



The Eigencurve is Proper

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The Eigencurve is Proper

A dissertation presented

by

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to

The Department of Mathematics

in partial fulfillment of the requirements

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Abstract

Coleman and Mazur constructed a rigid analytic curve $\mathcal{C}_{p,N}$, called the eigencurve, whose points correspond to all finite slope overconvergent p -adic eigenforms. We prove the conjecture that the eigencurve $\mathcal{C}_{p,N}$ is proper over the weight space for any prime p and tame level N .

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1. INTRODUCTION

1.1. **Coleman and Mazur’s question.** The study of p -adic families of modular forms dates back to Serre’s Eisenstein family [Ser]. It provides arithmetic information about the possible congruences between coefficients of modular forms (in different weights). In particular, the congruences of the constant terms give congruences between the special values of zeta-functions. The theory was further developed by Hida in [HidaA] and [HidaB], in which he constructed p -adic families of cuspidal Hecke eigenforms, varying analytically with weights. One can also attach families of Galois representations to the families of eigenforms. This later led to Mazur’s important work on Galois deformation theory. However, Hida’s construction only apply to ordinary modular forms.

This restriction on ordinary forms was resolved by Coleman and Mazur’s work [CM98] in which they constructed the p -adic *eigencurve* (of tame level 1). It is a rigid analytic curve \mathcal{C} whose points correspond to all finite slope overconvergent p -adic eigenforms, including the ordinary ones. There is a natural projection map $\pi : \mathcal{C} \rightarrow \mathcal{W}$ whose image characterizes the weights of the eigenforms. Here the rigid space \mathcal{W} is called the *weight space*. The construction is later generalized to any tame level N by Buzzard [Buz07].

Understanding the geometry of the eigencurve (or, more generally, eigenvarieties) can provide arithmetic information about p -adic modular forms. Since Coleman and Mazur’s work, many progresses have been made towards the geometry of the eigencurve, for example [Bel12], [BM12], [Buz03], [BuCa05], and [BK05]. However, some fundamental questions remain open, among which is the following conjecture known as “properness of the eigencurve”.

In [CM98], Coleman and Mazur raise the following question:

Does there exist a p -adic family of finite slope overconvergent eigenforms, parameterized by a punctured disk, that converges to an overconvergent eigenform at the puncture which is of infinite slope?

In other words, this is to ask whether the projection map $\pi : \mathcal{C} \rightarrow \mathcal{W}$ satisfies the *valuative criterion for properness*. We will show that the answer to this question is affirmative. More precisely, the main result of this thesis is the following theorem.

Theorem 1.1 (Properness of the Eigencurve). *Let $\mathcal{C}_{p,N}$ be the Coleman-Mazur eigencurve of tame level N , and let $\pi : \mathcal{C}_{p,N} \rightarrow \mathcal{W}$ denote the natural projection to the weight space. Let D be the closed unit disk over some finite extension L over \mathbb{Q}_p , and let D^* be the punctured disk with the origin removed. Suppose $h : D^* \rightarrow \mathcal{C}_{p,N}$ is a morphism of rigid analytic spaces such that $\pi \circ h$ extends to D . Then h extends to a morphism $\tilde{h} : D \rightarrow \mathcal{C}_{p,N}$ so that the following diagram commutes.*

$$\begin{array}{ccc}
 D^* & \xrightarrow{h} & \mathcal{C}_{p,N} \\
 \downarrow & \nearrow \tilde{h} & \downarrow \pi \\
 D & \xrightarrow{\quad} & \mathcal{W}
 \end{array}$$

We want to point out that although this property is named “properness of the eigencurve”, the projection π is actually not proper in the sense of rigid analytic geometry because it is of infinite degree. The readers should think of “proper” as shorthand for “proper locally in the domain”. We also remark that a stronger form of the conjecture is false. According to [CS04], there exists a pointwise sequence of finite slope eigenforms converging to an eigenform of infinite slope. Therefore, our statement must be formulated in the framework of rigid families.

In the past decade, some progresses have been made towards this problem. In [BuCa06], the properness was proved for $p = 2$ and $N = 1$. In [Cal08], the properness

was proved at integral weights in the center of \mathcal{W} . In a joint work with Ruocuhan Liu [DL13], we proved the conjecture at weights which are non p -adic Liouville numbers.

1.2. Strategy of the proof. To attack the main theorem, we study the associated family of Galois representations. More precisely, we look at the family of p -adic Galois representations on the punctured disk by pulling-back the natural Galois representations on the eigencurve. By restricting to $G_{\mathbb{Q}_p}$ -representations, we reduce the problem to a local question! We can then make use of powerful tools from p -adic Hodge theory and (φ, Γ) -modules. In particular, we make extensive use of the recent advances on p -adic Hodge theory in rigid analytic families (e.g., [BeCo08], [KL10], and [Bel13]).

Let us explain the idea more carefully. Let Σ be the finite set of places of \mathbb{Q} consisting of the infinite place and the primes dividing pN . By [CM98], the family of overconvergent eigenforms on the eigencurve give rise to a family of $G_{\mathbb{Q}, \Sigma}$ -representations on the normalization of $\mathcal{C}_{p, N}$ which interpolates the Galois representations associated to classical forms. Pulling back along h , we obtain a family of $G_{\mathbb{Q}, \Sigma}$ -representations V_{D^*} on the punctured disk D^* . From the construction, $\mathcal{C}_{p, N}$ can be regarded as an analytic subspace of $X_p \times \mathbb{G}_m$, where X_p is the deformation space of the pseudo-representations associated to all p -modular residue representations of $G_{\mathbb{Q}, \Sigma}$. We claim that the composition $D^* \rightarrow \mathcal{C}_{p, N} \rightarrow X_p \times \mathbb{G}_m \rightarrow X_p$ extends to a morphism $D \rightarrow X_p$. Consequently, V_{D^*} extends to a family of Galois representations V_D on the entire disk D .

Now we look at the family of local p -adic Galois representations (namely, restrict to $G_{\mathbb{Q}_p}$). Abusing the notation, we still call it V_D . Let V_D^* be its dual. For each point $x \in D$, we write V_x^* for the specialization of V_D^* at x . At every point x away from the puncture, the representation V_x^* is in fact *trianguline*. This is due to a result of Kisin [Kis03]. More precisely, let $\alpha_{\mathcal{C}}$ be the function of U_p -eigenvalues on the eigencurve and let $\alpha \in \mathcal{O}(D^*)^\times$ be the pullback of $\alpha_{\mathcal{C}}$. For any $x \in D^*$, the specialization V_x^*

has a crystalline period with Frobenius eigenvalue $\alpha(x)$. It is straightforward to see that α extends to an analytic function on the entire disk. We want to show that the specialization at the puncture point is also trianguline. In particular, we need to show that $\alpha(0) \neq 0$.

Up till now, we have reduced the original global conjecture to the following local question about family of p -adic Galois representations.

Local Question: *Suppose we have a family of $G_{\mathbb{Q}_p}$ -representations V_D^* of rank 2 on a closed unit disk D . Let α be a rigid analytic function on the disk which is invertible on the puncture disk D^* . Assume that for every point x on the punctured disk, we have $(\mathbf{B}_{\text{crys}}^+ \otimes V_x^*)^{G_{\mathbb{Q}_p}, \varphi=\alpha(x)} \neq 0$. In particular, every V_x^* is trianguline. We want to show that V_0 is also trianguline with nonzero Frobenius eigenvalue $\alpha(0)$.*

The proof of this local question consists of three steps. Remember that we need to cut out a crystalline period on the entire disk. The main idea is to first look at the de Rham period.

Step 1: Away from the puncture

On the punctured disk, we are able to prove that crystalline periods coincide with de Rham periods. More precisely, for any affinoid $\text{Sp}(R) \subset D^*$, we have $D_{\text{crys}}^+(V_R^*)^{\varphi=\alpha} = D_{\text{crys}}^+(V_R^*) = D_{\text{dR}}^+(V_R^*)$ are free of rank 1. The main ingredient of the proof is a variation of “finite slope subspace” originally constructed by Kisin. In [Kis03], it is proved that the crystalline and de Rham periods coincide on “ Y -small” affinoid subdomains of $\mathcal{C}_{p,N}$. The Y -smallness condition is later removed by Liu in his generalization of finite slope subspaces [Liu12].

Step 2: Passing to the puncture

We make use of the crucial fact that D_{dR}^+ commutes with flat base change. Applying base change from annuli to the disk implies $D_{\text{dR}}^+(V_D^*)$ is nonzero and has generic rank 1. Pick an element $e \in D_{\text{dR}}^+(V_D^*)$ whose specialization at 0 is nonzero. Note that e is de Rham on the disk and crystalline on the punctured disk. This forces e to be crystalline on the disk.

Moreover, we know that $\varphi e = \alpha e$ on the entire disk. Therefore, the nonzero image of e under specialization $D_{\text{crys}}^+(V_D^*) \otimes S/\mathfrak{m}_0 \hookrightarrow D_{\text{crys}}^+(V_0^*)$ gives us the desired crystalline period.

Step 3: Conclusion

That we had constructed is not just a crystalline period in V_0^* , but a family of crystalline period e ; namely, a global triangulation. This implies that $(V_0, \alpha(0))$ lives in the space $X_p \times \mathbb{G}_m$ and is a (p -adic) limit of points on $\mathcal{C}_{p,N}$. However, the eigencurve $\mathcal{C}_{p,N}$ is an analytic subspace of $X_p \times \mathbb{G}_m$. Therefore, the point $(V_0, \alpha(0))$ also lives on the eigencurve!

1.3. Structure of the thesis. We first review some of the background material in Chapter 2. In section 2.1, we recap some basic rigid analytic geometry as we will later study rigid analytic families of p -adic Galois representations. In section 2.2 and 2.3, we give an overview on p -adic Hodge theory and the theory of (φ, Γ) -modules.

Chapter 3 contributes to the study of rigid analytic families of p -adic Galois representations. Especially, we introduce recent advances on p -adic Hodge theory and (φ, Γ) -modules in rigid analytic families. We end the chapter with the flat base change theorem for the de Rham functor D_{dR}^+ .

In Chapter 4, we review the definition of overconvergent modular forms and briefly sketch Coleman-Mazur's construction of the eigencurve. We also recall the construction of the natural family of Galois representations on (the normalization) of the eigencurve.

In Chapter 5, we study finite slope subspace. The notion was originally defined by Kisin in [Kis03], and later generalized by Liu in [Liu12]. We adopt the language of Liu as well as his result on the existence of finite slope subspaces. Then we list some of the main results in [Liu12] which play a central role in the proof of the main theorem.

In chapter 6, we prove the main theorem. The proof consists of three parts. In section 6.1, we show that the natural family of $G_{\mathbb{Q},\Sigma}$ -representations on D^* extends to the entire disk. In section 6.2, we focus on the local question by restricting to $G_{\mathbb{Q}_p}$ -representations. Finally, in section 6.3, we finish the proof by finding a crystalline period at the puncture point. In particular, this yields $\alpha(0) \neq 0$.

2. PRELIMINARIES

2.1. Rigid analytic geometry. Throughout the thesis, we study objects in rigid analytic families; namely, objects which live on rigid spaces. The main purpose of rigid analytic geometry is to provide a notion of analytic spaces over non-archimedean fields, serving as an analogue of complex manifolds over \mathbb{C} . The standard reference for rigid geometry is [BGR].

2.1.1. Tate algebras and affinoid algebras. Let k be a complete non-archimedean field, namely, a field complete with respect to a norm $|\cdot| : k \rightarrow \mathbb{R}_{\geq 0}$ satisfying $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in k$. For example, the field \mathbb{Q}_p of p -adic numbers is a complete non-archimedean field equipped with the normalized norm given by $|p| = p^{-1}$.

