Arithmetic Properties of Moduli Spaces and Topological String Partition Functions of Some Calabi-Yau Threefolds

Arithmetic Properties of Moduli Spaces and Topological String Partition Functions of Some Calabi-Yau Threefolds

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Jie Zhou

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Dissertation Advisor:
Professor Shing-Tung Yau

Author:
Jie Zhou

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Abstract

This thesis studies certain aspects of the global properties, including geometric and arithmetic, of the moduli spaces of complex structures of some special Calabi-Yau threefolds (B-model), and of the corresponding topological string partition functions defined from them which are closely related to the generating functions of Gromov-Witten invariants of their mirror Calabi-Yau threefolds (A-model) by the mirror symmetry conjecture.

For the mirror families (B-model) of the one-parameter families (A-model) of $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$ with varying Kähler structures, the bases are the moduli spaces of complex structures of the corresponding mirror Calabi-Yaus. We identify them with certain modular curves by studying the Picard-Fuchs systems and periods of the corresponding mirror families. In particular, the singular points on the moduli spaces correspond to the cusps and elliptic points on the modular curves.

We take the BCOV holomorphic anomaly equations with boundary conditions as the defining equations for the topological string partition functions. Using polynomial recursion and the above identification, we interpret the boundary conditions as regularity conditions for modular forms and express the equations purely in terms of the language of modular form theory. This turns the problem of solving the equations into a combinatorial problem. We also solve for the first few topological string partition functions genus by genus recursively in terms of almost-holomorphic modular forms. Assuming the validity of mirror symmetry conjecture, we prove a version of integrality for the Gromov-Witten

invariants of the original non-compact Calabi-Yau threefolds (A-model) as a consequence of the modularity of the partition functions.

Motivated by the results for the aforementioned non-compact Calabi-Yaus, we construct triples of differential rings on the moduli spaces of complex structures for some one-parameter families of compact Calabi-Yau threefolds (B-model), e.g., the quintic mirror family, in a systematic way. These rings are defined from the Picard-Fuchs equations and special Kähler geometry on the moduli spaces. They share structures similar to the triples of rings of modular forms, quasi-modular forms and almost-holomorphic modular forms defined on modular curves. Moreover, the topological string partition functions are Laurent polynomials in the generators of the differential rings.

Contents

List of Tables

List of Figures

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To my families

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Introduction

Calabi-Yau (CY) manifolds have very rich structures and have been intensively studied in mathematics and physics since they were introduced. The studies of CYs have inspired a lot of new subjects, especially those related to mirror symmetry and string theory. For nice reviews on the history and developments of mirror symmetry and its applications, see e.g., Greene (1996); Witten (1988); Cox and Katz (2000); Klemm (2003); Hori *et al.* (2003); Mariño (2006); Alim (2012) .

One of the most interesting problems is to "count" the number of holomorphic curves inside a given CY 3-fold $\check{X}$. These numbers are formulated rigorously as the Gromov-Witten invariants, which are defined as certain integrals over the moduli spaces of stable maps into the target space $\check{X}$. The study of Gromov-Witten invariants attracts a lot of attention and is closely related to the studies of quantum cohomology, Donaldson-Thomas invariants, integrable systems, quasi-modular forms, variation of Hodge structures, wall-crossing phenomena, etc.

For some special CYs like hypersurfaces and complete intersections in toric varieties, the Gromov-Witten invariants can be computed by using the localization technique in Kontsevich (1994), topological vertex in Aganagic:2003db, etc. However, for more general CYs, directly computing these invariants is a difficult problem in mathematics, partially due to the fact that the moduli space of stable maps into the CY $\check{X}$ is very complicated. For a detailed review on this subject, see Cox and Katz (2000); Hori *et al.* (2003) and references therein.

0.1 Counting curves via mirror symmetry

The mirror symmetry conjecture provides a way to bypass this difficulty. Given a smooth CY 3-fold $\check{X}$, one puts it in a family of CY 3-folds $\check{\pi} : \check{\mathcal{X}} \to \check{\mathcal{M}}$ by varying its complexified Kähler structure while fixing the complex structure, where $\check{\mathcal{M}}$ is the moduli space of complexified Kähler structures[1] of $\check{X}$ whose dimension is $h^{1,1}(\check{X})$. The generating function of genus g Gromov-Witten invariants gives a function defined on the moduli space

$$\check{F}_g(\check{t}) = \sum_{\beta \in H_2(\check{X},\mathbb{Z})} \langle e^{\omega} \rangle_{g,\beta} \,, \tag{1}$$

where

$$\omega = \sum_{i=1}^{h^{1,1}(\check{X})} \check{t}^i \omega_i \,, \quad \langle \omega_{i_1} \omega_{i_2} \cdots \omega_{i_k} \rangle_{g,\beta} = \prod_{j=1}^{k} \mathrm{ev}_{i_j}^* \omega_{i_j} \cap [\overline{\mathcal{M}_{g,k}(\check{X},\beta)}]^{\mathrm{vir}} \,.$$

Here

- $\{\omega_i\}_{i=1}^{h^{1,1}(\check{X})}$ are the generators for the Kähler cone in the moduli space $\check{\mathcal{M}}$ of Kähler structure of $\check{X}$;
- $\check{t} = \{\check{t}^i\}_{i=1}^{h^{1,1}(\check{X})}$ are local coordinates on the moduli space $\check{\mathcal{M}}$ centered around the so-called large volume limit given by $\check{t}^i = \infty$, $\forall i$;
- $[\overline{\mathcal{M}_{g,k}(\check{X},\beta)}]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli space $\mathcal{M}_{g,k}(\check{X},\beta)$ of stable maps of genus g and class β.
- $\mathrm{ev}_{i_j}, j = 1, 2, \cdots k$ are the evaluation maps: $\mathcal{M}_{g,k}(\check{X},\beta) \to \check{X}$.

An alternative way to write the above generating function $F_g(\check{t})$ in which the Gromov-Witten invariants appear more naturally is the following

$$\check{F}_g(\check{t}) = \sum_{\beta \in H_2(\check{X},\mathbb{Z})} N_{g,\beta}(\check{X}) e^{2\pi i \int_\beta \omega} \,. \tag{2}$$

Note that in this formula $N_{g,\beta}(\check{X})$ is independent of $\check{t}$ but only depends on $\check{X}$, this results from the fact that the Gromov-Witten invariants are deformation invariant, see e.g., McDuff

[1] Throughout the thesis, we shall simply call it Kähler structure by abuse of language.

and Salamon (1994); Ruan and Tian (1995).

The mirror symmetry conjecture predicts that for the CY 3-fold family (A model) $\check{\pi} : \check{\mathcal{X}} \to \check{\mathcal{M}}$, there exists another family (B model) of CY 3-folds $\pi : \mathcal{X} \to \mathcal{M}$ satisfying the following properties.

Mirror symmetry conjecture

- The moduli space $\check{\mathcal{M}}$ of Kähler structures of $\check{X}$ is identified with the moduli space $\mathcal{M}$ of complex structures of X. This implies in particular that $h^{1,1}(\check{X}) = \dim \check{\mathcal{M}} = \dim \mathcal{M} = h^{2,1}(X)$.

- There exist distinguished coordinates $\check{t} = \{\check{t}^i\}_{i=1}^{h^{1,1}(\check{X})}$ on $\check{\mathcal{M}}$ and $t = \{t^i\}_{i=1}^{h^{2,1}(X)}$ on $\mathcal{M}$, called canonical coordinates in Bershadsky *et al.* (1994), so that the map $\check{t} = t$ gives the identification $\check{\mathcal{M}} \cong \mathcal{M}$. This map is called the mirror map. In practice, one first matches some distinguished singular points on the moduli spaces, e.g., the large volume limit on $\check{\mathcal{M}}$ and the large complex structure limit on $\mathcal{M}$, then one identifies neighborhoods of these singular points by matching the Kähler normal coordinates (see Section 3.3.1) on the moduli spaces.

- For each genus g, there is a function $F_g(t)$ defined on the moduli space $\mathcal{M}$ so that under the mirror map, it is identical to $\check{F}_g(\check{t})$.

- Moreover, topological string theory implies that the more natural objects one should be looking at on both sides are some non-holomorphic objects $\check{\mathcal{F}}^{(g)}(\check{t}, \bar{\check{t}})$ and $\mathcal{F}^{(g)}(t, \bar{t})$ which are again identical under the mirror map. These quantities are called topological string partition functions. The "holomorphic limit" (see Section 3.3.1) of the normalized partition functions give rise to the quantities $\check{F}_g(\check{t})$ and $F_g(t)$, respectively.

Remark. The holomorphic limit, firstly defined in Bershadsky *et al.* (1993, 1994), is a very important concept in higher genus mirror symmetry and will be used frequently in this thesis, so we would like to mention this concept here, the full details could be found in Section 3.3.1.

Roughly speaking, the holomorphic limit of a non-holomorphic function $f(z, \bar{z})$ defined on a complex manifold M parametrized by z is obtained as follows. First, one thinks of the coordinates $(z, \bar{z})$ as independent coordinates in a formal neighborhood of the diagonal $\Delta : M \to M \times \overline{M}$, where $\overline{M}$ is the complex manifold equipped with the complex structure opposite to that on M. It can be proved that under the diagonal map Δ, M is real inside $M \times \overline{M}$. Hence any analytic function in $(z, \bar{z})$ on M can be analytically continued to an analytic function on $M \times \overline{M}$. On the diagonal, $\bar{z}$ is the honest complex conjugate of z, but away from the diagonal they are independent. Then the holomorphic limit of f based at the point z_*, denoted by $\lim_{\bar{z}=\bar{z}_*}$ hereafter in this thesis, is obtained by taking the degree zero part of the Laurent series expansion of the function $f(z, \bar{z})$ with respect to $\bar{z}$ and centered at $\bar{z}_*$.

To illustrate this, we discuss an example $f(z, \bar{z}) = z + \bar{z}$. The holomorphic limit of f at the base point $z_* = 0$ is computed as follows

$$f(z, \bar{z}) = z + 1 \cdot (\bar{z} - 0), \quad \lim_{\bar{z}=0} f = z\,,$$

while the holomorphic limit at $z_* = 1$ is computed to be

$$f(z, \bar{z}) = (z + 1) + 1 \cdot (\bar{z} - 1), \quad \lim_{\bar{z}=1} f = z + 1\,.$$

Two trivial observations are in order. First, the holomorphic limit of a non-holomorphic function based at any point is a holomorphic (more precisely, meromorphic) function, as the name suggests. Second, in general, holomorphic limits of the same non-holomorphic function, but based at different points, are not related by analytic continuation.

Finding the functions $\mathcal{F}^{(g)}(t, \bar{t})$ is sometimes much easier than computing the generating functions $\check{F}_g(\check{t})$ since the former satisfy some recursive differential equations called holomorphic anomaly equations in Bershadsky *et al.* (1993, 1994), as will be reviewed in Section 0.2 below. These differential equations and the corresponding boundary conditions were derived from physics, but can be formulated purely in terms of mathematical language[2].

The general idea of counting curves via mirror symmetry is as follows (see Figure 1 below). First one takes the BCOV holomorphic anomaly equations with boundary conditions as the defining equations for the topological string partition functions $\mathcal{F}^{(g)}(t,\bar{t})$. Then one tries to solve for them from the equations. After that one normalizes them and takes the holomorphic limit at the large complex structure limit on $\mathcal{M}$ to get $F_g(t)$. Finally one uses the mirror map $\check{t}=t$ which matches the large volume limit on $\check{\mathcal{M}}$ with the large complex structure limit to obtain $\check{F}_g(\check{t})$, and then to extract the Gromov-Witten invariants $N_{g,\beta}(\check{X})$ from Eq. (2).

In this way, via mirror symmetry, counting curves in the CY 3-fold $\check{X}$ is translated into solving differential equations on the moduli space $\mathcal{M}$ of its mirror manifold X.

0.2 Holomorphic anomaly equations and polynomial recursion

The mirror map matching the large volume limit on $\check{\mathcal{M}}$ with the large complex structure limit on $\mathcal{M}$ has been well understood in the literature. The difficult part in the above procedure in counting curves via mirror symmetry is to solve for the topological string partition functions from the BCOV holomorphic anomaly equations. In this section, we shall review these equations, and explain the polynomial recursion technique which was developed in Yamaguchi and Yau (2004); Alim and Länge (2007) to solve the equations.

Special Kähler geometry on deformation space

Consider a family $\pi:\mathcal{X}\to\mathcal{M}$ of CY 3-folds over a variety $\mathcal{M}$ parametrized by the complex coordinate system $z=\{z^i\}_{i=1}^{\dim\mathcal{M}}$. For a generic $z\in\mathcal{M}$, the fiber $\mathcal{X}_z$ is a smooth CY 3-fold. We also assume that $\dim\mathcal{M}=h^1(\mathcal{X}_z,T\mathcal{X}_z)$ for a generic $\mathcal{X}_z$, where $T\mathcal{X}_z$ is the holomorphic tangent bundle of $\mathcal{X}_z$. In the following, we shall use the notation X to denote a generic fiber $\mathcal{X}_z$ in the family without specifying the point z.

[2]See Li (2011); Costello and Li (2012) for recent developments on rigorously defining the topological string partition functions and BCOV holomorphic anomaly equations.

$$
\begin{tikzcd}
 & \text{Mirror Symmetry} & \\
\substack{\text{A-model}\\ \text{(symplectic geometry)}} & & \substack{\text{B-model}\\ \text{(complex geometry)}} \\
\check{X} & & X \\
\check{\pi} : \check{\mathcal{X}} \to \check{\mathcal{M}} & & \pi : \mathcal{X} \to \mathcal{M} \\
\check{\mathcal{M}} & & \mathcal{M} \\
\check{t} & \text{mirror map} \arrow[l] \arrow[r] & t \\
\check{F}_g(\check{t}) & \text{Gromov-Witten} \arrow[l, bend right] & F_g(t) \\
\check{\mathcal{F}}^{(g)}(\check{t}, \bar{\check{t}}) \arrow[u, "\substack{\text{normalization}\\ \text{holomorphic limit}}"'] & \text{BCOV holomorphic anomaly equations} \arrow[r, bend left] & \mathcal{F}^{(g)}(t, \bar{t}) \arrow[u, "\substack{\text{normalization}\\ \text{holomorphic limit}}"]
\end{tikzcd}
$$

Figure 1: *Big picture of counting curves via mirror symmetry*

In many examples discussed in this thesis, the smooth CY 3-fold X is toric in nature, and the variety $\mathcal{M}$ will be the moduli space of complex structures of X which can be constructed torically.

The variation of complex structures on X can be studied by looking at the periods Π of the family according to the general theory of variation of Hodge structures, see Griffiths and Schmid (1975); Carlson *et al.* (2003); Voisin (2002, 2007). They are defined to be the integrals $\int_C \Omega_z$, where $C \in H_3(\mathcal{X}_z, \mathbb{Z})$ and $\Omega = \{\Omega_z\}$ is a holomorphic section of the Hodge line bundle $\mathcal{L} = \mathcal{R}^0\pi_*\Omega^3_{\mathcal{X}|\mathcal{M}}$ on $\mathcal{M}$. They satisfy a differential equation system $\mathcal{L}_{\mathrm{CY}}\,\Pi = 0$ called the Picard-Fuchs system which is induced from the Gauss-Manin connection on the Hodge bundle $\mathcal{H} = \mathcal{R}^3\pi_*\underline{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{M}} = \mathcal{R}^3\pi_*\Omega^\bullet_{\mathcal{X}|\mathcal{M}}$. The base $\mathcal{M}$ of the family is equipped with the Weil-Petersson metric whose Kähler potential K is determined from

$$e^{-K(z,\bar{z})} = i\int_{\mathcal{X}_z} \Omega_z \wedge \overline{\Omega}_z \,. \tag{3}$$

The metric $G_{i\bar{j}} = \bar{\partial}_{\bar{j}}\partial_i K$ is the Hodge metric induced from the Hermitian metric $h(\Omega,\Omega) = i^3\int_X \Omega\wedge\overline{\Omega}$ on the Hodge line bundle $\mathcal{L}$. This metric is called a special Kähler metric, see Strominger (1990); Freed (1999). Among its other properties, it satisfies the following "special geometry relation"

$$-R_{i\bar{j}}{}^{k}{}_{l} = \partial_{\bar{j}}\Gamma^k_{il} = \delta^k_l G_{i\bar{j}} + \delta^k_i G_{l\bar{j}} - C_{ilm}\overline{C}^{mk}_{\bar{j}}\,, \quad i,\bar{j},k,l = 1,2,\cdots \dim\mathcal{M}\,, \tag{4}$$

where

$$C_{ijk}(z) = -\int_{\mathcal{X}_z} \Omega_z \wedge \partial_i\partial_j\partial_k\Omega_z \tag{5}$$

is the so-called Yukawa coupling and

$$\overline{C}^{mk}_{\bar{j}} = e^{2K}G^{k\bar{k}}G^{m\bar{m}}\overline{C}_{\bar{j}\bar{k}\bar{m}}\,. \tag{6}$$

Note that by definition $C_{ijk}\in\Gamma(\mathcal{M},\mathrm{Sym}^3 T^*\mathcal{M}\otimes\mathcal{L}^2)$ and it is symmetric with respect to the sub-indices i,j,k. Integrating Eq. (5), one then gets the "integrated special geometry relation"

$$\Gamma^k_{ij} = \delta^k_j K_i + \delta^k_i K_j - C_{ijm}S^{mk} + s^k_{ij}\,, \tag{7}$$

where S^{mk} is defined to be a solution to

$$\bar{\partial}_{\bar{n}}S^{mk} = \overline{C}^{mk}_{\bar{n}}\,, \tag{8}$$

and s^k_{ij} could be any holomorphic quantity. There is a natural covariant derivative D acting on sections of the Hodge bundle

$$\mathcal{H} = \mathcal{R}^3\pi_*\underline{\mathbb{C}}\otimes\mathcal{O}_{\mathcal{M}} = \mathcal{L}\oplus\mathcal{L}\otimes T\mathcal{M}\oplus\overline{\mathcal{L}\otimes T\mathcal{M}}\oplus\overline{\mathcal{L}}\,. \tag{9}$$

It is induced from the Chern connection associated to the Weil-Petersson metric and the connection on $\mathcal{L}$ induced by the Hermitian metric h. See Strominger (1990); Bershadsky *et al.* (1994); Freed (1999); Hosono (2008) for details on this.

Among the singular points on the moduli space $\mathcal{M}$, there is a distinguished point called the large complex structure limit. It is mirror to the large volume limit on the mirror side (A side) and is usually a maximally unipotent monodromy point. It plays the following special role in genus zero mirror symmetry initiated by Candelas *et al.* (1991). Assume the large complex structure limit is given by $z = 0$. Near this point, the solutions to the Picard-Fuchs system $\mathcal{L}_{\mathrm{CY}}\,\Pi = 0$ could be obtained by the Frobenius method and have the following form

$$(X^0, X^a, P_a, P_0) = X^0(1, t^a, \partial_{t^a}F(t), 2F - t^a\partial_{t^a}F(t)), \quad a = 1, 2\cdots \dim\mathcal{M}\,, \tag{10}$$

where near $z = 0$,

$$X^0(z) \sim 1 + \mathcal{O}(z)\,, \tag{11}$$

$$t^a \sim \frac{1}{2\pi i}\log z^a + \text{regular functions}\,. \tag{12}$$

The functions $t = \{t^a\}_{a=1}^{\dim\mathcal{M}}$ give local coordinates on the punctured moduli space $\mathcal{M} - \{z = 0\}$. They are in fact the canonical coordinates based at the large complex structure limit, see Section 3.3.1 for details. The existence of the holomorphic function $F(t)$, called prepotential, and the above particular form in Eq. (10) for the periods result from the special Kähler geometry on $\mathcal{M}$. The same form for the periods also holds elsewhere on the moduli space.

The normalized (so that after normalization it gives a section of $\mathcal{L}^0$) Yukawa coupling in the t coordinates is then given by

$$C_{t^at^bt^c} = \partial^3_{t^at^bt^c}F = \frac{1}{(X^0)^2}C_{z^iz^jz^k}\frac{\partial z^i}{\partial t^a}\frac{\partial z^j}{\partial t^b}\frac{\partial z^k}{\partial t^c} \in \Gamma(\mathcal{M}, \mathrm{Sym}^3 T^*\mathcal{M}\otimes\mathcal{L}^0)\,. \tag{13}$$

Genus zero mirror symmetry predicts that under the mirror map $\check{t} = t$, one has

$$C_{t^at^bt^c} = \kappa_{abc} + \sum_{\beta\in H_2(\check{X},\mathbb{Z})}^{\infty} d_ad_bd_cN_{0,d_ad_bd_c}e^{2\pi i d_a t^a}\,, \tag{14}$$

where $\kappa_{abc} = \omega_a\cup\omega_b\cup\omega_c$ are the classical triple intersection numbers of the mirror manifold $\check{X}$ of X, $d_a = \int_\beta \omega_a$, and $N_{0,d_ad_bd_c}$ is the same as the quantity $N_{0,\beta}$ in Eq. (2). This prediction has been checked for many CY 3-fold families and matches the results obtained by directly computing $N_{0,\beta}$ using techniques from the A-side (e.g., the localization technique in Givental

(1997); Lian *et al.* (1997)).

Holomorphic anomaly equations with boundary conditions and polynomial recursion

The genus g topological string partition function $\mathcal{F}^{(g)}$ as defined in Bershadsky *et al.* (1993, 1994) is a (smooth) section of the line bundle $\mathcal{L}^{2-2g}$ over $\mathcal{M}$, it is shown to satisfy the following holomorphic anomaly equation:

$$\begin{aligned}
\bar{\partial}_{\bar{\imath}}\partial_j\mathcal{F}^{(1)} &= \frac{1}{2}C_{jkl}\overline{C}_{\bar{\imath}}^{kl} + (1-\frac{\chi}{24})G_{j\bar{\imath}}\,, && (15)\\
\bar{\partial}_{\bar{\imath}}\mathcal{F}^{(g)} &= \frac{1}{2}\overline{C}_{\bar{\imath}}^{jk}\left(\sum_{r=1}^{g-1}D_j\mathcal{F}^{(r)}D_k\mathcal{F}^{(g-r)} + D_jD_k\mathcal{F}^{(g-1)}\right), \quad g\geq 2\,, && (16)
\end{aligned}$$

where χ is the Euler characteristic of the mirror manifold $\check{X}$ of the CY 3-fold X. It turns out that any higher genus $\mathcal{F}^{(g)}$ can be determined recursively from these up to addition by a holomorphic function, as we shall discuss now.

Recursion in terms of propagators and polynomial structure A solution of these recursion equations is given in terms of Feynman rules in Bershadsky *et al.* (1994). The propagators S^{ij}, S^i, S for these Feynman rules are defined by the following:

$$\bar{\partial}_{\bar{\imath}}S^{ij} = \overline{C}_{\bar{\imath}}^{ij}\,, \qquad \bar{\partial}_{\bar{\imath}}S^{j} = G_{i\bar{\imath}}S^{ij}, \qquad \bar{\partial}_{\bar{\imath}}S = G_{i\bar{\imath}}S^{i}. \tag{17}$$

By definition, the propagators S, S^i and S^{ij} are sections of the bundles $\mathcal{L}^{-2}\otimes\mathrm{Sym}^m T\mathcal{M}$ with $m=0,1,2$, respectively. The vertices of the Feynman rules are given by the functions $\mathcal{F}^{(g)}_{i_1\cdots i_n} = D_{i_1}\cdots D_{i_n}\mathcal{F}^{(g)}$.

Integrating the genus one holomorphic anomaly equation Eq. (15) and using the integrated special geometry relation Eq. (7), we get

$$\bar{\partial}_{\bar{\imath}}\partial_j\mathcal{F}^{(1)} = \bar{\partial}_{\bar{\imath}}\left(-\frac{1}{2}\partial_j\log\det G + (\frac{h^{2,1}(X)+3}{2} - \frac{\chi}{24})K_j\right)\,. \tag{18}$$

In Yamaguchi and Yau (2004); Alim and Länge (2007) it was shown, using Eq. (18) and again Eq. (7), that the holomorphic anomaly equations in Eq. (16) for $g\geq 2$ can be put into the following form

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(g)} = \bar{\partial}_{\bar{i}}\mathcal{P}^{(g)}\,, \tag{19}$$

where $\mathcal{P}^{(g)}$ is a polynomial of the generators S^{ij}, S^{i}, S, K_i with the coefficients being holomorphic quantities which might have poles. The proof relies on the fact that these generators form a differential ring. More precisely, the derivatives of the generators are given by Alim and Länge (2007) as follows:

$$\begin{aligned}
D_i S^{jk} &= \delta_i^j S^k + \delta_i^k S^j - C_{imn} S^{mj} S^{nk} + h_i^{jk}\,,\\
D_i S^{j} &= -C_{imn} S^m S^{jn} + 2\delta_i^j S + h_i^{jk} K_k + h_i^j\,,\\
D_i S &= -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h_i^{mn} K_m K_n + h_i^m K_m + h_i\,,\\
D_i K_j &= -K_i K_j + C_{ijm} S^{mn} K_n - C_{ijm} S^m + h_{ij}\,,
\end{aligned} \tag{20}$$

where $h_i^{jk}, h_i^j, h_i, h_{ij}$ are holomorphic quantities. The generators S^{ij}, S^i, S can be solved from Eq. (17) or determined from Eq. (20), up to addition by holomorphic quantities[3]. In fact, a set of solutions whose holomorphic limits are vanishing was explicitly given by Alim *et al.* (2010); Hosono (2008) in terms of geometric quantities. Hence Eq. (20) does not actually give a differential ring due to the existence of these holomorphic quantities and their derivatives. To make it a genuine ring, one needs to include all of the derivatives of these holomorphic functions, see Hosono (2008).

In Yamaguchi and Yau (2004); Alim *et al.* (2013), for some one-parameter CY 3-fold families these holomorphic functions and their derivatives are packaged together by making use of the special Kähler geometry on the moduli space, and are essentially Laurent polynomials of the Yukawa couplings in Eq. (5). Then one gets a differential ring with finitely many generators, including the non-holomorphic generators S^{ij}, S^i, S, K_i and the holomorphic Yukawa couplings. We shall discuss their further properties in more details in Chapter 3.

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[3]Some of the holomorphic quantities $h_i^{jk}, h_i^j, h_i, h_{ij}$ can not be uniquely determined since the above equations Eq. (20) are derived by integrating some equations, see Alim and Länge (2007); Alim *et al.* (2010) for the analysis of the degrees of freedom of their choices.

Now we justify the structure in Eq. (19) by induction, following Alim and Länge (2007). First for genus one case we have already had Eq. (18). Plugging in the integrated special geometry relation Eq. (7), we obtain

$$\bar{\partial}_{\bar{\imath}}\partial_j \mathcal{F}^{(1)} = \bar{\partial}_{\bar{\imath}} \left(\frac{1}{2} C_{jkl} S^{kl} + (1 - \frac{\chi}{24}) K_j \right) . \tag{21}$$

Now note that the non-holomorphicity of the topological string partition functions only comes from the non-holomorphic generators and thus the anti-holomorphic derivative on the left-hand side of the holomorphic anomaly equation Eq. (16) can be replaced by derivatives with respect to these generators. Furthermore, one can make a change of generators following Alim and Länge (2007)

$$\begin{aligned} \tilde{S}^{ij} &= S^{ij}, \\ \tilde{S}^{i} &= S^{i} - S^{ij} K_j, \\ \tilde{S} &= S - S^{i} K_i + \frac{1}{2} S^{ij} K_i K_j, \\ \tilde{K}_i &= K_i \, . \end{aligned} \tag{22}$$

The differential ring structure among these new non-holomorphic generators follows from Eq. (20) easily. Replacing the $\bar{\partial}_{\bar{\imath}}$ derivative in Eq. (16) by derivatives with respect to the new non-holomorphic generators, then using Eq. (21) and the definitions Eq. (17), one gets, see Alim and Länge (2007),

$$\begin{aligned} \bar{\partial}_{\bar{\imath}} \mathcal{F}^{(g)} &= \overline{C}_{\bar{\imath}}^{jk} \left(\frac{\partial \mathcal{F}^{(g)}}{\partial S^{jk}} - \frac{1}{2} \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^{k}} \tilde{K}_j - \frac{1}{2} \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^{j}} \tilde{K}_k + \frac{1}{2} \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}} \tilde{K}_j \tilde{K}_k \right) + G_{\bar{\imath}j} \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{K}_j} \\ &= \frac{1}{2} \overline{C}_{\bar{\imath}}^{jk} \left(\sum_{r=1}^{g-1} D_j \mathcal{F}^{(r)} D_k \mathcal{F}^{(g-r)} + D_j D_k \mathcal{F}^{(g-1)} \right) . \end{aligned} \tag{23}$$

Assuming the independence of $\overline{C}_{\bar{i}}^{jk}$ and $G_{\bar{i}j}$, then one gets two sets of equations:

$$\begin{aligned}\frac{\partial \mathcal{F}^{(g)}}{\partial S^{jk}} &- \frac{1}{2}\frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^{k}}\tilde{K}_j - \frac{1}{2}\frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^{j}}\tilde{K}_k + \frac{1}{2}\frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}}\tilde{K}_j\tilde{K}_k \\ &= \sum_{r=1}^{g-1} D_j\mathcal{F}^{(r)} D_k\mathcal{F}^{(g-r)} + D_j D_k \mathcal{F}^{(g-1)}\,, \qquad (24)\\ \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{K}_j} &= 0\,. \qquad (25)\end{aligned}$$

Eq. (19) then follows from the above two equations and Eq. (20).

The polynomial structure given in Eq. (19) also allows to determine the non-holomorphic part $\mathcal{P}^{(g)}$ of $\mathcal{F}^{(g)}$ genus by genus recursively from Eq. (24) as polynomials of the new non-holomorphic generators $\tilde{S}^{ij}, \tilde{S}^{i}, \tilde{S}, \tilde{K}_i$ or equivalently the odd ones:

$$\mathcal{P}^{(g)} = \mathcal{P}^{(g)}(S^{ij}, S^{i}, S, K_i)\,. \qquad (26)$$

Moreover, the coefficients of the monomials in these non-holomorphic generators are explicit Laurent polynomials in the holomorphic generators, with the coefficients of the monomials in the non-holomorphic and holomorphic generators being universal constants. These constants come from the Feynman diagram interpretation in Bershadsky *et al.* (1994), or equivalently, the combinatorics from recursion. They are independent of the geometry under consideration. For example, in Bershadsky *et al.* (1994), it is worked out that for any geometry the highest power of S^{ij} in the genus two partition function always takes the form $\frac{1}{8}C_{ijp}C_{qmn}S^{ij}S^{pq}S^{mn} + \frac{1}{12}C_{ipm}C_{jqn}S^{ij}S^{pq}S^{mn} + \cdots$.

Holomorphic ambiguities and boundary conditions In the above discussions only the local geometric properties of the moduli space $\mathcal{M}$ are used.

According to Eq. (19), $\mathcal{F}^{(g)}$ is only determined up to addition by a holomorphic function $f^{(g)}$ called holomorphic ambiguity:

$$\mathcal{F}^{(g)} = \mathcal{P}^{g}(S^{ij}, S^{i}, S, K_i) + f^{(g)}\,. \qquad (27)$$

Boundary conditions on the (global) moduli space are needed to fix the holomorphic ambiguity $f^{(g)}$. What are commonly used are the asymptotic behaviors of $\mathcal{F}^{(g)}$ near the singular points on the moduli space $\mathcal{M}$. There are basically three types of singularities, as summarized in Table 1. One must be reminded that in practice to identify these singularities

Table 1: *Types of singularities on moduli space*

type	characteristic property	mirror
large complex structure limit	maximally unipotent monodromy	large volume limit
conifold point	vanishing cycle	$*$
orbifold point	finite monodromy	$*$

is more involved than by simply looking at the characteristic properties. Another thing one needs to pay particular attention to is that in some special examples, the conifold point and the orbifold point really stand for geometries like conifold CY and CYs with extra symmetries in the moduli space, respectively, but in general there is no good reason that these points must lie in the geometric phase.

The boundary conditions of $\mathcal{F}^{(g)}$ at the large complex structure limit and at the conifold point $\mathcal{M}$, see Bershadsky *et al.* (1993, 1994); Ghoshal and Vafa (1995); Antoniadis *et al.* (1995); Huang and Klemm (2007); Huang *et al.* (2009), are described as follows. These conditions are satisfied by the normalized holomorphic limits of $\mathcal{F}^{(g)}$ based at the corresponding points. More precisely, the boundary condition at the large complex structure limit (LCSL) is given by

$$\begin{aligned} \lim_{\mathrm{LCSL}} \frac{1}{(X^0)^{2-2\cdot 1}} \mathcal{F}^{(1)} &= -\frac{1}{24} t^i \int_{\check{X}} c_2(T\check{X}) \omega_i + \mathcal{O}(e^{2\pi i t})\,, \\ \lim_{\mathrm{LCSL}} \frac{1}{(X^0)^{2-2g}} \mathcal{F}^{(g)} &= (-1)^g \frac{\chi(\check{X})}{2} \frac{|B_{2g} B_{2g-2}|}{2g(2g-2)(2g-2)!} + \mathcal{O}(e^{2\pi i t}), \quad g \geq 2\,, \end{aligned} \tag{28}$$

where B_{2g}, B_{2g-2} are the Bernoulli numbers and $t = \{t^i\}_{i=1}^{h^{2,1}(X)}$ are the ratios of periods near the large complex structure limit as defined in Eq. (10). The boundary condition at the conifold locus (CON) determined by $\Delta_j(z)=0, j=1,2\cdots m$, is given by

$$\begin{aligned}\lim_{\mathrm{CON}} \frac{1}{(X^0_{\mathrm{CON}})^{2-2\cdot 1}}\mathcal{F}^{(1)} &= -\frac{1}{12}\log t_c^j + \text{regular}\,,\\ \lim_{\mathrm{CON}} \frac{1}{(X^0_{\mathrm{CON}})^{2-2g}}\mathcal{F}^{(g)} &= \frac{c^{g-1}B_{2g}}{2g(2g-2)(t_c^j)^{2g-2}} + \text{regular} \quad \text{for some } c, \quad g\geq 2\,, \end{aligned} \tag{29}$$

where again B_{2g} is the Bernoulli number, m is the number of components for the discriminant, and $X^0_{\mathrm{CON}}, X^{j,\text{vanishing}}_{\mathrm{CON}}, t_c^j = X^{j,\text{vanishing}}_{\mathrm{CON}}/X^0_{\mathrm{CON}}$ are the regular non-vanishing period, regular vanishing period, normalized vanishing period near the conifold locus $\Delta_j=0$, respectively. In the literature Eq. (29) is called the gap condition due to the fact that the sub-leading singular terms in between are vanishing.

The holomorphic ambiguity $f^{(g)}$ can then be fixed (at least in principle) by making the ansatz

$$f^{(g)}(z) = \sum_{j=1}^{m} \frac{h_j^{(g)}(z)}{\Delta_j^{2g-2}}\,, \tag{30}$$

where $h_j^{(g)}(z)$ is a polynomial in z of degree $\leq (2g-2)\deg\Delta_j$, and then applying the above boundary conditions to $\mathcal{F}^{(g)} = \mathcal{P}^{(g)} + f^{(g)}$.

Challenges This approach works well for some non-compact and compact CY 3-folds, see e.g., Bershadsky *et al.* (1993, 1994); Marino and Moore (1999); Katz *et al.* (1999); Klemm and Zaslow (1999); Klemm *et al.* (2005); Yamaguchi and Yau (2004); Klemm and Marino (2008); Huang *et al.* (2009); Aganagic *et al.* (2008); Huang and Klemm (2007); Alim and Länge (2007); Grimm *et al.* (2007); Alim *et al.* (2010); Haghighat *et al.* (2008); Haghighat and Klemm (2010); Sakai (2011); Alim and Scheidegger (2012); Alim (2012); Klemm *et al.* (2012). The Gromov-Witten invariants obtained this way via mirror symmetry match the results from computations on the A side. But there exist some challenges in this approach.

1. The global mathematical (e.g., analytical) properties of the generators which involve derivatives of the periods and ratios of periods are not clearly understood, so it is not convenient to study further the global properties of the functions $\mathcal{F}^{(g)}$ and F_g using the expressions one gets from this approach. In particular, it is not clear how the

functions F_g converge and whether the functions F_g have any symmetry or modularity.

2. It is not clear how to analytically continue non-holomorphic quantities and how the different holomorphic limits based at different points of the functions $\mathcal{F}^{(g)}$ are related. This gives rise to difficulties in applying the boundary conditions[4]. This is illustrated in Figure 2 below.

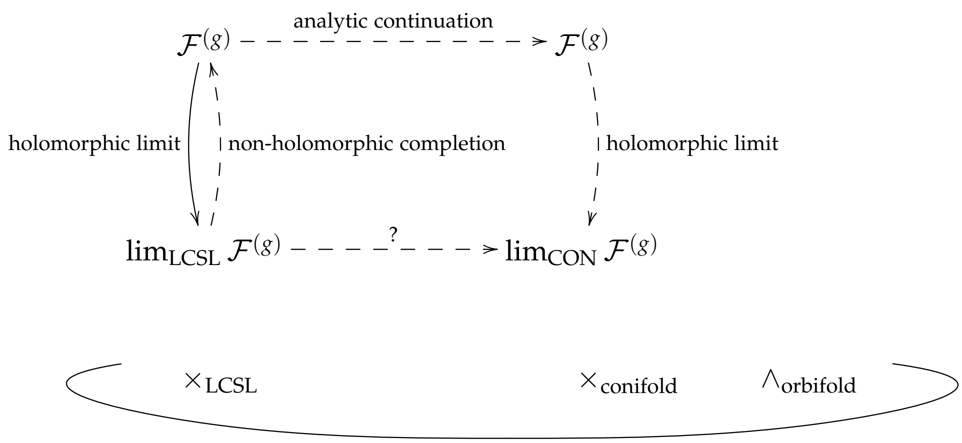

Figure 2: *Applying boundary conditions using analytic continuation*

3. Despite the fact it might fail for some CYs, the ansatz made in Eq. (30) is not rigorously justified.

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[4] In practice, to find the holomorphic limit of $\mathcal{F}^g$, what one usually do is to compute the holomorphic limit of propagators from Eq. (17) or Eq. (20) and then use the polynomial structure. This then requires a good understanding of the canonical coordinates near the singular points at which we want to apply boundary conditions to $\mathcal{F}^g$. In the literature, the holomorphic limit of the geometric quantities at the conifold point are computed by using the statement that the vanishing period serves as the canonical coordinate. To really compute the Kähler potential and thus determine the canonical coordinate require a suitable choice for the basis of the periods so that they become integrals with respect to symplectic homology basis, this was done numerically by analytic continuation from such a choice at the large complex structure limit in e.g., Huang *et al.* (2009).

It is not clear to the author how to see that the vanishing period must be a canonical coordinate basing on first principles instead of using numerical computations, but it appears that the argument in Appendix 3 in Huang *et al.* (2009) which explains that the holomorphic limit of the propagators are independent of the choice for the section Ω of $\mathcal{L}$ is relevant in understanding this. See also Hosono and Lian (1996); Freed (1999); Kapranov (1999) for related discussions.

0.3 Arithmetic properties of moduli spaces and of partition functions

Modularity of topological string partition functions

To resolve the challenges mentioned above, we studied in the joint work Alim *et al.* (2013) the arithmetic properties of the moduli space $\mathcal{M}$ which help explore the global properties like modularity of the topological string partition functions. Right now we can only deal with some special families of one-parameter non-compact CY 3-folds with $\mathcal{M} \cong \mathbb{P}^1$. The main idea is sketched in the following and will be elaborated in Chapter 2:

1. We identify the moduli space $\mathcal{M}$ with a certain modular curve $X_\Gamma = \Gamma\backslash\mathcal{H}^*$ by studying their variations of Hodge structures (simply, periods), where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. Then we express the geometric quantities (metric tensors, connections, curvatures, derivatives of periods, Yukawa couplings, etc.) in terms of generators of the ring of almost-holomorphic modular forms. Moreover, the polynomial part $\mathcal{P}^{(g)}$, and thus the full $\mathcal{F}^{(g)}$ can be shown to be an almost-holomorphic modular form of weight 0. This allows to rewrite the holomorphic anomaly equations in terms of the language of modular form theory.

