Papers on and Around the Access Problem

Three Papers on and around the Access Problem

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SHARON BERRY
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Thesis advisor: Edward Hall Sharon Berry

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Abstract

The three papers which make up this dissertation form part of a larger project, which aims to solve the 'access problem' for realism about mathematics by providing a clear and plausible example of what a satisfying explanation of human accuracy about objective mathematical facts could look like. They fit into this project as follows.

The first paper argues that one cannot explain our use of good mathematical axioms and inference rules merely by saying that any syntactically consistent (mathematical) proof procedures we had accepted would have given meaning to our mathematical vocabulary in such a way as to ensure their own correctness.

The second paper outlines my core two-part strategy for solving the access problem: 1) reduce the problem of accounting for human access to mathematical facts to the problem of accounting for access to combinatorial possibility 2) use the fact that human beings face pressure to correctly predict and explain the behaviour of concrete objects to explain our possession of good methods of reasoning about combinatorial possibility. It largely focuses on part 2) of this story.

The third paper develops part 1) of the two-part story above. It motivates and helps defend a (loosely) neo-carnapian explanation for mathematicians' freedom to introduce new kinds of objects, like the complex numbers and the sets. If this neo-carnapian explanation is correct, then there's an easy route from the kind of accuracy about combinatorial possibility discussed in the second paper to accuracy about more familiar mathematical topics.

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For my parents Edward Berry and Lishar Huang Berry.

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1

Malament-Hogarth Machines and Tait's Axiomatic Conception of Mathematics

1.1 Introduction

In his 2001 paper, *Beyond the Axioms: the question of objectivity in mathematics*[28], William Tait advances an axiomatic conception of mathematics on which provability constitutes the sole criterion for mathematical truth. According to this view, proof is ultimately the only source of epistemic justification for mathematical beliefs. As Tait puts it, "the assertion of a mathematical proposition is warranted only by a proof of it"[28].

In this paper I will argue that Tait's axiomatic conception of mathematics implies that it is not only practically, but also metaphysically, impossible to be justified in believing a mathematical statement without being justified in believing that statement is provable. I will then show that there are possible courses of experience which would justify acceptance of a mathematical statement without justifying belief that this statement is provable.

If it succeeds, my argument does two things. First, it provides a direct counterexample to Tait's claims about the relationship between proof and justification in mathematics. There turn out to be kinds of evidence which suffice to justify mathematical beliefs, but which would not provide any such justification if Tait's view were correct. Second, it establishes an independently interesting conclusion: nothing in the nature of mathematics or justification precludes our empirically discovering the right answer to questions which are independent of our axioms.

1.2 Tait's Axiomatic Approach to Mathematics

1.2.1 The Axiomatic Approach

At the beginning of *Beyond the Axioms*[28] Tait criticizes forms of realism which make mathematics 'speculative' in the sense that, "even the most elementary computations, deductions and propositions must answer to a reality which we, at best, can only partially comprehend and about which we could be wrong."[28] Instead, he proposes an axiomatic approach to mathematics, on which any consistent mathematical practice we adopt would give meaning to our mathematical expressions in such a way as to ensure the truth whatever statements the practice instructs us to accept.

This account claims to be a form of realism, in the sense that it takes mathematical statements to be literally true and to stand in no need of paraphrase. However, it denies that our choice of mathematical axioms is answerable to an independent reality in the sense indicated by the quote above. As a result, foundational worries about whether our most fundamental axioms might be false are necessarily unfounded. We can acquire particular false mathematical beliefs if, for example, a slip of the pencil leads us to falsely believe that a statement is derivable from our axioms, but it would be impossible for statements which we accept as axioms to themselves be false. The only way our axioms can fail to express truths is if they are inconsistent. Therefore, the traditional realist's problem of accounting for human knowledge of mathematics reduces to a problem of explaining our ability to choose consistent axioms.

1.2.2 Contrast With More Conventional Approaches

To appreciate how radical Tait's proposal that all consistent mathematical practices would express truths is, it will be important to distinguish the notion of consistency which Tait invokes from a more demanding notion, coherence, which is relevant to contemporary discussions of structuralism, plenitudinous platonism and other similar philosophical interpretations of mathematics. To this end it is important to clarify what Tait means by proof and mathematical practice.

In Tait's sense, a mathematical practice is a norm telling mathematicians to accept certain premises (the axioms) and derive further conclusions from them. Note that in this sense our mathematical practice reflects our full judgement of what proofs establish their conclusions, not merely what can be proved from arbitrary axioms. While some might be inclined to call various infinitary objects proofs, Tait's usage makes it clear that not only must proofs be finitary but also algorithmically verifiable. Indeed, Tait assumes Gödel's incompleteness theorem applies to our mathematical practice[1] so we can infer our mathematical practice is algorithmically describable and doesn't include any non-computable notion of consequence or specification of axioms that might evade the assumptions of Gödel's theorem.

With this in mind, I will say that a mathematical practice is *syntactically consistent* if one cannot derive both a statement and it's negation. I will also say that a mathematical practice is *coherent* if it is (mathematically) possible to

[1] "But, of course, completeness fails and must fail. Nor is the essential incompleteness due simply to Gödel's incompleteness theorem."[28].

simultaneously satisfy its axioms[2]. When the allowed inferences are clear from context I will speak of theories being syntactically consistent or coherent. Note that the axioms under consideration here need not be first-order, which, as we will see, forces these two notions to come apart.

Many philosophers with structuralist, plenitudinous platonist or fictionalist leanings are attracted to the idea that only (and all[3]) coherent theories are legitimate topics for mathematical investigation. Accordingly, they accept that all coherent mathematical practices would lead mathematicians to express only truths if they were adopted. However, they do not accept Tait's stronger claim that all syntactically consistent practices would lead mathematicians to express only truths, because they think that syntactically consistent theories need not be coherent. Specifically, consistency and coherence come apart in cases where our mathematical axioms employ powerful vocabulary (such as the language of second order logic) which can constrain the behavior mathematical structures in ways that go beyond our mathematical practice. Accordingly, there can be syntactically consistent mathematical practices which nonetheless fail to be coherent.

These considerations are more than hypothetical as most most mathematicians (as well as philosophers of mathematics) appear to have non-first-order beliefs about mathematical subjects (number theory, analysis, set theory etc.) which they take to constrain the meaning of their terms. For instance, they believe the numbers are as small as possible while satisfying certain basic principles of arithmetic. However, no consistent collection of first-order axioms can fully express this idea. Any first-order theory that describes the

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[2]For instance, the existence of a model demonstrates a theory (collection of axioms) is consistent but, when working in strong logics, not all coherent collections of axioms will have a set model, e.g., second order set theory describes the full universe of sets and can't be captured by any set model inside that universe.

[3]In the case of the plenitudinous platonist I am speaking loosely with regard to the claim that all coherent theories are acceptable topics for investigation. For appeal to quantifier restriction, broad limits on abstraction or something more is needed to deal with Boolos' point that not all internally coherent descriptions of mathematical objects are compatible with one another.

natural numbers will also be satisfied by some non-standard model including infinite 'numbers.' Thus, it would seem, that our real axioms for the numbers go beyond what is first-order expressible, in ruling out these spurious infinitary 'numbers.' Indeed, as we will see in the section below, there are very general reasons for thinking that no mathematical practice can capture all the facts that necessitated by such powerful descriptions.

Accordingly there appears to be a definite sense in which syntactically consistent extensions of our mathematical practice can nonetheless fail to be coherent by making claims which conflict with what these rich descriptions of the numbers require. Tait rejects this line of reasoning by denying that we have any such practice-transcendent grip on the intended structure of the numbers. He denies that there can be any "grounds upon which a proposition, undecided by our present axioms, is nonetheless really true or really false."[28] Thus, he denies that claims like, 'the numbers are as small as possible while satisfying ...' can take on a richer meaning which requires the truth (or falsehood) of statements that our mathematical practice doesn't let us prove or refute.

Motivated by roughly Wittgenstinean concerns about rule following, he criticizes various proposals about what our understanding of these richer requirements could consist of. He points out that, for example, any definition we give of the intended structure of the natural numbers will itself use terms that need to be antecedently understood, that any mental picture we associate with these words will need interpretation and that facts about the neural mechanisms which underly our actual use of mathematical terms cannot ground a distinction between correct use and psychologically natural and ingrained by incorrect use. Instead, he concludes that that our mathematical beliefs can't express any practice-transcendent grasp of the intended structure of mathematical objects.

This skepticism puts Tait in a position to accept the shocking claim that any syntactically consistent theory, including any syntactically consistent extension of the axioms which we allow to figure in our reasoning about the numbers, could express a truth if we chose to extend our mathematical practice in the appropriate way. In cases where a claim about the structure of the numbers is not provable or refutable from axioms, he holds that we are *free to stipulate either answer*. So, for example, Tait must accept that we are free to stipulate the truth or falsity of any number theoretic statements which are genuinely independent of our mathematical practice. Indeed, in the case of the Continuum Hypothesis (a set theoretic statement which can be shown to be independent of the generally accepted ZFC axioms of set theory) he explicitly says, "Until we determine it, CH [the contiuum hypothesis] is ... indeterminate"[28] and there may be equally good "directions in which our conception of set could develop"[28] which would require us to adopt axioms which imply either the continuum hypothesis or its negation.

1.2.3 Independence and the Axiomatic Conception of Mathematics

One might think that any theory which takes mathematical practices to play as direct a role in determining mathematical truth as Tait's does would face problems with Gödelian incompleteness. However, Tait, while accepting that the premises of the Incompleteness theorem apply to the system of axioms and inference rules which constitute human proof practices, evades the obvious problems.

For example, allowing that some mathematical statements will be neither provable nor refutable from axioms we are inclined to accept does not prevent Tait from making sense of mathematicians' use of the law of the excluded middle. Insofar as all statements of the form $\varphi \vee \neg\varphi$ are provable in classical logic, and classical logic forms part of accepted mathematical proof procedures, these statements will come out to be true on Tait's account - even in cases where neither φ nor $\neg\varphi$ is provable. Furthermore, Tait suggests that in taking our mathematical practice to ensure the truth of $\varphi \vee \neg\varphi$ we should our take it to ensure the truth of statements of the form 'φ is true or φ is false' as well. Thus, Tait accepts all the same claims as the standard Platonist about principles derived from the law of the excluded middle.

The difference between Tait's view and more conventional views only emerges when we consider the possibility of introducing new mathematical axioms. Platonists, structuralists and some fictionalists will rely on the categoricity of our non-first-order description of the numbers[4] to conclude that our current understanding of the intended structure of the numbers already suffices to determine a definite answer to the question of whether any number theoretic mathematical claim is true, whether we know it or not. As a result, we are not free to sharpen our mathematical notions by stipulating an answer to this question.

In contrast, as we saw above, Tait holds that there are no grounds on which a statement which is independent of our axioms could be right rather than wrong. He denies that we have any proof-transcendent grasp of the intended structure of the numbers which determines a right answer to statements independent of our axioms. As a result we are free to stipulate the truth value of independent mathematical statements at will. Thus, although Tait and the more mainstream philosophers above will both assert that exactly one of φ and $\neg\varphi$ are correct, Tait will go further and say that we we are free to stipulate which of these statements is correct.

1.2.4 Warrant

Finally, Tait's axiomatic understanding of mathematics leads him to advance an epistemic constraint on mathematical justification which will be the main focus of this paper. He claims that, "proof is the sole criterion for for mathematical truth"[28] in the sense that, "the assertion of a mathematical proposition is warranted only by a proof of it"[28]. I think this claim requires a little unpacking. One might object that we routinely assert mathematical claims without having access to a proof, for example, in response to looking at a calculator [20] or listening to an expert. However, both of these examples can be easily accommodated by charitably interpreting Tait to say that proof is the sole *source* of mathematical justification, in the sense that only proof or reason to think a claim is provable can justify a mathematical assertion. When we learn a

[4] That is, on the fact that these descriptions uniquely determine a mathematical structure.

mathematical claim by talking to someone who has examined a proof, we gain evidence that the relevant claim is provable. Similarly, when we check a mathematical claim via calculator we gain evidence that the claim is provable.

Once understood as above, the idea that "proof is the sole criterion for mathematical truth"[28] gains strong motivation from Tait's larger claims about axiom choice, in the following way. Suppose there were (contra the hypothesis above) some independent source of mathematical knowledge, which allowed us to learn mathematical propositions without any appeal to the possibility of proving these statements. Any such independent source of mathematical knowledge would have to involve awareness of some "grounds" besides provability from the axioms, upon which a mathematical proposition could be true rather than false. But such grounds would seem to provide "grounds upon which a proposition, undecided by our present axioms, is nonetheless really true or really false."[28] And this in turn, would seem to provide a sense in which the addition of consistent axioms could be wrong. In particular, if some proof-independent method of mathematical learning could teach us that an independent sentence φ was true, this would seem to provide a sense in which adding the negation of φ to our axioms could be consistent but wrong.

Now, the above argument that proof is the only ultimate source of justification for mathematical claims is a purely a priori one, and makes no appeal to contingent facts. Thus if it succeeds, it establishes not only the absence but the *metaphysical impossibility* of alternative sources of mathematical knowledge. In the remainder of this paper I will argue that certain conceivable courses of experience would justify belief in a mathematical statement without justifying belief that this statement was provable.

1.3 Malament-Hogarth Machines and Independent Sentences

My argument begins with the consideration of Π^0_1 sentences, that is, simple statements in the language of arithmetic which are writable in the form $(\forall n)F(n)$

where $F(n)$ contains only bounded quantifiers[5].

There are Gödelian reasons for thinking that some Π_1^0 sentences will be independent of our mathematical practice[25]. The incompleteness theorem applies to any collection of mathematical statements, such as those which could be derived using a particular mathematical practice, which is syntactically consistent, algorithmically enumerable, and sufficiently powerful to capture certain basic facts of number theory. It tells us that, any such collection will fail to include both some Π_1^0 sentence and it's negation. In particular, our mathematical practice will fail to specify the truth or falsity of some Π_1^0 sentence[6].

I will argue that experience with certain physical hyper-computers could justify belief in the truth of such independent sentences. David Malament and Mark Hogarth have pointed out that certain solutions to the equations of general relativity would allow a person (the operator) and a computer to take different paths through space-time in such a way that the following strange thing happens: no matter how long it takes for the computer to signal its result, the operator will receive that signal within a bounded amount of time, e.g., the operator would receive the result within a single day no matter how many steps occur in the computation[16].

A person exploiting this set up would be able to 'compute' things that a Turing machine cannot. Assuming limitations on memory, power and reliability can be overcome, such a person could seek evidence for the truth of an arbitrary Π_1^0 sentence φ by programming the computer to check all of φ's instances. For example, if they were interested in in the Goldbach conjecture, they would first program a computer to check that 4 is the sum of two primes, 6 is the some of

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[5]Thus, for example, the Goldbach conjecture states that every even number greater than 2 is writeable as the sum of two primes. This qualifies as a Π_1^0 sentence because it requires that $\forall n$ $n = 2$ or n is odd or $\exists x \leq n \exists y \leq n$ and x is prime and y is prime and $x + y = n$, where the property of being prime is itself expressible using only bound quantifiers.

[6]We can assume that the practice in question allows one to reject any sentence which, if added to the list of allowable assumptions, would generate a contradiction. Thus, we can choose to add some Π_1^0 sentence or it's negation as an assumption while keeping our practice syntactically consistent.

two primes and so on, signaling back if it ever finds an even number that isn't the sum of two primes. Then they would then launch this computer on a path such that any signal sent by the computer would reach them within a day. If they do not receive a signal within this day (and if there is no stage at which the computer fails to transition as required by its program) then we can infer the Goldbach conjecture is true. Call the whole system consisting of the computer, means to launch the computer through a suitable region of space-time and the signaling mechanism a Malament-Hogarth (or MH) machine. MH machines can be made to test every Π^0_1 sentence using the strategy just described[7].

Discussion of Malament-Hogarth machines in the literature has centered on questions about the physical possibility of the space-time structure needed for an MH machine as well as whether MH machines would count as computers. For my purposes, however, all that will matter is whether it would be metaphysically possible for some course of experience to justify the belief that one was dealing with such a machine. I will argue that, contra Tait's claim, experiences as of dealing with an MH machine can provide a source of (epistemic) justification for asserting independent Π^0_1 sentences which does not depend on reason to think

[7]The notion of an MH machine which I am invoking here may be slightly unusual in the following sense. There are two different ways of precisifying the idea that a computer can run forever and hence check every instance of a true Π^0_1 statement. On the one hand, one can require merely that the MH machine computer goes through the successor of each stage that it goes through (unless and until its program tells it to stop). On the other hand one can require that the computer goes through exactly ω many stages and not more. Probably the standard notion in the literature requires an MH machine to go through exactly ω stages, so that it reports back if **and only if** some Π^0_1 sentence is true. However, since one thing that's at issue here is whether we can think thoughts which distinguish a unique structure ω from various 'nonstandard models'. Accordingly, it may appear question-begging to assume that we can understand, much less be justified in believing, the claim that some physical system is an MH machine in any sense which requires it to go through *exactly* ω stages. Fortunately for us, the existence of a machine that goes through *at least* ω stages and doesn't find a counterexample suffices to justify the corresponding Π^0_1 sentence. After all, such a machine still fails to find any genuinely finite counterexample. Thus, I will define MH machine in the weaker sense indicated above, and note that the metaphysical possibility of some MH machine that justifies belief in an independent Π^0_1 sentences is sufficient for my purposes.

that these claims are provable[8].