Definition 2.1. The *Tate algebra* over k of n variables is defined as

$$T_n = T_n(k) = k\langle X_1, \dots, X_n \rangle := \left\{ \sum_J a_J X^J \mid |a_J| \rightarrow 0 \text{ as } J \rightarrow \infty \right\}.$$

Here the index J runs over multi-index $(j_1, \dots, j_n) \in \mathbb{Z}_{\geq 0}^{\oplus n}$ and “ X^J ” stands for $X_1^{j_1} \cdots X_n^{j_n}$. The condition “ $J \rightarrow \infty$ ” means $\sum_{i=1}^n j_i \rightarrow \infty$.

Definition 2.2. The *Gauss norm* on T_n is defined by

$$\left| \sum_J a_J X^J \right| := \sup_J |a_J|.$$

Note that the Gauss norm makes T_n into a Banach k -algebra.

Definition 2.3. A *k -affinoid algebra* is a k -algebra such that $A \cong T_n(k)/I$ for some Tate algebra $T_n(k)$ and some ideal $I \subset T_n(k)$.

Note that the ideals $I \subset T_n$ are closed Banach subspaces of T_n . Thus we can equip $A \cong T_n/I$ with the quotient norm and make it a Banach k -algebra. If we have two different presentations of A , the induced Banach norms will be equivalent.

2.1.2. *Rigid spaces.* Let A be a k -affinoid algebra. In this section, we study the spectrum $X = \mathrm{Sp}(A)$ equipped with a Grothendieck topology and a structure sheaf \mathcal{O}_X . They are called *affinoid spaces*.

Definition 2.4. Let A be a k -affinoid algebra. Define the *affinoid space* $X = \mathrm{Sp}(A)$ to be the set of maximal ideals of A . For any maximal ideal $x \in X$ and any $f \in A$, we write $f(x)$ for the image of f in A/x . Using this language, we define the *spectral (semi-)norm* $|\cdot|_{\mathrm{sp}}$ on A by setting $|f|_{\mathrm{sp}} = \sup_{x \in X} |f(x)|$.

Definition 2.5. Let A be a k -affinoid algebra and let $X = \mathrm{Sp}(A)$ be the associated affinoid space. A subset $U \subset X$ is called an *affinoid subdomain* if there exists a k -affinoid algebra A' and a morphism $A \rightarrow A'$ such that the induced map $\alpha : \mathrm{Sp}(A') \rightarrow \mathrm{Sp}(A)$ factors through U and the following universal property holds:

For any morphism $A \rightarrow B$ of k -affinoid algebras such that the induced map $\alpha' : \mathrm{Sp}(B) \rightarrow \mathrm{Sp}(A)$ factors through U , the map α' uniquely factors through $\alpha : \mathrm{Sp}(A') \rightarrow \mathrm{Sp}(A)$.

Lemma 2.6. *Let $U \subset X$ be an affinoid subdomain as above and let $\alpha : \mathrm{Sp}(A') \rightarrow \mathrm{Sp}(A)$ be corresponding map. Then α is injective and it induces a bijection $\mathrm{Sp}(A') \xrightarrow{\sim} U$.*

Using Lemma 2.6, we may always identify the affinoid subdomain U with the set $\mathrm{Sp}(A')$. In particular, every affinoid subdomain has a structure of affinoid space. The structure is unique up to a canonical isomorphism. Moreover, if $U \subset X$ and $V \subset U$ are affinoid subdomains, so is $V \subset X$. For later use, we point out the following fact: if $\mathrm{Sp}(A') \hookrightarrow \mathrm{Sp}(A)$ is an affinoid subdomain, then the morphism $A \rightarrow A'$ is flat.

Examples of affinoid subdomains include those called rational subdomains. Let $X = \mathrm{Sp}(A)$ and let $f_0, f_1, \dots, f_n \in A$ with no common zeros. Consider the subset

$$X\left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\right) = \{x \in X \mid |f_i(x)| \leq |f_0(x)| \text{ for all } i = 1, \dots, n\} \subset X$$

which is called a *rational subdomain* of X . One can show that rational subdomains are indeed affinoid subdomains. It is a deep theorem that every affinoid subdomain can be written as a finite union of rational subdomains.

Definition 2.7. Let $X = \mathrm{Sp}(A)$ be an affinoid space. An *affinoid covering* of X is a finite collection of affinoid subdomains U_1, \dots, U_n such that $X = \bigcup_{i=1}^n U_i$. The collection of affinoid subdomains and affinoid coverings gives a Grothendieck topology on X , called the *weak G-topology*.

We define a structure (pre-)sheaf \mathcal{O}_X on $X = \mathrm{Sp}(A)$. For any affinoid subdomain $U = \mathrm{Sp}(A') \rightarrow X$, we simply set $\mathcal{O}_X(U) = A'$. The presheaf \mathcal{O}_X is indeed a sheaf with respect to the weak G-topology on X . The pair (X, \mathcal{O}_X) forms a locally ringed space.

Finally, we can define the general notion of rigid spaces by gluing the affinoid spaces together.

Definition 2.8. A *rigid analytic k -space* is a locally ringed space (X, \mathcal{O}_X) equipped with a Grothendieck topology such that X admits a covering $X = \bigcup X_i$ where $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid space for all i .

2.2. p -adic Hodge theory. In this section, we review some basic definitions from p -adic Hodge theory. In particular, we review the construction of Fontaine's period rings, which can be used to classify different subcategories of p -adic Galois representations. The main reference for this section is [Ber02].

2.2.1. Some perfect rings. Let us first fix some notations. Let K be a finite extension of \mathbb{Q}_p and let $G_K = \mathrm{Gal}(\overline{K}/K)$. Let $\chi : G_K \rightarrow \mathbb{Z}_p^\times$ be the cyclotomic character. Moreover, choose a compatible system of primitive p -power roots of unity $(\zeta_{p^n})_{n \geq 0}$. Namely, each ζ_{p^n} is a primitive p^n -th root of unity and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Write $K_n = K(\zeta_{p^n})$ for each $n \geq 1$ and let $K_\infty = \bigcup_{n \geq 1} K_n$. We also write $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma = \Gamma_K = \mathrm{Gal}(K_\infty/K)$.

Consider a field of characteristic p defined by $\tilde{\mathbf{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p$; namely, $\tilde{\mathbf{E}}$ consists of elements of the form $(x^{(i)})_{i \geq 0}$ with $x^{(i)} \in \mathbb{C}_p$ and $(x^{(i+1)})^p = x^{(i)}$ for each i . For example, the element $\varepsilon = (\zeta_{p^n})_{n \geq 0}$ is in $\tilde{\mathbf{E}}$. Let $\tilde{\mathbf{E}}^+$ be the subset of those $x = (x^{(i)}) \in \tilde{\mathbf{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. One can show that $\tilde{\mathbf{E}}$ is an algebraically closed field of characteristic p . Nevertheless, we can define a valuation $v_{\mathbf{E}}$ on $\tilde{\mathbf{E}}$ by setting $v_{\mathbf{E}}(x) = v_p(x^{(0)})$. Then $\tilde{\mathbf{E}}$ is complete with respect to this valuation and $\tilde{\mathbf{E}}^+$ is the valuation ring. There is also a natural Frobenius map φ on $\tilde{\mathbf{E}}$ defined by $\varphi((x^{(i)})) = ((x^{(i)})^p)$, namely, raising each component to the p -th power. Similarly, there is a natural G_K -action by acting component-wise.

Let $\tilde{\mathbf{A}} = W(\tilde{\mathbf{E}})$ and $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$. These are perfect rings of characteristic 0. We define a topology on $\tilde{\mathbf{A}}$ which is referred as the *weak topology*. First, we choose an element $\tilde{p} \in \tilde{\mathbf{E}}^+$ with $\tilde{p}^{(0)} = p$. Let $[\tilde{p}]$ be its Teichmüller lift. We define the weak topology on $\tilde{\mathbf{A}}$ by specifying the following sets to be a basis of open neighborhoods around 0:

$$U_{m,n} = p^m \tilde{\mathbf{A}} + [\tilde{p}]^n \tilde{\mathbf{A}}^+$$

where $m, n \geq 0$. The weak topology on $\tilde{\mathbf{A}}^+$ is given by the subspace topology.

We can also define natural Frobenius and G_K -actions on $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^+$. More precisely, we define the Frobenius map by

$$\varphi\left(\sum_{i=0}^{\infty} p^i [x_i]\right) = \sum_{i=0}^{\infty} p^i [x_i^p]$$

and for each $g \in G_K$, we define

$$g\left(\sum_{i=0}^{\infty} p^i [x_i]\right) = \sum_{i=0}^{\infty} p^i [g(x_i)].$$

Both actions are continuous with respect to the weak topology and preserve the subring $\tilde{\mathbf{A}}^+$.

Finally we define $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}[1/p]$ and $\tilde{\mathbf{B}}^+ = \tilde{\mathbf{A}}^+[1/p]$.

2.2.2. *Period rings \mathbf{B}_{HT} , \mathbf{B}_{dR} , and \mathbf{B}_{crys} .* In this section, we recall the constructions of Fontaine's period rings, especially the Hodge-Tate, de Rham, and crystalline period rings.

First of all, the Hodge-Tate period ring is defined as $\mathbf{B}_{\text{HT}} = \mathbb{C}_p[t, t^{-1}]$. It is equipped with a grading given by the powers of t and a natural G_K -action given by $g(t) := \chi(g)t$. Clearly, the G_K -action preserves grading. Moreover, we know that $\mathbf{B}_{\text{HT}}^{G_K} = K$.

Now we construct the de Rham period ring \mathbf{B}_{dR} . Consider a map $\theta : \tilde{\mathbf{B}}^+ \rightarrow \mathbb{C}_p$ defined by $\theta(\sum p^i [x_i]) = \sum p^i x_i^{(0)}$. This map is continuous and G_K -equivariant. The kernel of θ is a principal ideal generated by $[\tilde{p}] - p$. We define $\mathbf{B}_{\text{dR}}^+ = \varprojlim_i \tilde{\mathbf{B}}^+ / \ker(\theta)^i$. In particular, we obtain a map $\theta : \mathbf{B}_{\text{dR}}^+ \rightarrow \mathbb{C}_p$. Note that each quotient $\tilde{\mathbf{B}}^+ / \ker(\theta)^i$ has a natural structure of p -adic Banach space. This allows us to equip \mathbf{B}_{dR}^+ with p -adic Fréchet topology. The natural G_K -action on $\tilde{\mathbf{B}}^+$ induces a natural G_K -action on \mathbf{B}_{dR}^+ . It is straightforward to check that the G_K -action is continuous with respect to the topology defined above.

Recall that $\varepsilon = (\zeta_{p^n})_{n \geq 0}$ is an element in $\tilde{\mathbf{E}}^+$ and let $[\varepsilon]$ be its Teichmüller lift. Consider an element $t \in \mathbf{B}_{\text{dR}}^+$ defined by

$$t := \log([\varepsilon]) = \log(1 + ([\varepsilon] - 1)) = \sum_{n=1}^{\infty} \frac{(1 - [\varepsilon])^n}{n}.$$

One checks that t indeed converges to an element in \mathbf{B}_{dR}^+ . It turns out t is a generator of $\ker(\theta) \subset \mathbf{B}_{\text{dR}}^+$ and \mathbf{B}_{dR}^+ is t -adically complete. Define $\mathbf{B}_{\text{dR}} = \mathbf{B}_{\text{dR}}^+[1/t]$ and define a filtration on \mathbf{B}_{dR} by setting $\text{Fil}^i \mathbf{B}_{\text{dR}} = t^i \mathbf{B}_{\text{dR}}^+$. Then the associated graded ring is just \mathbf{B}_{HT} . Furthermore, the G_K -action on \mathbf{B}_{dR} preserves the filtration. We also know that $\mathbf{B}_{\text{dR}}^{G_K} = K$.