2. The large complex structure limit and conifold point on the moduli space $\mathcal{M}$ are identified with cusps on the modular curve X_Γ, which are in turn related by the Fricke involution (a.k.a. Atkin-Lehner involution). At the cusps, taking the holomorphic limit is equivalent to taking the "constant term map" for almost-holomorphic modular forms in Kaneko and Zagier (1995) which sends an almost-holomorphic modular form to a quasi-modular form, while the non-holomorphic completion becomes the modular completion which takes a quasi-modular form to an almost-holomorphic modular form. Fricke involution then relates the topological string partition functions and their holomorphic limits at the two singular points on $\mathcal{M}$. This is illustrated in Figure 3.

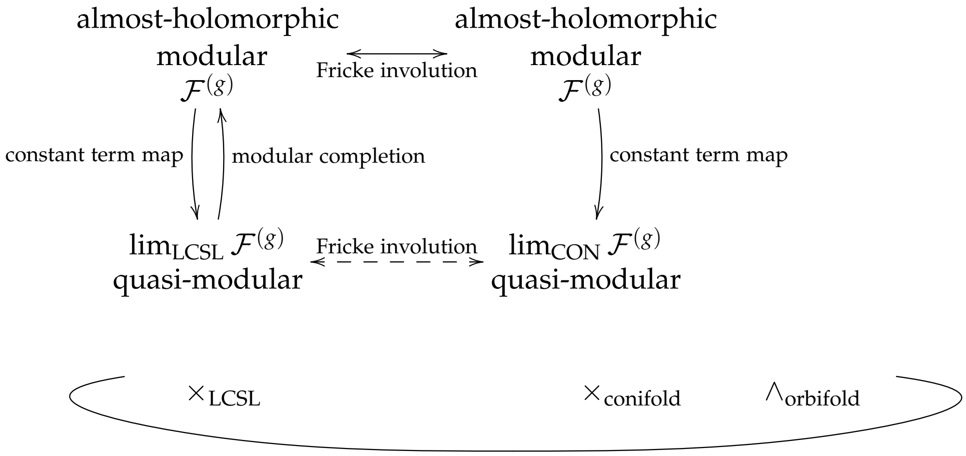

Figure 3: *Applying boundary conditions using modularity*

3. The holomorphic ambiguity $f^{(g)}$, as a modular function on the modular curve, must be a rational function in the generators of the ring of modular forms. The boundary conditions for $\mathcal{F}^{(g)}$ determines the form that $f^{(g)}$ can take, which then gives a natural ansatz for $f^{(g)}$.

We want to point out that similar works using modularity to solve for $\mathcal{F}^g$ existed in the literature, e.g., Mohri (2002); Huang and Klemm (2007); Aganagic *et al.* (2008); Hosono (2008); Haghighat *et al.* (2008). Our work differ from the previous works in that we study the arithmetic structures of the moduli space more closely by working out explicitly the identification between the moduli space and the modular curve, and by finding the Fricke involution which exchanges cusps to relate boundary conditions at the large complex structure limit and at the conifold point. This allows us to understand the BCOV holomorphic anomaly equations and the boundary conditions in terms of modular form theory and turns the problem of solving the holomorphic anomaly equations into a combinatorial problem (see Section 2.4 and Appendix B).

We proved in the joint work Alim *et al.* (2013) that

Theorem 0.1. *Consider the mirror families of the one-parameter families of Calabi-Yau threefolds $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$, respectively.*

1. *For each mirror family $\pi : \mathcal{X} \to \mathcal{M}$, the moduli space $\mathcal{M}$ is a modular curve $X_\Gamma = \Gamma\backslash\mathcal{H}^*$, where the modular group Γ is $\Gamma_0(3)$, $\Gamma_0(4)$, $\Gamma_0(3)$, $\Gamma_0(2)$, $\Gamma(1)^*$, respectively.*

2. *For each family, the solutions to the holomorphic anomaly equations with the boundary conditions, if they exist, are almost-holomorphic modular forms of weight zero with respect to the corresponding modular group.*

Basing on the techniques developed in the joint work Alim *et al.* (2013), I shall prove the following result in this thesis.

Theorem 0.2. *For the mirror family $\pi : \mathcal{X} \to \mathcal{M}$ of the $K_{\mathbb{P}^2}$ family, for any genus g, the solution to the holomorphic anomaly equation with boundary conditions exists and is unique. In particular, it is an almost-holomorphic modular form of weight zero with respect to the modular group $\Gamma_0(3)$.*

This family, as one of the simplest families of non-compact CY 3-folds, was intensively studied in the literature, see e.g., Chiang *et al.* (1999); Klemm and Zaslow (1999); Aganagic *et al.* (2008); Haghighat *et al.* (2008); Alim *et al.* (2010). In particular, in Aganagic *et al.* (2008), some arithmetic aspects of the moduli space $\mathcal{M}$ were studied by looking at the monodromy group of the family $\pi : \mathcal{X} \to \mathcal{M}$ and the first few partition functions were obtained. My work differs from the previous works in that after making identification between the moduli space and the modular curve, I used the Fricke involution on the modular curve to analyze the boundary conditions for the holomorphic anomaly equations, which makes the computations easier and the rigorous proofs for the existence and uniqueness possible.

Furthermore, in Alim *et al.* (2013) we solved for the first few partition functions for the mirror CY 3-folds of $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$ genus by genus recursively and explicitly expressed them in terms of the generators of the rings of almost-holomorphic modular forms for the corresponding modular groups, see also Chapter 2. Using the mirror symmetry conjecture, we then got the generating functions of the Gromov-Witten invariants of $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$ in terms of quasi-modular forms. They paralleled closely the known results in the literature e.g., Dijkgraaf (1995); Kaneko and Zagier (1995); Li (2011, 2012); Milanov and Ruan (2011) for elliptic curves and Mohri (2002) for other geometries. These results were checked to produce correct Gromov-Witten invariants up to degree 10 which could be found in e.g., Klemm and Zaslow (1999); Katz *et al.* (1999).

As an easy consequence of the modularity and mirror symmetry, we shall prove in this thesis the following integrality result about the Gromov-Witten invariants $\{N_{g,d}\}_{d=1}^{\infty}$.

Corollary 0.3. *Assume mirror symmetry conjecture is valid for the mirror families $\pi : \mathcal{X} \to \mathcal{M}$ in Theorem 0.1. Then for each family, for any genus g, there exists a number $C_g \in \mathbb{Z}$ so that $C_g N_{g,d} \in \mathbb{Z}, \forall\, d \geq 1$.*

Before we proceed, we now discuss briefly the $K_{\mathbb{P}^2}$ example to summarize the various layers of structures on the moduli space $\mathcal{M}$ we have mentioned so far. In this case, the moduli space satisfies $\mathcal{M} \cong \mathbb{P}^1$. There are three singular points on the moduli space, corresponding to the large complex structure limit, conifold, orbifold, respectively. We choose the complex coordinate α on $\mathcal{M}$ suitably so that $\alpha = 0, 1, \infty$ gives the above three points. As shall be shown in Chapter 2, the modular curve is $X_\Gamma = \Gamma \backslash \mathcal{H}^*$ with $\Gamma = \Gamma_0(3)$. The local geometric, global geometric and arithmetic structures on the moduli space $\mathcal{M}$ are displayed in Table 2 and illustrated graphically in Figure 4.

Table 2: *Layers of structures on moduli space*

space	role	singularities	application
$T_*\mathcal{M}$ tangent space	deformation space local geometric	$\alpha = 0, \cdots$	genus 0 mirror symmetry
M complex analytic space	moduli variety global geometric	$\alpha = 0, 1, \infty$	solving for $\mathcal{F}^{(g)}$
$X_{\Gamma_0(3)}$ arithmetic locally symmetric variety	modular variety arithmetic	α =Hauptmodul $[\tau] = [i\infty], [0], [\exp \frac{2\pi i}{3}]$	modularity of $\mathcal{F}^{(g)}$

Differential rings constructed from special Kähler geometry on moduli spaces

In the above approach in understanding the modularity of topological string partition functions, the first step is to identify the moduli space with certain modular variety. For the special non-compact examples mentioned above, we used two properties: 1. the period

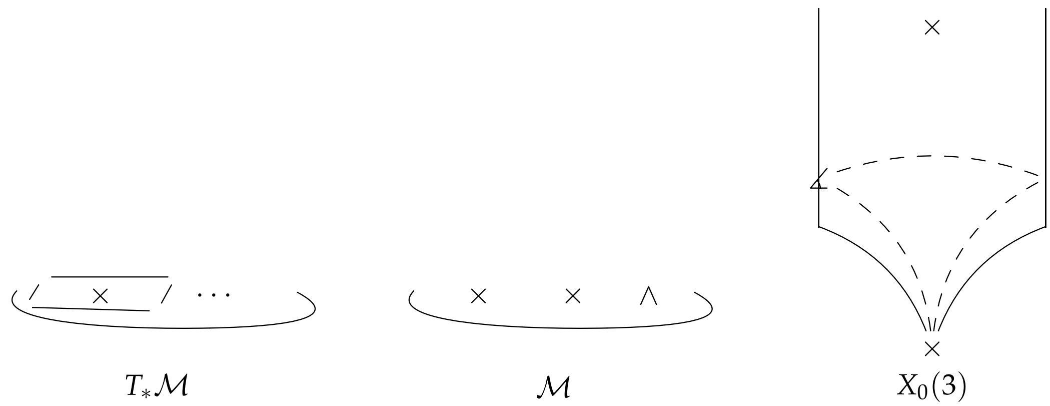

Figure 4: *Illustration of structures on moduli space*

domain (after quotient) is a modular curve; 2. the period map gives an isomorphism from the moduli space to the (quotient of) period domain. Neither of this is true for more general CYs, see e.g., Griffiths and Schmid (1975); Carlson *et al.* (2003); Debarre (2012) and references therein. The first is because in general the period domain is not a Hermitian symmetric domain (non-classical case). The second is because a global Torelli type theorem is lacking[5]. In fact, for general CY 3-folds the moduli spaces of complex structures are known not to be any arithmetic locally symmetric variety, so the classic modular form theory does not apply directly.

The lessons we learned from studying the aforementioned non-compact CYs tell us that there is a dictionary between the geometric quantities S^{ij}, S^i, S, K_i and the modular objects when the moduli space is known to be a modular curve. This is the origin that the differential ring Eq. (20) closes under the covariant derivative D, since the ring of the geometric quantities S^{ij}, S^i, S, K_i is essentially identical to the ring of almost-holomorphic modular forms in this case. For more general cases, although we are lacking a ring of almost-holomorphic modular forms attached to the moduli space, the ring of geometric quantities S^{ij}, S^i, S, K_i can always be defined without using or relying on the existence of the arithmetic

[5]See Usui (2008) for a recent result on generic Torelli for the mirror quintic family which belongs to the non-classical case.

structure of the moduli space. This inspires us to study the differential ring generated by S^{ij}, S^i, S, K_i. We hope that understanding the global, or even more, automorphic properties of this ring will help understand the properties of the topological string partition functions and thus of the Gromov-Witten invariants of the mirror CYs, or even the arithmetic of the CY manifolds themselves.

In the joint work Alim *et al.* (2013), we studied such differential rings, called special polynomial rings, for some one-parameter families of CY 3-folds. We showed that the rings have gradings playing the role of modular weights and that the partition functions are Laurent polynomials of the generators of the differential rings.

In Zhou (2013), I developed a systematic way to construct triples of graded rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ defined on the deformation spaces for some particular one-parameter families of CY 3-folds. This work was based on the joint work Alim *et al.* (2013). It also followed closely the lines of thoughts in Ceresole *et al.* (1993a,b); Kaneko and Zagier (1995); Aganagic *et al.* (2008); Yamaguchi and Yau (2004); Alim and Länge (2007); Hosono (2008). The new addition to the previous works is that the rings are constructed from the Picard-Fuchs system of the CY 3-fold families and the Weil-Petersson geometry on the moduli spaces in such a way that the analogue between these rings and the rings of modular objects is more explicit. This work will be explained in Section 3 of the thesis. The main results are as follows.

Theorem 0.4. *For some special one-parameter families $\pi : \mathcal{X} \to \mathcal{M}$ of Calabi-Yau threefolds, e.g., the quintic mirror family, there exist graded rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ satisfying the following properties.*

1. *The rings are generated by the periods solved from the Picard-Fuchs equation, the connections of the Weil-Petersson metric on $\mathcal{M}$, the connections on the Hodge line bundle $\mathcal{L} = R^0\pi_*\Omega^3_{\mathcal{M}|\mathcal{X}}$, and their derivatives.*

2. *The rings are bi–graded by (k, m). There exist differential operators $D, \hat{D}$ such that $(\widetilde{\mathcal{R}}, D)$, $(\widehat{\mathcal{R}}, \hat{D})$ become graded differential rings. Taking derivatives $D, \hat{D}$ will not change the grading m and increases the grading k by 2.*

3. *For the families in Theorem 0.1, the rings $\mathcal{R}, \widetilde{\mathcal{R}}$ coincide with the rings of modular forms and*

quasi-modular forms for the corresponding modular groups. The grading k coincides with the modular weight.

Moreover, a parallelism between the triples $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ defined on the moduli space $\mathcal{M}$ and the triple $(M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma))$ defined on the modular curve X_Γ is made: the same way that they are constructed, the similarity between the set of operations (non-holomorphic completion and modular completion; holomorphic limit and "constant term map" Kaneko and Zagier (1995)). These are illustrated in Figure 5 below. They provide evidences that indeed the graded rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ are analogues[6] of the rings of modular objects $(M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma))$.

$$
\begin{tikzcd}
\widehat{\partial_\tau} \circlearrowright \widehat{M}(\Gamma) \arrow[d, shift right, "\text{constant term map}"'] \\
\partial_\tau \circlearrowright \widetilde{M}(\Gamma) \arrow[u, shift right, bend right, "\text{modular completion}"'] \\
M(\Gamma) \arrow[u, "\partial_\tau"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\widehat{\mathcal{R}} \circlearrowleft \hat{D} \arrow[d, shift right, "\text{holomorphic limit}"'] \\
\widetilde{\mathcal{R}} \circlearrowleft D \arrow[u, shift right, bend right, "\text{non-holomorphic completion}"'] \\
\mathcal{R} \arrow[u, "D"']
\end{tikzcd}
$$

$$\pi_\Gamma : \mathcal{E}_\Gamma \to X_\Gamma \qquad\qquad \pi : \mathcal{X} \to \mathcal{M}$$

Figure 5: *Analogues between rings defined from special Kähler geometry and rings of modular objects*

The structure of this thesis is as follows.

In Chapter 1, we shall review some preliminaries on modular curves and modular

[6]The topological string partition functions are polynomials in these generators. Due to the interpretation of topological string partition functions as generating functions of Gromov-Witten invariants and the polynomial structure, these generators themselves may contain some enumerative information. To some extent, physics (topological string theory) predicts the existence of a theory of modular forms which may or may not be related to the one discussed here. See also Hosono (2008); Movasati (2011) for related works.

forms. In particular, we shall work out the full details about the generators for the rings of almost-holomorphic modular forms for the modular groups $\Gamma_0(N)$ with $N = 1^*, 2, 3, 4$ whose forms suit our purpose in studying the modularity of topological string partition functions. We shall also discuss the Fricke involution acting on the modular curves and on the rings of almost-holomorphic modular forms.

In Chapter 2, we study the arithmetic of the moduli spaces of complex structures for the mirror families $\pi : \mathcal{X} \to \mathcal{M}$ of the one-parameter CY 3-fold families $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$. We identify the moduli spaces with modular curves, and then apply the modular form theory to solve the BCOV holomorphic anomaly equations. We shall prove the main theorems and corollary outlined earlier in this section.

In Chapter 3, we shall construct the triples of differential rings on the moduli space of certain CY 3-fold families, and study the analogues between these triples and the triples of rings of modular forms, quasi-modular forms and almost-holomorphic modular forms.

We conclude with some discussions in Chapter 4.

Chapter 1

Preliminaries on modular forms[1]

We shall briefly review some basic concepts about modular curves, modular forms and quasi-modular forms, which will be relevant in subsequent chapters, in Section 1.1 and Section 1.2 of this chapter. More details can be found in Diamond and Shurman (2005); Zagier (2008) and the references therein. We then give the full details for the generators of the rings of almost-holomorphic modular forms for the Hecke subgroups $\Gamma_0(N), N = 1^*, 2, 3, 4$ in Section 1.3. In Section 1.4 we highlight an involution, the Fricke involution, which acts on the modular curves and thus on modular forms and exchanges their expansions at two different cusps.

1.1 Modular groups and modular curves

The generators and relations for the group $\mathrm{SL}(2,\mathbb{Z})$ are given by the following:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = -I, \quad (ST)^3 = -I. \tag{1.1}$$

---

[1]This chapter is based on the joint work Alim *et al.* (2013) with Murad Alim, Emanuel Scheidegger and Shing-Tung Yau, and my paper Zhou (2013) which grew out of the discussions with the authors of Alim *et al.* (2013).

We will consider in the following the genus zero congruence subgroups called Hecke subgroups of $\Gamma(1) = \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, c \equiv 0 \mod N \right\} < \Gamma(1) \tag{1.2}$$

with $N = 2,3,4$. A further subgroup that we will consider in the sequel is the unique normal subgroup in $\Gamma(1)$ of index 2 which is often denoted by $\Gamma_0(1)^*$, it will be discussed in more detail in Section 1.3.2 and Section 2.2.1. By abuse of notation, we write $N = 1^*$ when listing it together with the other groups $\Gamma_0(N), N = 2,3,4$.

The group $\mathrm{SL}(2,\mathbb{Z})$ acts on the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \, \mathrm{Im}\tau > 0\}$ by fractional linear transformations:

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,.$$

The quotient space $Y_0(N) = \Gamma_0(N)\backslash\mathcal{H}$ is a non-compact orbifold with certain punctures corresponding to the cusps and orbifold points corresponding to the elliptic points under the action of the group $\Gamma_0(N)$. By filling the punctures, one then gets a compact orbifold $X_0(N) = \overline{Y_0(N)} = \Gamma_0(N)\backslash\mathcal{H}^*$ where $\mathcal{H}^* = \mathcal{H} \cup \{i\infty\} \cup \mathbb{Q}$. The orbifold $X_0(N)$ can be equipped with the structure of a Riemann surface. The signature for the group $\Gamma_0(N)$ and the two orbifolds $Y_0(N), X_0(N)$ could be represented by $\{p, \mu; \nu_2, \nu_3, \nu_\infty\}$, where p is the genus of $X_0(N)$, μ is the index of $\Gamma_0(N)$ in $\Gamma(1)$, and ν_i are the numbers of $\Gamma_0(N)$-equivalent elliptic points or parabolic points of order i. The signatures for the groups $\Gamma_0(N)$, $N = 1^*, 2, 3, 4$ are listed in Table 1.1 (see e.g., Rankin (1977)). The fundamental domains for

Table 1.1: *Signatures for the groups* $\Gamma_0(N), N = 1^*, 2, 3, 4$

N	ν_2	ν_3	ν_∞	μ	p
1^*	0	1	2	2	0
2	1	0	2	3	0
3	0	1	2	4	0
4	0	0	3	6	0

these groups are depicted in Figure 1.1.

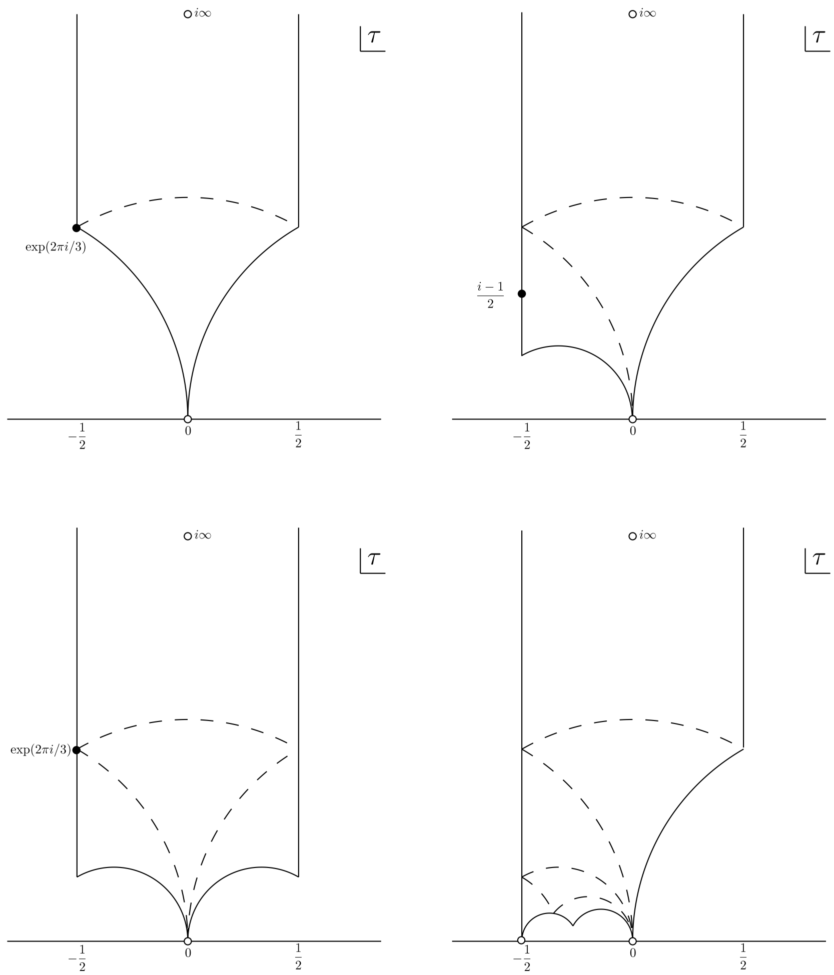

Figure 1.1: *Fundamental domains for* $\Gamma_0(N), N = 1^*, 2, 3, 4$, *respectively. The empty and full circles stand for cusps and elliptic points, respectively.*

The space $X_0(N)$ is called a modular curve and is the moduli space of pairs (E, C), where E is an elliptic curve and C is a cyclic subgroup of order N of the group of the N–torsion points $E_N \cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$. It classifies each cyclic N-isogeny $\phi : E \to E/C$ up to isomorphism, see for example Husemöller (2004); Diamond and Shurman (2005).

Similarly, we can define the modular curve $X_\Gamma = \Gamma\backslash\mathcal{H}^*$ associated to a general subgroup Γ of finite index in $\Gamma(1)$. We refer the reader to Diamond and Shurman (2005) for more details on this.

For a large class of non-compact CY 3-folds, the relevant geometry of the mirror manifolds are captured by the so-called mirror curves, see Hori and Vafa (2000). In what follows we shall only consider the cases where the mirror curves are elliptic curves. These already include many interesting examples such as the mirrors of the canonical bundle of $\mathbb{P}^2$ and the canonical bundles of the del Pezzo surfaces dP_n, $n = 5, 6, 7, 8$ discussed in this thesis. See for instance Lerche *et al.* (1997); Chiang *et al.* (1999); Katz *et al.* (1999); Mohri (2002) for more examples.

As we shall discuss in greater detail later in Section 2.2.1, for the one-parameter families of canonical bundle of $\mathbb{P}^2$ and the canonical bundles of the del Pezzo surfaces dP_n, $n = 5, 6, 7$, the bases of the corresponding families of mirror curves are the modular curves $X_0(N)$ with $N = 3, 4, 3, 2$, respectively. The canonical bundle of dP_8 is exceptional in the sense the base of the corresponding mirror curve family is not a modular curve of the form $X_0(N)$. It is given by $\Gamma_0(1)^* \backslash \mathcal{H}^*$, where $\Gamma_0(1)^*$ is the subgroup of $\Gamma(1)$ mentioned earlier. This base is a copy of $\mathbb{P}^1$ parametrized by a particularly chosen coordinate function z, and is a $2:1$ cover of the j–plane $\mathbb{P}^1$ realized by the map $j(z) = 1/z(1 - 432z)$. In the following we shall denote the base of this family of elliptic curves by $X_0(1^*)$. See Lian and Yau (1996a); Klemm *et al.* (1996); Maier (2009) for more discussions on this group and the corresponding elliptic curve family.

1.2 Modular forms, quasi-modular forms and almost-holomorphic modular forms

We proceed by recalling some basic concepts in modular form theory following Diamond and Shurman (2005). In the following, we shall use the notation Γ for a general congruence subgroup of finite index in $\Gamma(1)$. In particular, we can take Γ to be the modular group $\Gamma_0(N)$ described above and discuss the modular form theory associated to it.

Modular functions

A meromorphic modular function with respect to the a subgroup Γ of finite index in $\Gamma(1)$ is a meromorphic function $f : X_\Gamma \to \mathbb{P}^1$. Consider the restriction of f to $Y_\Gamma = \Gamma\backslash\mathcal{H}$. Since the restriction is meromorphic, the function f can be lifted to a meromorphic function f on $\mathcal{H}$. Then one gets a function $f : \mathcal{H} \to \mathbb{P}^1$ such that

(i) $f(\gamma\tau) = f(\tau), \quad \forall\gamma \in \Gamma\,.$

(ii) f is meromorphic on $\mathcal{H}$.

(iii) f is "meromorphic at the cusps" in the sense that the function

$$f|_\gamma : \tau \mapsto f(\gamma\tau) \tag{1.3}$$

is meromorphic at $\tau = i\infty$ for any $\gamma \in \Gamma(1)$.

The third condition requires more explanation. For any cusp class[2] $[r] \in \Gamma\backslash\mathcal{H}^*$with respect to the modular group Γ, one chooses a representative $r \in \mathbb{Q} \cup \{i\infty\}$. Then it is easy to see that one can find an element $\gamma_* \in \Gamma(1)$ so that $\gamma_* : i\infty \mapsto r$. The above condition means that the function defined by $\tau \mapsto f \circ \gamma_*(\tau)$ is meromorphic near $\tau = i\infty$ and that the function f is declared to be "meromorphic at the cusp $[r]$" if this condition is satisfied.

To summarize, a meromorphic modular function with respect to the modular group is a meromorphic function $f : \mathcal{H} \to \mathbb{P}^1$ satisfying the above properties on modularity, meromorphicity, and growth condition at the cusps.

Modular forms

Similarly, we can define a (meromorphic) modular form of weight k with respect to the group Γ to be a (meromorphic) function $f : \mathcal{H} \to \mathbb{P}^1$ satisfying the following conditions:

(i) $f(\gamma\tau) = j_\gamma(\tau)^k f(\tau), \quad \forall\gamma \in \Gamma\,$, where j is called the automorphy factor and is defined by

[2]We use the notation $[\tau]$ to denote the equivalence class of $\tau \in \mathcal{H}^*$ under the group action of Γ on $\mathcal{H}^*$.

$$j : \Gamma \times \mathcal{H} \to \mathbb{C}, \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto j_\gamma(\tau) := (c\tau + d)\,.$$

(ii) f is meromorphic on $\mathcal{H}$.

(iii) f is "meromorphic at the cusps" in the sense that the function

$$f|_\gamma : \tau \mapsto j_\gamma(\tau)^{-k} f(\gamma\tau) \tag{1.4}$$

is meromorphic at $\tau = i\infty$ for any $\gamma \in \Gamma(1)$.

We will need to be able to take roots of modular forms. For this purpose one introduces a function $v : \Gamma \to \mathbb{C}$, called multiplier system of weight k for Γ, such that $|v(\gamma)| = 1$ and $v(\gamma_1\gamma_2) = w(\gamma_1, \gamma_2)v(\gamma_1)v(\gamma_2)$. Here, $w(\gamma_1, \gamma_2)$ are numbers in $\{\pm 1\}$ making $v(\gamma)(c\tau + d)$ into an automorphy factor. Replacing the automorphy factor by $j_\gamma(\tau) = v(\gamma)(c\tau + d)$ in Eq. (1.4), one defines modular forms with respect to a multiplier system, see for example Rankin (1977) for details.

Note that the first condition above can be more conveniently rephrased in terms of the slash operator $|_\gamma : f \mapsto f|_\gamma$ in Eq. (1.3) to be

$$f|_\gamma = f, \quad \forall \gamma \in \Gamma\,. \tag{1.5}$$

The space of holomorphic modular forms for Γ forms a graded ring and is denote by $M_*(\Gamma)$.

We now also review the definitions of quasi-modular forms and almost-holomorphic modular forms following Kaneko and Zagier (1995); Zagier (2008).

Quasi-modular forms

A (meromorphic) quasi-modular form of weight k with respect to the group Γ is a (meromorphic) function $f : \mathcal{H} \to \mathbb{P}^1$ satisfying the following conditions:

(i) There exist meromorphic functions $f_i, i = 0, 1, 2, 3, \ldots, k-1$ such that

$$f(\gamma\tau) = j_\gamma(\tau)^k f(\tau) + \sum_{i=0}^{k-1} c^{k-i} j_\gamma(\tau)^i f_i(\tau)\,, \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,. \tag{1.6}$$

(ii) f is meromorphic on $\mathcal{H}$.

(iii) f is "meromorphic at the cusps" in the sense that the function

$$f|_\gamma : \tau \mapsto j_\gamma(\tau)^{-k} f(\gamma\tau) \tag{1.7}$$

is meromorphic at $\tau = i\infty$ for any $\gamma \in \Gamma(1)$.

The space of quasi-modular forms for Γ forms a graded differential ring and is denote by $\widetilde{M}_*(\Gamma)$.

Almost-holomorphic modular forms

An almost-holomorphic modular form of weight k with respect to the group Γ is a smooth function $f(\tau, \bar{\tau})$ defined on $\mathcal{H}$ satisfying the following conditions:

(i) $f(\gamma\tau, \overline{\gamma\tau}) = j_\gamma(\tau)^k f(\tau, \bar{\tau}), \quad \forall\gamma \in \Gamma.$

(ii) f is smooth on $\mathcal{H}$.

(iii) For any $\gamma \in \Gamma(1)$, the function

$$f|_\gamma : (\tau, \bar{\tau}) \mapsto j_\gamma(\tau)^{-k} f(\gamma\tau, \overline{\gamma\tau}) \tag{1.8}$$

grows at most polynomially in $\frac{1}{\mathrm{Im}\tau}$ as $\frac{1}{\mathrm{Im}\tau} \to 0$.

The space of almost-holomorphic modular forms for Γ forms a graded differential ring and is denote by $\widehat{M}_*(\Gamma)$.

An almost-holomorphic modular form has the following structure, see Kaneko and Zagier (1995),

$$f(\tau, \bar{\tau}) = \sum_{m=0}^{[k/2]} f_m(\tau) Y^m,\ Y = \frac{1}{\mathrm{Im}\tau}\,, \tag{1.9}$$

where $f_m(\tau), m = 0, 1, 2, \cdots [k/2]$ are meromorphic functions on $\mathcal{H}$.

As shown in Kaneko and Zagier (1995), one has the ring isomorphism $\widetilde{M}_*(\Gamma) = M_*(\Gamma) \otimes \mathbb{C}[E_2]$, where E_2 is the Eisenstein series defined by

$$E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \sigma_1(d) q^d,\; q = e^{2\pi i \tau},\; \sigma_1(d) = \sum_{k:\, k|d} k.$$

Moreover, there is a ring isomorphism $\widehat{M}_*(\Gamma) \to \widetilde{M}_*(\Gamma)$ defined by $f(\tau, \bar{\tau}) \mapsto f_0(\tau)$, where f_0 is the function f_m in Eq. (1.9) with $m = 0$. If one regards Y as a formal variable, then this is the "constant term map" obtained by taking the limit $Y = \frac{1}{\mathrm{Im}\tau} \to 0$ (which could be induced from $\bar{\tau} \to \infty$, by thinking of $\bar{\tau}$ as a complex coordinate independent of τ). The inverse map takes a quasi-modular form to an almost-holomorphic modular form. We shall call this map "modular completion" in this thesis.

1.3 Rings of quasi-modular forms and almost-holomorphic modular forms

In this section, we shall show the explicit computations of the rings of quasi-modular forms and almost-holomorphic modular forms for the modular groups $\Gamma_0(N), N = 1^*, 2, 3, 4$. Before introducing these we recall the familiar case for the full modular group $\Gamma(1)$.

1.3.1 Rings for the full modular group $\Gamma(1)$

Take the group Γ to be the full modular group $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$. Then $M_*(\Gamma(1)) = \mathbb{C}[E_4, E_6]$, where E_4, E_6 are the familiar Eisenstein series defined by

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{d=1}^{\infty} \sigma_3(d) q^d,\; q = e^{2\pi i \tau},\; \sigma_3(d) = \sum_{k:\, k|d} k^3\,, \\ E_6(\tau) &= 1 - 504 \sum_{d=1}^{\infty} \sigma_3(d) q^d,\; q = e^{2\pi i \tau},\; \sigma_5(d) = \sum_{k:\, k|d} k^5\,. \end{aligned}$$

The Eisenstein series E_2 is a quasi-modular form for $\Gamma(1)$ since it transforms according to

$$E_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2 E_2(\tau) + \frac{12}{2\pi i} c(c\tau+d), \quad \forall\, \tau \in \mathcal{H}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)\,.$$

Recall that

$$\frac{1}{\mathrm{Im}\frac{a\tau+b}{c\tau+d}} = (c\tau+d)^2 \frac{1}{\mathrm{Im}\tau} - 2ic(c\tau+d), \quad \forall\, \tau \in \mathcal{H}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)\,.$$

we know the modular completion of the quasi-modular form $E_2(\tau)$ is

$$\widehat{E_2}(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\tau}\,. \tag{1.10}$$

It transforms according to

$$\widehat{E_2}(\gamma\tau,\overline{\gamma\tau}) = (c\tau+d)^2 \widehat{E_2}(\tau,\bar{\tau}),\, \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)\,.$$

Then $M_*(\Gamma(1)) = \mathbb{C}[E_4, E_6], \widetilde{M}_*(\Gamma(1)) = \mathbb{C}[E_2, E_4, E_6], \widehat{M}_*(\Gamma(1)) = \mathbb{C}[\widehat{E_2}, E_4, E_6]$. Hence the non-holomorphicity of generators is closely related to almost-holomorphic modularity. This will be a key property for later discussions. Moreover, the latter two rings carry differential ring structures given by

$$\begin{aligned} \partial_\tau E_2 &= \frac{1}{12}(E_2^2 - E_4),\, \partial_\tau E_4 = \frac{1}{3}(E_2 E_4 - E_6),\, \partial_\tau E_6 = \frac{1}{2}(E_2 E_6 - E_4^2)\,, && (1.11) \\ \widehat{\partial_\tau} \widehat{E}_2 &= \frac{1}{12}(\widehat{E}_2^2 - E_4),\, \widehat{\partial_\tau} E_4 = \frac{1}{3}(\widehat{E}_2 E_4 - E_6),\, \widehat{\partial_\tau} E_6 = \frac{1}{2}(\widehat{E}_2 E_6 - E_4^2)\,, && (1.12) \end{aligned}$$

where $\partial_\tau := \frac{1}{2\pi i}\frac{\partial}{\partial\tau} : \widetilde{M}_k(\Gamma(1)) \to \widetilde{M}_{k+2}(\Gamma(1))$ and $\widehat{\partial_\tau} = \partial_\tau + \frac{k}{12} \cdot \frac{-3}{\pi \mathrm{Im}\tau} : \widehat{M}_k(\Gamma(1)) \to \widehat{M}_{k+2}(\Gamma(1))$. Eq. (1.11) in the above is known as the Ramanujan identities and Eq. (1.12) is easily derived from Eq. (1.11) and the definition of $\widehat{E}_2$ in Eq. (1.10).

1.3.2 Rings for the Hecke subgroups $\Gamma_0(N), N = 1^*, 2, 3, 4.$

We now consider the genus zero modular curves $X_0(N)$ with $N = 1^*, 2, 3, 4$ and discuss the corresponding rings of quasi-modular forms and almost-holomorphic modular forms.

The explicit generators for these rings in terms of η or θ or Eisenstein series are known in the literature (see for instance Maier (2009, 2011) for a collection of results), but for our later purpose in applying them to study topological string partition functions, we shall reconstruct them below from geometric quantities, following the discussion in Maier (2009).

The relevant data giving the ring of quasi-modular forms as well as the modular parameter τ are captured by the periods ω_0 and ω_1 of the corresponding families of elliptic curves described in the following, see Lian and Yau (1996a); Klemm *et al.* (1996); Mohri (2002); Maier (2009, 2011). The families of elliptic curves are given by $\pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N) = \Gamma_0(N)\backslash\mathcal{H}^*$ with $N = 1^*, 2, 3, 4$, where $\mathcal{E}_{\Gamma_0(N)}$ is the elliptic modular surface described in Kodaira (1963); Shioda (1972) by

$$\begin{aligned} \mathcal{E}_{\Gamma_0(N)} \quad & := (\Gamma_0(N) \rtimes \mathbb{Z}^2)\backslash(\mathcal{H}^* \times \mathbb{C}) \,, \\ \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (m,n)\right) \quad & : (\tau, z) \mapsto \left(\frac{a\tau+b}{a\tau+d}, \frac{z+m\tau+n}{c\tau+d}\right), \quad \forall\, \gamma \in \Gamma_0(N)\,. \end{aligned} \tag{1.13}$$

The explicit equations, j–invariants and Picard-Fuchs operators of these families could be found in e.g., Lian and Yau (1996a); Klemm *et al.* (1996). In the following we shall only display the Picard-Fuchs operators

$$\mathcal{L}_{\text{elliptic}} = \theta^2 - \alpha(\theta + \frac{1}{r})(\theta + 1 - \frac{1}{r}), \quad \theta = \alpha\frac{\partial}{\partial\alpha}\,, \tag{1.14}$$

where $r = 6, 4, 3, 2$ for $N = 1^*, 2, 3, 4$, respectively. The parameter α is the complex coordinate on the space $\mathcal{M}$ in which the Picard-Fuchs equation takes the above particular form. Thinking of the base space $\mathcal{M}$ as the genus zero modular curve $X_0(N)$, it is then a modular function (called Hauptmodul) for the modular group $\Gamma_0(N)$. Each of these Picard-Fuchs equations has three regular singularities located at $\alpha = 0, 1, \infty$ on the corresponding modular curve. They correspond to the cusp classes $[i\infty], [0]$ and (or) the elliptic point according to Table 1.1. Moreover, the parameter α on $\mathcal{M}$ is such that the set $\{\alpha = 0, 1\}$ coincides with the set of cusp classes $\{[\tau] = [i\infty], [0]\}$. This can be seen by looking at the indicial equations of the Picard-Fuchs equations which tell the types of singularities.

For the Picard-Fuchs equation Eq. (1.14), we define

$$\beta := 1 - \alpha \tag{1.15}$$

and choose a basis of solutions to be

$$\omega_0(\alpha) = {}_2F_1(\frac{1}{r}, 1 - \frac{1}{r}, 1; \alpha), \quad \omega_1(\alpha) = \frac{i}{\sqrt{N}}\, {}_2F_1(\frac{1}{r}, 1 - \frac{1}{r}, 1; \beta)\,, \tag{1.16}$$

then one has

$$\tau(\alpha) = \frac{\omega_1(\alpha)}{\omega_0(\alpha)} = \frac{i}{\sqrt{N}} \frac{{}_2F_1(\frac{1}{r}, 1 - \frac{1}{r}, 1; \beta)}{{}_2F_1(\frac{1}{r}, 1 - \frac{1}{r}, 1; \alpha)}\,, \tag{1.17}$$

where ${}_2F_1(\frac{1}{r}, 1 - \frac{1}{r}, 1; \alpha)$ is a Gauss hypergeometric function (see Erdélyi *et al.* (1981)).

Using the above basis of solutions and analytic continuation of hypergeometric functions (see Erdélyi *et al.* (1981)), one can easily verify that the monodromies are given by, see e.g., Mohri *et al.* (2001); Mohri (2002)

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T\,, M_1 = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix} = -ST^N S, M_\infty = M_1 M_0\,, \tag{1.18}$$

respectively, where S and T are the standard generators for $\mathrm{SL}(2,\mathbb{Z})$ in Eq. (1.1). Hence the monodromy group is $\Gamma_0(N)$ if $N \neq 1$, and coincides with the modular group. For $N = 1$, the monodromy group is $\Gamma(1)$. However, the base is the modular curve $X_0(1^*) = \Gamma(1^*)\backslash\mathcal{H}^*$, see Section 2.2.1 for details.

