk of radius s_n^{-1} with $z_n \to s_*^{-1}$, hence by definition of s_n and equation 18 one also has $F_n(z_n) = 0$, but since F has real coefficients then also its conjugate $\overline{z_n}$ is a zero of F_n, hence in the limit s_*^{-1} is a real, non-negative zero of F with multiplicity two, but this is a contradiction because the derivative $F'(t)$ also has real, non-negative coefficients so it does not vanish on the interval $[0, 1)$. This proves the claim

$$\lim_{n\to\infty} s_n^{-1} = s^{-1}$$

and continuity of entropy follows.

Things get a bit more complicated when some iterate $f_c^n(0)$ maps to the α fixed point. In this case, the iterates of θ under the doubling map hit the boundary of the partition, hence its itinerary is no longer stable under perturbation. However, a simple check proves that even in this case the coefficients for the function $\Delta_n(t)$ still converge to the coefficients of $\Delta(t)$. Indeed, if n is the smallest step k such that $f_c^k(c) = \alpha$, then for each $k \geq n$ we have $\epsilon(f_c^k(c)) = \hat{\chi}_0(f_c^k(c)) = \chi_2(f_c^k(c)) = 1$. On the other hand, as θ_n^- tends to θ^-, the itinerary of the critical value with respect to the partition $I_0 \cup I_1 \cup \cdots \cup I_q$ approaches a preperiodic cycle of period q, where the period is either $(\overline{I_2, I_2, \ldots, I_2, I_3, I_1})$ or $(\overline{I_1, I_2, I_2, \ldots, I_2, I_3})$. In both cases one can check by explicit computation that the coefficients in the power series expansion of $\Delta_n(t)$ converge to the coefficients of $\Delta(t)$. □

16.3. **Piecewise linear models.** Let us now provide a complementary point of view on the problem of finding the entropy of a tree map, by explicitly constructing a semiconjugacy to a piecewise linear model. Since the results are not needed for the rest of the proof of the main theorem, we will not give complete proofs. Suppose there is a semiconjugacy χ of f onto a piecewise linear model $\phi : T' \to T'$ acting on a finite tree T' homeomorphic to T. That is, $\chi : T \to T'$ is a continuous map such that $\chi \circ f = \phi \circ \chi$.

The tree T can be written as a finite union of intervals $T = \bigcup_{k=1}^n I_k$ on which f is monotone. Each $x \in T$ has a well-defined itinerary $\mathbf{itin}(x) \in \{1, \ldots, n\}^{\mathbb{N}}$ obtained by keeping track of which part of the tree the orbit of x visits:

$$\mathbf{itin}(x) = \{s_i\}_{i \geq 0} \qquad s_i = k \Leftrightarrow f^i(x) \in I_k.$$

Let us call *kneading sequence* the itinerary of the critical value. Similarly, the map $\phi : T' \to T'$ can be defined as a piecewise linear map with derivative of constant absolute value λ. That is, we can write $T' = \bigcup_{k=1}^n J_k$ as a union of intervals with $J_k = \chi(I_k)$, and on each J_k the dynamics has the form

$$\phi_{|J_k}(x) := \epsilon_k \lambda x + a_k$$

with $\epsilon_k \in \{\pm 1\}$ and $a_k \in \mathbb{R}$. This way, the points on T' can also be given an itinerary with respect to the partition $\bigcup_{k=1}^{n} J_k$. In order to construct the semiconjugacy, we want to map x to the point $x' \in T'$ with the same itinerary, i.e. such that

$$\mathbf{itin}_f(x) = \mathbf{itin}_\phi(x').$$

The semiconjugacy maps $x \in T$ to $x' = K(x, \lambda)$ given by the formula:

$$K(x, \lambda) = -\sum_{k=1}^{\infty} a_{s(k)} \epsilon_{s(1)} \cdots \epsilon_{s(k)} \lambda^{-k}$$

where $\{s(1), s(2), \dots, \}$ is the itinerary of x.

The value of the entropy can be computed by imposing that the critical value maps to the critical value. This yields an equation in λ which depends only on the kneading sequence. The largest positive real solution $\lambda \geq 1$ is the growth rate of f, i.e. $\lambda = e^{h_{top}(f)}$.

Example

Let $f(z)$ be a quadratic polynomial along the principal vein in the 1/3-limb of the Mandelbrot set (the vein constructed by Douady-Branner). Let us consider the tree T obtained by joining the α fixed point with the tips at angles 1/4, 1/2 and 0. The tree is a tripod, it is forward invariant in the sense that $f(T) \subseteq T$ and it contains the Hubbard tree. Moreover, the entropy of the map restricted to T is the same as the entropy of the restriction to the Hubbard tree.

The piecewise linear model is made of 4 pieces (labeled as $1, 2, 3, 4$ in the picture on the left).

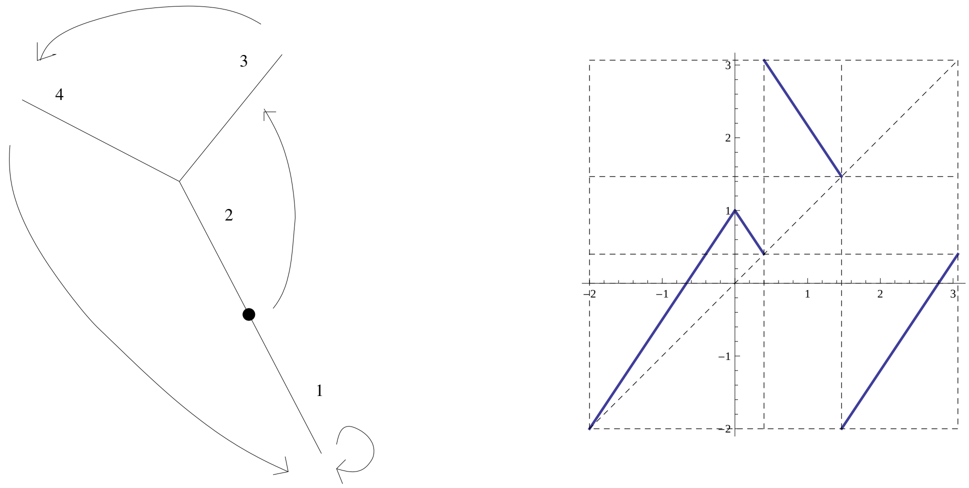


FIGURE 12. On the left: the combinatorics of the piecewise linear tree model for f. On the right: the graph of the piecewise linear model, represented as a (discontinuous) interval map.

By separating the identifications at the α fixed point, we get the following model for the piecewise linear map ϕ (Figure 12, right):

$$\phi(x) := \begin{cases} \lambda x + 1 & \text{if } \frac{1}{1-\lambda} \leq x \leq 0 \\ -\lambda x + 1 & \text{if } 0 < x \leq \frac{1}{1+\lambda} \\ -\lambda x + \frac{\lambda^2-\lambda+2}{\lambda^2-\lambda} & \text{if } \frac{1}{1+\lambda} < x \leq \frac{\lambda^2-\lambda+2}{\lambda^3-\lambda} \\ \lambda x + \frac{\lambda^2+3}{1-\lambda^2} & \text{if } \frac{\lambda^2-\lambda+2}{\lambda^3-\lambda} < x \leq \frac{\lambda^2+\lambda+2}{\lambda^3-\lambda}. \end{cases}$$

Otherwise said, the coefficients of ϕ are the following:

$$\begin{array}{ll} \epsilon_1 = +1 & a_1 = 1 \\ \epsilon_2 = -1 & a_2 = 1 \\ \epsilon_3 = -1 & a_3 = \frac{\lambda^2-\lambda+2}{\lambda^2-\lambda} \\ \epsilon_4 = +1 & a_4 = \frac{\lambda^2+3}{1-\lambda^2}. \end{array}$$

Since in our model the critical value has coordinate $x = 1$, then the entropy is given by the equation

$$K(c, \lambda) = 1.$$

Note that the kneading sequence can be computed in terms of external angles. Indeed, given an external angle θ of a ray landing on the critical value c, the itinerary of c is given by the itinerary of θ under the doubling map with respect to the partition:

$$\begin{aligned}
P_1 &= [5/8, 1/8] \\
P_2 &= (1/8, 1/7] \cup \{2/7\} \cup [4/7, 5/8) \\
P_3 &= (1/7, 2/7) \\
P_4 &= (2/7, 4/7).
\end{aligned}$$

For instance, if the external angle of the critical point is $1/5$, then its itinerary is $\overline{(3, 4, 1, 2)}$ hence the image of the critical value is

$$K(c, \lambda) = \frac{\frac{\lambda^2-\lambda+2}{\lambda^2-\lambda}\lambda^{-1} + \frac{\lambda^2+3}{1-\lambda^2}\lambda^{-2} + \lambda^{-3} - \lambda^{-4}}{1-\lambda^{-4}} = 1$$

hence once gets as a solution the growth rate $\lambda \cong 1.39534$.

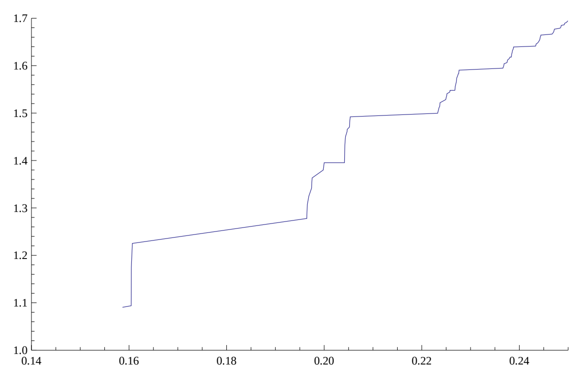


FIGURE 13. A picture of the entropy along the Douady-Branner vein.

17. Combinatorial surgery

The goal of this section is to transfer the result about the real line to the principal veins $v_{p/q}$; in order to do so, we will define a surgery map (inspired by the construction of Branner-Douady [BD] for the 1/3-limb) which carries the combinatorial principal vein in the real limb to the combinatorial principal vein in the p/q-limb.

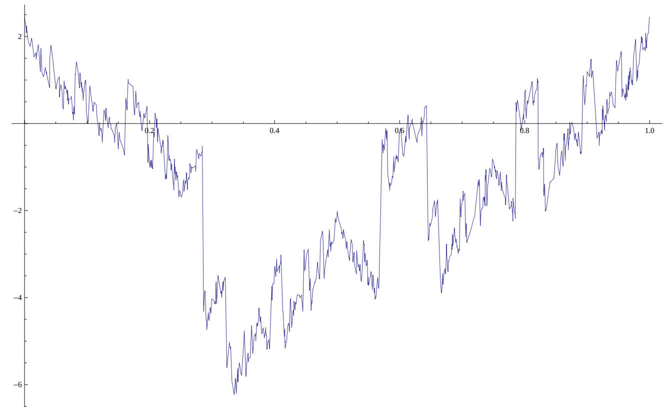


FIGURE 14. The function $K(\theta, \lambda)$ as a function of the external angle θ, for fixed $\lambda = 1.39534$. The restriction of $K(\theta, \lambda)$ to the set of angles of rays landing on the tree is the semiconjugacy to the piecewise linear model.

17.1. **Orbit portraits.** Let $0 < p < q$, with p, q coprime. There exists a unique set $C_{p/q}$ of q points on the unit circle which is invariant for the doubling map D and such that the restriction of D on $C_{p/q}$ preserves the cyclic order of the elements and acts as a rotation of angle p/q. That is $C_{p/q} = \{x_1, \dots, x_q\}$, where $0 \leq x_1 < x_2 < \dots < x_q < 1$ are such that $D(x_i) = x_{i+p}$ (where the indices are computed mod q).

The p/q-*limb* in the Mandelbrot set is the set of parameters c for which the set of angles of rays landing on the α fixed point in the dynamical plane for f_c is precisely $C_{p/q}$ (for a reference, see [Mi1]). In Milnor's terminology, the set $C_{p/q}$ is an *orbit portrait*: we shall call it the α *portrait*.

Given p/q, there are exactly two rays landing on the intersection of the p/q-limb with the main cardioid: let us denote these two rays as θ_0 and θ_1. The angle θ_0 can be found by computing the symbolic coding of the point p/q with respect to the rotation of angle p/q on the circle and using the following partition:

$$A_0 := \left(0, 1 - \frac{p}{q}\right] \quad A_1 := \left(1 - \frac{p}{q}, 1\right].$$

For instance, if $p/q = 2/5$, we have that the orbit is $(2/5, 4/5, 1/5, 3/5, 0)$, hence the itinerary is $(0, 1, 0, 0, 1)$ and the angle is $\theta_0 = 0.\overline{01001} = 9/31$. The other angle θ_1 is obtained by the same algorithm but using the partition:

$$A_0 := \left[0, 1 - \frac{p}{q}\right) \quad A_1 := \left[1 - \frac{p}{q}, 1\right)$$

(hence if $p/q = 2/5$, we have the itinerary $(0,1,0,1,0)$ and $\theta_1 = 0.\overline{01010} = 10/31$.) Let us denote as Σ_0 the first $q-1$ binary digits of the expansion of θ_0, and Σ_1 the first $q-1$ digits of the expansion of θ_1.

17.2. **The surgery map.** Branner and Douady [BD] constructed a continuous embedding of the 1/2-limb of the Mandelbrot set into the 1/3-limb, by surgery in the dynamical plane. The image of the real line under this surgery map is a continuous arc inside the Mandelbrot set, joining the parameter at angle $\theta = 1/4$ with the cusp of $\mathcal{M}$. Let us now describe, for each p/q-limb, the surgery map on a combinatorial level.

In order to construct the surgery map, let us first define the following coding for external angles: for each $\theta \neq \frac{1}{3}, \frac{2}{3}$, we set

$$A_{p/q}(\theta) := \begin{cases} 0 & \text{if } 0 \leq \theta < \frac{1}{3} \\ \Sigma_0 & \text{if } \frac{1}{3} < \theta < \frac{1}{2} \\ \Sigma_1 & \text{if } \frac{1}{2} \leq \theta < \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} < \theta < 1. \end{cases}$$

Then we can define the following map on the set of external angles:

Definition 17.1. *Let $0 < p < q$, with p, q coprime. The* combinatorial surgery map *$\Psi_{p/q} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is defined on the set of external angles as follows.*

- *If θ does not land on a preimage of the α fixed point (i.e. $D^k(\theta) \neq \frac{1}{3}, \frac{2}{3}$ for all $k \geq 0$), we define $\Psi_{p/q}(\theta)$ as the number with binary expansion*

$$\Psi_{p/q}(\theta) := 0.s_1 s_2 s_3 \dots \qquad \text{with } s_k := A_{p/q}(D^k(\theta)).$$

- *Otherwise, let h be the smallest integer such that $D^h(\theta) \in \{\frac{1}{3}, \frac{2}{3}\}$. Then we define*

$$\Psi_{p/q}(\theta) := 0.s_1 s_2 \ldots s_{h-1} s_h$$

with $s_k := A_{p/q}(D^k(\theta))$ *for* $k < h$ *and* $s_h := \begin{cases} \overline{\Sigma_0 1} & \text{if } D^h(\theta) = \frac{1}{3} \\ \overline{\Sigma_1 0} & \text{if } D^h(\theta) = \frac{2}{3}. \end{cases}$

Intuitively, the surgery takes the Hubbard tree of a real map, which is a segment, breaks it into two parts $[c, \alpha]$ and $[\alpha, f(c)]$ and maps them to two different branches of a q-pronged star (see Figure 15).

