



New Boundary Conditions for (AdS_3)

Citation

Compère, Geoffrey, Wei Song, and Andrew Strominger. 2013. New Boundary Conditions for (AdS_3) . *Journal of High Energy Physics* 2013, no. 5: 152.

Published Version

doi:10.1007/jhep05(2013)152

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New Boundary Conditions for AdS₃

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Abstract

New chiral boundary conditions are found for quantum gravity with matter on AdS₃. The associated asymptotic symmetry group is generated by a single right-moving $U(1)$ Kac-Moody-Virasoro algebra with $c_R = \frac{3\ell}{2G}$. The Kac-Moody zero mode generates global left-moving translations and equals, for a BTZ black hole, the sum of the total mass and spin. The level is positive about the global vacuum and negative in the black hole sector, corresponding to ergosphere formation. Realizations arising in Chern-Simons gravity and string theory are analyzed. The new boundary conditions are shown to naturally arise for warped AdS₃ in the limit that the warp parameter is taken to zero.

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1 Introduction

In a seminal paper [1] Brown and Henneaux found consistent boundary conditions for quantum gravity (plus matter) on AdS₃ and showed that they imply the semiclassical states form representations of left and right Virasoro algebras with $c_{L,R} = \frac{3\ell}{2G}$, where G is Newton's constant and ℓ is the AdS₃ radius. The Virasoro symmetries act infinitesimally on the coordinates t^\pm of the boundary cylinder as

$$\delta t^+ = \epsilon^+(t^+), \quad \delta t^- = \epsilon^-(t^-). \tag{1.1}$$

Importantly, Brown-Henneaux boundary conditions admit all BTZ black holes [2, 3]. Generalizations and modifications are analyzed in *e.g.* [4–10].

The analysis of [1] considers only parity-invariant boundary conditions. In this paper we drop this restriction and find a new set of non-chiral but self-consistent boundary conditions for AdS₃ quantum gravity (plus matter) admitting BTZ black holes. The asymptotic symmetry generators form a

single right-moving $c_R = \frac{3\ell}{2G}$ noncompact $U(1)$ Kac-Moody-Virasoro algebra and (1.1) is replaced by

$$\delta t^+ = \epsilon(t^+), \quad \delta t^- = \sigma(t^+). \quad (1.2)$$

Note that the diffeomorphisms of *both* t^- and t^+ depend only on t^+ . The zero mode $\delta t^- = \text{constant}$ of the Kac-Moody symmetry is a global left translation.

We further compute the (nonzero but normalization-dependent) Kac-Moody level. A salient feature is that it is positive around the global vacuum, but negative in the black hole sector. This negative level is related to fact that the vector fields generating the Kac-Moody symmetry are spacelike in the black hole sector. This can be thought of as a type of ergosphere and entails negative energy modes and superradiant-type instabilities of the black holes. These instabilities are a simple versions of the ones encountered in a variety of contexts including AdS_2 fragmentation [11] and Kerr superradiance. We hope that the current work will provide a simple useful context in which to study them.

The discovery of these new boundary conditions was sparked by a circle of recent investigations [12–25] involving warped AdS_3 spacetimes, near-horizon extreme Kerr, string compactifications and their potential dual warped CFT_{2S} - all of which have in common global $SL(2, \mathbb{R}) \times U(1)$ symmetries. Asymptotic boundary conditions have been discussed in these contexts [13–15, 18, 21, 22, 26–30] which contain one left Virasoro or one right Kac-Moody Virasoro but none with both a simultaneous left and right Virasoro. In contemplating this puzzle we realized that if consistent right Kac-Moody Virasoro boundary conditions exist for warped AdS_3 , they should persist in the limit that the warping is deformed to zero and ordinary AdS_3 reappears. We find this is indeed the case: we spell out the new AdS_3 boundary conditions, verify their consistency, compute the central extensions and work out several examples.