1.4 Believing in MH machines

It seems fairly clear that experiences which justified the belief that one was dealing with a genuine MH machine could thereby suffice to justify believing the mathematical result indicated by the that machine. After all dealings with a calculator can justify mathematical beliefs, and a MH machine merely performs infinitely many such computations. All that remains to show is that some experience could justify the belief that one was dealing with an MH machine. To this end, let us consider, in detail, what is required to justify such a belief.

First, one must believe that one's universe has the right kind of space-time structure. Could any experience justify this belief? I take the existence of actual evidence-heavy debates about whether the laws of physics are compatible with the needed space-time features to suggest that it can [7, 9]. Moreover, the history

[8]Another route to my desired conclusion begins with the idea that the mere existence of a tradition of working with a given system of axioms and proof procedures and failing to find a contradiction can gives us reason to believe in the consistency of that proof system. This reason to believe that our proof practice as a whole is consistent gives us reason to believe in the truth of a sentence which is not derivable in that system, namely the arithmetical consistency sentence for that system. Personally, I am not at all unfriendly to this route to establishing the conclusion that there can be unprovable truths.

However, there is a significant line of worry in the literature about whether merely using a system without encountering a contradiction can give us reason to believe that that system is consistent, or whether the reason it gives us can be sufficient to let us qualify as having knowledge. Following Frege, some philosophers have argued the numbers differ from one another so radically that "in the absence of proof, we should not expect numbers (in general) to share any interesting properties."[11] and hence that dealings with any number of finite cases where some number has failed to code a proof of $0 = 1$ in ZFC can never provide us with any justification at all for the belief that some (untried) number fails to code a proof of contradiction in ZFC. Less radically, it is sometimes argued that dealings with particular cases always provides us with a biased sample - with knowledge of what holds for small numbers and short proofs, and that such knowledge provides no basis for justified generalization to the claim that all numbers have a certain property or that no larger proof is possible.[22]

Accordingly I propose to bypass these issues by providing a counterexample to Tait's thesis which has no dependence on the claim that a history of safe use of a mathematical theory can provide justification sufficient for knowledge of the claim that that theory is consistent.

of physics gives a straightforward picture of the kind of evidence which would suffice to justify such a belief.

Second, one must believe that a computer (or machine if you prefer) can be constructed with access to sufficient memory and power. Turing machines, as mathematical abstracta, are allowed to use an unbounded amount of memory and operate for indefinitely many stages. However, the ordinary computers that we build only have access to a finite amount of memory and power. These facts present a problem since an unbounded amount of memory and indefinitely many operations are needed to check arbitrary Π^0_1 sentences.

A traditional answer to worries about energy in discussions of the physical possibility of (memory-limited versions of) MH machines draws on the fact that the computer doing the computations travels infinitely far: one could build the computer to harvest energy as it travels[4]. In a universe like that theorized by some early twentieth century astronomers, matter appears spontaneously in empty space at a certain rate. This would provide a guaranteed source of energy which the machine could harvest on its journey. Similarly, the computer could draw on this energy (converting to matter as needed) to construct further memory cells as necessary.

Alternately, in a 'gunky' universe which allows for complexity at an arbitrarily small scale there is a more elegant solution to concerns about power and memory. With the same strategy used by the electronics industry (shrink components to pack more functionality into a smaller, and hence more energy efficient, package) the computer could continually replace itself by a more efficient copy using less energy but with more memory. Provided one increases efficiency at a sufficient rate the total energy needed would be finite.

Third, one needs to believe that the traveling computer one has constructed is sufficiently accurate to perform as designed throughout its journey. Suppose that the computer being launched has some constant (independent) probability $\varepsilon > 0$ of making an error at every given stage in the computation. Then probability that the computer makes it through n stages without failure is $(1 - \varepsilon)^n$. Thus, the probability of completely correct performance goes to 0 as n

goes to infinity. So it would seem that the probability that a Malament-Hogarth machine has worked as intended when verifying a Π^0_1 sentence should be 0.

However, we can apply the same strategy we used above to make our MH machine as reliable as desired. As well as increasing in capability over time we engineer our computer to improve its reliability as well. Well known techniques in circuit design redundancy and error correction can be used to arbitrarily reduce the probability of an error. By increasing reliability sufficiently quickly we can make the overall chance of machine malfunction arbitrarily small. In this way, dealings with an MH machine can produce degrees of justification which come arbitrarily close to one's justification for believing one's overall physical theory is correct.

Alternately, if you don't like the idea of engineering an indefinitely self-improving computer as above, one can also circumvent worries about energy and error-rates by appeal to possible physical laws which directly constrain the behavior of physical objects that can be used to build a computer. One can imagine an MH machine computer whose basic components were fundamental particles whose behavior was completely determined by fundamental physical laws. In this way there seem to be conceivable and elegant systems of fundamental physical laws which would imply the perfect functioning of various simple building blocks for an MH machine, and hence the perfect functioning of the MH machine as a whole. As no truly bizarre physical laws are required for this scenario, we have every reason to believe that some course of experience would justify concluding that one had built an MH machine in such a manner.

Of course, neither of the accounts above eliminate the possibility of systemic error, e.g, the epistemic possibility that the general physical theory which you used to calculate error rates when designing the MH machine could be incorrect. I have provided some reason for thinking that one's justification for believing one had launched a genuine MH machine could approach one's justification for accepting our most certain physical theories. But, if mathematical knowledge required a substantially different and higher standard of justification than physical knowledge, it might seem that this degree of justification could never suffice to underwrite mathematical knowledge. As a result one might worry that dealings with an MH machine could never provide *sufficient* justification to ground mathematical knowledge.

Note, however, that we cannot say that mathematical knowledge requires certainty on pain of ruling out mathematical knowledge by testimony. And it would seem that beliefs about the physical structure of space and the components in an MH machine can acquire justification on par with one's justification for believing a credible mathematical witness. Thus, it would seem that experiences can provide us with at least the degree of justification which suffices to grant us mathematical knowledge in cases where we learn we new mathematical truths by accepting credible mathematical testimony.

In light of these considerations, I conclude that a suitable course of experience could justify someone in believing that they had built a working MH machine and thereby in believing any Π^0_1 sentence which this machine appeared to verify.

1.5 Responses

Now let us return to Tait. I have argued above that dealings with a Malament-Hogarth machine could justify asserting Π^0_1 sentences, even when those sentences are unprovable. This is in direct conflict with the central tenants of Tait's account of mathematics which, as we saw, require that (evidence for) proof be the only possible source of justification for asserting a mathematical claim.

I will conclude by considering some responses to the line of argument above.

1.5.1 Epistemic vs. Pragmatic Justification

First, defenders of Tait might resist my claim that experiences as of dealing with MH machines can provide *epistemic* reason for accepting Π^0_1 sentences. Perhaps such experiences don't show that the relevant Π^0_1 statement currently expresses a truth, but only provide pragmatic reasons to study 'systems of number theory' where this sentence is accepted as an axiom.

Although immediately unintuitive, this way of understanding the relationship between observations as of MH machines and number theoretic facts can be motivated somewhat by considering an analogy to geometry. The great success of Euclidean geometry in describing space[9] made it practically useful and convenient for the Greeks to study mathematical systems which included the parallel postulate. It is plausible that adopting (or rejecting) the parallel postulate for reasons like these does not involve learning that any antecedently understood mathematical statement expresses a truth. Instead, it involves pragmatically choosing to study a given mathematical system, because facts about this system appear to have a certain desirable relationship to facts about the external world. At first glance the role which I have argued that MH machines can play in justifying number-theoretic beliefs can seem similar to the role of physical applications in motivating the choice of axioms for geometry.

However, I think this kind of defense is ultimately quite difficult to maintain. Accepting it would require us to reject a certain aspect of mainstream mathematical practice (or at least, mainstream mathematical belief-revision dispositions) as irrational. If experience just makes it rational to study systems in which a certain axiom is true, we ought not to conclude that a mathematical statement is false in response to failures of these applications, but only (at most) that other mathematical systems deserve attention as well. And in the case of geometry this is exactly what happened. When experience motivated the study of non-euclidean geometry we did not say euclidean geometry was wrong but only that other kinds of geometry were worth studying as well. In contrast, in the case of arithmetic our dispositions to revise beliefs are quite different. Learning that an MH machine 'verified' a Π^0_1 sentence would make people say that they were wrong to ever believe its negation. Thus, in contrast with the geometrical case such an experience would lead us to dismiss our previous beliefs as wrong about the numbers rather than right about some other system.

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[9]At least near earth for the kind of low-tech uses that can ignore relativistic effects

1.5.2 Exception for Π^0_1 Sentences?

Second, some readers may feel that the counterexample presented in this paper is not very deep, because it depends on exploiting special features of the simplest possible kind of independent statements - independent Π^0_1 sentences. Thus, one might think that although I have presented a genuine counterexample to Tait's view as stated, my objection can easily be handled by a simple modification of Tait's view which takes Π^0_1 sentences to be a special exception to his general claim that there can be no "grounds upon which a proposition, undecided by our present axioms, is nonetheless really true or really false." I will now argue that no such quick fix solution can be given, without giving up central parts of Tait's view.

Π^0_1 sentences are special in the following sense: if a Π^0_1 sentence is consistent then it is true. Because Π^0_1 sentences have the form $\forall x \varphi(x)$ where $\varphi(x)$ is quantifier free, if such a sentence is false then there is some particular number n such which constitutes a counterexample. Since φ lacks any unbounded quantifiers, this latter statement will be provable via basic arithmetic rules. As a result, uncontroversial set theoretic reasoning about the numbers allows us to prove that any Π^0_1 sentence of arithmetic which is independent from our overall theory (or even just the part of it summarized by the Peano Axioms) must be true[18].

More traditionally realist readers (like myself) will be inclined to think that these considerations point out a clear ground upon which a mathematical sentence which is independent of our axioms can be right rather than wrong. Objective facts about derivability in formal systems like PA combine with our expectations about the relationships between arithmetical sentences and derivability to ensure that independent Π^0_1 sentences are "really true" despite our inability to prove this fact.

However, I do not think that Tait could accept the above argument. The argument above crucially turns on taking our beliefs about the numbers to latch on to objective proof-transcendent facts about derivability in formal systems, and make the truth or falsehood of undecidable sentences reflect these objective facts.

Allowing this immediately threatens Tait's core motivating idea that our choice of axioms in mathematics is not a matter of speculating about some independent partly understood subject matter.

If one allows that there are such objective and determinate proof-transcendent facts about derivability in formal systems, and that the meanings of our words can latch on to these facts then these facts about provability would seem to constitute a legitimate subject matter for investigation. Which sentences are provable from which formal systems? Thus, there would seem to be a portion of mathematics at least (the study of derivability in formal systems) where "even the most elementary computations, deductions and propositions" *are* answerable to "a reality which we, at best, can only partially comprehend and about which we could be wrong." Thus, I think Tait must deny that our talk about consistency and derivability latches on to any such proof transcendent facts about derivability[10].

Furthermore, it is not clear that Tait can accept the above account of provability for Π°_1 sentences while maintaining that we are free to stipulate right answers to *any* quantified sentences in the language of arithmetic. If considerations of consistency are sufficient to provide a sense in which all Π°_1 sentences are "really" right or wrong (even in cases where they are not derivable or refutable) then similar considerations allow us to ground the truth-value of all arithmetic sentences. Thus, in attempting to make a special exception for Π°_1 sentences we end up extinguishing any role for stipulation in arithmetic and

[10]Admittedly, adopting this line of response raises serious problems of its own. For example, what sense are we to make of Tait's own talk of consistency when saying that, e.g., inconsistency debars an axiom system from giving meaning to our mathematical claims? If claims about consistency are only determined to have a particular truth value by being derived in some axiom system, what axiom system is relevant to Tait's claim? If the relevant axiom system is the total collection of mathematical claims we are inclined to accept, there's a prima facie problem. This system (presumably) cannot prove its own consistency[12]. If there is not a finite demonstrable inconsistency in our axioms, then the question of whether the total collection of axioms that we are inclined to accept determine a consistent and hence true mathematical system, or an inconsistent (and hence meaningless) one will turn out to have the same status as independent Π°_1 sentences. This seems like an odd consequence. It also seems odd that it in stipulating facts about arithmetic we could thereby determine facts about what alternative choices of axioms would have been meaningful.

instead force it to answer to an independent reality.

The key point to note is that there is a direct relationship between whether a sentence with $n+1$ quantifier alternations is true and the facts about what true (or false) statements with n quantifier alternations can be proved from it. In particular, a sentence that begins with a universal quantifier and includes $n+1$ quantifier alternations followed by some formula containing only bounded quantifiers (called Π^0_{n+1} sentence) is true if and only if it adding it as an axiom to Peano Arithmetic does not allow one to prove some false Σ^0_n sentence, (i.e., some statement that begins with an existential quantifier and includes n quantifier alternations followed by a formula with only bounded quantifiers). To see why this is so, consider the example of arbitrary Π^0_2 sentence $\forall x \exists y \psi(x, y)$ where ψ is a formula in the language of arithmetic with only bounded quantifiers. This sentence is false iff $\exists x \forall y \neg\psi(x, y)$ is true, i.e, if there is some number n such that $\forall y \neg\psi(\overbrace{S \circ S \cdots \circ S}^{n}(0), y)$ is true. But, from $\forall x \exists y(x, y)$ one can derive (over Peano arithmetic) the negation of each such instance. Thus, an arbitrary Π^0_2 sentence is false if and only if adding it as an axiom allows you to derive some false Σ^0_1 sentence.

As a result, if we are only free to stipulate new mathematical axioms in a way that honors our current expectations about the relationship between provability and truth in arithmetic, then if our hand is forced with regard to Σ^0_n sentences it will be forced with regard to Π^0_{n+1} sentences as well. But, noting this fact allows us to inductively show that our lack of any freedom to choose truth values for bounded sentences permeates all the way up, and we are never free to choose how to settle the truth value of any quantified sentences in the language of arithmetic. We can inductively fix mandatory truth values for all such sentences as follows:

- $\Pi^0_0 = \Sigma^0_0$ sentences are true iff they are provable in PA.
- Σ^0_{n+1} sentences are true iff the Π^0_{n+1} sentences which form their negations are false
- Π^0_{n+1} sentences are true iff adding them to Peano Arithmetic as an axiom

does not allow you to derive some false instance, i.e., a false Σ^0_n sentence.

Finally, even if Tait could somehow motivate making a special exception for Π^0_1 sentences but not more complex sentences of arithmetic, he would still run into a version of the problem presented by MH machines. Hogarth has demonstrated that variants of the MH machine can be constructed to check arbitrary sentences in the language of number theory [16]. The key idea is to consider a Malament-Hogarth machine which spawns other MH machines. Thus, to check the truth of a $\forall x \exists y \varphi(x, y)$ sentence one builds a computer which checks whether $\exists y \varphi(1, y)$, by building and launching a standard MH machine computer which is set to look for a y such that $\exists y \varphi(1, y)$, and radio back to the main computer if it ever finds one. If the main computer doesn't receive a signal within the relevant interval, it decides that $\neg \exists y \varphi(1, y)$ so it has found a counterexample, and it radios back that $\forall x \exists y \varphi(x, y)$ must be false. If it does receive a signal from the child computer it decides that $\exists y \varphi(1, y)$ is true and proceeds to check whether $\exists y \varphi(2, y)$ and all other instances in the same way, using the property of MH machines to accomplish all this checking in bounded time for the operator. Thus, if we do not hear back from the master computer within the relevant interval we can conclude that the relevant $\forall x \exists y \varphi(x, y)$ sentence is true. A similar extension of the machinery allows one to check the truth of all arithmetic sentences. Thus, it would seem that the possibility of getting (partly empirical) justification for believing arithmetical statements which are independent of all axioms we are inclined to accept is not limited to Π^0_1 sentences.

1.6 Conclusion

In this paper I have argued that experiences as of creating a Malament-Hogarth machine could provide epistemic justification for believing independent mathematical sentences, justification which does not appeal to any reason to think these statements are provable from axioms we are disposed to accept.

If this is correct it constitutes a counterexample to Tait's claim that proof is the only possible source of warrant for asserting a mathematical claim. And insofar as we have seen that central features of Tait's axiomatic understanding of mathematics lead him to the conclusion that only proof can justify mathematical claims, reason to deny this epistemic claim casts doubt on this theory as a whole.

Our discussion of MH machines also provides reasons for doubting projectivist and pragmatist approaches to truth in arithmetic. We have seen in the pages above that we expect the right answers to questions about the numbers to be reflected by certain constraints on how it is metaphysically possible for infinite physical systems like MH machines to behave. Insofar as this is the case, we are not free to pragmatically stipulate right answers. This result is interesting in itself and has implications for many philosophical interpretations of mathematics, though none so dramatic as the implications it has for Tait.