One then defines a triple following Borwein and Borwein (1991); Berndt *et al.* (1995); Maier (2009),

$$A(\alpha) = \omega_0(\alpha), \quad B(\alpha) = (1-\alpha)^{\frac{1}{r}} A(\alpha), \quad C(\alpha) = \alpha^{\frac{1}{r}} A(\alpha)\,. \tag{1.19}$$

These functions A, B, C are possibly multi-valued on the modular curve $X_0(N)$ and have divisors as follows

$$\mathrm{div} A = \frac{1}{r}(\alpha = \infty),\, \mathrm{div} B = \frac{1}{r}(\alpha = 1),\, \mathrm{div} C = \frac{1}{r}(\alpha = 0)\,. \tag{1.20}$$

This fact is the key in Alim *et al.* (2013) in analyzing the singularities of the topological string partition functions as solutions to the holomorphic anomaly equations, as we shall see later in Chapter 2.

Remark 1.1. In the following, by abuse of notation we shall use interchangeably the notation $A(\tau)$ to denote the function $A(\alpha(\tau))$ and thus $A(\alpha) = A(\tau(\alpha))$, where $\tau(\alpha)$ is defined as in Eq. (1.17) and $\alpha(\tau)$ is the Hauptmodul.

Note that the quantities A, B, C satisfy the equation $A^r = B^r + C^r$ and that $\alpha = C^r/A^r$. We now define further the quantity[3]

$$E = \partial_\tau \log C^r B^r, \quad \partial_\tau := \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\,. \tag{1.21}$$

It turns out that the ring generated by A, B, C, E is closed under the derivative ∂_τ.

Theorem 1.2. *For each of the elliptic curve families $\pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4$ with $r = 6, 4, 3, 2$ respectively, the following identities hold:*

$$\begin{aligned}
\partial_\tau A &= \frac{1}{2r}A(E + \frac{C^r - B^r}{A^r}A^2)\,,\\
\partial_\tau B &= \frac{1}{2r}B(E - A^2)\,,\\
\partial_\tau C &= \frac{1}{2r}C(E + A^2)\,,\\
\partial_\tau E &= \frac{1}{2r}(E^2 - A^4)\,.
\end{aligned} \tag{1.22}$$

Proof. These identities follow from Eqs. (1.27), (1.28), (1.29) below. □

The ring generated by A, B, C, E has an obvious grading denoted by k below: the gradings assigned to A, B, C, E are $1, 1, 1, 2$, taking the derivative ∂_τ will increase the grading by 2. Similar to Eq. (1.12) in the full modular group case, one gets the following

Theorem 1.3. *For each of the elliptic curve families $\pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4$ with $r = 6, 4, 3, 2$ respectively, defining $\widehat{E} = E + \frac{r}{6}\frac{-3}{\pi \mathrm{Im}\tau}$ and $\widehat{\partial_\tau} = \partial_\tau + \frac{k}{12}\frac{-3}{\pi \mathrm{Im}\tau}$, then the following*

[3]Throughout this thesis, we shall take $\partial_\tau := \frac{1}{2\pi i}\frac{\partial}{\partial\tau}$ as defined here.

identities hold:

$$
\begin{aligned}
\widehat{\partial_\tau} A &= \frac{1}{2r} A(\widehat{E} + \frac{C^r - B^r}{A^r} A^2)\,, \\
\widehat{\partial_\tau} B &= \frac{1}{2r} B(\widehat{E} - A^2)\,, \\
\widehat{\partial_\tau} C &= \frac{1}{2r} C(\widehat{E} + A^2)\,, \\
\widehat{\partial_\tau} \widehat{E} &= \frac{1}{2r} (\widehat{E}^2 - A^4)\,.
\end{aligned}
\tag{1.23}
$$

Proof. Assume that the desired non-holomorphic quantity $\widehat{E}$ is given by $\widehat{E} = E + \Delta E$ with $\Delta E = \lambda \frac{-3}{\pi \mathrm{Im}\tau}$ for some constant λ. Then it is easy to see with $\widehat{\partial_\tau} = \partial_\tau + \frac{k}{2r}\Delta E$, the first three identities follow from Theorem 1.2. Solving for the constant λ from the last identity, we then get $\lambda = \frac{r}{6}$. Thus the conclusion follows. $\square$

Remark 1.4. The results in Theorem 1.2 were known a few decades ago in the literature, see e.g., Maier (2011)for a review. In fact, for each of these cases, one can find the θ or η-expressions of the quantities A, B, C, E and prove the formulas by checking the θ or η-expressions. The relations between these generators and the Eisenstein series E_2, E_4, E_6 are also known, see e.g., Maier (2009), Maier (2011), Alim *et al.* (2013) for a collection of these results. One could then, for example, use the Eisenstein series expression of E to obtain the almost-holomorphic modular form $\widehat{E} = E + \Delta E$, with

$$
\Delta E = \frac{2}{N+1} \frac{-3}{\pi \mathrm{Im}\tau}, \; N = 1^*, 2, 3, \quad \Delta E = \frac{1}{3} \frac{-3}{\pi \mathrm{Im}\tau}, \; N = 4\,. \tag{1.24}
$$

These agree with the above choices $\Delta E = \frac{r}{6} \frac{-3}{\pi \mathrm{Im}\tau}$. But this method could not be generalized to CY 3-fold families.

For later use, we shall display the η-expansions of A, B, C here (again $N = 1^*$ case is exceptional) in Table 1.2. From these expressions one can get in particular the η-expansion of the Hauptmodul $\alpha = C^r / A^r$. Moreover, the choice for the Hauptmodul α satisfies the property that $\alpha([i\infty]) = 0, \alpha([0]) = 1$, while the rest of the branch point on the modular curve, as shown in Table 1.1, gives $\alpha = \infty$. This is summarized in Table 1.3. We leave the other expressions of the functions A, B, C, E to Appendix A. More details on the arithmetic

Table 1.2: *η-expansions of A, B, C for $\Gamma_0(N), N = 1^*, 2, 3, 4$*

N	A	B	C	E
1^*	$E_4(\tau)^{\frac{1}{4}}$	$\left(\frac{E_4(\tau)^{\frac{3}{2}}+E_6(\tau)}{2}\right)^{\frac{1}{6}}$	$\left(\frac{E_4(\tau)^{\frac{3}{2}}-E_6(\tau)}{2}\right)^{\frac{1}{6}}$	$\frac{E_2(\tau)+NE_2(N\tau)}{N+1}$
2	$\frac{(2^6\eta(2\tau)^{24}+\eta(\tau)^{24})^{\frac{1}{4}}}{\eta(\tau)^2\eta(2\tau)^2}$	$\frac{\eta(\tau)^4}{\eta(2\tau)^2}$	$2^{\frac{3}{2}}\frac{\eta(2\tau)^4}{\eta(\tau)^2}$	$\frac{E_2(\tau)+NE_2(N\tau)}{N+1}$
3	$\frac{(3^3\eta(3\tau)^{12}+\eta(\tau)^{12})^{\frac{1}{3}}}{\eta(\tau)\eta(3\tau)}$	$\frac{\eta(\tau)^3}{\eta(3\tau)}$	$3\frac{\eta(3\tau)^3}{\eta(\tau)}$	$\frac{E_2(\tau)+NE_2(N\tau)}{N+1}$
4	$\frac{(2^4\eta(4\tau)^8+\eta(\tau)^8)^{\frac{1}{2}}}{\eta(2\tau)^2} = \frac{\eta(2\tau)^{10}}{\eta(\tau)^4\eta(4\tau)^4}$	$\frac{\eta(\tau)^4}{\eta(2\tau)^2}$	$2^2\frac{\eta(4\tau)^4}{\eta(2\tau)^2}$	$\frac{E_2(\tau)-2E_2(2\tau)+4E_2(4\tau)}{3}$

Table 1.3: *The values of the Hauptmodul α at the branch points for $\Gamma_0(N), N = 1^*, 2, 3, 4$, where $*$ depends on the modular group.*

$[\tau]$	$[i\infty]$	$[0]$	$*$
α	0	1	∞

aspects can be found in Zagier (2008); Maier (2009, 2011).

Remark 1.5. Strictly speaking, the ring generated by A, B, C, E above does not form a differential ring due to the negative powers in Eq. (1.22). However, it is easy to see that by choosing suitable powers of these generators one can indeed get a ring. For example, in the $r = 6, 4, 3, 2$ cases, one can choose $A^4, B^6 - C^6, E$; A^2, B^4, E; A, B^3, E and A, B^2, E, respectively.

The ring generated by A, B, C, E is not exactly[4] the ring of quasi-modular forms for $\Gamma_0(N)$. For example, in the case $N = 3$, the ring of quasi-modular forms with the non-trivial Dirichlet character χ_{-3} is $\widetilde{M}_*(\Gamma_0(3), \chi_{-3}) = \mathbb{C}[A, B^3, E] \cong \mathbb{C}[A, F = C^3 - B^3, E]$, where $\chi_{-3}(d) = \left(\frac{-3}{d}\right)$ is the Legendre symbol. The corresponding differential structure is given by

$$\begin{aligned}
\partial_\tau A &= \frac{1}{2}(EA + F)\,,\\
\partial_\tau F &= \frac{1}{2}(EF + A^5)\,,\\
\partial_\tau E &= \frac{1}{6}(E^2 - A^4)\,.
\end{aligned}$$

One also has $\widetilde{M}_*(\Gamma_0(2)) = \mathbb{C}[A^2, B^4, E]$, $\widetilde{M}_*(\Gamma_0(4), \chi_{-4}) = \mathbb{C}[A, B^2, E]$, etc. See e.g., Mohri *et al.* (2001); Mohri (2002); Maier (2009, 2011) for details.

[4]The author would like to thank Professor Don Zagier for discussions on these.

However, in all of the discussions in this thesis we shall not use directly the transformations in the corresponding modular group $\Gamma_0(N)$, but use only the differential equations they satisfy. Moreover, in the applications to the studies of the holomorphic anomaly equations and the topological string partition functions, eventually only elements in the above rings of quasi-modular forms ($r = 6$ case is exceptional) will be involved, see Alim *et al.* (2013).

For the reasons mentioned above, by abuse of language, we shall call the following rings $\mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}]$, $\mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E]$ and $\mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, \hat{E}]$ to be the rings of modular forms, quasi-modular forms, almost-holomorphic modular forms for $\Gamma_0(N)$, and denote them by $M_*(\Gamma_0(N))$, $\widetilde{M}_*(\Gamma_0(N))$, $\widehat{M}_*(\Gamma_0(N))$ respectively. We shall also call the gradings "modular weights" which could be negative.

Since later we shall need to generalize the construction to some CY 3-fold families using the Picard-Fuchs systems, we shall reproduce below the details in constructing the graded differential rings $(\widetilde{M}_*(\Gamma_0(N)), \partial_\tau)$ using properties of the Picard-Fuchs equations from Eq. (1.14).

We start from the following observation.

Proposition 1.6. *For each of the elliptic curve families $\pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4$, one has $\partial_\tau \alpha = \alpha \beta A^2$, where as before $\beta = 1 - \alpha, \partial_\tau = \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$.*

Proof. For ease of notation, first we write the Picard-Fuchs operator in Eq. (1.14) as

$$\mathcal{L}_{\text{elliptic}} = \theta^2 - \alpha(\theta + c_1)(\theta + c_2), \quad \text{with} \;\; c_1 = \frac{1}{r}, \, c_2 = 1 - \frac{1}{r}\,, \tag{1.25}$$

and define

$$\tilde{\mathcal{L}}_{\text{elliptic}} := \left((\theta + \theta \log \omega_0)^2 - \alpha(\theta + \theta \log \omega_0 + c_1)(\theta + \theta \log \omega_0 + c_2) \right),$$

Then we have

$$\omega_0 \, \tilde{\mathcal{L}}_{\text{elliptic}} \frac{\Pi}{\omega_0} = \mathcal{L}_{\text{elliptic}} \, \Pi = 0 \;\; \text{for a period } \Pi \, .$$

In particular, we have $\tilde{\mathcal{L}}\, 1 = 0$, $\tilde{\mathcal{L}}_{\text{elliptic}}\, \tau = 0$. Subtracting $\tilde{\mathcal{L}}_{\text{elliptic}}\, 1$ from $\tilde{\mathcal{L}}_{\text{elliptic}}\, \tau$, we then get

$$\beta\theta^2\tau + (2\beta\theta\log\omega_0 - \alpha(c_1+c_2))\,\theta\tau = 0\,.$$

That is,

$$\theta\log(\omega_0^2\,\theta\tau) - (c_1+c_2)\frac{\alpha}{\beta} = \theta\log(\omega_0^2\,\theta\tau) + (c_1+c_2)\theta\log\beta = 0\,.$$

Solving this first order differential equation for $\omega_0^2\,\theta\tau$, we obtain

$$\theta\tau = \frac{c}{\beta^{c_1+c_2}\omega_0^2} = \frac{c}{\beta\omega_0^2}\,.$$

for some constant c. By looking at the leading terms in α of both sides as $\alpha\to 0$, we can then find that $c=\frac{1}{2\pi i}$. Hence $\partial_\tau\alpha = \alpha\beta A^2$ as claimed. $\square$

In what follows, we shall call the modular function α the algebraic modulus for the modular curve, while τ the transcendental modulus for the modular curve. The above formula then gives a differential equation relating the algebraic and transcendental moduli.

Recall the definitions of B, C which implies that $\frac{\alpha}{\beta} = \frac{C^r}{B^r}$, we then get

Corollary 1.7. *For each of the elliptic curve families $\pi_{\Gamma_0(N)}: \mathcal{E}_{\Gamma_0(N)} \to X_0(N), N = 1^*, 2, 3, 4$, the following is true:*

$$A^2 = \frac{\partial_\tau\alpha}{\alpha\beta} = \partial_\tau\log\frac{\alpha}{\beta} = \partial_\tau\log\frac{C^r}{B^r}\,. \tag{1.26}$$

Using the definition $E = \partial_\tau\log C^r B^r$ in Eq. (1.21), we have

$$\partial_\tau B = \frac{1}{2r}B(E - A^2), \quad \partial_\tau C = \frac{1}{2r}C(E + A^2)\,. \tag{1.27}$$

From $A^r = B^r + C^r$, we can easily get

$$\partial_\tau A = \frac{1}{2r}A(E + \frac{C^r - B^r}{A^r}A^2)\,. \tag{1.28}$$

Using the Picard-Fuchs equation Eq. (1.25) satisfied by A:

$$\beta(\theta A)^2 - (c_1+c_2)\alpha\theta A - c_1c_2\alpha A = 0\,,$$

we obtain

$$\partial_\tau^2 \log A = (\partial_\tau \log A)^2 + (c_1 + c_2 - 1)\alpha\beta A^2 \partial_\tau \log A + c_1 c_2 \alpha\beta A^4 \,.$$

This second order differential equation of A will become a first order differential equation of E since Eq. (1.28) says that $E = 2r\partial_\tau \log A - (\alpha - \beta)A^2$, one then gets

$$\partial_\tau E = \frac{1}{2r}E^2 + (2c_1c_2 r\alpha\beta - \frac{1}{2r}(\alpha-\beta)^2 - 2\alpha\beta)A^4 = \frac{1}{2r}E^2 - \frac{1}{2r}(\alpha+\beta)^2 A^4 = \frac{1}{2r}(E^2 - A^4)\,. \tag{1.29}$$

1.4 Fricke involution

Each of the modular curves $X_0(N)$ with $N = 1^*, 2, 3, 4$, as a covering of the j–plane $\Gamma(1)\backslash\mathcal{H}^*$, has three branch points. According to Table 1.1, they are two distinguished cusps given by $[i\infty] = [1/N]$ and $[0] = [1/1]$. The third branch point is a cubic elliptic point, quadratic elliptic point, cubic elliptic point and a cusp for $N = 1^*, 2, 3, 4$, respectively. The Fricke involution (a.k.a. Atkin-Lehner involution) is defined by

$$W_N : \tau \mapsto -\frac{1}{N\tau}\,. \tag{1.30}$$

It exchanges these two cusps and fixes the third branch point, see Figure 1.2 for an example.

Recall that the modular curve $X_0(N)$ is the moduli space of enhanced elliptic curves (E, C), where C is an order N subgroup of the N-torsion group $E_N \cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$. Using this interpretation and the association that

$$\begin{aligned} &\mathcal{H} \to \{\text{isomorphism classes of}(E, C)\} \\ &\tau \mapsto \left[\left(\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau), \langle\frac{1}{N}\rangle\right)\right]\,, \end{aligned}$$

the Fricke involution acts by sending (E, C) to $(E/C, E_N/C)$.

From the explicit expression of the Hauptmodul α which follows from Table 1.2, it turns

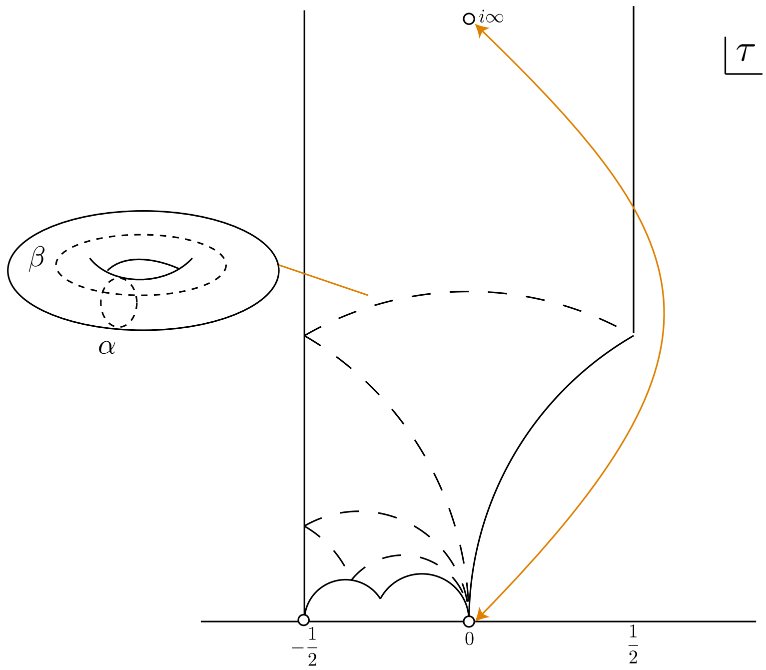

Figure 1.2: *Fricke involution on the fundamental domain for* $\Gamma_0(4)$

out that, see e.g., Maier (2009), the Fricke involution maps the Hauptmodul

$$\alpha(\tau) \quad \text{to} \quad \beta(\tau) := \alpha(-\frac{1}{N\tau}) = 1 - \alpha(\tau)\,. \tag{1.31}$$

Note that this is consistent with Eq. (1.17) in the sense that

$$\tau(\alpha) \mapsto \tau(\beta) = \frac{i}{\sqrt{N}} \frac{{}_2F_1(\frac{1}{r}, 1-\frac{1}{r}, 1; 1-\beta)}{{}_2F_1(\frac{1}{r}, 1-\frac{1}{r}, 1; \beta)} = \frac{i}{\sqrt{N}} \frac{{}_2F_1(\frac{1}{r}, 1-\frac{1}{r}, 1; \alpha)}{{}_2F_1(\frac{1}{r}, 1-\frac{1}{r}, 1; 1-\alpha)} = -\frac{1}{N\tau(\alpha)}\,. \tag{1.32}$$

The Fricke involution acts on the ring of quasi-modular forms according to

$$\begin{aligned}
A(-\frac{1}{N\tau}) &= \frac{\sqrt{N}}{i}\tau\, A(\tau)\,,\\
B(-\frac{1}{N\tau}) &= \frac{\sqrt{N}}{i}\tau\, C(\tau)\,,\\
C(-\frac{1}{N\tau}) &= \frac{\sqrt{N}}{i}\tau B(\tau)\,,\\
E(-\frac{1}{N\tau}) &= N\tau^2 E(\tau) + \frac{12}{2\pi i}\frac{2N\tau}{N+1}, \qquad N = 1^*, 2, 3\,,\\
E(-\frac{1}{N\tau}) &= N\tau^2 E(\tau) + \frac{12}{2\pi i}\frac{2N\tau}{6}, \qquad N = 4\,.
\end{aligned} \tag{1.33}$$

For all cases $N = 1^*, 2, 3, 4$, the non-holomorphic completion $\widehat{E}(\tau, \bar{\tau})$ transforms according

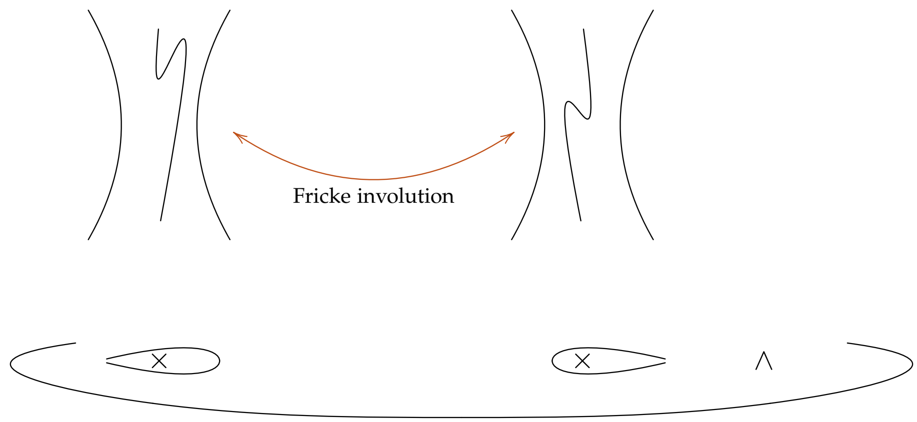

Figure 1.3: *Fricke involution on the family of elliptic curves*

to:

$$\widehat{E}\Big(-\frac{1}{N\tau},\overline{(-\frac{1}{N\tau})}\Big)=N\tau^2\widehat{E}(\tau,\bar{\tau})\,. \tag{1.34}$$

The above transformations have a nicer form by making use of the slash operator $|_\gamma$ as follows [5]. Take the following representative for W_N:

$$W_N=\frac{1}{\sqrt{N}}\begin{pmatrix}0 & 1\\ -N & 0\end{pmatrix}\in \mathrm{SL}(2,\mathbb{R})\,, \tag{1.35}$$

and define the slash operator on an almost-holomorphic modular form f by

$$f|_{W_N}=(\sqrt{N}\tau)^{-k}f(W_N\tau,\overline{W_N\tau})\,, \tag{1.36}$$

then we get $W_N^2=-I$ and

$$A|_{W_N}=iA,\quad B|_{W_N}=iC,\quad C|_{W_N}=iB,\quad \hat{E}|_{W_N}=\hat{E}\,. \tag{1.37}$$

Through out this thesis, we shall use both the Fricke involution Eqs. (1.33), (1.34) computationally and Eq. (1.37) conceptually.

[5]The author wants to thank Professor Don Zagier for suggesting this, which will be important later when translating the BCOV holomorphic anomaly equations with boundary conditions purely in terms of the language of modular form theory.

Besides the mathematical consequences, the Fricke involution also has an interpretation as a duality in physics, as explained in the joint work Alim *et al.* (2013) and shall be mentioned briefly in Section 2.6 in the next chapter.

Chapter 2

Topological string partition functions for some non-compact CY geometries in terms of modular forms[1]

In this chapter, we consider the mirror families (B model) $\pi : \mathcal{X} \to \mathcal{M}$ of the one-parameter families (A model) of the CY 3-folds $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$. We shall study the connections between the family $\pi : \mathcal{X} \to \mathcal{M}$ of CY 3-folds and the family of elliptic curves $\pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N)$, as described in Eq. (1.13). By comparing their periods, we shall identify the moduli space $\mathcal{M}$ with the modular curve $X_0(N)$, and singular points on $\mathcal{M}$ with the cusp classes and elliptic points on $X_0(N)$. After that we express the geometric quantities (e.g., connections, non-holomorphic generators S^{ij}, S^i, S, K_i and holomorphic Yukawa couplings) defined on the moduli space $\mathcal{M}$ in terms of almost-holomorphic modular forms defined on $X_0(N)$. Combining with the polynomial recursion, we can then determine the non-holomorphic part $\mathcal{P}^{(g)}$ in $\mathcal{F}^{(g)} = \mathcal{P}^{(g)} + f^{(g)}$. Then we use the Fricke involution to analyze the boundary conditions and to fix the holomorphic ambiguity $f^{(g)}$. Finally we conclude with some interesting observations and consequences of the structure of the

[1]Based on the joint work Alim *et al.* (2013) with Murad Alim, Emanuel Scheidegger and Shing-Tung Yau.

topological string partition functions $\mathcal{F}^{(g)}$ as almost-holomorphic modular forms.

The structure of this chapter is as follows. In Section 2.1 we shall review the geometries of the non-compact CY 3-folds that we shall consider. In Section 2.2 we make the identification between the moduli spaces and modular curves and discuss the Fricke involution on the singular points. We apply the results in Section 2.2 to solve for the first few topological string partition functions genus by genus recursively in Section 2.3. Then in Section 2.4 we use the Fricke involution to interpret the boundary conditions as regularity conditions for almost-holomorphic modular forms and to rewrite the holomorphic anomaly equations purely in terms of the language of modular form theory. After that in Section 2.5 we prove a version of integrality result for Gromov-Witten invariants of the geometries under consideration. We mention the interpretation of Fricke involution as a physics duality in Seiberg-Witten theory in Section 2.6. We conclude this chapter with some discussions and questions for future directions in Section 2.7.

2.1 Non-compact CY 3-fold geometries

The non-compact geometries we shall consider have all been studied before using different methods.

We start with a detailed discussion of $K_{\mathbb{P}^2}$ which denotes the canonical bundle $\mathcal{O}(-3) \to \mathbb{P}^2$ of $\mathbb{P}^2$, and its mirror. Higher genus topological string partition functions on this model have been studied in a number of works using different techniques, see for example Klemm and Zaslow (1999); Katz *et al.* (1999). The use of a different set of quasi-modular forms for this example was considered in Aganagic *et al.* (2008). The generators of Alim and Länge (2007), as reviewed in Eq. (20) in the Introduction, were used in Haghighat *et al.* (2008); Alim *et al.* (2010) for higher genus computations. Our new addition to these previous discussions consists of the explicit identification of the ring of quasi-modular forms of $\Gamma_0(3)$ which is adapted to this specific moduli space as well as its translation to the ring of generators S^{ij}, S^i, S, K_i in Alim and Länge (2007). Furthermore, this example serves as a testing ground for the duality of topological partition functions which turns out to exchange the large complex structure and the conifold loci.

The other non-compact geometries we consider are canonical bundles K_{dP_n} of del Pezzo surfaces dP_n, $n = 5, 6, 7, 8$ and their mirrors. These were considered in the physical context of non-critical string theories. For the purpose of our work, see Lerche *et al.* (1997) and references therein. Higher genus computations using the holomorphic anomaly equations and enumerative information from the A-model for these geometries were considered in Katz *et al.* (1999).

Now we shall review the details on the geometries of these non-compact CY 3-folds. The A-model geometries are $K_{\mathbb{P}^2}$ and $K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$, where dP_n is the del Pezzo surface obtained from blowing up $\mathbb{P}^2$ at n points[2]. We take the Kähler structures of the non-compact CY 3-folds $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$ to be the ones induced by $-K_B$, where $B = \mathbb{P}^2, \mathrm{dP}_n, n = 5, 6, 7, 8$ are the corresponding surfaces. Taking dP_6 for example, it is obtained from $\mathbb{P}^2$ by blowing up 6 points (and thus is not toric), one has

$$-K_{\mathrm{dP}_6} = -K_{\mathbb{P}^2} - \sum_{i=1}^{6} E_i \,,$$

where the E_is, $i = 1, 2 \cdots 6$ are the exceptional curve classes. Since $-K_{\mathrm{dP}_6}$ is ample, one can use the linear system to embed dP_6 into a projective space

$$| - K_{\mathrm{dP}_6}| : \mathrm{dP}_6 \to \mathbb{P}\check{H}^0(\mathrm{dP}_6, -K_{\mathrm{dP}_6}) \,,$$

as a degree 3 hypersurface in $\mathbb{P}^3$. The class $-K_{\mathrm{dP}_6}$ is then induced by the hyperplane class of $\mathbb{P}^3$ by pulling it back using the embeding.

The mirror CY families (B-model) are families of non-compact CY 3-folds. The explicit equations and Picard-Fuchs equations of these families are given in Lerche *et al.* (1997); Chiang *et al.* (1999) and will be discussed soon in the following.

[2]The sub-index n stands for the number of points in $\mathbb{P}^2$ which are blown up to create dP_n. The degree of this surface is $9 - n$.

By abuse of language, in the following, sometimes we shall call the mirror CY families local $\mathbb{P}^2$ and local $\mathrm{dP}_n, n = 5,6,7,8$ if no confusion arises.

2.1.1 Local $\mathbb{P}^2$

Now we shall discuss the mirror CY 3-fold family $\pi : \mathcal{X} \to \mathcal{M}$ of the $K_{\mathbb{P}^2}$ family, constructed in e.g., Chiang *et al.* (1999); Hori and Vafa (2000). We refer the interested readers to Lerche *et al.* (1997); Alim *et al.* (2013) and references therein for the detailed discussions on other families. For each z on $\mathcal{M}$, the fiber $\mathcal{X}_z$ of the non-compact CY 3-fold family is itself a conic fibration given by, see Hori and Vafa (2000),

$$\mathcal{X}_z : uv - H(x,y;z) = 0, \quad (u,v,x,y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2\,,$$

where $H(x,y;z) = y^2 - (x+1)y - 27zx^3$. The degeneration locus of this conic fibration is the elliptic curve $\mathcal{E}_z : H(x,y;z) = 0$. It is called the mirror curve in the literature since the whole mirror CY 3-fold geometry can be constructed from knowing only the information of this curve. Then as z moves in $\mathcal{M}$, one gets the Hori-Vafa mirror curve family $\pi_{\mathrm{HV}} : \mathcal{E}_{\mathrm{HV}} \to \mathcal{M}$.

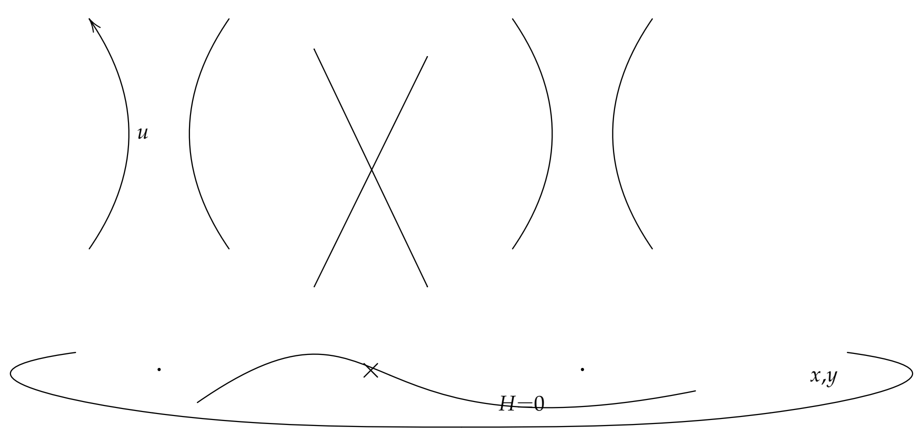

Figure 2.1: *Mirror CY 3-fold as a conic fibration*

Remark 2.1. Another way to construct the mirror in the literature is to use the toric duality Batyrev (1994). One considers Candelas *et al.* (1994b); Chiang *et al.* (1999) an elliptic fibration $\check{Y}$ over $\mathbb{P}^2$. The compact CY 3-fold $\check{Y}$ can be regarded as a complete intersection in a toric variety whose mirror is a compact CY 3-fold Y. The moduli space of Kähler structures of $\check{Y}$ is of dimension two: roughly speaking they correspond to the Kähler classes induced from the Poincare dual of the base $\mathbb{P}^2$ and of the surface ruled over the hyperplane section in the base $\mathbb{P}^2$. Denote these two classes by ω_H, ω_L respectively. Then a generic Kähler class is given by $\omega = t_1\omega_H + t_2\omega_L$. In the limit $t_1 \to \infty$ (called large fiber limit in Chiang *et al.* (1999)), the compact geometry $\check{Y}$ becomes effectively $\check{X} = K_{\mathbb{P}^2}$. One can also work out the same limit in the geometry Y, then one gets that the degeneration locus is in fact the Hesse cubic curve

$$x_1^3 + x_2^3 + x_3^3 - z^{-\frac{1}{3}}x_1x_2x_3 = 0, \quad j(z) = \frac{(1+216z)^3}{z(1-27z)^3}\,. \tag{2.1}$$

Again as α moves in $\mathcal{M}$, one gets an elliptic curve family $\pi_{\text{Hesse}} : \mathcal{E}_{\text{Hesse}} \to \mathcal{M}$. It turns out that this elliptic curve family is 3-isogenous to Hori-Vafa mirror curve family $\pi_{\text{HV}} : \mathcal{E}_{\text{HV}} \to \mathcal{M}$, see e.g., Husemöller (2004) for details. In particular, they have the same Picard-Fuchs equations. For our later purposes, the Hori-Vafa mirror curve family $\pi_{\text{HV}} : \mathcal{E}_{\text{HV}} \to \mathcal{M}$ will work perfectly.

2.1.2 Local del Pezzos

The non-compact CY 3-fold families mirror to $K_{\mathrm{dP}_n}, n = 5,6,7,8$ families are given in e.g., Lerche *et al.* (1997); Chiang *et al.* (1999) as follows:

$$\begin{aligned}
n=5 \;\; &: \;\; \mathbb{P}^4[-1,1,1,1,1,1][2,2] : x^{-2} + z_1^2 + z_2^2 + z_3^2 - z^{-\frac{1}{r}}xz_3z_4z_5 = 0\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad z_3^2 + z_4^2 - z^{-\frac{1}{r}}z_1z_2 = 0\,,\\
n=6 \;\; &: \;\; \mathbb{P}^4[-1,1,1,1,1][6] : x^{-3} + y^3 + w^3 + z_1^3 + z_2^3 - z^{-\frac{1}{r}}xyz_1z_2w = 0\,,\\
n=7 \;\; &: \;\; \mathbb{P}^4[-1,2,1,1,1][6] : x^{-4} + y^2 + w^4 + z_1^4 + z_2^4 - z^{-\frac{1}{r}}xyz_1z_2w = 0\,,\\
n=8 \;\; &: \;\; \mathbb{P}^4[-1,3,2,1,1][6] : x^{-6} + y^2 + w^3 + z_1^6 + z_2^6 - z^{-\frac{1}{r}}xyz_1z_2w = 0\,,
\end{aligned} \tag{2.2}$$

where the numbers r are given by $2,3,4,6$ for $n = 5,6,7,8$, respectively.

2.2 Moduli spaces as modular curves

For the non-compact geometries mentioned above, there are two ways to identify the moduli spaces with modular curves. One can look at the equations of the geometries, and then seek for the arithmetic properties of the bases of the families; or one could study the Hodge-theoretic aspects by looking at how the periods of the varieties vary. We shall only mention the former very briefly since in general it requires a lot of knowledge on the (equations for) families, and shall emphasize the latter since it is more convenient for computational purposes and requires less on the detailed information on the explicit equations for the families.

2.2.1 Geometric correspondence

For the local $\mathbb{P}^2$ family, we have seen in the previous section that the moduli space $\mathcal{M}$ is on the one hand the base of the non-compact CY 3-fold family, and is on the other hand the base of the elliptic curve families $\pi_{\mathrm{HV}} : \mathcal{E}_{\mathrm{HV}} \to \mathcal{M}$ and $\pi_{\mathrm{Hesse}} : \mathcal{E}_{\mathrm{Hesse}} \to \mathcal{M}$. However, it is a standard fact from Eq. (1.13) that the base of the Hesse cubic curve family Eq. (2.1) is exactly $X_0(3) \cong \Gamma_0(3)\backslash\mathcal{H}^*$. Therefore, we must have $\mathcal{M} \cong X_0(3)$.

Similarly, for the local del Pezzo geometries $K_{\mathrm{dP}_n}, n = 5,6,7,8$, the corresponding equations for the elliptic curve families (called elliptic curves of E_n type, $n = 5,6,7,8$ respectively), are obtained from the equations for the CY 3-folds by getting rid of the $-1,1$ entries and the corresponding monomials in Eq. (2.2). The j-invariants, as well as the Picard-Fuchs operators are summarized here, see Lian and Yau (1996a); Lerche *et al.* (1997); Klemm *et al.* (1996); Chiang *et al.* (1999) for more details. In the following, $\theta = z\frac{\partial}{\partial z}$.

$$E_5 : \begin{cases} x_1^2 + x_3^2 - z^{-\frac{1}{4}} x_2 x_4 &= 0 \\ x_2^2 + x_4^2 - z^{-\frac{1}{4}} x_1 x_3 &= 0 \end{cases} \qquad j(z) = \frac{(1+224z+256z^2)^3}{z(1-16z)^4}, \quad \mathcal{L}_{\mathrm{elliptic}} = \theta^2 - 4z(2\theta+1)^2. \tag{2.3}$$

The base of this family of elliptic curves is the modular curve $X_0(4)$. It has three singular points: two cusp classes $[i\infty], [0]$ corresponding to $z = 0, 1/16$ respectively; and the cusp class $[1/2]$ corresponding to $z=\infty$.

$$E_6:\ x_1^3+x_2^3+x_3^3-z^{-\frac{1}{3}}x_1x_2x_3=0\,,\quad j(z)=\frac{(1+216z)^3}{z(1-27z)^3}\,,\quad \mathcal{L}_{\text{elliptic}}=\theta^2-3z(3\theta+1)(3\theta+2)\,. \tag{2.4}$$

The base of this family of elliptic curves is the modular curve $X_0(3)$. It has three singular points: two cusp classes $[i\infty],[0]$ corresponding to $z=0,1/27$ respectively; and the cubic elliptic point $[\rho]$ corresponding to $z=\infty$, where $\rho=\exp(2\pi i/3)$.

$$E_7:\ x_1^4+x_2^4+x_3^2-z^{-\frac{1}{4}}x_1x_2x_3=0\,,\quad j(z)=\frac{(1+192z)^3}{z(1-64z)^3}\,,\quad \mathcal{L}_{\text{elliptic}}=\theta^2-4z(4\theta+1)(4\theta+3)\,. \tag{2.5}$$

The base of this family of elliptic curves is the modular curve $X_0(2)$. It has three singular points: two cusp classes $[i\infty],[0]$ corresponding to $z=0,1/64$ respectively; and the quadratic elliptic point $[(i-1)/2]=ST([i])$ corresponding to $z=\infty$.

$$E_8:\ x_1^6+x_2^3+x_3^2-z^{-\frac{1}{6}}x_1x_2x_3=0\,,\quad j(z)=\frac{1}{z(1-432z)}\,,\quad \mathcal{L}_{\text{elliptic}}=\theta^2-12z(6\theta+1)(6\theta+5)\,. \tag{2.6}$$

The base of this family of elliptic curves is the curve $X_0(1^*)$. It has three singular points: two cusp classes $[i\infty],[0]$ corresponding to $z=0,1/432$ respectively; and the cubic elliptic point $[\rho]$ corresponding to $z=\infty$.

The relations between the Picard-Fuchs operators of the above elliptic curves of E_n type and the elliptic curve families in Eq. (1.14) are related by $\alpha=\kappa_N z$ according to Table 2.1.

Table 2.1: *Arithmetic numbers in the Picard-Fuchs equations of elliptic curves of E_n type, $n=5,6,7,8$.*

n	5	6	7	8
N	4	3	2	1^*
r	2	3	4	6
κ_N	16	27	64	432

Note that in the above we did not claim that the corresponding elliptic curves families are isomorphic, and we don't need that since all we concern is essentially that the bases of these elliptic curve families are the same modular curve and that the Picard-Fuchs equations take the same particular form.

2.2.2 Hodge-theoretic correspondence

Now we study the periods of the CY 3-fold families. As we shall see, we will get the same identifications as above. Moreover, we can also figure out the exact relations between the periods of the CY 3-folds and the periods for the corresponding elliptic curves that attached to them Hodge-theoretically (which are the same as the mirror curves as they should be).

Direct computation shows that, see Lerche *et al.* (1997); Chiang *et al.* (1999), in all cases above, the Picard-Fuchs operators have the following form:

$$\mathcal{L}_{\mathrm{CY}} = \mathcal{L}_{\mathrm{elliptic}} \circ \theta = \left(\theta^2 - \alpha(\theta + \frac{1}{r})(\theta + 1 - \frac{1}{r})\right) \circ \theta, \quad \theta = \alpha \frac{\partial}{\partial \alpha}, \, \alpha = \kappa_N z \,, \tag{2.7}$$

where the operators $\mathcal{L}_{\mathrm{elliptic}}$ are exactly the same as the ones for the elliptic curve families in Eq. (1.14) and κ_N are the same numbers in Table 2.1.