It remains to define the crystalline period rings $\mathbf{B}_{\text{crys}}^+$ and \mathbf{B}_{crys} . Indeed, $\mathbf{B}_{\text{crys}}^+$ is a subring of \mathbf{B}_{dR}^+ consisting of elements whose expansions satisfy certain growth conditions. Recall that $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \tilde{\mathbf{E}}^+$. We write $\varepsilon_1 = (\zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots) \in \tilde{\mathbf{E}}^+$ and define $\omega = ([\varepsilon] - 1)/([\varepsilon_1] - 1) \in \tilde{\mathbf{B}}^+$. One can check that $\theta(\omega) = 0$ and thus ω

is a generator of $\ker(\theta)$. It turns out every element in $\mathbf{B}_{\mathrm{dR}}^+$ can be written as a sum $\sum_{i=0}^{\infty} a_i \omega^i$ with $a_i \in \tilde{\mathbf{B}}^+$. We define

$$\mathbf{B}_{\mathrm{crys}}^+ = \left\{ x = \sum_{i=0}^{\infty} a_i \frac{\omega^i}{i!} \text{ where } a_i \in \tilde{\mathbf{B}}^+ \text{ and } a_i \rightarrow 0 \right\} \subset \mathbf{B}_{\mathrm{dR}}^+$$

and define $\mathbf{B}_{\mathrm{crys}} = \mathbf{B}_{\mathrm{crys}}^+[1/t]$. We have $\mathbf{B}_{\mathrm{crys}}^{G_K} = K_0$, the maximal unramified subextension of K/\mathbb{Q}_p .

There is a natural filtration on $\mathbf{B}_{\mathrm{crys}}$ induced by the inclusion $\mathbf{B}_{\mathrm{crys}} \subset \mathbf{B}_{\mathrm{dR}}$. Moreover, we can put Frobenius structures on $\mathbf{B}_{\mathrm{crys}}^+$ and $\mathbf{B}_{\mathrm{crys}}$ induced from the natural Frobenius $\varphi : \tilde{\mathbf{B}}^+ \rightarrow \tilde{\mathbf{B}}^+$. We remark that such attempt fails for $\mathbf{B}_{\mathrm{dR}}^+$. For example, $1/([\tilde{p}^{1/p}] - p) \in \mathbf{B}_{\mathrm{dR}}^+$ but $\varphi(1/([\tilde{p}^{1/p}] - p)) = 1/([\tilde{p}] - p) \notin \mathbf{B}_{\mathrm{dR}}^+$.

2.2.3. The functors D_* . For any of the period rings $\mathbf{B}_* \in \{\mathbf{B}_{\mathrm{HT}}, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{crys}}\}$ constructed above, we consider the functor D_* from the category of finite dimensional \mathbb{Q}_p -representations of G_K to the category of K -vector spaces (or, K_0 -vector space in the crystalline case) defined as follows. Let V be such a finite dimensional \mathbb{Q}_p -representation; namely, a finite dimensional \mathbb{Q}_p -vector space equipped with a \mathbb{Q}_p -linear continuous action of G_K . Define

$$D_*(V) := (\mathbf{B}_* \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Note that \mathbf{B}_{HT} has an extra structure of grading which makes $D_{\mathrm{HT}}(V)$ a graded K -vector space. Similarly, $D_{\mathrm{dR}}(V)$ is naturally a filtered K -vector space, and $D_{\mathrm{crys}}(V)$ is a K_0 -vector space equipped with a semi-linear Frobenius φ and a filtration on $D_{\mathrm{crys}}(V) \otimes_{K_0} K$. An element of $D_{\mathrm{HT}}(V)$ (resp., $D_{\mathrm{dR}}(V)$, $D_{\mathrm{crys}}(V)$) is called a *Hodge-Tate period* (resp., *de Rham period*, *crystalline period*) of V .

We say that V is *Hodge-Tate* (resp., *de Rham*, *crystalline*) if $\dim_K D_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p} V$ (resp., $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$, $\dim_{K_0} D_{\mathrm{crys}}(V) = \dim_{\mathbb{Q}_p} V$). We have the

following inclusion relation between the categories of these Galois representations.

$$\{ \text{crystalline repn.} \} \subset \{ \text{de Rham repn.} \} \subset \{ \text{Hodge-Tate repn.} \}$$

Remark 2.9. There is another interpretation of Hodge-Tate representations. A \mathbb{Q}_p -representation of G_K is Hodge-Tate if V has a decomposition $\mathbb{C}_p \otimes V = \bigoplus_{i=1}^d \mathbb{C}_p(k_i)$ as $\mathbb{C}_p[G_K]$ -modules. Here the integers k_1, \dots, k_d are called *Hodge-Tate weights*.

Remark 2.10. Those subcategories of Galois representations arise naturally in the study of number theory. For example, for a (classical) modular form f of weight $k \geq 1$, the associated $G_{\mathbb{Q}_p}$ -representation V_f is in fact crystalline with Hodge-Tate weight 0 and $k - 1$.

2.3. (φ, Γ) -modules. Roughly speaking, (φ, Γ) -modules are linear algebraic objects equipped with semi-linear Frobenius and Γ -action. The classical theory of (φ, Γ) -modules is introduced by Fontaine as modules over non-perfect rings \mathbf{B}_K . The theory was later established by Cherbonnier-Colmez over overconvergent rings \mathbf{B}_K^\dagger , and then by Berger over Robba rings $\mathbf{B}_{\text{rig}, K}^\dagger$. In particular, there are equivalences of categories between p -adic representations of G_K and categories of *étale (φ, Γ) -modules* over \mathbf{B}_K , \mathbf{B}_K^\dagger , and $\mathbf{B}_{\text{rig}, K}^\dagger$, respectively. We will adopt Berger's language for our purpose.

2.3.1. Some non-perfect rings. Consider the map $\mathbb{F}_p((T)) \rightarrow \tilde{\mathbf{E}}$ sending T to $\varepsilon - 1$. Let $\mathbf{E}_{\mathbb{Q}_p}$ denote the image of this map and let \mathbf{E} be the separable closure of $\mathbf{E}_{\mathbb{Q}_p}$ inside $\tilde{\mathbf{E}}$. Let \mathbf{E}^+ denote the subring of \mathbf{E} of elements with positive $v_{\mathbf{E}}$ -valuation.

Consider the map $\mathbb{Z}_p[[T]][[T^{-1}]] \rightarrow \tilde{\mathbf{A}}$ sending T to $\pi = [\varepsilon] - 1$. Let $\mathbf{A}_{\mathbb{Q}_p}$ denote the p -adic completion of the image of this map and let $\mathbf{B}_{\mathbb{Q}_p} = \mathbf{A}_{\mathbb{Q}_p}[1/p]$. Let \mathbf{B} be the p -adic completion of the maximal unramified extension of $\mathbf{B}_{\mathbb{Q}_p}$ inside $\tilde{\mathbf{B}}$. Let $\mathbf{A} = \mathbf{B} \cap \tilde{\mathbf{A}}$ and $\mathbf{A}^+ = \mathbf{B} \cap \tilde{\mathbf{A}}^+$. Note that $\mathbf{A}_{\mathbb{Q}_p}$ (resp., \mathbf{A}) is the Cohen ring of $\mathbf{E}_{\mathbb{Q}_p}$ (resp., \mathbf{E}).

There is a natural G_K -action on \mathbf{A} . We define $\mathbf{A}_K := \mathbf{A}^{H_K}$ and $\mathbf{B}_K := \mathbf{A}_K[1/p]$. When $K = \mathbb{Q}_p$, this definition coincides with the one defined above. There is also a Frobenius action on \mathbf{A} and \mathbf{A}^+ induced from the one on $\tilde{\mathbf{A}}$. However, since \mathbf{A} and \mathbf{A}^+ are not perfect rings, the Frobenius maps are no longer surjective.

2.3.2. *Some overconvergent rings.* For $s \geq 0$, consider the following subring of $\tilde{\mathbf{A}}$:

$$\tilde{\mathbf{A}}^{\dagger,s} = \left\{ x = \sum_{i=0}^{\infty} p^i [x_i] \in \tilde{\mathbf{A}} \mid v_{\mathbf{E}}(x_i) + \frac{psi}{p-1} \geq 0, \lim_{i \rightarrow \infty} (v_{\mathbf{E}}(x_i) + \frac{psi}{p-1}) = \infty \right\}$$

and put $\tilde{\mathbf{B}}^{\dagger,s} = \tilde{\mathbf{A}}^{\dagger,s}[1/p]$. Define

$$\mathbf{A}^{\dagger,s} = \mathbf{A} \cap \tilde{\mathbf{A}}^{\dagger,s}, \quad \mathbf{B}^{\dagger,s} = \mathbf{B} \cap \tilde{\mathbf{B}}^{\dagger,s}.$$

Taking the union over all s , we obtain the “overconvergent rings”:

$$\mathbf{A}^{\dagger} = \cup_{s \geq 0} \mathbf{A}^{\dagger,s}, \quad \mathbf{B}^{\dagger} = \cup_{s \geq 0} \mathbf{B}^{\dagger,s}.$$

All these rings possess natural G_K -actions. Hence we can take Galois invariants:

$$\mathbf{A}_K^{\dagger,s} := (\mathbf{A}^{\dagger,s})^{H_K}, \quad \mathbf{B}_K^{\dagger,s} := (\mathbf{B}^{\dagger,s})^{H_K}$$

and

$$\mathbf{A}_K^{\dagger} := (\mathbf{A}^{\dagger})^{H_K}, \quad \mathbf{B}_K^{\dagger} := (\mathbf{B}^{\dagger})^{H_K}.$$

Clearly, $\mathbf{A}^{\dagger,s}$, $\mathbf{B}^{\dagger,s}$, \mathbf{A}^{\dagger} , and \mathbf{B}^{\dagger} have a natural action of $\Gamma \simeq G_K/H_K$. There are also natural Frobenius actions on them induced from the ones on $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$. Nevertheless, we can define a norm w_s on $\mathbf{B}^{\dagger,s}$ by setting $w_s(x) = \inf_i (v_{\mathbf{E}}(x_i) + \frac{psi}{p-1})$. Then $\mathbf{B}^{\dagger,s}$ is complete with respect to this norm.

Now we introduce the Robba rings. Note that, for each $s' > s$, $w_{s'}$ is also a norm on $\mathbf{B}^{\dagger,s}$. Therefore, it makes sense to consider the Fréchet completion of $\mathbf{B}^{\dagger,s}$ with respect to the family of norms $\{w_{s'}\}_{s' \geq s}$. The resulting ring is denoted by $\mathbf{B}_{\text{rig}}^{\dagger,s}$. Similarly, we can consider the family of norms $\{w_{s'}\}_{s' \geq s}$ on $\mathbf{B}_K^{\dagger,s}$ and let $\mathbf{B}_{\text{rig},K}^{\dagger,s}$ to be

the Fréchet completion. By continuity, $\mathbf{B}_{\text{rig}}^{\dagger,s}$ and $\mathbf{B}_{\text{rig},K}^{\dagger,s}$ are equipped with natural Γ and Frobenius actions extending the ones on $\mathbf{B}^{\dagger,s}$ and $\mathbf{B}_K^{\dagger,s}$.