2

Mathematical Knowledge and Combinatorial Possibility: A New Strategy for Solving the Access Problem

2.1 Introduction

Claims like 'there are numbers between 5 and 10' seem to assert the existence of abstract mathematical objects like numbers and sets. But, if we suppose that mathematics is really about such objects, it can seem like a mystery that human beings have such accurate mathematical beliefs. Given that we cannot directly perceive or have causal contact with mathematical objects like the numbers or the sets, why should there be any match at all between facts about these objects and our beliefs? In the absence of any suitable explanatory mechanism, the idea that we have systematically true beliefs about mathematical objects can seem as mysterious as the idea that we have systematically true beliefs about day-to-day events in remote Nepalese villages[1]. As a result, philosophers who accept the existence of mathematical objects face a prima facie problem about how to account for substantial human accuracy about mathematics without positing a miracle[2]. This classic concern is called the access problem.

In this paper, I propose a new approach to the access problem. On the one hand, I will show how we can characterize what it takes for mathematical objects to exist in terms of a primitive modal notion which I call combinatorial possibility[3]. This reduces the problem of accounting for human access to mathematical facts to the problem of accounting for access to combinatorial possibility. On the other hand, I will provide a story that explains our access to accurate methods of reasoning about combinatorial possibility by means of our interaction with concrete objects.

Let me emphasize that the resulting account does not make mathematical knowledge empirical. Although experience with the concrete world is not needed to justify mathematical beliefs, I will argue that our dealings with concrete objects can nonetheless help explain our possession of good a priori methods of reasoning about combinatorial possibility, along the following lines.

As inquirers we try to predict and explain the behavior of concrete objects. There are more and less economical ways of doing so. When we are dealing with sufficiently diverse and plentiful collections of concrete objects, the most economical explanations will often appeal to a combination of general principles which are expected to constrain how any objects can be related by any relations,

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[1] See [10], pg 26 for a related formulation which has heavily influenced my own under- standing of the access problem.

[2] See [1] for an influential early formulation of the problem

[3] This proposal expands on work by Putnam, Hellman and Parsons [15, 23, 24] who have each explored various ways of giving intuitively correct truth conditions for claims about mathematical objects in terms of modal facts about what patterns of relationships are broadly logically coherent in a certain suitable sense.

and specific physical or metaphysical laws whose application is restricted to certain particular kinds of objects or relations. This push to predict and explain the behavior of concrete objects by appeal to facts about combinatorial possibility can be leveraged to explain the accuracy of our reasoning about more powerful claims involving combinatorial possibility, such as those which are needed to make sense of claims in the language of set theory and number theory.

After providing some background on the nature of the access problem in §2.2 , I explain my notion of combinatorial possibility in §3.4. I then outline an account of mathematical objects on which knowledge of combinatorial possibility can suffice to explain accuracy about mathematical objects[4]. In §2.4 I demonstrate some mechanisms by which interactions with concrete objects can provide (limited but substantial) access to facts about about combinatorial possibility. Finally, in §2.5 I discuss how the resulting view avoids problems for major existing accounts and in §2.6 I address some worries about the amount of mathematical knowledge which my story is in a position to explain.

2.2 On the access problem and existing approaches to solving it

Let us begin by saying a bit more about the nature of the access problem and the problems which face existing approaches to solving it. The access problem is at heart an explanatory demand. In raising an access worry one attempts to point out a kind of internal tension between a philosopher's account of some practice (such as mathematical, moral or aesthetic reasoning) and their best picture of our place in the world.

Certain kinds of correlations seem to 'cry out for explanation' in a way that makes a theory less attractive to the extent that it cannot account for them. Thus, for example, a theory of fundamental physics which posited two fundamental

[4] For reasons of space, I will focus in this paper on showing how to account for human knowledge of facts about the numbers, but see the the appendix for discussion of how the same story can be extended to account for knowledge of the sets and other mathematical objects.

constants that exactly agreed with one another but did not allow for any unifying explanation of this fact would look less attractive than an alternative theory which made sense of the same data while not positing any such brute coincidence. Another kind of correlation which can cry out for explanation is the agreement between people's beliefs about some subject and the facts about that subject. When a philosophical account takes our success at some practice to involve this kind of correlation between human psychology and objective facts, but appears to rule out any possible explanation, this view faces an access problem.

Accordingly, to say that a theory which holds that mathematical claims describe some subject matter S faces an access problem is to argue, roughly, as follows.

1. No plausible causal or explanatory mechanism could connect human beliefs to the facts about subject matter S.
2. Given (1) it would be miraculous if human beings it would be miraculous if human beings had a substantial body of largely true beliefs about S
3. Human beings *do* have a substantial body of largely true beliefs about whatever the subject matter of mathematics is.

Conclusion: We have a good reason to think the subject matter of mathematics is not S.

Getting clear on the essentially explanatory (rather than justificatory) nature of the access problem has both positive and negative consequences with regard to solving it. On the negative side, it shows that one cannot hope to dissolve the access problem merely by getting rid of contentious empiricist or foundationalist assumptions about justification, e.g., the assumption that all legitimate knowledge must be justified by appeal to some limited set of foundational facts about experience, logic and analytic truths. For, even if we allow that a priori justification for mathematical beliefs, and knowledge of mathematics, requires nothing more than being disposed to find a coherent web of mathematical truths obvious, we can still ask for an explanation of how we came to be in the fortunate position of finding truths (rather than falsehoods or meaningless statements) obvious.

On the positive side, it means that solving the access problem does not require providing any additional justification for mathematical claims. Rather, a philosopher can solve their access problem merely by providing some suitable story which connects human psychology with objective mathematical facts in such a way as to make human accuracy about mathematics look non-mysterious. They do not even need to endorse the relevant story. This story just needs to be plausible enough (once combined with our current knowledge of mathematical practice) to defeat the impression that no naturalistic explanation of human accuracy with respect to abstract mathematical objects is possible.

Existing attempts to solve the access problem can be roughly classified in three groups: reductive rationalist, projectivist and empiricist. I will devote the remainder of this section to situating my own proposal in this paper with regard to these major existing lines of attack.

Reductive rationalist proposals attempt to show that mathematical knowledge can be gained by deploying some faculty of a priori insight like logical deduction or Gödelian concept inspection[5]. The problem is that, if human accuracy about mathematics seems prima facie mysterious, human accuracy about whatever abstract and powerful logical or conceptual facts the rationalist invokes to explain our mathematical knowledge seem equally mysterious.

Projectivist proposals try to explain our tendency to be right about mathematics by arguing that nothing would qualify as accepting basic methods of

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[5]For an example of a contemporary reductive rationalist proposal see Hartry Field's fictionalism[10]. If his version of fictionalism is correct, mathematicians' ability to institute and correctly discuss mathematical fictions only requires accuracy about a kind of second order logical coherence. However, our knowledge of this coherence poses an access problem itself. See also Wright and Hale's neo-logicism [Bob Hale, 2001] and Balaguer's plenitudinous platonism [Balaguer, 2001]. Such demonstrations accord well with the idea that mathematical knowledge is a priori and may illuminate connections between mathematics and other subjects. However, they are often held to provide no satisfactory response to, or lasting therapy for, intuitive worries about human access to mathematics[Jenkins, 2008] [Parsons, 2007].

mathematical reasoning which are wrong. They hold that any (syntactically consistent) norms for reasoning about mathematical objects which we accept as governing what premises and inferences can occur in a legitimate mathematical proof would determine the meaning of our mathematical terms in such a way as to ensure its own accuracy[6]. These accounts face problems because there are possible mathematical practices which would never allow us to derive a contradiction, but are nonetheless intuitively incoherent and hence allow for the deduction of some false or meaningless claim[7].

Finally, existing **Empiricist** proposals attempt to explain human accuracy about mathematics by invoking some connection between mathematical facts and the perceivable world around us, such that observation or scientific reasoning about the perceptible world can justify our mathematical beliefs. These accounts are attractive because they promise to provide a non-buck-passing explanation for partial human accuracy with respect to a realm of objective mathematical facts. However, accepting an empiricist view of mathematics requires one to deny intuitive verdicts about what justifies particular mathematical beliefs.

Existing empiricist accounts have also faced problems as to how dealings with concreta could be the original source of mathematical justification and knowledge. Mill famously connected mathematics to experience by saying that truths of arithmetic like $2 + 3 = 5$ mean something about what happens when you physically collect objects together. But philosophers, such as Frege, have found this implausible[8].

Quine argues that we are justified in accepting those mathematical objects that

[6]For an example of a projectivist view see Tait[Tait, 2001] and also, to a lesser degree, Maddy [Maddy, 2000].

[7]See section 2.5 for more details.

[8]For one thing, the laws of arithmetic seem to apply to things like poems which it doesn't make sense to talk of physically combining. Therefore, arithmetic can't be about what happens when you physically combine things. Second, the facts of arithmetic don't always track what happens when you combine objects. For instance, when you combine 2 liters of one liquid and 3 liters of another, you will sometimes get more or less than five liters of liquid, depending on the chemical properties of the liquids involved[Frege, 1980].

are indispensable to our best physical theories. However, serious questions have been raised about whether quantification over mathematical objects is indispensable[Field, 1980b] and even if so whether quantification over any particular mathematical objects is indispensable. Also, there are doubts about whether this kind of account can justify our degree of mathematical knowledge. Empiricist accounts which take experiences to simultaneously justify and explain our mathematical beliefs also seem ill-equipped to make sense of the possibility that substantial portions of dispositions to accept mathematical beliefs might be relatively innate, and can thus seem unattractively committed to pre-judging open questions in empirical psychology.

My task in this paper will be to articulate a new approach to the access problem which avoids all of the problems mentioned above.

2.3 Combinatorial Possibility

Now that this background is in place, we can turn to the details of my proposal. I will begin this section by developing the key notion of combinatorial possibility. I will then discuss how appeals to combinatorial possibility can be used to paraphrase claims about mathematical objects.

2.3.1 Introducing the Notion of Combinatorial Possibility

In a nutshell, combinatorial possibility is possibility with regard to the most general laws of how any objects can be related by any relations. To motivate this idea, consider the following sentence.

SNOBS There are finitely many people at the party, but for each partygoer there is some other partygoer that owns more books on Proust than they do.

One can know that SNOBS is false without doing any psychology or investigating the metaphysics of owning or the nature of being a book. SNOBS simultaneously demands that both that there are only finitely many partygoers, and that for each partygoer, A, there is some other partygoer, B, such that B bears the 'owns' relation to more items in the extension of 'book on Proust' than A[9]. This demand seems to violate very general constraints on how *any* objects can be related by *any* relations. Thus it seems to require something which is broadly logically, or structurally, impossible. However, because the notion 'finite' cannot be expressed purely in terms of first order logical connectives, we cannot capture this idea by saying that SNOBS demands something which is logically impossible in the familiar sense associated with first order logic.

I take consideration of examples like SNOBS to suggest that we have a grip on a more general sense in which the broadly logical structure of what a sentence demands can ensure that the sentence is false. We have some notion of how it would be in principle possible for any relations to apply to any objects, and we are able to discern that none of these in principle possible choices for how relations like 'owns' and 'is a book on Proust' could apply would suffice to make SNOBS true. Thus, even without considering anything particular about the relation of owning or the property of being a book on Proust, we can see that SNOBS must be false.

I will refer to this notion as combinatorial possibility. With combinatorial possibility, we ignore any limits on the size of the relevant domains. Even if the actual world contained only finitely many objects, it would still be combinatorially possible for there to be infinitely many objects. We also treat all relations as arbitrary *n*-ary relations. For example, if we formalize a sentence using predicates raven(x) and vegetable(x) we will consider the structural possibility of scenarios where a single object falls in the extension of both 'raven' and 'vegetable,' despite the fact that it is presumably metaphysically impossible for any one object to be both a raven and vegetable.

[9]To avoid undesirable complexity we don't provide the full translation of the concepts finitely and more in purely combinatorial vocabulary at this point but note that unlike in first order logic such a translation does exist.

A sentence written in a first order relational language[10] will be combinatorial possible iff it is first order consistent. However, we can also apply it to sentences that use more powerful resources to describe the relationships between objects.

In particular, we can evaluate the combinatorial possibility of states of affairs which are themselves described by appeal to facts about what is combinatorially possible while fixing the applications of certain designated predicates. First, we can describe the way that predicates apply to objects in the actual world by saying things like the following.

KITTENS: Given what kittens and blankets there are, it is combinatorially impossible that each kitten slept on a different blanket last night.

This sentence will be true iff there are more kittens than blankets. We can then evaluate whether whether KITTENS and other descriptions like it require something which is combinatorially possible (as presumably what KITTENS requires is). The result is a kind of nesting of combinatorial possibility operators, which allows us to ask, for example, whether it would be combinatorially possible for some properties F and G to apply in such a way that it would be combinatorially impossible for any (potentially new) objects satisfying the predicate R to bear some further relationship $\psi[F, G, R]$ to the Fs and the Gs.

Expanding the range of sentences which can be evaluated for logical possibility in this way allows us to formulate intuitively meaningful claims which cannot be expressed purely by talking about first order logical possibility. For example, we will see that one can formalize claims like SNOBS, using only first order logical connectives and the kind of nested appeals to combinatorial possibility mentioned above. Once so formalized, we can see that SNOBS requires something which is, indeed, combinatorially impossible. Now let us turn to the task of providing suitable paraphrases for mathematical statements by appeal to facts about combinatorial possibility.

[10] By relational language I mean a language not containing function or constant symbols.

2.3.2 Primitiveness

In order to make unambiguous claims about combinatorial possibility, I will introduce a subscripted $\Diamond_{P,Q}$ operator which can be read as 'given the facts about how P and Q apply, it is combinatorially possible that...'. As usual, $\Box_{P,Q}$ will abbreviate will abbreviate $\neg\Diamond_{P,Q}\neg$ and can be read as 'given the facts about how P and Q apply, it is combinatorially necessary that...'. I will then describe the semantics for these operators more formally, by showing how claims about combinatorial possibility relate to more familiar claims about the existence of set theoretic models.

I should note, however, that I relate facts about combinatorial possibility to facts about set theoretic models for purposes of exposition alone. Given enough time, an understanding of combinatorial possibility could be built from the ground up, just as an understanding of the primitive notions of set theory is built up in standard mathematics courses[11]. If anything, the fact that talk about all 'in principle possible' ways of choosing sets comes naturally when one is trying to explain the intended width of the hierarchy of sets suggests that our understanding of combinatorial possibility (or a suitably close notion) is prior to our understanding of set theory.

Some readers might worry that facts about combinatorial possibility are dependent on facts about set theoretic models in a way that would make appealing to them in an account of the nature of mathematical objects circular. However, I want to suggest that facts combinatorial possibility require no grounding in the existence of objects of any kind, be they physical objects or abstract set theoretic models. In this regard, the view I am suggesting is analogous to a standard position on metaphysical possibility: taking metaphysical possibility to be primitive, as opposed to analyzing it in terms of the actual existence of Lewissian possible worlds [21]. Just as a realist about possible worlds

[11]Just as with set theory one could start with axioms and inference procedures about combinatorial possibility that suffice to establish sufficient mathematical foundations. However, also as with set theory, such axioms and inference procedures couldn't capture all truths about combinatorial possibility.

can give accurate truth conditions for claims about metaphysical possibility in terms of possible worlds, a realist about mathematical objects can give accurate truth conditions for the claims about combinatorial possibility in terms of sets[12]. But, we need not think about the metaphysical possibility of there being, say, a pink elephant as obtaining in virtue of the actual existence of a Lewissian possible world containing a pink elephant or any other object. Similarly, I want to suggest that there are primitive, fundamental modal facts about what patterns of relationships between objects are combinatorially possible, and that these facts do not require the existence of any witnessing objects to make them true[13].

2.3.3 Formalism

I will now describe the semantics for these operators more formally, by showing how claims about combinatorial possibility relate to more familiar claims about the existence of set theoretic models.

Start with a standard first order language $\mathscr{L}$ which lacks function and constant symbols but includes a countable collection of relation symbols of every arity. I will explain how truth conditions for a larger language $\mathscr{L}$' whose sentences include the diamond operator[14] A sentence in the language of combinatorial

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[12]At least, a realist could give accurate truth conditions for claims about combinatorial possibility in terms of set theory, if we ignore the problem of the limited size of the hierarchy of sets.

[13]See section 5 of the introduction to [10] for an argument that we must take some notion of broadly logical coherence/possibility along these lines as a modal primitive whose existence is not grounded in the existence of any abstract objects.