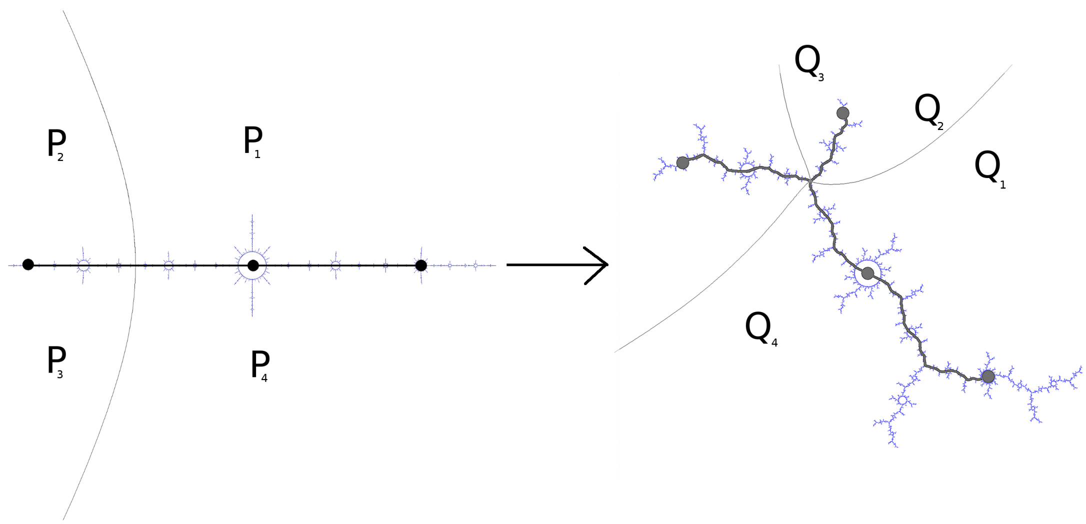


FIGURE 15. The surgery map $\Psi_{1/3}$. The original tree (left) is a segment, which gets "broken" at the α fixed point and a new branch is added so as to form a tripod (right). External rays belonging to the sectors P_1, P_2, P_3, P_4 are mapped to sectors Q_1, Q_2, Q_3, Q_4 respectively.

The image of $1/2$ under $\Psi_{p/q}$ is the external angle of the "tip of the highest antenna" inside the p/q-limb and is denoted as $\tau_{p/q} := \Psi_{p/q}(1/2) = 0.\Sigma_1$.

Let us now fix a rotation number p/q and denote the surgery map $\Psi_{p/q}$ simply as Ψ.

Lemma 17.2. *The map Ψ is strictly increasing (hence injective), in the sense that if $0 \leq \theta < \theta' < 1$, then $0 \leq \Psi(\theta) < \Psi(\theta') < 1$.*

Proof. Let us consider the partitions $P_1 := [0, 1/3), P_2 := [1/3, 1/2), P_3 := [1/2, 2/3)$, $P_4 := [2/3, 1)$ and $Q_1 := [0, 0.\overline{0\Sigma_1})$, $Q_2 := [0.\overline{\Sigma_0 1}, 0.\Sigma_1)$, $Q_3 := [0.\Sigma_1, 0.\overline{\Sigma_1 0})$, $Q_4 := [0.\overline{1\Sigma_0}, 1)$. It is elementary (even though a bit tedious) to check that the map Ψ respects the partitions, in the sense that $\Psi(P_i) \subseteq Q_i$ for each $i = 1, 2, 3, 4$. Indeed, we know

$$\begin{array}{rcl} D(P_1) & \subseteq & P_1 \cup P_2 \cup P_3 \\ D(P_2) & = & P_4 \\ D(P_3) & = & P_1 \\ D(P_4) & \subseteq & P_2 \cup P_3 \cup P_4 \end{array}$$

so the binary expansion of any element $\Psi(\theta)$ is represented by an infinite path in the graph

$$\begin{tikzcd}
& \Sigma_0 \arrow[dr, leftrightarrow] & \\
0 \arrow[loop left] \arrow[ur] \arrow[dr, leftrightarrow] & & 1 \arrow[loop right] \arrow[dl] \\
& \Sigma_1 &
\end{tikzcd}$$

Let us now check for instance that $\Psi(P_1) \subseteq Q_1$. Indeed, if $\theta \in P_1$ then in the above graph the coding of $\varphi(\theta)$ starts from 0 and hence by looking at the graph can be either of the form

$$\Psi(\theta) = 0.(0\Sigma_1)^k 0^n \Sigma_0 \cdots < 0.\overline{0\Sigma_1} \qquad k \geq 0, n \geq 1$$

or

$$\Psi(\theta) = 0.(0\Sigma_1)^k 0^n \Sigma_1 \cdots < 0.\overline{0\Sigma_1} \qquad k \geq 0, n \geq 2$$

so in both cases $0 \leq \Psi(\theta) < 0.\overline{0\Sigma}$ and the claim is proven.

Then, given $0 \leq \theta < \theta' < 1$, let k the smallest integer such that $D^k(\theta)$ and $D^k(\theta')$ lie in two different elements of the partition $\bigcup_i P_i$. Since the map D^k is increasing and the preimage of 0 lies on the boundary of the partition, we have $D^k(\theta) \in P_i$ and

$D^k(\theta') \in P_j$ with $i < j$, so $\Psi(D^k(\theta)) < \Psi(D^k(\theta'))$ because the first one belongs to Q_i and the second one to Q_j, hence we have

$$\Psi(\theta) = 0.s_1 s_2 \dots s_{k-1} \Psi(D^k(\theta)) < 0.s_1 s_2 \dots s_{k-1} \Psi(D^k(\theta')) = \Psi(\theta').$$

$\square$

We can also define the map Ψ on the set of real leaves by defining the image of a leaf to be the leaf joining the two images (if $\ell = (\theta_1, \theta_2)$, we set $\Psi(\ell) := (\Psi(\theta_1), \Psi(\theta_2))$). From the previous lemma it follows monotonicity on the set of leaves:

Lemma 17.3. *The surgery map $\Psi = \Psi_{p/q}$ is strictly increasing on the set of leaves. Indeed, if $\{0\} \leq \ell_1 < \ell_2 \leq \{1/2\}$, then $\{0\} \leq \Psi(\ell_1) < \Psi(\ell_2) \leq \{\tau_{p/q}\}$.*

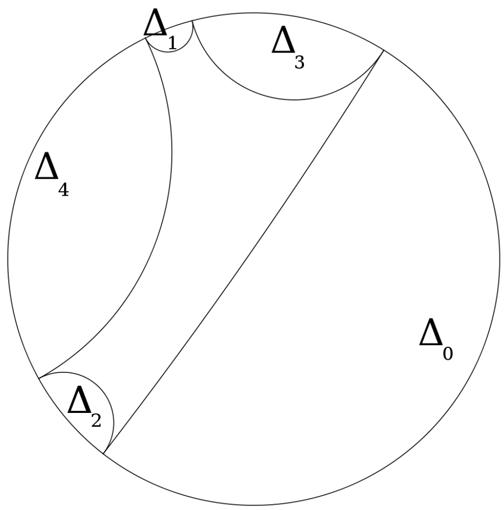
Let us now denote by $\Theta_0 := 0.\overline{1\Sigma_0}$ and $\Theta_1 := 0.\overline{0\Sigma_1}$ the two preimages of θ_0 and θ_1 which lie in the portrait $C_{p/q}$. Note that $D(\Theta_i) = \theta_i$ for $i = 0, 1$.

17.3. **Forbidden intervals.** The leaves (θ_0, θ_1) and (Θ_0, Θ_1) divide the circle in three parts. Let us denote by Δ_0 the part containing 0, and as Δ_1 the part containing $\tau_{p/q}$. Moreover, for $2 \leq i \leq q-1$, let us denote $\Delta_i := D^{i-1}(\Delta_1)$. With this choice, the intervals $\Delta_0, \Delta_1, \dots, \Delta_{q-1}$ are the connected components of the complement of the α portrait $C_{p/q}$.

Let us also denote by $\hat{C}_{p/q} := C_{p/q} + \frac{1}{2}$ the set of angles of rays landing on the preimage of the α fixed point, and $\hat{\Delta}_i := \Delta_i + \frac{1}{2}$ for $0 \leq i \leq q-1$, so that $\hat{\Delta}_0, \hat{\Delta}_1, \dots, \hat{\Delta}_{q-1}$ are the connected components of the complement of $\hat{C}_{p/q}$.

The *forbidden interval* $I_{p/q}$ is then defined as

$$I_{p/q} := \bigcup_{i=1}^{q-2} \hat{\Delta}_i.$$



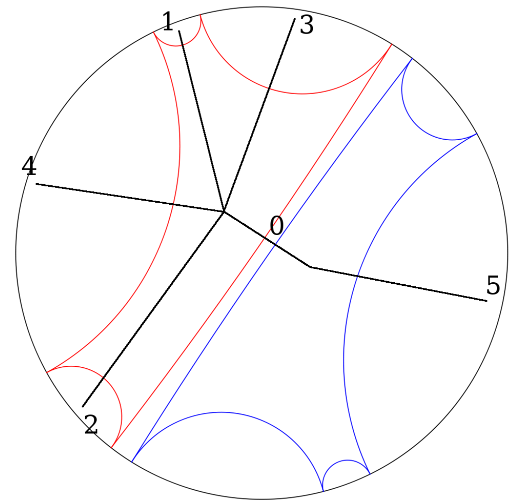


FIGURE 16. Left: the α fixed portrait $C_{p/q}$ when $p/q = 2/5$, with the complementary intervals Δ_i. Right: The portraits $C_{p/q}$ and $\hat{C}_{p/q}$, with the Hubbard tree drawn as dual to the lamination. The numbers indicate the position of the iterates of the critical value.

The name "forbidden interval" arises from the fact that this interval is avoided by the trajectory of an angle landing on the Hubbard tree of some parameter on the vein $v_{p/q}$. Indeed, the following characterization is true:

Proposition 17.4. *Let $\ell \in P(\tau_{p/q})$ be the characteristic leaf of a parameter c on the principal vein $v_{p/q}$, with $\ell = (\theta^-, \theta^+)$, and let $J := (D^{q-1}(\theta^-), D^{q-1}(\theta^+))$ the interval delimited by $D^{q-1}(\ell)$ and containing* 0*. Then the set of rays landing on the Hubbard tree of c is characterized as*

$$H_c := \{\theta \in S^1 \ : \ D^n(\theta) \notin I_{p/q} \cup J \quad \forall n \geq 0\}.$$

Proof. It follows from the description of H_c in Proposition 9.3 together with the fact that the Hubbard tree is a q-pronged star. $\square$

The explicit characterization also immediately implies that the sets H_c are increasing along principal veins:

Proposition 17.5. *Let $\ell < \ell'$ be the characteristic leaves of parameters c, c' which belong to the principal vein $v_{p/q}$.*

(1) Then we have the inclusion

$$H_c \subseteq H_{c'};$$

(2) if T_c and $T_{c'}$ are the respective Hubbard trees, we have

$$h_{top}(f_c\mid_{T_c}) \leq h_{top}(f_c\mid_{T_{c'}}).$$

Proof. (1) Let J be the interval containing 0 delimited by $D^{q-1}(\ell)$, and J' the interval delimited by $D^{q-1}(\ell')$. Since $\ell < \ell' < \{\tau_{p/q}\}$, one has $\{0\} < D^{q-1}(\ell') < D^{q-1}(\ell)$, so $J' \subseteq J$. If $\theta \in H_c$, then by Proposition 17.4 its orbit avoids $I_{p/q} \cup J$, hence it also avoids $I_{p/q} \cup J'$ so it must belong to $H_{c'}$.

(2) From (1) and Theorem 7.1,

$$h_{top}(f_c\mid_{T_c}) = \text{H.dim } H_c \cdot \log 2 \leq \text{H.dim } H_{c'} \cdot \log 2 = h_{top}(f_{c'}\mid_{T_{c'}}).$$

□

Monotonicity of entropy along arbitrary veins is proven, for postcritically finite parameters, in Tao Li's thesis [TaoL]. Recently, a very elegant argument for monotonicity along veins without the restriction to postcritically finite parameters was found by Tan Lei.

17.4. **Surgery in the dynamical and parameter planes.** The usefulness of the surgery map comes from the fact that it maps the real vein in parameter space to the other principal veins, and also the Hubbard trees of parameters along the real vein to Hubbard trees along the principal veins. As we will see in this subsection, the correspondence is almost bijective.

Let Z denote the set of angles which never map to the endpoints of fixed leaf $\ell_0 = (1/3, 2/3)$:

$$Z := \{\theta \in S^1 \ : \ D^n(\theta) \neq 1/3, 2/3 \ \ \forall n \geq 0\}.$$

Moreover, we denote by Ω the set of angles which never map to either the forbidden interval $I_{p/q}$ or the α portrait $C_{p/q}$:

$$\Omega := \{\theta \in \Delta_0 \cup \Delta_1 \ : \ D^n(\theta) \notin I_{p/q} \cup C_{p/q} \ \ \forall n \geq 0\}.$$

It is easy to check the following

Lemma 17.6. *The map Ψ is continuous on Z, and the image $\Psi(Z)$ is contained in Ω. Given $\theta \in \Omega$, let $0 = n_0 < n_1 < n_2 < \dots$ be the return times of θ to $\Delta_0 \cup \Delta_1$. Then the map*

$$\Phi(\theta) := 0.s_0 s_1 s_2 \dots \qquad \text{with } s_k = \begin{cases} 0 & \text{if } D^{n_k}(\theta) \in [0, \Theta_1) \cup (\theta_0, \tau_{p/q}) \\ 1 & \text{if } D^{n_k}(\theta) \in [\tau_{p/q}, \theta_1) \cup (\Theta_0, 1) \end{cases}$$

defined on Ω is an inverse of Ψ, in the sense that $\Phi \circ \Psi(\theta) = \theta$ for all $\theta \in Z$.