Finally, taking the union over all s , we obtain the *Robba rings* $\mathbf{B}_{\text{rig},K}^{\dagger} := \cup_{s \geq 0} \mathbf{B}_{\text{rig},K}^{\dagger,s}$. Clearly, it has natural Γ - and Frobenius actions on it. We remark that the elements in the Robba ring have a more explicit description when K/\mathbb{Q}_p is unramified. (This in particular suffices for our use because we will be mainly interested in the case $K = \mathbb{Q}_p$.) When K/\mathbb{Q}_p is unramified, let \mathcal{B}_K^s denote the ring of bounded rigid analytic functions on the half-open annulus $\{0 < v_p(T) \leq 1/s\}$. Then there is an isomorphism

$$\mathcal{B}_K^s \rightarrow \mathbf{B}_K^{\dagger,s}$$

sending $f(T)$ to $f(\pi)$. Taking the Fréchet completion of the union $\cup_s \mathbf{B}_K^{\dagger,s}$, we know that $\mathbf{B}_{\text{rig},K}^{\dagger}$ can be identified with the ring of all analytic functions on some half-open annulus with outside radius 1. The Frobenius action is given by $\varphi(T) = (1+T)^p - 1$ and the Γ -action is given by $\gamma(T) = (1+T)^{\chi(\gamma)} - 1$ for each $\gamma \in \Gamma$.

We remark that the above explicit description can be extended to the case when K/\mathbb{Q}_p is ramified. But the isomorphism will be non-canonical.

2.3.3. (φ, Γ) -modules and p -adic Galois representations.

Definition 2.11. A (φ, Γ) -module over $\mathbf{B}_{\text{rig},K}^{\dagger}$ is a finite free $\mathbf{B}_{\text{rig},K}^{\dagger}$ -module D equipped with semi-linear φ - and Γ -actions, commuting with each other, and such that $\varphi(D)$ generates D .

The theory of (φ, Γ) -modules is particularly useful in the study of p -adic Galois representations. To every p -adic representation of G_K , we can associate a (φ, Γ) -module over $\mathbf{B}_{\text{rig},K}^{\dagger}$ which translates a Galois representation into a linear algebraic object. This allows us to construct examples and perform explicit calculations.

More precisely, the process above turns a Galois representation into an *étale* (φ, Γ) -module. To define the notion of étaleness, we first review Kedlaya's slope theory of φ -modules.

Definition 2.12. Let B be a ring equipped with a Frobenius action φ . (We will apply the construction to $\tilde{\mathbf{B}}$, \mathbf{B}_K^\dagger , and $\mathbf{B}_{\text{rig},K}^\dagger$.)

- (a) A φ -module over B is a finite free B -module equipped with a semi-linear φ -action such that $\varphi(M)$ generates M .
- (b) For any rational number $r = a/b \in \mathbb{Q}$ with coprime integers $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 1}$, we define a φ -module M_r over B of rank b as follows. Let e_1, \dots, e_b be a basis of M_r . The φ -action on M_r is determined by $\varphi(e_i) = e_{i+1}$ ($1 \leq i \leq b-1$) and $\varphi(e_b) = p^a e_1$. This φ -module M_r is called an *elementary* φ -module of slope r .

The theory of Dieudonné-Manin decomposition says that every φ -module M over $\tilde{\mathbf{B}}$ has can be decomposed as a direct sum of φ -modules isomorphic to elementary ones, namely, $M \simeq \bigoplus_i M_{r_i}$.

Definition 2.13. A φ -module M over \mathbf{B}_K^\dagger is *isocline of slope r* if in the Dieudonné-Manin decomposition of $M \otimes_{\mathbf{B}_K^\dagger} \tilde{\mathbf{B}}$, all r_i 's are equal to r .

As for φ -modules over the Robba rings $\mathbf{B}_{\text{rig},K}^\dagger$, Kedlaya proves the following slope decomposition theorem.

Theorem 2.14. *Let D be a φ -module over $\mathbf{B}_{\text{rig},K}^\dagger$. There is a unique filtration*

$$0 = D_0 \subset D_1 \subset \dots \subset D_n = D$$

such that for each i , we have $D_i/D_{i-1} \simeq M_i \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_{\text{rig},K}^\dagger$ for some φ -module M_i over \mathbf{B}_K^\dagger isocline of slope r_i and the slopes satisfy $r_1 < r_2 < \dots < r_n$.

Definition 2.15. A φ -module D over $\mathbf{B}_{\text{rig},K}^\dagger$ is called *étale* if in the slope decomposition of D we have $n = 1$ and $r_1 = 0$. Namely, it comes from a φ -module isocline of slope 0.

Now we come back to the discussion of (φ, Γ) -modules. Clearly, a (φ, Γ) -module is just a φ -module over $\mathbf{B}_{\text{rig},K}^\dagger$ equipped with an extra Γ -action.

Definition 2.16. A (φ, Γ) -module D over $\mathbf{B}_{\text{rig},K}^\dagger$ is called *étale* if it is étale as a φ -module.

Theorem 2.17. *There is an equivalence of categories:*

$$D_{\text{rig}}^\dagger : \{p\text{-adic representations of } G_K\} \longrightarrow \{\text{étale } (\varphi, \Gamma)\text{-modules over } \mathbf{B}_{\text{rig},K}^\dagger\}$$

sending V to $D_{\text{rig}}^\dagger(V) = (\mathbf{B}_{\text{rig}}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}$. The inverse is given by $D \mapsto (\mathbf{B}_{\text{rig}}^\dagger \otimes_{\mathbf{B}_{\text{rig},K}^\dagger} D)^{\varphi=1}$.

3. FAMILIES OF p -ADIC GALOIS REPRESENTATIONS

In this section, we study p -adic Galois representations in rigid analytic families. In particular, we extend the study of p -adic Hodge theory and (φ, Γ) -modules into family versions.

First, let us clarify some notations. Let G be a topological group and let X be a rigid analytic space over \mathbb{Q}_p . By a *family of p -adic representations* of G on X , we mean a locally free \mathcal{O}_X -module V_X equipped with an \mathcal{O}_X -linear continuous action of G . If $X = \mathrm{Sp}(S)$ is an affinoid space, a family of p -adic representation is also called an *S -linear G -representation*, denoted by V_S . Moreover, If $\mathrm{Sp}(R) \subset \mathrm{Sp}(S)$ is an affinoid subdomain and V_S is a family of p -adic representation over $\mathrm{Sp}(S)$, we use V_R to denote the base change of V_S from S to R . Finally, for every point $x \in \mathrm{Sp}(S)$, we write V_x for the specialization $V_S \otimes_S k(x)$ of V_S at x .

3.1. p -adic Hodge theory in families. In this section, we give a brief review on various p -adic Hodge theoretic functors for families of p -adic representations of $G_{\mathbb{Q}_p}$. We refer the reader to [BeCo08] and [KL10] for more details.

3.1.1. The functors D_* in families. Let S be a \mathbb{Q}_p -affinoid algebra, and let V_S be an S -linear $G_{\mathbb{Q}_p}$ -representation.

Let $\mathbf{B}_{\mathrm{HT}}^+$, $\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{B}_{\mathrm{crys}}^+$ be the Hodge-Tate, de Rham and crystalline period rings used in p -adic Hodge theory. For each $k > 0$, $\mathbf{B}_{\mathrm{dR}}^+/(t^k)$ is naturally a \mathbb{Q}_p -Banach space. This gives a Fréchet topology on

$$\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_k \mathbf{B}_{\mathrm{dR}}^+/(t^k).$$

So we can define $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_k S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}^+/(t^k)$. We can also define $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{HT}}^+$ and $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{crys}}^+$ as $\mathbf{B}_{\mathrm{HT}}^+$, $\mathbf{B}_{\mathrm{crys}}^+$ have natural \mathbb{Q}_p -Banach space structures. For an S -linear

representation V_S of $G_{\mathbb{Q}_p}$, following [BeCo08], we set

$$D_{\text{HT}}(V_S) = ((S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{HT}}) \otimes_S V_S)^{G_{\mathbb{Q}_p}},$$

$$D_{\text{dR}}^+(V_S) = ((S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+) \otimes_S V_S)^{G_{\mathbb{Q}_p}},$$

and

$$D_{\text{crys}}^+(V_S) = ((S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+) \otimes_S V_S)^{G_{\mathbb{Q}_p}}.$$

Taking an direct limit, we can define $D_{\text{dR}}(V_S)$. More precisely, we define

$$\begin{aligned} D_{\text{dR}}(V_S) &= \bigcup_i \text{Fil}^{-i} D_{\text{dR}}(V_S) = \bigcup_i ((S \widehat{\otimes}_{\mathbb{Q}_p} t^{-i} \mathbf{B}_{\text{dR}}^+) \otimes V_S)^{G_{\mathbb{Q}_p}} \\ &= \bigcup_i \varprojlim_k ((S \widehat{\otimes}_{\mathbb{Q}_p} (t^{-i} \mathbf{B}_{\text{dR}}^+ / t^k)) \otimes V_S)^{G_{\mathbb{Q}_p}}. \end{aligned}$$

The module $D_{\text{crys}}(V_S)$ can be defined similarly.

3.1.2. *Flat base change for D_{dR}^+ .* The following proposition ensures the flat base change property of the functor $D_{\text{dR}}^+(\cdot)$.

Proposition 3.1. *Let V_S be an S -linear $G_{\mathbb{Q}_p}$ -representation. If $f : S \rightarrow S'$ is a flat morphism of \mathbb{Q}_p -affinoid algebras, then*

$$D_{\text{dR}}^+(V_S) \otimes_S S' \xrightarrow{\sim} D_{\text{dR}}^+(V \otimes_S S').$$

Proof. This is proved in the proof in [Bel13, Proposition 5.30]. □

3.2. (φ, Γ) -modules in rigid analytic families.

3.2.1. *The (φ, Γ) -module functor D_{rig}^+ .* In [BeCo08], Berger and Colmez construct the family version of overconvergent (φ, Γ) -modules functor for free S -linear representations. This functor is later generalized to general S -linear representations by Kedlaya and the second author in [KL10].

More precisely, let V_S be an S -linear $G_{\mathbb{Q}_p}$ -representation of rank d . For sufficiently large s , one can construct a locally free $S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\mathbb{Q}_p}^{\dagger,s}$ -module $D^{\dagger,s}(V_S)$ of rank d such that for any $x \in M(S)$, $D^{\dagger,s}(V_S) \otimes_S S/\mathfrak{m}_x$ is naturally isomorphic to $D^{\dagger,s}(V_x)$. We set

$$(S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\mathbb{Q}_p}^{\dagger}) = \bigcup_s S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\mathbb{Q}_p}^{\dagger,s}$$

and define

$$D^{\dagger}(V_S) = (S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\mathbb{Q}_p}^{\dagger}) \otimes_{S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\mathbb{Q}_p}^{\dagger,s}} D^{\dagger,s}(V_S) = \bigcup_s D^{\dagger,s}(V_S).$$

This is a locally free $S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\mathbb{Q}_p}^{\dagger}$ -module of rank d and specializes to $D^{\dagger}(V_x)$ for any $x \in M(S)$. Moreover, $D^{\dagger}(V_S)$ is equipped with commuting semilinear φ, Γ -actions. This makes $D^{\dagger}(V_S)$ an *étale* (φ, Γ) -module over $S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\mathbb{Q}_p}^{\dagger}$ in the sense of [KL10, Definition 2.8].