This immediately tells that the moduli spaces are the modular curves $X_0(N)$ since the variation of the periods for the CY 3-fold families can be fully captured by the variation of the periods for the elliptic curve families with Picard-Fuchs operator being $\mathcal{L}_{\mathrm{elliptic}}$. In fact, the periods of CY 3-folds are related to integrals on the curves over certain chains by integration along the non-compact direction in the conic fibrations, related works can be found in e.g., Lerche *et al.* (1997); Hosono (2004).

2.2.3 Singular points on the moduli spaces

The singularities for the CY 3-fold families $\alpha = 0, 1, \infty$ are worked out to be the so-called large complex structure limit, conifold point and orbifold point, see Lerche *et al.* (1997); Chiang *et al.* (1999). Therefore, according to Table 1.3, we can get the types of singularities on the moduli spaces as shown in Table 2.2.

According to the above identification $\mathcal{M} \cong X_0(N)$, we can see that since the Fricke

Table 2.2: *Types of singularities on $\mathcal{M} \cong X_0(N)$ for $N = 1^*, 2, 3, 4$, where $*$ depends on the modular group.*

Type	α	$[\tau]$
large complex structure limit	0	$[i\infty]$
conifold point	1	$[0]$
orbifold point	∞	$*$

involution exchanges the two cusps $[i\infty]$ and $[0]$, it will then exchange the large complex structure limit with the conifold point. This will be very useful later in applying the boundary conditions to solve the BCOV holomorphic anomaly equations.

2.2.4 Periods for the CY 3-fold families

We now consider the periods of the CY 3-folds near the large complex structure limit $\alpha = 0$. The solutions to the Picard-Fuchs equations could be obtained using the Frobenius method and are given by Meijer-G functions, see Erdélyi *et al.* (1981); Diaconescu and Gomis (2000). According to the structure Eq. (10) for the periods in special Kähler geometry, the solutions to $\mathcal{L}_{\mathrm{CY}}\Pi = 0$ are given by

$$X^0 = 1, \quad t \sim \frac{1}{2\pi i}\log\alpha + \mathcal{O}(\alpha^0), \quad F_t \tag{2.8}$$

for some holomorphic function F. The quantity t, defined to be the ratio of the two periods above, will be called special coordinate hereafter and play a key role in the whole discussion (see Section 2.3.1 and Section 3.3.1 below).

Since $\mathcal{L}_{\mathrm{CY}} = \mathcal{L}_{\mathrm{elliptic}} \circ \theta$, we know the derivatives of these periods must be periods of the corresponding elliptic curves we have found geometrically or Hodge-theoretically earlier. Hence we can choose the basis suitably so that[3]

$$\theta t = \frac{1}{2\pi i}\omega_0, \quad \kappa^{-1}\theta F_t = \frac{1}{2\pi i}\omega_1 = \frac{1}{2\pi i}\tau\omega_0 \tag{2.9}$$

are the periods of the elliptic curve given in Eq. (1.16), respectively, where κ is the classical

[3]For simplification, in the following we shall sometimes ignore the $2\pi i$ factors, so then, for example, $\theta t = \omega_0$.

triple intersection of $\check{X}$. It is $-\frac{1}{3}$ for $\check{X} = K_{\mathbb{P}^2}$ ad $n-9$ for $K_{\mathrm{dP}_n}, n = 5,6,7,8$. Then

$$\tau = \frac{\omega_1}{\omega_0} = \frac{\kappa^{-1}\theta F_t}{\theta t} = \kappa^{-1} F_{tt}\,. \tag{2.10}$$

This definition was motivated in Aganagic *et al.* (2008) to establish the modularity for non-compact CY 3-folds and is very important in understanding the arithmetic properties of special Kähler geometry and topological string partition functions, as we shall see later.

2.3 Solving BCOV holomorphic anomaly equations genus by genus recursively

Now we are ready to apply the results on modular forms and Fricke involution in Chapter 1 to solve for the topological string partition functions. First we shall express the geometric quantities, including Yukawa couplings, propagators, etc., in terms of almost-holomorphic modular forms. This then gives the non-holomorphic part $\mathcal{P}^{(g)}$ in $\mathcal{F}^{(g)} = \mathcal{P}^{(g)} + f^{(g)}$. Then we use the Fricke involution to analyze the boundary conditions and to fix $f^{(g)}$.

In this section, we shall work out the full details for the genus two case for $K_{\mathbb{P}^2}$. The other cases are similar to this and will be discussed briefly. We shall also give the proofs of Theorem 0.1 and Theorem 0.2 sketched in the Introduction part, which are inspired by the computations for the genus two case.

2.3.1 Local $\mathbb{P}^2$

Genus 0

It is known that (see Chiang *et al.* (1999), and also Proposition 3.6 and Remark 3.7) the Yukawa coupling $C_{\alpha\alpha\alpha}$, as defined in Eq. (5), is given by

$$C_{\alpha\alpha\alpha} = \frac{\kappa}{\alpha^3(1-\alpha)}\,, \tag{2.11}$$

where $\kappa = -\frac{1}{3}$ is again the classical triple intersection number for $K_{\mathbb{P}^2}$. Note that $X^0 = 1$, it follows that

$$C_{ttt} = \frac{1}{(X^0)^2} \frac{\kappa}{\alpha^3(1-\alpha)} (\frac{\partial \alpha}{\partial t})^3 = \frac{\kappa}{(\theta t)^3 (1-\alpha)} \,. \tag{2.12}$$

Recall from Eq. (2.9) and Eq. (1.19) that $\theta t = \omega_0 = A, 1-\alpha = \beta = \frac{B^r}{A^r}$ with $r = 3$, we can get from Table 1.2 that

$$C_{ttt} = \frac{\kappa}{B^3} = -\frac{1}{3} \frac{\eta(3\tau)^3}{\eta(\tau)^9} \tag{2.13}$$

is a modular form of weight -3 with respect to the modular group $\Gamma_0(3)$.

Dictionary between propagators and almost-holomorphic modular forms

Holomorphic limits of metric and connections

The Kähler potential of the Weil-Petersson metric is determined from Eq. (3) and Eq. (10) and is given by

$$e^{-K} = i X^0 \overline{X^0} \left(2\overline{F(t)} - 2F(t) + (t^a - \bar{t}^a)(F_a + \overline{F_a}) \right) , \tag{2.14}$$

where according to mirror symmetry and Eq. (2) the prepotential $F(t)$ has the form

$$F(t) = \frac{\kappa_{abc}}{3!} t^a t^b t^c + Q(t) + \sum_d N_{0,d} e^{dt} \,, \tag{2.15}$$

with $Q(t)$ being a quadratic polynomial of $t = \{t^a\}$. Following Ferrara and Louis (1992); Bershadsky *et al.* (1994), rewriting the above equation as

$$e^{-K(z,\bar{z})} = X^0 \bar{X}^0 e^{-\mathrm{K}(t,\bar{t})}, \quad e^{-\mathrm{K}(t,\bar{t})} = i \left(2\overline{F(t)} - 2F(t) + (t^a - \bar{t}^a)(F_a + \overline{F_a}) \right) \,, \tag{2.16}$$

one then gets

$$K_{z^i} = -\partial_{z^i} \log X^0 + \mathrm{K}_{t^a} \frac{\partial t^a}{\partial z^i}, \quad \Gamma^{z^k}_{z^i z^j} = \frac{\partial z^k}{\partial t^a} \frac{\partial}{\partial z^i} \frac{\partial t^a}{\partial z^j} + \frac{\partial z^k}{\partial t^c} \Gamma^{t^c}_{t^a t^b} \frac{\partial t^a}{\partial z^j} \frac{\partial t^b}{\partial z^j} \,, \tag{2.17}$$

where $\Gamma^{t^c}_{t^a t^b}$ is computed in the metric given by the new Kähler potential $\mathrm{K}(t, \bar{t})$. Then at the large complex structure limit $z_* = 0$, one can show that the special coordinates $t = \{t^a\}$ are the canonical coordinates (see Section 3.3.1) satisfying

$$\partial_{t^I} \mathrm{K}_{t^a}|_{\bar{z}=\bar{z}_*} = 0, \quad \partial_{t^I} \Gamma^{t^c}_{t^a t^b}|_{\bar{z}=\bar{z}_*} = 0, \quad \forall I \text{ s.t. } |I| \geq 0\,. \tag{2.18}$$

This then implies that one has the following holomorphic limits at the large complex structure limit:

$$\lim_{\mathrm{LCSL}} K_{z^i} = -\partial_{z^i} \log X^0, \quad \lim_{\mathrm{LCSL}} \Gamma^{z^k}_{z^i z^j} = \frac{\partial z^k}{\partial t^a} \frac{\partial}{\partial z^i} \frac{\partial t^a}{\partial z^j}\,. \tag{2.19}$$

The above discussions apply to general families of compact or non-compact CY 3-folds.

In the current non-compact one-modulus case, we get

$$\lim_{\mathrm{LCSL}} K_\alpha = 0, \quad \lim_{\mathrm{LCSL}} \Gamma^\alpha_{\alpha\alpha} = \frac{\partial\alpha}{\partial t} \frac{\partial}{\partial\alpha} \frac{\partial t}{\partial\alpha} = \partial_\alpha \log \partial_\alpha t\,. \tag{2.20}$$

It turns out that the above results in the local coordinate $x = \ln \alpha$ defined on the puncture moduli space $\mathcal{M} - \{\alpha = 0\}$ are cleanest:

$$C_{xxx} = \frac{\kappa}{1-\alpha} \tag{2.21}$$

$$\lim_{\mathrm{LCSL}} K_x = 0, \quad \lim_{\mathrm{LCSL}} \Gamma^x_{xx} = \theta \log A\,. \tag{2.22}$$

Solving for holomorphic limit of propagators from special geometry relation

In this case, the integrated special geometry relation Eq. (7) gives

$$\Gamma^x_{xx} = 2K_x - C_{xxx} S^{xx} + s^x_{xx}\,.$$

Taking the holomorphic limit at the large complex structure limit, one gets

$$\lim_{\mathrm{LCSL}} \Gamma^x_{xx} = 2 \lim_{\mathrm{LCSL}} K_x - C_{xxx} \lim_{\mathrm{LCSL}} S^{xx} + s^x_{xx}\,.$$

This implies that

$$\lim_{\mathrm{LCSL}} S^{xx} = C^{-1}_{xxx} (2 \cdot 0 - \lim_{\mathrm{LCSL}} \Gamma^x_{xx} + s^x_{xx})\,.$$

From Proposition 1.6 and Eq. (1.2) we get

$$\lim_{\mathrm{LCSL}} \Gamma^x_{xx} = \theta \log A = \theta\tau\,\partial_\tau \log A = \frac{1}{\beta A^2} \frac{1}{2r} (E + \frac{C^r - B^r}{A^r} A^2)\,.$$

Taking the choice $s^x_{xx} = \frac{1}{2r}\frac{C^r - B^r}{A^r} = \frac{1}{6}\frac{C^3 - B^3}{A^3}$, then we obtain

$$\lim_{\text{LCSL}} S^{xx} = -\frac{1}{2r\kappa}\frac{E}{A^2} = -\frac{1}{2}\frac{E}{A^2}\,. \tag{2.23}$$

The holomorphic limits of S^x, S at the large complex structure limit can be easily solved from Eq. (20) that they satisfy, and the solutions are not unique as mentioned before in Introduction. In particular, we can take the following choice as in Alim and Länge (2007); Alim *et al.* (2010):

$$\lim_{\text{LCSL}} S^x = \lim_{\text{LCSL}} S = 0\,. \tag{2.24}$$

With this particular choice, the holomorphic functions h^x_x, h_x, h_{xx} are identically zero, while h^{xx}_x is determined from the first line in Eq. (20):

$$D_x S^{xx} = 2S^x - C_{xxx} S^{xx} S^x + h^{xx}_x\,.$$

Again taking the holomorphic limit at the large complex structure limit, we get

$$h^{xx}_x = \lim_{\text{LCSL}} h^{xx}_x = -\frac{1}{12}\frac{A^3}{B^3}\,. \tag{2.25}$$

Recovering non-holomorphic propagators from modularity

So far we have computed the holomorphic limits of the generators S^{xx}, S^x, S, K_x at the large complex structure limit $\alpha = 0$, they turn out to be quasi-modular forms. The non-holomorphic counterpart should reduce to them in the holomorphic limit, which is given by $\bar{t} = i\infty$ according to Eq. (2.8).

We observe that near the large complex structure limit, we actually have

$$\tau \sim t = \frac{1}{2\pi i}\log \alpha + \text{regular}$$

according to Eq. (2.10). Then the holomorphic limit is the same as $\frac{1}{\mathrm{Im}\tau} \to 0$, that is, the "constant term map" in Kaneko and Zagier (1995) in modular form theory. This suggests that the non-holomorphic quantities can be obtained from the inverse of the constant term map, that is, the modular completion of their holomorphic limits. Note that the non-holomorphic completion does not have to be identical to the modular completion. However, for many purposes, eventually we only care about the holomorphic limits (e.g., in extracting the Gromov-Witten invariants under the mirror map). Therefore, we can safely assume that near the large complex structure limit, taking the holomorphic limit of geometric quantities is equivalent to taking the constant term map of almost-holomorphic modular forms, while taking the non-holomorphic completion is equivalent to taking the modular completion from quasi-modular forms to almost-holomorphic modular forms, as illustrated in Figure 3 in Introduction.

Therefore, from Eqs. (2.21), (2.23), (2.24), we get

$$S^{xx} = \frac{1}{2}\frac{\hat{E}}{A^2}, \quad S^x = S = 0, \quad K_x = 0\,. \tag{2.26}$$

Moreover, the non-holomorphic completions of the holomorphic function C_{xxx} and holomorphic ambiguities (e.g., s^x_{xx}, h^{xx}_x) should be themselves. This is consistent with the fact that they are honest modular forms rather than quasi-modular forms, that is, they do not change upon modular completion.

Genus 1

From Eq. (18) in Introduction, we get

$$\partial_j \mathcal{F}^{(1)} = -\frac{1}{2}\partial_j \log\det G + (\frac{h^{2,1}(X)+3}{2} - \frac{\chi}{24})K_j + \partial_j \log|f^{(1)}|^2 \tag{2.27}$$

for some holomorphic function $f^{(1)}$. Using Eq. (2.21), we then get for the current one-modulus case, near the large complex structure limit,

$$\lim_{\text{LCSL}} \partial_\alpha \log\det G = \partial_\alpha \log\frac{1}{\alpha}\theta t = \partial_\alpha \log(\frac{1}{\alpha}A) = -1 + \text{regular}\,. \tag{2.28}$$

Applying the boundary conditions Eqs. (28), (29) to this case, we obtain $\partial_\alpha f^{(1)} = \partial_\alpha \log \alpha^b (1-\alpha)^a$ with $b + \frac{1}{2} = -\frac{c_2(\check{X})}{24}, a = -\frac{1}{12}$. Therefore, the holomorphic limit of $\partial_x \mathcal{F}^{(1)}$ is given by

$$\lim_{\text{LCSL}} \partial_x \mathcal{F}^{(1)} = -\frac{1}{2}\partial_x \log\theta t + 0 + \partial_x \log \alpha^{b+\frac{1}{2}}(1-\alpha)^a\,. \tag{2.29}$$

Since for $\check{X} = K_{\mathbb{P}^2}$, $c_2(\check{X}) = 2$, we are led to

$$\lim_{\text{LCSL}} \partial_x \mathcal{F}^{(1)} = -\frac{1}{2}\partial_x \log \theta t - \frac{1}{12}\partial_x \log \alpha(1-\alpha) = \frac{1}{2} - \frac{1}{12}\frac{EA}{B^3}\,. \tag{2.30}$$

Note that this quantity is regular at the orbifold point $\alpha = \infty$ where A vanishes and B, C, E are regular according to Eq. (1.20). This is the key property later in making use of the regularity condition at the orbifold to prove the existence and uniqueness theorems of the solutions to the BCOV holomorphic anomaly equations.

Note that only the first derivative of $\mathcal{F}^{(1)}$ is physical, that is, when extracting the genus one Gromov-Witten invariants, one would have to consider the genus one stable maps with one marked point. The corresponding generating function is given by

$$\begin{aligned}
\lim_{\text{LCSL}} \partial_t \mathcal{F}^{(1)} &= \partial_t x \lim_{\text{LCSL}} \partial_x \mathcal{F}^{(1)} = \frac{1}{A}\left(\frac{1}{2} - \frac{1}{12}\frac{EA}{B^3}\right),\\
\lim_{\text{LCSL}} \partial_\tau \mathcal{F}^{(1)} &= -\frac{1}{2}\partial_\tau \log \eta(\tau)\eta(3\tau) = -\frac{1}{12}E\,.
\end{aligned}$$

Up to addition by an anti-holomorphic function, we have from Eq. (2.29) and Table 1.2 that

$$\lim_{\text{LCSL}} \mathcal{F}^{(1)} = -\frac{1}{2}\log \theta t - \frac{1}{12}\log \alpha(1-\alpha) = -\frac{1}{12}\log B^3 C^3 = -\frac{1}{2}\log \eta(\tau)\eta(3\tau)\,.$$

It follows that their non-holomorphic completions are given by

$$\begin{aligned}
\mathcal{F}^{(1)} &= -\frac{1}{2}\log \sqrt{Im\tau}\eta(\tau)\overline{\eta(\tau)}\sqrt{Im 3\tau}\eta(3\tau)\overline{\eta(3\tau)}\,,\\
\partial_\tau \mathcal{F}^{(1)} &= -\frac{1}{12}\hat{E}\,.
\end{aligned}$$

These agree with the existing results in the literature, e.g., Mohri *et al.* (2001); Mohri (2002); Aganagic *et al.* (2008); Haghighat *et al.* (2008).

Higher genus

Polynomial part

With the above choices for the generators S^{xx}, S^x, S, K_x, polynomial recursion tells that the non-holomorphic part $\mathcal{P}^{(2)}$ in Eq. (26) is given by the following polynomial

$$\mathcal{P}^{(2)} = \frac{5}{24}(C_{xxx})^2(S^{xx})^3 - \frac{3}{8}C_{xxx}s^x_{xx}(S^{xx})^2 + \frac{1}{8}(\partial_x C_{xxx})(S^{xx})^2 + \frac{1}{4}C_{xxx}h^{xx}_x S^{xx}\,. \tag{2.31}$$

Plugging in the corresponding quantities, we then get

$$\mathcal{P}^{(2)} = \frac{1}{1728}\frac{\hat{E}(6A^4 - 9A^2\hat{E} + 5\hat{E}^2)}{B^6}\,. \tag{2.32}$$

Ansatz for holomorphic ambiguity

Now the holomorphic ambiguity $f^{(2)}$ is a meromorphic function on $\mathcal{M} \cong X_0(3)$, it must be a rational function of the Hauptmodul, that is, it is a modular function of weight zero. Therefore, it is given by a ratio of two modular forms of the same weight. We know from the boundary conditions that $\mathcal{F}^{(2)}$ can only be singular at the singularities $\alpha = 0, 1, \infty$. Moreover, at the large complex structure limit $\alpha = 0$ and at the orbifold point $\alpha = \infty$, the holomorphic limits are conjecturally equivalent to the generating functions of Gromov-Witten invariants and of orbifold Gromov-Witten invariants of the mirror manifold, respectively. Hence it is regular at the two singular points $\alpha = 0, \infty$ on the moduli space. Therefore, the denominator can only involve the generator B according to the structure for the divisors in Eq. (1.20).

We shall show later that the gap condition at the conifold point tells that it must be B^6. Of course it could be smaller, but one can always multiply both the numerator and denominator by certain powers of B so that the denominator is B^6.

We must point out that regularity at the orbifold is a very strong condition, and it is no longer true for local del Pezzos, as we shall see in the next section.

It follows then that the ambiguity $f^{(2)}$ has the form

$$f^{(2)} = \frac{1}{1728}\frac{c_0 A^6 + c_1 A^3 B^3 + c_2 B^6}{B^6}\,. \tag{2.33}$$

Boundary condition at LCSL

We now apply the boundary conditions to solve for the constants c_0, c_1, c_2.

The condition at the large complex structure in Eq. (28) tells that

$$\lim_{\alpha=0} \lim_{\mathrm{LCSL}} \mathcal{F}^{(2)} = (-1)^2 \frac{\chi}{2} \frac{|B_4 B_2|}{4 \cdot 2 \cdot 2!} \,. \tag{2.34}$$

For the current case[4], $\chi = \chi(K_{\mathbb{P}^2}) = 3$. From the expressions of A, B, C in terms of hypergeometric series in Eq. (1.19) or the η-function expansions in Table 1.2 we know that $\lim_{\alpha=0} A = 1, \lim_{\alpha=0} B = 1, \lim_{\alpha=0} \lim_{LCSL} \hat{E} = \lim_{\alpha=0} E = 1$. This then gives a linear relation among c_0, c_1, c_2:

$$\frac{1}{12^3}(2 + c_0 + c_1 + c_2) = (-1)^2 \frac{\chi}{2} \frac{|B_4 B_2|}{4 \cdot 2 \cdot 2!} = -\frac{\chi}{2^3 \cdot 5 \cdot 12^2} \,. \tag{2.35}$$

Vanishing period for gap condition

Now we shall apply the gap condition at the conifold point in Eq. (29). Note that the vanishing period $t_c(\beta)$ can be easily solve as the vanishing period of $\mathcal{L}_{\mathrm{CY}}$ written in the $\beta = 1 - \alpha$ coordinate and has the following form as a series

$$t_c(\beta) = \beta + \mathcal{O}(\beta^2) \,. \tag{2.36}$$

We can then invert this series to get

$$\beta(t_c) = t_c + \mathcal{O}(t_c^2) \,. \tag{2.37}$$

A different normalization of $t_c(\beta)$ will correspond to a different c in the gap condition in Eq. (29).

---

[4] In fact, the number χ is irrelevant in extracting the Gromow-Witten invariants using mirror symmetry. Hence in this thesis we sometimes do not plug in the value for χ in the results for topological string partition functions.

Fricke involution

Since we have written down the full non-holomorphic quantity $\mathcal{F}^{(2)} = \mathcal{P}^{(2)} + f^{(2)}$, we need to find its holomorphic limit based at the conifold point $\beta = 0$. In principle, one needs to find the $(\beta, \bar{\beta})$ expression of the non-holomorphic quantity $\mathcal{F}^{(2)}$ centered near $(\beta, \bar{\beta}) = (0,0)$. It is difficult to do this by analytic continuation in the $(\alpha, \bar{\alpha})$ space, starting from its expression near $(\alpha, \bar{\alpha}) = (0,0)$. We proceed by making use of the Fricke involution on the modular forms in Eqs. (1.33), (1.34) which imply that

$$\begin{aligned}
A(\alpha) &= \frac{i}{\sqrt{N}\tau} A(\beta)\,, \\
B(\alpha) &= (1-\alpha)^{\frac{1}{r}} A(\alpha) = \frac{i}{\sqrt{N}\tau} \beta^{\frac{1}{r}} A(\beta) = \frac{i}{\sqrt{N}\tau} C(\beta)\,, \\
C(\alpha) &= \alpha^{\frac{1}{r}} A(\alpha) = \frac{i}{\sqrt{N}\tau} (1-\beta)^{\frac{1}{r}} A(\beta) = \frac{i}{\sqrt{N}\tau} B(\beta)\,, \\
\hat{E}(\alpha, \bar{\alpha}) &= -\left(\frac{i}{\sqrt{N}\tau}\right)^2 \hat{E}(\beta, \bar{\beta})\,,
\end{aligned} \tag{2.38}$$

where as before, $\alpha = \alpha(\tau), \beta = \beta(\tau) = 1 - \alpha(\tau), \tau = \tau(\alpha) = \frac{\omega_1(\alpha)}{\omega_0(\alpha)}, A(\alpha) = A(\alpha(\tau)) = A(\tau)$ and similarly for other quantities.

Plugging this into the formula $\mathcal{F}^{(2)}$, we then get

$$\begin{aligned}
\mathcal{F}^{(2)} &= \mathcal{P}^{(2)} + f^{(2)} \\
&= \frac{1}{1728} \frac{\hat{E}(\alpha,\bar{\alpha})(6A(\alpha)^4 - 9A(\alpha)^2 \hat{E}(\alpha,\bar{\alpha}) + 5\hat{E}(\alpha,\bar{\alpha})^2)}{B(\alpha)^6} \\
&\quad + \frac{1}{1728} \frac{c_1 A(\alpha)^6 + c_2 A(\alpha)^3 B(\alpha)^3 + c_3 B(\alpha)^6}{B(\alpha)^6} \\
&= \frac{1}{1728} \frac{-\hat{E}(\beta,\bar{\beta})(6A(\beta)^4 - 9A(\beta)^2 \hat{E}(\beta,\bar{\beta}) + 5\hat{E}(\beta,\bar{\beta})^2)}{C(\beta)^6} \\
&\quad + \frac{1}{1728} \frac{c_0 A(\beta)^6 + c_1 A(\beta)^3 C(\beta)^3 + c_2 C(\beta)^6}{C(\beta)^6}\,.
\end{aligned}$$

The last equality in the above follows from the fact $\mathcal{F}^{(2)}$ has modular weight 0 so that the $\frac{i}{\sqrt{N}\tau}$ factors are canceled out.

Remark 2.2. The holomorphic limit of the quantity $\hat{E}(\beta, \bar{\beta})$ is worked out as follows. Recall that the conifold point $\beta = 0$ corresponding to $\tau(\alpha) = 0$ or equivalently $\tau(\beta) = -\frac{1}{N\tau(\alpha)} = i\infty$.

Then from Eq. (1.21) and Eq. (1.6), we obtain

$$
\begin{aligned}
& \lim_{\bar{\beta}=0} \hat{E}(\beta,\bar{\beta}) \\
= & \lim_{\bar{\beta}=0} \left(E(\beta) + \frac{1}{2}\frac{-3}{\pi \mathrm{Im}\tau(\beta)} \right) = \lim_{\tau(\beta)=i\infty} \left(E(\beta) + \frac{1}{2}\frac{-3}{\pi \mathrm{Im}\tau(\beta)} \right) \\
= & E(\beta) = \partial_{\tau(\beta)} \log C^r(\beta) B^r(\beta) \\
= & \partial_{\tau(\beta)}\beta \, \partial_\beta \log C^r(\beta) B^r(\beta) \,, \\
= & \alpha\beta A^2(\beta) \, \partial_\beta \log C^r(\beta) B^r(\beta) \,.
\end{aligned}
$$

This then tells that β expansion of the holomorphic limit $E(\beta)$ at the conifold point.

This can alternatively be derived in the τ coordinate as follows. Denote $\tau_D = W_N \tau = -\frac{1}{3\tau}$, then

$$
\begin{aligned}
\hat{E}(\tau,\bar{\tau}) &= \partial_\tau \log C^r(\tau) B^r(\tau) + \frac{1}{2}\frac{-3}{\pi \mathrm{Im}\tau} \,, \\
&= \partial_\tau \tau_D \, \partial_{\tau_D} \log(\frac{\sqrt{N}\tau_D}{i})^{2r} C^r(\tau_D) B^r(\tau_D) + \frac{1}{2}\frac{-3}{\pi \mathrm{Im}\tau} \,, \\
&= \frac{1}{N\tau^2} \left(\partial_{\tau_D} \log C^r(\tau_D) B^r(\tau_D) + 2r \partial_{\tau_D} \log \tau_D + N\tau^2 \frac{1}{2}\frac{-3}{\pi \mathrm{Im}\tau} \right) \,, \\
&= \frac{1}{N\tau^2} \left(\partial_{\tau_D} \log C^r(\tau_D) B^r(\tau_D) + \frac{1}{2}\frac{-3}{\pi \mathrm{Im}\tau_D} \right) \,.
\end{aligned}
$$

Comparing with the desired transformation

$$
\hat{E}(\tau,\bar{\tau}) = \frac{1}{N\tau^2} \left(E(\tau_D) + \frac{1}{2}\frac{-3}{\pi \mathrm{Im}\tau_D} \right) \,,
$$

we get

$$
E(\tau_D) = \partial_{\tau_D} \log C^r(\tau_D) B^r(\tau_D) = \partial_{\tau(\beta)}\beta \, \partial_\beta \log C^r(\beta) B^r(\beta) \tag{2.39}
$$

as above. This implies in particular

$$
E(\beta) = (-\sqrt{3}\tau)^2 E(\alpha) + \frac{1}{2}\frac{12}{2\pi i}(-\sqrt{3})(-\sqrt{3}\tau) \,, \tag{2.40}
$$

as it should be as a quasi-modular form which transforms according to Eq. (1.33). We can also see that between the two different forms written in the τ coordinate and the Hauptmodul, i.e., α coordinate, the former is more symmetric than the latter. Moreover, the latter also tells the analytic continuation of $E(\alpha)$ on $\mathcal{M}$ which is parametrized by the complex coordinate α. This is not easily accessed without using modularity.

Solving for the ambiguity

Recall from Eq. (1.20) that $\mathrm{div} C(\alpha) = \frac{1}{3}(\alpha = 0)$, we know $\mathrm{div} C(\beta) = \frac{1}{3}(\beta = 0)$. On the other hand, from Eq. (2.37) we know that the gap condition implies $\mathrm{div} \lim_{\mathrm{CON}} \mathcal{F}^{(2)} = (-2)\mathrm{div}(t_c) = (-2)\mathrm{div}(\beta)$. It follows then the exponent of B in the denominator of $f^{(2)}$ must be $2/\frac{1}{3} = 6$.

Now we plug in all of the series expansions for $A(\beta), C(\beta), E(\beta)$ into $\lim_{\mathrm{CON}} \mathcal{F}^{(2)}$ and use the series expansion Eq. (2.37), we get

$$\lim_{\mathrm{CON}} \mathcal{F}^{(2)} = \frac{-20 + c_0}{1728 t_c^2} + \frac{14 + 11c_0 + 9c_1}{15552 t_c} + \text{regular}\,. \tag{2.41}$$

The gap condition, with $c = 3$ fixed by other means[5], then gives two additional equations involving only c_0, c_1:

$$\begin{aligned} \frac{-20 + c_0}{1728} &= c \frac{B_4}{(2\cdot 2)(2\cdot 2 - 2)} = -\frac{c}{240} = -\frac{1}{80}\,, & (2.42)\\ \frac{14 + 11c_0 + 9c_1}{15552} &= 0\,. & (2.43) \end{aligned}$$

Solving the linear equations Eqs. (2.35), (2.42), we obtain

$$c_0 = -\frac{8}{5}, \quad c_1 = \frac{2}{5}, \quad c_2 = \frac{-8 - 3\chi}{10} = -\frac{17}{10}\,. \tag{2.44}$$

Therefore, we have obtained $\mathcal{F}^{(2)}$ in terms of almost-holomorphic modular forms:

$$\begin{aligned} \mathcal{F}^{(2)} &= \mathcal{P}^{(2)} + f^{(2)} & (2.45)\\ &= \frac{1}{1728}\frac{\hat{E}(6A^4 - 9A^2\hat{E} + 5\hat{E}^2)}{B^6} + \frac{1}{1728}\frac{-\frac{8}{5}A^6 + \frac{2}{5}A^3B^3 - \frac{17}{10}B^6}{B^6}\,. & (2.46) \end{aligned}$$

Similarly, one can solve for the other higher genus ones recursively. But the computation

[5] For example, this can be determined by using one further condition at the LCSL like genus 2 degree 1 Gromov-Witten invariant. Once it has been fixed, one can use it for higher genus computations in Eq. (29) without any other additional conditions.

gets unmanageable very quickly. For example, using Mathematica, we get for genus three

$$\begin{aligned}
&\mathcal{F}^{(3)}\\
&=\frac{\left(-2532A^{10}+3444A^{7}B^{3}-1140A^{4}B^{6}+48AB^{9}\right)\hat{E}}{1244160B^{12}}\\
&+\frac{\left(3516A^{8}-3708A^{5}B^{3}+732A^{2}B^{6}\right)\hat{E}^{2}}{1244160B^{12}}\\
&+\frac{\left(-2645A^{6}+1900A^{3}B^{3}-120B^{6}\right)\hat{E}^{3}}{1244160B^{12}}\\
&+\frac{\left(1200A^{4}-420AB^{3}\right)\hat{E}^{4}}{1244160B^{12}}-\frac{25A^{2}\hat{E}^{5}}{82944B^{12}}+\frac{5\hat{E}^{6}}{82944B^{12}}\\
&+\frac{5359A^{12}-8864A^{9}B^{3}+4160A^{6}B^{6}-496A^{3}B^{9}+2(8-3\chi)B^{12}}{8709120B^{12}}\,.
\end{aligned}$$

2.3.2 Proofs of Theorem 0.1 and Theorem 0.2

Before we proceed to the discussions of local del Pezzos, we now make a pause and prove Theorem 0.1 and Theorem 0.2 basing on the ideas presented above. Both of them follow easily from the arithmetic structure of the moduli spaces as modular curves. The statements of the theorems are recalled below and followed by proofs.

Theorem 2.3. *Consider the mirror families of the one-parameter families of Calabi-Yau threefolds* $K_{\mathbb{P}^2}, K_{dP_n}, n=5,6,7,8$, *respectively.*

(i) For each mirror family $\pi:\mathcal{X}\to\mathcal{M}$, *the deformation space* $\mathcal{M}$ *is a modular curve* $X_\Gamma=\Gamma\backslash\mathcal{H}^*$, *where the modular group* Γ *is* $\Gamma_0(3)$, $\Gamma_0(4)$, $\Gamma_0(3)$, $\Gamma_0(2)$, $\Gamma(1)^*$, *respectively.*

(ii) For each family, the solutions to the holomorphic anomaly equations with the boundary conditions, if they exist, are almost-holomorphic modular forms of weight zero with respect to the corresponding modular group.

Proof. The first part follows from the identification of the moduli spaces with modular curves we made in Section 2.2.

Polynomial recursion in Eq. (26) says that the non-holomorphic part $\mathcal{P}^{(g)}$ of $\mathcal{F}^{(g)}=\mathcal{P}^{(g)}+f^{(g)}$ is a polynomial of the non-holomorphic generator S^{xx} and holomorphic generators

$C_{xxx}, s^x_{xx}, h^{xx}_x$, and their derivatives.[6] The holomorphic ambiguity $f^{(g)}$ if a rational function of the Hauptmodul, hence it is a modular function. Since all of the geometric quantities involved are almost-holomorphic modular forms, $\mathcal{F}^{(g)}$, if it exists as a solution to the BCOV holomorphic anomaly equation, itself must be an almost-holomorphic modular form with respect to the corresponding modular group which depends on the geometry. This proves the second part of the theorem. □

Theorem 0.2 follows from the use of Fricke involution in applying the gap condition in Eq. (29).

Theorem 2.4. *For the mirror family $\pi : \mathcal{X} \to \mathcal{M}$ of the $K_{\mathbb{P}^2}$ family, for any genus g, the solution to the holomorphic anomaly equation with boundary conditions exists and is unique. In particular, it is an almost-holomorphic modular form of weight zero with respect to the modular group $\Gamma_0(3)$.*

Proof. We only need to show that the following set of boundary conditions

- leading-term contribution of $\lim_{\mathrm{LCSL}} \mathcal{F}^{(g)}$ in Eq. (28),
- gap condition for $\lim_{\mathrm{CON}} \mathcal{F}^{(g)}$ in Eq. (29),
- regularity of $\mathcal{F}^{(g)}$ at the orbifold point

give exactly the desired conditions[7]: if there are fewer conditions, then the unknowns is under-determined; if there are more, the system is over-determined and one needs to check consistency of the boundary conditions.

The proof is based on the computations for the $g = 2$ case.

First, by induction, we can easily see that $\mathcal{P}^{(g)}$ has the following structure: the non-holomorphic part is a rational function in the generators $A, B^3, \hat{E}$ whose denominator is

[6]In fact, as mentioned in Introduction and shall be proved in Chapter 3, for this case one can choose a minimal set of non-holomorphic and holomorphic generators so that the derivatives of the generators are also (Laurent) polynomials of these generators.

[7]This is explained to the author by Professor Don Zagier. The author thanks him for his very helpful and inspiring discussions on this and for encouraging me to put the results on mathematically firm ground. The author also thanks his colleague Teng Fei for discussions on extracting the existence and uniqueness theorems from the computations.

$(B^3)^{2g-2}$. [8]

That $\mathcal{F}^{(g)}$ is singular only at the conifold tells that the denominator of $\mathcal{F}^{(g)}$ can only be of the form B^{n_g} for some integer n_g. Now we use the facts on the divisor of B and the asymptotic behavior of the vanishing period t_c near the conifold point to determine the exponent in the denominator. Recall $\mathrm{div}B = \frac{1}{3}(\beta = 0)$ and $t_c(\beta) = \beta + \mathcal{O}(\beta^2)$, according to the gap condition which says that $\mathcal{F}^{(g)} \sim \frac{1}{t_c^{2g-2}} + \mathcal{O}(t_c^0)$, we then get $n_g = (2g-2)/\frac{1}{3} = 6g-6$.

It follows then that the numerator must be a modular form of weight $6g-6$ in the ring of modular forms $M_*(\Gamma_0(3), \chi_{-3}) = \mathbb{C}[A, B^3]$. The dimension is given by $(6g-6)/3 = 2g-2$. Hence the numerator must be a polynomial in A^3, B^3 and have the form $\sum_{k=0}^{2g-2} c_k (A^3)^{2g-2-k}(B^3)^k$, where $c_0, c_1, c_2, \cdots c_{2g-2}$ are under-determined constants.

Using the asymptotic behavior $A^3, B^3 \sim 1 + \mathcal{O}(\alpha)$ we know the condition at the large complex structure limit in Eq. (28) gives a linear equation

$$c_0 + c_1 + c_2 + \cdots c_{2g-2} = \text{some constant determined from Eq. (28)}\,. \tag{2.47}$$

Now we use the the Fricke involution as in Eq. (2.38), then we have near the conifold point $\beta = 0$

$$\frac{c_k (A^3)^{2g-2-k}(B^3)^k}{(B^3)^{2g-2}} \sim \frac{1}{t_c^{2g-2-k}} + \text{sub-leading terms}, \; k = 0, 1, 2, \cdots 2g-2\,. \tag{2.48}$$

By comparing the leading terms in the gap condition, we get an equation

$$c_0 = \text{some constant}\,.$$

This then determines c_0. By comparing the next sub-leading terms, we get an equation for c_2,

$$(\text{some number})c_0 + c_1 = \text{some constant}\,.$$

This then determines c_1. Proceeding like this, one can then determine $c_0, c_1, c_2, \cdots c_{2g-3}$. We finally use the condition given in Eq. (2.47) to fix c_{2g-2}.

[8]For example, the leading term with highest power in $S^{xx} = \hat{E}/2A^2$ is proportional to $(C_{xxx})^{2g-2}(S^{xx})^{3g-3} = (\kappa^{2g-2}/2^{3g-3})(\hat{E}^{3g-3}/(B^3)^{2g-2})$.

Hence according to the structure in Eq. (2.48), the coefficient matrix for the equations of $c_0, c_1, c_2, \cdots, c_{2g-2}$ is a $(2g-2+1) \times (2g-2+1)$ lower triangular matrix with diagonal entries being 1s. This is why the solution $(c_0, c_1, c_2, \cdots, c_{2g-2})$ to the linear equations from the boundary conditions exists and is unique. $\square$

2.3.3 Local del Pezzos

Using the identification of the moduli spaces with modular curves in Section 2.2, the same strategy as above can determine the topological string genus by genus in terms of almost-holomorphic modular forms. In the following, we shall only display the results and omit the details.

