Tetrahedral Colloidal Clusters from Random Parking of Bidisperse Spheres

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Using experiments and simulations, we investigate the clusters that form when colloidal spheres stick irreversibly to—or “park” on—smaller spheres. We use either oppositely charged particles or particles labeled with complementary DNA sequences, and we vary the ratio \( \alpha \) of large to small sphere radii. Once bound, the large spheres cannot rearrange, and thus the clusters do not form dense or symmetric packings. Nevertheless, this stochastic aggregation process yields a remarkably narrow distribution of clusters with nearly 90% tetrahedra at \( \alpha = 2.45 \). The high yield of tetrahedra, which reaches 100% in simulations at \( \alpha = 2.41 \), arises not simply because of packing constraints, but also because of the existence of a long-time lower bound that we call the “minimum parking” number. We derive this lower bound from solutions to the classic mathematical problem of spherical covering, and we show that there is a critical size ratio \( \alpha_c = (1 + \sqrt{2}) / 2 \), close to the observed point of maximum yield, where the lower bound equals the upper bound set by packing constraints. The emergence of a critical value in a random aggregation process offers a robust method to assemble uniform clusters for a variety of applications, including metamaterials.

Understanding the geometry of clusters formed from small particles is a fundamental problem in condensed matter physics, with implications for phenomena ranging from nucleation [1] to self-assembly [2]. Colloidal particles are a useful experimental system for studying cluster geometry and its relation to phase behavior [3] for several reasons: they are large enough to be directly observed using optical microscopy; their assembly can be understood in terms of geometry [4,5]; and they can be driven to cluster by a variety of controllable interactions, including capillary forces [2], depletion [6], fluctuation-induced forces [7], or DNA-mediated attraction [8]. Colloidal clusters are also useful materials in their own right. They can be used, for example, as building blocks for isotropic optical metamaterials known as metafluids [9–11]. Tetrahedral clusters are of particular interest for metafluids since the tetrahedron is the simplest cluster with isotropic dipolar symmetry [9]. An unsolved challenge for this application is to determine the interactions and conditions that enable assembly of bulk quantities of highly symmetric, uniform clusters such as tetrahedra.

With this motivation in mind, we study experimentally the geometry and size distribution of binary clusters formed when small colloidal spheres are mixed with an excess of large spheres that stick irreversibly and randomly to their surfaces [Fig. 1(a)]. An obvious way to control the cluster geometry in such binary systems is to vary the size ratio. One might expect that at certain ratios the particles could arrange into dense clusters or “spherical packings”—arrangements of spheres around a central sphere that maximize surface density [12–14]. Such packings have long been used in modeling the microstructure of dense, disordered atomic systems [15,16]. But unlike atoms, colloidal particles can stick irreversibly, such that two particles bound to a third show no motion relative to one another. This type of binding occurs frequently in strongly interacting, monodisperse colloidal suspensions, which

FIG. 1. (a) Two colloidal sphere species are mixed together to form clusters. (b) Oppositely charged polystyrene spheres cluster due to electrostatic attraction. Optical micrograph shows a tetramer (\( N = 4 \)). (c) Polystyrene spheres labeled with complementary DNA strands (not to scale) cluster due to DNA hybridization. Optical micrograph shows a trimer (\( N = 3 \)); the small, central sphere is fluorescent.
consequently form fractal aggregates instead of dense glasses [17,18]. Similarly, in the binary systems we study, the irreversible and stochastic process of sticking precludes the formation of dense or symmetric packings. The large spheres park, rather than pack, on the surfaces of the small spheres.

Surprisingly, this random and nonequilibrium process can produce clusters of uniform size. Our experiments show that at a size ratio \( \alpha = R_{\text{big}}/R_{\text{small}} = 2.45 \), where \( R_{\text{big}} \) and \( R_{\text{small}} \) are the sphere radii, nearly all of the clusters contain four large spheres stuck to a smaller sphere (Table I). In these experiments we use a 100:1 stoichiometric ratio of the two sphere species, statistically ensuring that each cluster contains only one small sphere surrounded by two or more larger spheres. After waiting several days for the average cluster size to saturate, we measure the distribution of \( N \), the number of large spheres bound to each small sphere [19]. We do not count single large spheres, nonspecifically aggregated clusters of large spheres, or clusters with multiple small spheres. While there are many isolated large spheres due to the high stoichiometric ratio, the latter two types of cluster are rare.

The \( N = 4 \) tetramers that we observe are not dense packings or, in general, symmetric arrangements. As can be seen from the images in Fig. 1, there is space between the large particles, and the resulting tetrahedra are irregular. Moreover, the ratio \( \alpha = 2.45 \) is well below the value \( \alpha = 4.44 \) found by Miracle et al. [20] for efficient tetrahedral packing in binary atomic clusters. In fact, at \( \alpha = 4.29 \), closer to this bound, we see much smaller clusters and few tetrahedra. The sparsity of large spheres in the clusters is a result of the irreversible, nonequilibrium, random binding: once the big particles stick to the smaller ones, we do not see them detach or move relative to one another. We expected such a stochastic process to lead to a much broader distribution of clusters. At other values of \( \alpha \) it does (Table I), but at \( \alpha = 2.45 \) we obtain 90% tetramers.