Proposition 17.7. *The surgery map $\Psi = \Psi_{p/q}$ maps the real combinatorial vein bijectively onto the principal combinatorial vein $P(\tau_{p/q})$ in the p/q-limb, up to a countable set of prefixed parameters; indeed, one has the inclusions*

$$P(\tau_{p/q}) \setminus \bigcup_{n \geq 0} D^{-n}(C_{p/q}) \subseteq \Psi(P(1/2)) \subseteq P(\tau_{p/q}).$$

Proof. Let $m \in P(1/2)$ be a minor leaf, and M_1, M_2 its major leaves. By the criterion of Proposition 3.3, all the elements of the forward orbit of m have disjoint interior, and their interior is also disjoint from m, M_1 and M_2, so the set of leaves $\{D^n(m) \ : \ n \geq 0\} \cup \{M_1, M_2\}$ (which may be finite or infinite) is totally ordered, and they all lie between $\{0\}$ and $\{1/2\}$. Indeed, they are all smaller than m, which is also the shortest leaf of the set. Now, by Lemma 17.3, the set

$$\{\Psi(D^n(m)) \ : \ n \geq 0\} \cup \{\Psi(M_1), \Psi(M_2)\}$$

is also totally ordered, and all its elements have disjoint interior and lie between $\{0\}$ and $\Psi(m)$. Note that all leaves smaller than $\ell_0 := (1/3, 2/3)$ map under Ψ to leaves smaller than (Θ_0, Θ_1), and all leaves larger than ℓ_0 map to leaves larger than $\Psi(\ell_0) = (\theta_0, \theta_1)$. Note moreover that if a leaf $\mathcal{L}$ is larger than (θ_0, θ_1), then its length increases under the first $q-1$ iterates (i.e. until it comes back to Δ_0):

$$\ell(D^k(\mathcal{L})) = 2^k \ell(\mathcal{L}) \qquad 0 \leq k \leq q-1.$$

As a consequence, the shortest leaf in the set

$$S := \{D^n(\Psi(m)) \ : \ n \geq 0\} \cup \{\Psi(M_1), \Psi(M_2)\}$$

is $\Psi(m)$, and its images all have disjoint interiors, hence by Proposition 3.3 we have that $\Psi(m)$ belongs to QML, and it is smaller than $\tau_{p/q}$ by monotonicity of Ψ. Conversely, any leaf ℓ of $P(\tau_{p/q})$ whose endpoints never map to the fixed orbit portrait $C_{p/q}$ belongs to Ω, hence $\Psi(\ell)$ is well-defined and, since Ψ preserves the ordering, it belongs to $P(1/2)$ by Proposition 3.3. $\square$

Proposition 17.8. *Let $c \in [-2, 1/4]$ be a real parameter, with characteristic leaf ℓ, and let c' be a parameter with characteristic leaf $\ell' = \Psi(\ell)$. Moreover, let us set $\tilde{H}_{c'} := H_{c'} \cap (\Delta_0 \cup \Delta_1) \setminus \bigcup_n D^{-n}(C_{p/q})$. Then the inclusions*

$$\tilde{H}_{c'} \subseteq \Psi(H_c) \subseteq H_{c'}$$

hold. As a consequence,

$$\text{H.dim } \Psi(H_c) = \text{H.dim } H_{c'}.$$

Proof. Let $\theta \in H_c$ and $\ell := (\theta, 1-\theta)$ be its associated real leaf and let ℓ_c the postcharacteristic leaf for f_c. Let us first assume $D^n(\theta) \neq 1/3, 2/3$ for all n. Then by Lemma 17.6 $\Psi(\theta)$ lies in Ω, so its orbit always avoids $I_{p/q}$. Moreover, by Proposition 9.3

$$D^n(\ell) \geq \ell_c \qquad \text{for all } n \geq 0.$$

Then, by monotonicity of the surgery map (Lemma 17.3)

$$\Psi(D^n(\ell)) \geq \Psi(\ell_c) \qquad \text{for all } n \geq 0.$$

Moreover, given $N \geq 0$ either

$$D^N(\Psi(\ell)) \notin \Delta_0 \cup \Delta_1$$

or one can write

$$D^N(\Psi(\ell)) = \Psi(D^n(\ell))$$

for some integer n, so the orbit of $\Psi(\theta)$ always avoids the interval delimited by the leaf $\Psi(\ell_c)$, hence by Proposition 9.3 we have $\Psi(\theta) \in H_{c'}$. The case when $D^n(\theta)$ hits $\{1/3, 2/3\}$ is analogous, except that the leaf ℓ is eventually mapped to the leaf (θ_0, θ_1) which belongs to the α portrait.

Conversely, let $\theta' \in \tilde{H}_{c'}$ and ℓ' be its corresponding leaf. Then by Proposition 17.4 it never maps to $I_{p/q}$, so by Lemma 17.6 there exists $\theta \in Z$ such that $\theta' = \Psi(\theta)$. Let $\ell := (\theta, 1-\theta)$ be its corresponding real leaf. Moreover, also by Proposition 17.4 all iterates of ℓ' are larger than $\Psi(\ell_c)$, so by monotonicity of the surgery map all iterates of ℓ are larger than ℓ_c, so, by Proposition 9.3, θ lies in H_c. The equality of dimensions arises from the fact that for $2 \leq i \leq q-1$ one has

$$H_{c'} \cap \Delta_i = D^{q-1}(H_{c'} \cap \Delta_1)$$

and the doubling map preserves Hausdorff dimension. $\square$

Finally, we need to check that the surgery map behaves well under renormalization. Indeed we have the

Lemma 17.9. *Let W be a real hyperbolic component, and Ψ the surgery map. Then for each $\theta \in \mathcal{R}$,*

$$\Psi(\tau_W(\theta)) = \tau_{\Psi(W)}(\theta)$$

where $\Psi(W)$ is the hyperbolic component whose endpoints are the images via surgery of the endpoints of W.

Proof. Let $\theta = 0.\theta_1\theta_2\ldots$ be the binary expansion of θ. Denote as $\theta^- = 0.\overline{S_0}$, $\theta^+ = 0.\overline{S_1}$ the angles of parameter rays landing at the root of W, and as $\Theta^- := \Psi(\theta^-) = 0.\overline{T_0}$ and $\Theta^+ := \Psi(\theta^+) = 0.\overline{T_1}$ the angles landing at the root of $\Psi(W)$. Finally, let $p := |S_0|$ denote the the period of W. Then $\tau_W(\theta)$ has binary expansion

$$\tau_W(\theta) = 0.S_{\theta_1}S_{\theta_2}\ldots$$

By using the fact that either $\theta^- \leq \theta^+ < 1/3$ or $2/3 < \theta^- \leq \theta^+$, one checks that for each $0 \leq k < p$, the points

$$D^k(0.S_{\theta_1}S_{\theta_2}\ldots)$$

and

$$D^k(0.\overline{S_{\theta_1}})$$

lie in the same element of the partition $\bigcup_{i=1}^4 P_i$. As a consequence, by definition of the surgery map Ψ, we get that

$$\Psi(\tau_W(\theta)) = 0.T_{\theta_1}T_{\theta_2}\ldots$$

and the claim follows.

□

17.5. **Proof of Theorem 15.1.**

Definition 17.10. *The set $\mathcal{D}_{p/q}$ of* dominant parameters along $v_{p/q}$ *is the image of the set of (real) dominant parameters $\mathcal{D}$ under the surgery map:*

$$\mathcal{D}_{p/q} := \Psi_{p/q}(\mathcal{D}).$$

We can now use the surgery map to transfer the inclusion of the Hubbard trees of real maps in the real slice of the Mandelbrot set to an inclusion of the Hubbard trees in the set of angles landing on the vein in parameter space.

Proposition 17.11. *Let $c \in v_{p/q}$ be a parameter along the vein with non-renormalizable combinatorics, and c' another parameter along the vein which separates c from the main cardioid (i.e. if ℓ and ℓ' are the characteristic leaves, $\ell' < \ell \leq \{\tau_{p/q}\}$). Then there exists a piecewise linear map $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that*

$$F(\tilde{H}_{c'}) \subseteq P_c.$$

Proof. Let $\theta \in [0, \tau_{p/q}]$ be a characteristic angle for c. Let us first assume that the forward orbit of θ never hits $C_{p/q}$. Then by Proposition 17.7 there exists an angle $\theta_R \in [0, 1/2] \cap \mathcal{R}$ such that $\theta = \Psi(\theta_R)$, and by Lemma 17.9 θ_R is not renormalizable. Then, by Proposition 11.1, there exist a $\theta'_R < \theta_R$ arbitrarily close to θ_R (and by continuity of Ψ we can choose it so that $\Psi(\theta'_R)$ lands on the vein closer to c than c') and a piecewise linear map F_R of the circle such that

$$F_R(H_{\theta'_R}) \subseteq P_{\theta_R}. \tag{19}$$

We claim that the map $F := \Psi \circ F_R \circ \Psi^{-1}$ satisfies the claim. Indeed, if $\xi \in [0, 1/2)$ recall that the map F_R constructed in Proposition 11.1 has the form

$$F_R(\xi) = s + \xi \cdot 2^{-N}$$

where s is a dyadic rational number and N is some positive integer. Thus, $D^N(F_R(\xi)) = \xi$, so also

$$\Psi(\xi) = \Psi(D^N(F_R(\xi))) = D^M(\Psi(F_R(\xi)))$$

for some integer M. Thus we can write for $\xi \in H_{\theta'_R} \cap Z$

$$\Psi(F_R(\xi)) = t + \Psi(\xi) \cdot 2^{-M}$$

where t is a dyadic rational number, and t and M only depend on s and the element of the partition $\bigcup P_i$ to which ξ belongs. Thus we have proven that $F = \Psi \circ F_R \circ \Psi^{-1}$ is piecewise linear. Now, by Proposition 17.8, eq. (19), and Proposition 17.7 we have the chain of inclusions

$$\Psi \circ F_R \circ \Psi^{-1}(\tilde{H}_{c'}) \subseteq \Psi \circ F_R(H_{\theta'_R}) \subseteq \Psi(P_{\theta_R}) \subseteq P_c.$$

Finally, if the forward orbit of θ hits $C_{p/q}$, then by density one can find an angle $\tilde{\theta} \in (\theta', \theta)$ such that its forward orbit does not hit $C_{p/q}$, and apply the previous argument to the parameter $\tilde{c}$ with characteristic angle $\tilde{\theta}$, thus getting the inclusion

$$F(\tilde{H}_{c'}) \subseteq P_{\tilde{c}} \subseteq P_c.$$

□

Proof of Theorem 15.1. Let c be a parameter along the vein $v_{p/q}$. Then by Theorem 5.5

$$\frac{h_{top}(f_c \mid_{T_c})}{\log 2} = \text{H.dim } H_c.$$

We shall prove that the right hand side equals H.dim P_c. Now, since $P_c \subseteq H_c$, it is immediate that

$$\text{H.dim } P_c \leq \text{H.dim } H_c$$

hence we just have to prove the converse inequality. Let us now assume $c \in v_{p/q}$ non-renormalizable. Then by Proposition 17.11 for each $c' \in [0, c]$ we have the inclusion

$$F(\tilde{H}_{c'}) \subseteq P_c$$

so, since F is linear hence it preserves Hausdorff dimension, we have

$$\text{H.dim } H_{c'} = \text{H.dim } \tilde{H}_{c'} \leq \text{H.dim } P_c$$

and as a consequence

$$\text{H.dim } P_c \geq \sup_{c' \in [0,c]} \text{H.dim } H_{c'}$$

where $[0, c]$ is the segment of the vein $v_{p/q}$ joining 0 with c. Now by continuity of entropy (Theorem 16.5)

$$\sup_{c' \in [0,c]} \text{H.dim } H_{c'} = \text{H.dim } H_c$$

hence the claim is proven for all non-renormalizable parameters along the vein. Now, the general case follows as in the proof of Theorem 8.3 by successively renormalizing and using the formulas of Proposition 12.2. $\square$

So far we have worked with the combinatorial model for the veins, which conjecturally coincide with the set of angles of rays which actually land on the vein. Finally, the following proposition proves that the vein and its combinatorial model actually have the same dimension, independently of the MLC conjecture.

Proposition 17.12. *Let $c \in v_{p/q} \cap \partial\mathcal{M}$ and ℓ its characteristic leaf. Let*

$$\overline{P}_c := \{\theta \in S^1 \ : \ R_M(\theta) \text{ lands on } v \cap [0, c]\}$$

be the set of angles of rays landing on the vein v closer than c to the main cardioid, and

$$P_c := \{\theta \in S^1 \ : \ \theta \text{ is endpoint of some } \ell' \in QML, \ \ell' \leq \ell\}$$

its combinatorial model. Then the two sets have equal dimension:

$$\text{H.dim } \overline{P}_c = \text{H.dim } P_c.$$

Proof. Fix a principal vein $v_{p/q}$, and let τ_W be the tuning operator relative to the hyperbolic component of period q in $v_{p/q}$; moreover, denote as τ the tuning operator relative to the hyperbolic component of period 2. Let P_c^{fr} the set of angles which belong to the P_c with finitely renormalizable combinatorics; then Proposition 3.2 yields the inclusions

$$\text{H.dim } P_c^{fr} \subseteq \text{H.dim } \overline{P}_c \subseteq \text{H.dim } P_c$$

hence to prove the proposition it is sufficient to prove the equality

$$\text{H.dim } P_c^{fr} = \text{H.dim } P_c.$$

Let now $c_n := \tau_W(\tau^n(-2))$ the tips of the chain of nested baby Mandelbrot sets which converge to the Feigenbaum parameter in the p/q-limb, and let ℓ_n be the characteristic leaf of c_n. Then if H.dim $P_c > 0$, there exists a unique $n \geq 1$ such that $\ell_n < \ell \leq \ell_{n-1}$, hence by monotonicity and by Theorem 15.1 we know

$$\text{H.dim } P_c \geq \text{H.dim } P_{c_n} = \frac{1}{2^n q}.$$

Now, each element of P_c is either of the form $\tau_W \tau^{n-1}(c')$ with c' non-renormalizable, or of the form $\tau_W(\tau^{n-1}(\tau_V(c')))$ where V is some hyperbolic window of period larger than 2. However, we know by Proposition 12.2 that the image of $\tau_W \circ \tau^{n-1} \circ \tau_V$ has Hausdorff dimension at most $\frac{1}{q \cdot 2^{n-1} \cdot 3} <$ H.dim P_c, hence one must have

$$\text{H.dim } P_c = \text{H.dim } \{\theta \in \overline{P}_c \ : \ \theta = \tau_W \tau^{n-1}(\theta'), \ \theta' \text{ non-renormalizable}\} \leq \text{H.dim } P_c^{fr}$$

which yields the claim. $\square$

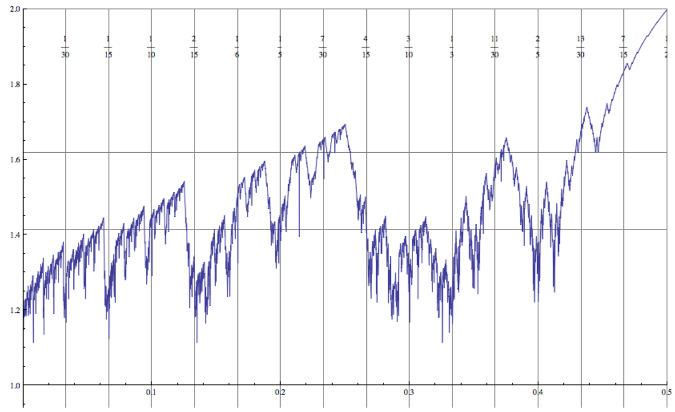


Figure 17. Entropy of Hubbard trees as a function of the external angle (by W. Thurston).

18. Further developments

18.1. **Thurston's point of view.** The results of the thesis relate to recent work of W. Thurston, who looked at the entropy of Hubbard trees as a function of the external angle. Indeed, every external angle θ of the Mandelbrot set combinatorially determines a lamination (see section 3) and the lamination determines an abstract Hubbard tree, of which we can compute the entropy $h(\theta)$.

Thurston produced very interesting pictures, suggesting that the complexity of the Mandelbrot set is encoded in the combinatorics of the Hubbard tree, and the variation in entropy reflects the geometry of $\mathcal{M}$.

In this sense, Theorems 1.6 and 1.8 contribute to this program: in fact, the entropy grows as one goes further from the center of $\mathcal{M}$ (see also [TaoL]), and our results make precise the relationship between the increase in entropy and the increased hairiness of the Mandelbrot set.

Bruin and Schleicher [BS] recently proved that entropy is continuous as a function of the external angle.

Note that Thurston's approach is in some sense dual to ours, since we look at the variation of entropy along the veins, i.e. from "inside" the Mandelbrot set as opposed to from "outside" as a function of the external angle.

We point out that the idea of the pseudocenter described in the introduction (see also sections 13 and 14) seems also to be fruitful to study the entropy of the Hubbard tree as a function of the external angle: indeed, we conjecture that the maximum of the entropy on any wake is achieved precisely at its pseudocenter. Let us denote by $h(\theta)$ the entropy of the Hubbard tree corresponding to the parameter of external angle θ.

Conjecture 18.1. *Let $\theta_1 < \theta_2$ be two external angles whose rays $R_M(\theta_1)$, $R_M(\theta_2)$ land on the same parameter in the boundary of the Mandelbrot set. Then the maximum of entropy on the interval $[\theta_1, \theta_2]$ is attained at its pseudocenter:*

$$\max_{\theta \in [\theta_1, \theta_2]} h(\theta) = h(\theta^*)$$

where θ^* is the pseudocenter of the interval $[\theta_1, \theta_2]$.