It is important to note that the semiclassical consistency of the chiral boundary conditions found here does not guarantee the existence of any non-trivial fully quantum theory obeying these boundary conditions. So far no clear example is known. Nevertheless it appears to us that the semiclassical self-consistency is highly non-trivial and motivates the search for such quantum theories. A potential warped stringy example is discussed in [24], and one may attempt to impose the new chiral boundary conditions on any string compactification to AdS_3 . Other candidates are supplied by the chiral AdS_3 Chern-Simon gravity solved semiclassically in section 3, and its

holographic Liouville dual analyzed in [31]. The quantum versions of these theories remains to be studied.

In a companion paper [31], we consider the problem from a dual purely two-dimensional perspective of (classical) Liouville gravity. Chiral gauge conditions for the two-dimensional metric are found for which the residual symmetry is generated by a single right-moving Virasoro-Kac-Moody, as opposed to the familiar two Virasoros found in conformal gauge. The action, equations of motion and Dirac brackets of the resulting nonlinear chiral theory of Liouville gravity are identical to those of the AdS₃ Chern-Simons gravity with new boundary conditions constructed in this paper. Hence two-dimensional chiral Liouville theory and AdS₃ Chern-Simons gravity with the new boundary conditions are equivalent.

The paper is organized as follows. In section 2 we specify the new boundary conditions, show that the charges are finite and integrable and derive the chiral Kac-Moody-Virasoro asymptotic symmetry algebra including the central extensions. In section 3 we construct the simplest example by applying the boundary conditions to the $SL(2, \mathbb{R})_R \times SL(2, \mathbb{R})_L$ Chern-Simons formulation of pure Einstein gravity on AdS₃. The application of Brown-Henneaux boundary conditions is well-known [32] to yield a single non-chiral Liouville scalar with central charges $c_{R,L} = \frac{3\ell}{2G}$. The new boundary conditions yield instead two right-moving chiral scalars with a single $c_R = \frac{3\ell}{2G}$ Virasoro-Kac-Moody, which together form a chiral version of Liouville gravity [31]. The Kac-Moody level is also computed as a function of the current normalization. In section 4 the realization of the new boundary conditions in weak-field perturbation theory is presented. We perturbatively construct the lowest weight representations for a scalar field coupled to gravity. The lowest weight state is a single scalar particle in the center of AdS₃. The descendants are arbitrary excited single scalar particle states surrounded by a gas with arbitrary numbers of right-moving boundary gravitons and right-moving boundary photons. In section 5 we discuss a compactification of string theory to warped AdS₃ which has been analyzed in several papers [21, 22, 24, 33]. Consistent boundary conditions are given for general warp parameter, and then shown to reduce precisely to the new chiral boundary conditions in the limit that the warp parameter is taken to zero.

2 Asymptotic symmetry group analysis

In this section we present the new boundary conditions and derive the asymptotic symmetry algebra.

2.1 New boundary conditions

Let $t^\pm = t \pm \phi$ where $\phi \sim \phi + 2\pi$. We impose the following asymptotically AdS₃ boundary conditions on the 3d metric

$$\begin{aligned}
 g_{rr} &= \frac{\ell^2}{r^2} + O(r^{-4}), \\
 g_{r\pm} &= O(r^{-3}), \\
 g_{+-} &= -\frac{\ell^2 r^2}{2} + O(r^0), \\
 g_{++} &= \partial_+ \bar{P}(t^+) \ell^2 r^2 + O(r^0), \\
 g_{--} &= 4G\ell\Delta + O(r^{-1}),
 \end{aligned} \tag{2.1}$$

where Δ is a fixed constant and $\partial_+ \bar{P}(t^+)$ is periodic. In Fefferman-Graham coordinates, any three-dimensional Einstein metric admits the expansion

$$ds^2 = \ell^2 \frac{dr^2}{r^2} + \ell^2 r^2 \left(g_{ab}^{(0)} + \frac{1}{r^2} g_{ab}^{(2)} + O(r^{-3}) \right) dx^a dx^b. \tag{2.2}$$