[14]The well formed formulas of $\mathscr{L}$' can be specified in a familiar way by the following combination of recursive conditions:

1. If φ is a formula of first order logic with equality in the language containing the relation symbols R^k_n but no function or constant symbols then $\varphi \in \mathscr{L}$
2. If $\varphi \in \mathscr{L}$ then $\exists x\varphi$, $\forall x\varphi$ and $\neg\varphi$ are in $\mathscr{L}$
3. If φ, ψ in $\mathscr{L}$ then $\varphi \vee \psi$ and $\varphi \& \psi$ are in $\mathscr{L}$
4. If $P_1 \ldots P_n$ is a finite list of relation symbols chosen from the R^k_n and $\varphi \in \mathscr{L}$ is a well formed formula with no free variables, then $\Diamond_{P_1 \ldots P_n}\varphi$ and $\Box_{P_1 \ldots P_n}\varphi$ are in $\mathscr{L}$

possibility is a formula in the language of combinatorial possibility without any free variables[15] are built up, given facts about how the relations expressed by relation symbols in $\mathscr{L}$ apply to the world.

If φ is a sentence in our base language not containing $\Diamond$, then a sentence of the form $\Diamond\varphi$ will be true iff it is combinatorially possible for there to be objects over which the quantifiers in φ range, and relations interpreting the relation symbols in φ, which make φ true. We can express this idea in terms of set theory by saying that $\Diamond\varphi$ is true if there is some set theoretic model interpreting the relation symbols in φ which makes φ true. Note, however, that the combinatorial possibility of scenarios involving very large collections may not be witnessed by the existence of a set model because of limitations on the size of set models[16].

Sentences can also contain subscripted $\Diamond$ operators. A sentence of the form $\Diamond_{R_1,\ldots,R_m}\varphi$ is true if it is combinatorially possible to make φ true while leaving the extensions of the relation symbols $R_1, \ldots, R_m$ unchanged. In terms of set theory (with the same caveat about size) we can capture this idea by first defining what it takes for a sentence involving combinatorial vocabulary to be true relative to a model as follows.

> $\Diamond_{R_1,\ldots,R_m}\varphi$ is true relative to a model $\mathscr{M}$ just if there is another model $\mathscr{M}'$ which assigns the same sets of tuples[17] to the extensions of $R_1, \ldots, R_m$ as $\mathscr{M}$ and makes φ true[18].

As usual a sentence is a wff without free variables.

[15]For reasons of elegance that will become clear later on, I will also take the language of combinatorial possibility to include the sentence $\bot$.

[16]For example, if you are an actualist about set theory, and think that we are in a position to give a description of the hierarchy of sets which uniquely describes its intended structure, then the combinatorial possibility of some objects satisfying that categorical description will not be witnessed by any model within the hierarchy of sets.

[17]Remember, the extension of an n-ary predicate R is the set of tuples $\langle a_0, \ldots, a_{n-1}\rangle$ such that $R(a_0, \ldots, a_{n-1})$ holds.

[18]In doing this $\mathscr{M}$ can alter the extension of predicates which are not among the specified $R_1, \ldots, R_m$ and change the total size of the domain.

Intuitively, we can think of $\Diamond_{R_1,\ldots,R_m}\varphi$ as being true relative to the model $\mathscr{M}$ if it is possible to satisfy φ while letting $\mathscr{M}$ fix the interpretation of the relation symbols $R_1, \ldots, R_m$. Note that the definition above implicitly defines what it takes for a sentence involving any number of nested uses of the subscripted diamond operator to be true relative to a model[19].

We can then use this notion of truth relative to a model to describe how the truth values for arbitrary sentences in $\mathscr{L}$ are systematically determined by facts about how the atomic predicates in $\mathscr{L}$ apply to the world. One can think of the meaning of atomic predicates subscripted in $\Diamond_{R_1,\ldots,R_m}\varphi$ as combining with facts about the actual world to determine which set theoretic models correctly reflect the state of the actual world with regard to $R_1, \ldots, R_m$. We can then say that a statement of the form $\Diamond_{R_1,\ldots,R_m}\varphi$ in $\mathscr{L}$' is true iff it is true relative to some model which agrees with the actual world regarding the application of $R_1, \ldots, R_m$[20].

Finally, it is worth noting that pure statements about combinatorial possibility (statements in which every subscripted $\Diamond$ occurs within the scope of some non-subscripted $\Diamond$) are true if and only if φ is true relative to the empty model or indeed any other model at all. As a result, reliably evaluating such statements about pure combinatorial possibility does not require tracking any contingent facts about the actual world, and (in terms of set theory) we can say that a sentence $\Diamond$ is true iff it is true relative to the empty model.

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[19]One can check this by inducting on the level of nested $\Diamond \ldots$ clauses.

[20]Exactly how and to what extent the meanings of atomic predicates combine with facts about a given possible world to determine which set theoretic models suitably correspond to actual state of the world is, of course, a vexed philosophical question which I will not attempt to say anything about here. I only want to note that my account of how truth conditions are built up for sentences involving the 3 is compatible with a range of different opinions about how tightly or loosely the state of the world constrains which set theoretic models qualify as adequately reflecting the actual application of predicates to the world. For example, some accounts of vagueness would hold that each possible world actually corresponds to a set of models witnessing the various acceptable of vague predicates like 'bald' or (more adventurously) various meanings for the quantifiers in our language. If you hold such a view, then the story given here explains when and how vagueness about the applications of predicates percolates through to corresponding vagueness in the truth conditions for subscripted $\Diamond \ldots$ statements.

2.3.4 Describing The Numbers Using Combinatorial Possibility

Existing set theoretic arguments show that one can uniquely describe the intended structure of the numbers by saying that they satisfy a collection of axioms called PA* together with full induction. PA* is the conjunction of the usual axioms of Peano Arithmetic, excepting the induction schema, in relational form. That is, instead of representing successor (S), plus, and times as a function symbols we instead treat them as relation symbols[21]. This leaves us with a first order statement PA* in terms of the relations $<, S, +, *$ that captures everything about the natural numbers except the principle of induction a.k.a. the least number principle.

It remains to express the unrestricted principle of induction. Now if we could find some formula I expressing full induction we could translate a (relational) sentence φ about the natural numbers into a statement $\Diamond(\text{PA}^* \wedge I \wedge \varphi)$ about combinatorial possibility with the appropriate truth conditions. Induction is usually expressed in the language of second order logic as follows]:

$$(\forall X)\left([0 \in X \wedge (\forall n)(n \in X \supset S(n) \in X)] \supset (\forall n)\mathbb{N} \implies (n \in X)\right)$$

We can express the same idea in terms of combinatorial possibility by saying that it would be mathematically impossible (fixing the facts about what is a number and a successor) for a one place relation, say, P, to apply to apply to 0 (the unique number that isn't a successor) and to the successor of each object that it applies to without applying to all the numbers. Now we can define the formula I as follows:

$$I = \Box_{\mathbb{N},S}\neg\left([P(0) \wedge (\forall x)(\forall y)\left(P(x) \wedge S(x,y) \implies P(y)\right) \implies (\forall x)(\mathbb{N}(x) \implies P(x))]\right)$$

[21]Thus, instead of writing $S(x) = y$ or $x + y = z$ we would write $S(x,y)$ or $+(x,y,z)$. We also add axioms asserting that the relations $S, +, *$ act as functions and that there is a unique x which is not the successor of any y. This last axiom defines 0 and allows us to replace any mention of the constant symbol 0 with such a paraphrase. Writing the axioms for arithmetic in relational form is standardly done in mathematical logic and involves no loss of power in this case.

Where $P(0)$ is shorthand for $(\exists z)(\forall w)\ (\mathbb{N}(z) \wedge \neg S(w, z) \wedge P(z))$.

Now, as it is combinatorially possible for objects to be related in the way that the numbers are supposed to be, the statement $\Diamond(\mathrm{PA}^* \wedge I)$ will be true. Moreover, since $\mathrm{PA}^* \wedge I$ asserts that the extension of N and the relations $<, S, +, *$ on that extension behave as the natural numbers with full induction $\Diamond(\mathrm{PA}^* \wedge I \wedge \varphi)$ will be true for a formula φ in the language of number theory just if φ is true of the natural numbers.

We can further show that this approach suffices to determine intuitively correct truth conditions for all statements in the language of number theory. As it is combinatorially possible for objects to be related in the way that the numbers are related, the statement $\Diamond(\mathrm{PA}^* \wedge I)$ will be true. Moreover, since $\mathrm{PA}^* \wedge I$ asserts that the extension of N and the relations $<, S, +, *$ on that extension behave as the natural numbers with full induction $\Diamond(\mathrm{PA}^* \wedge I \wedge \varphi)$ will be true for a formula φ in the language of number theory just if φ is true of the natural numbers. It is also worth noting that, since only the objects-and-relations structure of the terms number and successor are being used in the story above, we can give the same truth conditions by substituting any other one and two place relations such as cat and admires. This removes any potential confusion as to whether our definition of number is in some sense circular. A similar strategy of using appeals to combinatorial possibility to do the work of second order logic can be used to paraphrase statements of set theory[22].

The fact that one can provide intended truth conditions for statements of number theory in terms of combinatorial possibility is important because it lets us ground facts about number existence in facts about combinatorial possibility without giving up the intuitive idea that all number theoretic questions in the language of number theory have definite right answers. Many attempts to understand mathematical truths as analytic consequences of some kind of implicit definition of the numbers have failed at this point. For example, we can't understand mathematicians' claim that φ as saying that φ is derivable from any

[22]See the relevant section of my dissertation (available by request) for more details.

finite collection of claims in standard first-order logic without sacrificing the above intuition. In contrast, like characterizations in terms of second order logic, we can give a description of the numbers in terms of combinatorial possibility which has all truths in the language of number theory as combinatorially necessary consequences. This is possible because, unlike first-order logical consequence, the notion of combinatorially necessary consequence cannot be characterized by providing any exhaustive algorithm which derives all consequences of the relevant kind.

I will now argue that thinking about what it takes for there to be a number in terms of facts about combinatorial possibility gives us a good starting point to explain how we access the (necessarily incomplete) collection of facts about numbers mathematicians are able to derive. By the above argument, the problem of access to the numbers reduces to a problem of explaining how we can have substantial (but incomplete) knowledge about combinatorial possibility. Given knowledge of the kind of existence conditions for the numbers provided above, we can learn number theoretic facts by learning appropriate facts about combinatorial possibility. Thus, we can solve the problem of access to mathematical if we can account for our ability to reason correctly about combinatorial possibility.

2.4 Knowledge of Combinatorial Possibility

Now let us turn to the final task of showing that there is no access problem with respect to combinatorial possibility. To appreciate the nature and difficulty of the task at hand, one should note that, even though combinatorial possibility intuitively captures facts about what might be called logical coherence, these are not themselves first order logical facts. We said above that a first order logical statement requires something combinatorially possible i� it is syntactically consistent[23]. However, it is important to note that these facts about the

[23]That is, $\Diamond\varphi$ iff no contradiction can be derived from φ using the inference rules of first order logic.

syntactical consistency of first order logical sentences are not themselves truths of pure first order logic, in the sense that they could be derived using only the introduction and elimination rules for the first order logical connectives $\&, \vee, \neg, \exists, \forall$. Nor, does the mere ability to derive first order logical consequences provide a reliable method of recognizing that a collection of sentences is (first order) consistent[24]. Thus, even if we suppose that someone has full knowledge of first order logic, more is needed to account for their ability to deduce facts about combinatorial possibility.

In this section I will outline two ways for facts about concrete objects to 'kick back' and correct our general methods of reasoning about combinatorial possibility. I will suggest that, just as our scientific observations push us to correct general scientific theories whose content goes beyond the mere facts observed, so too our experience with concrete objects can lead us to valid methods of reasoning about combinatorial possibility that generalize what we learn from particular cases.

Note that even though combinatorial possibility intuitively captures facts about what might be called logical coherence these are not themselves first order logical facts. Indeed, the fact that a collection of axioms is first order consistent is not itself a truth of pure first order logic in the sense that it can be derived from these facts[25]. Thus even if we suppose that someone has full knowledge of logic, more is needed to account for their ability to deduce that any claims involving the diamond of combinatorial possibility.

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[24]Because the rules for first order logic happen to be complete one could in principle verify a first order theory was consistent by exhaustively searching through all proofs whose premises come from the theory and showing that none of them yield a contradiction, but this requires checking an infinite number of cases, so it is clearly not how we arrive at beliefs about broadly logical possibility.

[25]Because the rules for first order logic happen to be complete one could in principle arrive at beliefs about combinatorial impossibility by exhaustively searching through all proofs from given premises and showing that none of them prove 0=1 but this requires checking a countable infinity of cases, so it is clearly not how *we* arrive at beliefs about broadly logical possibility.

2.4.1 Inference from φ to $\Diamond\varphi$

First, and most obviously, there's the inference from φ to $\Diamond\varphi$. If you falsely think that it is combinatorially impossible for any objects to be related in a certain way, this belief can be corrected by learning that some objects are actually so related. This provides a way for principles which say that too few things are combinatorially possible to be corrected. Imagine, for example, that you aren't sure whether the state of affairs described by some mathematical hypothesis involving relations *P*, *Q*, and *R* is combinatorially possible. If I then point out that the relations of friendship, nephew-hood and having been in military service together apply in just this way to the royal family of Sweden, this will get you to accept that the scenario in question is, indeed, combinatorially possible.

More generally, knowledge that some scenario is physically or metaphysically possible will also allow for the inference that it is combinatorially possible, and hence yield $\Diamond\varphi$ beliefs in much the same way. For, no physically or metaphysically possible scenario can be ruled out by the most general principles of how any objects can be related by any relations. Hence, we can reliably infer that what's physically or metaphysically possible is also combinatorially possible. So, if you know that it would be physically possible to have $1,000$ different chess boards set with chess pieces, each in a different configuration, then you can reliably infer that it must be combinatorially possible to have $1,000$ different chess boards set with pieces in different configurations[26]. In this way, the same pressures which cause us to have accurate beliefs about physical possibility can help give us accurate beliefs about combinatorial possibility.

2.4.2 Generalization from $\neg\varphi$ to $\neg\Diamond\varphi$

Conversely, the need to explain certain regularities in what's actual can push us toward the conclusion that certain states of affairs are combinatorially impossible. Suppose, for example, that someone thought it was combinatorially possible for 9

[26] Of course, one must first properly formalize the claim to write it in terms of combinatorial possibility but this is not difficult.

items to differ from one another in which of three properties they have, e.g., for 9 people to choose different combinations of sundae toppings from a sundae bar containing three toppings. This person would have to explain the striking law-like regularity that, regardless of the type of items and properties, we never wind up observing more than 8 such objects. They would have to postulate new physical regularities to explain why apparently random processes of flipping three coins never generated the forbidden 9th possible outcome. Furthermore, whatever physical explanation is chosen would have to apply at every physical scale we can observe, from relationships between the tiniest particles to relationships between planets and stars. Also, they would have to explain why the same regularity held, in apparently exactly the same way, with regard to much less concrete subject matter like poems or countries. Try as you may, you will never manage to think up a poem with 9 different stanzas, each of which differs from the all the others in regard to which of three poetic themes it mentions.

More generally, knowledge that some scenario is physically or metaphysically possible will also allow for the inference that it is combinatorially possible. For, no physically or metaphysically possible scenario can be ruled out by considerations of combinatorial possibility. Hence, we can reliably infer that what's physically or metaphysically possible is also combinatorially possible. So, if you know that it would be physically possible to have 1,000 different chess boards set with chess pieces, each in a different configuration, then you can reliably infer that it must be combinatorially possible to have 1, 000 different chess boards set with pieces in different configurations[27]. In this way, the same pressures which cause us to have accurate beliefs about physical possibility can help give us accurate beliefs about combinatorial possibility.

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[27]Of course, one must first properly formalize the claim to write it in terms of combinatorial possibility but this is not difficult.

2.4.3 Generalizations about $\Diamond\varphi$

Finally, the usefulness of being able to anticipate that certain states of affairs *are* combinatorial possible can press us to accept positive generalizations about combinatorial possibility[28]. Many plans we can verbally represent will be ones that can be discarded as impossible to realize purely on the ground of general combinatorial principles. As a result we face pressure to recognize various general principles which allow us to recognize plans which at least pass the low bar of being combinatorially possible. If you are disposed to form (and make promises based on) a logically unsatisfiable plan for how to divide up the spoils of the mammoth hunt, this will get you in trouble. Similarly you will fail to recognize some logically possible and mutually satisfactory division of spoils that would have gotten everyone to go in on the mammoth hunt you will be failing to live up to your evolutionary potential.

Generalizations of this kind involve allow one to deduce the combinatorial possibility of scenarios which one has never observed to be actual. Nonetheless they are subject to correction by experience in three ways. First they generalize initial data points which are known to be combinatorially possible. Second insofar as they imply new predictions about what states of affairs are combinatorially possible they can be corrected by coming into conflict with well-entrenched and motivated negative general principles about combinatorial possibility. Third they can come into conflict with ones belief about what combinatorial possibilities are physically possible. For example you may believe that all combinatorially possible ways for having arrows connecting 10 dots are physically realizable, so if a theory predicts that it is combinatorially possible to connect 10 dots in a way that satisfies a certain property but no one is ever able to actually create the relevant diagram, this can create pressure to reject the relevant general theory.