The high yield of tetramers occurs in two experimental systems with different types of interactions. In both systems the interactions are specific, strong, and short-ranged, and the particles do not rearrange once bound. In the first system the clustering is driven by electrostatic interactions. We mix large, positively-charged particles with small, negatively-charged particles, as shown in Fig. 1(b). To adjust \( \alpha \), we use several different particle sizes [19]. We add salt to reduce the Debye length to approximately 3 nm, small enough to ensure that the interaction range does not significantly influence the effective particle size. In the second system the clustering is driven by hybridization of grafted DNA strands [19]. As shown in Fig. 1(c), we mix small and large spheres labeled with complementary DNA oligonucleotides [21]. We work well below the DNA melting temperature so that the attractive interaction is many times the thermal energy [22].

To better understand why the distribution is sharply peaked at \( N = 4 \) for \( \alpha = 2.45 \), we use simulations and analytical techniques that account for the irreversibility of the aggregation process. Our simulations use a “random parking” algorithm [23–26] to model the formation of clusters. The algorithm involves attaching large spheres to randomly selected positions on the surface of a small sphere, subject to a no-overlap constraint [19]. We do not model the finite range of the interactions, which in both experimental systems is small compared to the particle size, or the diffusion of the particles prior to binding. In accord with experimental observations, the particles are not allowed to rearrange once bound. We repeat the process numerically to obtain distributions of cluster sizes as a function of a single parameter, \( \alpha \).

The simulations find a 100% yield of tetramers at the size ratio \( \alpha \approx 2.41 \). As in the experiments, the large particles in these tetramers are not densely packed, and the clusters are therefore distorted tetrahedra. We also find that while the yield of any particular cluster can be maximized by varying \( \alpha \) [Fig. 2(a)], the yield approaches 100% only for dimers (\( N = 2 \)) and tetramers (\( N = 4 \)). Interestingly, the yield curve for tetramers has a cusp at its peak, showing that the size ratio \( \alpha_c \) at the maximum is a mathematical critical point.

The simulated distributions agree well with those found experimentally [Figs. 2(b) and 2(c)] for both electrostatic and DNA-mediated interactions. For instance, at \( \alpha = 2.45 \) with electrostatic interactions, we find a sharply peaked distribution consisting almost entirely of tetramers. This value of \( \alpha \) is close to but not precisely at the critical value, so a small yield of trimers is predicted and observed experimentally. In contrast, at \( \alpha = 1.90 \) we find a mixture of mostly \( N = 4 \) and \( N = 5 \) clusters in both the DNA system and simulations. Some discrepancy arises between the simulated and experimental histograms because the yield curves in Fig. 2(a) are steep: a slight error in the effective size ratio can shift the cluster distribution. Nevertheless, the random sphere parking model successfully reproduces both the large yield of tetrahedra near \( \alpha_c \) and the details of the measured histograms at various other \( \alpha \).

That we can reproduce the same phenomenon in two different experimental systems and in a one-parameter model suggests that the critical size ratio \( \alpha_c \) has a universal,

<table>
<thead>
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<th>Size ratio ( \alpha )</th>
<th>1.94</th>
<th>2.45</th>
<th>3.06</th>
<th>4.29</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 6 )</td>
<td>6.3</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>39.2</td>
<td>0.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( N = 4 )</td>
<td>54.4</td>
<td>90.2</td>
<td>18.6</td>
<td>0.7</td>
</tr>
<tr>
<td>( N = 3 )</td>
<td>0.0</td>
<td>6.6</td>
<td>69.9</td>
<td>35.9</td>
</tr>
<tr>
<td>( N = 2 )</td>
<td>0.0</td>
<td>0.8</td>
<td>10.9</td>
<td>51.0</td>
</tr>
<tr>
<td>( N = 1 )</td>
<td>0.0</td>
<td>0.8</td>
<td>0.6</td>
<td>11.1</td>
</tr>
<tr>
<td>( N = 0 )</td>
<td>0.0</td>
<td>0.8</td>
<td>0.0</td>
<td>1.3</td>
</tr>
</tbody>
</table>
have been equally inconsiderate, leaving spaces between
the centers of any two big spheres must be at least
2Rbig. Consider then a sphere of radius (Rsmall + Rbig) that
circumscribes the centers of the parked spheres. If this
sphere is completely covered with N circles of radius
2Rbig, it will be impossible to add an (N + 1)th large
sphere. We are led naturally to the spherical covering
problem, a problem with a rich history in mathematics.
Like spherical packings, spherical coverings are solutions
to an extremum problem: they are arrangements of N
points on a sphere that minimize the largest distance
between any location on the sphere surface and the closest
point [13]. But unlike spherical packings, spherical cover-
ings need not correspond to arrangements of nonoverlap-
ing spheres. We therefore solve for the minimum parking
curve by examining the solutions to the spherical covering
problem [27] at each N and manually verifying that they
correspond to nonoverlapping configurations [19].