(2.1) then reduces to

$$\begin{aligned}
 g_{--}^{(0)} &= 0, & g_{++}^{(0)} &= \partial_+ \bar{P}(t^+), & g_{-+}^{(0)} &= g_{+-}^{(0)} = -\frac{1}{2}, \\
 g_{--}^{(2)} &= \frac{4G}{\ell} \Delta,
 \end{aligned} \tag{2.3}$$

with subleading terms unconstrained by the boundary conditions. This can be compared to the standard Brown-Henneaux boundary conditions [1] in Fefferman-Graham coordinates

$$g_{--}^{(0)} = g_{++}^{(0)} = 0, \quad g_{-+}^{(0)} = g_{+-}^{(0)} = -\frac{1}{2}, \tag{2.4}$$

with subleading terms unconstrained. The new boundary conditions differ in several respects from these. They are chiral since they do not treat left

and right movers symmetrically. The leading order part of the metric $g_{++}^{(0)}$ is allowed to fluctuate with right-movers while the subleading part of the metric $g_{--}^{(2)}$ is restricted to be a fixed constant. In both cases, the boundary metric $g_{ab}^{(0)}$ is restricted to be Ricci-flat, $R_{(0)} = 0$.

Note that an alternative version of the boundary conditions exists where Δ is allowed to be varied while $\bar{P}(t^+)$ is periodic. We discuss these alternative boundary conditions separately in Appendix B.

2.2 Variational principle

We consider gravitational theories whose metrics are asymptotically governed by the vacuum Einstein equation. After the addition of the standard Gibbons-Hawking term and counterterms, the variation of the action is given by

$$\delta S_0 = \frac{\ell}{16\pi G} \int d^2x \sqrt{-g_{(0)}} g_{(2)}^{ab} \delta g_{(0)ab} \quad (2.5)$$

where indices are raised with the boundary metric $g^{(0)ab}$. The total action is defined as S_0 complemented by a chiral boundary term,

$$S = S_0 + \frac{\Delta}{4\pi} \int dt^+ dt^- \sqrt{-g_{(0)}} g_{(0)}^{--}. \quad (2.6)$$

With the boundary conditions (2.1), the variation of the action (2.5) is given by

$$\delta S = -\frac{1}{2\pi} \int dt^+ dt^- \partial_+ \bar{P} \delta \Delta = 0. \quad (2.7)$$

2.3 Nonlinear solutions

For pure 3D Einstein gravity with no matter, the Fefferman-Graham expansion terminates after fourth order [34] :

$$ds^2 = \ell^2 \frac{dr^2}{r^2} + \ell^2 r^2 (g_{(0)ab} + \frac{g_{(2)ab}}{r^2} + \frac{g_{(4)ab}}{r^4}) dx^a dx^b \quad (2.8)$$

where $g_{(4)ab} = \frac{1}{4} g_{(2)ac} g^{(0)cd} g_{(2)db}$ and $g_{(2)ab}$ is constrained by the equations of motion. It is then a straightforward exercise to solve Einstein's equations with the boundary conditions (2.1). The general solution obeying the

boundary conditions can be written as

$$\begin{aligned}
ds^2 &= \frac{\ell^2}{r^2} dr^2 - \ell^2 r^2 dt^+ (dt^- - \partial_+ \bar{P}(t^+) dt^+) \\
&+ 4G\ell \left[\bar{L}(t^+) (dt^+)^2 + \Delta (dt^- - \partial_+ \bar{P}(t^+) dt^+)^2 \right] \\
&- \frac{16G^2 \Delta}{r^2} \bar{L}(t^+) dt^+ (dt^- - \partial_+ \bar{P}(t^+) dt^+).
\end{aligned} \tag{2.9}$$