[28]By positive and negative generalizations I mean things like general axioms and inference rules (or ways of reasoning with mental pictures) which allow us to deduce new claims about combinatorial possibility which go beyond those motivated by the fairly direct contact with experience discussed in the previous two subsections.

2.4.4 Three Just-So-Stories

Now that we have these ways for facts about com- binatorial possibility to 'kick back' and correct our beliefs about combinatorial possibility in mind, let us turn to the task at hand: explaining how creatures like us could have gotten correct methods of reasoning about combinatorial possibility.

Recall that the access problem is at heart an explanatory demand. Philosophers who take mathematics to describe mind independent abstract objects seem, thereby, committed to positing some kind of supernatural assistance shaping our mathematical intuitions[29] or a profound coincidence whereby we just randomly happened to get intuitions that correctly match the facts about an independent realm of abstract mathematical objects.

I will now attempt to defeat this impression by giving three examples of how perfectly ordinary processes could account for the (general) accuracy of our (necessarily incomplete) armchair reasoning about combinatorial possibility - and thus for our access to facts about mathematical objects via the story above.

My first and simplest just-so story appeals to conscious suggestion and correction by experience within a person's lifetime. It may seem strange to imagine broadly scientific generalizations from experience extending and correcting our reasoning about combinatorial possibility as above. Wouldn't such correction by experience have to leave traces in the phenomenology or justification of our reasoning about combinatorial possibility? Yet we don't appeal to any kind of memories of past experiences with concrete objects when reasoning about what patterns of relation- ships between objects are combinatorially possible.

I will argue that we actually have independent reason to believe that experience can suggest and correct mathematical beliefs without leaving any such traces in our mathematical practice. Think about the development of the kinds of hunches that guide mathematical research. Mathematicians don't choose which

[29] For instance, something along the lines of Platonic recollection or Descartes benevolent deity shaping our intuitions.

proof strategies to try by tossing coins, but rather on the basis of less than fully confident hunches about how certain things should turn out. They have some ideas of what proof strategies are likely to work out or not, antecedent to actually trying them. These ideas develop and improve over time. So it seems highly plausible that past re- search experiences are causally involved in leading them to have the right hunches. Nonetheless, when they say that this strategy looks promising or that one looks unpromising, they don't consciously summon up cases of similar strategies being tried or otherwise appeal to claims about their history and past experiences for justification[30].

A second just so story takes our development of good general methods of reason- ing about combinatorial possibility to have happened more slowly, over the course of the whole of human history rather than a single individuals lifetime. Here, our reasoning about combinatorial possibility is subject to a process of generalization and correction in response to experience, much as in the story

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[30]Alternatively, consider the following, more interesting example of correction by experience leading people to accept propositions without them being disposed to cite experience in justifying those propositions:

In the Monty Hall game show, there's a car behind one door, and goats behind two others. You are asked to pick a door, and then Monty opens another door to reveal a goat. Now you have the opportunity to switch doors or to stay with the one you originally chose. When presented with the Monty Hall problem, many people initially feel attracted to two incompatible ways of analyzing the relevant probabilities: one which says that switching doors will improve your odds, and one which says that it will not. A recent NY Times article on cognitive illusions about the Monty Hall problem, linked to a computer simulation of the contestant's dilemma, which kept statistics for the results of switching vs. not switching in order to convince readers that its arguments for switching were correct [29].

I take it that a number of readers actually switched verdicts about the Monty Hall case response to playing with the above computer simulation. Such readers would presumably not cite these experiences as part of the justification when analyzing probabilities in the correct way in the future. Yet the experience of playing around with a computer simulation played a crucial role causal role in leading them to adopt good a priori methods of reasoning about probability. Playing with the computer simulation did not merely give the time or inspiration to produce additional mathematical argument using methods which they already accepted a priori. Rather, it lead them to prefer one method for doing such a priori analysis to another.

Thus it appears that we have good empirical reasons to believe that there are at least some cases where experience can help causally explain why we came to accept accurate a priori methods of reasoning, despite this experience not being remembered or cited as necessary justification for later a priori arguments using these methods.

above. However, each individual person gets most of their principles for reasoning about combinatorial possibility by picking them up from the people around them. Individuals only very rarely suggest revising the methods of reasoning about combinatorial possibility which are generally accepted by those around them. Once suggested, theories are adopted when they seem to elegantly predict and explain regularities in what's actual and rejected if and when they fail to do so. Thus our general methods of reasoning about combinatorial possibility get developed and corrected over the course of the history of ideas.

Finally, a third possible just-so story takes the methods of correction mentioned above to function on an evolutionary level. If reasoning about combinatorial possibility were 'hardwired' in some sense that precluded correction by experience, it could be hardwired in a number of different ways. There might be a brain system dedicated to reasoning about combinatorial possibility. Or there might be a general brain process that nudges us towards beliefs that certain things are physically possible, and then a conscious process of inference to the best explanation which leads us to distinguish some of these methods of reasoning as combinatorially necessary principles[31]. Though evolution may not care about elegance and theoretical beauty in quite the sense that we do, mental resources are expensive and those methods of reasoning that could be encoded in the simplest manner and handle the most general situations would be favored.

Ultimately, it's an empirical question if one or some combination of these stories is right. All one needs to do to defeat the access problem is to provide one coherent story explaining our beliefs that's compatible with what we know now. I have attempted to provide a range of different just-so stories in order to to show that my solution doesn't depend on any particular plausibility judgement about

[31] See Spelke's experiments with infants, [27] for an example of the kind of data which might suggest that some reasoning about what patterns of relationships between objects are combinatorially possible are relatively innate. Further results along these lines might suggest that children have good methods of reasoning about combinatorial possibility before they are in a position to do much personal experimentation with concrete objects, or hear good methods of reasoning advocated in the classroom.

the importance of nature, nurture or individual experience in determining these kinds of beliefs. The key idea of this paper - that the dealings with concreta can explain our possession of good a priori methods of reasoning about combinatorial possibility and thus (in turn) our accuracy with regard to mathematical objects - can be realized by a number of different scientific stories about the development of human cognition.

2.5 Improvement on existing accounts

Now that my proposal is on the table I'd like to return to the problems which beset existing approaches to the access problem, and note how my story purports to avoid them.

First, recall that reductive rationalist proposals faced a buck-passing problem. In order to capture intended truth conditions for mathematical claims they needed to posit substantive a priori access to objective proof transcendent facts about logical possibility, truth in the fiction or some other notion filling this role. But access to these facts seemed to pose just as much of an access problem as a priori access to mathematical facts. My story about mathematical knowledge resembles these proposals in taking mathematical knowledge to involve a priori access to proof transcendent facts; I hold that our knowledge of mathematical facts arises from our acceptance of good a priori methods of reasoning about combinatorial possibility. However, it addresses the buck-passing objection which typically besets these rationalist proposals by showing how to provide a substantive story about how our access to combinatorial possibility facts can be explained.

Next we saw that projectivist approaches face a problem about denying the apparent possibility of being fully syntactically consistent but wrong about mathematics. In contrast, my theory explains how mathematical proof practices which don't allow one to derive a contradiction can nonetheless be incoherent in a way that ensures that some of the claims which they allow you to derive are false or meaningless. If a community accepted the existence conditions for 'the numbers' in terms of combinatorial possibility as provided above while adding the Gödel sentence for their proof practice they would be accepting an incoherent system of claims[32]. It then explains why we tend not to be wrong in this way by showing how to tell a substantive causal about how bad general methods of reasoning about mathematics can be corrected.

Finally, empiricist accounts require us to accept some very revisionary claims about the nature of mathematical justification. My proposal resembles these accounts in appealing to dealings with concrete objects to explain our accuracy about combinatorial possibility (and thence about mathematics). However, it avoids empiricism's revisionary conclusions about justification, by taking these dealings with concrete objects to causally explain our possession of accurate a priori methods of reasoning about mathematics instead of justifying particular mathematical beliefs.

Empiricist accounts also seem committed to pre-judging certain otherwise open psychological questions about the innateness of our methods of mathematical reasoning. In contrast, we have seen that my story about correction by experience can be realized at a variety of different levels: by evolutionary selection on innate tendencies as well as conscious belief revision and cultural selection in the course of intellection history. As a result, my account has no problem making sense of relatively innate mathematical knowledge, should empirical results turn out to support this.

Finally, existing versions of empiricism about mathematics face problems over the use of concreta to justify our mathematical premises, rather than merely to explain our tendency to accept true rather than false statements as premises. For example, Mill's empiricism ties mathematical facts to specific physical acts of

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[32]Here proof practice refers to the computable process which we settle on as an acceptable way to infer new claims about combinatorial possibility from old ones. As our proof practice for combinatorial possibility will necessarily be incomplete mere syntactic consistency doesn't imply coherence. Since the existence conditions for the numbers are categorical they must be incoherent with either the Gödel statement for this proof procedure or it's negation and the form of the Gödel statement means that it must be the negation that is coherent. As a result they either accept false or meaningless claims.

combining by saying that claims about addition simply mean claims about the result of certain kinds of combining. But as Frege pointed out [11], this seems unmotivated in that arithmetic seems to apply in exactly the same way to objects like poems which it doesn't make sense to physically combine, and the results of physically combining objects like dew drops don't necessarily match the facts about sums. Abstract facts about combinatorial possibility have direct consequences: certain claimed patterns of relationships between objects can never be actual. But, unlike claims about physical combining which apply to all objects, these facts don't entail any kind of intuitively contingent claim what will happen when you do a certain experiment. Experiments like counting are generally reliable and a good place to start in assessing whether there are, say, 5 people at a Sundae bar each of whom have chosen differently from two available toppings. But assessing how many people there are by counting is like assessing whether all the ravens in a zoo are black by looking at them. Unlucky circumstances where people split and fuse when you try to count them or your rivals dust black ravens with flour when you are about to see them, are not absolutely ruled out by the fact that it is combinatorially impossible for 5 people to choose differently from a Sundae bar with two toppings or the fact that all the ravens in the zoo are black[33].

Quine has, more influentially, argued that we are justified in believing in mathematical objects because statements about these objects play an indispensable role in our best physical theories. However, serious questions have been raised as to whether quantification over mathematical objects is really indispensable[Field, 1980b] and, if so, whether appeals this indispensability can be used to support anything like the nature and degree of mathematical knowledge which we actually have. But, taking dealings with experience to

[33]Kitcher's neo-Millian story avoids these problems only by abstracting the notion of what an ideal being could combine until one arrives at a notion that behaves very similarly to combinatorial possibility. However, once this notion is doing all the work, there seems to be no need for contentious and intuitively irrelevant claims about ideal beings. Why think of our knowledge as coming from the fact that we can apply a predicate some way, rather than from the knowledge that a predicate applies a certain way [19].

causally explain our use of accurate general methods of reasoning about logico-mathematical possibility rather than to directly justify assertions about mathematical objects puts us in a position to avoid these worries. One need not suppose that quantifying over any particular type of mathematical object is indispensable to scientific theorizing. Indeed, one suppose that a suitably complex and clever nominalistic paraphrase could entail all the desired physical consequences of our best scientific theories without even appealing to the combinatorial possibility of objects with the structure of the numbers or the sets. One only needs to suppose that the relevant facts about combinatorial possibility are derivable from some general methods of reasoning about logico-mathematical possibility which were practically useful in predicting and explaining the behavior of concrete objects[34]

2.6 Underdetermination by evidence worries

Now that my picture is on the table, I will consider a family of related objections arising from the following simple idea: we causally interact with relatively small finite collections but the nominalist must appeal to facts about the combinatorial possibility of much larger collections when describing the intended behavior of the numbers (and the sets). This creates a worry that the theory is underdetermined by evidence. In this section I will articulate and respond to a number of different ways of spelling out this worry.

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[34]Similar problems arise for what is perhaps the closest view to mine in the existing literature, Hellman's modal structuralism [15]. Hellman agrees that mathematical knowledge is gained by insight into a kind of broadly logical possibility. However, he invokes the Quinean indispensability of such objects, or at least their broadly logical possibility, to explain our access to them. Accordingly, Hellman faces all the worries about accepting revisionary empiricist claims about the justification of mathematics, the in principle dispensability of appeals to powerful mathematical structures, and compatibility with innate inclinations which beset Quine's view.

2.6.1 Generalization From Cases

An extreme position one might take is that generalization from cases is completely unreliable with regard to mathematics[35]. However, taking this position raises a serious problem about how to make sense of the fact that (as discussed above) mathematicians frequently use hunches developed from past experience, judgements of general plausibility or theoretical attractiveness and the results of computational searches[36] to guide their research.

If we want to make sense of these aspects of mathematical practice, we must also admit that knowledge of particular cases can be a reliable corrective to mathematical beliefs. Thus, it can't be the case that something about mathematics makes the kind of elegant generalization from cases we find in the sciences utterly unreliable when applied to the mathematical realm. Such reliability may never amount to infallibility. But, insofar as we want to leave room for mathematical knowledge by testimony, we cannot suppose that mathematical knowledge requires any such extreme degree of accuracy. Thus, the extreme position that dealings with cases can never explain the accuracy of mathematical intuitions seems hard to sustain.

2.6.2 A Gap Between the Finite and the Infinite?

Next, you might worry that the story suggested above cannot explain the *degree* of mathematical knowledge which we have. One might think that it has the power to explain our access to facts about the combinatorial possibility of finite collections, but it cannot explain access to the kind of facts about combinatorial possibility for infinite collections, such as those quantified over in number theory. With regard to this worry I would like to note two things.

First, there arguably are countably infinite collections of physical objects

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[35]Frege seems to have thought this about arithmetic [11] pg. 16

[36]Of course they do not do this naively. If they already know that counterexamples would have to be huge they wouldn't change their judgements because no small counterexamples were found.

which sense perception and scientific inference to the best explanation gives us access to. Consider, for example, the stretches of space along the path of an arrow, or the stretches of time during which the arrow is traveling. Thus, concreta arguably can 'kick back' to correct our reasoning about combinatorial possibility in the infinite case as well as the finite.

Second, even if you don't accept that our experience with physical objects directly provides access to some infinite collections, reasoning about how it would be combinatorially possible for the physical objects that actually exist to be supplemented with an infinite collection of abstract objects can be very useful to stating elegant laws which predict and explain the behavior of these objects.

Consider the task of predicting what physical inscriptions of series of letters one will ever encounter. In making these predictions it can be helpful to consider the actual physical inscriptions that exist as existing alongside a larger system of abstract objects ('strings') which witness all combinatorially possible ways putting together finitely many letter inscriptions chosen from the relevant finite alphabet. Even if all the string inscriptions we encounter are relatively short, there's pressure for them to recognize the combinatorial possibility of string inscriptions of arbitrary finite size arising from closure principles. Many closure principles which smoothly predict the facts about what short strings are physically possible will have the consequence that very long strings are combinatorially possible - even strings which are too long to physically realize given the number of fundamental particles in the universe. Take, for example, the principle that if it is combinatorially possible to have a string of letter inscriptions with a certain property, then it is combinatorially possible to have a 'doubled' string which concatenates two copies of that string. [Accordingly, a scenario which supposes that abstract objects exist witnessing all combinatorially possible choices of how to concatenate numbers will be a scenario in which we allow there to be infinitely many different objects.] Our beliefs about combinatorial possibility for infinite collections can be tested and corrected by the consequences they have for the infinite collection of all strings and thereby what physical string inscriptions are possible.

Thus, I think there is no problems about how dealing with finite numbers of physical objects could have lead us to recognize the combinatorial possibility of various infinite collections, as well as various further facts about what these infinite collections would have to be like. From these initial 'data points' (together with our knowledge of finite collections) the mechanisms of correction discussed in section 4 can then explain how we came to accept correct general principles which have largely correct conclusions about what such infinite collections must be like.

2.6.3 Combinatorial Possibility and Large Collections

A related but more moderate worry about explaining our access to facts about combinatorial possibility and large collections, concerns the size of the collections which we would need to consider. Our dealings with objects in the world tend to involve finite (and sometimes countable) collections such as the number of gingerbread cookies on a table. Yet, providing a nominalist paraphrase for statements of set theory requires evaluating claims about the combinatorial possibility of scenarios involving vast numbers of objects. Thus, one might worry that principles of reasoning which are shaped to elegantly predict and explain what is combinatorially possible for small collections cannot support the degree of combinatorial (and hence mathematical) knowledge which we actually have. A critic might advance the following analogy: saying that elegant generalization from knowledge of finite and countable collections yields principles which accurately describe the larger collections considered in pure mathematics is like saying that inference to the best explanation plus observations of birds in New Mexico allows us to learn about birds in Canada as well. Presumably, in the ornithological case, we need to go gather more data in order to get true beliefs about birds in Canada. But, in the mathematical case, we can't gather more data. Thus, facts about combinatorial possibility with respect to larger collections can appear to remain permanently a mystery.

I want to respond to this worry by accepting the analogy and claiming that it actually fits the current state of human knowledge with regard to facts about the higher infinite rather well[37]. Even in the case of birds, we can know some things about birds in Canada just by inference to the best explanation from the facts about the birds in New Mexico. If we discovered tomorrow that some new island that had never yet been visited by explorers contained birds, I think we would reasonably expect many facts to carry over: any birds on that island would have DNA, that they would have hollow bones etc. Our expectations about the new island would just be very sparse and less confident relative to our beliefs about birds in locations that we have observed.