The topological invariants for the corresponding A-model non-compact CY 3-folds can be found in Lerche *et al.* (1997); Chiang *et al.* (1999)

$$\kappa = n - 9\,, \quad c_2 = -12 + 2(9-n)\,, \quad c_3 = \chi = -2h(E_n)\,, \quad h(E_n) = 8, 12, 18, 30\,.$$

The following numbers in Table 2.1 will also be used a lot throughout the discussion

$$n = 5, 6, 7, 8; \quad N = 4, 3, 2, 1; \quad r = 2, 3, 4, 6\,. \tag{2.49}$$

Genus 0

As the local $\mathbb{P}^2$ case, for local dP_n we have

$$C_{\alpha\alpha\alpha} = \frac{\kappa}{\alpha^3(1-\alpha)}, \quad C_{xxx} = \frac{\kappa}{(1-\alpha)}, \quad C_{ttt} = \frac{\kappa}{(\theta t)^3(1-\alpha)} = \kappa \frac{A^{r-3}}{B^r}\,.$$

The quantity C_{ttt} is a modular form of weight -3 since $\theta t = A$ has modular weight 1, and $1 - \alpha = \beta = \frac{B^r}{A^r}$ is a modular function.

Genus 1

Near the large complex structure limit $\alpha = 0$, the holomorphic limit of genus one partition function is given by

$$\lim_{\text{LCSL}} \mathcal{F}^{(1)} = -\frac{1}{2}\log \theta t + \log(1-\alpha)^a \alpha^{b+\frac{1}{2}} \,. \tag{2.50}$$

The universal constant a is given by $a = -\frac{1}{12}$, while $b + \frac{1}{2} = -\frac{c_2}{24}$. Plugging in these numbers, we get

$$\begin{aligned} \lim_{\text{LCSL}} \mathcal{F}^{(1)} &= -\frac{1}{2}\log A + \log\left(B^{-\frac{r}{12}} C^{-\frac{rc_2}{24}} / A^{-\frac{r}{12}-\frac{rc_2}{24}}\right), \\ \lim_{\text{LCSL}} \partial_\tau \mathcal{F}^{(1)} &= -\frac{1}{4r}E + \frac{1}{2}A^2\left(\frac{1}{12} - \frac{c_2}{24} - (\frac{1}{12} + \frac{c_2}{24})(C^r - B^r)A^{-r}\right). \end{aligned}$$

Now we will consider the singular behavior of the quantity $F^{(1)}_{\text{orb}}$ which is defined to be the analytic continuation of $\lim_{\text{LCSL}} \mathcal{F}^{(1)}$ to the orbifold point $\alpha = \infty$. In each case above, according to Eq. (1.20) we have that near the orbifold point,

$$\theta t = \omega_0 \sim \alpha^{-\frac{1}{r}}(1 + \mathcal{O}(\alpha^{-\frac{1}{r}}))\,. \tag{2.51}$$

Hence

$$F^{(1)}_{\text{orb}} \sim -\frac{1}{12}\log(\alpha^{-\frac{1}{r}})^6(1+\mathcal{O}(\alpha^{-\frac{1}{r}}))(1-\alpha)\alpha^{\frac{c_2}{2}} \sim -\frac{1}{12}\log\alpha^{-\frac{6}{r}+1+\frac{c_2}{2}}\,. \tag{2.52}$$

Changing to the local coordinate $\psi = \alpha^{-\frac{1}{r}}$ near the orbifold point, we then have

$$F^{(1)}_{\text{orb}} \sim -\frac{1}{12}\log\psi^{6-r(1+\frac{c_2}{2})} = -\frac{1}{12}\log\psi^{h(E_n)}\,. \tag{2.53}$$

The numbers $h(E_n) = 6 - r(1+\frac{c_2}{2})$ for $n = 5, 6, 7, 8$ cases are given by $8, 12, 18, 30$, respectively, they are the dual Coxeter numbers of the Lie algebra E_n, see Chiang *et al.* (1999); Lerche *et al.* (1997). Due to the singular behavior of the genus one partition function, the higher genus ones will also be singular from recursion. This higher genus singularity also appears in the ambiguities determined in Katz *et al.* (1999).

Propagators in terms of almost-holomorphic modular forms

We make the following choices of non-holomorphic generators and ambiguity in the integrated special geometry relation Eq. (7) :

$$S^x = S = K_x = 0\,,$$
$$s^x_{xx} = \frac{1}{2r}\frac{\alpha-\beta}{\beta}\,.$$

With these choices, one has

$$\begin{aligned} S^{xx} &= -\frac{1}{2r\kappa}\frac{E}{A^2} = \frac{1}{2rN}\frac{E}{A^2}\,, \\ h^{xx}_x &= \frac{1}{(2r)^2\kappa\beta}\,. \end{aligned}$$

One can also choose other ambiguities s^x_{xx} which give different S^{xx} and h^{xx}_x. For example, taking $s^x_{xx} = 2\partial_x \log \beta^a \alpha^{b+\frac{1}{2}} = 2\theta \log \beta^{-\frac{1}{12}}\alpha^{-\frac{c_2}{24}}$, we then have $\lim_{\mathrm{LCSL}} S^{xx} = 2C^{-1}_{xxx} \lim_{\mathrm{LCSL}} \partial_x \mathcal{F}^{(1)}$. This is singular at the orbifold for local de Pezzos and regular for local $\mathbb{P}^2$, as mentioned above.

Boundary conditions

As the initial condition in solving the BCOV holomorphic anomaly equations, the singularity at the orbifold of the (derivative of) genus one topological string partition will a priori bring singularities to higher genus partition functions.

This is in contrast to the local $\mathbb{P}^2$ case in which $6 - r(1+\frac{c_2}{2}) = 0$ and the genus one partition function is not singular at the orbifold point. Note that local $\mathbb{P}^2$ and local dP_6 have the same modular groups, the same choice for the propagators and almost the same Yukawa couplings, but different genus one partition functions which result in completely different higher genus partition functions with different singular behaviors.

Since at this moment the precisely singular behavior of $\mathcal{F}^{(g)}$ is not fully understood, the boundary condition at the orbifold point is not clear to us. Therefore, we need to solve for the topological string partition functions from the following boundary conditions:

- leading term (degree 0) contribution of $\lim_{\mathrm{LCSL}} \mathcal{F}^{(g)}$, and possibly higher degree contributions

- gap condition for $\lim_{\mathrm{CON}} \mathcal{F}^{(g)}$,

For example, for local dP_6, the same procedure as in the local $\mathbb{P}^2$ case tells that the denominator for $\mathcal{F}^{(g)}$ is $A^{3g-3}B^{6g-6}$. Hence the number of under-determined coefficients is at least $3g-3+1$ (the ambiguity $f^{(2)}$ might be more singular) as the dimension of the vector space of weight $(9g-9)$ modular forms is $3g-3$. For the other geometries, by induction, one can show similarly that the number of under-determined coefficients is at least $3g-3+1$.

The gap condition gives $2g-2$ linear equations on the coefficients, combing the leading term of $\lim_{\mathrm{LCSL}} \mathcal{F}^{(g)}$, one has in total $2g-2+1$ linear equations.

In practice, for lower genus partition functions, one can get other conditions from the information on lower degree Gromov-Witten invariants. This could be provided from direct computation from the A-side, or (partially) from the Castelnuovo's bound (see Huang *et al.* (2009)). However, it is not clear how many lower degree contributions to the generating function of Gromov-Witten invariants $\lim_{\mathrm{LCSL}} \mathcal{F}^{(g)}$ would give linear independent equations among the under-determined constants when combined with the known $2g-2+1$ conditions. This presents a difficulty in proving a similar result to Theorem 0.2 for the local del Pezzos.

Genus 2

Using the boundary condition at the large complex structure limit and the gap condition at the conifold with $c = -1/\kappa$, as well as the Castelnuovo's bound, we can solve for the following genus two topological string partition functions:

- Local dP_5:

$$-\frac{A^6+70A^4B^2+228B^6+8A^2B^4(-3+2\chi)}{92160A^2B^4}+\frac{\left(9A^4-468A^2B^2+1440B^4\right)E}{165888A^2B^4}+\frac{\left(-27A^2-594B^2\right)E^2}{165888A^2B^4}+\frac{5E^3}{6144A^2B^4}$$

- Local dP_6:

$$-\frac{16A^6-4A^3B^3+B^6(488+27\chi)}{155520B^6}+\frac{\left(6A^7-24A^4B^3+96AB^6\right)E}{15552A^3B^6}+\frac{\left(-9A^5-24A^2B^3\right)E^2}{15552A^3B^6}+\frac{5E^3}{15552B^6}$$

- Local dP_7:

$$\frac{-133A^8+302A^4B^4-4B^8(501+16\chi)}{368640B^8}+\frac{\left(549A^{10}-972A^6B^4+4860A^2B^8\right)E}{663552A^4B^8}+\frac{\left(-459A^8-810A^4B^4\right)E^2}{663552A^4B^8}+\frac{5A^2E^3}{24576B^8}$$

- Local dP_8:

$$\frac{-299A^{12}+634A^6B^6-36B^{12}(60+\chi)}{207360B^{12}}+\frac{\left(75A^{16}-60A^{10}B^6+420A^4B^{12}\right)E}{41472A^6B^{12}}+\frac{\left(-33A^{14}-42A^8B^6\right)E^2}{41472A^6B^{12}}+\frac{5A^6E^3}{41472B^{12}}$$

Due to the limit of space, we shall not include the lengthy expressions for the genus 3 results here. They could be found in the joint work Alim *et al.* (2013).

As a consistency check, we have checked by Mathematica that all of the quasi-modular functions we have obtained for local $\mathbb{P}^2$ and local $\mathrm{dP}_n, n=5,6,7,8$ reproduce the integral Gopakumar-Vafa invariants (up to degree 10) listed in the literature, e.g., Katz *et al.* (1999).

2.4 Holomorphic anomaly equations in terms of modular form theory language

In the previous section, we used the Fricke involution computationally. Essentially we used Eq. (2.38) to obtain the analytic continuation from the large complex structure limit to the conifold point in the $(\alpha,\bar{\alpha})$ space of the non-holomorphic topological string partition functions. It is realized by the action of Fricke involution on the almost-holomorphic modular forms.

In this section we now try to understand it more conceptually. We shall interpret the gap condition as a certain regularity condition at the cusp in the language of modular form theory. Using this we can turn the problem of solving the BCOV holomorphic anomaly equations with boundary conditions into a combinatorial problem.

First we digest a little bit on the condition “meromorphic at the cusps" in the definition of meromorphic modular forms in Section 1.2. A modular function is "meromorphic at the cusp [0]" means that first we choose

$$\gamma_* = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

to map the infinity cusp $[\tau] = [i\infty]$ to the cusp $[\tau] = [0]$, then we require the function

$$f|_{\gamma_*} : \tau \mapsto f(\gamma_* \tau) \tag{2.54}$$

to be meromorphic near the cusp $\tau = i\infty$. The nicest case is that the function $f|_{\gamma_*}$ is a function of $q_\tau = \exp 2\pi i \tau$, so that there exists some function $\hat{f}$ with $f|_{\gamma_*}(\tau) = \hat{f}(q_\tau)$. Then the meromorphicity condition becomes a condition on the q_τ expression of $\hat{f}$.

As an analogue, the gap condition at the cusp $[\tau] = [0]$ for $\Gamma_0(N), N = 1^*, 2, 3, 4$ discussed above, means the following. First, one chooses a transformation sending $i\infty$ to 0, which turns out not to be an element in $\mathrm{SL}(2,\mathbb{Z})$ but the Fricke involution

$$W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

so that $\lim_{\mathrm{LCSL}} \mathcal{F}^{(g)}|_{W_N}$ satisfies the gap condition

$$\lim_{\mathrm{LCSL}} \mathcal{F}^{(g)}|_{W_N} = \frac{c^{g-1} B_{2g}}{2g(2g-2)(t_c^*)^{2g-2}} + \mathcal{O}((t_c^*)^0)\,,$$

where t_c^* is the Fricke involution of the vanishing period t_c defined near the conifold. This function is obtained as follows. Recall that $t_c(\beta) = \beta + \mathcal{O}(\beta^2)$ satisfies the property that $\theta t_c = \alpha \partial_\alpha t_c$ is a period of the elliptic curve since it is a solution to $\mathcal{L}_{\text{elliptic}}$. Therefore, it is a linear combination of the two periods of the elliptic curve family:

$$\theta t_c = a\omega_0 + b\omega_1 \quad \text{for some constants } a, b\,. \tag{2.55}$$

The asymptotic behaviors of the above functions near $\beta = 0$ are given by:

$$
\begin{aligned}
t_c &= \beta + \mathcal{O}(\beta^2)\,. \\
\omega_0 &\sim \log\beta + \text{regular}\,, \\
\omega_1 &= \frac{i}{\sqrt{N}}(1 + \mathcal{O}(\beta))\,.
\end{aligned}
$$

This tells that

$$\theta t_c = -\frac{\sqrt{N}}{i}\omega_1 = -\,{}_2F_1(\frac{1}{r}, 1-\frac{1}{r}, 1; \beta)\,. \tag{2.56}$$

Recall Eq. (1.6), we then know that t_c is determined from

$$\partial_\tau t_c = \beta A^2 \cdot -\frac{\sqrt{N}}{i}\omega_1 = -\frac{\sqrt{N}\tau}{i}\beta A^3, \quad t_c|_{\tau=0} = 0\,. \tag{2.57}$$

Applying Fricke involution to the above equation and the boundary condition, we then get

$$\partial_\tau t_c^* = \alpha A^3 = \frac{C^r}{A^{r-3}}, \quad t_c^*|_{\tau=i\infty} = 0\,. \tag{2.58}$$

This then allows us to rephrase the BCOV holomorphic anomaly equation purely in terms of the language of modular form theory. [9]

In the following, we shall only discuss the local $\mathbb{P}^2$ case. As before, taking $S^{xx} = \frac{1}{2}\frac{\hat{E}}{A^2}, S^x = S = K_x = 0$, and using the fact that $\Gamma^x_{xx} = \frac{1}{2r\beta}\frac{\hat{E}}{A^2} + \frac{\alpha-\beta}{2r\beta}, \partial_x = \frac{1}{\partial_\tau x}\partial_\tau = \frac{1}{\beta A^2}\partial_\tau$, we can see that Eq. (24)

$$\partial_{S^{xx}}\mathcal{F}^{(g)} = \frac{1}{2}\left(\sum_{r=1}^{g-1}\partial_x\mathcal{F}^{(r)}\partial_x\mathcal{F}^{(g-r)} + (\partial_x - \Gamma^x_{xx})\partial_x\mathcal{F}^{(g-1)}\right)$$

now becomes

$$\partial_{\hat{E}}\mathcal{F}^{(g)} = \frac{1}{4}\frac{1}{B^6}\left(\sum_{r=1}^{g-1}\partial_\tau\mathcal{F}^{(r)}\partial_\tau\mathcal{F}^{(g-r)} + \partial_\tau^2\mathcal{F}^{(g-1)} - \frac{3}{2r}(\hat{E} - A^2)\partial_\tau\mathcal{F}^{(g-1)}\right)\,.$$

---

[9] See for example Hosono *et al.* (1999); Mohri (2002); Hosono (2002); Sakai (2011, 2013) for related works in different directions.

The boundary conditions in Eqs. (28), (29) become

$$\begin{aligned}\lim_{\bar{\tau}=i\infty}\mathcal{F}^{(g)} &= (-1)^g\frac{\chi}{2}\frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}+\mathcal{O}(q_\tau^1)\,,\\ \lim_{\bar{\tau}=i\infty}\mathcal{F}^{(g)}|_{W_N} &= \frac{c^{g-1}B_{2g}}{2g(2g-2)(t_c^*)^{2g-2}}+\mathcal{O}(q_\tau^0)\,,\end{aligned}$$

where $r=3,\chi=3,c=3$.

To make full use of polynomial structure, we consider the quantity $\mathcal{F}_g := B^{6g-6}\mathcal{F}^{(g)}$. It is then an almost-holomorphic modular form of weight $6g-6$ and satisfies

$$\begin{aligned}\partial_{\hat{E}}\mathcal{F}_g &= \frac{1}{4}\sum_{r=1}^{g-1}(\partial_\tau+(1-r))\mathcal{F}_r(\partial_\tau+1-(g-r))\mathcal{F}_{g-r}\\ &\quad+\frac{1}{4}\left(\partial_\tau+(2-g)(\hat{E}-A^2)\right)^2\mathcal{F}_{g-1}-\frac{1}{8}\left(\partial_\tau+(2-g)(\hat{E}-A^2)\right)\mathcal{F}_{g-1}\,.\end{aligned}$$

Combining with the derivatives of almost-holomorphic modular forms in Eq. (1.23) and the boundary conditions explained above, solving for the weight $6g-6$ almost-holomorphic anomaly equations $\mathcal{F}_g$ then becomes a combinatorial problem. More details can be found in Appendix B.

2.5 Integrality of Gromov-Witten invariants[10]

Assuming that mirror symmetry conjecture is true, we can prove the integrality result for the Gromov-Witten variants for the non-compact CY geometries (A-model) mentioned in Corollary 0.3 in Introduction, by using the modularity of the topological string partition functions.

Corollary 2.5. *Assume the mirror symmetry conjecture is valid for the families in Theorem 0.1. Then for each family, for any genus g, there exists a number $C_g \in \mathbb{Z}$ so that $C_g N_{g,d} \in \mathbb{Z}, \forall d \geq 1$.*

In the following we shall present the proof for the local $\mathbb{P}^2$ case, the proofs for other cases are almost identical to this one.

[10]The author thanks Professor Chiu-Chu Melissa Liu for discussions on this.

For the genus zero part, the Yukawa coupling has the form[11]

$$2\pi i\frac{\partial\tau}{\partial t} = -3\cdot C_{ttt} = -3(-\frac{1}{3}+\sum_{d=1}^{\infty}\frac{n_{0,d}d^3q_t^d}{1-q_t^d})\,, q_t = \exp t\,.$$

Integrating this equation, and using the asymptotic behaviors of τ and t as $t\mapsto -\infty$, we then get, see Mohri *et al.* (2001); Stienstra (2005),

$$q_\tau = \exp 2\pi i\tau = (-q_t)\prod(1-q_t^d)^{3d^2 n_{0,d}}\,. \tag{2.59}$$

The integrality for $n_{0,d}$ for toric Calabi-Yau 3-folds is proved in Peng (2007), this implies that the q_t expansion of q_τ is integral, that is,

$$q_\tau \in -q_t(1+\mathbb{Z}[[q_t]])\,. \tag{2.60}$$

Now the holomorphic limit of the (normalized) topological sting partition function $\mathcal{F}^{(g)}$ solved from the BCOV holomorphic anomaly equations, denoted by F_g below, are rational functions in the generators in the ring of quasi-modular forms $\widehat{M}_*(\Gamma_0(3),\chi_{-3}) = \mathbb{Q}[A,B^3,E]$ for $\Gamma_0(3)$. From polynomial recursion we can see that

$$F_g \in \frac{1}{C_g}\mathbb{Z}[[A,B^3,B^{-3},E]]\,, \tag{2.61}$$

where the constant C_g is an integer depending on the genus g and the geometry of the CY 3-fold family. This constant is usually very big, it results from the universal coefficients in the polynomial recursion (see e.g., Eq. (2.31)) and the boundary conditions (see e.g., Eq. (2.34)). For example, for the present case,

$$C_2 = 17280, C_3 = 8709120,\cdots$$

The explicit expressions of the quasi-modular forms A,B^3,B^{-3},E in terms of θ or η or Eisenstein series are known to be integral, see Appendix A. That is, we have

$$A,B^3,B^{-3},E\in\mathbb{Z}[[q_\tau]])\,. \tag{2.62}$$

[11]Note that here the normalization $t=\log(-z)+\cdots$ is taken, where $z=0$ gives the large complex structure limit on the moduli space, see e.g., Diaconescu and Gomis (2000), Mohri *et al.* (2001).

Now the proof of Corollary 2.5 follows easily from the above observations.

Proof. Combining Eqs. (2.60), (2.62), (2.61), we obtain

$$C_g F_g \in \mathbb{Z}[[q_t]] . \tag{2.63}$$

On the other hand, by the mirror symmetry conjecture, one has from Eq. (2) that

$$F_g = \sum_{d=0}^{\infty} N_{g,d} q_t^d , \tag{2.64}$$

where $N_{g,d}, d = 0, 1, 2 \cdots$ are the genus g degree d Gromov-Witten invariants of the CY 3-fold $\check{X} = K_{\mathbb{P}^2}$. From Eq. (2.63) and Eq. (2.64) it follows immediately that

$$C_g N_{g,d} \in \mathbb{Z},\ d = 0, 1, 2 \cdots$$

□

There is a conjecture about the integrality of the so-called Gopakumar-Vafa invariants $n_{g,d}$ defined in Gopakumar and Vafa (1998a,b), which are related to the Gromov-Witten invariants $N_{g,d}$ by a multiple-cover formula, see Aspinwall and Morrison (1993); Manin (1995); Voisin (1996); Katz *et al.* (1999). For example, at genus 0 and 1, one has

$$N_{0,d} = \sum_{k|d} \frac{n_{0,d/k}}{k^3}, \quad N_{1,d} = \sum_{k|d} \frac{1}{k} (\frac{1}{12} n_{0,d/k} + n_{1,d/k}) .$$

However, the integrality of $C_g N_{g,d}$ does not follow from these multiple-cover formulas and the conjectural integrality of Gopakumar-Vafa invariants. Therefore Corollary 2.5 is not completely trivial.

Remark 2.6. In the above proof, we used the interpretation of C_{ttt} as the generating function of genus 0 Gopakumar-Vafa invariants (instanton numbers) when written in q_t expansion:

$$2\pi i \frac{\partial \tau}{\partial t} = -3C_{ttt} = -3(\kappa + \sum_{d=1}^{\infty} \frac{n_{0,d} d^3 q_t^d}{1 - q_t^d}) .$$

It leads to some interesting arithmetic properties of quantities when written in the q_τ

coordinate. For example, recall that

$$C_{ttt} = \frac{\kappa}{B^3} = -\frac{1}{3}\frac{\eta(3\tau)^3}{\eta(\tau)^9}\,, \tag{2.65}$$

then we have the following nice expressions:

$$\begin{aligned}
2\pi i\frac{\partial\tau}{\partial t} &= \frac{1}{B^3} = \frac{\eta(3\tau)^3}{\eta(\tau)^9}\,,\\
\frac{1}{2\pi i}\frac{\partial t}{\partial\tau} &= B^3 = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9\sum_{n\geq 1}\chi_{-3}(n)\frac{n^2 q_\tau^n}{1-q_\tau^n}\,.
\end{aligned}$$

Therefore, integrating the above equation, one gets the following expression of the local coordinate $q_t \sim \alpha + \text{regular}(\alpha)$ on the moduli space which essentially gives the mirror map

$$q_t = (-q_\tau)\prod(1-q_\tau^n)^{9n\chi_{-3}(n)}\,.$$

It is related to the Mahler measure discussed in e.g., Mohri *et al.* (2001); Stienstra (2005).

2.6 Seiberg-Witten curve family and physics interpretation of Fricke involution

In this section, we shall consider the Legendre curve family and the Jacobi curve family. The bases of these curve families are isogenous modular curves. The former is $X(2) = \Gamma(2)\backslash\mathcal{H}^*$, while the latter is $X_0(4) = \Gamma_0(4)\backslash\mathcal{H}^*$. We shall show that in this case the Fricke involution gives the Seiberg-Witten S-duality in Seiberg and Witten (1994).

2.6.1 Legendre family and Jacobi family

First we shall review the basic properties of the two elliptic curve families which are used in Seiberg and Witten (1994) to model the $4d$ $\mathcal{N}=2$ supersymmetric gauge theory with $SU(2)$ gauge group.

The Legendre family is given by

$$y^2 = x(x-1)(x-\lambda), \quad j(\lambda) = 2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}\,. \tag{2.66}$$

It is the elliptic modular surface in Eq. (1.13) associated to the modular group $\Gamma(2)$ given by $\pi_{\Gamma(2)} : \mathcal{E}_{\Gamma(2)} \to X(2) = \Gamma(2)\backslash\mathcal{H}^*$. The Picard-Fuchs operator attached to this elliptic curve family is the hypergeometric operator

$$\lambda(1-\lambda)\partial_\lambda^2 + (1-2\lambda)\partial_\lambda - \frac{1}{4}\,. \tag{2.67}$$

Some of the special values including the cusps are summarized in the following table

Table 2.3: *Special values of the Hauptmodul for* $\Gamma(2)$

τ	$0, 1, i\infty \in [0]$	$[i]$	$[\rho] = [\exp \frac{2\pi i}{3}]$
λ	$1, \infty, 0$	$-1, \frac{1}{2}, 2$	ρ, ρ^2
j	∞	1728	0

The group of Deck transformation is given by $\mathrm{PSL}(2,\mathbb{Z})/\Gamma(2)$ and is identified with $\mathrm{Perm}(3)$ according to the exact sequence

$$0 \to \Gamma(2) \to \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{Perm}(3) \to 0\,. \tag{2.68}$$

More precisely, this group is generated by the elements whose images in $\mathrm{Perm}(3)$ are identified according to the following

$$\langle 1 \rangle = \lambda,\ \langle 0,1 \rangle = 1-\lambda,\ \langle 0,\infty \rangle = 1/\lambda,\ \langle 0,1,\infty \rangle = 1/(1-\lambda),\ \langle 0,\infty,1 \rangle = (1-\lambda)/\lambda\,, \tag{2.69}$$

where the function $f(\lambda)$ on the right hand side means the transformation $\lambda \mapsto f(\lambda)$. For example, the transformation $\tau \mapsto -\frac{1}{\tau} \in \mathrm{PSL}(2,\mathbb{Z})/\Gamma(2)$ exchanges $0, \infty$ and fixes $1, -1, \in [1]$, hence according to the above table, the corresponding Deck transformation on λ is

$$\lambda(-\frac{1}{\tau}) = 1 - \lambda(\tau)\,. \tag{2.70}$$

Similarly, $\lambda(1-\frac{1}{\tau}) = 1 - \frac{1}{\lambda(\tau)}$. Note that the two elements $\tau \mapsto -\frac{1}{\tau}$, $\tau \mapsto 1 - \frac{1}{\tau}$ generates $\mathrm{PSL}(2,\mathbb{Z})$ and thus the corresponding cosets generates the Deck group. In particular, the S-transformation induces the transformation $\lambda \to 1-\lambda$ on the Hauptmodul for the modular curve $X(2)$.

Now we proceed to the Jacobi family, it is different form the elliptic curve family in Eq. (1.13) associated to the modular group $\Gamma_0(4)$. But the base of this family could be identified with the modular curve $X_0(4)$. More precisely, the equation of the family and the corresponding j-invariant is given by

$$y^2 = (1-x^2)(1-\alpha_J x^2),\; j(\alpha_J) = 2^4 \frac{(\alpha_J^2 + 14\alpha_J + 1)^3}{\alpha_J(1-\alpha_J)^4}\,. \tag{2.71}$$

In fact, the j-invariants: $\alpha_J \mapsto j(\alpha_J)$ and $\alpha_4 \mapsto j(\alpha_4)$ are based on the same function, where α_4 is the Hauptmodul for the modular curve $X_0(4)$ we defined in Section 1.2.

Remark 2.7. Note that this does not imply that $\alpha_J = \alpha_4$. They could be related by Deck transformation. Even if they are chosen so that $\alpha_4(\tau) = \alpha_J(\tau)$, one can not say that the elliptic modular surface is isomorphic to the Jacobi model, since that $j \circ \phi = j$, where ϕ is the base change, does not imply that the two families are isomorphic.[12] Another way to see that is to note that the Picard-Fuchs equation associated to this family is not the same as the one for the E_5 family in Eq. (2.2), which is same as that for the elliptic curve family in Eq. (1.13) associated to the modular group $\Gamma_0(4)$.

The bases of the Legendre model and Jacobi model are the two modular curves $X(2)$ and $X_0(4)$ parametrized by the corresponding Hauptmodul λ, α_4, as discussed in Klemm *et al.* (1995). The corresponding modular groups are isomorphic, in fact, they are conjugate to each other:

$$\Gamma_0(4) \to \Gamma(2): \quad \tau \mapsto 2\tau. \tag{2.72}$$

The Hauptmoduln are related by

$$\alpha_4(\tau) = \lambda(2\tau). \tag{2.73}$$

In terms of the elliptic modulus κ, one has

$$\alpha_4(\tau) = \kappa^2(\tau) = \frac{\theta_2^4}{\theta_3^4}(2\tau), \quad \lambda(\tau) = \frac{4\kappa}{1+\kappa^2}(\tau) = \frac{\theta_2^4}{\theta_3^4}(\tau). \tag{2.74}$$

Denote by τ_2, τ_4 the natural transcendental coordinates for the modular curves $\Gamma(2), \Gamma_0(4)$,

[12]See the constant j–invariant family which is not trivial in Miranda (1989).

respectively. Then we know from Eq. (2.72) that $\tau_2 = 2\tau_4$. Therefore, the S-transformation on $X(2)$ $\tau_2 \mapsto -\frac{1}{\tau_2}$ which gives $\lambda(-\frac{1}{\tau_2}) = 1 - \lambda(\tau_2)$ is realized as

$$\tau_4 \mapsto -\frac{1}{4\tau_4} \,. \tag{2.75}$$

This is exactly the Fricke involution (4-isogeny) on the modular curve $X_0(4)$:

$$\alpha_4(-\frac{1}{4\tau_4}) = \lambda(-\frac{2}{4\tau_4}) = \lambda(-\frac{1}{\tau_2}) = 1 - \lambda(\tau_2) = 1 - \alpha_4(\tau_4). \tag{2.76}$$

The two periods of the Legendre model are given by the solutions to the hypergeometric equations above

$$\omega_0(\tau_2) = \;_2F_1(\frac{1}{2}, \frac{1}{2}, 1; \lambda(\tau_2)), \, \omega_1(\tau_2) = i \;_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1 - \lambda(\tau_2)), \, \tau_2 = \frac{\omega_1(\tau_2)}{\omega_0(\tau_2)}. \tag{2.77}$$

On the other hand, the corresponding pull back of the Legendre model using $\mathbb{P}_\alpha \to \mathbb{P}_\lambda$, (defined locally according to the uniformization in terms of κ, and is induced by $\tau_4 \mapsto \tau_2 = 2\tau_4$) gives another elliptic curve family. The above base change of the Legendre family has the same Picard-Fuchs equation as the one attached to the E_5 family and to the elliptic curve family Eq. (1.13) associated to the modular group $\Gamma_0(4)$. By tautology, the periods for this family are

$$\omega_0(\tau_4) = \;_2F_1(\frac{1}{2}, \frac{1}{2}, 1; \alpha(\tau_4)), \, \omega_1(\tau_4) = \frac{i}{\sqrt{4}} \;_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1 - \alpha(\tau_4)), \, \tau_4 = \frac{\omega_1(\tau_4)}{\omega_0(\tau_4)}. \tag{2.78}$$

2.6.2 Seiberg-Witten curve family

We shall not discuss the physics details of Seiberg-Witten theory, but will only mention the geometry and the arithmetic of the moduli spaces involved. For a review on the physics story, see e.g., Lerche (1997); Klemm (1997); Donagi (1997). The Seiberg-Witten curve family Seiberg and Witten (1994) was introduced as a geometric model to meet the following properties (on monodromy, symmetries) required from physics (pure $4d$ $\mathcal{N} = 2$

supersymmetric Yang-Mills with gauge group $G = SU(2)$)

$$u = \Lambda^2, \text{ massless monopole}, a_D = 0, \tau = 0\,, \tag{2.79}$$

$$u = -\Lambda^2, \text{ massless dyon}, 2a - a_D = 0\,, \tag{2.80}$$

$$u = \infty,\ \nu_\infty = 2A, \text{ two cycles shrinking, not stable degeneration}, \ \tau = i\infty\,, \tag{2.81}$$

where u is the parameter of the base $\mathcal{M}$ of the desired elliptic curve family, called quantum moduli space of vacua in Seiberg-Witten theory. The quantities a, a_D are the integrals of some meromorphic differential (Seiberg-Witten differential) over the homology cycles of the elliptic curves and are called the Seiberg-Witten periods, while $\tau = \frac{\partial a_D}{\partial a}$ gives the modular parameter for the elliptic curve and is the coupling constant of the Seiberg-Witten theory. Note that the coefficients $2, -1$ in the vanishing integral $2a - a_D$ at $u = -\Lambda^2$, called charges of the massless dyon in physics term, depend on the normalization of the homology cycles.

Legendre Family

The Seiberg-Witten curve family Seiberg and Witten (1994) $\pi_{sw,2} : \mathcal{E}_{sw,2} \to \mathcal{M}$ given by

$$E_{sw,2} :\ y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u_2),\ j(u_2, \Lambda^2) = 2^6 \frac{(3\Lambda^4 + u_2^2)^3}{\Lambda^4 (u_2^2 - \Lambda^4)^2} \tag{2.82}$$

satisfies the desired properties above. This curve family is easily seen to be related to the Legendre family. Now we identify the correct normalization and the relation between the parameters u on $\mathcal{M}$ and λ on $X(2)$.

Naively, one has

$$\lambda = \frac{u_2 + \Lambda^2}{2\Lambda^2}\,. \tag{2.83}$$

But this identity between u_2, λ is determined only up to Deck group transformation on λ. The one that gives the correct normalization values of τ is

$$\frac{u_2}{\Lambda^2} = -1 + \frac{2}{\lambda}. \tag{2.84}$$

This leads to the special values for the Hauptmodul λ as shown in Table 2.4. The S-transformation $\tau \mapsto -\frac{1}{\tau}$ exchanges $\lambda \mapsto 1 - \lambda$ and thus exchanges $u = \Lambda^2, \infty$, corresponding

Table 2.4: *Special values of the Legendre family as Seiberg-Witten curve family*

τ	$0, 1, i\infty \in [0]$	$[i]$	$[\rho]$
λ	$1, \infty, 0$	$-1, \frac{1}{2}, 2$	ρ, ρ^2
j	∞	1728	0
u	$\Lambda^2, -\Lambda^2, \infty$	$*$	$*$

to $\tau = 0$ and $\tau = i\infty$, this is exactly the S-duality in physics.

Jacobi Family

The Seiberg-Witten curve family in Seiberg and Witten (1994) $\pi_{sw,4} : \mathcal{E}_{sw,4} \to \mathcal{M}$ is given by

$$z + \frac{\Lambda^4}{z} = 2(x^2 - u_4)\,, \tag{2.85}$$

or equivalently

$$E_{sw,4} : y^2 = (x^2 - u_4)^2 - \Lambda^4,\; j(u_4) = 2^6 \frac{(4u_4^2 - 2\Lambda^4)^3}{\Lambda^8(u_4^2 - \Lambda^4)}\,. \tag{2.86}$$

It also satisfies the desired properties above. This family is identical to the Jacobi family with

$$\alpha_J = \frac{u_4 \pm \Lambda^2}{u_4 \mp \Lambda^2}\,, \tag{2.87}$$

up to Deck transformation. According to $\lambda(\tau_2) = \frac{2\Lambda^2}{u_2(\tau_2)+\Lambda^2}$, one has

$$\alpha(\tau_4) = \frac{2\Lambda^2}{u_2(\tau_2) + \Lambda^2}. \tag{2.88}$$

This leads to the following identification

$$u_2(\tau_2) = u_4(\tau_4),\; \alpha_4(\tau_4) = \frac{2\Lambda^2}{u_2(\tau_2) + \Lambda^2} = 1 - \alpha_J = \frac{2\Lambda^2}{u_4(\tau_4) + \Lambda^2}. \tag{2.89}$$

Note that for the Seiberg-Witten curve family $\mathcal{E}_{sw,2}$ (Legendre family), the massless dyon ($2a - a_D = 0$) occurs at $u_2(\tau_2) = -\Lambda^2$ where $\tau_2 = \frac{\partial_{u_2} a_D(u_2)}{\partial_{u_2} a(u_2)}$. However, for the Seiberg-Witten curve family $\mathcal{E}_{sw,4}$ (Jacobi family), this point is given by $u_4(\tau_4) = -\Lambda^2$ where $\tau_4 = \frac{\partial_{u_4} a_D(u_4)}{\partial_{u_4} a(u_4)}$. Since $\tau_2(u_2) = 2\tau_4(u_4)$ and the cusp on $X_0(4)$ is $\tau_4 = \frac{1}{2}$, the corresponding τ_2 value on the modular curve $X(2)$ is $\tau_2 = 2\tau_4 = 1$. This matches the result in Table 2.4.

2.7 Conclusions and discussions

In this chapter, we studied the arithmetic structures of the moduli spaces of certain special CY 3-folds and the corresponding topological string partition functions. We discussed some consequences of these arithmetic structures. We also mentioned the physics interpretations of the Fricke involution as a duality in Seiberg-Witten theory.

The whole discussion relies heavily on the identification of the moduli spaces with modular varieties. Right now we can only deal with those non-compact CY 3-fold families whose mirror curves are the elliptic curves families of $E_n, n = 5, 6, 7, 8$ type. In particular, the corresponding mirror curve needs to be of genus one[13]. For example, just as what we did to solve the topological string partition functions for $K_{\mathbb{P}^2} = K_{\mathbb{P}^2[1,1,1]}$, the same idea works for the non-compact geometries $K_{WP^2[1,1,2]}, K_{WP^2[1,2,3]}$ whose mirror curve family are the same as the corresponding del Pezzos in Eq. (2.2), see e.g., Mohri *et al.* (2001) for related work.

If the mirror curve is not of genus one or the mirror elliptic curve family is not of $E_n, n = 5, 6, 7, 8$ type, then we don't know how to apply the ideas at this moment. For example, for the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, the mirror curve has genus zero (see Section 3.4 for more discussions on this family whose generating functions of Gromov-Witten invariants have been known both in mathematics and physics a long time ago), the procedure of using modularity to solve for $\mathcal{F}^{(g)}$s we used above does not seem to apply. Another interesting case is that when the mirror curve has higher genus, for example, for the toric CY varieties used to engineer $4d$ $\mathcal{N} = 2$ $SU(n)$ gauge theories in Katz *et al.* (1997). Presumably, more complicated modular varieties and modular forms than the classical ones (elliptic, Siegel, Hilbert) will appear on the mirror side, see e.g., Aganagic *et al.*

[13] A convenient way to figure out the genus of the mirror curve for a toric CY 3-fold is to think of it as the thickening of the 1-skeleton of the toric diagram, see e.g., Bouchard and Sulkowski (2012), hence the genus is the number of loops in the toric diagram.

(2008) for related discussions on this.

It would be interesting to find more general examples of CY families, compact or non-compact, which are parametrized by modular varieties, and try to use the chain of ideas presented in this chapter to study the CY varieties themselves. Existing works about such CYs and toplogical string partition functions for some of them could be found in e.g., the following papers and references therein:

- CY 2-fold or 3-fold families that are related directly or indirectly to elliptic curves or K3s: of Borcea-Voisin type Borcea (1992); Voisin (1993); Borcea (1997); Abe and Sato (1997); Rohde (2009, 2010); Garbagnati and van Geemen (2010); Dillies (2012); Debarre (2012); Garbagnati (2013); Cattaneo and Garbagnati (2013); by Kummer construction Borcea (1992); Cynk and Hulek (2007); and some other constructions Yui (2001, 2003, 2004); Hulek and Verrill (2006); Meyer (2005); Hulek *et al.* (2006); Schimmrigk (2007); Top and Yui (2007); Yui (2012); Debarre (2012), etc.

- Families of lattice polarized K3 surfaces: Beauville *et al.* (1985); Nagura and Sugiyama (1995); Aspinwall and Morrison (1994); Dolgachev (1996); Clingher and Doran (2006); Smith (2007); Whitcher (2009); Clingher and Doran (2012); Elkies and Kumar (2012), etc.

- Families of K3 fibrations or $K3 \times T^2/\mathbb{Z}^2$: Ferrara *et al.* (1995); Kachru and Vafa (1995); Marino and Moore (1999); Klemm *et al.* (2005); Klemm and Marino (2008); Grimm *et al.* (2007); Haghighat and Klemm (2010), etc.

- Families of elliptic fibrations: Candelas *et al.* (1994a,b); Aganagic *et al.* (2008); Alim and Scheidegger (2012); Klemm *et al.* (2012), etc.

- Other geometries: Ceresole *et al.* (1993a,b); Debarre (2012), etc.

Another potential application of the arithmetic structure on the moduli space would be to solve the master anomaly equations, discussed in e.g., Bershadsky *et al.* (1994); Witten (1993); Dijkgraaf *et al.* (2002); Verlinde (2004); Gunaydin *et al.* (2006), on the modular curve.