In terms of Fefferman-Graham coefficients, one has

$$\begin{aligned}
g_{--}^{(0)} &= 0, & g_{+-}^{(0)} &= -\frac{1}{2}, & g_{++}^{(0)} &= \partial_+ \bar{P}, \\
g_{--}^{(2)} &= \frac{\Delta}{k} & g_{+-}^{(2)} &= -\frac{\Delta}{k} \partial_+ \bar{P}, & g_{++}^{(2)} &= \frac{1}{k} (\bar{L}(t^+) + \Delta (\partial_+ \bar{P})^2),
\end{aligned} \tag{2.10}$$

where we defined for convenience

$$k \equiv \frac{\ell}{4G}. \tag{2.11}$$

In the special case of vanishing $\partial_+ \bar{P}$ and $\bar{L}(t^+) = \bar{\Delta}$ the solution becomes

$$\frac{ds^2}{\ell^2} = \frac{dr^2}{r^2} - r^2 dt^+ dt^- + \frac{\bar{\Delta}}{k} (dt^+)^2 + \frac{\Delta}{k} (dt^-)^2 - \frac{\Delta \bar{\Delta}}{k^2 r^2} dt^+ dt^-, \tag{2.12}$$

which is just the BTZ black hole with $\ell M = \Delta + \bar{\Delta}$ and $J = \Delta - \bar{\Delta}$. As we will see below the general solution can be interpreted as a BTZ black hole dressed by gases of boundary gravitons and photons represented by the non-zero modes of $\bar{L}(t^+)$ and $\bar{P}(t^+)$.

Of course when matter sources are present, the solutions are modified in the interior. In the following we assume that the matter sources fall off sufficiently rapidly so that (2.2)-(2.10) can be used in the asymptotic analysis.

2.4 Asymptotic symmetry algebra

This phase space of metrics is preserved under the action of the asymptotic symmetry algebra. This consists of a right-moving Virasoro algebra and a ‘‘crossover’’ right-moving $U(1)$ current algebra whose zero mode is the left-moving generator ∂_- , and is generated by

$$\xi_R(\epsilon) = \epsilon(t^+) \partial_+ - \frac{r}{2} \epsilon'(t^+) \partial_r + (\textit{subleading}), \tag{2.13}$$

$$\eta(\sigma) = \sigma(t^+) \partial_- + (\textit{subleading}). \tag{2.14}$$

Note that asymptotically the vector field η goes from spacelike to timelike when Δ crosses zero. We will see in the following that this implies a sign change in the level of the associated current algebra.

The canonical infinitesimal charges associated with these generators as defined in [35–37] can be directly integrated on the phase space. They are given by

$$Q_{\xi_R} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \epsilon(t^+) (\bar{L}(t^+) - \Delta(\partial_+ \bar{P}(t^+))^2), \quad (2.15)$$

$$Q_\eta = \frac{1}{2\pi} \int_0^{2\pi} d\phi \sigma(t^+) (\Delta + 2\Delta \partial_+ \bar{P}(t^+)). \quad (2.16)$$

The charges are finite and conserved and determined up to a constant background value. Here, we set the zero mode of the charge Q_η to Δ in order to reproduce the usual charge when Δ is allowed to vary. When $\partial_+ \bar{P} = \delta \partial_+ \bar{P} = 0$, ∂_+ is canonically associated with $\bar{\Delta}$ and ∂_- with Δ . For the BTZ black hole, the energy is $M = Q_{\partial_t} = \frac{1}{\ell} (Q_{\partial_+} + Q_{\partial_-}) = \frac{\bar{\Delta} + \Delta}{\ell}$ and the angular momentum is $J = Q_{-\partial_\phi} = -Q_{\partial_+} + Q_{\partial_-} = \Delta - \bar{\Delta}$.

Setting $\epsilon = e^{int^+}$, $\sigma = e^{int^+}$ we define $\bar{\mathcal{L}}_n = Q_{\xi_R}$, $\bar{\mathcal{P}}_n = Q_\eta$. The Dirac bracket between these generators is given by

$$i\{\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\} = (m-n)\bar{\mathcal{L}}_{m+n} + \frac{c_R}{12} m^3 \delta_{m,-n}, \quad (2.17)$$

$$i\{\bar{\mathcal{L}}_m, \bar{\mathcal{P}}_n\} = -n\bar{\mathcal{P}}_{m+n}, \quad (2.18)$$

$$i\{\bar{\mathcal{P}}_m, \bar{\mathcal{P}}_n\} = \frac{k_{KM}}{2} m \delta_{m,-n}. \quad (2.19)$$