But, this is just what happens with regard to our knowledge of what's combinatorially necessary with regard to large collections: as one goes from claims about finite collections, to countable collections (like the numbers), to uncountable collections (like the sets) our beliefs do get more sparse and less confident. For example, the continuum hypothesis[38] (CH) is a fairly simple

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[37]Here it may, again, it may be helpful to contrast my theory with Quine's. Although both theories take correction by experience to play an important role in accounting for mathematical knowledge, my theory and Quine's have very different consequences for what one should say about cases where mathematical facts have little or no bearing on ones expectations for experiences with concrete objects. On Quine's view we are justified in believing in whatever mathematical objects we quantify over in our best physical theories. So *absence* of relevant physical experience motivates negative claims about the height of the hierarchy of sets. Thus, insofar as our dealings with the physical world don't require us to consider higher regions of the hierarchy of sets, the Quinean picture suggests that Occam's razor should push us to reject the existence of sets at these higher levels. Thus, on this picture lack of empirical pushback on reasoning about the sets motivates definite negative conclusions about them.

In contrast, my proposal says that large sets of various kind exist if and only if this is a combinatorially necessary consequence of a certain ontologically exuberant hypothesis H describing the intended behavior of the hierarchy of sets. (Recall that the hierarchy of sets is supposed to contain, at each level, sets corresponding to every possible way or choosing from the sets below, and to contain levels which extend - in some sense - as far as possible.) But there is no intuitive pressure, analogous to Occam's razor, to say that fewer rather than more objects would be needed to satisfy the hypothesis H. Thus, on my view the divorce between certain claims about the higher infinite and experience motivates pessimism about how much we will be able to learn about the behavior of large sets, whereas on Quine's it motivates active denial that there are any such sets.

[38]The continuum hypothesis states that there are no sets whose cardinality is intermediate between the cardinality of the real numbers and that of the natural numbers. See [17] pg 176-

question involving sets of (relatively) small infinite size, yet it is known that both the truth and the falsity of CH are compatible with ZFC. Our knowledge about what *large* collections of objects and relations are combinatorially possible is also less confident than our knowledge of what finite and countable collections of objects are combinatorially possible. Sociologically, mathematicians are frequently much more confident in their claims about numbers, sets of numbers and sets of sets of numbers than in the distinctive claims of set theory about what much larger patterns of mathematical objects would have to be like.

Thus, I think this last worry points to something that is an attractive feature rather than a flaw of the account at hand: it explains why we have so relatively little knowledge of what's combinatorially possible with respect to large collections, and hence relatively little knowledge of the corresponding facts about higher set theory.

2.7 Conclusion

In this paper I have proposed a two-part strategy for solving the access problem. First we can characterize what it takes for mathematical objects like numbers and sets to exist in terms of combinatorial possibility. This reduces the problem of accounting for human access to facts about mathematical objects to a problem of accounting for human accuracy in our general methods of reasoning about combinatorial possibility.

Second, we can solve the resulting accuracy problem by considering the role of reasoning about combinatorial possibility in our attempts to predict and explain the behavior of concrete objects. I have suggested a number of ways in which the push to predict and explain the common behavior of all concrete objects by appeal to general principles about combinatorial possibility can be leveraged to explain the fact that we employ correct general principles of combinatorial possibility when reasoning about pure mathematics.

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186 for the proof that CH is independent of the Zermelo-Fraenkel axioms.

3

Quantifier Variance and Mathematicians' Freedom

3.1 Introduction

Philosophers of mathematics have been much struck by mathematicians' apparent freedom to introduce new kinds of mathematical objects, such as complex numbers, sets and the objects and arrows of category theory[1]. In this

[1]For example in a recent AJP paper Julian Cole writes,"Reflecting on my experiences as a research mathematician, three things stand out. First, the frequency and intellectual ease with which I endorsed existential pure mathematical statements and referred to mathematical entities. Second, the freedom I felt I had to introduce a new mathematical theory whose variables ranged over any mathematical entities I wished, provided it served a legitimate mathematical purpose. And third, the authority I felt I had to engage in both types of activities. Most mathematicians will recognize these features of their everyday mathematical lives. Observations like my first are made frequently..."

paper, I explore a way of using recent work on quantifier variance to explain this easy access to foundational facts about new kinds of mathematical objects.

In [5] David Chalmers suggests a way of describing truth conditions for sentences in languages with quantifiers that are more ontologically profligate than our own by appealing to facts about certain set theoretic models. I will show that this proposal can be reformulated in such a way as to free it of arbitrary limitations on size by replacing appeals to set theory with appeals to an (independently motivated) notion of broadly logical possibility. After amending Chalmers' proposal in this way, I use it to provide a theory of how stipulative definitions can change the meanings of our quantifiers. Combining this theory with the popular structuralist intuition that mathematics is somehow 'the science of structure' yields a simple explanation for mathematicians' freedom to stipulate. Distinctive features of the resulting theory allow it to avoid known problems for existing attempts to explain this phenomenon.

3.2 Motivating the project

3.2.1 The problem of incompatible stipulations

Contemporary mathematical practice appears to be quite cavalier in allowing mathematicians to introduce new kinds of mathematical objects. New kinds of objects (such as the sets, the real numbers and the complex numbers) can be introduced merely by providing something like a coherent description of a certain mathematical structure. However, making sense of this fact poses a philosophical challenge along the following lines.

One cannot explain mathematicians' freedom to stipulate by saying that *all* coherent descriptions of mathematical structures are true because, as Boolos emphasized[2], not all coherent descriptions of mathematical structures are compatible with one another. Uzquiano gives an interesting example of such internally coherent but conflicting descriptions in his discussion of the

[2] [2]

relationship between full mereology and set theory[30]. Although it seems coherent to say that every object belongs to a set, and it seems coherent to say that every plurality of objects has a mereological fusion, Uzquiano shows one cannot *conjoin* both claims with certain natural axioms of set theory and meriology on pain of logical contradiction[3].

Accordingly philosophers who accept mathematicians' freedom to stipulate face pressure to tell some more complex story about what distinguishes good from bad attempts to introduce new types of mathematical objects and how mathematicians' cavalier manner of proceeding can be a reliable way of forming true beliefs.

3.2.2 Families of Existing Approaches

A number of different ways of reconciling mathematicians' freedom to stipulate with the point above have been considered in the literature. These approaches can be grouped into families as follows[4].

1. Plenitudinous approaches (e.g. Plenitudinous Platonism, Actualist Set Theoretic Foundationalism)
2. Institutional/Social Constructive Approaches (e.g. Julien Cole)
3. Metasemantic approaches:
    - Hypotheticalist approach to mathematics (e.g. Hellman)
    - Neo-Carnapian/Quantifier Variance

[3]Note that Uzquiano argues for a conflict between set theory and the claim that *all* objects have merological fusions i.e., all objects *including abstract ones like sets*. There is no conflict between set theory and commitment to allowing arbitrary mereological fusions of, say, arbitrary physical objects.

[4]I omit fictionalism from this list of explanatory strategies because fictionalists would deny the explanandum: that mathematicians manage to reliably form true beliefs via making ontologically cavalier stipulations. However close analogs to everything which I say about hypotheticalist views would appear to apply to fictionalism.

Plenitudinous approaches, like classic Set Theoretic Foundationalism and neo-Fregeanism, interpret mathematicians as talking about a fixed but large universe of mathematical objects. They articulate generous 'limits of abstraction' such that all *acceptable* coherent characterizations of mathematical structures will be compatible. Thus, for example, in the case of standard set theoretic foundationalism, acceptable stipulations will be those which have a 'standard' model in the hierarchy of sets. They then argue that all such acceptable mathematical stipulations will express truths because they truly describe suitable portions of a plentiful mathematical universe.

Plenitudinous views face the objection that their choice of *which* limits to impose can seem unmotivated. For example, in the case of standard set theoretic foundationalism, this worry takes the following form: if the hierarchy of sets has some definite height, why doesn't the mathematical structure one would get by adding a layer of classes to this hierarchy constitute an acceptable object for mathematical investigation?

Institutional/Social Constructive approaches like Julien Cole's[6] take mathematical objects to be 'institutional' facets of reality which "exist in virtue of collective agreement" and are, in some sense, created and "sustained in existence by a relevant group of people collectively recognizing or accepting their existence." Just as lawyers can bring companies into being, so too mathematicians can ensure the existence of some suitable collection of mathematical objects just by choosing to accept certain existential claims about such objects.

Adopting an institutional approach allows one to avoid the worries about drawing an unmotivated distinction which beset Plenitudinous accounts. For on this approach *social facts about mathematicians' choices* explain why mathematical objects exist corresponding to some coherent descriptions of putative mathematical objects but not others. Thus, for example, in Uziquiano's case of sets and mereological fusions, philosophers who accept an institutional account of mathematical objects can appeal to the fact that the mathematical community happens to have developed set theory but not mereology to explain why there are objects satisfying Uziquiano's set theoretic axioms but not his meriological ones.

However institutional approaches face a problem about how to make sense of incompatible stipulations. For they take acts of accepting mathematical posits to help ground the existence of mathematical objects, rather than (for example) merely to change the meaning of the positer's words in such a way as to ensure that utterances of certain sentences will henceforth express truths. Accordingly, (it seems that) the total history of the world with regard to mathematical stipulation will need to determine a *single unified domain* of mathematical objects whose behavior determines the truth of all meaningful mathematical statements. But, what happens if different communities of mathematicians get isolated from one another and adopt incompatible stipulations characterizing different kinds of mathematical objects, e.g., sets and mereological fusions as in Uzquiano's case above? One might try to say that objective mathematical reality corresponds to the coherent parcel of mathematical stipulations which gets adopted by the *most* people in the total history of the world. But adopting this proposal appears to generate the absurd consequence that gaining knowledge of the existence of mathematical objects requires gaining knowledge that no future group of people who are inclined to accept incompatible stipulations will ever gain sufficient political power[5].

Metasemantic views deny that all true mathematical utterances are made true by portions of a single universe. Instead they appeal to factors like context dependence and meaning change to explain how making any one of a range of different (potentially incompatible) stipulations characterizing mathematical structures would still have lead one to express truths. At least two major variants of this strategy have been employed in the literature.

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[5]Questions have been raised about how to square such proposals with the idea that mathematical statements can be timelessly and necessarily true. However proponents of these approaches, have attempted to address this worry by noting that standard acts of social construction (such as founding a company or granting an individual some important social status) can take effect retroactively. For example, Cole notes that sports authorities can retroactively rule that a player has been on the 'injured list' for the past two days, and he suggests that mathematical authorities can similarly rule that sets and numbers exist timelessly and amodally.

Hypotheticalist forms of the metasemantic approach hold that the true logical form of a mathematical sentence 'φ' is something like a conditional claim 'if D then φ', where D combines all the descriptions of intended structures of mathematical objects currently in play. These views face the objection that taking mathematical existence claims to have such a different logical structure and semantics and from existence claims about ordinary and scientific objects can seem ad hoc and (ceterus paribus) unattractive. As Stanford Encyclopedia puts it, "the language of mathematics strongly appears to have the same semantic structure as ordinary non-mathematical language... the following two sentences appear to have the same simple semantic structure of a predicate being ascribed to a subject:

(4) Evelyn is prim.

(5) Eleven is prime.

This appearance is also borne out by the standard semantic analyses proposed by linguists and semanticists."[6].

Finally, **Neo-carnapian** approaches hold that coherent stipulations introducing new kinds of mathematical objects can change the meanings of the quantifiers "∃" and "∀", in such a way as to ensure their own truth. Thus, for example, introducing complex numbers might change the meaning of our quantifiers so as to make the sentence 'there is a number which is the square root of -1' go from expressing a false proposition to expressing a true proposition.

On this account, one can reliably form true beliefs by (doing something like) introducing arbitrary stipulative definitions which coherently extend our current mathematical practice by introducing new kinds of mathematical objects. There are a wide range of different quantifier-like senses which expressions like "∃" can take on, where these senses need not be amount to quantifier restrictions of any unique fundamental sense of the quantifier. Coherent mathematical stipulations have the power to shift which of these meanings "∃" has in our current context, in such a way as to ensure that the stipulated sentences will now express truths in

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[6]This example comes from [3] pg. 288 via [22], but the point goes back to [1]

this context.

Neo-carnapian accounts avoid the problem for hypothetical accounts noted above. They honor our desire for a uniform account of the meaning and logical form of existence claims about mathematical objects and grammatically similar existence claims involving scientific and ordinary objects like holes, shadows, electrons etc. For they say that a single notion of existence is relevant to different ordinary language existence claims "Evelyn is prim." and "Eleven is prime." in any given context, though future choices to start talking in terms of new kinds of mathematical objects (or sociological objects like countries or literary objects like genres) may change exactly which notion of existence this is [7].

Neo-carnapian accounts also avoid institutional views' problems with incompatible mathematical stipulation. For suppose that different communities like (say) Russian and American mathematicians during the cold war made different, internally coherent but apparently incompatible, choices of which mathematical objects to talk in terms of. The neo-carnapian approach allows us to say that both communities would still express truths: the Russians' mathematical stipulations will give their inscriptions of "$\exists$" one meaning ($\exists_R$) while the Americans' stipulations will give their inscriptions of "$\exists$" another ($\exists_A$)[8].

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[7]Relatedly, one should note that that using quantifier variance does not require one to accept that normal English employs verbally different expressions corresponding to at least two different quantifier senses (a metaphysically natural and demanding one and a laxer one), so that it might be true to say things bad-sounding things like "composite objects exist but they do not really exist" in certain contexts. With regard to any particular context we can fully agree with David Lewis that, "The several idioms of what we call 'existential' quantification are entirely synonymous and interchangeable. It does not matter whether you say 'some things are donkeys' or 'there are donkeys' or 'donkeys exist'...whether true or whether false all three statements stand or fall together.".

[8]Accepting a neo-carnapian view also lets us avoid awkwardness institutional views face in the capturing the standard claim that mathematical objects exist necessarily and at all times. For the neo-carnapian makes no claim that mathematicians' acts of stipulation create or sustain mathematical objects in existence. Rather they say that these acts of mathematical stipulations introducing M give 'there are' a meaning on 'there are Ms' express a truth at all times in all possible worlds. See [8] for details.

Finally, neo-carnapian approaches avoid the problems about unmotivated distinctions which beset plenitudinous approaches by appealing to variant quantifier senses which are not mere restrictions of a maximal fundamental one. Accepting this kind of quantifier variance allows one to say that all coherent mathematical posits would succeed in changing the meaning of one's quantifiers in such a way as to ensure their own truth – even if it is not the case that all such posits can be understood as describing portions of a single large universe.

One might think that giving a neo-carnapian explanation of mathematicians' freedom to stipulate commits one to a strong carnapian position on metaontology: that all ontological debates are somehow defective. However it is important to note that this is not so. For philosophers on both sides of recent debates about the defectiveness of ontology have been motivated to allow *non-metaphysicians* substantial freedom to deploy variant senses of the quantifier which do not amount to mere quantifier restrictions of some maximally natural unrestricted notion of existence.

Friends of ontology like Sider have used quantifier variance (between the street and the philosophy room) to capture the intuition that plumbers' utterances like 'there's a hole in a sink' can express uncontroversially true statements, despite the fact that there's a deep open question about what exists in the more fundamental sense relative to the metaphysics room. They say that there is a unique maximally natural sense of the quantifier which ontologists aim to exploit, and that it is a deep open question whether holes exist in this sense. However they allow that there is also a different (potentially non-maximally-ontologically insightful) sense which the quantifier can take on in the context of ordinary life/plumbing discussions, on which sentences like 'There are more holes in this pipe than that one' can uncontroversially express truths.

Foes of ontology like Hersh have invoke quantifier variance (between different acceptable precisifications of language uses on the street) to explain how ontological discussions about whether certain kinds of objects exist can be defective. They combine quantifier variance with either a parity claim that all different senses of the quantifier are on par, or rejection of appeals to the relative naturalness of quantifiers as meaningless. Accordingly they maintain that we cannot succeed in giving the quantifier a special maximally natural meaning in the metaphysics room which comes apart from ordinary practice, and that in cases where ordinary practice is undecided between different variant senses of the quantifier there there may be no right answer to existence questions.

Thus both the friends and foes of ontology have been inclined allow non-metaphysicians substantial freedom to deploy variant senses of the quantifier which do not amount to mere quantifier restriction of some maximally natural unrestricted notion of existence. And merely allowing *this* role for Quantifier Variance suffices to let us explain mathematicians' freedom to introduce new objects by saying that mathematical stipulations shift the sense of our quantifiers (in ordinary non-mathematical contexts) in such a way as to ensure their own truth].