The study of the entropy of Hubbard trees of polynomials is a very recent field, thus many questions are completely open. In the following, we present a few observations.

18.2. **Galois conjugates.** If c is a postcritically finite parameter, then its Hubbard tree is a finite tree and its dynamics can be encoded by a finite Markov chain. The topological entropy is then just the leading eigenvalue of the transition matrix of the Markov chain, and the characteristic polynomial of such a matrix also has other roots. Hence, for each parameter one can consider the set of all Galois conjugates of the entropy, and take the union all such sets over all possible postcritically finite quadratic polynomials. Thurston started the exploration of such object and produced the picture in Figure 18 [Th2].

Note that because of renormalization the picture is closed under taking n^{th}-roots, which immediately proves that the set accumulates on the unit circle. Also, by using the entropy as a parameter, one can prove that the part of the picture outside the unit disk is path-connected.

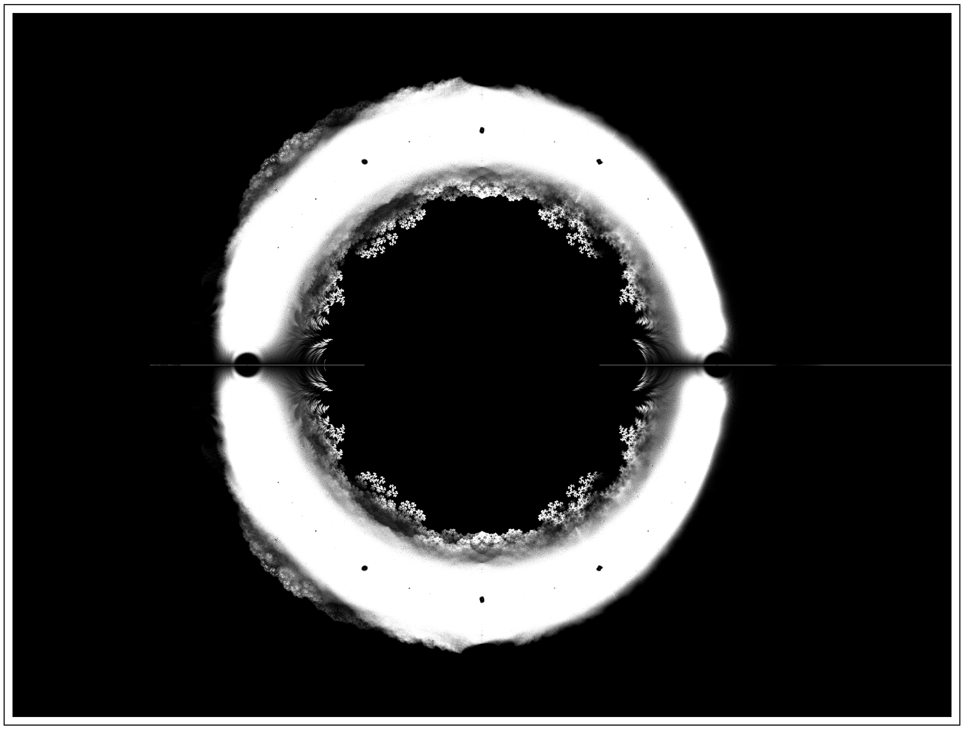


Figure 18. Galois conjugates of entropies of real quadratic maps.

A similar construction can be done for arbitrary veins. Namely, given a vein v one can consider all postcritically finite parameters on v, and construct the Markov chain for each Hubbard tree and then plot the union of all the roots of the characteristic polynomials. Here we show the pictures for the principal veins in the 1/3, 1/5 and 1/11-limbs (Figures 19, 20 and 21).

It would be very interesting to explain the fractal structure of such pictures, as well as studying the examples which produce the Galois conjugates of smallest absolute value.

18.3. **A combinatorial bifurcation measure.** The monotonicity statement in Tao Li's thesis allows us to define a transverse measure on the quadratic minor lamination QML. Let $\ell_1 < \ell_2$ be two ordered leaves of QML, corresponding to two parameters c_1 and c_2, and let γ be a tranverse arc connecting ℓ_1 and ℓ_2. Then one can assign the measure of the arc γ to be the difference between the entropy of the two Hubbard

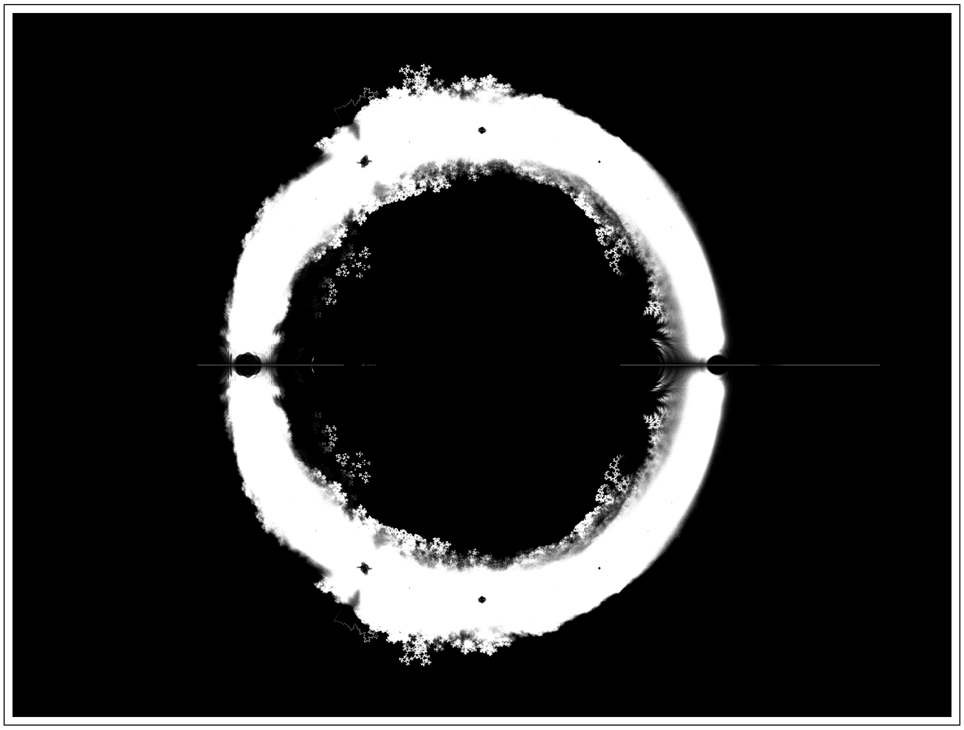


FIGURE 19. Galois conjugates of entropies of maps along the vein $v_{1/3}$.

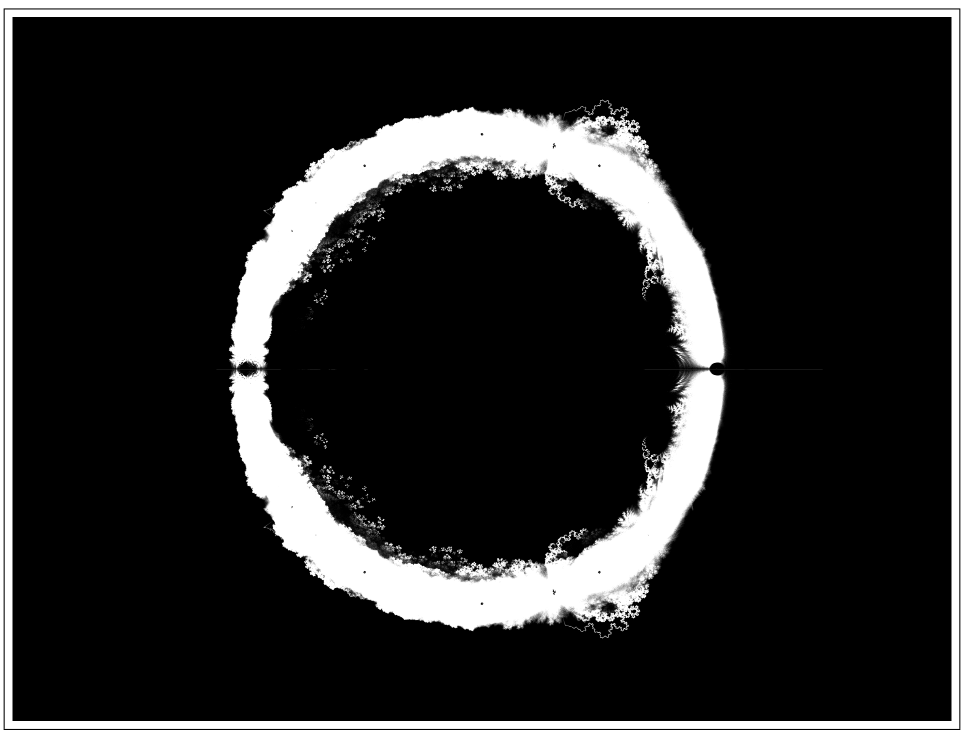


FIGURE 20. Galois conjugates of entropies of maps along the vein $v_{1/5}$.

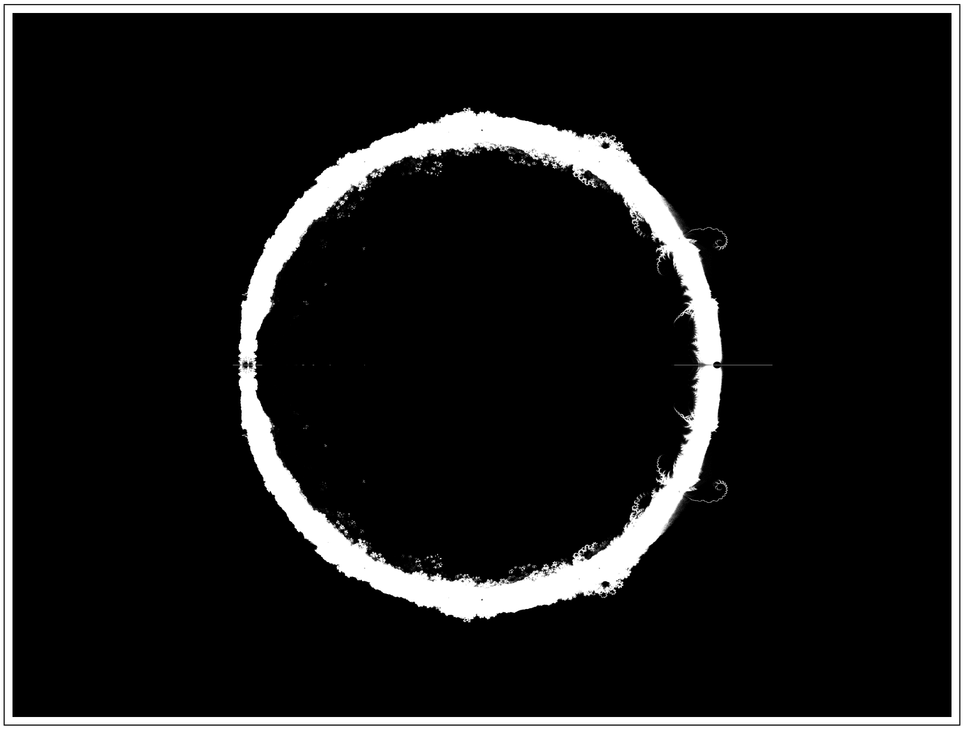


FIGURE 21. Galois conjugates of entropies of maps along the vein $v_{1/11}$.

trees:

$$\mu(\gamma) := h(f_{c_2} \mid_{T_{c_2}}) - h(f_{c_1} \mid_{T_{c_1}}).$$

By Tao Li's and our results, such a measure can be interpreted as a *transverse bifurcation measure*: in fact, as one crosses more and more leaves from the center of the Mandelbrot set to the periphery, i.e. as the map f_c undergoes more and more bifurcations, one picks up more and more measure. The measure can also be interpreted as the derivative of the entropy in the direction transverse to the leaves: note also that, since period doubling bifurcations do not change the entropy, μ is non-atomic.

The dual to the lamination is an $\mathbb{R}$-tree, and the transverse measure μ defines a metric on such a tree. By pushing it forward to the actual Mandelbrot set, one endows the union of all veins in $\mathcal{M}$ with the structure of a metric $\mathbb{R}$-tree. It would be very interesting to analyze the properties of such transverse measure, and also comparing it to the other existing notions of bifurcation measure.

19. DYNAMICS OF α-CONTINUED FRACTIONS

It is a well-known fact that the continued fraction expansion of a real number can be analyzed in terms of the dynamics of the interval map $G(x) := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$, known as the *Gauss map*. A generalization of this map is given by the family of *α-continued fraction transformations* T_α, which will be the object of study of the present section. For each $\alpha \in [0,1]$, the map $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ is defined as $T_\alpha(0) = 0$ and, for $x \neq 0$,

$$T_\alpha(x) := \frac{1}{|x|} - c_{\alpha,x}$$

where $c_{\alpha,x} = \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$ is a positive integer.

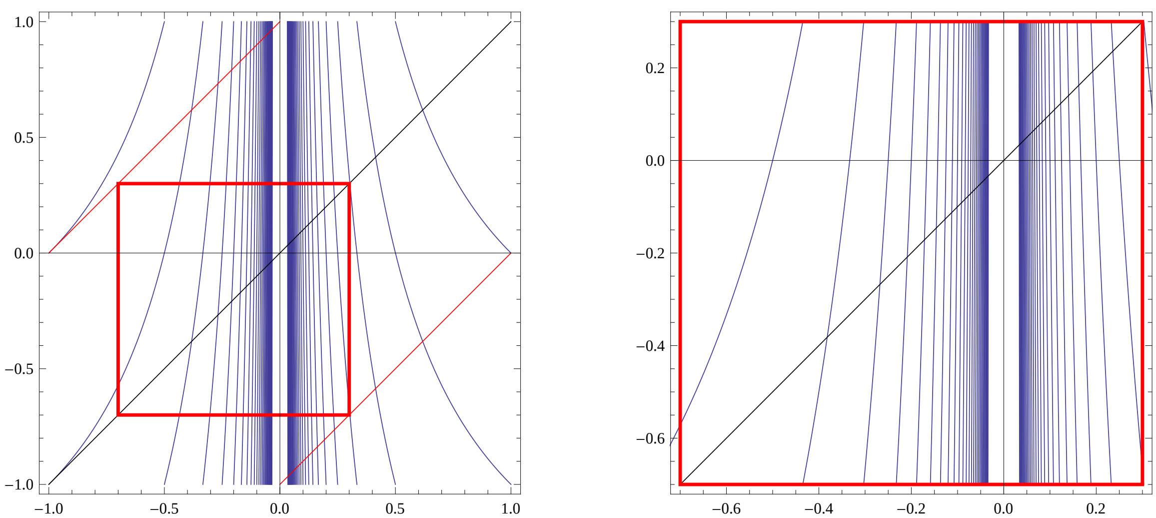


FIGURE 22. The graph of the α-continued fraction maps T_α. To the left: the graphs of $1/|x| - c$ for c integer. Each value of α determines a square of unit side length, which we take as the domain of T_α. To the right: the graph of T_α for $\alpha = 3/10$.

Each of these maps is associated to a different continued fraction expansion algorithm, and the family T_α interpolates between maps associated to well-known expansions: $T_1 = G$ is the usual Gauss map which generates regular continued fractions, while $T_{1/2}$ is associated to the *continued fraction to the nearest integer*, and T_0 generates the *by-excess continued fraction* expansion. For more about α-continued fraction expansions, their metric properties and their relations with other continued fraction expansions we refer to [Na], [Sc], [IK]. This family has also been studied in relation to the Brjuno function [MMY], [MCM].