For sufficiently large s , we define

$$D_{\text{rig}}^{\dagger,s}(V_S) = (S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}) \otimes_{S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}} D^{\dagger,s}(V_S).$$

We set

$$S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger} = \bigcup_s S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s},$$

and define

$$D_{\text{rig}}^{\dagger}(V_S) = (S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger}) \otimes_{S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger,s}} D_{\text{rig}}^{\dagger,s}(V_S) = \bigcup_s D_{\text{rig}}^{\dagger,s}(V_S).$$

Then $D_{\text{rig}}^{\dagger}(V_S)$ is an *étale family of* (φ, Γ) -module over $S\widehat{\otimes}_{\mathbb{Q}_p}\mathbf{B}_{\text{rig},\mathbb{Q}_p}^{\dagger}$ in the sense of [KL10, Definition 6.3].

3.2.2. *The functors D_{dif}^+ and D_{Sen} .* The construction of (φ, Γ) -modules helps to construct the family version of the functors D_{dif}^+ and D_{Sen} . The same construction for a single point recovers the original constructions in [Fon04] and [Sen73].

Recall that for $0 < s \leq r_n = p^{n-1}(p-1)$, one has the localization map

$$\iota_n : \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^{\dagger, s} \rightarrow \mathbb{Q}_p(\zeta_{p^n})[[t]].$$

This induces a continuous map $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^{\dagger, s} \rightarrow S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})[[t]]$. Define

$$D_{\text{dif}}^{+, n}(V_S) = (S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})[[t]]) \otimes_{\iota_n, S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^{\dagger, s}} D_{\text{rig}}^{\dagger, s}(V_S).$$

It is clear that $D_{\text{dif}}^{+, n}(V_S)$ is a locally free $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})[[t]]$ -module of rank d equipped with a semilinear Γ -action.

Abusing the notation, we still denote by ι_n the natural map $\iota_n : D_{\text{rig}}^{\dagger, s}(V_S) \rightarrow D_{\text{dif}}^{+, n}(V_S)$. We define

$$D_{\text{dif}}^+(V_S) = \bigcup_n D_{\text{dif}}^{+, n}(V_S).$$

Moreover, we define

$$D_{\text{Sen}}^n(V_S) = D_{\text{dif}}^{+, n}(V_S)/(t)$$

and we set

$$D_{\text{Sen}}(V_S) = \bigcup_n D_{\text{Sen}}^n(V_S).$$

In particular, $D_{\text{Sen}}^n(V_S)$ is a locally free $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$ -module of rank d . Clearly, the constructions of D_{dif}^+ and D_{Sen} are compatible with specializations because the construction of D_{rig}^{\dagger} is.

On $D_{\text{Sen}}(V_S)$, we can construct ‘‘Sen operator’’ generalizing the one constructed in [Sen80]. Pick a topological generator $\gamma \in \Gamma$ which is close enough to 1. Consider the operator

$$\Theta_{\text{Sen}} := \frac{\log \gamma}{\log_p \chi(\gamma)}.$$

This definition is independent of the choice of γ . The roots of the characteristic polynomial (called *Sen polynomial*) of Θ_{Sen} are called the *Hodge-Tate-Sen weights* of V_S . We remark that the specialization of the Hodge-Tate-Sen weights at a point $x \in \text{Sp}(S)$ coincide with the *generalized Hodge-Tate weights* of V_x studied by Sen [Sen80].

The following theorem compare the constructions from (families version) p -adic Hodge theory and those from (families of) (φ, Γ) -modules. It generalizes the comparison results in [Sen73], [Fon04], and [Ber02] to rigid families.

Theorem 3.2. [Bel13, Theorem 5.9, Theorem 5.11, Theorem 5.13] *For an S -linear representation V_S of $G_{\mathbb{Q}_p}$, we have $D_{\text{HT}}(V_S) = \bigoplus_{i \in \mathbb{Z}} (D_{\text{Sen}}(V_S) \cdot t^i)^\Gamma$, $D_{\text{dR}}^+(V_S) = D_{\text{dif}}^+(V_S)^\Gamma$ and $D_{\text{crys}}^+(V_S) = (D_{\text{rig}}^\dagger(V_S))^\Gamma$.*

4. THE EIGENCURVE

The notion of families of p -adic modular forms was first studied by Serre [Ser]. The rough idea is as follows. Let $A = \mathbb{Q}_p\langle T \rangle$ and let $D = \mathrm{Sp}(A)$ be the p -adic rigid (closed) unit disk. In particular, A is the ring of rigid analytic functions on D . For any $a \in A$ and any integer k , viewed as a point in D , we write $a(k)$ for the evaluation of a at k . If we look at q -expansions, a p -adic family of modular eigenforms on D is a formal expansion

$$f = \sum_{n \geq 0} a_n q^n \in A[[q]]$$

such that for each integer $k \in D$, the specialization

$$f_k = \sum_{n \geq 0} a_n(k) q^n \in \mathbb{C}_p[[q]]$$

is an eigenform of weight k . In [Ser], Serre studied the family of p -adic Eisenstein series. The upshot is that it provides arithmetic information about the possible congruences between coefficients of modular forms (in different weights). In particular, the congruences of the constant terms give congruences between the special values of zeta-functions.

The theory was further developed by Hida in [HidaA] and [HidaB], in which he constructed p -adic families of cuspidal Hecke eigenforms, varying analytically with weights. One can also attach families of Galois representations to the families of eigenforms. This later led to Mazur's important work on Galois deformation theory. However, Hida's construction only apply to ordinary modular forms. This restriction on ordinary forms was resolved by Coleman and Mazur's work [CM98] in which they constructed the *eigencurve*. It is a rigid analytic curve \mathcal{C} whose points parameterize all (finite slope) overconvergent p -adic eigenforms, including the ordinary ones. There is a natural map $\pi : \mathcal{C} \rightarrow \mathcal{W}$ whose image characterizes the weights of the eigenforms. Here the rigid space \mathcal{W} is called the *weight space*. The construction is later generalized

to any tame level N by Buzzard in [Buz07]. Understanding the geometry of the eigencurve (or, more generally, eigenvarieties) can provide arithmetic information about p -adic modular forms.

4.1. Overconvergent modular forms. In this section, we review the constructions of overconvergent modular forms, which are first studied by Coleman [Col97].

We first fix some notations. Let N be a positive integer prime to p . For each $m \geq 1$, let $X = X_1(Np^m)$ be the rigid analytic modular curve over \mathbb{Q}_p of level $\Gamma_1(Np^m)$, and let $\mathcal{E} \rightarrow X$ denote the universal semiabelian scheme. Let $e : X \rightarrow \mathcal{E}$ be the identity section. Consider the line bundle $\omega := e^*\Omega_{\mathcal{E}/X}^1$ over X , namely, the sheaf of invariant differentials. We also write $X_1(Np^m)(0)$ for the ordinary locus of $X_1(Np^m)$.

Inspired by Serre's work, Katz defined the notion of p -adic modular forms [Kat73], which are p -adic interpolations of classical modular forms. Roughly speaking, a p -adic modular form of level $\Gamma_1(Np^m)$ and weight k is a section of ω^k defined on the ordinary locus $X_1(Np^m)(0)$. Each p -adic modular form has a q -expansion at ∞ . For the precise definition, please refer to [Kat73].

In [Col97], Coleman studied overconvergent modular forms, which are p -adic modular forms converging beyond the ordinary locus. He showed that a large class of overconvergent modular forms live in a p -adic family. One can also work with general weights by p -adically interpolates classical (integral) weights.

We start with the notion of the weight space.

Definition 4.1. The *weight space* of tame level N is a rigid analytic space $\mathcal{W} = \mathcal{W}_{p,N}$ over \mathbb{Q}_p whose K -points are given by $\mathcal{W}(K) = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times, K^\times)$.

We have a natural embedding $\mathbb{Z} \hookrightarrow \mathcal{W}(\mathbb{Q}_p)$ by sending integer k to the character $a \mapsto a^k$. In particular, the classical weights can be viewed as points in the weight space. One can show that \mathcal{W} is isomorphic to disjoint union of finitely many open unit disks.

Coleman's definition of overconvergent modular forms treat integral weights and non-integral weights separately. The definition of overconvergent modular forms of integral weights are geometric, while he uses p -adic Eisenstein series and a trick to define the forms for general weights.

For weight $\kappa = k \in \mathbb{Z}$, the construction is as follows. Let $0 < w < p/(p+1)$ be a rational number. Let K/\mathbb{Q}_p be a finite extension containing an element with valuation w . Let E_{p-1} be the normalized Eisenstein series of level 1 and weight $p-1$. Note that E_{p-1} is a lift of the Hasse invariant. Consider the following open subset of $X_1(N)$:

$$X_1(N)(w) := \{x \in X_1(N) \mid |E_{p-1}| \geq p^{-w}\} \subset X_1(N),$$

the *strict neighborhood of width w* of the ordinary locus of $X_1(N)$. Moreover, consider the modular curve $X(N, p)$ of level $\Gamma_1(N) \cap \Gamma_0(p)$. Then the natural projection $X(N, p) \rightarrow X_1(N)$ has a section given by canonical subgroup. Define subset $X_1(Np)(w) \subset X_1(Np)$ by

$$X_1(Np)(w) := X_1(Np) \times_{X(N, p)} X_1(N)(w).$$

Now we define a *w -overconvergent modular form* of weight k and level $\Gamma_1(Np)$ to be an element of $H^0(X_1(Np)(w), \omega^{\otimes k})$. An *overconvergent modular form* is nothing but a w -overconvergent modular form for some $w > 0$. The notion of overconvergent modular forms for level $\Gamma_1(Np^m)$ can be defined similarly. We remark that every overconvergent modular form has q -expansions at every cusp in $X_1(Np^m)(w)$.

Now we treat the case when the weight $\kappa \in \mathcal{W}(K)$ is not an integer. The definition uses Serre's p -adic Eisenstein series E_κ (ref. [Col97]). In particular, for any $\kappa \in \mathcal{W}$, the Eisenstein series E_κ has a q -expansion at the cusp ∞ . Let f be a p -adic modular form of weight κ . Then f is called a *w -overconvergent modular form* of weight κ if f/E_κ extends to a section of $\mathcal{O}_{X_1(Np^m)}$ over $X_1(Np^m)(w)$. By an *overconvergent*

modular form of weight κ we mean a w -overconvergent form for some $w > 0$.

Finally, we remark that there is a geometric definition given by Andreatta-Iovita-Stevens in [AIS12], in which they constructed explicitly a rigid analytic bundle ω^κ on $X_1(Np^m)(w)$ whose sections are the w -overconvergent modular forms.

4.2. The Coleman-Mazur eigencurve. For a newform f , let ρ_f denote the p -adic Galois representation of $G_{\mathbb{Q}}$ associated to f . Recall that a p -modular residual representation of tame level N is a two dimensional $G_{\mathbb{Q}}$ -representation \bar{V} over $\bar{\mathbb{F}}_p$ which is isomorphic to the reduction of ρ_f for some newform f of level $\Gamma_1(Np^m)$ with $m \geq 1$. The Eichler-Shimura relation implies that the $G_{\mathbb{Q}}$ -action on \bar{V} factors through $G_{\mathbb{Q},\Sigma}$.