This is not to say that neo-carnapian approaches face no objections. One serious worry about such approaches (which will be the main focus of this paper) concerns the intelligibility of the kind of powerful variant quantifier senses which the neo-carnapian explanation needs to posit. Nearly everyone will allow that expressions like 'there is' can sometimes take on a *restricted* sense, as when someone says "all the beers are in the fridge". However many philosophers are inclined doubt the intelligibility of appeals to alternative quantifier-like senses for "∃" which are not mere restrictions of some fundamental, most generous sense of the quantifier. For example Wright and Hale claim not to understand what such senses of the quantifier "just what ...the postulated variant quantifier meanings [are] supposed to be."[13], and challenge philosophers who appeal to quantifier variance to flesh out their view by providing an explanation of, "why the allegedly different quantifiers which can all be expressed by the words 'there are' are quantifiers and ... how they differ in meaning."[13] [Insofar as we saw that one of the main benefits of the neo-carnapian approach over plenitudinous views arises from the fact that one is not forced to posit a single universe within which standard models for all acceptable characterizations of mathematical objects fit inside, it will be important to defend this claim.]

I think some ideas from David Chalmers' 'Ontological Antirealism'[5] provide an attractive strategy for answering Wright and Hale's general worry. In the section below I will review a (simplified) version of Chalmers' proposal. I will then note that, unfortuantely, Chalmers' story does not allow us to describe *enough* variation in quantifier meaning for the needs of the neo-carnapian explanation of mathematicians' freedom just described. The rest of the paper will then be devoted to developing a theory of how mathematical stipulation works and what variant senses are available to the quantifier which is up to the task at hand.

3.3 Chalmers' Proposal and the Problem of Size

Chalmers' description of variant quantifier senses begin with a kind of Fregean idea about how to explain the meaning of a logical connective. We can describe what variant senses of the quantifier might be like by describing how these variant quantifier senses could systematically contribute to the truth conditions for propositions in which they figure[9].

We can describe variant quantifier senses $\exists_c$ by associating them with "furnishing functions" F_c. Furnishing functions pair metaphysically possible worlds with set theoretic models. For each possible world w, the set theoretic model $F_c(w)$ specifies how many objects will count as 'existing' in the relevant sense $\exists_c$ at that possible world, and how various properties will apply to these objects. Thus, for example, if it is true to say in shop class that there are holes, then the furnishing function associated with English as spoken in shop class will associate the actual world with a set theoretic model which assigns the property HOLE a non-empty extension.

Thinking about furnishing functions allows us to use our own *current* sense of

[9] Here I am assuming that one can satisfactorally explain the meaning of a logical connective merely by expalining how it systematically contributes to the truth or falsity of sentences in which it figures. If one denies this it is hard to see how we could give any explanation for the meaning of $\wedge$, $\vee$ etc.

the quantifier to systematically describe truth conditions for utterances employing alternative senses of the quantifier, as follows. The truth or falsehood of a proposition φ employing some alternative quantifier meaning $\exists_c$ at a possible world w can be determined by starting with the set theoretic model $f_c(w)$'s domain and extensions for atomic properties and relations, and then applying standard recursive rules for the logical connectives. Thus, for example, a proposition of the form $\exists_c F(x) \& G(x)$ will be true at exactly those possible worlds w such that $f_c(w)$ assigns some object to the extension of both properties F and G.

Chalmers also suggests that there are constraints on "admissible" furnishing functions, such that not all furnishing functions correspond to genuine alternative senses of the quantifier. He doesn't say much about what the constraints would be, but the key idea is that we want to allow that varying the sense of the quantifier can change the extensions associated with properties in some ways but not in other ways. Thus, for example, quantifier senses which allow arbitrary mereological fusions can differ from those which don't with regard to how many objects have the property RED at various possible worlds(in e.g. counting the mereological fusion of two red balls as an additional red object). But perhaps there is is no variant quantifier sense which allows a single object to count as being both RED and GREEN ie. no sense whose whose furnishing function assigns some object to the extensions of both RED and GREEN at some possible world.[foonote re: slight abuse of notation here]

Using the machinery above allows us to concretely describe possible quantifier-like meanings for "$\exists$" which are *not* mere restrictions on whatever sense of the quantifier we are currently using. Note, for example, that merely imposing quantifier restrictions can never make a proposition of the form "$\exists x F(x) \& G(x)$"[10] go from being false to being true. In contrast, we *can* decribe such changes in truth value using Chalmers' model theoretic strategy. We simply consider furnishing functions which associate the actual world with a set

[10] where F and G are atomic predicates

theoretic model which assigns some object to the extension of both properties F and G. The fact that there is actually no overlap in the objects which have properties F and G does not prevent there from being such a set theoretic model which (in effect) assigns these properties overlapping extensions.

In this way, Chalmers' strategy allows us to use whatever Wright and Hale take to be the fundamental unrestricted sense of the quantifier to describe alternative quantifier senses which are not mere restrictions of this sense. Suppose, for example, that we are currently speaking a language in which there are not arbitrary mereological fusions (so "there is something which has a part that's a nose and a part that's an Eiffel tower" expresses a falsehood). Plausibly we cannot describe contexts in which this sentence comes out true merely by appealing to the possibility of quantifier restriction, for the reasons noted above. However we can describe a sutiable alternate quantifier sense for "there is" to take on in this context using Chalmers' strategy[11].

Thus I think Chalmers' story provides a clear and satisfying response to Wright and Hale's worry. It explains how the above variant quantifier senses differ from one another; if $\exists_1$ and $\exists_2$ are quantifier senses associated with suitably different suitably furnishing functions, then the propositions $\exists_1 F(x) \& G(x)$ and $\exists_2 F(x) \& G(x)$ will be true at different sets of possible worlds. It also provides a clear sense in which these variant quantifier senses are all quantifier-like. For within any language that uses one of the variant senses proposed for "$\exists$" reasoning in accordance with standard first order logical inference rules will always be truth preserving. Thus all changes in quantifier meaning between the variants which Chalmers describes will preserve "$\exists$"'s usual inferential role.

Unfortunately, however, Chalmers' method does not permit us to describe *sufficient* quantifier variation for the needs of the neo-Carnapian explanation of mathematicians' freedom to stipulate sketched above. The problem is that

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[11]Similarly, if one supposes that the fundamental language makes it true to say that there are mereological fusions but not fictional objects so "there is something which is both a mountain and created by J.R.R. Tolken" comes out false, we will be able to use Chalmers' strategy to describe a meaning which the quantifier could take on which this sentence comes out true.

Chalmers specifies how each quantifier sense contributes to truth conditions for sentences by associating possible worlds with set theoretic models. This prevents us from using his method to explain how (intuitively acceptable) mathematical stipulations which require the existence of mathematical structures too large to have models within the hierarchy of sets can change the meaning of the "∃" in such a way as to ensure their own truth[12]. Accordingly Chalmers' method will not suffice to describe the kind of quantifier variance which the neo-carnapian explanation of mathematicians' freedom to stipulate needs.

3.4 Combinatorial Possibility

In this paper I will outline an alternative method for describing the kind of variant quantifier senses which the neo-carnapian explanation wants to associate with mathematical posits. This proposal will use something like Chalmers' strategy for describing alternate quantifier senses. However it will attempt to avoid problems of size by replacing appeal to set theoretic models with appeal to an intuitive and independently-motivated notion of broadly logical possibility and exploiting our possession of principles which categorically describe the intended structure of some new kinds of objects which we might choose to talk in terms of to describe the behavior of the variant quantifier sense which "∃" would take on, should we choose to talk in terms of such objects.

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Admittedly, by Skolem's theorem one can find a countable model for any (consistent) first-order theory in a countable language. However, intuitively it appears that we can mean the standard model of the numbers, which can't be specified in first order logic. Accordingly many philosophers have found it appealing to allow the use of more powerful logical vocabulary, such as second order logical quantification, which (unlike the vocabulary of first order logic) has the power to uniquely characterize the intended structure of the natural numbers. Accordingly, if we follow Chalmers' idea of taking logical vocabulary (including this powerful logical vocabulary) to apply in a straightforward way within the relevant domains, then appealing to Skolem's theorem won't guarantee the existence of suitable model. Furthermore, if you think that you can use this rich logical vocabulary to categorically describe the hierarchy of sets itself, then it must be possible to formulate coherent claims (corresponding to acceptable mathematical stipulations) which don't admit a set model.

In the philosophy of mathematics literature, many people have taken an interest in an expressively powerful notion of logical possibility/coherence[13] with something like the following three features.

- It differs from mere syntactic consistency in that descriptions employing powerful second-order vocabulary (or the equivalent) will qualify as requiring something logically impossible because these descriptions cannot be satisfied by any pattern of relationships between objects, despite the fact that no contradiction can be derived from them via first order logic or any formal system of inferences which we currently accept.

- It is fundamentally modal, in the sense that facts about logical possibility are not taken to require grounding in the existence of anything like a model within the hierarchy of sets[14].

- It is relativisable, in the sense that one can talk about what is logically possible 'given' the facts about how certain predicates apply to the actual world. Thus, for example, we can say things like, "Given what kittens and baskets there are, it is not logically possible that each kitten is sleeping in a distinct basket" which will be true if and only if there are at least as many baskets as kittens in the actual world[15].

This notion of broadly logical possibility has been used for a number of purposes.

- to characterize acceptable mathematical posits

- to provide hypotheticalist paraphrases for statements of applied mathematics e.g.

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[13]See, for example, Hellman [15] and Shapiro[26]. The details of my presentation are most influenced by Hellman.

[14]Indeed if we take the hierarchy of sets to have a definite structure and supposed that we are able to uniquely describe the intended structure of the hierarchy of sets then this description will be something which is broadly logically possible but has no model inside the hierarchy of sets

[15]Hellman certainly needs something like this. It is less clear that Shapiro does.

- to respond to the Burelli-Forti paradox by reinterpreting set theory as an investigation of broadly logical extendability[16].

I will call this notion of broadly logical possibility/coherence 'combinatorial' possibility[17].

I propose that we can use this notion to describe the behavior of alternative quantifier senses, in much the way that Chalmers uses facts about furnishing functions and set theoretic models. Hypotheticalists like Hellman use statements about combinatorial possibility to articulate what they claim is the true logical form of mathematical statements (and thus show that these are not ontologically committal). I propose to use similar statements as a tool for using the current meaning of "$\exists$" to describe alternative quantifier senses, by systematically describing truth conditions for sentences in contexts where "$\exists$" takes on these senses[18]

Rather than saying, "there are numbers" expresses a proposition whose true structure is a statement about combinatorial posibility (and hence the objections to failure of uniform semantics noted above) we allow that this statement has the same logical form as "there are birds". However we say that acts of stipulative definition can change the meaning which "there are" takes on when either of the above sentences is uttered (outside the metaphysics room). We then invoke the statements about combinatorial possibility merely as a useful tool for systematically capturing the truth conditions for such existence claims in different contexts.

The story about stipulative definitions' power to change quantifier meaning which I am going to propose takes the following form. It describes how certain

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[16]See [14] and [15]

[17]Although doubts about the intelligibility of appeals to combinatorial possibility have sometimes been raised, I think one can make a strong case that facts about combinatorial possibility deserve approximately as good standing as notions from set theory. See the appendix to Paper 2 for many more details on the language of combinatorial possibility and set theory.

[18]Arguably this strategy aligns well with Putnam's claim that mathematical existence statements are true, but just that its also possible to take a modal perspective on them.

explicit acts of stipulation can change the way that expressions like "there is" and "∃" interact with *words that name* properties and relations (e.g. predicates and relation symbols if we are speaking some interpretation of the language of first order logic) to determine the truth or falsity of sentences. It makes no claims about when such changes preserve or modify the meaning of these property and relation expressions. This is an important feature because it allows us to defend the key claim needed for our neo-carnapian explanation (that certain stipulative definitions can change the meanings of our quantifier expressions in such a way as to ensure their own truth[19]), without having to decide vexed questions about when acts of stipulative definition also count as changing the meaning of property and relation expressions. For example, it allows us to say that stipulative definitions introducing 'holes' can succeed in changing the meaning of the quantifier expressions in such a way as to make sentences like "there are at least 1,000 things located in Grant's tomb" express truths, without deciding whether such stipulations count as changing the meaning of "located in" – or even whether there is a determinate right answer to this question.

3.5 Stipulative Definition and Quantifier Variance

Unfortunately natural languages are far too large and messy for me to explicitly describe the effects of stipulation in such languages within the constraints of this paper. Instead, I will consider a syntactically simpler formal language which retains various relevant features of English. I will describe how acts of stipulative definition can change the meanings of expressions in this languages by describing how they effect truth conditions for sentences, and let the reader extend this picture to the messier case of natural language by analogy.

I will consider a formal language $\mathscr{L}$ which supplements the first order logical connectives with subscriptable combinatorial possibility operators $\Box_{P_1 \ldots P_n}$ and

[19]And leave us with an overall language that has various good properties, such as preserving the truth-preserving-ness of the inference rules of standard first order logical.

$\Diamond_{P1\ldots Pn}$[20][21].

Definition. *Start by fixing some countable infinite collection R^k_n of relation symbols for $k, n \in \omega$ with R^k_n of arity $k > 0$. We can now define the formal language $\mathscr{L}$ to be the smallest class with the following properties:*

1. *If φ is a formula of first order logic with equality in the language containing the relation symbols R^k_n but no function or constant symbols then $\varphi \in \mathscr{L}$*
2. *If $\varphi \in \mathscr{L}$ then $\exists x\varphi$, $\forall x\varphi$ and $\neg\varphi$ are in $\mathscr{L}$*
3. *If φ, ψ in $\mathscr{L}$ then $\varphi \vee \psi$ and $\varphi \& \psi$ are in $\mathscr{L}$*
4. *If $P_1 \ldots P_n$ is a finite list of relation symbols chosen from the R^k_n and $\varphi \in \mathscr{L}$ then $\Diamond_{P_1\ldots P_n}\varphi$ and $\Box_{P_1\ldots P_n}\varphi$ are in $\mathscr{L}$*
5. *If φ is a quantifier free sentence in $\mathscr{L}$ then $\Diamond_m$ is in $\mathscr{L}$*

As usual, a sentence in $\mathscr{L}$ is a formula in $\mathscr{L}$ without any free variables[22].

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I will then appeal to a notion of interpretations of $\mathscr{L}$. These interpretations associate sentences in the formal language of $\mathscr{L}$ with truth conditions[23]. In order

[20]See the appendix on combinatorial possibility for an argument that all structures in mainstream mathematics can be categorically described in the language of combinatorial possibility (except perhaps for the appeals to the hierarchy of sets which there are independent reasons (related to the Burelli-Forti paradox) for thinking may be better understood in a potentialist fashion rather than as appeals to any definite logically possible structure of objects.

[21]The language of first order logic might seem like an obvious choice for the this purpose. However, as noted above, one cannot uniquely describe the intended structure of various mathematical objects, like the natural numbers, using (standard interpretations of) the first order logical connectives alone. Hence my choice of $\mathscr{L}$. Arguably the same work could be done by adding other notions like second order quantifiers or plural quantifiers to our language instead of the above combinatorial possibility opperators.

[22]The free variables in φ are given by exactly the same rules as the free variables for a first order language i.e. the free variables of $\Diamond_{\mathscr{L}}\varphi$ are the same as the free variables of φ.

[23]Since my aim is to give a simple model I am restricting attention to interpretations of $\mathscr{L}$ which lacking all expressions like 'here', 'now' 'for all we know' which might prevent us from associating sentences directly with sets of possible world.

to model the fact that stipulative definitions can give meaning to hitherto meaningless words, I will think of these interpretations as designating finitely many of the above relation symbols in $\mathscr{L}$ as meaningful, and then determining truth conditions for all well formed-sentences which only use meaningful relation symbols outside the $\Diamond$ and $\Box$ operators[24] – in a way that preserves the inference rules associated with $\exists$ and $\forall$ and the meaning of all other logical connectives. To simplify matters further, I will only discuss the effects of stipulations made in interpretations of $\mathscr{L}$ on which some a range of atomic predicates $P_1 \ldots P_n$ behave like an exhaustive list of the different 'kinds' of objects, in the sense that "$\forall x P_1(x) \vee \ldots P_n(x)$" comes out true. With this range of interpretations of $\mathscr{L}$ in mind, let us turn to the question of how stipulative definitions can change meanings.

When one gives a stipulative definition, one typically wants to use the existing meaning of some terms to give new meaning to other terms. Accordingly, one might be inclined to represent acts of stipulative definition with ordered pairs of the form $\langle S, N \rangle$ where S is a sentence which the stipulative definition aims to make true, and N is a finite list of relation symbols which the stipulative definition aims to define (or redefine)[25]. The presence of N in this representation is helpful in capturing acts of stipulative re-definition. For it allows us to distinguish the terms which a stipulative definition aims to define from those which it uses to do the defining.

However, if we want to allow stipulative definitions which introduce holes and locate them in space, without permitting these stipulations to change facts about how 'located at' applies to previously-recognized objects like tables and particles, we will want to recognize two more dimensions along which acts of stipulative definition can vary. I will add a term Q which captures whether a given stipulation is 'empowered' to change the meaning of the quantifier or not. I will also add a

[24]For these purposes I am taking the predicates $P_1..P_n$ subscripted in an expression of the form $\Diamond_{P_1..P_n}\varphi$ operator to qualify as 'outside' the diamond.