Every T_α has infinitely many branches, and, for $\alpha > 0$, all branches are expansive and T_α admits an invariant probability measure absolutely continuous with respect to Lebesgue measure. Hence, each T_α has a finite measure-theoretic entropy $h(\alpha)$ with respect to such invariant measure: the entropy of the map T_α is proportional to the speed of convergence of the corresponding expansion algorithm (known as *α-euclidean algorithm*) [BDV], and to the exponential growth rate of the partial quotients in the α-expansion of typical values [NN].

Nakada [Na], who first investigated the properties of this family of continued fraction algorithms, gave an explicit formula for $h(\alpha)$ for $\frac{1}{2} \leq \alpha \leq 1$, from which it is evident that entropy displays a phase transition phenomenon when the parameter equals the golden mean $g := \frac{\sqrt{5}-1}{2}$ (see also Figure 23, left):

$$h(\alpha) = \begin{cases} \frac{\pi^2}{6\log(1+\alpha)} & \text{for } \frac{\sqrt{5}-1}{2} < \alpha \leq 1 \\ \frac{\pi^2}{6\log\frac{\sqrt{5}+1}{2}} & \text{for } \frac{1}{2} \leq \alpha \leq \frac{\sqrt{5}-1}{2}. \end{cases} \tag{20}$$

Several authors have studied the behavior of the metric entropy of T_α as a function of the parameter α ([Ca], [LM], [NN], [KSS]); in particular Luzzi and Marmi [LM] first produced numerical evidence that the entropy is continuous, although it displays many more (even if less evident) phase transition points and it is not monotone on the interval $[0, 1/2]$. Subsequently, Nakada and Natsui [NN] identified a dynamical condition that forces the entropy to be, at least locally, monotone: indeed, they noted that for some parameters α, the orbits under T_α of α and $\alpha - 1$ collide after a number of steps, i.e. there exist N, M such that:

$$T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1) \tag{21}$$

and they proved that, whenever the *matching condition* (21) holds, $h(\alpha)$ is monotone on a neighbourhood of α. They also showed that h has *mixed monotonic behavior* near the origin: namely, for every $\delta > 0$, in the interval $(0, \delta)$ there are intervals on which $h(\alpha)$ is monotone, others on which $h(\alpha)$ is increasing and others on which $h(\alpha)$ is decreasing.

In [CT] it is proven that the set of parameters for which (21) holds actually has full measure in parameter space. Moreover, such a set is the union of countably many open intervals, called *maximal quadratic intervals*. Each maximal quadratic interval I_r is labeled by a rational number r and can be thought of as a stability domain in parameter space: indeed, the number of steps M, N it takes for the orbits to collide is the same for each $\alpha \in I_r$, and even the symbolic orbit of α and $\alpha - 1$ up to the collision is fixed (compare to *mode-locking* phenomena in the theory of circle maps). For this reason, the complement of the union of all I_r is called the *bifurcation set* or *exceptional set* $\mathcal{E}$.

Numerical experiments [LM], [CMPT] show the entropy function $h(\alpha)$ displays self-similar features: the main goal of this section is to prove such self-similar structure by exploiting the self-similarity of the bifurcation set $\mathcal{E}$.

The way to study the self-similar structure was suggested to us by the unexpected isomorphism between $\mathcal{E}$ and the real slice of the boundary of the Mandelbrot set (Theorem 1.1). In the family of quadratic polynomials, Douady and Hubbard [DH] described the small copies of the Mandelbrot set which appear inside the large Mandelbrot set as images of *tuning operators*: we define a similar family of operators using the dictionary of section 14.

Our construction is the following: we associate, to each rational number r indexing a maximal interval, a *tuning map* τ_r from the whole parameter space of α-continued fraction transformations to a subset W_r, called *tuning window*. Note that τ_r also maps the bifurcation set $\mathcal{E}$ into itself. A tuning window W_r is called *neutral* if the alternating sum of the partial quotients of r is zero. Let us define a *plateau* of a real-valued function as a maximal, connected open set where the function is constant.

In the following sections we will prove Theorem 1.10 of the introduction, namely that the function h is constant on every neutral tuning window W_r, and every plateau of h is the interior of some neutral tuning window W_r. Even more precisely, we will characterize the set of rational numbers r such that the interior of W_r is a plateau (see Theorem 23.14). A particular case of the theorem is the following recent result [KSS]:

$$h(\alpha) = \frac{\pi^2}{6\log(1+g)} \qquad \forall \alpha \in [g^2, g],$$

and (g^2, g) is a plateau (i.e. h is not constant on $[t, g]$ for any $t < g^2$).

On non-neutral tuning windows, instead, entropy is non-constant and h reproduces, on a smaller scale, its behavior on the whole parameter space $[0, 1]$. Let us reformulate Theorem 1.11 of the introduction:

Theorem 19.1. *If h is increasing on a maximal interval I_r, then the monotonicity of h on the tuning window W_r reproduces the behavior on the interval $[0, 1]$, but with reversed sign: more precisely, if I_p is another maximal interval, then*

(1) h is increasing on $I_{\tau_r(p)}$ iff it is decreasing on I_p;

(2) h is decreasing on $I_{\tau_r(p)}$ iff it is increasing on I_p;

(3) h is constant on $I_{\tau_r(p)}$ iff it is constant on I_p.

If, instead, h is decreasing on I_r, then the monotonicity of I_p and $I_{\tau_r(p)}$ is the same.

As a consequence, we can also completely classify the local monotonic behavior of the entropy function $\alpha \mapsto h(\alpha)$:

Theorem 19.2. *Let α be a parameter in the parameter space of α-continued fractions. Then:*

(1) if $\alpha \notin \mathcal{E}$, then h is monotone on a neighbourhood of α;

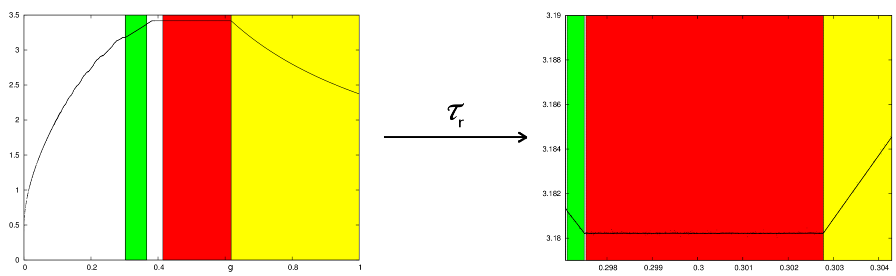


FIGURE 23. An illustration of Theorem 19.1 is given in the picture: on the left, you see the whole parameter space $[0,1]$, and the graph of h. Colored strips correspond to three maximal intervals. On the right, x ranges on the tuning window $W_{1/3} = [\frac{5-\sqrt{3}}{22}, \frac{\sqrt{3}-1}{2})$ relative to $r = 1/3$. Maximal intervals on the left are mapped via τ_r to maximal intervals of the same color on the right. As prescribed by Theorem 19.1, the monotonicity of h on corresponding intervals is reversed. Note that in the white strips (even if barely visible on the right) there are infinitely many maximal quadratic intervals.

(2) if $\alpha \in \mathcal{E}$, then either

(i) α is a phase transition*: h is constant on the left of α and strictly monotone (increasing or decreasing) on the right of α;*

(ii) α lies in the interior of a neutral tuning window*: then h is constant on a neighbourhood of α;*

(iii) otherwise, h has mixed monotonic behavior *at α, i.e. in every neighbourhood of α there are infinitely many intervals on which h is increasing, infinitely many on which it is decreasing and infinitely many on which it is constant.*

Note that all cases occur for infinitely many parameters: more precisely, (1) occurs for a set of parameters of full Lebesgue measure; (2)(i) for a countable set of parameters; (2)(ii) for a set of parameters whose Hausdorff dimension is positive, but smaller than $\frac{1}{2}$; (2)(iii) for a set of parameters of Hausdorff dimension 1. Note also that all phase transitions are of the form $\alpha = \tau_r(g)$, i.e. they are tuned images of the phase transition at $\alpha = g$ which is described by formula (20). The largest parameter for which (2)(iii) occurs is indeed $\alpha = g^2$, which is the left endpoint of the neutral tuning window $W_{1/2}$. Moreover, there is an explicit algorithm to decide, whenever α is a quadratic irrational, which of these cases occurs.

20. Background and definitions

The continued fraction expansion of a number

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

will be denoted by $x = [a_1, a_2, \dots]$, and the n^{th} convergent of x will be denoted by $\frac{p_n}{q_n} := [a_1, ..., a_n]$. Often we will also use the compact notation $x = [S]$ where $S = (a_1, a_2, \dots)$ is the (finite or infinite) string of partial quotients of x.

If S is a finite string, its length will be denoted by $|S|$. A string A is a *prefix* of S if there exists a (possibly empty) string B such that $S = AB$; A is a *suffix* of S if there exists a (possibly empty) string B such that $S = BA$; A is a *proper suffix* of S if there exists a non-empty string B such that $S = BA$.

20.1. Fractal sets defined by continued fractions.

We can define an action of the semigroup of finite strings (with the operation of concatenation) on the unit interval. Indeed, for each S, we denote by $S \cdot x$ the number obtained by appending the string S at the beginning of the continued fraction expansion of x; by convention the empty string corresponds to the identity.

We shall also use the notation $f_S(x) := S \cdot x$; let us point out that the Gauss map $G(x) := \left\{\frac{1}{x}\right\}$ acts as a shift on continued fraction expansions, hence f_S is a right inverse of $G^{|S|}$ ($G^{|S|} \circ f_S(x) = x$). It is easy to check that concatenation of strings corresponds to composition $(ST) \cdot x = S \cdot (T \cdot x)$; moreover, the map f_S is increasing if $|S|$ is even, decreasing if it is odd. It is not hard to see that f_S is given by the formula

$$f_S(x) = \frac{p_{n-1}x + p_n}{q_{n-1}x + q_n} \tag{22}$$

where $\frac{p_n}{q_n} = [a_1, \ldots, a_n]$ and $\frac{p_{n-1}}{q_{n-1}} = [a_1, \ldots, a_{n-1}]$. The map f_S is a contraction of the unit interval: indeed, by taking the derivative in the previous formula and using the relation $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ (see [IK]), $f_S'(x) = \frac{(-1)^n}{(q_{n-1}x+q_n)^2}$, hence

$$\frac{1}{4q(S)^2} \leq |f_S'(x)| \leq \frac{1}{q(S)^2} \qquad \forall x \in [0,1] \tag{23}$$

where $q(S) = q_n$ is the denominator of the rational number whose continued fraction expansion is S.

A common way of defining Cantor sets via continued fraction expansions is the following:

Definition 20.1. *Given a finite set $\mathcal{A}$ of finite strings of positive integers, the* regular Cantor set *defined by $\mathcal{A}$ is the set*

$$K(\mathcal{A}) := \{x = [W_1, W_2, \ldots] \ : \ W_i \in \mathcal{A} \ \forall i \geq 1\}.$$

For instance, the case when the alphabet $\mathcal{A}$ consists of strings with a single digit gives rise to sets of continued fractions with *restricted digits* [He].

An important geometric invariant associated to a fractal subset K of the real line is its *Hausdorff dimension* H.dim K. In particular, a regular Cantor set is generated by an *iterated function system*, and its dimension can be estimated in a standard way (for basic properties about Hausdorff dimension we refer to Falconer's book [Fa], in particular Chapter 9).

Indeed, if the alphabet $\mathcal{A} = \{S_1, \ldots, S_k\}$ is not redundant (in the sense that no S_i is prefix of any S_j with $i \neq j$), the dimension of $K(\mathcal{A})$ is bounded in terms of the smallest and largest contraction factors of the maps f_W ([Fa], Proposition 9.6):

$$\frac{\log N}{-\log m_1} \leq \text{H.dim } K(\mathcal{A}) \leq \frac{\log N}{-\log m_2} \tag{24}$$

where $m_1 := \inf_{\substack{W\in\mathcal{A} \\ x\in[0,1]}} |f'_W(x)|$, $m_2 := \sup_{\substack{W\in\mathcal{A} \\ x\in[0,1]}} |f'_W(x)|$, and N is the cardinality of $\mathcal{A}$.

20.2. **Maximal intervals and matching.** Let us now relate the previous construction to the dynamics of α-continued fractions. The main result of [CT] is that for all parameters α belonging to a maximal quadratic interval I_r, the orbits of α and $\alpha - 1$ under the α-continued fraction transformation T_α coincide after a finite number of steps, and this number of steps depends only on the usual continued fraction expansion of the pseudocenter r:

Theorem 20.2 ([CT], Theorem 3.1)**.** *Let I_r be a maximal quadratic interval, and $r = [a_1, \ldots, a_n]$ with n even. Let*

$$N = \sum_{i \text{ even}} a_i \qquad M = \sum_{i \text{ odd}} a_i. \tag{25}$$

Then for all $\alpha \in I_r$,

$$T_\alpha^{N+1}(\alpha) = T_\alpha^{M+1}(\alpha - 1). \tag{26}$$

Equation (26) is called *matching condition.* Notice that N and M are the same for all α which belong to the open interval I_r. Indeed, even more is true, namely the symbolic orbits of α and $\alpha - 1$ up to steps respectively N and M are constant over all the interval I_r ([CT], Lemma 3.7). Thus we can regard each maximal quadratic interval as a stability domain for the family of α-continued fraction transformations, and the complement $\mathcal{E}$ as the bifurcation locus.

One remarkable phenomenon, which was first discovered by Nakada and Natsui ([NN], Theorem 2), is that the matching condition locally determines the monotonic behavior of $h(\alpha)$:

Proposition 20.3 ([CT], Proposition 3.8)**.** *Let I_r be a maximal quadratic interval, and let N, M be as in Theorem 20.2. Then:*

(1) if $N < M$, the entropy $h(\alpha)$ is increasing for $\alpha \in I_r$;

(2) if $N = M$ it is constant on I_r;

(3) if $N > M$ it is decreasing on I_r.

21. Tuning

Let us now define *tuning operators* acting on parameter space, inspired by the analogy with complex dynamics. We will then see how such operators are responsible for the self-similar structure of the entropy.

21.1. **Tuning windows.** Let $r \in \mathbb{Q}_E$ be the pseudocenter of the maximal interval $I_r = (\alpha_1, \alpha_0)$; if $r = [S_0] = [S_1]$ are the even and odd expansions of r, then $\alpha_i = [\overline{S_i}]$ $(i = 0, 1)$. Let us also set $\omega := [S_1\overline{S_0}]$ and define the tuning window generated by r as the interval

$$W_r := [\omega, \alpha_0).$$

The value α_0 will be called the *root* of the tuning window. For instance, if $r = \frac{1}{2} = [2] = [1, 1]$, then $\omega = [2, \overline{1}] = g^2$ and the root $\alpha_0 = [\overline{1}] = g$.

The following proposition describes in more detail the structure of the tuning windows: a value x belongs to $B(\omega) \cap [\omega, \alpha_0]$ if and only if its continued fraction is an infinite concatenation of the strings S_0, S_1.

Proposition 21.1. *Let $r \in \mathbb{Q}_E$, and let $W_r = [\omega, \alpha_0)$. Then*

$$\mathcal{B}(\omega) \cap [\omega, \alpha_0] = K(\Sigma)$$

where $K(\Sigma)$ is the regular Cantor set on the alphabet $\Sigma = \{S_0, S_1\}$.

For instance, if $r = \frac{1}{2}$, then $W_{\frac{1}{2}} = [g^2, g)$, and $\mathcal{B}(g^2) \cap [g^2, g]$ is the set of numbers whose continued fraction expansion is an infinite concatenation of the strings $S_0 = (1, 1)$ and $S_1 = (2)$.