Let $R_{\bar{V}}$ be the universal deformation ring of the pseudo-representation associated to \bar{V} , and let $r_{\bar{V}}$ be the associated rank 2 universal pseudo-representation. Let $X_{\bar{V}} = \mathrm{Sp}R_{\bar{V}}[1/p]$ be the rigid analytic space over \mathbb{Q}_p associated to $R_{\bar{V}}$. For a newform f of level $\Gamma_1(Np)$, if the reduction of ρ_f is isomorphic to \bar{V} , and if the U_p -eigenvalue α_f of f is nonzero, it gives rise to a modular point (ρ_f, α_f^{-1}) in $X_{\bar{V}} \times \mathbb{G}_m$. Let $X_p = \coprod X_{\bar{V}}$ where \bar{V} runs through all (finitely many) p -modular tame level N residual representations. By the work of Coleman and Mazur [CM98], one may regard the eigencurve $\mathcal{C}_{p,N}$ of tame level N as an analytic subspace of $X_p \times \mathbb{G}_m$, whose reduction is equal to the Zariski closure of all such modular points (ρ_f, α_f^{-1}) . As a result of the construction, we see that the \mathbb{C}_p -points of $\mathcal{C}_{p,N}$ correspond bijectively to finite slope overconvergent p -adic eigenforms of tame level N . We remark that $\mathcal{C}_{p,N}$ contains not only forms of level $\Gamma_1(Np)$, but also all forms of level $\Gamma_1(Np^m)$ for $m \geq 1$.

From this construction using deformation rings, it is not obvious that $\mathcal{C}_{p,N}$ is indeed a curve. To prove more geometric properties of the eigencurve, Coleman and Mazur give another construction using Hecke algebras. They proved that the new construction is a curve and is same as the previous construction. The new method

uses the Fredholm theory for the compact U_p -operators on the p -adic Banach space of overconvergent modular forms. For details, please refer to [CM98].

4.3. Families of Galois representations on the eigencurve. In this section, we construct a natural family of p -adic Galois representations on the normalization of the eigencurve.

Let r be the universal pseudo-representations on X_p . We obtain a rank 2 pseudo-representation of $G_{\mathbb{Q},\Sigma}$ on the eigencurve $\mathcal{C}_{p,N}$ by pulling back r through $\mathcal{C}_{p,N} \subset X_p \times \mathbb{G}_m$. Let $\tilde{\mathcal{C}}_{p,N}$ be the normalization of $\mathcal{C}_{p,N}$. By [CM98, Theorem 5.1.2] (see the remark after it), any rank 2 pseudo-representations of $G_{\mathbb{Q},\Sigma}$ on a smooth rigid analytic curve over \mathbb{Q}_p can be converted naturally to a family of p -adic representations of $G_{\mathbb{Q},\Sigma}$. Thus there exists a family of $G_{\mathbb{Q},\Sigma}$ -representations $V_{\tilde{\mathcal{C}}}$ of rank 2 on $\tilde{\mathcal{C}}_{p,N}$ whose associated pseudo-representation is isomorphic to the pullback of the pseudo-representation on $\mathcal{C}_{p,N}$.

We still use $V_{\tilde{\mathcal{C}}}$ to denote its restriction to $G_{\mathbb{Q}_p}$. Let $\kappa \in \mathcal{O}(\tilde{\mathcal{C}})$ be the pull-back of the weight function on \mathcal{C} . We claim that it has generalized Hodge-Tate weights 0 and $\kappa - 1$. Indeed, this follows from the fact that the specializations V_z of $V_{\tilde{\mathcal{C}}}$ at classical points z are crystalline with Hodge-Tate weight 0, $k - 1$, and that the classical points are dense in $\tilde{\mathcal{C}}$. An immediate consequence is that $V_{\tilde{\mathcal{C}}}$ admits a Hodge-Tate period. Later, we will show that we can lift this Hodge-Tate period to a de Rham and even crystalline period.

Moreover, we write $\alpha \in \mathcal{O}(\tilde{\mathcal{C}})$ to be the pull-back of the U_p -eigenvalue and let $Z \subset \tilde{\mathcal{C}}$ be the subset of classical points z such that V_z has distinct crystalline Frobenius eigenvalues. Then the family of representations $V_{\tilde{\mathcal{C}}}$ forms a *weakly refined family* of p -adic Galois representations of $G_{\mathbb{Q}_p}$ (in the sense of [BelChe, Definition 4.2.7]) with $F = \alpha$, $\kappa_1 = 0$, $\kappa_2 = \kappa - 1$ and dense subset $Z \subset \tilde{\mathcal{C}}$.

5. FINITE SLOPE SUBSPACE

The notion of *finite slope subspace* was first introduced by Kisin. In [Kis03], the finite slope subspaces are formulated in order to understand the relations between de Rham periods and crystalline periods of the family of Galois representations on the eigencurve. More precisely, it is proved that the positive de Rham periods coincide with positive crystalline periods on “ Y -small” affinoid subdomains of the eigencurve. The definition is later enhanced by Liu [Liu12] by removing the Y -smallness condition imposed in Kisin’s original definition. This modification is important for our purpose because we need to compare the de Rham and crystalline periods on general affinoid subdomains. Therefore, we adopt the definitions as in [Liu12].

From now on, we take $K = \mathbb{Q}_p$. Let X be a separated and reduced rigid analytic space over some finite extension L of \mathbb{Q}_p . Let V_X be a family of p -adic representations of $G_{\mathbb{Q}_p}$ over X . We further assume that the Sen polynomial for V_X is of the form $TQ(T)$ for some $Q(T) \in \mathcal{O}_X[T]$. Let $\alpha \in \mathcal{O}(X)^\times$ be an invertible analytic function on X .

We need the following notation in the definition below. If $X' \subset X$ is an analytic subspace and h is an analytic function on X , we write X'_h for the non-vanishing locus of h in X' . In particular, if j is an integer, then $X'_{Q(j)}$ excludes exactly those points which has $-j$ as a Hodge-Tate weight.

Definition 5.1. Let (X, V_X, α) be a triple as above. An analytic subspace $X_{fs} \subset X$ is called a *finite slope subspace* of X with respect to the pair (V_X, α) if it satisfies the following two conditions:

- (1) For every integer $j \leq 0$, the subspace $(X_{fs})_{Q(j)}$ is Zariski-dense in X_{fs} .
- (2) For any affinoid algebra R over L and any morphism $g : \mathrm{Sp}(R) \rightarrow X$ which factors through $X_{Q(j)}$ for every integer $j \leq 0$, the morphism g factors through

X_{fs} if and only if the natural map

$$\iota_n : (D_{\text{rig}}^\dagger(V_R))^{\varphi=g^*(\alpha), \Gamma=1} \rightarrow (D_{\text{dif}}^{+,n}(V_R))^\Gamma$$

is an isomorphism for all sufficiently large n .

The uniqueness of the finite slope subspace is obvious. The following result guarantees the existence of finite slope subspace.

Theorem 5.2 ([Liu12], Theorem 3.3.1). *Given any triple (X, V_X, α) as above, the rigid analytic space X has a unique finite slope subspace X_{fs} .*

The finite slope subspace has the following important property which will play an important role in next section.

Theorem 5.3 ([Liu12], Theorem 3.3.4). *Let $\text{Sp}(S)$ be an affinoid subdomain of X_{fs} and let $|\cdot|_{\text{sp}}$ denote the spectral norm taken on S . Then for any $n \geq n(V_S)$ and $k > \log_p |\alpha^{-1}|_{\text{sp}}$, the natural map*

$$(D_{\text{rig}}^\dagger(V_S))^{\varphi=\alpha, \Gamma=1} \rightarrow (D_{\text{dif}}^{+,n}(V_S)/(t^k))^\Gamma$$

is an isomorphism.

Remark 5.4. Notice that we have $D_{\text{rig}}^\dagger(V_S)^\Gamma = D_{\text{crys}}^+(V_S)$ and $D_{\text{dif}}^+(V_S)^\Gamma = D_{\text{dR}}^+(V_S)$ by Theorem 3.2. This theorem indeed allows us to compare positive de Rham periods and crystalline ones on the affinoid $\text{Sp}(S)$.

Now we apply the above discussion to the family of Galois representations on the eigencurve. Recall that we have a family of 2-dimensional $G_{\mathbb{Q}, \Sigma}$ -representations on the normalization $\tilde{\mathcal{C}}$ of $\mathcal{C} = \mathcal{C}_{p,N}$. We still use $V_{\tilde{\mathcal{C}}}$ to denote the corresponding representation of $G_{\mathbb{Q}_p}$. Let $\alpha_{\mathcal{C}} \in \mathcal{O}(\mathcal{C}_{p,N})^\times$ be the function of U_p -eigenvalues and let $\alpha_{\tilde{\mathcal{C}}} \in \mathcal{O}(\tilde{\mathcal{C}}_{p,N})^\times$ be the pullback of $\alpha_{\mathcal{C}}$. Note that $V_{\tilde{\mathcal{C}}}$ has 0 as a Hodge-Tate-Sen weight. So we may write the Sen polynomial of $V_{\tilde{\mathcal{C}}}$ as $T(T - \kappa_{\tilde{\mathcal{C}}})$ for some $\kappa_{\tilde{\mathcal{C}}} \in \mathcal{O}(\tilde{\mathcal{C}}_{p,N})$.

Proposition 5.5. *The finite slope subspace associated to the pair $(V_{\tilde{\mathcal{C}}}, \alpha_{\tilde{\mathcal{C}}})$ is the normalized eigencurve $\tilde{\mathcal{C}}$ itself.*

Proof. This is one of the main result of [Liu12]. More precisely, the discussion in [Liu12, Section 5.4] shows that $V_{\tilde{\mathcal{C}}}$ forms a *weakly refined family* of p -adic Galois representations. Then by [Liu12, Proposition 5.1.4], the finite slope subspace of such family is the underlying space itself. \square

6. THE MAIN THEOREM

With all the preparations above, we can finally prove the main theorem.

Theorem 6.1 (Properness of the Eigencurve). *Let $\mathcal{C}_{p,N}$ be the Coleman-Mazur eigencurve of tame level N , and let $\pi : \mathcal{C}_{p,N} \rightarrow \mathcal{W}$ denote the natural projection to the weight space. Let D be the closed unit disk over some finite extension L over \mathbb{Q}_p , and let D^* be the punctured disk with the origin removed. Suppose $h : D^* \rightarrow \mathcal{C}_{p,N}$ is a morphism of rigid analytic spaces such that $\pi \circ h$ extends to D . Then h extends to a morphism $\tilde{h} : D \rightarrow \mathcal{C}_{p,N}$ compatible with $\pi \circ h$.*

As mentioned in the introduction section, we attack this problem by studying the associated family of Galois representations on the disk. Then we further reduce the problem to a local question by restricting to $G_{\mathbb{Q}_p}$ -representations. Here is the breakdown of the strategy:

- We look at the associated family of $G_{\mathbb{Q},\Sigma}$ -representations V_{D^*} on the punctured disk D^* . It is obtained by pulling-back the natural $G_{\mathbb{Q},\Sigma}$ -representations on the normalized eigencurve $\tilde{\mathcal{C}}_{p,N}$. In section 6.1, we show that V_{D^*} naturally extends to a representation V_D on the entire disk.
- In section 6.2, we reduce the problem to a local question by looking at the associated $G_{\mathbb{Q}_p}$ -representations, still denoted by V_{D^*} and V_D . Write $V_{D^*}^*$ and V_D^* for the dual of V_{D^*} and V_D , respectively. We claim that the finite slope subspace of $V_{D^*}^*$ is the punctured disk D^* itself. It in turn allows us to compare the de Rham periods and crystalline periods on the punctured disk. This is proved in section 6.2.1.
- Using an explicit calculation, in section 6.2.2 we show that the positive de Rham periods $D_{\text{dR}}^+(V_{D^*}^*)$ coincides with the positive crystalline periods $D_{\text{crys}}^+(V_{D^*}^*)$.