These equations are satisfied by the quantity

$$\mathcal{Z}_{\text{top}}(t,\bar{t}) = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2}\mathcal{F}^{(g)}(t,\bar{t})\right)\,, \tag{2.90}$$

where λ is the so-called topological string coupling constant, according to Bershadsky *et al.* (1994). The advantage of considering this quantity is that the equations it satisfies, called the master anomaly equations, consist of only two wave-like equations whose coefficients are geometric quantities easily computed from the special Kähler geometry on the moduli space. It seems that solving the master anomaly equations satisfied by $\mathcal{Z}_{\text{top}}(t,\bar{t})$ would be easier than solving the holomorphic anomaly equations for $\{\mathcal{F}^{(g)}\}_{g=0}^{\infty}$ genus by genus recursively, since the general structure for the latter is yet not known.

Chapter 3

Differential rings from special Kähler geometry [1]

3.1 Motivation and introduction

For the non-compact CY 3-fold families discussed in the previous chapter, to solve the holomorphic anomaly equations, we firstly identified the moduli spaces with modular curves, and then expressed the geometric quantities, in particular the propagators, as modular forms to figure out the non-holomorphic part. After that we used the Fricke involution to deal with the boundary conditions which helped fix the holomorphic ambiguities.

A natural hope is to extend the above procedure to more general geometries, e.g., the mirror quintic family [2]. But the period domain is rarely a Hermitian symmetric domain and thus the moduli space of (complex structures) of a CY 3-fold is not a modular variety, so the above chain of ideas does not apply directly.

However, what has been used is essentially the knowledge on the singularities of the

[1]This chapter is based on the joint work Alim *et al.* (2013) with Murad Alim, Emanuel Scheidegger and Shing-Tung Yau, and my paper Zhou (2013) which grew out of the discussions with the authors of Alim *et al.* (2013).

[2]There are some studies in this direction, see e.g., Candelas *et al.* (1991); Ceresole *et al.* (1993a,b); Movasati (2011); Doran *et al.* (2013).

generators which form the differential ring in Eq. (20) and on how to analytically continue the non-holomorphic generators. There is a chance that this could be figured out without relying on the existence of the identification between the moduli space and the modular variety.

3.1.1 Local $\mathbb{P}^2$ revisited

Now we shall explain how this could be done by looking at the local $\mathbb{P}^2$ example. Recall that the choice for the ambiguity s^x_{xx} in the integrated special geometry relation Eq. (7) we have made give rise to the non-holomorphic generator $S^{xx} = \hat{E}/2A^2$ which is an almost-holomorphic modular form. Moreover, using the differential ring structure Eq. (1.3) for the almost-holomorphic modular forms, we can deduce the holomorphic quantity h^{xx}_x in Eq. (20) which turns out to be a modular function. Knowing how the Fricke involution acts on the generators $A, B, C, \hat{E}$ of the ring $\widehat{M}_*(\Gamma_0(3), \chi_{-3})$ of almost-holomorphic modular forms tells how the non-holomorphic and holomorphic generators in Eq. (20) transform, or equivalently their analytic continuations from the large complex structure limit to the conifold point in the $(\alpha, \bar{\alpha})$ space.

In the following we shall discuss how to do this without using the generators A, B, C and the Fricke involution on $X_0(3)$ and thus on $\widehat{M}_*(\Gamma_0(3), \chi_{-3})$ in the first place.

First, we define the holomorphic quantities A, B, C from periods as in Eq. (1.19), and the quantity τ in terms of special geometry quantity according to Eq. (2.10). We then determine the holomorphic limit of the propagator S^{xx} from the integrated special geometry relation Eq. (7) up to addition by the holomorphic ambiguity $C^{-1}_{xxx} s^x_{xx}$. More precisely, from Eq. (2.21), we get

$$\lim_{\mathrm{LCSL}} \Gamma^x_{xx} = \partial_x \log \partial_x t = -C_{xxx} \lim_{\mathrm{LCSL}} S^{xx} + s^x_{xx} \,. \tag{3.1}$$

where s^x_{xx} is an arbitrary holomorphic function and is not tensorial as what happens to Γ^x_{xx}. We also choose $\lim_{\mathrm{LCSL}} S^x = \lim_{\mathrm{LCSL}} S = \lim_{\mathrm{LCSL}} K_x = 0$ for the current non-compact CY 3-

fold case. Proposition 1.6 then allows to compute tensor transformations from the coordinate $x = \ln \alpha$ to the special coordinate t defined near the large complex structure limit in Eq. (2.8), which satisfies $\partial_x t = \theta t = A$. In particular, we get $\lim_{\mathrm{LCSL}} S^{tt} = \lim_{\mathrm{LCSL}} S^{xx} (\frac{\partial t}{\partial x})^2$.

In this set-up, we defined quantities from only the properties of the Picard-Fuchs equations $\mathcal{L}_{\mathrm{CY}} = \mathcal{L}_{\mathrm{elliptic}} \circ \theta$ for the CY 3-fold family $\pi : \mathcal{X} \to \mathcal{M}$, and did not use explicitly the knowledge on the modular curve $X_0(3)$. But we have used the fact that the special coordinate t is the canonical coordinate (see Section 3.3.1) near the large complex structure limit, both to define the holomorphic limit and to derive the holomorphic limit of Γ^x_{xx} at the this point using Eq. (2.19).

Now we consider the ring formed by the generators $\lim_{\mathrm{LCSL}} S^{tt}, \theta t, C^{-1}_{ttt} = \kappa^{-1} \partial_\tau t = \kappa^{-1} \beta (\theta t)^3$, where $\beta = 1 - \alpha$. The differential ring structure Eq. (20) of the non-holomorphic generators and Proposition 1.6 give rise to the following differential structure among the holomorphic limits of the non-holomorphic and holomorphic generators,

$$
\begin{aligned}
\partial_\tau \lim_{\mathrm{LCSL}} S^{tt} &= -\lim_{\mathrm{LCSL}} S^{tt} \lim_{\mathrm{LCSL}} S^{tt} - \frac{1}{12\kappa} (\theta t)^4 && (3.2)\\
\partial_\tau \theta t &= C^{-1}_{ttt} \theta^2 t = -\lim_{\mathrm{LCSL}} S^{tt} \theta t + C^{-1}_{ttt} s^x_{xx} && (3.3)\\
\partial_\tau C^{-1}_{ttt} &= -3 C^{-1}_{ttt} \lim_{\mathrm{LCSL}} S^{tt} + C^{-1}_{ttt} (\theta t)^2 (\partial_x \log C^{-1}_{xxx}) + 3 s^x_{xx}\,, && (3.4)
\end{aligned}
$$

with $\partial_x \log C^{-1}_{xxx} = -\frac{\alpha}{\beta}$. More precisely, the first equation above follows from the first equation in Eq. (20); the second from the definition of the propagator S^{tt} from the integrated special geometry relation Eq. (7). The third one follows from the definition $C^{-1}_{ttt} = \kappa^{-1} \beta (\theta t)^3$, Proposition 1.6 and the second equation above.

To make the ring closed, we need to add the holomorphic quantity $\partial_x \log C_{xxx} = \frac{\alpha}{\beta}$. Now its derivative already lies in the ring of the above generators again according to Proposition 1.6:

$$
\partial_\tau \partial_x \log C_{xxx} = (\partial_x \log C_{xxx}) (\theta t)^2 \,.
$$

Moreover, there is a natural weight associated to these quantities: the weight for tensors associated to x, t, τ are given by $0, 1, -2$, respectively.

Remark 3.1. This agrees with the known modular weight for $\Gamma_0(3)$. In fact, choosing $s^x_{xx} = \frac{\alpha-\beta}{6\beta}$ in Eq. (2.21), we then have $\lim_{\mathrm{LCSL}} S^{tt} = \frac{1}{2}E, \theta t = A, C^{-1}_{ttt} = \kappa^{-1}B^3, \partial_x \log C_{xxx} = \frac{C^3}{B^3}$, and the differential ring structure Eq. (20) and Proposition 1.6 derived from special geometry on the moduli space $\mathcal{M}$ is exactly equivalent to the differential ring structure in Theorem 1.2 for the quasi-modular forms for $\Gamma_0(3)$.

As explained in Introduction, polynomial recursion tells that the topological string partition functions are Laurent polynomials of these generators of weight 0 due to the fact that they are functions of x rather than tensors (of course the weight is not the same as the degree, which is $2-2g$ due to the fact that $\mathcal{F}^{(g)}$ is a section of the line bundle $\mathcal{L}^{2-2g}$).

The suitable coordinates that are needed in applying the boundary conditions at the large complex structure limit and at the conifold point are given by $\alpha, \beta = 1-\alpha$, respectively. As we pointed out in Introduction, it is not easy to compute the non-holomorphic completions from the holomorphic limits without using the explicit structures of the rings of quasi-modular forms and almost-holomorphic modular forms. For example, from only the expression of $\lim_{\mathrm{LCSL}} S^{tt}$, one would not able to recover S^{tt} and thus can not proceed to obtain $\lim_{\mathrm{CON}} S^{tt}$ by taking the holomorphic limit of S^{tt} at the conifold point.

We now use the transformation $\alpha \leftrightarrow \beta$, to related $\lim_{\mathrm{LCSL}}$ and $\lim_{\mathrm{CON}}$: roughly speaking first we simply express all of the building blocks $\lim_{\mathrm{LCSL}} S^{tt}, \theta t, C^{-1}_{ttt}, \partial_x \log C_{xxx}$ as functions of α near the large complex structure limit $\alpha = 0$; then we replace α by β to get the expansion of the topological string partition functions in terms of β, that is, near the conifold point $\beta = 0$. This then allows to apply the gap condition to the resulting β expansion.

More precisely, $\theta t = A(\alpha), C^{-1}_{ttt} = \kappa^{-1}\beta A(\alpha)^3, \partial_x \log C_{xxx} = \frac{\alpha}{\beta}$ are easily seen to be map to corresponding quantities $A(\beta), \kappa^{-1}\alpha A(\beta)^3, \frac{\beta}{\alpha}$, respectively. The action on $\lim_{\mathrm{LCSL}} S^{tt}$ is more subtle, both because it is the holomorphic limit of a non-holomorphic function, and because that it is determined up to addition of a holomorphic function as can be seen from Eq. (3.1).

To understand this we first carry our some computations using the known modularity. The holomorphic limit of the integrated special geometry relation Eq. (3.1) reads

$$-\frac{E(\alpha)}{6\beta A^2(\alpha)} = -\theta_\alpha \log A(\alpha) + \frac{\alpha-\beta}{6\beta}\,. \tag{3.5}$$

Using the transformation law for the quasi-modular form $E(\beta)$, we get

$$\frac{E(\beta)}{6\alpha A^2(\beta)} = -\frac{E(\alpha)}{6\alpha A^2(\alpha)} + \frac{1}{2}\frac{12}{2\pi i}\frac{3\tau}{-3\tau^2}\frac{1}{6\alpha A^2(\alpha)}\,. \tag{3.6}$$

Plugging Eq. (3.5) into Eq. (3.6), we get

$$\frac{E(\beta)}{6\alpha A^2(\beta)} = -\theta_\beta \log A(\alpha) + \frac{\beta-\alpha}{6\alpha} + \frac{1}{2}\frac{12}{2\pi i}(-\frac{1}{\tau})\frac{1}{6\alpha A^2(\alpha)}\,. \tag{3.7}$$

We also know that from Proposition 1.6 that

$$\theta_\beta \log A(\beta) = \theta_\beta \log A(\alpha) - \frac{1}{2\pi i}\frac{1}{\tau}\frac{1}{\alpha A^2(\alpha)}\,. \tag{3.8}$$

The two equations Eqs. (3.7), (3.8) above tell that

$$\frac{E(\beta)}{6\alpha A^2(\beta)} = -\theta_\beta \log A(\beta) + \frac{\beta-\alpha}{6\alpha}\,. \tag{3.9}$$

Comparing the above equation with Eq. (3.5) we can see that the duality should acts as [3]

$$E(\alpha) \mapsto -E(\beta). \tag{3.10}$$

This is consistent with Eq. (2.38) that $A^2(\beta) = -N\tau^2 A^2(\alpha), \hat{E}(\beta,\bar{\beta}) = N\tau^2\hat{E}(\alpha,\bar{\alpha})$.

In retrospect, the difference about the sign can be explained in the following way: the integrated special geometry relation should not transform naively using $\alpha \leftrightarrow \beta$ since it involves the first derivative, which carries the information of the Jacobian and thus picks a sign under the transformation $\alpha \to \beta = 1-\alpha$. So although both A^2, E are quasi-modular forms of weight 2, they transform under different laws.

To summarize, remembering that E comes from a relation defined using the first deriva-

[3] What this really means is that $\lim_{\mathrm{LCSL}} \hat{E}$ and $-\lim_{\mathrm{CON}} \hat{E}$ are based on the same function, but with different arguments α, β respectively.

tive helps identify the correct sign under the transformation.

This then provides a solution in how to analytically continue $\lim_{\mathrm{LCSL}} S^{tt}$ without knowing modularity but by using the involution $\alpha \leftrightarrow \beta$. One does not know in the first place which choice of s^x_{xx} would give rise to the quantity $\lim_{\mathrm{LCSL}} S^{tt}$ that transforms according to $\lim_{\mathrm{LCSL}} S^{tt}(\alpha)$ to $-\lim_{\mathrm{CON}} S^{tt}(\beta)$ without using modularity[4]. In other words, a different choice for the modular function s^x_{xx} than above would result in a combination of the quasi-modular form E and a modular function which transform in different ways under the involution. For example, by choosing the holomorphic ambiguity s^x_{xx} so that Eq. (2.27) becomes $\partial_x \mathcal{F}^{(1)} = \frac{1}{2} C_{xxx} S^{xx}$, one can see by computations using the expressions in terms of quasi-modular forms that the transformation $\lim \lim_{\mathrm{LCSL}} S^{xx} \mapsto -\frac{\beta}{\alpha} \lim_{\mathrm{CON}} S^{xx}$ is not true. However, one can work out the transformation of the holomorphic quantity s^x_{xx} (it can be arbitrary, but one usually choose it to be a rational function) and the quantity $\lim_{\mathrm{LCSL}} \Gamma^x_{xx} = \theta \log A$, then the integrated special geometry relation Eq. (3.5) induces the correct transformation on $\lim_{\mathrm{LCSL}} S^{tt}$. Alternatively, one can use replace the generator $\lim_{\mathrm{LCSL}} S^{tt}$ by the quantity $\lim_{\mathrm{LCSL}} \Gamma^x_{xx}$ whose transformation law is easier.

Therefore, at least for the non-compact CY 3-fold examples discussed in Theorem 0.1, from

- The differential ring structure of the generators in Eq. (20) and Proposition 1.6;
- The map[5] $\alpha \mapsto \beta = 1 - \alpha$ exchanging the large complex structure limit and the conifold point which serves as a "duality"[6];

[4]In fact, using the known modularity, we have derived from Eq. (3.5) that the choice $s^x_{xx} = \frac{1}{2r} \frac{\alpha - \beta}{\beta}$ will do the job.

[5]The root of this symmetry lies in the fact that the hypergeometric equation Eq. (1.14) written in $\beta = 1 - \alpha$ coordinate takes the same form as the one written in α coordinate. Hence the analytic period $A(\alpha)$ near $\alpha = 0$ would give rise to the analytic period $A(\beta)$ near $\beta = 0$. This symmetry of the hypergeometric equation reflects the special properties of the underlying elliptic curve family.

[6]In retrospect, what is essential is the fact that the topological string partition functions are modular functions so that the τ factors can be canceled under the Fricke involution which induces the transformation. Otherwise one would have to deal with the extra τ factors which makes the transformation on generators difficult to access in terms of the algebraic coordinates α, β.

- The transformation law for the generators of the differential ring,

one can then solve the holomorphic anomaly equations without knowing the arithmetic of the moduli spaces.

Remark 3.2. Now we can compare the difference between the holomorphic limit of $\mathcal{F}^{(g)}$ at the conifold and the analytic continuation from its holomorphic limit at the large complex structure limit. This should be compare to the discussion in Remark 2.2 and goes back to the second item mentioned at the end of Section 0.2.

It suffices to figure out how the building blocks, i.e., the generators, transform. Assuming modularity, we know the analytic continuation of $E(\alpha)$ is given by

$$E(\alpha) = (-\sqrt{3}\tau_D)^2 E(\beta) + \frac{1}{2}\frac{12}{2\pi i}(-\sqrt{3})(-\sqrt{3}\tau_D), \quad \tau_D = W_N \tau\,. \tag{3.11}$$

Without using modularity, we will get the same equation, but now τ (and similarly τ_D) is now defined to be the ratio satisfying $\frac{\sqrt{N}}{i}\tau = \frac{A(\beta)}{A(\alpha)}$ with N being an unknown constant. Recall that the multiplication by $\frac{\sqrt{N}}{i}\tau$ factors will be canceled out due to the fact that $\mathcal{F}^{(g)}$ has weight zero, we shall get

$$E(\alpha) \mapsto -E(\beta) - \frac{1}{2}\frac{12}{2\pi i}\frac{1}{\tau_D}, \quad \tau_D = -\frac{1}{N\tau}\,. \tag{3.12}$$

While the correct result given by the Fricke involution should be (again forgetting the $\frac{\sqrt{N}}{i}\tau$ factors)

$$E(\alpha) = (\lim_{\mathrm{LCSL}} \hat{E})(\alpha) = -(\lim_{\mathrm{CON}} \hat{E})(\beta) = -E(\beta)\,. \tag{3.13}$$

Hence they are the same only after modulo the terms involving τ .

3.1.2 Constructing differential rings on moduli spaces

Experiences from dealing with the non-compact CY 3-folds suggest that studying the properties of the rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ might be useful in solving the holomorphic anomaly equations for more general CY 3-fold families, as explained in the previous section. This is the motivation of this work.

In this chapter, for certain one-parameter CY 3-fold family $\pi : \mathcal{X} \to \mathcal{M}$, we shall define a triple of graded rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$. The analogue of the modular variable τ is constructed according to Eq. (2.10) using the special Kähler geometry on the base $\mathcal{M}$, and the periods are solved from the Picard-Fuchs equation of the CY 3-fold family. This triple share similar structures and operations with the triple $(M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma))$ defined for the elliptic curve family $\pi_\Gamma : \mathcal{E}_\Gamma \to X_\Gamma$. In particular, they have gradings which play the role of modular weights.

For the one-parameter families of non-compact CY 3-folds with an elliptic curve sitting inside each fiber as discussed in Chapter 2, the base $\mathcal{M}$ of each CY 3-fold family $\pi : \mathcal{X} \to \mathcal{M}$ has been identified with a modular curve in Section 2.2. The triple of rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ we shall define are closely related to the known graded rings $(M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma))$ of modular objects.

Remark 3.3. The above idea of deriving/constructing modular forms from geometric quantities arising from Picard-Fuchs equations is not new. Given a genus zero subgroup Γ of finite index of the full modular group $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$, the generators of the ring of modular forms for Γ could be obtained starting from some θ or η-functions or Eisenstein series. Alternatively, one could parameterize the modular curve $X_\Gamma = \Gamma \backslash \mathcal{H}^*$ by a Hauptmodul α (generator of the rational functional field of the modular curve), and consider the periods which are solutions to the Picard-Fuchs equation attached to the elliptic modular surface Eq. (1.13) $\pi_\Gamma : \mathcal{E}_\Gamma \to X_\Gamma$ constructed from the modular group Γ. Knowing the relation between the Hauptmodul α and the normalized period τ of the elliptic curve family, one could then obtain the graded differential ring of quasi-modular forms $\widetilde{M}_*(\Gamma)$ by taking successive derivatives of the periods with respect to τ. This ring $\widetilde{M}_*(\Gamma)$ includes the ring of modular forms $M_*(\Gamma)$ and contains further elements which are not modular but quasi-modular in the sense of Kaneko and Zagier (1995). The quasi-modular forms in $\widetilde{M}_*(\Gamma)$ could be completed to modular forms by adding some non-holomorphic parts. Then one gets the graded differential ring of almost-holomorphic modular forms $\widehat{M}_*(\Gamma)$. See Chapter 1, Kaneko and Zagier (1995); Zagier (2008) and references therein for details about the construction.

The structure of this chapter is as follows. In Section 3.2, first we recall briefly the construction of the graded rings $M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma)$ for the elliptic curve families in Eq. (1.13) from their Picard-Fuchs equations in Eq. (1.14). Then we construct by analogy the graded rings $\mathcal{R}, \widetilde{\mathcal{R}}$ for certain one-parameter CY 3-fold families $\pi : \mathcal{X} \to \mathcal{M}$. In Section 3.3, we consider these rings using the properties of the special Kähler geometry on the deformation space $\mathcal{M}$. We make use of the canonical coordinates and holomorphic limit to lift the ring $\widetilde{\mathcal{R}}$ to a non-holomorphic ring $\widehat{\mathcal{R}}$, by relating them to the ring constructed by Yamaguchi and Yau (2004). The main results of this chapter are summarized in Section 3.3.5. Some of their applications in solving the BCOV holomorphic anomaly equations are also discussed in Section 3.3. We conclude with some discussions and questions in Section 3.4.

3.2 Differential rings from Picard-Fuchs equations

In this section, we shall explain how to obtain differential rings from the Picard-Fuchs equations for certain special one-parameter CY 3-fold families.

We recall that in the construction of the rings for quasi-modular forms and almost-holomorphic modular forms for the modular groups $\Gamma_0(N)$ in Section 1.3, the key property we used is the formula in Proposition 1.6 which could be thought of as a differential equation satisfied by the normalized period of the elliptic curves. As we shall see, for an one-parameter CY 3-fold family $\pi : \mathcal{X} \to \mathcal{M}$, the normalized period, that is, the special coordinate t, does not satisfy an analogous relation. However, the parameter τ on $\mathcal{M}$ satisfying a similar identity (see Proposition 3.8 below). To get the identity, one needs to make use of the properties of Yukawa couplings in different coordinates.

3.2.1 Yukawa couplings

In the following, we shall explain how to derive Proposition 1.6 by computing the Yukawa couplings for the purpose of later generalization.

Proposition 3.4. *The Yukawa coupling in the transcendental modulus τ, defined by $C_\tau = \int \Omega \wedge \partial_\tau \Omega$, satisfies $C_\tau = 1$.*

That is, there is no "quantum correction" added to the classical intersection number $\kappa = 1$, see Lian and Yau (1995, 1996a,b); Hosono (2008).

Proof. Take the local parameter of the (punctured) moduli space near the point $\alpha = 0$ to be τ. At the base point τ_*, the fiber of the family $\pi_{\Gamma_0(N)} : \mathcal{E}_{\Gamma_0(N)} \to X_0(N)$ is the torus $T_{\tau_*} = \mathbb{C}/\Lambda_{\tau_*}$, where $\Lambda_{\tau_*} = \mathbb{Z} \oplus \mathbb{Z}\tau_*$. We take the holomorphic top form on T_{τ_*} to be dz_{τ_*} which descends from $\mathbb{C}$. For any τ near the base point τ_*, the diffeomorphism sending T_{τ_*} to T_τ is given by $z_\tau = \frac{\tau - \bar{\tau}_*}{\tau_* - \bar{\tau}_*} z_{\tau_*} + \frac{\tau_* - \tau}{\tau_* - \bar{\tau}_*} \bar{z}_{\tau_*}$. From this, one can then see that $\partial_\tau \Omega_\tau|_{\tau_*} = \frac{1}{\tau_* - \bar{\tau}_*}(dz_{\tau_*} - d\bar{z}_{\tau_*})$. It follows then that $C_{\tau_*} = \int_{T_{\tau_*}} \frac{i}{2\mathrm{Im}\tau_*} dz_{\tau_*} \wedge d\bar{z}_{\tau_*} = 1$. □

One can also compute the Yukawa coupling in the algebraic modulus α as follows.

Proposition 3.5. *The Yukawa coupling in the algebraic modulus α, defined by $C_\alpha = \int \Omega \wedge \partial_\alpha \Omega$, satisfies $C_\alpha = \frac{1}{\alpha\beta}$.*

Proof. Recall that the Picard-Fuchs equation Eq. (1.25) tells that when integrated over cycles, one has $\theta^2 \Omega = (c_1 + c_2)\frac{\alpha}{\beta}\theta\Omega + c_1 c_2 \frac{\alpha}{\beta}\Omega$. Now we have

$$\begin{aligned}
\theta(\alpha C_\alpha) &= \theta \int \Omega \wedge \theta\Omega = \int \theta\Omega \wedge \theta\Omega + \int \Omega \wedge \theta^2 \Omega \\
&= 0 + \int \Omega \wedge \left((c_1 + c_2)\frac{\alpha}{\beta}\theta\Omega + c_1 c_2 \frac{\alpha}{\beta}\Omega \right) \\
&= 0 + (c_1 + c_2)\frac{\alpha}{\beta} \int \Omega \wedge \theta\Omega + 0 \\
&= (c_1 + c_2)\frac{\alpha}{\beta}(\alpha C_\alpha) = \frac{\alpha}{\beta}(\alpha C_\alpha)\,.
\end{aligned}$$

Solving αC_α from this equation, we get $\alpha C_\alpha = \frac{c}{\beta}$ from some constant c. We then fix this c by looking at the behaviors of both sides near $\alpha = 0$. This gives $c = 1$. Hence the conclusion follows. □

Having computed the Yukawa coupling in two different coordinates τ and α in Proposition 3.4 and Proposition 3.5, we can then derive the equation $\partial_\tau \alpha = (C_\alpha)^{-1}\omega_0^2$ given in

Proposition 1.6 between the transcendental modulus τ and the algebraic modulus α, from the following relation

$$C_\tau = \frac{1}{2\pi i}\frac{1}{\omega_0^2}\frac{\partial\alpha}{\partial\tau}C_\alpha\,. \tag{3.14}$$

3.2.2 Picard-Fuchs equations for CY 3-fold families

Motivated by the discussions on elliptic curve families, we shall work out similar rings $\mathcal{R}, \widetilde{\mathcal{R}}$ living on the deformation spaces of CY 3-folds with given Picard-Fuchs equations. As before we limit ourselves to the case $\dim \mathcal{M} = 1$. We shall start by computing the Yukawa couplings in different coordinates, then we derive an equation analogous to Proposition 1.6 between the complex coordinate α and a suitably chosen coordinate τ on the deformation space $\mathcal{M}$. After that we construct a ring out of special Kähler geometry quantities (connections, Yukawa couplings, etc.).

The Picard-Fuchs equations we are interested in are the ones in Eq. (2.7) for non-compact CY 3-fold families

$$\mathcal{L}_{\mathrm{CY}} = \mathcal{L}_{\mathrm{elliptic}}\circ\theta = \left(\theta^2 - \alpha(\theta+\frac{1}{r})(\theta+1-\frac{1}{r})\right)\circ\theta,\quad \theta = \alpha\frac{\partial}{\partial\alpha},\, \alpha = \kappa_N z\,, \tag{3.15}$$

We shall also study in detail the Picard-Fuchs equation for the compact CY 3-fold family which is given by the quintic mirror family in Candelas *et al.* (1991), with Picard-Fuchs equation

$$\mathcal{L}_{CY} = \theta^4 - \alpha(\theta+\frac{1}{5})(\theta+\frac{2}{5})(\theta+\frac{3}{5})(\theta+\frac{4}{5})\,, \tag{3.16}$$

where α is related to the parameters z, ψ in Candelas *et al.* (1991) by $\alpha = 5^5 z = \psi^{-5}$.

For all of the Picard-Fuchs equations in Eqs. (3.15), (3.16), they have three regular singularities located at $\alpha = 0, 1, \infty$ on the base $\mathcal{M}$, corresponding to the large complex structure limit, conifold point, orbifold point, respectively.

In the following we shall first consider slightly more general Picard-Fuchs equations before we specialize to the Picard-Fuchs Eqs. (3.15) (3.16) mentioned above.

Suppose the Picard-Fuchs operator for the family $\pi : \mathcal{X} \to \mathcal{M}$ of CY 3-folds is of the form

$$\mathcal{L}_{\mathrm{CY}} = \theta^4 - \alpha \prod_{i=1}^{4} (\theta + c_i) = (1-\alpha)\theta^4 - \alpha(\sigma_1 \theta^3 + \sigma_2 \theta^2 + \sigma_3 \theta + \sigma_4)\,, \tag{3.17}$$

where $\theta = \alpha \frac{\partial}{\partial \alpha}$ and σ_is are the symmetric polynomials of the constants c_1, c_2, c_3, c_4. For the quintic mirror family case, one has $(c_1, c_2, c_3, c_4) = (1/5, 2/5, 3/5, 4/5)$ and thus $\sigma_1 = 2$. As before, we shall denote $\beta = 1 - \alpha$.

The Yukawa coupling in the t coordinate is then given by $C_{ttt} = F_{ttt} = \kappa + \mathcal{O}(q_t)$ with $q_t = e^{2\pi i t}$ according to mirror symmetry. In the complex coordinate α, we have

Proposition 3.6. *The Yukawa coupling, defined by $C_{\alpha\alpha\alpha} = -\int \Omega \wedge \partial_\alpha^3 \Omega$, is given by $C_{\alpha\alpha\alpha} = \frac{\kappa}{\alpha^3 \beta}$.*

Proof. First due to Griffiths transversality, we have $\alpha^3 C_{\alpha\alpha\alpha} = -\int \Omega \wedge \theta^3 \Omega$. By integration by parts and Griffiths transversality, it follows that

$$\begin{aligned}
\theta(\alpha^3 C_{\alpha\alpha\alpha}) &= -\int \theta\Omega \wedge \theta^3 \Omega - \int \Omega \wedge \theta^4 \Omega \\
&= -\left(\theta \int \theta\Omega \wedge \theta^2 \Omega - \int \theta^2 \Omega \wedge \theta^2 \Omega \right) - \int \Omega \wedge (\sigma_1 \frac{\alpha}{\beta} \theta^3 \Omega + \cdots) \\
&= -\theta \left(\theta \int \Omega \wedge \theta^3 \Omega - \int \Omega \wedge \theta^3 \Omega \right) - 0 + \sigma_1 \frac{\alpha}{\beta} (\alpha^3 C_{\alpha\alpha\alpha}) \\
&= -\theta(\alpha^3 C_{\alpha\alpha\alpha}) + \sigma_1 \frac{\alpha}{\beta} (\alpha^3 C_{\alpha\alpha\alpha})\,.
\end{aligned}$$

Solving $\alpha^3 C_{\alpha\alpha\alpha}$ from this equation, we then get $C_{\alpha\alpha\alpha} = \frac{c}{\alpha^3 \beta}$. Using the fact that

$$\frac{1}{(X^0)^2} C_{\alpha\alpha\alpha} (\frac{\partial \alpha}{\partial t})^3 = C_{ttt} = \kappa + \mathcal{O}(q)\,, \tag{3.18}$$

we know $c = \kappa$. Hence the assertion follows. □

Remark 3.7. The Yukawa couplings for the non-compact CY 3-fold families discussed above also have the same form. Intuitively, the is because non-compact geometries can be regarded as certain limits of corresponding compact CY geometries whose Yukawa couplings take the above form.

Now that we have computed the Yukawa coupling in the special coordinate t and complex coordinate α, we shall find the analogue of Proposition 1.6. According to the definition[7] of τ in Eq. (2.10), near $\alpha = 0$ we have $\tau(\alpha) \sim \frac{1}{2\pi i}\log\alpha + \mathcal{O}(\alpha^0)$. Therefore, from

$$\frac{1}{(X^0)^2}C_{\alpha\alpha\alpha}(\frac{\partial\alpha}{\partial t})^3 = C_{ttt} = 2\pi i\kappa\frac{\partial\tau}{\partial t} = 2\pi i\kappa\frac{\partial\tau}{\partial\alpha}\frac{\partial\alpha}{\partial t}\,. \tag{3.19}$$

we obtain the following assertion.

Proposition 3.8.

$$\partial_\tau\alpha = \alpha\cdot\kappa(\alpha^3 C_{\alpha\alpha\alpha})^{-1}\cdot(X^0\theta t)^2 = \alpha\beta(X^0\theta t)^2, \quad \partial_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\,. \tag{3.20}$$

Note that the only places in which we have used the special Kähler geometry are in the definition Eq. (2.10) of τ in terms of F_{tt} and the limit of Eq. (3.18) as α goes to 0. But we could have defined τ as the quantity satisfying the equation $\frac{1}{(X^0)^2}C_{\alpha\alpha\alpha}(\frac{\partial\alpha}{\partial t})^3 = 2\pi i\kappa\frac{\partial\tau}{\partial t}$ and the condition $\lim_{\alpha\to 0}2\pi i\frac{\partial\tau}{\partial t} = 1$ without referring to the prepotential $F(t)$ and Yukawa coupling C_{ttt}, thus only the Picard-Fuchs equation and no special Kähler geometry is needed.

We shall now take Proposition 3.8 as the starting point to construct the analogue of the ring of quasi-modular forms. Motivated by the discussions of elliptic curve families, we define the following triple

$$A = X^0\theta t, \quad B = (1-\alpha)^{\frac{1}{r}}A, \quad C = \alpha^{\frac{1}{r}}A\,,$$

where r is some under-determined constant and does not show up in the final form of the ring $\widehat{\mathcal{R}}$ we shall consider later. Similarly we define

$$E = \partial_\tau\log C^r B^r = \partial_\tau\log\alpha\beta A^{2r} = (\alpha-\beta)A^2 + \partial_\tau\log A^{2r}\,. \tag{3.21}$$

Now thanks to Proposition 3.8, we get

$$A^2 = \frac{\partial_\tau\alpha}{\alpha\beta} = \partial_\tau\log\frac{\alpha}{\beta} = \partial_\tau\log\frac{C^r}{B^r}\,. \tag{3.22}$$

We also have the following relations among these generators following from the definitions

[7] From now on, to simplify notations, we shall take $t\sim\log\alpha + \mathcal{O}(\alpha^0)$, hence $\tau = \frac{1}{2\pi i}\kappa^{-1}F_{tt}$.

of A, B, C, E and Eq. (3.22),

$$\partial_\tau B = \frac{1}{2r} B(E - A^2), \quad \partial_\tau C = \frac{1}{2r} C(E + A^2)\,. \tag{3.23}$$

To get a closed ring, we need to prove A satisfies a differential equation with coefficients being holomorphic functions of α, β. Define

$$A' = X^0, \quad A'' = \theta t\,, \tag{3.24}$$

and denote $\mathcal{R} = \mathbb{C}[A'^{\pm 1}, A''^{\pm 1}, B^{\pm 1}, C^{\pm 1}]$. It turns out after adding $\partial_\tau^i A', \partial_\tau^i A'', i = 1, 2, 3$, the ring will close under the derivative ∂_τ. Note that the generator E is already included according to Eq. (3.21).

Proposition 3.9. *The ring $\widetilde{\mathcal{R}}$ generated by $\partial_\tau^i A', i = 0, 1, 2, 3; \partial_\tau^j A'', j = 0, 1, 2$ and B, C, B^{-1}, C^{-1}, is closed under the derivative ∂_τ.*

Proof. The Picard-Fuchs equation tells that if one defines

$$\tilde{\mathcal{L}}_{\mathrm{CY}} = (\theta + \theta \log X^0)^4 - \alpha \prod_{i=1}^{4} (\theta + \theta \log X^0 + c_i)\,,$$

then $X^0 \tilde{\mathcal{L}}_{\mathrm{CY}} \frac{\Pi}{X^0} = \mathcal{L}_{\mathrm{CY}} \Pi = 0$ for a period Π. In particular, one has $\mathcal{L}_{\mathrm{CY}} X^0 = 0$ and $\tilde{\mathcal{L}}_{\mathrm{CY}} \frac{X^1}{X^0} = \tilde{\mathcal{L}}_{\mathrm{CY}} t = 0$. The first equation $\mathcal{L}_{\mathrm{CY}} X^0 = 0$ tells that $\theta^4 X^0$ could be expressed as a polynomial of $\theta^i X^0, i = 0, 1, 2, 3$ with coefficients being rational functions of α, β. Using the relation $\theta = \beta^{-1}(X^0 \theta t)^{-2} \partial_\tau$ following from Proposition 3.8, we know that $\partial_\tau^4 X^0$ is a polynomial in $\partial_\tau^i X^0, 0, i = 1, 2, 3$; $\partial_\tau^j \theta t, j = 0, 1, 2, 3$ and B, C, B^{-1}, C^{-1}. Similarly, by considering the second equation $\tilde{\mathcal{L}}_{\mathrm{CY}} t = 0$, one sees that $\theta^3 \theta t$ and thus $\partial_\tau^3 \theta t$ is also contained in the ring as claimed. □

Remark 3.10. Note that when taking the derivative ∂_τ, negative powers of generators will appear. But as mentioned in Remark 1.5, to avoid them one only needs to choose a suitable set of generators carefully. In fact, in the final form of the graded ring $\widehat{\mathcal{R}}$ we shall consider below, we are going to make a specific choice of generators so that no negative powers will appear in the derivatives of the generators.

From Proposition 3.8 one can easily see that in fact the subring generated by $\partial_\tau^i A', i = 0,1,2,3; \partial_\tau^j A'', j = 0,1,2; \alpha^\pm, \beta^\pm$ is also closed under ∂_τ. We shall denote this differential subring by $(\widetilde{\mathcal{R}}^{sub}, \partial_\tau)$ in which the constant r does not show up.

Rings for non-compact CY 3-fold families

Now we consider the non-compact CY 3-fold families whose Picard-Fuchs equations Eq. (2.7) reduce to some third order differential equations of the form $\mathcal{L}_{\text{elliptic}} \circ \theta$.

One has $X^0 = 1$ and thus $A = \theta t = \omega_0$; moreover, the parameter $\tau = \frac{1}{2\pi i}\kappa^{-1}F_{tt} = \frac{1}{2\pi i}\kappa^{-1}\frac{\theta F_t}{\theta t}$ is equal to $\frac{\omega_1}{\omega_0}$, where ω_0, ω_1 are the periods of $\mathcal{L}_{\text{elliptic}}$ given in Eq. (1.16). Thus the parameter τ is the transcendental modulus of the elliptic curve sitting inside the CY 3-fold and lies on the upper half plane $\mathcal{H}$. Therefore, in these cases, one has $\mathcal{R} \cong \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}] = M_*(\Gamma_0(N)), \widetilde{\mathcal{R}} \cong \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E] = \widetilde{M}_*(\Gamma_0(N))$.

Gradings

There are two natural gradings, denoted by (k, m), on the ring $\widetilde{\mathcal{R}}$. The grading m indicates that the element is a section of $\mathcal{L}^m$ and will be called the degree. Recall that X^0 is a period of the form $\int_C \Omega$ and $C_{\alpha\alpha\alpha} = -\int_X \Omega \wedge \partial_\alpha^3 \Omega$, where Ω is a section of the Hodge line bundle $\mathcal{L} \to \mathcal{M}$, we can easily figure out the degree of the generators. The second grading, called the weight k, is motivated by the studies of elliptic curve families and non-compact CY 3-folds discussed above, in which τ is really parametrizing the upper half plane $\mathcal{H}$. We then defines the degrees and weights for the quantities $X^0, \theta t, B, C, \alpha, (\alpha^3 C_{\alpha\alpha\alpha})$ to be $(1,0), (0,1), (1,1), (1,1), (0,0), (0,2)$ respectively. Taking the derivative ∂_τ with respect to τ will not change the degree, but raise the weight by 2. Then we have the decomposition $\mathcal{R} = \oplus_{(k,m)} \mathcal{R}_{k,m}$. Similarly, there is a such decomposition for the graded differential ring $(\widetilde{\mathcal{R}}, \partial_\tau)$.

The above discussions suggests that the rings $\mathcal{R} = \mathbb{C}[(X^0)^{\pm 1}, (\theta t)^{\pm 1}, B^{\pm 1}, C^{\pm 1}]$, $\widetilde{\mathcal{R}} = \mathcal{R} \otimes \mathbb{C}[\partial_\tau^i X^0, i = 1,2,3; \partial_\tau^j \theta t, j = 1,2]$, defined on the deformation space $\mathcal{M}$, are the analogues of $M_*(\Gamma), \widetilde{M}_*(\Gamma)$ defined on the modular curve X_Γ, and the weight k plays the role of modular weight. The generators $\partial_\tau^i X^0, i = 1,2,3; \partial_\tau^j \theta t, j = 1,2$ should be considered as the analogue of quasi-modular forms. We shall give more evidences for this later.