21.2. **Tuning operators.** For each $r \in \mathbb{Q}_E$ we can define the *tuning map* $\tau_r : [0, 1] \to [0, r]$ as $\tau_r(0) = \omega$ and

$$\tau_r([a_1, a_2, \dots]) = [S_1 S_0^{a_1 - 1} S_1 S_0^{a_2 - 1} \dots]. \tag{27}$$

Note that this map is well-defined even on rational values (where the continued fraction representation is not unique); for instance, $\tau_{1/3}([3, 1]) = [3, 2, 1, 2, 1, 3] = [3, 2, 1, 2, 1, 2, 1] = \tau_{1/3}([4])$.

It will be sometimes useful to consider the action that τ_r induces on finite strings of positive integers: with a slight abuse of notation we shall denote this action by the same symbol τ_r.

Lemma 21.2. *For each $r \in \mathbb{Q}_E$, the map τ_r is strictly increasing (hence injective). Moreover, τ_r is continuous at all irrational points, and discontinuous at every positive rational number.*

The first key feature of tuning operators is that they map the bifurcation set into a small copy of itself:

Proposition 21.3. *Let $r \in \mathbb{Q}_E$. Then*

(i) $\tau_r(\mathcal{E}) = \mathcal{E} \cap W_r$, and τ_r is a homeomorphism of $\mathcal{E}$ onto $\mathcal{E} \cap W_r$;

(ii) $\tau_r(\mathbb{Q}_E) = \mathbb{Q}_E \cap W_r \setminus \{r\}$.

Let us moreover notice that tuning windows are nested:

Lemma 21.4. *Let $r, s \in \mathbb{Q}_E$. Then the following are equivalent:*

(i) $\overline{W_r} \cap \overline{W_s} \neq \emptyset$ with $r < s$;

(ii) $r = \tau_s(p)$ for some $p \in \mathbb{Q}_E$*;*

(iii) $\overline{W_r} \subseteq W_s$*.*

21.3. **Proofs.**

Proof of Lemma 21.2. Let us first prove that τ_r preserves the order between irrational numbers. Pick $\alpha, \beta \in (0,1) \setminus \mathbb{Q}$, $\alpha \neq \beta$. Then

$$\alpha := [P, a, a_2, a_3, ...], \qquad \beta := [P, b, b_2, b_3, ...]$$

where P is a finite string of positive integers (common prefix), and we may assume also that $a < b$. Then

$$\tau_r(\alpha) := [\tau_r(P), S_1, S_0^{a-1}, S_1, ...], \qquad \tau_r(\beta) := [\tau_r(P), S_1, S_0^{b-1}, S_1, ...].$$

Since $|S_0^{a-1}|$ is even and $S_1 << S_0$, we get $S_0^{a-1}S_1 << S_0^{b-1}S_1$, whence $S_1S_0^{a-1}S_1 >> S_1S_0^{b-1}S_1$. Therefore, since $|P| \equiv |\tau_r(P)| \mod 2$, we get that either $|P|$ is even, $\alpha > \beta$ and $\tau_r(\alpha) > \tau_r(\beta)$, or $|P|$ is odd , $\alpha < \beta$ and $\tau_r(\alpha) < \tau_r(\beta)$, so we are done. The continuity of τ_r at irrational points follows from the fact that if $\beta \in (0,1) \setminus \mathbb{Q}$ and x is close to β then the continued fraction expansions of x and β have a long common prefix, and, by definition of τ_r, then their images will also have a long prefix in common, and will therefore be close to each other. Finally, let us check that the function is increasing at each rational number $c > 0$. This follows from the property:

$$\sup_{\substack{\alpha \in \mathbb{R}\setminus\mathbb{Q} \\ \alpha < c}} \tau_r(\alpha) < \tau_r(c) < \inf_{\substack{\alpha \in \mathbb{R}\setminus\mathbb{Q} \\ \alpha > c}} \tau_r(\alpha). \tag{28}$$

Let us prove the left-hand side inequality of (28) (the right-hand side one has essentially the same proof). Suppose $c = [S]$, with $|S| \equiv 1 \mod 2$. Then every irrational $\alpha < c$ has an expansion of the form $\alpha = [S, A]$ with A an infinite string. Hence $\tau_r(\alpha) = [\tau_r(S), \tau_r(A)]$, and it is not hard to check that $\sup \tau_r(\alpha) = [\tau_r(S), S_1, \overline{S_0}] < [\tau_r(S)] = \tau_r(c)$. Discontinuity at positive rational points also follows from (28). $\square$

To prove Propositions 21.1 and 21.3 we first need some lemmas.

Lemma 21.5. *Let $r = [S_0] = [S_1] \in \mathbb{Q}_E$ and y be an irrational number with continued fraction expansion $y = [B, S_*, \dots]$, where B is a proper suffix of either S_0 or S_1, and S_* equal to either S_0 or S_1. Then $y > [S_1]$.*

Proof. If $B = (1)$ then there is hardly anything to prove (by Prop. 13.14, the first digit of S_1 is strictly greater than 1). If not, then one of the following is true:

(1) $S_0 = AB$ and A is a prefix of S_1 as well;

(2) $S_1 = AB$ and A is a prefix of S_0 as well.

By Proposition 13.14, in the first case we get that $BA \geq AB = S_0 >> S_1$, while in the latter $BA >> AB = S_1$; so in both cases $BA >> S_1$ and the claim follows. $\square$

Lemma 21.6. *Let $r \in \mathbb{Q}_E$, and $x, y \in [0,1] \setminus \mathbb{Q}$. Then*

$$G^k(x) \geq y \quad \forall k \geq 0$$

if and only if

$$G^k(\tau_r(x)) \geq \tau_r(y) \quad \forall k \geq 0.$$

Proof. Since τ_r is increasing, $G^k(x) \geq y$ if and only if $\tau_r(G^k(x)) \geq \tau_r(y)$ if and only if $G^{N_k}(\tau_r(x)) \geq \tau_r(y)$ for $N_k = |S_0|(a_1 + \cdots + a_k) + (|S_1| - |S_0|)k$.

On the other hand, if h is not of the form N_k, $G^h(\tau_r(x)) = [B, S_*, \dots]$ with B a proper suffix of either S_0 or S_1, and S_* equal to either S_0 or S_1. By Lemma 21.5 it follows immediately that

$$G^h(\tau_r(x)) > [S_1] \geq \tau_r(y).$$

$\square$

Proof of Proposition 21.1. Let us first prove that, if $x \in \mathcal{B}(\omega) \cap [\omega, \alpha_0]$ then $x = S \cdot y$ with $y \in \mathcal{B}(\omega) \cap [\omega, \alpha_0]$ and $S \in \{S_0, S_1\}$; then the inclusion

$$\mathcal{B}(\omega) \cap [\omega, \alpha_0] \subset K(\Sigma)$$

will follow by induction. If $x \in \mathcal{B}(\omega) \cap [\omega, \alpha_0]$ then the following alternative holds

$(x > r)$ $x = S_0 \cdot y$ and $S_0 \cdot y = x < \alpha_0 = S_0 \cdot \alpha_0$, therefore $y \leq \alpha_0$;

$(x < r)$ $x = S_1 \cdot y$ and $S_1 \cdot y = x > \omega = S_1 \cdot \alpha_0$, therefore $y \leq \alpha_0$;

Note that, since the map $y \mapsto S \cdot y$ preserves or reverses the order depending on the parity of $|S|$, in both cases we get to the same conclusion. Moreover, since $\mathcal{B}(\omega)$ is forward-invariant with respect to the Gauss map and $x \in \mathcal{B}(\omega)$, then $y = G^k(x) \in \mathcal{B}(\omega)$ as well, hence $y \in \mathcal{B}(\omega) \cap [\omega, \alpha_0]$.

To prove the other inclusion, let us first remark that every $x \in K(\Sigma)$ satisfies $\omega \leq x \leq \alpha_0$. Now, let $k \in \mathbb{N}$; either $G^k(x) \in K(\Sigma)$, and hence $G^k(x) \geq \omega$, or $G^k(x) = [B, S_*, ...]$ satisfies the hypotheses of Lemma 21.5, and hence we get that $y > [S_1] > \omega$. Since $G^k(x) \geq \omega$ holds for any k, then $x \in \mathcal{B}(\omega)$. $\square$

Proof of Proposition 21.3. (i) Recall the notation $W_r = [\omega, \alpha_0)$, and let $v \in \mathcal{E} \cap W_r$. By the inclusion of $\mathcal{E} \cap [\omega, 1]$ in $\mathcal{B}(\omega)$ we have $\mathcal{E} \cap W_r \subseteq \mathcal{B}(\omega) \cap [\omega, \alpha_0)$, hence, by Proposition 21.1, $v \in K(\Sigma)$. Moreover, $v < r$ because $\mathcal{E} \cap [r, \alpha_0) = \emptyset$. As a consequence, the continued fraction expansion of v is an infinite concatenation of strings in the alphabet $\{S_0, S_1\}$ starting with S_1. Now, if the expansion of v terminates with $\overline{S_0}$, then $G^k(v) = \omega$ for some k, hence v must coincide with $\omega = [S_1\overline{S_0}]$, so $v = \tau_r(0)$ and we are done. Otherwise, there exists some $x \in [0, 1)$ such that $v = \tau_r(x)$: then by Lemma 21.6 we get that

$$G^k(v) \geq v \ \ \forall k \geq 0 \Rightarrow G^k(x) \geq x \ \ \forall k \geq 0$$

which means x belongs to $\mathcal{E}$.

Viceversa, let us pick $x := \tau_r(v)$ with $v \in \mathcal{E}$. By definition of τ_r, $x \in W_r$. Moreover, since v belongs to $\mathcal{E}$, $G^n(v) \geq v$ for any n, hence by Lemma 21.6 also $\tau_r(v)$ belongs to $\mathcal{E}$. The fact that τ_r is a homeomorphism follows from bijectivity and compactness.

(ii) Let $p \in \mathbb{Q}_E$ and $I_p = (\alpha_1, \alpha_0)$ the maximal quadratic interval generated by p; by point (i) above also the values $\beta_i := \tau_r(\alpha_i)$, $(i = 0, 1)$ belong to $\mathcal{E} \cap W_r$. Since τ_r is strictly increasing, no other point of $\mathcal{E}$ lies between β_1 and β_0, hence $(\beta_1, \beta_0) = I_s$ for some $s \in \mathbb{Q}_E \cap [\omega, r)$. Since $\tau_r(p)$ is a convergent to both $\tau_r(\alpha_0)$ and $\tau_r(\alpha_1)$, then $\tau_r(p) = s$.

To prove the converse, pick $s \in \mathbb{Q}_E \cap [\omega, r)$ and denote $I_s = (\beta_1, \beta_0)$. Again by point (i), $\beta_i := \tau_r(\alpha_i)$ for some $\alpha_0, \alpha_1 \in \mathcal{E}$, and (α_1, α_0) is a component of the complement of $\mathcal{E}$, hence there exists $p \in \mathbb{Q}_E$ such that $I_p = (\alpha_1, \alpha_0)$. As a consequence, $s = \tau_r(p)$. $\square$

Proof of Lemma 21.4. Let us denote $W_s = [\omega(s), \alpha_0(s))$, $W_r = [\omega(r), \alpha_0(r))$, $W_p = [\omega(p), \alpha_0(p))$. Suppose (i): then, since the closures of W_r and W_s are not disjoint, $\omega(s) \leq \alpha_0(r)$. Moreover, $\omega(s) \in \mathcal{E}$ and $\mathcal{E} \cap (r, \alpha_0(r)] = \{\alpha_0(r)\}$, hence $\omega(s) \leq r$ because $\omega(s)$ cannot coincide with $\alpha_0(r)$, not having a purely periodic continued fraction expansion. Hence $r \in W_s$ and, by Proposition 21.3, there exists $p \in \mathbb{Q}_E$ such that $r = \tau_s(p)$.

Suppose now (ii). Then, since $r = \tau_s(p)$, also $\alpha_0(r) = \tau_s(\alpha_0(p)) \leq s < \alpha_0(s)$, and $\omega(r) = \tau_s(\omega(p)) \in W_s$, which implies (iii).

(iii) $\Rightarrow$ (i) is clear. $\square$

22. Tuning and monotonicity of entropy: proof of Theorem 19.1

Definition 22.1. *Let $A = (a_1, ..., a_n)$ be a string of positive integers. Then its* matching index $[\![A]\!]$ *is the alternating sum of its digits:*

$$[\![A]\!] := \sum_{j=1}^{n} (-1)^{j+1} a_j. \tag{29}$$

Moreover, if $r = [S_0]$ is a rational number between 0 and 1 and S_0 is its continued fraction expansion of even length, we define the matching index *of r to be*

$$[\![r]\!] := [\![S_0]\!].$$

The reason for this terminology is the following. Suppose $r \in \mathbb{Q}_E$ is the pseudocenter of the maximal quadratic interval I_r: then by Theorem 20.2, a matching condition (26) holds, and by formula (25)

$$[\![r]\!] = \sum_{j=1}^{n} (-1)^{j+1} a_j = M - N \tag{30}$$

where $r = [S_0]$ and $S_0 = (a_1, \ldots, a_n)$. This means, by Proposition (20.3), that the entropy function $h(\alpha)$ is increasing on I_r iff $[\![r]\!] > 0$, decreasing on I_r iff $[\![r]\!] < 0$, and constant on I_r iff $[\![r]\!] = 0$.

Lemma 22.2. *Let $r, p \in \mathbb{Q}_E$. Then*

$$[\![\tau_r(p)]\!] = -[\![r]\!][\![p]\!]. \tag{31}$$

Proof. The double bracket notation behaves well under concatenation, namely:

$$[\![AB]\!] := \begin{cases} [\![A]\!] + [\![B]\!] & \text{if } |A| \text{ even} \\ [\![A]\!] - [\![B]\!] & \text{if } |A| \text{ odd.} \end{cases}$$

Let $p = [a_1, ..., a_n]$ and $r = [S_0]$ be the continued fraction expansions of even length of $p, r \in \mathbb{Q}_E$; using the definition of τ_r we get

$$[\![\tau_r(p)]\!] = \sum_{j=1}^{n} (-1)^{j+1} \left([\![S_1]\!] - (a_j - 1)[\![S_0]\!]\right)$$

and, since $n = |A|$ is even, the right-hand side becomes $[\![S_0]\!] \sum_{j=1}^{n} (-1)^j a_j$, whence the thesis. $\square$

Definition 22.3. *A quadratic interval I_r is called* neutral *if* $[\![r]\!] = 0$*. Similarly, a tuning window W_r is called* neutral *if* $[\![r]\!] = 0$*.*

As an example, the rational $r = \frac{1}{2} = [2] = [1,1]$ generates the neutral tuning window $W_{1/2} = [g^2, g)$.

Proof of Theorem 19.1. Let I_r be a maximal quadratic interval over which the entropy is increasing. Then, by Theorem 20.2 and Proposition 20.3, for $\alpha \in I_r$, a matching condition (26) holds, with $M - N > 0$. This implies by (30) that $[\![r]\!] > 0$. Let now I_p be another maximal quadratic interval. By Proposition 21.3 (ii), $I_{\tau_r(p)}$ is also a maximal quadratic interval, and by Lemma 22.2

$$[\![\tau_r(p)]\!] = -[\![r]\!][\![p]\!].$$

Since $[\![r]\!] > 0$, then $[\![\tau_r(p)]\!]$ and $[\![p]\!]$ have opposite sign. In terms of the monotonicity of entropy, this means the following:

(1) if the entropy is increasing on I_p, then by (30) $[\![p]\!] > 0$, hence $[\![\tau_r(p)]\!] < 0$, which implies (again by (30)) that the entropy is decreasing on $I_{\tau_r}(p)$;
(2) if the entropy is decreasing on I_p, then $[\![p]\!] < 0$, hence $[\![\tau_r(p)]\!] > 0$ and the entropy is increasing on $I_{\tau_r}(p)$;
(3) if the entropy is constant on I_p, then $[\![p]\!] = 0$, hence $[\![\tau_r(p)]\!] = 0$ and the entropy is constant on $I_{\tau_r}(p)$.