Our analytical results for the bounds reveal why αc is a
special point: it is the only nontrivial point where the
calculated maximum and minimum parking curves come
together (Fig. 3). Analytically we find the location of the
critical value to be αc = (1 + √2) ≈ 2.41, very close to the
values where the experimental distributions are peaked.
At α slightly larger than this value, the minimum parking
configuration corresponds to two spheres placed at oppo-
site poles (Nmin = 2), and the maximum N is obtained by
first parking three large spheres next to one another, so that
there is room for one more sphere to park (Nmax = 4). At α
slightly smaller than αc, the big spheres can park along
orthogonal axes about the small sphere to make an octa-
hedron (Nmax = 6). The minimum N is obtained by placing
four spheres as far from each other as possible, so as to
make the addition of a fifth impossible (Nmin = 4). Thus,
as we increase α through αc, Nmax goes from 6 to 4 and
Nmin from 4 to 2, and the two curves become infinitesim-
ally close.

The parking process is therefore geometrically con-
strained to yield clusters with exactly N = 4 particles in
the limit α → αc. A simple geometric argument sheds
some light on this result. At αc, there is always room for
four large spheres to park. Parking more spheres requires
that at least three park precisely along a great circle of the
smaller particle, but the probability of this happening
randomly is zero. Thus irreversible binary aggregation, a
stochastic process, has a deterministic feature at the critical
size ratio: although the space between the large spheres

FIG. 2 (color online). (a) Yield curves, as determined by
simulations, for N-particle clusters, 2 ≤ N ≤ 8, where the criti-
cal size ratio αc is marked with a black line. Below are histo-
grams for (b) DNA-labeled particles (left) at α = 1.90 and
(c) charged particles (right) at α = 2.45, as observed in experi-
ments (colored bars) and as predicted from simulations (gray
bars). Error bars are 95% confidence intervals (Wilson score
interval method).

gEometrical origin. Intuitively, one might expect that it is
related to packing constraints on the large spheres. Other
theoretical studies of random sphere parking [23,24] have
calculated the maximum number of large spheres Nmax that
can fit around a small sphere at a given α. However, this
bound cannot by itself explain why the yield of tetramers
can reach 100% while that of other clusters, such as trimers
or hexamers, cannot. At a given α, it tells us only why no
clusters larger than Nmax(α) can form, but it says nothing
about the probability of forming smaller clusters with
different arrangements.

Therefore we also examine a different bound, one not
previously discussed in the context of random sphere park-
ing: the “minimum parking” curve Nmin(α). Nmin is the
smallest number of hard spheres that can be positioned on a
smaller sphere such that another sphere cannot fit. To
understand this bound, consider a simple, one-dimensional
analogy to car parking on a busy city street, where if a
space opens up that is large enough to fit a car, it is filled.
The minimum parking number occurs when all drivers
have been equally inconsiderate, leaving spaces between
their parked cars that are all slightly too small for another
car to fit. This lower bound is meaningful only at long
times, when all available parking spaces have been filled.
The long-time limit holds also for our experiments and
simulations, which we carry out until the average cluster
size has saturated.

Whereas the upper bound Nmax(α) is straightforwardly
related to solutions of the well-known spherical packing
problem [13,27], the calculation of the lower bound Nmin(α)
requires a different approach. In our clusters, the distance
between the centers of any two big spheres must be at least
2Rbig. Consider then a sphere of radius (Rsmall + Rbig) that
circumscribes the centers of the parked spheres. If this
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size ratio: although the space between the large spheres
large spheres after the tetramers have formed [29]. A large number of symmetric tetrahedra simply by shrinking the small spheres can vary, it may well be possible to form large quantities of symmetric tetrahedra simply by shrinking the small spheres after the tetramers have formed [29]. An additional step, such as density gradient centrifugation [2], will also be required to separate the assembled clusters from the many unbound large particles. Furthermore, although the yield will approach 100% only for dimers and tetramers, the yield of any $N$-particle cluster can be maximized by choosing the appropriate size ratio. For instance, the yield of octahedral clusters, also promising candidates for building metamaterials [10], may surpass 70% at $\alpha = 1.42$.

The size ratio in binary colloidal systems thus emerges as a valuable control parameter for directed self-assembly. Moreover, because it does not require precise control over the interactions, random parking offers a robust and simple way to make colloidal clusters that are more monodisperse than those prepared through other methods [2].

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**FIG. 3** (color online). $N_{\text{max}}$ (solid gray) and $N_{\text{min}}$ (black) as functions of $\alpha$. Cluster images show sphere configurations at discontinuities of these curves. Average cluster sizes from simulations (dashed gray line) and experiments (blue and red data points) are shown. We characterize the statistical dispersion in each distribution by the average absolute deviation from the median, indicated by dotted light gray lines for simulations and vertical bars for experiments.

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