[25]Thus, for example, in the standard definition of $+$, S would be "for all numbers n and m $n + 0 = n$ and $n + S(m) = S(n + m)$", and N would be $+$.

finite set E of relation symbols whose application can be extended provided that their application to currently-acknowledged objects does not change.

Thus, I will use quadruples of the form $\langle S, N, Q, E\rangle$ to represent different possible types of stipulative definition. Here

- S is a sentence which the stipulative definition aims to make true[26].
- N is a list of relation-symbols which are being wholly redefined by the stipulative definition.
- Q is 1 if the stipulation definition is empowered to change the meaning of the quantifier in attempting to make S true and 0 otherwise.
- E is a list of relation-symbols whose application may be extended by the stipulative definition.

I will characterize a range of 'acceptable' stipulative definitions to make while speaking a given interpretation I_0 of $\mathscr{L}$, and then describe how making such stipulations can change the meaning of terms in $\mathscr{L}$. In calling these stipulations acceptable, I do not mean to suggest that it would be wise - or even reasonable - to make such stipulations, but only that (as we will see below) one could succeed in learning that S by making this stipulation[27].

Informally speaking, a stipulative definition $\langle S, N, 1, E\rangle$ is acceptable iff two conditions are satisfied. First one can ensure that S expresses a truth just by:

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[26]Note that S may take the form of a conjunction of definition-like statements involving the terms in N with a finite list of principles whose truth is to be preserved when the application of predicates and relations-symbols in E is extended.

[27]In many cases there are strong pragmatic reasons for not making a stipulation despite the making it would provide knowledge of the above kind. For example, a stipulation might ensure the truth of S at the cost of potentially changing the meaning of various other terms in one's language that one can no longer be sure that many of the sentences which one previously accepted express truths. For example one can perhaps stipulate that the meaning of life is to be extended in such a way as to ensure that 'there is life on mars' express a truth. Doing so (arguably) provides a reliable method of gaining knowledge of whatever proposition is now expressed by 'there is life on mars' after one's stipulation. But this gain in knowledge comes at the cost of the loss of knowledge that e.g. various sentences universal generalizations about 'life' which one had previously accepted will continue to express truths.

increasing the total size of the universe, (suitably) extending the application of relation symbols in E, determining suitable extensions for the relation symbols in N and leaving the interpretation of the other relation symbols in our language unchanged.

Definition. *A stipulative definition* $\langle S, N, 1, E\rangle$ *is* ***acceptable*** *relative to* I_0 *iff*

$\Diamond_{R_1 \ldots R_n} \mathcal{E}_1 \& \ldots \mathcal{E}_n \& \widehat{S}$

is true on I_0*, where* $R_1 \ldots R_n$ *is a list of all meaningful atomic predicates which don't occur in N,* $E = \{E_1, \ldots E_n\}$*,* $E'_1 \ldots E'_n$ *is a list of otherwise unused relations where each* E'_i *has the same arity as* E_i*, for each k-place relation* E_i
$\mathcal{E}_i = \forall x_1 \ldots x_k [(P_1(x_1) \vee \ldots P_n(x_1)) \& \ldots \& (P_1(x_k) \vee \ldots P_n(x_k))] \rightarrow [E_i(x_1 \ldots x_k) \leftrightarrow E'_i(x_1 \ldots x_k)]$ *and the symbol* $\widehat{\varphi}$ *abbreviates the sentence you get by starting with* φ *and replacing each occurrence of some* E_i *with the corresponding* E'_i*.*

Now I claim that making an acceptable stipulation $\langle S, N, 1, E\rangle$ while speaking some interpretation I_0 of $\mathcal{L}$ changes the meaning of one's quantifier and relation symbols as follows. A sentence φ will be true under the new interpretation I_1 which is relevant to one's utterances after making this stipulation if it's a matter of combinatorially necessary that, were the universe (and the application of predicates in E) to be extended in such a way as to make S true, then φ would have to be true as well.

More formally, for all sentences φ:

φ is true on I_1 iff $\Box_{R_1 \ldots R_n}(\mathcal{E}_1 \& \ldots \mathcal{E}_n \& \widehat{S} \rightarrow \widehat{\varphi})$ is true on I_0

Thus, making any acceptable stipulation $\langle S, N, 1, E\rangle$ will ensure that S comes to express a (metaphysically necessary) truth.

If the above equivalence is correct, we can exploit it to describe the kind of alternative senses of the quantifier which the neo-carnapian explanation wants to posit, by describing how they contribute to the truth conditions of sentences.

Like Chalmers' set theoretic proposal, the above story explains the meaning of alternative senses for the quantifier by describing how these senses systematically contribute to truth conditions for a larger unit (in this case, a sentence). It also lets us explain why these alternative notions are quantifier-like, by noting that standard inference rules for the quantifiers will remain truth-preserving[28].

Unlike Chalmers' account however, the above story lets us describe senses of 'exists' which demand the 'existence' of mathematical structures which have no model within the mathematical structures we accept. Appealing to the fundamentally modal notion of combinatorial possibility allows us to crisply describe truth conditions for statements involving variant ontologically profligate senses of the quantifier. In particular, it allows us to explain how mathematical stipulations characterizing structures which are too large to have any (standard) models within the hierarchy of sets could nonetheless succeed in changing the meaning of our quantifiers in such a way as to ensure their own truth.

3.5.1 Examples

To make it plausible that the above theory provides a good model for the effects of acts of stipulation introducing new kinds of mathematical objects (and demonstrate how the various elements in it function), let me work through an example. Suppose that our base language does not 'talk in terms of' numbers. Suppose further that there are 11 rats.

One can give a categorical description of the intended structure of the numbers (how number(), S(), plus(), times() apply) in the language of combinatorial possibility - I will call this $PA_\Diamond$[29]. One can then use this to formulate an ontologically empowered stipulation which introduces talk of numbers as follows: $\langle NUMS, \{\text{number, S, plus, times}\}, 1, \emptyset\rangle$.

NUMS:
$PA_\Diamond \& \forall x[P_1(x) \vee P_2(x) \ldots P_n(x) \vee \text{number}(x)] \& \forall x[P_1(x) \vee P_2(x) \ldots P_n(x) \rightarrow \neg\text{number}(x) \& \neg\exists y S(x,y) \& \neg\exists y S(y,x) \& \ldots]$

[28] Note that these rules remain truth preserving even in cases where my theory does not generate classical truth conditions, but rather leaves truth-value gaps because the relevant stipulation fails to characterize a unique structure.

[29] See Paper 2 for one way of doing this

NUMS essentially says: there are numbers related to one another as per $PA_{\Diamond}$, all objects are are either numbers or of some type $P_1 \ldots P_n$, the numbers are distinct from all the different types of objects $P_1 \ldots P_n$ which we are currently talking in terms of, and S(), plus(,) times(,) only apply to the numbers.

Plausibly there will be an interpretation I_0 which attractively models our language such that $\langle NUMS, \{\text{number, S, plus, times}\}, 1, \emptyset \rangle$ qualifies as acceptable stipulative definition relative to I_0[30] . Let $\mathcal{M}(I_0)$ be the list of atomic predicate and relation symbols which our current interpreted language accepts as meaningful. Let $P_1 \ldots P_n$ be a list of predicates which behave like an exhaustive list of object types under I_0, as per the assumption above.

Now, the theory I have just outlined implies that stipulating $\langle NUMS, \{\text{number, S, plus, times}\}, 1, \emptyset \rangle$ while speaking I_0 would shift one to an interpretation I_1 of $\mathscr{L}$ such that:

φ is true in I_1 iff $\Box_{R_1 \ldots R_n}(\mathcal{E}_1 \& \ldots \mathcal{E}_n \& \widehat{NUMS} \rightarrow \widehat{\varphi})$ is true on I_0

How plausible is this as a description of the results of making a stipulative definition which introduces numbers?

Note that because $\langle NUMS, \{\text{number, S, plus, times}\}, 1, \emptyset \rangle$ is not permitted to extend any atomic relation symbols (the fourth term above is the $\emptyset$), the above equivalence works out to

φ is true on I_1 iff $\Box_{R_1 \ldots R_n}(\text{NUMS} \rightarrow \varphi)$ is true on I_0

Also note that $R_1 \ldots R_n = \mathcal{M}(I_0) - N$, so $R_1 \ldots R_n$ includes all the relation symbols like 'rat()' which the I_0 that models our our materialistic language treats as meaningful *except* for the expressions 'number()', 'S()', 'plus(,)', 'times(,)'[31], which the above stipulative definition attempts to define.

Now let us think about some different kinds of sentences:

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[30]The claim that $\langle NUMS, \{\text{number, S, plus, times}\}, 1, \emptyset \rangle$ is acceptable in I_0 a) it is combinatorially possible, given the facts about how all atomic predicates in current use apply, for there to be objects with the intended structure of the numbers and b) Combining *NUMS* with the requirement to hold fixed all the relations in $\mathcal{M}(I_0)$ but not {number, S, plus, times} suffices to pin down a unique structure for universe, and way for all henceforth-meaningful relation symbols (i.e. those in either $\mathcal{M}(I_0)$ or {number, S, plus, times}) to apply etc.

[31]if any these should happen to be in $\mathcal{M}(I_0)$

First consider a purely mathematical statements about the numbers which are not explicitly included in NUMS, like PRIMES: 'There are infinitely many prime numbers[32]' Because *NUMS* includes a categorical description of the numbers $PA_{\Diamond}$, it's combinatorially necessary that if number(), S(), plus(,) etc apply as per *NUMS* then there are infinitely many prime numbers. Thus we have $\Box(NUMS \rightarrow PRIMES)$. We also have the weaker statement that it is combinatoriallly necessary, given the application of any relations, that $NUMS \rightarrow PRIMES$. Thus we have $\Box_{R_1 \ldots R_n}(NUMS \rightarrow PRIMES)$ as desired.

More generally, note that this account explains how facts about objects introduced into our language by stipulative definition can, in some cases, outrun our ability to prove claims about these objects. Intuitively we'd like all statements in the language of number theory to have definite truth conditions. *NUMS* includes a categorical description of the intended structure of the numbers, so for every sentence φ in the language of number theory, either $\Box(NUMS \rightarrow \varphi)$ or $\Box(NUMS \rightarrow \neg\varphi)$. Accordingly the paraphrase indicated above does indeed ensure that for every sentence φ in the language of number theory either φ or $\neg\varphi$ comes out true.

Next consider a purely physical sentence like "$\exists x\ \mathrm{rat}(x)$". My theory predicts that this sentence will express a truth in I_1, as follows. There are eleven rats. So, given what rats there are, it's combinatorially necessary that $\exists x rat(x)$. Accordingly it's combinatorially necessary, given the facts about how 'rat()' and *a range of other relation sybols* in $R_1 \ldots R_n$ apply, that there $\exists x rat(x)$. Thus $\Box_{R_1 \ldots R_n} \exists x\ \mathrm{rat}(x)$ and $\Box_{R_1 .. rats() .. R_n}[NUMS \rightarrow \exists x\ \mathrm{rat}(x)]$

Finally consider the simple applied mathematical statement, 'There are a prime number of rats' (i.e., 'It would be combinatorially possible, fixing what rats and numbers there are, for the rats to be bijectively paired with an initial segment of the natural numbers up to some number $n - 1$ where n is prime). This statement comes out true because, in essence, the existence of 11 rats makes it broadly

[32]Strictly speaking this statement and 'There are a prime number of rats' are imprecise natural language descriptions requiring translation into $\mathscr{L}$.

logically necessary, given what rats there are, that if there are numbers as per NUMS as well, then it is combinatorially possible (given what rats and numbers there are) to pair up rats with the numbers below 11, or without 11 being prime[33].

Thus I think the story above yields intuitively appealing verdicts about how acts of ontologically empowered stipulation can change the meaning of our words in various contexts.

3.6 Explaining Mathematicians' Freedom To Stipulate

Now that the above model for how ontologically empowered stipulative definitions can change the meaning of our quantifiers (in a way that evades the size limits noted above), is on the table, let us return to the topic of explaining mathematicians' freedom to introduce new kinds of mathematical objects.

I want to propose that one can explain this freedom by saying that what mathematicians do in these circumstances *functions like* a process of explicit ontologically empowered stipulative definition and deduction. Admittedly, as Quine and others have emphasized, there can be some vagueness and arbitrariness about whether, for example, we act more like we *stipulated* that the numbers satisfy induction and then *derived* the least number principle or like we stipulated that the numbers satisfy the least number principle and then derived that they satisfy induction. Nonetheless noting that an explicit process of stipulation and deduction could produce just the kind of easy a priori access to existence claims about new mathematical objects which mathematicians seem to have has substantial power to dispel the apparent mystery of this easy access – even if one acknowledges the actual story is a little messier.

One might worry the neo-carnapian explanation of mathematicians' freedom to introduce new objects 'proves too much', leaving us with a mystery about why we don't have equally easy access to ordinary objects. However I think popular

[33]By 11 I mean, of course, the 11th successor of the number (0) which is not a successor of anything.

structuralist ideas about the nature and aim of mathematics provide a natural response to this worry.

In all areas, plumbing, sociology and mathematics, we are free to do something that functions like ontologically empowered stipulation: to say that whenever matter behaves like such-and-such there will qualify as being a hole, whenever people behave like so-and-so there will qualify as being a country etc. However I propose that our general ability to start talking in terms of new kinds of objects combines with some more distinctive facts about the nature and aims of mathematics to explain the impression that mathematicians have distinctively easy access to new kinds of objects.

Typically when starting to talk in terms of new kinds of objects (like diseases or countries) we aim to track or describe some kind of pattern in the contingent structure of the world and how previously existing terms apply. Thus we don't directly specify a single structure which we intend the newly introduced objects to have. Rather we specify how facts about how the existence of new kinds of objects are supposed to track facts about the application of old properties and relations. We may also make stipulations which under-determine certain facts about how objects of the new kind are to be individuated, in hopes of deferring to intrinsic structure in the world which makes e.g. one way of individuating diseases more natural than others.

Matters are different however, when we turn to the objects of pure mathematics: pure sets, numbers etc. Here our core aim is to investigate necessary truths about what structures (of some objects being related by some relations) are logico-mathematically coherent. As the popular structuralist slogan puts it. 'mathematics is the science of structure'[34].

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[34]For example, all (or nearly all) intuitively mathematical disputes (even disputes about questions like the continuum hypothesis which cannot be decided using standard proof procedures) can be cashed out in terms of pure questions in the language of combinatorial possibility (i.e. questions where all atomic predicate symbols occur inside at least one unsubscripted $\Box$ or $\Diamond$ operator) in a way that intuitively captures what participants are disagreeing about. In contrast, disputes which are intuitively non-mathematical, such as disputes about whether there are really numbers, or which numbers are to be identified with sets, are distinguished by the fact that

Given this fact about the nature of mathematical practice, our aim introducing new kinds of pure mathematical objects like complex numbers or pure sets is not to track facts about how other properties apply, but rather to 'witness' facts about logico-matheamtical possibility. We do this by talking in terms of objects which instantiate certain elegant and interesting descriptions of combinatorially possible structures. Accordingly it makes sense for mathematicians to directly specify the intended structure of mathematical objects[35] rather than merely specifying claims which relate existence facts about new mathematical objects to facts about the behavior of other kinds of objects. As a result, postulates introducing mathematical objects typically allow us to derive existence claims as immediate combinatorially necessary consequences of the fact that the newly introduced objects satisfy our stipulations, rather than merely letting us derive conditionals like 'if the world is like such-and-such then there are so and so holes'.

Thus, our general power to introduce ontologically empowered stipulative definitions yields knowledge of existence in mathematics much more than in other areas because mathematics' nature as the science of structure allows stipulative definitions which directly specify facts about the existence and behavior of new types of objects to be much more useful than they (typically) are in other area in other areas.

3.7 Conclusion

In this paper I have attempted to develop the neo-carnapian story which uses the idea that the quantifier can take on different meanings in different contexts to explain mathematicians' freedom to stipulate. A major source of objection to this approach in the literature was that no sense could be made of the kind of alternative, extremely ontologically profligate, senses for the quantifier which this

although they employ mathematical vocabulary, there is no statement of pure combinatorial possibility which the disputants would disagree about.

[35]Or to specify their structure via specifying an intended relationship between newly introduced types of objects other mathematical objects whose structure is directly specified.

account required one to posit. I suggested that Chalmers' (essentially) set theoretic description of variant quantifiers' contribution to the truth conditions for sentences sufficed to address worries about the intelligibility of ontologically profligate senses for the quantifier, but did not allow us to make sense of of the extremely ontologically profligate quantifier senses needed to explain mathematicians' freedom to stipulate. I then outlined a variant method for describing alternative ontologically profligate senses of the quantifier which replaced appeals to set theory with appeals to combinatorial possibility. This alternative sense of the quantifier allowed one to describe the behavior of a sufficiently broad range of alternative quantifier senses to vindicate the original idea that all coherent extensions of a given mathematical practice would express truths.

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Colophon

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