If, instead, the entropy is decreasing on I_r, then $[\![r]\!] > 0$, hence $[\![\tau_r(p)]\!]$ and $[\![p]\!]$ have the same sign, which similarly to the previous case implies that the monotonicity of entropy on I_p and $I_{\tau_r(p)}$ is the same. $\square$

Remark 22.4. *The same argument as in the proof of Theorem 19.1 shows that, if $r \in \mathbb{Q}_E$ with $[\![r]\!] = 0$, then the entropy on $I_{\tau_r(p)}$ is constant for each $p \in \mathbb{Q}_E$ (no matter what the monotonicity is on I_p).*

23. Plateaux: proof of Theorem 1.10

The goal of this section is to prove Theorem 23.14, which characterizes the plateaux of the entropy and has as a consequence Theorem 1.10 in the introduction. Meanwhile, we introduce the set of *untuned parameters* and *dominant parameters*.

23.1. **The importance of being Hölder.** The first step in the proof of Theorem 1.10 is proving that the entropy function $h(\alpha)$ is indeed constant on neutral tuning windows:

Proposition 23.1. *Let $r \in \mathbb{Q}_E$ generate a neutral maximal interval, i.e. $[\![r]\!] = 0$. Then the entropy function $h(\alpha)$ is constant on $\overline{W_r}$.*

By Remark 22.4, we already know that the entropy is locally constant on all connected components of $W_r \setminus \mathcal{E}$, which has full measure in W_r. However, since $W_r \cap \mathcal{E}$ has, in general, positive Hausdorff dimension, in order to prove that the entropy is actually constant on the whole W_r one needs to exclude a devil staircase behavior. We shall exploit the following criterion:

Lemma 23.2. *Let $f : I \to \mathbb{R}$ be a Hölder-continuous function of exponent $\eta \in (0, 1)$, and assume that there exists a closed set $C \subseteq I$ such that f is locally constant at all $x \notin C$. Suppose moreover* $\text{H.dim } C < \eta$*. Then f is constant on I.*

Proof. Suppose f is not constant: then by continuity $f(I)$ is an interval with non-empty interior, hence $\text{H.dim } f(I) = 1$. On the other hand, we know f is constant on the connected components of $I \setminus C$, so we get $f(I) = f(C)$, whence

$$\text{H.dim } f(C) = \text{H.dim } f(I) = 1.$$

But, since f is η-Hölder continuous, we also get (e.g. by [Fa], Proposition 2.3)

$$\text{H.dim } f(C) \leq \frac{\text{H.dim } C}{\eta}$$

and thus $\eta \leq \text{H.dim}\ C$, contradiction. $\square$

Let us know check the hypotheses of Lemma 23.2 are met in our case; the first one is given by the following

Theorem 23.3 ([Ti]). *For all fixed $0 < \eta < 1/2$, the function $\alpha \mapsto h(\alpha)$ is locally Hölder-continuous of exponent η on $(0, 1]$.*

We are now left with checking that the Hausdorff dimension of $\mathcal{E} \cap W_r$ is sufficiently small:

Lemma 23.4. *For all $r \in \mathbb{Q}_E$, an upper bound to the Hausdorff dimension of $\mathcal{E} \cap W_r$ is*

$$\text{H.dim}\ \mathcal{E} \cap W_r \leq \frac{\log 2}{\log 5} < 1/2.$$

Proof. Let $r \in \mathbb{Q}_E$, $r = [S_0] = [S_1]$ and $\overline{W_r} = [\omega, \alpha]$. By the inclusion of $\mathcal{E} \cap [\omega, 1]$ in $\mathcal{B}(\omega)$ and Proposition 21.1,

$$\mathcal{E} \cap W_r \subset \mathcal{B}(\omega) \cap [\omega, \alpha] = K(\Sigma), \qquad \text{with} \qquad \Sigma = \{S_0, S_1\}.$$

Note we also have $K(\Sigma) = K(\Sigma_2)$ with $\Sigma_2 = \{S_0S_0, S_1S_0, S_1S_0, S_1S_1\}$ and, by virtue of (23) we have the estimate

$$|f'_{S_iS_j}(x)| \leq \frac{1}{q(S_iS_j)^2}, \quad i, j \in \{0, 1\}.$$

On the other hand, setting $Z_0 = (1, 1)$ and $Z_1 = (2)$ we can easily check that

$$q(S_iS_j) \geq q(Z_iZ_j) = 5 \quad \forall i, j \in \{0, 1\};$$

whence $|f'_{S_iS_j}(x)| \leq \frac{1}{25}$ and, by formula (24), we get our claim. $\square$

Proposition 23.1 now follows from Lemma 23.2, Theorem 23.3 and Lemma 23.4.

23.2. **Untuned parameters.** The set of *untuned parameters* is the complement of all tuning windows:

$$UT := [0, g] \setminus \bigcup_{r \in \mathbb{Q} \cap (0,1)} W_r.$$

Note that, since $I_r \subseteq W_r$, $UT \subseteq \mathcal{E}$. Moreover, we say that a rational $a \in \mathbb{Q}_E$ is *untuned* if it cannot be written as $a = \tau_r(a_0)$ for some $r, a_0 \in \mathbb{Q}_E$. We shall denote by $\mathbb{Q}_{UT}$ the set of all $a \in \mathbb{Q}_E$ which are untuned. Let us start out by seeing that each pseudocenter of a maximal quadratic interval admits an "untuned factorization":

Lemma 23.5. *Each $r \in \mathbb{Q}_E$ can be written as:*

$$r = \tau_{r_m} \circ \cdots \circ \tau_{r_1}(r_0), \qquad \text{with } r_i \in \mathbb{Q}_{UT} \ \forall i \in \{0, 1, ..., m\}. \tag{32}$$

Note that m can very well be zero (when r is already untuned).

Proof. A straightforward check shows that the tuning operator has the following associativity property:

$$\tau_{\tau_p(r)}(x) = \tau_p \circ \tau_r(x) \qquad \forall p, r \in \mathbb{Q}_E, \ x \in (0, 1). \tag{33}$$

For $s = [a_1, ..., a_m] \in \mathbb{Q}_E$ we shall set $\|s\|_1 := \sum_1^m a_i$; this definition does not depend on the representation of s, moreover

$$\|\tau_p(s)\|_1 = \|p\|_1 \|s\|_1 \qquad \forall p, s \in \mathbb{Q}_E.$$

The proof of (32) follows then easily by induction on $N = \|r\|_1$, using the fact that $\max(\|p\|_1, \|s\|_1) \leq \|\tau_p(s)\|_1/2$. $\square$

As a consequence of the following proposition, the connected components of the complement of UT are precisely the tuning windows generated by the elements of $\mathbb{Q}_{UT}$:

Proposition 23.6. *The set $\overline{UT}$ is a Cantor set: indeed,*

(i)

$$UT = [0, g] \setminus \bigcup_{r \in \mathbb{Q}_{UT}} W_r;$$

(ii) if $r, s \in \mathbb{Q}_{UT}$ *with* $r \neq s$*, then* $\overline{W_r}$ *and* $\overline{W_s}$ *are disjoint;*

(iii) if $x \in \overline{UT} \setminus UT$*, then there exists* $r \in \mathbb{Q}_{UT}$ *such that* $x = \tau_r(0)$*.*

Proof. (i). It is enough to prove that every tuning window W_r is contained in a tuning window W_s, with $s \in \mathbb{Q}_{UT}$. Indeed, let $r \in \mathbb{Q}_E$; either $r \in \mathbb{Q}_{UT}$ or, by Lemma 23.5, there exists $p \in \mathbb{Q}_E$ and $s \in \mathbb{Q}_{UT}$ such that $r = \tau_s(p)$, hence $W_r \subseteq W_s$.

(ii). By Lemma 21.4, if the closures of W_r and W_s are not disjoint, then $r = \tau_s(p)$, which contradicts the fact $r \in \mathbb{Q}_{UT}$.

(iii). By (i) and (ii), $\overline{UT}$ is a Cantor set, and each element x which belongs to $\overline{UT} \setminus UT$ is the left endpoint of some tuning window W_r with $r \in \mathbb{Q}_{UT}$, which is equivalent to say $x = \tau_r(0)$. □

Lemma 23.7. *The Hausdorff dimension of* UT *is full:*

$$\text{H.dim } UT = 1.$$

Proof. By the properties of Hausdorff dimension,

$$\text{H.dim } \mathcal{E} = \max\{\text{H.dim } UT, \sup_{r \in \mathbb{Q}_{UT}} \text{H.dim } \mathcal{E} \cap W_r\}.$$

Now, by [CT], H.dim $\mathcal{E} = 1$, and, by Lemma 23.4, H.dim $\mathcal{E} \cap W_r < \frac{1}{2}$, hence the claim. □

23.3. **Dominant parameters.** Recall that a finite string of positive integers and even length is dominant if it is smaller than all its proper suffixes (Definition 10.4). A related definition is the following:

Definition 23.8. *A quadratic irrational* $\alpha \in [0, 1]$ *is a* dominant parameter *if its continued fraction expansion is of the form* $\alpha = [\overline{S}]$ *with* S *a dominant string.*

For instance, $(2,1,1,1)$ is dominant, while $(2,1,1,2)$ is not (it is not true that $(2,1,1,2) << (2)$). In general, all strings whose first digit is strictly greater than the others are dominant, but there are even more dominant strings (for instance $(3,1,3,2)$ is dominant).

Remark 23.9. *By Proposition 13.14, if S is dominant then $[S] \in \mathbb{Q}_E$.*

A very useful feature of dominant strings is that they can be easily used to produce other dominant strings:

Lemma 23.10. *Let S_0 be a dominant string, and B a proper suffix of S_0 of even length. Then, for any $m \geq 1$, $S_0^m B$ is a dominant string.*

Proof. Let Y be a proper suffix of $S_0^m B$. There are three possible cases:

(1) Y is a suffix of B, hence a proper suffix of S_0. Hence, since S_0 is dominant, $S_0 >> Y$ and $S_0^m B >> Y$.

(2) Y is of the form $S_0^k B$, with $1 \leq k < m$. Then by dominance $S_0 >> B$, which implies $S_0^{m-k} B >> B$, hence $S_0^m B >> S_0^k B$.

(3) Y is of the form $C S_0^k B$, with $0 \leq k < m$ and C a proper suffix of S_0. Then again the claim follows by the fact that S_0 is dominant, hence $S_0 >> C$.

□

Lemma 23.11. *A dominant string S_0 cannot begin with two equal digits.*

Proof. By definition of dominance, S_0 cannot consist of just $k \geq 2$ equal digits. Suppose instead it has the form $S_0 = (a)^k B$ with $k \geq 2$ and B non empty and which does not begin with a. Then by dominance $(a)^k B << B$, hence $a << B$ since B does not begin with a. However, this implies $aB << aa$ and hence $aB << (a)^k B = S_0$, which contradicts the definition of dominance because aB is a proper suffix of S_0. □

The reason why dominant parameters turn out to be so useful is that they can approximate untuned parameters. Indeed, by Proposition 10.6, the set of dominant parameters is dense in $UT \setminus \{g\}$. More precisely, every parameter in $UT \setminus \{g\}$ is limit point from the right of dominant parameters.

Proposition 23.12. *Every element $\beta \in \overline{UT} \setminus \{g\}$ is limit point of non-neutral maximal quadratic intervals.*

Proof. We shall prove that either $\beta \in UT \setminus \{g\}$, and β is limit point from the right of non-neutral maximal quadratic intervals, or $\beta = \tau_s(0)$ for some $s \in \mathbb{Q}_{UT}$, and β is limit point from the left of non-neutral maximal quadratic intervals.

If $\beta \in UT$ then, by Proposition 10.6, β is the limit point from the right of a sequence $\alpha_n = [\overline{A_n}]$ with A_n dominant. If $[\![A_n]\!] \neq 0$ for infinitely many n, the claim is proven. Otherwise, it is sufficient to prove that every dominant parameter α_n such that $[\![A_n]\!] = 0$ is limit point from the right of non-neutral maximal intervals. Let S_0 be a dominant string, with $[\![S_0]\!] = 0$, and let $\alpha := [\overline{S_0}]$. First of all, the length of S_0 is bigger than 2: indeed, if S_0 had length 2, then condition $[\![S_0]\!] = 0$ would force it to be of the form $S_0 = (a, a)$ for some a, which contradicts the definition of dominant. Hence, we can write $S_0 = AB$ with A of length 2 and B of positive, even length. Then, by Lemma 23.10, $S_0^m B$ is also dominant, hence $p_m := [S_0^m B] \in \mathbb{Q}_E$ by Remark 23.9. Moreover, $\alpha < p_m$ since $S_0 << B$. Furthermore, S_0 cannot begin with two equal digits (Lemma 23.11), hence $[\![A]\!] \neq 0$ and $[\![S_0^m B]\!] = [\![B]\!] = [\![S_0]\!] - [\![A]\!] \neq 0$. Thus the sequence I_{p_m} is a sequence of non-neutral maximal quadratic intervals which tends to β from the right, and the claim is proven.

If $\beta \in \overline{UT} \setminus UT$, then by Proposition 23.6 (iii) there exists $s \in \mathbb{Q}_{UT}$ such that $\beta = \tau_s(0)$. Since $\overline{UT}$ is a Cantor set and β lies on its boundary, β is the limit point (from the left) of a sequence of points of UT, hence the claim follows by the above discussion. $\square$

23.4. **Characterization of plateaux.**

Definition 23.13. *A parameter $x \in \mathcal{E}$ is* finitely renormalizable *if it belongs to finitely many tuning windows. This is equivalent to say that $x = \tau_r(y)$, with $y \in UT$. A parameter $x \in \mathcal{E}$ is* infinitely renormalizable *if it lies in infinitely many tuning windows W_r, with $r \in \mathbb{Q}_E$. Untuned parameters are also referred to as* non renormalizable.

We are finally ready to prove Theorem 1.10 stated in the introduction, and indeed the following stronger version:

Theorem 23.14. *An open interval $U \subseteq [0,1]$ of the parameter space of α-continued fraction transformations is a plateau for the entropy function $h(\alpha)$ if and only if it is the interior of a neutral tuning window $U = \overset{\circ}{W}_r$, with r of either one of the following types:*

$$(NR) \quad r \in \mathbb{Q}_{UT}, [\![r]\!] = 0 \qquad \text{(non-renormalizable case)}$$

$$(FR) \quad r = \tau_{r_1}(r_0) \text{ with } \begin{cases} r_0 \in \mathbb{Q}_{UT}, & [\![r_0]\!] = 0 \\ r_1 \in \mathbb{Q}_E, & [\![r_1]\!] \neq 0 \end{cases} \qquad \text{(finitely renormalizable case).}$$

Proof. Let us pick r which satisfies (NR), and let $W_r = [\omega, \alpha_0)$ be its tuning window. By Proposition 23.1, since $[\![r]\!] = 0$, the entropy is constant on $\overline{W_r}$. Let us prove that it is not constant on any larger interval. Since $r \in \mathbb{Q}_{UT}$, by Proposition 23.6, α_0 belongs to UT. If $\alpha_0 = g$, then by the explicit formula (20) the entropy is decreasing to the right of α_0. Otherwise, by Proposition 23.12, α_0 is limit point from the right of non-neutral maximal quadratic intervals, hence entropy is not constant to the right of α_0. Moreover, by Proposition 23.6, ω belongs to the boundary of UT, hence, by Proposition 23.12, it is limit point from the left of non-neutral intervals. This means that the interior of W_r is a maximal open interval of constance for the entropy $h(\alpha)$, i.e. a plateau.