- Finally, we wrap up the proof in section 6.3. More precisely, we find a nonzero de Rham period on the disk and show that it is indeed a crystalline period with nonzero Frobenius eigenvalue. This suffices to conclude the proof.

6.1. Families of Galois representations on the punctured disk. Let $\mathcal{C}_{p,N}$ be the Coleman-Mazur eigencurve of tame level N and let $\tilde{\mathcal{C}}_{p,N}$ be the normalization of $\mathcal{C}_{p,N}$. Recall that there exists a family of $G_{\mathbb{Q},\Sigma}$ -representations $V_{\tilde{\mathcal{C}}}$ of rank 2 on $\tilde{\mathcal{C}}_{p,N}$ whose associated pseudo-representation is isomorphic to the pullback of the pseudo-representation on $\mathcal{C}_{p,N}$ (see section 4.3).

Moreover, we may assume that the given map $h : D^* \rightarrow \mathcal{C}_{p,N}$ is dominant. (Otherwise, the image of h must be a single point by the connectedness of D^* , and the situation becomes trivial.) Since D^* is smooth, the map $h : D^* \rightarrow \mathcal{C}_{p,N}$ must factor through $\tilde{\mathcal{C}}_{p,N}$. Abusing the notation, we still denote the resulting map $D^* \rightarrow \tilde{\mathcal{C}}_{p,N}$ by h . Let V_{D^*} denote the pullback of $V_{\tilde{\mathcal{C}}}$ along h . The goal of this section is to show that the composition $u : D^* \rightarrow \mathcal{C}_{p,N} \hookrightarrow X_p \times \mathbb{G}_m \rightarrow X_p$ extends to a morphism on the entire disk D . Consequently, this implies that V_{D^*} extends to a family of $G_{\mathbb{Q},\Sigma}$ -representations on D .

Before proceeding, let us first make the following observation.

Lemma 6.2. *Let $F \in \mathcal{O}(D^*)$ be a bounded function on the punctured disk. Namely, there exists a constant C such that $|F(x)| \leq C$ for all $x \in D^*$. Then F extends uniquely to an element of $\mathcal{O}(D)$.*

Proof. The uniqueness is obvious. After scaling, we may assume $|F(x)| \leq 1$ for all $x \in D^*$. Let $D_{0,n} = \mathrm{Sp}(\mathbb{Q}_p\langle T, p^n T^{-1} \rangle)$ be the closed annulus with outside radius 1 and inside radius p^{-n} . By the assumption above, $|F| \leq 1$ on $D_{0,n}$ for any $n \geq 1$. This implies $F \in \mathbb{Z}_p\langle T, p^n T^{-1} \rangle$ for all $n \geq 1$. Therefore,

$$F \in \bigcap_{n \geq 1} \mathbb{Z}_p\langle T, p^n T^{-1} \rangle = \mathbb{Z}_p\langle T \rangle = \mathcal{O}(D).$$

□

Proposition 6.3. *The composition morphism $u : D^* \rightarrow \mathcal{C}_{p,N} \hookrightarrow X_p \times \mathbb{G}_m \rightarrow X_p$ extends to a morphism of rigid analytic spaces $\tilde{u} : D \rightarrow X_p$.*

Proof. Since D^* is connected, it maps to $X_{\bar{V}}$ for some p -modular residue representation \bar{V} . Note that $X_{\bar{V}}$ is the generic fibre of $\mathrm{Spf}(R_{\bar{V}})$. Thus for any $x \in D^*$ and $t \in R_{\bar{V}}$, we have $|u^*(t)(x)| = |t(u(x))| \leq 1$. By Lemma 6.2, $u^*(t)$ extends uniquely to an element of $\mathbb{Z}_p\langle T \rangle$. This gives us a continuous morphism $R_{\bar{V}} \rightarrow \mathbb{Z}_p\langle T \rangle$, yielding the desired extension. □

Using this, we can show that V_{D^*} extends to the entire disk D . Denote by r_{D^*} the pseudo-representation associated to V_{D^*} . From the construction of V_{D^*} , we see that r_{D^*} is isomorphic to the pullback of the universal pseudo-representation on X_p along u .

Corollary 6.4. *The family of $G_{\mathbb{Q},\Sigma}$ -representations V_{D^*} extends to D .*

Proof. Let $\tilde{u} : D \rightarrow X_p$ be the morphism given by Proposition 6.3. Pulling back the universal pseudo-representation on X_p along \tilde{u} , we obtain a pseudo-representation r_D of $G_{\mathbb{Q},\Sigma}$ on D which extends r_{D^*} . Note that D is a smooth rigid analytic curve over \mathbb{Q}_p . As explained earlier in section 4, by [CM98, Theorem 5.1.2] again, one can convert r_D to a family of $G_{\mathbb{Q},\Sigma}$ -representations on D . This gives the desired extension. □

From now on, we denote by V_D the extended family of p -adic representations of $G_{\mathbb{Q},\Sigma}$ on D given by Corollary 6.4.

6.2. The local question. In this section, we study the restriction of various $G_{\mathbb{Q},\Sigma}$ -representations to $G_{\mathbb{Q}_p}$. Namely, we reduce the original problem to a local question.

Recall that we have families of $G_{\mathbb{Q},\Sigma}$ -representations V_{D^*} , V_D and $V_{\bar{c}}$ on D^* , D and

$\tilde{\mathcal{C}}$, respectively. We use the same notations to denote their restrictions to $G_{\mathbb{Q}_p}$. We also write $V_{D^*}^*$, V_D^* , and $V_{\tilde{\mathcal{C}}}^*$ for their duals.

6.2.1. *Finite slope subspace of $V_{D^*}^*$.* Let $\alpha_{\mathcal{C}} \in \mathcal{O}(\mathcal{C}_{p,N})^\times$ be the function encoding the U_p -eigenvalues. Let $\alpha_{\tilde{\mathcal{C}}} \in \mathcal{O}(\tilde{\mathcal{C}}_{p,N})^\times$ be the pullback of $\alpha_{\mathcal{C}}$ and let $\alpha \in \mathcal{O}(D^*)^\times$ be the pullback of $\alpha_{\tilde{\mathcal{C}}}$ via h . Since the family of p -adic representations $V_{\tilde{\mathcal{C}}}$ has 0 as a Hodge-Tate weight, we may write the Sen polynomial of $V_{\tilde{\mathcal{C}}}^*$ as $T(T - \kappa_{\tilde{\mathcal{C}}})$ for some $\kappa_{\tilde{\mathcal{C}}} \in \mathcal{O}(\tilde{\mathcal{C}}_{p,N})$. Let $\kappa = h^*(\kappa_{\tilde{\mathcal{C}}}) \in \mathcal{O}(D^*)$. It follows that the Sen polynomial of $V_{D^*}^*$ is $T(T - \kappa)$. In particular, we can talk about the finite slope subspace associated to the triple $(D^*, V_{D^*}^*, \alpha)$.

Proposition 6.5. *The finite slope subspace $(D^*)_{fs}$ of the punctured disk D^* associated to $(V_{D^*}^*, \alpha)$ is D^* itself.*

Proof. We need to check that the triple $(D^*, V_{D^*}^*, \alpha)$ satisfies the conditions (1) and (2) of Definition 5.1. According to Proposition 5.5, the finite slope subspace associated to the pair $(V_{\tilde{\mathcal{C}}}^*, \alpha_{\tilde{\mathcal{C}}})$ is $\tilde{\mathcal{C}}$ itself. Hence, the triple $(\tilde{\mathcal{C}}, \alpha_{\tilde{\mathcal{C}}}, V_{\tilde{\mathcal{C}}}^*)$ satisfies the conditions (1) and (2). In particular, $\tilde{\mathcal{C}}_{(\kappa_{\tilde{\mathcal{C}}}-j)}$ is scheme-theoretically dense in $\tilde{\mathcal{C}}$ for every $j \leq 0$. Since h is dominant and D^* is of dimension 1, we deduce that $D_{(\kappa-j)}^* = h^{-1}(\tilde{\mathcal{C}}_{(\kappa_{\tilde{\mathcal{C}}}-j)})$ is scheme-theoretically dense in D^* . Thus the triple $(D^*, V_{D^*}^*, \alpha)$ satisfies the condition (1). Moreover, the triple $(D^*, V_{D^*}^*, \alpha)$ also satisfies the condition (2) because $D_{(\kappa-j)}^*$ maps to $\tilde{\mathcal{C}}_{(\kappa_{\tilde{\mathcal{C}}}-j)}$ for every $j \leq 0$. This concludes the proof. \square

Proposition 6.6. *For any affinoid subdomain $\mathrm{Sp}(R)$ of D^* and any $k > \log_p |\alpha^{-1}|_{\mathrm{sp}}$ with $|\cdot|_{\mathrm{sp}}$ being the spectral norm on R , the natural map*

$$(\mathrm{D}_{\mathrm{rig}}^\dagger(V_R^*))^{\varphi=\alpha, \Gamma=1} \rightarrow (\mathrm{D}_{\mathrm{dif}}^+(V_R^*)/(t^k))^\Gamma$$

is an isomorphism. Furthermore, $(\mathrm{D}_{\mathrm{rig}}^\dagger(V_R^))^{\varphi=\alpha, \Gamma=1}$ is a locally free R -module of rank 1.*

Proof. Since the finite slope subspace of D^* is itself, it follows immediately from Theorem 5.3 that the given map is an isomorphism. Note that $\mathrm{Sp}(R)$ is smooth of dimension 1. One can deduce that the R -module $(D_{\mathrm{dif}}^+(V_R^*)/(t^k))^\Gamma$ is locally free as it is finite and torsion free. Moreover, by [Liu12, Corollary 1.5.6], the natural map

$$(D_{\mathrm{dif}}^+(V_R^*)/(t^k))^\Gamma \otimes_R k(x) \rightarrow (D_{\mathrm{dif}}^+(V_x^*)/(t^k))^\Gamma$$

is an isomorphism for any $x \in \mathrm{Sp}(R)$ with non-integral weight. It is clear that the right hand side is of $k(x)$ -dimension 1. Since the subset of points with non-integral weights is Zariski dense in $\mathrm{Sp}(R)$, we conclude that $(D_{\mathrm{dif}}^+(V_R^*)/(t^k))^\Gamma$ is a locally free R -module of rank 1, and so is $(D_{\mathrm{rig}}^\dagger(V_R^*))^{\varphi=\alpha, \Gamma=1}$. \square

Corollary 6.7. *For any affinoid subdomain $\mathrm{Sp}(R)$ of D^* , the natural map*

$$D_{\mathrm{crys}}^+(V_R^*)^{\varphi=\alpha} \rightarrow D_{\mathrm{dR}}^+(V_R^*)$$

is an isomorphism. Furthermore, they are locally free R -modules of rank 1.

Proof. By the previous proposition, the natural map

$$(D_{\mathrm{rig}}^\dagger(V_R^*))^{\varphi=\alpha, \Gamma=1} \rightarrow (D_{\mathrm{dif}}^+(V_R^*)/(t^k))^\Gamma$$

is an isomorphism for all sufficiently large k . Taking an inverse limit, we obtain natural isomorphism

$$(D_{\mathrm{rig}}^\dagger(V_R^*))^{\varphi=\alpha, \Gamma=1} \rightarrow D_{\mathrm{dif}}^+(V_R^*)^\Gamma = \varprojlim_k (D_{\mathrm{dif}}^+(V_R^*)/(t^k))^\Gamma.$$

Finally, we conclude by applying Theorem 3.2 which says $D_{\mathrm{crys}}^+(V_R^*) = D_{\mathrm{rig}}^\dagger(V_R^*)^\Gamma$ and $D_{\mathrm{dR}}^+(V_R^*) = D_{\mathrm{dif}}^+(V_R^*)^\Gamma$. \square

6.2.2. *De Rham periods vs crystalline periods.* By Corollary 6.7 in the previous section, we know that $D_{\mathrm{dR}}^+(V_R^*) = D_{\mathrm{crys}}^+(V_R^*)$ for any affinoid $\mathrm{Sp}(R) \subset D^*$; namely, the de

Rham periods and the crystalline periods coincide on the punctured disk. The goal of this section is to show $D_{\text{dR}}^+(V_D^*) = D_{\text{crys}}^+(V_D^*)$.