As explained earlier, one can get a smaller differential ring $\widetilde{\mathcal{R}}^{sub}$. It turns out that using special Kähler geometry of the deformation space $\mathcal{M}$, one may further reduce the number of generators in $\widetilde{\mathcal{R}} - \mathcal{R}$. For example, for the quintic mirror family case considered below, the sequence $\partial_\tau^i \theta t, i = 0,1,2$ could be reduced to $\partial_\tau^i \theta t, i = 0,1$ as discussed in Lian and Yau (1996a); Hosono and Lian (1996); Yamaguchi and Yau (2004). This is proved using the fact that t is the canonical coordinate on the deformation space $\mathcal{M}$ (more than just being the ratio of two periods), which we now turn to finally.

3.3 Differential rings from special Kähler geometry

In this section, we shall use properties of the special Kähler geometry on $\mathcal{M}$ to reduce the number of generators in $\widetilde{\mathcal{R}}$, and more importantly to define $\widehat{\mathcal{R}}$ as the "non-holomorphic completion" of $\widetilde{\mathcal{R}}$.

We first start by reviewing some basic properties about the canonical coordinates and the notion of holomorphic limit which will be important later. The discussions on these concepts apply to multi-parameter CY 3-fold families.

3.3.1 Canonical coordinates and holomorphic limits

On a Kähler manifold M, according to Bershadsky *et al.* (1994), the canonical coordinates $t = \{t^i\}_{i=1,2,\cdots \dim M}$ around the base point p are defined to be the holomorphic coordinates such that

$$\partial_{t^I} K_i|_p = 0 = \partial_{t^I} \Gamma_{ij}^k|_p, \quad \forall\, i,j,k = 1,2\cdots \dim M\,, \tag{3.25}$$

where I is a multi-index and $\partial_{t^I} = \partial_{t^{i_1}} \partial_{t^{i_2}} \cdots \partial_{t^{i_m}}$, $m = |I| \geq 0$. Note that the first equation is only a condition on the choice of the Kähler potential which transforms under the rule

$K \mapsto K + f + \bar{f}$, where f is purely holomorphic.

These coordinates are studied elsewhere in different contexts, for example Kapranov (1999); Higashijima and Nitta (2001); Higashijima *et al.* (2002); Gerasimov and Shatashvili (2004). They are the normal coordinates for the Kähler geometry and can be constructed using the holomorphic exponential map as in Kapranov (1999).

Exponential map and Gaussian normal coordinates

Now we shall recall some basic facts from Riemannian geometry. Given a Riemannian manifold M with the metric G_{ij}, the Gaussian normal coordinates base at the point $p \in M$ could be obtained in two ways: either as a coordinate system centered around p such that $\mathrm{Sym}(\partial_I \Gamma^k_{ij})|_p = 0\,, |I| \geq 0$, where $\mathrm{Sym}(\partial_I \Gamma^k_{ij})$ means the symmetrization of $\partial_I \Gamma^k_{ij}$ with respect to the sub-indices $I \cup \{i, j\}$; or as linear coordinates on the tangent vector space $T_p M$ defined by the exponential map $\exp_p : T_p M \to M$. Using the second view point, we get the following description: suppose a point q in a small neighborhood of p on M is on the geodesic $\gamma(s) = \exp_p(sv)$, where $|v| = 1$, and s is the arc-length parameter. Assume $q = \exp_p(sv)$ for some s and fix a coordinate system $x = \{x^i\}$ near p on M, then the Gaussian normal coordinates $\xi = \{\xi^i\}$ of $q = \exp_p(sv)$ are related to the coordinates $x = \{x^i\}$ by using the equations for the geodesic:

$$x^i(\exp_p(sv)) = x^i(p) + s\xi^i - \sum_{N=2}^{\infty} \frac{1}{N!} \Gamma^i_N|_p s^N \xi^N\,, \tag{3.26}$$

where $\Gamma^i_N := \nabla_{N-\{i_1,i_2\}} \Gamma^i_{i_1 i_2}$ are computed in $x = \{x^i\}$ coordinates, and N is a multi-index as before.

Holomorphic exponential map and canonical coordinates on Kähler manifolds

Now assume M is a Kähler manifold whose Kähler potential is $K(z, \bar{z})$, where $z = \{z^i\}_{i=1,2\cdots \dim M}$ is a complex coordinate system. Suppose the base point p is taken to be $(z_*, \bar{z}_*)$. From the second equation in Eq. (3.25), one can solve for t , see Higashijima and Nitta (2001); Higashijima *et al.* (2002); Gerasimov and Shatashvili (2004), and get the the holomorphic analogue of Eq. (3.26):

$$t^i(z;z_*,\bar{z}_*) = K^{i\bar{j}}(z_*,\bar{z}_*;z_*,\bar{z}_*)(K_{\bar{j}}(z,\bar{z}_*;z_*,\bar{z}_*) - K_{\bar{j}}(z_*,\bar{z}_*;z_*,\bar{z}_*))\,, \tag{3.27}$$

where a function f defined near the base point $(z_*,\bar{z}_*)$ is denoted by $f(z,\bar{z};z_*,\bar{z}_*)$. The holomorphic function $f(z,\bar{z}_*;z_*,\bar{z}_*)$ means the degree zero part in the Laurent expansion of the function $f(z,\bar{z};z_*,\bar{z}_*)$ in $\bar{z}$ centered at $\bar{z}_*$, where one thinks of $(z,\bar{z})$ as independent coordinates. This will be explained in Remark 3.11 below using holomorphic exponential map.

The canonical coordinates can not be defined in terms of geodesics in the Riemannian geometry since the exponential map is in general not holomorphic. However, there is a nice construction of holomorphic exponential map which gives rise to these canonical coordinates, see Kapranov (1999). To define the holomorphic exponential map, we first regard the complex manifold M as a Riemannian manifold and thus get the map $\exp_p^{\mathbb{R}} : T_p^{\mathbb{R}}M \to M$. This also defines the Gaussian normal coordinates ξ. Thinking of $T_p^{\mathbb{R}}M$ as a complex vector space equipped with the complex structure induced by the complex structure on M, then in general the map $\exp_p^{\mathbb{R}} : (\xi,\bar{\xi}) \mapsto (z(\xi,\bar{\xi}),\bar{z}(\xi,\bar{\xi}))$ is not holomorphic. Now with the assumption that the metric $G_{i\bar{j}}(z,\bar{z})$ is analytic in $z,\bar{z}$, we can analytically continue the map $\exp_p^{\mathbb{R}}$ to the corresponding complexifications $T_p^{\mathbb{C}}M, M_{\mathbb{C}} = M \times \overline{M}$, where $\overline{M}$ is the complex manifold with opposite complex structure as M.

The coordinates on the complexifications $T_p^{\mathbb{C}}M, M_{\mathbb{C}} = M \times \overline{M}$ are given by (ξ,η) and (z,w) respectively, they are the analytic continuation of the coordinates $(\xi,\bar{\xi}),(z,\bar{z})$ from $T_p^{\mathbb{R}}M \hookrightarrow T_p^{\mathbb{C}}M, \Delta : M \hookrightarrow M_C = M \times \overline{M}$ respectively, where $\Delta : M \to M \times \overline{M}, p \mapsto (p,\bar{p})$ is the diagonal embedding. Here the underlying point of $\bar{p}$ is really the same as p, but we have used the barred notation to indicated that it is a point on the complex manifold $\overline{M}$.

Since the Christoffel symbols $\Gamma_{ij}^k(z,\bar{z})$ are analytic in $(z,\bar{z})$, we know that the map $\exp_p^{\mathbb{C}} : (\xi,\eta) \mapsto (z(\xi,\eta),w(\xi,\eta))$ is analytic, that is, holomorphic in (ξ,η). Moreover, the map $\exp_p^{\mathbb{C}}$ defines a local bi-holomorphism from a small neighborhood around the point 0 inside $T_p^{\mathbb{C}}M$ to a small neighborhood of the point $(p,\bar{p})$ inside $M_{\mathbb{C}}$. One claims that

$\exp_p^{\mathbb{C}}|_{T^{1,0}M}$ gives a holomorphic map $T_p^{1,0}M \to M$ and is a local bi-holomorphism from a small neighborhood of $0 \in T_pM$ to a small neighborhood of $p \in M$. To show that it maps $T_p^{1,0}M$ to M, we only need to show that $w \circ \exp_p^{\mathbb{C}}|_{T_p^{1,0}M} = w(\bar{p})$, that is, $w(\xi, \eta)|_{\eta=0} = w(\bar{p})$. Recall that $\bar{z}$ and thus w satisfies the equation for the geodesic equation

$$\frac{d^2}{ds^2}\bar{z}^k + \Gamma^{\bar{k}}_{\bar{i}\bar{j}}\frac{d\bar{z}^{\bar{i}}}{ds}\frac{d\bar{z}^{\bar{j}}}{ds} = 0,\ \frac{d\bar{z}^{\bar{k}}}{ds}(0) = \bar{\xi}^{\bar{k}} = 0,\ \bar{z}(0) = \bar{z}(\bar{p})\,.$$

It is easy to see that $w(s) = w(\bar{p})$ is one and thus the unique solution to the differential equation. Therefore, $w \circ \exp_p^{\mathbb{C}}(\xi, \eta = 0) = w(\bar{p})$ as desired. Since $z(\xi, \eta)$ is holomorphic in both ξ, η, we know $z(\xi, \eta = 0)$ is holomorphic in ξ. The same reasoning for the exponential map $\exp_p^{\mathbb{R}}$ shows that it is locally a bi-holomorphism.

Hence one gets a holomorphic exponential map $\exp_p^{\mathrm{hol}} := \exp_p^{\mathbb{C}}|_{T_p^{1,0}M} : T_p^{1,0}M \to M$. We now denote the coordinate ξ on $T_p^{1,0}M$ by t, this is then the canonical coordinates desired since the equation satisfied by t which is similar to Eq. (3.26) implies the second equation in Eq. (3.25). This can be checked by direct computations.

The exponential maps $\exp_p^{\mathbb{R}}$ and $\exp_p^{\mathrm{hol}}$ are contrasted as follows:

$$\begin{aligned}
\exp_p^{\mathbb{R}} &= \exp_p^{\mathbb{C}}|_{T_p^{\mathbb{R}}M} = \exp_p^{\mathbb{C}}|_{T_p^{1,0}M \oplus \overline{T_p^{1,0}M}}\,,\\
\exp_p^{\mathrm{hol}} &= \exp_p^{\mathbb{C}}|_{T_p^{1,0}M} = \exp_p^{\mathbb{C}}|_{j(T_p^{1,0}M) = T_p^{1,0}M \oplus \{0\}}\,.
\end{aligned}$$

where $T_p^{1,0}M \oplus \overline{T_p^{1,0}M}$ means the image of the map $T_p^{1,0}M \to T_p^{1,0}M \oplus T_p^{0,1}M$, $v \mapsto (v, v^*)$, where v^* is the complex conjugate of v; and $j(T_p^{1,0}M)$ is the image of the map $j : T_p^{1,0}M \mapsto T_p^{1,0}M \oplus T_p^{0,1}M$, $v \mapsto (v, 0)$.

Holomorphic limit

The holomorphic limit of any function $f(z, \bar{z})$ based at z_* is defined as follows. First one analytically continues the map f to a map defined on $M_{\mathbb{C}}$. Using the fact that $\exp_p^{\mathbb{C}}$ is a local diffeomorphism from $T_p^{\mathbb{C}}M$ to $M_{\mathbb{C}}$, we get $\hat{f} = f \circ \exp_p^{\mathbb{C}} : T_p^{\mathbb{C}}M \to \mathbb{C}$. The holomorphic limit of $f(z, \bar{z})$ is given by $\hat{f} \circ j : T_p^{1,0}M \to T_p^{\mathbb{C}}M \to \mathbb{C}$.

From now on, to maintain consistency with the notations used in the literature, we shall use $(z,\bar{z}),(t,\bar{t})$ for $(z,w),(\xi,\eta)$ when considering holomorphic limits, if no confusion arises. In the following, sometimes we shall drop the notations $z_*, \bar{z}_*$ for the base point if it is clear from the context.

Remark 3.11. In the canonical coordinates t on the Kähler manifold M, the holomorphic limit is described by $f\circ \exp^{\mathrm{hol}}_{z_*} = \hat{f}|_{j(T^{1,0}_{z_*}M)} : T^{1,0}_{z_*}M\times\{0\}\to\mathbb{C}, t\mapsto f\circ\exp^{\mathrm{hol}}_{z_*}(t)$. In terms of an arbitrary local coordinate system z on M, taking the holomorphic limit of the function $f(z,\bar{z})$ at the base point z_* is the same as keeping the degree zero part of the Laurent expansion of $f(z,\bar{z})$ with respect to $\bar{z}$, where the center of the Laurent expansion is $\bar{z}_*$. That is, it is the evaluation map $ev_{\bar{z}_*}: f(\bullet,\bullet)\mapsto f(\bullet,\bar{z}_*)$. This is the limit that is used in the study of topological string theory in Bershadsky *et al.* (1993, 1994).

One thing that needs to be taken extra care of is the holomorphic limit of $\det G$ appearing in computing the topological string partition functions. One has $G_{z^i\bar{z}^{\bar{j}}} = G_{t^a\bar{t}^{\bar{b}}}\frac{\partial t^a}{\partial z^i}\frac{\partial \bar{t}^{\bar{b}}}{\partial \bar{z}^{\bar{j}}}$, $i,\bar{j},a,\bar{b} = 1,2,\cdots\dim M$ and $\log\det G_{z^i\bar{z}^{\bar{j}}} = \log\det G_{t^a\bar{t}^{\bar{b}}} + \log\det\frac{\partial t^a}{\partial z^i} + \log\det\frac{\partial \bar{t}^{\bar{b}}}{\partial \bar{z}^{\bar{j}}}$. Since only the holomorphic derivative of $\log\det G_{z^i\bar{z}^{\bar{j}}}$ will appear in the topological string partition functions (and also in the ring $\widehat{\mathcal{R}}$ we shall construct below), the purely anti-holomorphic term can be ignored. Moreover, from Eq. (3.25) one can see that $\log\det G_{t^a\bar{t}^{\bar{b}}}(t,\bar{t}) = \log\det G_{t^a\bar{t}^{\bar{b}}}(t_*,\bar{t}_*)$ is independent of t. Therefore, when computing $\log\det G_{z^i\bar{z}^{\bar{j}}}$ one can effectively exclude the purely anti-holomorphic term and the term $\log\det G_{t^a\bar{t}^{\bar{b}}}(t,\bar{t})$, then one only needs to take the holomorphic limit of the term $\log\det\frac{\partial t^a}{\partial z^i}$. This could also be seen from Eq. (3.27), which implies that

$$\frac{\partial t^i}{\partial z^k}(z,\bar{z}_*) = K^{i\bar{j}}(z_*,\bar{z}_*)K_{k\bar{j}}(z,\bar{z}_*)\,. \tag{3.28}$$

Therefore, in the coordinate system z, the holomorphic limit of the metric $G_{k\bar{j}}$, denoted by $\lim G_{k\bar{j}}$, is given by

$$\lim G_{k\bar{j}}(z,\bar{z}) = G_{k\bar{j}}(z,\bar{z}_*) = \frac{\partial t^i}{\partial z^k}(z)G_{i\bar{j}}(z_*,\bar{z}_*)\,. \tag{3.29}$$

Variation of the holomorphic exponential map and canonical coordinates

The holomorphic exponential map $\exp_p^{\mathrm{hol}}$ does not depend holomorphically on the base point z_*, see Kapranov (1999). The canonical coordinates thus also have non-holomorphic dependence, as we shall also see below in some examples. This is due to the fact that the space $T^{1,0}_{z_*}M$ changes non-holomorphically when z_* moves in M: that is, $\frac{\partial}{\partial \bar{z}_*}\pi_{J_{z_*}} \neq 0$, where $\pi_{J_{z_*}} = \frac{1}{2}(id - iJ_{z_*})$ is the projection from $T^{\mathbb{C}}_{z_*}M$ to $T^{1,0}_{z_*}M$. For a more precise discussion on this, see Kapranov (1999).

Take M to be the base $\mathcal{M}$ of the Calabi-Yau threefold family $\pi : \mathcal{X} \to \mathcal{M}$ and think of $T^{1,0}_{z_*}\mathcal{M}$ as a Lagrangian in $T^{\mathbb{C}}_{z_*}\mathcal{M}$, this then fits in the frame work of geometric quantization and is related to the base-point independence of the total free energy $\mathcal{Z}_{\mathrm{top}} = \sum_{g=0}^{\infty} \lambda^{2g-2}\mathcal{F}^{(g)}$ of the topological string theory for the family, as studied in Witten (1993). The background (base-point) independence of $\mathcal{Z}$ tells that it satisfies some wave-like equations on $\mathcal{M}$ arising from geometric quantization. These equations are shown in Witten (1993) to be equivalent to the master anomaly equations for $\mathcal{Z}$ in Bershadsky *et al.* (1994) which are identical to the set of holomorphic anomaly equations for the topological string partition functions $\{\mathcal{F}^{(g)}\}_{g=0}^{\infty}$.

3.3.2 Examples of canonical coordinates

In this section we shall compute the canonical coordinates for some Kähler manifolds.

Example 3.12 (Fubini-Study metric)**.** Consider the Fubini-Study metric defined on $\mathbb{P}^1$

$$\omega_{\mathrm{FS}} = \frac{i}{2}\frac{1}{(1+|z|^2)^2} dz \wedge d\bar{z}\,,$$

with Kähler potential $K = \ln(1+|z|^2)$. It follows then

$$K_z = \frac{\bar{z}}{(1+|z|^2)}, \quad K_{z\bar{z}} = \frac{1}{(1+|z|^2)^2}, \quad \partial_z^N K_{\bar{z}} = \frac{(-1)^{N+1}N!\bar{z}^{N-1}}{(1+|z|^2)^{N+1}}, \, N \geq 1\,.$$

At the point p represented by $z_* = 0$, we can see that $\partial_z^N K|_p = 0 = \partial_z^N K_{z\bar{z}}|_p, N \geq 1$. Hence z is the canonical coordinate based at $z_* = 0$. To find the canonical coordinate at a generic point p represented by z_*, we apply Eq. (3.27) and get

$$t(z; z_*, \bar{z}_*) = (1 + |z_*|^2)^2 \left(\frac{z}{(1 + z\bar{z}_*)} - \frac{z_*}{(1 + z_*\bar{z}_*)} \right) .$$

In particular, at $z_* = 0$, this coincides with z. The non-holomorphic dependence on the base point can be easily seen from this formula.

Example 3.13 (Poincare metric)**.** Consider the $\mathrm{SL}(2, \mathbb{Z})$ invariant metric

$$\omega = \frac{i}{2} K_{\tau\bar{\tau}} d\tau \wedge d\bar{\tau} = \frac{1}{y^2} dx \wedge dy$$

on the Poincare upper half plane $\mathcal{H}$, where $e^{-K} = \frac{\tau - \bar{\tau}}{i}$, $\tau = x + iy$. Straightforward computations show that

$$K_{\bar{\tau}} = \frac{1}{\tau - \bar{\tau}}, \; K_{\tau\bar{\tau}} = -\frac{1}{(\tau - \bar{\tau})^2} .$$

It follows that the canonical coordinate based at p given by τ_* is

$$t(\tau; \tau_*, \bar{\tau}_*) = -(\tau_* - \bar{\tau}_*)^2 \left(\frac{1}{\tau - \bar{\tau}_*} - \frac{1}{\tau_* - \bar{\tau}_*} \right)$$

In particular, if one takes the base point $\tau_* = i\infty$, then the canonical coordinate t coincides with the complex coordinate on $\mathcal{H}$ from the embedding $\mathcal{H} \hookrightarrow \mathbb{C}$.

Example 3.14 (Weil-Petersson metric for elliptic curve family)**.** Taking the elliptic curves parametrized by $\mathcal{H}$. As in the proof of Proposition 3.4, one takes the holomorphic top form $\Omega_\tau = dz_\tau$ on T_τ. Using the diffeomorphism from the fiber T_τ to the fiber T_{τ_*}

$$z_\tau = \frac{\tau - \bar{\tau}_*}{\tau_* - \bar{\tau}_*} z_{\tau_*} + \frac{\tau_* - \tau}{\tau_* - \bar{\tau}_*} \bar{z}_{\tau_*} ,$$

one can compute the Kähler potential for the Weil-Peterson metric from

$$e^{-K(\tau, \bar{\tau}; \tau_*, \bar{\tau}_*)} = i \int_{T_\tau} \Omega_\tau \wedge \overline{\Omega}_\tau = \frac{\tau - \bar{\tau}}{i} .$$

This is then the Poincare metric on the upper half plane considered in the above example.

Example 3.15. Suppose on the Kähler manifold M there exists complex coordinates $z = \{z^i\}$

and a holomorphic function $F(z)$, so that the Kähler form is given by

$$\omega = \frac{\sqrt{-1}}{2}\mathrm{Im}\tau_{ij}\, dz^i \wedge d\bar{z}^j = \sqrt{-1}\partial\bar{\partial}K\,,$$

where $K = \frac{1}{2}\mathrm{Im}(w_i\bar{z}^i), w_i(z) = \partial_{z^i}F(z)$, $\tau_{ij}(z) = \partial_{z^i}\partial_{z^j}F(z)$. The canonical coordinates are then given by

$$t^i(z;z_*,\bar{z}_*) = \frac{1}{\tau_{ij}(z_*) - \bar{\tau}_{ij}(\bar{z}_*)}(w_j(z;z_*,\bar{z}_*) - w_j(z_*;z_*,\bar{z}_*) - \bar{\tau}_{jk}(\bar{z}_*)(z^k - z^k_*))\,.$$

Manifolds satisfying these properties are studied in detail in Freed (1999).

3.3.3 Special Kähler metric on deformation spaces

Now we take M to be the base of the family $\pi : \mathcal{X} \to \mathcal{M}$ of CY 3-folds. Assume that $\dim \mathcal{M} = h(= h^{2,1}(X))$.

Fixing a section Ω of the the Hodge line bundle $\mathcal{L} \to \mathcal{M}$ and choosing a symplectic basis $\{A^I, B_J\}_{I,J=0,1,\cdots h}$ for $H_3(X,\mathbb{Z})$, then the periods are given by Eq. (10):

$$(\int_{A^0}\Omega, \int_{A^a}\Omega, \int_{B^a}\Omega, \int_{B_0}\Omega) = (X^0, X^a, \mathcal{F}_a, \mathcal{F}_0) = X^0(1, t^a, F_{t^a}, 2F - t^a F_{t^a})\,,$$

where $a = 1, 2, \cdots h$ and $\mathcal{F}(X^I)$ is a homogeneous holomorphic function of X of degree 2, see Strominger (1990); Bershadsky *et al.* (1994). Here the function F is defined by $(X^0)^{-2}\mathcal{F}$ and the sub-indices mean derivatives with respect to corresponding coordinates. Using the fact that the prepotential $F(t)$ has the form $F(t) = \frac{\kappa_{abc}}{3!}t^a t^b t^c + Q(t) + \sum_d N_d e^{dt}$, where $Q(t)$ is a quadratic polynomial of $t = \{t^a\}$, it can be shown that the special coordinates $\{t^a(z;z_*,\bar{z}_*)\}_{a=1}^h$ defined near the large complex structure limit $z_* = 0$ are the canonical coordinates based at z_* by checking that Eq. (3.25) is satisfied, see Bershadsky *et al.* (1994) for details. As discussed in Eq. (2.19) in Section 2.3.1, one has the following holomorphic limits:

$$\lim_{\mathrm{LCSL}} K_{z^i} = -\partial_{z^i}\log X^0, \quad \lim_{\mathrm{LCSL}} \Gamma^{z^k}_{z^i z^j} = \frac{\partial z^k}{\partial t^a}\frac{\partial}{\partial z^i}\frac{\partial t^a}{\partial z^j}\,. \tag{3.30}$$

In the remaining of this work we will only consider the holomorphic limit based at the large complex structure $z_* = 0$ which is given by $\bar{t} = \overline{i\infty}$, and simply denote this limit by lim without specifying the base point. This limit is interesting since it is in this particular limit that the topological string partition functions on a CY 3-fold X are identical (under the mirror map) to the generating functions of Gromov-Witten invariants of its mirror manifold $\check{X}$, as explained in Introduction.

3.3.4 Ring of Yamaguchi-Yau and the construction of the triple

In this section, we shall construct the ring $\widehat{\mathcal{R}}$. We shall review the construction of a ring in Yamaguchi and Yau (2004) for the quintic mirror family. The purpose is to reduce the number of generators for the algebra $\widetilde{\mathcal{R}}$ defined above and also find its non-holomorphic completion $\widehat{\mathcal{R}}$.

The construction of Yamaguchi-Yau says that the anti-holomorphic dependence of the normalized topological string partition functions $F^{(g)} = (X^0)^{2g-2}\mathcal{F}^{(g)}$ is encoded in the generators

$$\theta^i \log e^{-K}, i=1,2,3, \quad \theta \log \det G\,,$$

while the coefficients are polynomials of

$$\theta \log \alpha^3 C_{\alpha\alpha\alpha} = \theta \log \frac{\kappa}{\beta} = \frac{\alpha}{\beta}\,.$$

More precisely, according to the Picard-Fuchs equation Eq. (3.16) and the definition Eq. (3), one has

$$\mathcal{L}_{\mathrm{CY}}\, e^{-K} = \left(\theta^4 - \alpha \prod_{i=1}^{4}(\theta + c_i)\right) e^{-K} = 0\,, \tag{3.31}$$

where $c_i = i/5, i = 1,2,3,4$. This then implies that $\theta^4 e^{-K}$ is a polynomial of $\theta^i e^{-K}, i = 1,2,3$ and $\frac{\alpha}{\beta}$. The special geometry relation Eq. (4) implies that

$$\partial_\alpha \bar{\partial}_{\bar{\alpha}} \Gamma^{\alpha}_{\alpha\alpha} = \partial_\alpha G_{\alpha\bar{\alpha}} - \partial_\alpha (e^{2K} G^{\alpha\bar{\alpha}} G^{\alpha\bar{\alpha}} \overline{C}_{\bar{\alpha}\bar{\alpha}\bar{\alpha}})\,. \tag{3.32}$$

It follows then

$$\begin{aligned}&\bar{\partial}_{\bar{\alpha}}\{(\partial_\alpha \Gamma^{\alpha}_{\alpha\alpha} + (\Gamma^{\alpha}_{\alpha\alpha})^2 - 2\Gamma^{\alpha}_{\alpha\alpha}\partial_\alpha K\\ &-4\partial_\alpha K_\alpha + 2(\partial_\alpha K)^2 + (\partial_\alpha \log C_{\alpha\alpha\alpha})(2\partial_\alpha K - \Gamma^{\alpha}_{\alpha\alpha})\} = 0\,.\end{aligned}$$

Hence we know

$$\partial_\alpha \Gamma^\alpha_{\alpha\alpha} + (\Gamma^\alpha_{\alpha\alpha})^2 - 2\Gamma^\alpha_{\alpha\alpha}\partial_\alpha K$$
$$-4\partial_\alpha K_\alpha + 2(\partial_\alpha K)^2 + (\partial_\alpha \log C_{\alpha\alpha\alpha})(2\partial_\alpha K - \Gamma^\alpha_{\alpha\alpha}) = f_\alpha$$

for some holomorphic quantity f_α. Taking the holomorphic limit of the left hand side, according to Eq. (3.30), we get

$$\partial^2_\alpha \log \frac{\partial t}{\partial \alpha} + (\partial_\alpha \log \frac{\partial t}{\partial \alpha})^2 + 2\partial_\alpha \log \frac{\partial t}{\partial \alpha}\partial_\alpha \log X^0 + 4\partial_\alpha \partial_\alpha \log X^0$$
$$+2(\partial_\alpha \log X^0)^2 + (\partial_\alpha \log C_{\alpha\alpha\alpha})(-2\partial_\alpha\partial_\alpha \log X^0 - \partial_\alpha \log \frac{\partial t}{\partial \alpha}) = f_\alpha \,.$$

The holomorphic quantity was fixed in Lian and Yau (1996a); Yamaguchi and Yau (2004) (see also Hosono (2008)) to be $\frac{1-\frac{12}{5}\alpha}{\alpha^2\beta}$.

One can also replace the coordinate α in Eq. (3.32) by $x = \ln \alpha$ defined locally on the punctured deformation space, then we get

$$\theta^2 \log G_{x\bar{x}} + (\theta \log G_{x\bar{x}})^2 - 2\theta \log G_{x\bar{x}}\theta K$$
$$-4\theta^2 K + 2(\theta K)^2 + (\theta \log C_{xxx})(2\theta K - \theta \log G_{x\bar{x}}) = f_x \,, \tag{3.33}$$

where $\theta = \partial_x = \alpha\frac{\partial}{\partial\alpha}, C_{xxx} = \alpha^3 C_{\alpha\alpha\alpha} = \frac{\kappa}{\beta}, \theta \log C_{xxx} = \frac{\alpha}{\beta}$ and f_x is another holomorphic quantity. Now we take the holomorphic limit of the above identity and get

$$\theta^2 \log \theta t + (\theta \log \theta t)^2 + 2\theta \log \theta t \; \theta \log X^0$$
$$4\theta^2 \log X^0 + 2(\theta \log X^0)^2 + (\theta \log C_{xxx})(-2\theta \log X^0 - \theta \log \theta t) = f_x \,, \tag{3.34}$$

with

$$f_x = \frac{2}{5}\frac{\alpha}{\beta} \,.$$

Note that f_x is not a tensor. Therefore, as shown in Yamaguchi and Yau (2004), one gets the following Yamaguchi-Yau ring

$$\mathcal{R}_{YY} = \mathbb{C}[\theta^i \log e^{-K}, i = 1,2,3; \Gamma^x_{xx} = \theta \log G_{x\bar{x}}, \theta \log C_{xxx} = \frac{\alpha}{\beta}] \,. \tag{3.35}$$

Note that

$$\theta\theta\log C_{xxx}=\theta\frac{\alpha}{\beta}=\frac{\alpha}{\beta^2}=\theta\log C_{xxx}(\theta\log C_{xxx}+1)\,, \tag{3.36}$$

then the ring $\mathcal{R}_{YY}$ is closed under taking the derivative θ. The generators of this ring $(\mathcal{R}_{YY},\theta)$ are essentially $K_x, K_{xx}, K_{xx}, \Gamma^x_{xx}, \theta\log C_{xxx}$.

However, it is not convenient to directly interpret this as the analogue of the ring of almost-holomorphic modular forms. For this reason, we connect this ring $(\mathcal{R}_{YY},\theta)$ to $(\widetilde{\mathcal{R}},\partial_\tau)$.

Due to Eq. (3.34), and the relation between the derivatives θ and ∂_τ given by $\theta=\beta^{-1}(X^0\theta t)^{-2}\partial_\tau$ which follows from Proposition 3.8, we know that the set of generators for $\widetilde{\mathcal{R}}$ could be reduced to $\partial^i_\tau X^0, i=0,1,2,3; \partial^j_\tau\theta t, j=0,1; B^\pm, C^\pm$. Recall that $\mathcal{R}=\mathbb{C}[(X^0)^{\pm1},(\theta t)^{\pm1},B^{\pm1},C^{\pm1}]$, then one can see that

$$\widetilde{\mathcal{R}}\ \ =\mathbb{C}[\partial^i_\tau\log X^0, i=1,2,3;\partial^j_\tau\log\theta t, j=1;\alpha,\beta]\otimes\mathcal{R}\,.$$

Recall Eq. (2.17), we get the following

$$\begin{aligned}&\theta\log e^{-K(x,\bar{x})}=\theta\log(X^0\bar{X}^0e^{-\mathrm{K}(t,\bar{t})})=\theta\log X^0+\theta\log e^{-\mathrm{K}(t,\bar{t})}\\&\theta\log G_{x\bar{x}}=\theta\log(\theta t\ \bar{\theta}t\ G_{t\bar{t}})=\theta\log\theta t+\theta\log G_{t\bar{t}}\,.\end{aligned}$$

their holomorphic limits are

$$\lim\ \theta\log e^{-K(x,\bar{x})}=\theta\log X^0,\quad \lim\ \theta\log G_{x,\bar{x}}=\theta\log\theta t\,,$$

Therefore, the holomorphic limit of the ring $\mathcal{R}_{YY}$ is given by

$$\lim\mathcal{R}_{YY}=\mathbb{C}[\theta^i\log X^0, i=1,2,3;\theta\log\theta t,\frac{\alpha}{\beta}=\theta\log C_{xxx}] \tag{3.37}$$

That is, the generators $\theta^i\log X^0, i=1,2,3;\theta\log\theta t$ in $\widetilde{\mathcal{R}}_{0,0}$ are equivalent to the holomorphic limits of the non-holomorphic generators in $\mathcal{R}_{YY}$. It follows then that

$$\widetilde{\mathcal{R}}=\lim\mathcal{R}_{YY}\otimes\mathcal{R}\,. \tag{3.38}$$

This motivates us to define the non-holomorphic completion $\widehat{\mathcal{R}}$ of $\widetilde{\mathcal{R}}$ as

$$\widehat{\mathcal{R}} = \mathcal{R}_{YY} \otimes \mathcal{R}\,. \tag{3.39}$$

Moreover, $F^{(g)} \in \mathcal{R}_{YY} \subseteq \widehat{\mathcal{R}}_{0,0}$, where $\mathcal{R}_{YY}$ and $\widehat{\mathcal{R}}_{0,0}$ are only differed by the holomorphic generators of degree and weight zero.

3.3.5 Summary of results

In summary, in Section 3.2 we have constructed $(\widetilde{\mathcal{R}}, \partial_\tau)$ as a graded differential ring which is an analogue of the ring of quasi-modular forms. In this section we then used special Kähler geometry to refine the structure of the generators for the ring and to get

$$\begin{aligned}
\mathcal{R} &= \mathbb{C}[(X^0)^{\pm 1}, (\theta t)^{\pm 1}, B^{\pm 1}, C^{\pm 1}]\,,\\
\widetilde{\mathcal{R}} &= \mathcal{R} \otimes \mathbb{C}[\partial_\tau^i \log X^0, i = 1,2,3; \partial_\tau \log \theta t]\,,\\
\widehat{\mathcal{R}} &= \mathcal{R} \otimes \mathbb{C}[\partial_\tau^i \log e^{-K}, i = 1,2,3; \partial_\tau \log \det G_{x\bar{x}}]\,.
\end{aligned}$$

Recall the structure of the graded rings $(M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma))$ defined for $\pi_\Gamma : \mathcal{E}_\Gamma \to X_\Gamma$

$$\begin{aligned}
&\partial_\tau : M_*(\Gamma) \to \widetilde{M}_*(\Gamma)\,,\\
&\text{“modular completion"}: \quad \widetilde{M}_*(\Gamma) \to \widehat{M}_*(\Gamma) \subseteq \widetilde{M}_*(\Gamma)[Y], \qquad Y = \frac{1}{12}\frac{-3}{\pi \mathrm{Im}\tau}\,,\\
&\text{“constant term map"} \;\; Y \to 0: \; \widehat{M}_*(\Gamma) \to \widetilde{M}_*(\Gamma)\,,\\
&\partial_\tau : \widetilde{M}_k(\Gamma) \to \widetilde{M}_{k+2}(\Gamma)\,,\\
&\widehat{\partial_\tau} = \partial_\tau + kY : \widehat{M}_k(\Gamma) \to \widehat{M}_{k+2}(\Gamma)\,.
\end{aligned}$$

From Eq. (2.17), we know

$$\partial_\tau \log e^{-K(x,\bar{x})} = \partial_\tau \log(X^0 \bar{X}^0 e^{-\mathrm{K}(t,\bar{t})}) = \partial_\tau \log X^0 + \partial_\tau \log e^{-\mathrm{K}(t,\bar{t})} \tag{3.40}$$

$$\partial_\tau \log G_{x\bar{x}} = \partial_\tau \log(\theta t\; \bar{\theta} \bar{t}\; G_{t\bar{t}}) = \partial_\tau \log \theta t + \partial_\tau \log G_{t\bar{t}}\,. \tag{3.41}$$

Define $Y_1 = \partial_\tau \log G_{t\bar{t}}, Y_2 = -\partial_\tau \log e^{-\mathrm{K}(t,\bar{t})}$, then we have the following analogue between $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ defined for $\pi : \mathcal{X} \to \mathcal{M}$ and $(M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma))$ defined for $\pi_\Gamma : \mathcal{E}_\Gamma \to X_\Gamma$:

$$\begin{aligned}
&\partial_\tau : \mathcal{R} \to \widetilde{\mathcal{R}}\,,\\
&\text{“non-holomorphic completion"}: \quad \widetilde{\mathcal{R}} \to \widehat{\mathcal{R}} \subseteq \widetilde{\mathcal{R}}[Y_1, Y_2]\\
&\text{“holomorphic limit"} \;\; Y_1, Y_2 \to 0: \;\; \widehat{R} \to \widetilde{R}\,,\\
&D_\tau = \partial_\tau + k\partial_\tau \log \theta t + m(\partial_\tau \log X^0) : \widetilde{R}_{k,m}(\Gamma) \to \widetilde{R}_{k+2,m}(\Gamma)\,,\\
&\widehat{D_\tau} = \partial_\tau + k\partial_\tau \log G_{x\bar{x}} + m(-\partial_\tau \log e^{-K}) : \widehat{R}_{k,m} \to \widehat{R}_{k+2,m}\,.
\end{aligned}$$

where the operators $D_\tau, \widehat{D}_\tau$ come from the covariant derivative $\partial_x + k\Gamma^x_{xx} + mK_x$ on sections of $\mathrm{Sym}^{\otimes k} T\mathcal{M} \otimes \mathcal{L}^m$.

The above construction for $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ could also be formally applied to the elliptic curve families in Eq. (1.14), see Hosono (2008). The Weil-Petersson metric is determined from $e^{-K(\alpha,\bar{\alpha})} = i\omega_0\overline{\omega_0}(\tau - \bar{\tau}) = i\omega_0\overline{\omega_0}e^{-\mathrm{K}(\tau,\bar{\tau})}$. The quantities Y_1, Y_2 are now computed to be $\frac{-2}{12}\frac{-3}{\pi \mathrm{Im}\,\tau}$ and $\frac{-1}{12}\frac{-3}{\pi \mathrm{Im}\,\tau}$, respectively. The triple $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ coincides with the triple $(M_*(\Gamma), \widetilde{M}_*(\Gamma), \widehat{M}_*(\Gamma))$, as well as the maps among the members in the triple.

For the non-compact CY 3-fold families in Eq. (3.15), one has $X^0 = 1$ and $\theta t = A$. The rings $(\mathcal{R}, \widetilde{\mathcal{R}})$ coincide with $(M_*(\Gamma), \widetilde{M}_*(\Gamma))$, as mentioned earlier in this chapter. But the explicit forms for Y_1, Y_2 are not easy to compute in these cases[8].

It is easy to see that one should be able to apply the same construction for the quintic mirror family to construct triples $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ for other one-parameter CY 3-fold families whose Picard-Fuchs equation takes the form in Eq. (3.17) with $\sum_{i=1}^4 c_i = 2$. The only thing that needs to be checked is that the holomorphic quantity f_x in Eq. (3.34) is contained in $\mathbb{C}[B^{\pm 1}, C^{\pm 1}]$. In fact, for many CY families it is rational, see Lian and Yau (1996a); Hosono and Lian (1996); Yamaguchi and Yau (2004).[9] We shall not discuss the details in this work.

[8]This is because the Picard-Fuchs equation for a non-compact CY 3-fold family has only three periods, and the Kähler potential of the Weil-Petersson metric cannot be computed directly as for the compact cases. One needs to compactify Chiang *et al.* (1999) the non-compact CY 3-fold to a compact CY geometry, and then do computations there, after that one takes the decompactification limit of corresponding quantities carefully.

[9]The author thanks Professor Shinobu Hosono for email correspondences and telling him the references on

3.3.6 Special geometry polynomial rings and their applications

Most of the generators in $\widehat{\mathcal{R}}$ obtained from the elements in $\mathcal{R}_{YY}$ have weight zero. In Alim *et al.* (2013), a set of the non-holomorphic, positive weight generators for $\widehat{\mathcal{R}}$ are chosen so that no negative powers of the generators appear upon taking the derivative $\widehat{D}$. The particular form of the ring $\widehat{\mathcal{R}}$ is termed the special polynomial ring in Alim *et al.* (2013). For completeness, in the following we shall review the construction of the generators therein.