Now, suppose that r satisfies condition (FR), with $r = \tau_{r_1}(r_0)$. By the (NR) case, the interior of W_{r_0} is a plateau, and W_{r_0} is limit point from both sides of non-neutral intervals. Since τ_{r_1} maps non-neutral intervals to non-neutral intervals and is continuous on $\mathcal{E}$, then W_r is limit point from both sides of non-neutral intervals, hence its interior is a plateau.

Suppose now U is a plateau. Since $\mathcal{E}$ has no interior part, there is $r \in \mathbb{Q}_E$ such that I_r intersects U, hence, by Proposition 20.3, $[\![r]\!] = 0$ and actually $I_r \subseteq U$. Then, by Lemma 23.5 one has the factorization

$$r = \tau_{r_n} \circ \cdots \circ \tau_{r_1}(r_0)$$

with each $r_i \in \mathbb{Q}_{UT}$ untuned (recall n can possibly be zero, in which case $r = r_0$). Since the matching index is multiplicative (eq. (31)), there exists at least one r_i with zero meatching index: let $j \in \{0, \ldots, n\}$ be the largest index such that $[\![r_j]\!] = 0$. If $j = n$, let $s := r_n$: by the first part of the proof, the interior of W_s is a plateau, and it intersects U because they both contain r (by Lemma 21.4, r belongs to the interior of W_s), hence $U = \overset{\circ}{W}_s$, and we are in case (NR).

If, otherwise, $j < n$, let $s := \tau_{r_n} \circ \cdots \circ \tau_{r_{j+1}}(r_j)$. By associativity of tuning (eq. (33)) we can write

$$s = \tau_{s_1}(s_0)$$

with $s_0 := r_j$ and $s_1 := \tau_{r_n} \circ \cdots \circ \tau_{r_{j+2}}(r_{j+1})$. Moreover, by multiplicativity of the matching index (eq. (31)) $[\![s_1]\!] \neq 0$, hence s falls into the case (FR) and by the first part of the proof the interior of W_s is a plateau. Also, by construction, r belongs to the image of τ_s, hence it belongs to the interior of W_s. As a consequence, U and $\overset{\circ}{W}_s$ are intersecting plateaux, hence they must coincide. $\square$

24. Classification of local monotonic behavior

Lemma 24.1. *Any non-neutral tuning window W_r contains infinitely many intervals on which the entropy $h(\alpha)$ is constant, infinitely many over which it is increasing, and infinitely many on which it is decreasing.*

Proof. Let us consider the following sequences of rational numbers

$$s_n := [n, 1]$$

$$t_n := [n, n]$$

$$u_n := [n+1, n, 1, n].$$

It is not hard to check (e.g. using Proposition 13.14) that s_n, t_n, u_n belong to $\mathbb{Q}_E$. Moreover, by computing the matching indices one finds that, for $n > 2$, the entropy $h(\alpha)$ is increasing on I_{s_n}, constant on I_{t_n} and decreasing on I_{u_n}. Since W_r is non-neutral, by Theorem 19.1 τ_r either induces the same monotonicity or the opposite one, hence the sequences $I_{\tau_r(s_n)}, I_{\tau_r(t_n)}$ and $I_{\tau_r(u_n)}$ are sequences of maximal quadratic intervals which lie in W_r and display all three types of monotonic behavior. $\square$

Proof of Theorem 19.2. Let $\alpha \in [0, 1]$ be a parameter. If $\alpha \notin \mathcal{E}$, then α belongs to some maximal quadratic interval I_r, hence $h(\alpha)$ is monotone on I_r by Proposition 20.3, and by formula (30) the monotonicity type depends on the sign of $[\![r]\!]$.

If $\alpha \in \mathcal{E}$, there are the following cases:

(1) $\alpha = g$. Then α is a phase transition as described by formula (20);
(2) $\alpha \in UT \setminus \{g\}$. Then, by Proposition 23.12, α is limit point from the right of non-neutral tuning windows, and by Lemma 24.1 each non-neutral tuning window contains infinitely many intervals where the entropy is constant, increasing or decreasing; the parameter α has therefore mixed monotonic beahaviour.

(3) α is finitely renormalizable. Then one can write $\alpha = \tau_r(y)$, with $y \in UT$. There are three subcases:

(3a) $[\![r]\!] \neq 0$, and $y = g$. Since τ_r maps neutral intervals to neutral intervals and non-neutral intervals to non-neutral intervals, the phase transition at $y = g$ is mapped to a phase transition at α.

(3b) $[\![r]\!] \neq 0$, and $y \neq g$. Then, by case (2) y is limit point from the right of intervals with all types of monotonicity, hence so is α.

(3c) If $[\![r]\!] = 0$, then by using the untuned factorization (Lemma 23.5) one can write

$$\alpha = \tau_{r_m} \circ \cdots \circ \tau_{r_0}(y) \qquad r_i \in \mathbb{Q}_{UT}.$$

Let now $j \in \{0, \ldots, m\}$ be the largest index such that $[\![r_j]\!] = 0$. If $j = m$, then α belongs to the neutral tuning window W_{r_m}: thus, either α belongs to the interior of W_{r_m} (which means by Proposition 23.1 that the entropy is locally constant at α), or α coincides with the left endpoint of W_{r_m}. In the latter case, α belongs to the boundary of UT, hence by Proposition 23.12 and Lemma 24.1 it has mixed behavior. If $j < m$, then by the same reasoning as above $\tau_{r_j} \circ \cdots \circ \tau_{r_0}(y)$ either lies inside a plateau or has mixed behavior, and since the operator $\tau_{r_m} \circ \cdots \circ \tau_{r_{j+1}}$ either respects the monotonicity or reverses it, also α either lies inside a plateau or has mixed behavior.

(4) α is infinitely renormalizable, i.e. α lies in infinitely many tuning windows. If α lies in at least one neutral tuning window $W_r = [\omega, \alpha_0)$, then it must lie in its interior, because ω is not infinitely renormalizable. This means, by Proposition 23.1, that h must be constant on a neighbourhood of α. Otherwise, α lies inside infinitely many nested non-neutral tuning windows W_{r_n}. Since the sequence the denominators of the rational numbers r_n must be unbounded,

the size of W_{r_n} must be arbitrarily small. By Lemma 24.1, in each W_{r_n} there are infinitely many intervals with any monotonicity type and α displays mixed behavior.

$\square$

Note that, as a consequence of the previous proof, α is a phase transition if and only if it is of the form $\alpha = \tau_r(g)$, with $r \in \mathbb{Q}_E$ and $[\![r]\!] \neq 0$, hence the set of phase transitions is countable. Moreover, the set of points of $\mathcal{E}$ which lie in the interior of a neutral tuning window has Hausdorff dimension less than 1/2 by Lemma 23.4.

Finally, the set of parameters for which there is mixed behavior has zero Lebesgue measure because it is a subset of $\mathcal{E}$. On the other hand, it has full Hausdorff dimension because such a set contains $UT \setminus \{g\}$, and by Lemma 23.7 UT has full Hausdorff dimension.

References

[BdC] M. Baillif, A. de Carvalho, *Piecewise linear model for tree maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **11** (2001), no. 12, 3163–3169.

[BCIT] C. Bonanno, C. Carminati, S. Isola, G. Tiozzo, *Dynamics of continued fractions and kneading sequences of unimodal maps*, Discrete Contin. Dyn. Syst. **33** (2013), no. 4, 1313–1332.

[BDV] J. Bourdon, B. Daireaux, B. Vallée, *Dynamical analysis of α-Euclidean algorithms*, J. Algorithms, **44** (2002), no. 1, 246–285.

[Bi] P. Billingsley, *Ergodic theory and information*, John Wiley & Sons Inc., New York, 1965, xiii+195.

[Bo] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414.

[BD] B. Branner, A. Douady, *Surgery on complex polynomials*, Holomorphic dynamics (Mexico, 1986), Lecture Notes in Math. **1345**, 11–72, Springer, Berlin, 1988.

[BS] H. Bruin, D. Schleicher, *Hausdorff dimension of biaccessible angles for quadratic polynomials*, preprint arXiv:1205.2544.

[CMPT] C. Carminati, S. Marmi, A. Profeti, G. Tiozzo, *The entropy of α-continued fractions: numerical results*, Nonlinearity **23** (2010), 2429–2456.

[CT] C. Carminati, G. Tiozzo, *A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations*, Ergodic Theory Dynam. Systems **32** (2012), no. 4, 1249–1269.

[CT2] C. Carminati, G. Tiozzo, *The bifurcation locus for numbers of bounded type*, preprint arXiv:1109.0516.

[CT3] C. Carminati, G. Tiozzo, *Tuning and plateaux for the entropy of α-continued fractions*, Nonlinearity **26** (2013), 1049–1070.

[Ca] A. Cassa, *Dinamiche caotiche e misure invarianti*, Tesi di Laurea, University of Florence, 1995.

[Cas] J. Cassaigne, *Limit values of the recurrence quotient of Sturmian sequences*, Theoret. Comput. Sci. **218** (1999), no. 1, 3–12.

[CFS] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften **245**, Springer-Verlag, New York, 1982.

[dMvS] W. de Melo and S. van Strien, *One-dimensional Dynamics*, Springer-Verlag, Berlin, Heidelberg, 1993.

[Do1] A. Douady, *Algorithms for computing angles in the Mandelbrot set*, in *Chaotic dynamics and fractals (Atlanta, Ga., 1985)*, Notes Rep. Math. Sci. Engrg. **2**, 155–168, Academic Press, Orlando, 1986.

[Do2] A. Douady, *Descriptions of compact sets in* **C**, in *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, 429–465, Publish or Perish, Houston, 1993.

[Do3] A. Douady, *Topological entropy of unimodal maps: monotonicity for quadratic polynomials*, in *Real and complex dynamical systems (Hillerød, 1993)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **464**, 65–87, Kluwer, Dordrecht, 1995.

[DH] A. Douady, J. H. Hubbard, *Étude dynamique des polynômes complexes*, Publications Mathématiques d'Orsay **84**, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.

[DH2] A. Douady, J. H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4), **18**, (1985), no. 2, 287–343.

[Fa] K. Falconer, *Fractal geometry: mathematical foundations and applications*, Wiley, Chichester, 1990.

[Fu] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.

[He] D. Hensley, *A polynomial time algorithm for the Hausdorff dimension of continued fraction Cantor sets*, J. Number Theory, **58** (1996), 1, 9–45.

[Hu] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, in *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, 467–511, Publish or Perish, Houston, 1993.

[IK] M. Iosifescu, C. Kraaikamp *Metrical theory of continued fractions.* Mathematics and its Applications **547**, Kluwer Academic Publishers, Dordrecht, 2002.

[KSS] C. Kraaikamp, T. A. Schmidt, W. Steiner, *Natural extensions and entropy of α-continued fractions*, Nonlinearity **25** (2012), 2207–2243.

[Le] F. Ledrappier, *Some relations between dimension and Lyapounov exponents*, Comm. Math. Phys. **81** (1981), no. 2, 229–238.

[LeYo] F. Ledrappier, L.-S. Young, *The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (1985), no. 3, 540–574.

[LvS] G. Levin, S. van Strien, *Local connectivity of the Julia set of real polynomials*, Ann. of Math. (2) **147** (1998), no. 3, 471–541.

[LM] L. Luzzi, S. Marmi, *On the entropy of Japanese continued fractions*, Discrete Contin. Dyn. Syst. **20** (2008), 673–711.

[Ly] M. Yu. Lyubich, *Entropy of analytic endomorphisms of the Riemann sphere*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 4, 83–84, translated in Functional Anal. Appl. **15** (1981), no. 4, 300302.

[Ly2] M. Yu. Lyubich, *Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture*, Ann. of Math. (2) **149** (1999), no. 2, 319–420.

[Ma] A. Manning, *A relation between Lyapunov exponents, Hausdorff dimension and entropy*, Ergodic Theory Dynam. Systems **1** (1981), no. 4, 451–459.

[MMY] S. Marmi, P. Moussa, J.-C. Yoccoz, *The Brjuno functions and their regularity properties*, Comm. Math. Phys. **186** (1997), no. 2, 265–293.

[Mc] C. T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies **135**, Princeton University Press, Princeton, NJ, 1994.

[Mi1] J. Milnor, *Periodic orbits, external rays and the Mandelbrot set: an expository account*, Astérisque **261** (2000), 277-333.

[Mi2] J. Milnor, *Local connectivity of Julia sets: expository lectures*, in *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser. **274**, 67–116, Cambridge Univ. Press, Cambridge, 2000.

[MT] J. Milnor, W. Thurston, *On iterated maps of the interval*, Dynamical systems (College Park, MD, 1986–87), Lecture Notes in Math., 1342, 465–563, Springer, Berlin, 1988.

[MS] M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), no. 1, 45–63.

[MCM] P. Moussa, A. Cassa, S. Marmi, *Continued fractions and Brjuno functions*, Continued fractions and geometric function theory (CONFUN) (Trondheim, 1997), J. Comput. Appl. Math. **105** (1999), 403–415.

[Na] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426.

[NN] H. Nakada, R. Natsui, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity **21** (2008), 1207–1225.

[Po] C. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften **299**, Springer-Verlag, Berlin, 1992.

[Ri] J. Riedl, *Arcs in Multibrot Sets, Locally Connected Julia Sets and Their Construction by Quasiconformal Surgery*, PhD Thesis, TU München, 2001.

[Sa] R. Salem, *On some singular monotone functions which are strictly increasing*, TAMS **53** (1943), 427–439.

[Sch] D. Schleicher, *On Fibers and Renormalization of Julia Sets and Multibrot Sets*, Stony Brook IMS preprint 1998/13b, available on arXiv:math/9902156 [math.DS].

[Sc] F. Schweiger, *Ergodic theory of fibred systems and metric number theory*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1995.

[Sm] S. Smirnov, *On supports of dynamical laminations and biaccessible points in polynomial Julia sets*, Colloq. Math. **87** (2001), no. 2, 287–295.

[Th1] W. Thurston, *On the Geometry and Dynamics of Iterated Rational Maps*, in D. Schleicher, N. Selinger, editors, *Complex dynamics: families and friends* , 3–137, A K Peters, Wellesley, MA, 2009.

[Th2] W. Thurston, *Entropy in dimension one*, preprint 2011.

[TanL] Tan Lei, *Similarity between the Mandelbrot set and the Julia sets*, Commun. Math. Phys. **134** (1990), 587–617.

[TaoL] Tao Li, *A monotonicity conjecture for the entropy of Hubbard trees*, PhD Thesis, SUNY Stony Brook, 2007.

[Ti] G. Tiozzo, *The entropy of α-continued fractions: analytical results*, to appear in Ann. Sc. Norm. Pisa Cl. Sci. (2014).

[Wi] S. Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

[Yo] L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynam. Systems **2** (1982), no. 1, 109–124.

[Za1] S. Zakeri, *Biaccessibility in quadratic Julia sets*, Ergodic Theory Dynam. Systems. **20** (2000), no. 6, 1859–1883.

[Za2] S. Zakeri, *External Rays and the Real Slice of the Mandelbrot Set*, Ergodic Theory Dynam. Systems **23** (2003), 637–660.

[Za3] S. Zakeri, *On biaccessible points of the Mandelbrot set*, Proc. Amer. Math. Soc. **134** (2006), no. 8, 2239–2250.

[Zd] A. Zdunik, *On biaccessible points in Julia sets of polynomials*, Fund. Math. **163** (2000), no. 3, 277–286.