Let us first fix some notations which will be used in the rest of the thesis. Let $S = \mathbb{Q}_p\langle T \rangle$. For any $n \geq 0$ (resp. $n' > n \geq 0$), let $S_n = \mathbb{Q}_p\langle p^{-n}T \rangle$ (resp. $S_{n,n'} = \mathbb{Q}_p\langle p^{-n}T, p^{n'}T^{-1} \rangle$). Let V_n (resp. $V_{n,n'}$) be the restriction of V_D on $\text{Sp}(S_n)$ (resp. $\text{Sp}(S_{n,n'})$).

Definition 6.8. Let A be a \mathbb{Q}_p -Banach algebra.

- (i) For any $n \geq 0$, define the Banach algebra $A\langle p^{-n}T \rangle$ to be the ring of formal power series $\sum_{i \in \mathbb{N}} a_i T^i$ with $a_i \in A$ and such that $|a_i|p^{-ni} \rightarrow 0$ as $i \rightarrow \infty$. It is equipped with a Banach norm $|\sum_{i \in \mathbb{N}} a_i T^i| = \sup |a_i|p^{-ni}$.
- (ii) For any $n' > n \geq 0$, define the Banach algebra $A\langle p^{-n}T, p^{n'}T^{-1} \rangle$ to be the ring of Laurent series $\sum_{i \in \mathbb{Z}} a_i T^i$ with $a_i \in A$ and such that $|a_i|p^{-ni} \rightarrow 0$ as $i \rightarrow \infty$ and $|a_i|p^{-n'i} \rightarrow 0$ as $i \rightarrow -\infty$. It is equipped with a Banach norm $|\sum_{i \in \mathbb{Z}} a_i T^i| = \max\{\sup |a_i|p^{-ni}, \sup |a_i|p^{-n'i}\}$.

Using the facts that the elements $\{p^{-ni}T^i\}_{i \in \mathbb{N}}$ form an orthonormal basis of S_n and that $\{p^{-ni}T^i, p^{n'(i+1)}T^{-i-1}\}_{i \in \mathbb{N}}$ form an orthonormal basis of $S_{n,n'}$, we deduce the following lemma.

Lemma 6.9. *Let A be a \mathbb{Q}_p -Banach algebra. For any $n \geq 0$, we have natural identification of Banach algebras*

$$\eta_{n,A} : S_n \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T \rangle.$$

Similarly, for any $n' > n \geq 0$, we have natural identification of Banach algebras

$$\eta_{n,n',A} : S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T, p^{n'}T^{-1} \rangle.$$

Definition 6.10. Let $A = \varprojlim_{j \in J} A_j$ be a Fréchet algebra where A_j 's are \mathbb{Q}_p -Banach algebras.

- (i) Define the Fréchet algebra $A\langle p^{-n}T \rangle$ to be the inverse limit of Banach algebras $A_j\langle p^{-n}T \rangle$.
- (ii) For any $n' > n \geq 0$, define the Fréchet algebra $A\langle p^{-n}T, p^{n'}T^{-1} \rangle$ to be the inverse limit of Banach algebras $A_j\langle p^{-n}T, p^{n'}T^{-1} \rangle$.

Note that the natural inclusions $A_j\langle p^{-n}T \rangle \hookrightarrow A_j[[T]]$ induces an injective map

$$A\langle p^{-n}T \rangle = \varprojlim_{j \in J} A_j\langle p^{-n}T \rangle \hookrightarrow \varprojlim_{j \in J} A_j[[T]] = A[[T]].$$

Thus one may naturally identify $A\langle p^{-n}T \rangle$ as a subring of $A[[T]]$. Similarly, one can naturally identify $A\langle p^{-n}T, p^{n'}T^{-1} \rangle$ as a subset of $A[[T, T^{-1}]]$, which is not a ring!

Lemma 6.11. *We adopt the notations as in Definition 6.10. For any $n \geq 0$, we have natural identification of Fréchet algebras*

$$\eta_{n,A} : S_n \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T \rangle.$$

Similarly, for any $n' > n \geq 0$, we have natural identification of Fréchet algebras

$$\eta_{n,n',A} : S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} A \xrightarrow{\sim} A\langle p^{-n}T, p^{n'}T^{-1} \rangle.$$

Proof. This follows from first applying the previous lemma to the \mathbb{Q}_p -Banach algebras A_j and then taking inverse limits. \square

In particular, Lemma 6.9 applies to $A = \mathbf{B}_{\text{crys}}^+$ and Lemma 6.11 applies to $A = \mathbf{B}_{\text{dR}}^+ = \varprojlim \mathbf{B}_{\text{dR}}^+/(t^i)$.

Lemma 6.12. (i) *For any $n \geq 0$, the continuous map $\mathbf{B}_{\text{crys}}^+ \rightarrow \mathbf{B}_{\text{dR}}^+$ induces a natural inclusion $\mathbf{B}_{\text{crys}}^+\langle p^{-M}T \rangle \hookrightarrow \mathbf{B}_{\text{dR}}^+\langle p^{-M}T \rangle$.*

(ii) *For any $n' > n \geq 0$, the continuous map $\mathbf{B}_{\text{crys}}^+ \rightarrow \mathbf{B}_{\text{dR}}^+/(t^i)$ induces a natural inclusion $\mathbf{B}_{\text{crys}}^+\langle p^{-n}T, p^{n'}T^{-1} \rangle \hookrightarrow \mathbf{B}_{\text{dR}}^+\langle p^{-n}T, p^{n'}T^{-1} \rangle$.*

Proof. By the commutative diagram

$$\begin{array}{ccc}
\mathbf{B}_{\text{crys}}^+ \langle p^{-n}T \rangle & \longrightarrow & \mathbf{B}_{\text{dR}}^+ \langle p^{-n}T \rangle \\
\downarrow & & \downarrow \\
\mathbf{B}_{\text{crys}}^+ [[T]] & \longrightarrow & \mathbf{B}_{\text{dR}}^+ [[T]],
\end{array}$$

we see that the composition $\mathbf{B}_{\text{crys}}^+ \langle p^{-n}T \rangle \rightarrow \mathbf{B}_{\text{dR}}^+ \langle p^{-n}T \rangle \rightarrow \mathbf{B}_{\text{dR}}^+ [[T]]$ is injective. Hence the natural map $\mathbf{B}_{\text{crys}}^+ \langle p^{-n}T \rangle \rightarrow \mathbf{B}_{\text{dR}}^+ \langle p^{-n}T \rangle$ is injective. The proof of (ii) is similar. \square

As a consequence of Lemma 6.12, we may naturally identify $S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$ (resp. $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$) as a subring of $S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$ (resp. $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$).

Lemma 6.13. *For any $x \in S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$, if its image in $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}^+$ belongs to $S_{n,n'} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$, then $x \in S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$.*

Proof. By the previous lemmas, we may regard all the rings involved as subsets of $\mathbf{B}_{\text{dR}}^+ [[T, T^{-1}]]$. It follows from the assumption that

$$x \in \mathbf{B}_{\text{dR}}^+ \langle p^{-n}T \rangle \cap \mathbf{B}_{\text{crys}}^+ \langle p^{-n}T, p^{n'}T^{-1} \rangle \subseteq \mathbf{B}_{\text{dR}}^+ [[T]] \cap \mathbf{B}_{\text{crys}}^+ \langle p^{-n}T, p^{n'}T^{-1} \rangle = \mathbf{B}_{\text{crys}}^+ \langle p^{-n}T \rangle.$$

This is to say $x \in S_n \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+$. \square

Finally, we show that the crystalline and de Rham periods coincide on the entire disk.

Corollary 6.14. $D_{\text{crys}}^+(V_D^*) = D_{\text{dR}}^+(V_D^*)$.

Proof. Applying Corollary 6.7 to $R = S_{0,1}$, we have $D_{\text{crys}}^+(V_{0,1}^*) = D_{\text{dR}}^+(V_{0,1}^*)$. Using the previous lemma, we deduce that $D_{\text{dR}}^+(V_D^*) \subseteq D_{\text{crys}}^+(V_D^*)$. Indeed, any $x \in D_{\text{dR}}^+(V_D^*)$ must also live in $D_{\text{crys}}^+(V_{0,1}^*) = (S_{0,1} \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+ \otimes V_D^*)^{G_{\mathbb{Q}_p}}$. Thus $x \in (S_0 \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}^+ \otimes V_D^*)^{G_{\mathbb{Q}_p}} = D_{\text{crys}}^+(V_D^*)$. The inclusion from the other direction is obvious. \square

6.3. Proof of the main theorem. Applying Proposition 3.1 to the flat base change from S to $S_{0,1}$, we obtain isomorphism

$$D_{\text{dR}}^+(V_D^*) \otimes_S S_{0,1} \xrightarrow{\sim} D_{\text{dR}}^+(V_{0,1}^*).$$

By Corollary 6.7, we see that $D_{\text{dR}}^+(V_{0,1}^*)$ is a locally free $S_{0,1}$ -module of rank 1. In particular, this implies $D_{\text{dR}}^+(V_D^*) \neq 0$.

Pick a nonzero element $e \in D_{\text{dR}}^+(V_D^*)$. By dividing a suitable power of T , we may assume that the specialization e_0 of e at the puncture 0 is nonzero. Note that $e \in D_{\text{crys}}^+(V_D^*)$ by Corollary 6.14. Moreover, the image of e in $D_{\text{crys}}^+(V_{0,1}^*)$ belongs to $D_{\text{crys}}^+(V_{0,1}^*)^{\varphi=\alpha}$ by Corollary 6.7. That is, $\varphi(e) = \alpha e$ on $\text{Sp}(S_{0,1})$. Note that the norms of the U_p -eigenvalues of overconvergent p -adic eigenforms are less than or equal to 1. By Lemma 6.2, we must have $\alpha \in \mathcal{O}(D)$. Since $\text{Sp}(S_{0,1})$ is Zariski dense in D , this forces $\varphi(e) = \alpha e$ on the entire disk. In particular, $\varphi(e_0) = \alpha(0)e_0$. Since φ is injective on $D_{\text{crys}}^+(V_0^*)$, we conclude that $\alpha(0) \neq 0$. Thus $\alpha \in \mathcal{O}(D)^\times$.

Now we construct a map $\tilde{h} : D \rightarrow X_p \times \mathbb{G}_m$ of rigid analytic spaces by sending x to $(\tilde{u}(x), \alpha(x)^{-1})$ where \tilde{u} is given by Proposition 6.3. It is clear that $\tilde{h}|_{D^*} = h$. Since $\mathcal{C}_{p,N}$ is an analytic subspace of $X_p \times \mathbb{G}_m$, $\tilde{h}^{-1}(\mathcal{C}_{p,N})$ is an analytic subspace of D containing D^* . This forces $\tilde{h}^{-1}(\mathcal{C}_{p,N}) = D$, making \tilde{h} the desired extension of h . The proof of the main theorem is now complete.

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