First notice that the set of generators in $\widehat{\mathcal{R}}$ could be chosen to be $X^0\hat{D}^i \log e^{-K}, i = 1, 2, 3$; $\theta t\ \hat{D} \log\det G_{x\bar{x}}$. This is equivalent to the set of generators S^{xx}, S^x, S, K_x in Eq. (20). The reason is as follows. From the integrated special geometry relation Eq. (7)

$$\Gamma^x_{xx} = 2K_x - C_{xxx}S^{xx} + s^x_{xx} \tag{3.42}$$

we know that up to multiplication and addition by K_x and holomorphic quantities, S^{xx} is essentially $\Gamma^x_{xx} = \theta \log\det G_{x\bar{x}}$. The first or last equation in Eq. (20) tells that S^x is essentially $\partial_x K_x$, and the second tells that S is $\partial_x^2 K_x$. Moreover, the derivatives of the generators in $\mathcal{R}_{YY}$ coincide with the covariant derivatives for the generators S^{xx}, S^x, S, K_x in Eq. (20).

Now a nice set of generators for the special geometry polynomial ring $\widehat{\mathcal{R}}$ can be chosen as follows. First one makes the following change of generators in Eq. (22) following Alim and Länge (2007)

$$\tilde{S}^{tt} = S^{tt}, \quad \tilde{S}^t = S^t - S^{tt}K_t, \quad \tilde{S} = S - S^tK_t + \frac{1}{2}S^{tt}K_tK_t, \quad \tilde{K}_t = K_t\,.$$

Then as in Eq. (2.10) one defines $\tau = \frac{1}{2\pi i}\kappa^{-1}\partial_t F_t$ which gives $\frac{\partial\tau}{\partial t} = \frac{1}{2\pi i}\kappa^{-1}C_{ttt}$. After that one forms the following quantities on the deformation space $\mathcal{M}$:

$$\begin{aligned} &K_0 = \kappa C_{ttt}^{-1}\,(\theta t)^{-3}\,, \quad G_1 = \theta t\,, \qquad K_2 = \kappa C_{ttt}^{-1}\tilde{K}_t\,, \\ &T_2 = \tilde{S}^{tt}\,, \qquad\qquad\quad T_4 = C_{ttt}^{-1}\tilde{S}^t\,, \quad T_6 = C_{ttt}^{-2}\tilde{S}\,, \end{aligned}$$

where the propagators $\tilde{S}^{tt}, \tilde{S}^t, \tilde{S}$ are normalized by suitable powers of X^0 so that they are

this.

sections of $\mathcal{L}^0$. That is, they have degree zero. The weights of these generators are the sub-indices they carry. It follows that the derivatives of the generators of $\widehat{\mathcal{R}}$ given in Eq. (20) now become, see Alim *et al.* (2013),

$$\begin{aligned}
\partial_\tau K_0 &= -2K_0\,K_2 - K_0^2\,G_1^2\,(\tilde{h}^{\alpha}_{\alpha\alpha\alpha} + 3(s^{\alpha}_{\alpha\alpha}+1))\,,\\
\partial_\tau G_1 &= 2G_1\,K_2 - \kappa G_1\,T_2 + K_0 G_1^3(s^{\alpha}_{\alpha\alpha}+1)\,,\\
\partial_\tau K_2 &= 3K_2^2 - 3\kappa K_2\,T_2 - \kappa^2 T_4 + K_0^2\,G_1^4 k_{\alpha\alpha} - K_0\,G_1^2\,K_2\,\tilde{h}^{\alpha}_{\alpha\alpha\alpha}\,,\\
\partial_\tau T_2 &= 2K_2\,T_2 - \kappa T_2^2 + 2\kappa T_4 + \kappa^{-1}K_0^2 G_1^4 \tilde{h}^{\alpha}_{\alpha\alpha}\,,\\
\partial_\tau T_4 &= 4K_2 T_4 - 3\kappa T_2\,T_4 + 2\kappa T_6 - K_0\,G_1^2\,T_4 \tilde{h}^{\alpha}_{\alpha\alpha\alpha} - \kappa^{-1}K_0^2\,G_1^4\,T_2 k_{\alpha\alpha} + \kappa^{-2}K_0^3\,G_1^6 \tilde{h}_{\alpha\alpha}\,,\\
\partial_\tau T_6 &= 6K_2\,T_6 - 6\kappa T_2\,T_6 + \frac{\kappa}{2}T_4^2 - \kappa^{-1}K_0^2\,G_1^4\,T_4\,k_{\alpha\alpha} + \kappa^{-3}K_0^4\,G_1^8 \tilde{h}_\alpha - 2\,K_0\,G_1^2\,T_6 \tilde{h}^{\alpha}_{\alpha\alpha\alpha}\,,
\end{aligned} \tag{3.43}$$

where $\partial_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}$. The quantities $\tilde{h}^{\alpha}_{\alpha\alpha\alpha}, s^{\alpha}_{\alpha\alpha}, k_{\alpha\alpha}, \tilde{h}^{\alpha}_{\alpha\alpha\alpha}, \tilde{h}^{\alpha}_{\alpha\alpha}, \tilde{h}_{\alpha\alpha}, \tilde{h}_\alpha$ are some holomorphic quantities. It turns out that they are polynomials of an additional generator $C_0 = \theta \log C_{xxx} = \frac{\alpha}{\beta}$ with

$$\partial_\tau C_0 = C_0(C_0+1)G_1^2\,. \tag{3.44}$$

These explicit polynomials for the quintic mirror family case could be found in Alim *et al.* (2013) and are omitted here.

The generators can be used to simplify the holomorphic anomaly equations as follows. As mentioned earlier in the previous section, one has $F^{(g)} := (X^0)^{2g-2}\mathcal{F}^{(g)} \in \mathcal{R}_{YY} \subseteq \widehat{\mathcal{R}}_{0,0}$. The holomorphic anomaly equations then become, see Alim *et al.* (2013),

$$\begin{aligned}
&\frac{\partial F^{(g)}}{\partial T_2} - \frac{1}{\kappa}\frac{\partial F^{(g)}}{\partial T_4}\,K_2 + \frac{1}{\kappa^2}\frac{\partial F^{(g)}}{\partial T_6}K_2^2 = \frac{1}{2}\sum_{r=1}^{g-1}\partial_t F^{(g-r)}\,\partial_t\,F^{(r)} + \frac{1}{2}\partial_t^2 F^{(g-1)}\,,\\
&\frac{\partial F^{(g)}}{\partial K_2} = 0\,,
\end{aligned}$$

where $\partial_t = (X^0)^{-2}(C_0+1)(\theta t)^{-3}\partial_\tau$.

Example 3.16. Consider the local $\mathbb{P}^2$ example. As mentioned earlier in Section 3.2, we have that $\mathcal{M} \cong X_0(3), \widetilde{\mathcal{R}} \cong \widetilde{M}_*(\Gamma_0(3)) = \mathbb{C}[A^{\pm1}, B^{\pm1}, C^{\pm1}, E]$. In this case one can consistently choose the generators so that $T_4 = T_6 = K_2 = 0, T_2 = \frac{\hat{E}}{2}$ with $\partial_t = \kappa^{-1}C_{ttt}\partial_\tau = B^{-3}\partial_\tau$. Then the holomorphic anomaly equations simplify greatly. In particular, the equation for the holomorphic limit of $F^{(g)} = (X^0)^{2g-2}\mathcal{F}^{(g)}$ at the large complex structure limit, denoted by $F_g \in \widetilde{\mathcal{R}}_{0,0} \subseteq \mathbb{C}[A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, E]$, becomes the one in Section 2.4

$$\partial_E F_g = \frac{1}{4B^6}\left(\sum_{r=1}^{g-1} \partial_\tau F_{g-r} \partial_\tau F_r - \frac{E - A^2}{2} \partial_\tau F_{g-1} + \partial_\tau \partial_\tau F_{g-1}\right).$$

It would be interesting to apply the transformation $\alpha \mapsto \beta = 1 - \alpha$ according to the rules mentioned in Section 3.1 to the quintic mirror family case and see whether it would give a duality on topological string partition functions.

3.4 Conclusions and discussions

We constructed the graded rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ on the deformation space $\mathcal{M}$ from the periods of the Picard-Fuchs equation and special Kähler geometry on the deformation space. A parallelism between these rings and the rings $M_*(\Gamma), \widetilde{M}(\Gamma), \widehat{M}(\Gamma)$ was made: the way they were constructed; the relations among the members of the triple of rings. We further showed that in some special cases the rings $(\mathcal{R}, \widetilde{\mathcal{R}})$ are equivalent to the rings of modular quantities $(M_*(\Gamma), \widetilde{M}(\Gamma))$. These give some evidences that indeed the graded rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$ are analogues of the rings of modular objects $M_*(\Gamma), \widetilde{M}(\Gamma), \widehat{M}(\Gamma)$. These relations, as well as the non-holomorphic and holomorphic generators, are summarized in Figure 5 in Introduction and the following table.

Table 3.1: *Generators of the ring for compact and non-compact CY 3-fold geometries, where for non-compact geometries special choices for ambiguities are made.*

	compact geometries	non-compact geometries	Local $\mathbb{P}^2$ and local $\mathrm{dP}_n, n = 5,6,7,8$
non-holomorphic	$S^{xx} \sim \Gamma^x_{xx}$	$S^{xx} \sim \Gamma^x_{xx}$	$S^{xx} = \frac{1}{2}\frac{\hat{E}(\tau)}{A^2(\tau)}$
	$S^x \sim K_{xx}$	$\lim_{\mathrm{LCSL}} S^x = 0$	$\lim_{\mathrm{LCSL}} S^x = 0$
	$S \sim K_{xxx}$	$\lim_{\mathrm{LCSL}} S = 0$	$\lim_{\mathrm{LCSL}} S = 0$
	K_x	$\lim_{\mathrm{LCSL}} K_x = 0$	$\lim_{\mathrm{LCSL}} K_x = 0$
holomorphic	$A' = X^0$	$A' = X^0 = 1$	$A' = X^0 = 1$
	$A'' = \theta t, B, C$	$A'' = \theta t, B, C$	$A'' = A(\tau), B(\tau), C(\tau)$

We also discussed some of their applications in solving the holomorphic anomaly equations.

In the above construction of the triple of graded rings $(\mathcal{R}, \widetilde{\mathcal{R}}, \widehat{\mathcal{R}})$, the parameter $\tau = \frac{1}{2\pi i}\kappa^{-1}F_{tt}$ defined in Eq. (2.10) on the deformation space $\mathcal{M}$ was introduced to match the known modularity in the non-compact examples, see Aganagic *et al.* (2008). There are a number of interesting questions raised in Alim *et al.* (2013) about this quantity τ which we would like to address here and wish to pursue in the future.

Variation of Hodge structures

For the family $\pi : \mathcal{X} \to \mathcal{M}$ of non-compact CY 3-folds discussed in Chapter 2, the parameter τ is exactly the transcendental modulus for elliptic curve $\mathcal{E}_\alpha$ sitting inside the CY 3-fold $\mathcal{X}_\alpha$. It is the normalized period for the elliptic curve and lies on the upper half plane. This results from the fact that the vector space of the periods $(1, t, F_t)$ of $\mathcal{X}_\alpha$ is closed under the monodromy, and upon taking derivatives these periods become $(0, \omega_0, \omega_1)$, where the latter two are the two periods of $\mathcal{E}_\alpha$. In other words, while the three periods $(1, t, F_t)$ characterizes the variation of complex structure of the CY 3-fold, the quantities $(\theta t, \theta F_t)$ characterizes the variation of complex structure of the elliptic curve sitting inside it.

However, for a general one-parameter compact CY 3-fold family, e.g., the quintic mirror family, the vector space of periods $(X^0, X^0 t, X^0 F_t)$ is not invariant under the monodromy group. It is not clear what the geometric meaning or Hodge-theoretic meaning of $\tau = \frac{1}{2\pi i}\kappa^{-1}F_{tt}$ is.

Enumerative content of τ [10]

For the particular non-compact geometries in Eq. (2.7), the F_gs solved in e.g., Alim *et al.* (2013) from the holomorphic anomaly equations are explicit quasi-modular functions in τ (see also Aganagic *et al.* (2008) for related work). Whether the q_τ expansions of the

[10]The author thanks Murad Alim, Yaim Cooper and Shing-Tung Yau for discussions on this.

topological string partition functions have any enumerative content and how the q_t and q_τ expansions are related beg an explanation.

Recall that the mirror symmetry conjecture predicts for the CY 3-fold family $\pi : \mathcal{X} \to \mathcal{M}$, the holomorphic limit $F_g = \lim (X^0)^{2g-2}\mathcal{F}^{(g)}$ at the large complex structure limit is identical to the generating function of genus g Gromov-Witten invariants of $\check{X}$, that is, because of the interpretation of Eq. (2), we have

$$F_g(t) = \sum_{d=0}^{\infty} N_{g,d}^{\mathrm{GW}} q_t^d , \quad q_t = e^t \,, \tag{3.45}$$

where $N_{g,d}^{GW}$ denotes the genus g degree d Gromov-Witten invariants of the mirror manifold $\check{X}$. In particular, the prepotential $F(t)$ is given by

$$F(t) = \frac{\kappa}{3!} t^3 + \sum_{d=1}^{\infty} N_{g=0,d}^{\mathrm{GW}} q_t^d \,,$$

then $\tau = \frac{1}{2\pi i}\kappa^{-1} F_{tt}$ is the function determined from

$$2\pi i \tau = t + \kappa^{-1} \sum_{d=1}^{\infty} N_{g,d}^{\mathrm{GW}} d^2 q_t^d ,\, q_t = e^t \,. \tag{3.46}$$

As discussed in Section 2.5, this implies that

$$q_\tau = \exp 2\pi i \tau = (-q_t)(1 + \mathcal{O}(q_t)) \,. \tag{3.47}$$

It is natural to expect that there should be an enumerative problem associated to τ in the sense

$$F_g(\tau) = \sum_{d=0}^{\infty} N_{g,d}^{\mathrm{hyp}} q_\tau^d \,, \tag{3.48}$$

where like the Gromov-Witten invariants $N_{g,d}^{\mathrm{GW}}$, the numbers $N_{g,d}^{\mathrm{hyp}}$ may hypothetically count certain kind of invariants. Comparing the Eq. (3.48) with Eq. (3.45), we can then find the "multiple-cover formula" relating $N_{g,d}^{\mathrm{GW}}$ and $N_{g,d}^{\mathrm{hyp}}$ according to Eq. (3.47).

We don't have answers to any of these questions, and shall only display some examples below.

Example 3.17 (Resolved conifold)**.** Consider the resolved conifold which is the total space of

$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ and is a non-compact CY 3-fold. The Picard-Fuchs operator of the mirror CY family is given by, see e.g., Forbes and Jinzenji (2005) and references therein,

$$\mathcal{L}_{\mathrm{CY}} = \theta(\frac{\alpha}{1-\alpha})^{-1}\theta^2 . \tag{3.49}$$

Near the large complex structure limit given by $\alpha = 0$, a basis of the periods could be chosen to be

$$X^0 = 1, \quad t = \ln(-\alpha), \quad F_t \sim (\ln(-\alpha))^2 + \mathcal{O}(\alpha^0) . \tag{3.50}$$

Therefore near $\alpha = 0$, one has $t = \ln \alpha$ and thus $q_t = \alpha$. Moreover, the genus zero Gromov-Witten invariants are, see Aspinwall and Morrison (1993); Manin (1995); Voisin (1996),

$$N^{\mathrm{GW}}_{0,d} = \frac{1}{d^3} , \tag{3.51}$$

and the prepotential is

$$F(t) = \frac{\kappa}{3!}t^3 + \sum_{d=1}^{\infty} N^{\mathrm{GW}}_{0,d} q_t^d = \frac{\kappa}{3!}t^3 + \sum_{d=1}^{\infty} \frac{1}{d^3} q_t^d = \frac{\kappa}{3!}t^3 + \mathrm{Li}_3(q_t) . \tag{3.52}$$

This implies in particular that

$$C_{ttt} = \kappa + \sum_{d=1}^{\infty} q_t^d = \kappa + \frac{q_t}{1-q_t} . \tag{3.53}$$

The function τ then satisfies

$$2\pi i \tau = \kappa^{-1} F_{tt} = t + \kappa^{-1} \sum_{d=0}^{\infty} \frac{1}{d} q_t^d = t - \kappa^{-1} \ln(1-q_t) . \tag{3.54}$$

Note that κ can not be determined by studying the periods and is ambiguous. Consideration in physics in Vafa (2001) tells that a natural choice is $\kappa = 1$. In the following, we shall take this choice.

Remark 3.18. From Eq. (3.54) one can see that τ is itself the generating function of the sequence of numbers $\frac{1}{d} = d^2 N^{\mathrm{GW}}_{0,d}, d = 1, 2, \cdots$. These numbers appear in the study of the stable-quotient invariants defined in Marian *et al.* (2011) with

$$d^2 N^{\mathrm{GW}}_{0,d} = \int_{[Q_{0,2}(\mathbb{P}^1,d)]^{vir}} e(Ob) \cup ev_1^* H \cup ev_2^* H , \tag{3.55}$$

where Ob is the obstruction bundle in the construction of stable-quotient invariants, and the two insertions which give rise to $ev_1^*H \cup ev_2^*H$ are required for the stability in genus 0.

For higher genus partition functions, it is well known that, see Faber and Pandharipande (1999),

$$N_{g,d}^{\mathrm{GW}} = d^{2g-3} N_{g,1}^{\mathrm{GW}} = \frac{|B_{2g}|}{2g(2g-2)!} d^{2g-3}, \tag{3.56}$$

$$F_g = \frac{|B_{2g}|}{2g(2g-2)!} \mathrm{Li}_{3-2g}(q_t) \quad \text{in particular,} \quad F_1 = -\frac{1}{12} \log(1-q_t)\,. \tag{3.57}$$

To extract the numbers $N_{g,d}^{\mathrm{hyp}}$ associated to τ, we make use of Eq. (3.54) which gives rise to

$$2\pi i \tau = t - \ln(1-q_t), \quad q_\tau = \frac{q_t}{1-q_t}, \quad q_t = \frac{q_\tau}{1+q_\tau}\,.$$

Now from Eq. (3.53), one gets

$$C_{ttt} = \kappa t + \frac{q_t}{1-q_t} = \kappa \ln \frac{q_\tau}{1+q_\tau} + q_\tau = 2\pi i \kappa \tau + q_\tau - \kappa \sum_{k=1}^{\infty} (-1)^k q_\tau^k\,. \tag{3.58}$$

It follows that

$$N_{0,d}^{\mathrm{hyp}} = \kappa, \quad d=0, \quad N_{0,d}^{\mathrm{hyp}} = 1+\kappa, \quad d=1, \quad N_{0,d}^{\mathrm{hyp}} = (-1)^{d+1}\kappa, \quad d \geq 2\,.$$

For the generating function $\partial_t F_1$, we get

$$\partial_t F_1 = \frac{1}{12} \frac{q_t}{1-q_t} = \frac{1}{12} q_\tau\,. \tag{3.59}$$

This then tells that

$$N_{1,d}^{\mathrm{hyp}} = \frac{1}{12},\, d=1, \quad N_{1,d}^{\mathrm{hyp}} = 0,\, d \geq 2\,.$$

For higher genus partition functions, we have

$$\sum_{d=1}^{\infty} N_{g,d}^{\mathrm{hyp}} q_\tau^d = \frac{|B_{2g}|}{2g(2g-2)!} \mathrm{Li}_{3-2g}(q_\tau(1+q_\tau)^{-1}) = \frac{|B_{2g}|}{2g(2g-2)!} \theta_{q_t}^{2g-3} q_\tau\,. \tag{3.60}$$

Since $\theta_{q_t} := q_t \frac{\partial}{\partial q_t} = (1+q_\tau)\theta_{q_\tau}$, one can then find $N_{g,d}^{\mathrm{hyp}}$ by direct computations. For any

$g \geq 2$, the first few invariants with $d = 1, 2, 3 \cdots$ are listed as follows:

$$N_{g,d}^{\text{hyp}}: \qquad 1,\ -2 + 4^{2-g},\ 6 - 3 * 2^{5-2g} + 2 * 9^{2-g}\,,$$
$$-24 + 3 * 2^{9-4g} - 8 * 3^{5-2g} + 9 * 4^{3-g},$$
$$120\left(1 - 2^{8-4g} - 2^{5-2g} + 5^{3-2g} + 2 * 9^{2-g}\right) \cdots$$

Chapter 4

Conclusions and future directions

This thesis studied the arithmetic aspects of the moduli spaces of some CY 3-folds and some of their applications.

In this first part of the thesis, we considered the CY 3-fold families $K_{\mathbb{P}^2}, K_{\mathrm{dP}_n}, n = 5, 6, 7, 8$ whose mirror curve families are the elliptic curve families of E_n type. We then used the roles of the moduli spaces of the mirror CY 3-folds as the bases of the corresponding elliptic curve families and concluded that they are modular curves. Under the identification, the singular points on the moduli space of CY 3-folds become the cusps and elliptic points. Moreover, the Fricke involution exchanges the two cusp classes and fixes the rest of branch point on the modular curve. From the perspective of CY 3-folds, it exchanges the large complex structure limit with the conifold point. This then allows to analyze the boundary conditions in the BCOV holomorphic anomaly equation and use some elementary properties of the modular form theory to solve for the topological string partition functions genus by genus recursively. We then proved an integrality result of the Gromov-Witten invariants of the CY 3-folds under the assumption that mirror symmetry conjecture holds true in these cases. We also mentioned the physics interpretation of the Fricke involution as the Seiberg-Witten S-duality in Seiberg-Witten theory.

In the second part of the thesis, we constructed some differential rings from the Picard-Fuchs equation and the special Kähler geometry on the deformation spaces for some one-parameter CY 3-fold families. There rings enjoy very similar properties to those of the rings of quasi modular forms and almost-holomorphic modular forms defined for elliptic curve families parametrized by modular curves.

The main difficulties in understanding the arithmetic properties of the moduli spaces of more general CY 3-folds suited for our purposes lie in the following: 1. global Torelli type theorems for more general CY 3-folds are lacking (see e.g., Carlson *et al.* (2003); Voisin (2002, 2007); Debarre (2012) and references therein), which in some sense makes it difficult to transform the properties obtained via studying variation of (mixed) Hodge structures to global properties of the moduli spaces of CY 3-folds; 2. some analytic aspects of the geometric quantities constructed from the periods for the CY 3-folds and the connections of the Weil-Petersson metric on the special Kähler manifolds are not yet fully understood. For example, it is not clear what the zeros and poles of the periods are on the (image under the period map of) the moduli space.

We hope to study more general CY 3-folds than studied in this thesis (e.g., elliptic fibrations) and apply the ideas to solve for the corresponding topological string partition functions in the future. It would also be interesting to apply some physics ingredients to construct and study automorphic forms defined the moduli spaces, this might shed some light in understanding the geometry and arithmetic of the CY 3-folds themselves.

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Appendix A

Modular forms

We summarize the modular forms that appear in this thesis.

A.1 θ, η and Eisenstein

We define[1]

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2} e^{2\pi i (n+a)(z+b)} \,. \tag{A.1}$$

The following labels are given to the θ-functions:

$$\theta_1(z,\tau) = \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n z} \,, \tag{A.2}$$

$$\theta_2(z,\tau) = \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}n^2} e^{2\pi i n z} \,, \tag{A.3}$$

$$\theta_3(z,\tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n z} \,, \tag{A.4}$$

$$\theta_4(z,\tau) = \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n z} \,. \tag{A.5}$$

[1] In the literature the choice for q is a matter of convention, in this thesis we take $q = \exp 2\pi i\tau$.

We define the following θ-constants:

$$\theta_2(\tau) = \theta_2(0,\tau), \quad \theta_3(\tau) = \theta_3(0,\tau), \quad \theta_4(\tau) = \theta_2(0,\tau). \tag{A.6}$$

We also use the other version of the θ-constants following Zagier (2008)

$$\theta_F(\tau) = \theta_2(0,2\tau), \quad \theta(\tau) = \theta_3(0,2\tau), \quad \theta_M(\tau) = \theta_2(0,2\tau). \tag{A.7}$$

One can associate to any lattice Λ a theta function,

$$\Theta_\Lambda(\tau) = \sum_{x\in\Lambda} e^{2\pi i\cdot\frac{1}{2}||x||^2\tau}, \tag{A.8}$$

see Zagier (2008) and references therein for details on this.

The η-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}\sum_{n=1}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}}. \tag{A.9}$$

It transforms according to

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau). \tag{A.10}$$

The Eisenstein series are defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n}, \tag{A.11}$$

where B_k denotes the k-th Bernoulli number. E_k is a modular form of weight k for $k > 2$ and even. The discriminant form and the j–invariant are given by

$$\Delta(\tau) = \frac{1}{1728}\left(E_4(\tau)^3 - E_6(\tau)^2\right) = \eta(\tau)^{24}, \quad j(\tau) = 1728\frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}. \tag{A.12}$$

The following equalities are used a lot throughout our discussions

$$\partial_\tau \log\eta(\tau) = \frac{1}{24}E_2(\tau), \tag{A.13}$$

$$\partial_\tau \log\sqrt{\mathrm{Im}\,\tau}|\eta(\tau)|^2 = \frac{1}{24}\widehat{E_2}(\tau,\bar\tau). \tag{A.14}$$

where again by ∂_τ we mean $\frac{1}{2\pi i}\frac{\partial}{\partial\tau}$.

The following identities are useful

$$\begin{aligned}
\theta_3^4(\tau) &= \theta_2^4(\tau)+\theta_4^4(\tau)\,,\\
2\theta_3^2(2\tau) &= \theta_3^2(\tau)+\theta_4^2(\tau)\,,\\
\theta_4^2(2\tau) &= \theta_3(\tau)\theta_4(\tau)\,,\\
2\theta_2(2\tau)\theta_3(2\tau) &= \theta_2^2(\tau)\,,\\
2E_4(\tau) &= \theta_2^8(\tau)+\theta_3^8(\tau)+\theta_4^8(\tau)\,,\\
2E_6(\tau) &= (\theta_2^4(\tau)+\theta_4^4(\tau))(\theta_3^4(\tau)+\theta_4^4(\tau))(\theta_4^4(\tau)-\theta_2^4(\tau))\,,\\
256\Delta(\tau) &= \theta_2^8(\tau)\theta_3^8(\tau)\theta_4^8(\tau)\,.
\end{aligned}$$

A.2 Modular forms for $\Gamma_0(N), N = 1^*, 2, 3, 4.$

In the following we give the θ-expansions, see e.g., Mohri (2002); Zagier (2008); Maier (2009, 2011), for the generators of the ring of modular forms and their relations to the Eisenstein series for the groups $\Gamma_0(N)$, with $N = 1^*, 2, 3, 4$:

$$N=1^*: \quad A(\tau) = \Theta_{E_8}^{\frac{1}{4}}(\tau)\,. \tag{A.15}$$

$N=2$:

$$A(\tau) = \Theta_{D_4}^{\frac{1}{2}}(\tau) = \left(\frac{1}{4}(\theta_3^4(\tau)+\theta_4^4(\tau))^2\right)^{\frac{1}{4}} = \left((\theta_2^4(2\tau)+\theta_3^4(2\tau))^2\right)^{\frac{1}{4}}\,, \tag{A.16}$$

$$B(\tau) = \theta_3(\tau)\theta_4(\tau) = \theta_4^2(2\tau)\,, \tag{A.17}$$

$$C(\tau) = 2^{-\frac{1}{2}}\theta_2^2(\tau) = 2^{\frac{1}{2}}\theta_2(2\tau)\theta_3(2\tau)\,. \tag{A.18}$$

$N = 3:$

$$A(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2-mn+n^2} = \Theta_{A_2}(\tau) = \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau), \tag{A.19}$$

$$B(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} e^{2\pi i \frac{m-n}{3}} q^{m^2+mn+n^2} = 1 - 9\sum_{n=1}^{\infty} \chi_{-3}(n)\frac{n^2 q^n}{1-q^n}, \tag{A.20}$$

$$C(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+mn+n^2+m+n} = \frac{1}{2}(A(\frac{\tau}{3}) - A(\tau)), \tag{A.21}$$

$$= \vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\,\vartheta\begin{bmatrix}1/3\\0\end{bmatrix}(0,3\tau) + \vartheta\begin{bmatrix}2/3\\0\end{bmatrix}(0,\tau) + \vartheta\begin{bmatrix}1/6\\0\end{bmatrix}(0,3\tau). \tag{A.22}$$

$$N = 4: \quad A(\tau) = \Theta_{A_1\oplus A_1}(\tau) = \theta_3^2(2\tau), \quad B(\tau) = \theta_4^2(2\tau), \quad C(\tau) = \theta_2^2(2\tau). \tag{A.23}$$

The generators for the $N = 4$ case should be compared to the ring of even weight modular forms with respect to the principal congruence group $\Gamma(2)$, which is generated by any two of $\theta_3^4(\tau), \theta_2^4(\tau), \theta_4^4(\tau)$ since $\theta_3^4(\tau) = \theta_2^4(\tau) + \theta_4^4(\tau)$. Note that the group $\Gamma(2)$ is isomorphic to $\Gamma_0(4)$. There are also some nice relations among these generators and the ordinary Eisenstein series E_4, E_6:

$$N = 2: \quad B^4 + 4C^4 = E_4, \quad A^2(B^4 - 8C^4) = E_6. \tag{A.24}$$

$$N = 3: \quad A^4 + 8AC^3 = E_4, \quad A^6 - 20A^3C^3 - 8C^6 = E_6. \tag{A.25}$$

$$N = 4: \quad B^4 + 16B^2C^2 + 16C^4 = E_4, \quad B^6 - 30B^4C^2 - 96B^2C^4 - 64C^6 = E_6. \tag{A.26}$$

We also have

$$A^2(\tau) = \partial_\tau \log\frac{C(\tau)^r}{B(\tau)^r} = \frac{1}{N-1}(NE_2(N\tau) - E_2(\tau)). \tag{A.27}$$

Appendix B

Holomorphic anomaly equations for local $\mathbb{P}^2$ in terms of modular form theory language

Notation and convention:

- $\partial_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial \tau} = q\frac{\partial}{\partial q}, \quad q = \exp 2\pi i \tau,$
- $F^{(g)} = \lim_{\mathrm{LCSL}} \mathcal{F}^{(g)}, \quad F_g = B^{6g-6}F^{(g)},$
- Bernoulli numbers: $B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42...$

The ring of quasi-modular forms for $\Gamma_0(3)$ is

$$\widetilde{M}_*(\Gamma_0(3), \chi_{-3}) = \mathbb{C}[A, B^3, E] \cong \mathbb{C}[A, F = C^3 - B^3, E]\,.$$

The differential ring structure is

$$\begin{aligned}
\partial_\tau A &= \frac{1}{6}(EA + F)\,,\\
\partial_\tau F &= \frac{1}{2}(EF + A^5)\,,\\
\partial_\tau E &= \frac{1}{6}(E^2 - A^4)\,.
\end{aligned}$$

The Fricke involution $W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ acts by

$$A|_{W_N} = iA, F|_{W_N} = iF, E|_{W_N} := E\,,$$

where strictly speaking, one only has $\hat{E}|_{W_N} := \hat{E}$. But for computational purposes in which one only cares about holomorphic limits eventually, one could pretend as if $E|_{W_N} = E$, as explained in Chapter 3.

Holomorphic anomaly equations for $F^{(g)}$ and F_g are

$$\begin{aligned}
\partial_E F^{(g)} &= \frac{1}{4}\frac{1}{B^6}\left(\sum_{r=1}^{g-1}\partial_\tau F^{(r)}\partial_\tau F^{(g-r)} + \partial_\tau^2 F^{(g-1)} - \frac{1}{2}(E-A^2)\partial_\tau F^{(g-1)}\right)\,,\\
\partial_E F_g &= \frac{1}{4}\sum_{r=1}^{g-1}(\partial_\tau + (1-r))F_r\;(\partial_\tau + 1-(g-r))F_{g-r}\\
&\quad +\frac{1}{4}\left(\partial_\tau + (2-g)(E-A^2)\right)^2 F_{g-1} - \frac{1}{8}\left(\partial_\tau + (2-g)(E-A^2)\right) F_{g-1}\,.
\end{aligned}$$

The boundary conditions are

$$\begin{aligned}
F^{(g)} &= (-1)^g\frac{\chi}{2}\frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!} + \mathcal{O}(q^1)\,,\\
F^{(g)}|_{W_N} &= \frac{3^{g-1}B_{2g}}{2g(2g-2)(t_c^*)^{2g-2}} + \mathcal{O}(q^0)\,,\\
F_g|_{W_N} &= (-3)^{g-1}\frac{B_{2g}}{2g(2g-2)}\left(\frac{C^3}{t_c^*}\right)^{2g-2} + (C^3)^{2g-2}\mathcal{O}(q^0)\,,
\end{aligned}$$

where the precise value $\chi = \chi(K_{\mathbb{P}^2}) = 3$ is not very important in the whole computations, in particular it does not affect the Gromov-Witten invariants that we extract from mirror symmetry. The quantity t_c^* is defined by

$$\partial_\tau t_c^*(q) = C^3(\tau), \quad t_c(q)|_{q=0} = 0\,.$$

The q-series expansions of the related modular quantities are as follows:

$$
\begin{aligned}
A(\tau) &= \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau) = \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+mn+n^2} \\
&= 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + 12q^{19} + 12q^{21} + \cdots, \\
-F(\tau) &= B^3(\tau) - C^3(\tau) = \left(\frac{\eta(\tau)^3}{\eta(3\tau)}\right)^3 - \left(3\frac{\eta(3\tau)^3}{\eta(\tau)}\right)^3 \\
&= 1 - 36q - 54q^2 - 252q^3 - 468q^4 - 432q^5 \\
&\quad -702q^6 - 1800q^7 - 918q^8 - 2196q^9 - 2592q^{10} \\
&\quad -2160q^{11} - 3276q^{12} - 6120q^{13} - 2700q^{14} - 5616q^{15} \\
&\quad -7380q^{16} - 5184q^{17} - 6534q^{18} - 13032q^{19} - 5616q^{20} + \cdots, \\
C^3(\tau) &= \left(3\frac{\eta(3\tau)^3}{\eta(\tau)}\right)^3 = 27(q + 3q^2 + 9q^3 + 13q^4 + 24q^5 \\
&\quad +27q^6 + 50q^7 + 51q^8 + 81q^9 + 72q^{10} \\
&\quad +120q^{11} + 117q^{12} + 170q^{13} + 150q^{14} + 216q^{15} \\
&\quad +205q^{16} + 288q^{17} + 243q^{18} + 362q^{19} + 312q^{20} + \cdots), \\
E(\tau) &= \frac{3E_2(3\tau) + E_2(\tau)}{4} \\
&= 1 - 6q - 18q^2 - 42q^3 - 42q^4 - 36q^5 \\
&\quad -126q^6 - 48q^7 - 90q^8 - 150q^9 - 108q^{10} \\
&\quad -72q^{11} - 294q^{12} - 84q^{13} - 144q^{14} - 252q^{15} \\
&\quad -186q^{16} - 108q^{17} - 450q^{18} - 120q^{19} - 252q^{20} + \cdots, \\
t_c^*(q) &= 27(q + \frac{3}{2}q^2 + 3q^3 + \frac{13}{4}q^4 + \frac{24}{5}q^5 \\
&\quad +\frac{9}{2}q^6 + \frac{50}{7}q^7 + \frac{51}{8}q^8 + 9q^9 + \frac{36}{5}q^{10} \\
&\quad +\frac{120q^{11}}{11} + \frac{39}{4}q^{12} + \frac{170}{13}q^{13} + \frac{75}{7}q^{14} + \frac{72}{5}q^{15} \\
&\quad +\frac{205}{16}q^{16} + \frac{288}{17}q^{17} + \frac{27}{2}q^{18} + \frac{362}{19}q^{19} + \frac{78}{5}q^{20} + \cdots).
\end{aligned}
$$

The first few results for the topological string partition functions are shown as follows:

$$
\begin{aligned}
F^{(1)} &= -\frac{1}{2}\log\eta(q)\eta(q^3),\ \partial_\tau F^{(1)} = -\frac{1}{12}E\,, \\
F^{(2)} &= \frac{E(5E^2-9EA^2+6A^4)}{1728B^6} + \frac{-\frac{8}{5}A^6+\frac{2}{5}A^3B^3+\frac{-8-3\chi}{10}B^6}{1728B^6}\,, \\
F^{(3)} &= \frac{5359A^{12}-8864A^9B^3+4160A^6B^6-496A^3B^9+2(8-3\chi)B^{12}}{8709120B^{12}} \\
&+\frac{\left(-2532A^{10}+3444A^7B^3-1140A^4B^6+48AB^9\right)E}{1244160B^{12}} \\
&+\frac{\left(3516A^8-3708A^5B^3+732A^2B^6\right)E^2}{1244160B^{12}} \\
&+\frac{\left(-2645A^6+1900A^3B^3-120B^6\right)E^3}{1244160B^{12}} \\
&+\frac{\left(1200A^4-420AB^3\right)E^4}{1244160B^{12}} - \frac{25A^2E^5}{82944B^{12}} + \frac{5E^6}{82944B^{12}}.
\end{aligned}
$$

$$
\begin{aligned}
F_1 &= -\frac{1}{2}\log\eta(q)\eta(q^3), \\
\partial_\tau F_1 &= -\frac{1}{12}E\,, \\
17280F_2 &= 50E^3-90A^2E^2+60A^4E-14A^6-2A^3F\,, \\
8709120F_3 &= 525E^6-2625A^2E^5+(6930A^4+1470AF)E^4 \\
&-(12075A^6+6230A^3F+210F^2)E^3 \\
&+(12915A^8+10416A^5F+1281A^2F^2)E^2 \\
&-(7623A^{10}+8190A^7F+1869A^4F^2+42AF^3)E \\
&+1905A^{12}+2538A^9F+854A^6F^2+62A^3F^3\,.
\end{aligned}
$$

The gap conditions are checked as follows:

$$
\begin{aligned}
F^{(2)}|_{W_N} &= -\frac{1}{58320q^2}+\frac{1}{19440q}-\frac{1}{19440}+\frac{37q}{29160}-\frac{19q^2}{4860}+\frac{q^3}{240}-\frac{17q^4}{5832}+\frac{7q^5}{243}+\cdots\,, \\
\frac{3^{2-1}\cdot-\frac{1}{30}}{4\cdot 2}\Big(\frac{1}{(t_c^*)^2} &= -\frac{1}{58320q^2}+\frac{1}{19440q}-\frac{1}{77760}-\frac{7q}{58320}+\frac{241q^2}{1555200}+\frac{q^3}{518400}-\frac{14977q^4}{130636800}+\frac{227q^5}{21772800}+\cdots\,, \\
F_2|_{W_N} &= \frac{1}{80}+\frac{3q}{80}+\frac{3q^2}{20}-\frac{57q^3}{80}-\frac{21q^4}{10}-\frac{411q^5}{40}+\cdots\,, \\
(-3)^{1}\frac{-\frac{1}{30}}{4\cdot 2}\Big(\frac{C^3}{t_c^*}\Big)^2 &= \frac{1}{80}+\frac{3q}{80}+\frac{39q^2}{320}+\frac{21q^3}{160}+\frac{1617q^4}{6400}+\frac{453q^5}{6400}+\cdots
\end{aligned}
$$

$$F^{(3)}|_{W_N} = \frac{1}{59521392q^4} - \frac{1}{9920232q^3} + \frac{1}{5668704q^2} + \frac{19}{119042784q} - \frac{103}{132269760} - \frac{49q}{2834352} + \cdots,$$

$$\frac{3^{3-1}\frac{1}{42}}{6\cdot 4}\frac{1}{(t_c^*)^4} = \frac{1}{59521392q^4} - \frac{1}{9920232q^3} + \frac{1}{5668704q^2} + \frac{19}{119042784q} - \frac{529}{529079040} + \frac{43q}{39680928} + \cdots,$$

$$F_3|_{W_N} = \frac{1}{112} + \frac{3q}{56} + \frac{57q^2}{224} + \frac{159q^3}{224} + \frac{4233q^4}{2240} - \frac{2991q^5}{560} - \frac{450q^6}{7} + \cdots,$$

$$(-3)^2\frac{\frac{1}{42}}{6\cdot 4}\left(\frac{C^3}{t_c^*}\right)^4 = \frac{1}{112} + \frac{3q}{56} + \frac{57q^2}{224} + \frac{159q^3}{224} + \frac{15879q^4}{8960} + \frac{6747q^5}{2240} + \frac{70209q^6}{12544} + \cdots$$