The central charge and level of the current algebra are given by

$$c_R = \frac{3\ell}{2G}, \quad k_{KM} = -4\Delta. \quad (2.20)$$

The Kac-Moody level $k_{KM} = k$ is positive near the AdS₃ vacuum $\Delta = -k/4$ but becomes negative in the presence of a $\Delta > 0$ black hole. At the classical level, this means that boundary photons lower the right moving energy (\mathcal{L}_0) of the black hole, potentially indicating an instability. This is closely related to the facts that for nonzero $\partial_+ \bar{P}$ the vector field ∂_t is in general no longer everywhere timelike and $\eta(\sigma)$ is spacelike. This region may be thought of as an ergosphere. In this regard AdS₃ with the new boundary

conditions may provide useful insight into the more complicated Kerr ergosphere and associated superradiant instabilities. We hope to understand this better and expect the recent analysis of [25] may be relevant.

Defining the boson ψ by

$$\psi(t^+, t^-) = 2\Delta(-t^- + \bar{P}(t^+)), \quad (2.21)$$

one finds

$$\bar{\mathcal{P}}_n = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{int^+} \partial_+ \psi + \delta_n^0 \Delta, \quad (2.22)$$

and

$$\bar{\mathcal{L}}_n = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{int^+} (\bar{L}(t^+) + \frac{1}{k_{KM}} (\partial_+ \psi)^2), \quad (2.23)$$

which is the standard Sugawara formula for the stress tensor.

3 AdS₃ Chern-Simons gravity

In this section we derive the dual warped CFT₂ associated with the new boundary conditions to pure gravity on AdS₃ in the Chern-Simons formulation and verify that it contains the Virasoro-Kac-Moody structure predicted by the asymptotic symmetry analysis. We follow the well-known analysis for the Brown-Henneaux case [32], which was the first construction of a CFT₂ with $c_{R,L} = \frac{3\ell}{2G}$. For simplicity we restrict here to the classical (large level) limit.

We note that pure 3D Einstein gravity appears unlikely to be a fully consistent nonperturbative quantum theory of gravity with any boundary conditions because, among other reasons, it does not have enough degrees of freedom to account for black hole entropy. Nevertheless it does appear to model a subsector of more complete gravitational theories and the analysis here gives insight into the structure of our boundary conditions.

3.1 Chern-Simons formalism

Three dimensional Einstein gravity with a negative cosmological constant can be formulated as $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ Chern-Simons gauge theory, with the action [38, 39]

$$S_E[A, \bar{A}] = S_k[A] + S_{-k}[\bar{A}] \quad (3.1)$$

where

$$\begin{aligned}
S_k[A] &= \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) - \frac{k}{4\pi} \int_{\Sigma} d^2x \text{Tr}(A_t A_\phi) \\
&= -\frac{k}{4\pi} \int_{\mathcal{M}} d^3x \text{Tr}(-\dot{A}_\phi A_r + \dot{A}_r A_\phi - 2A_t F_{\phi r})
\end{aligned} \tag{3.2}$$

and

$$k = \frac{\ell}{4G}. \tag{3.3}$$

Here Σ is the boundary of \mathcal{M} at spatial infinity. The equations of motion are given by

$$F \equiv dA + A \wedge A = 0, \quad \bar{F} \equiv d\bar{A} + \bar{A} \wedge \bar{A} = 0 \tag{3.4}$$

where

$$A = \omega_\alpha{}^\beta + \frac{1}{\ell} e_\alpha{}^\beta, \quad \bar{A} = \omega_\alpha{}^\beta - \frac{1}{\ell} e_\alpha{}^\beta. \tag{3.5}$$

Here and elsewhere unbarred quantities refer to the left sector, and barred quantities to the right sector. More details on our conventions can be found in Appendix A.

3.2 The new boundary conditions

In terms of triads the new AdS_3 boundary conditions (2.1) in Fefferman-Graham gauge become

$$\begin{aligned}
\frac{e^{(+)}}{\ell} &= r(dt^- - \partial_+ \bar{P}(t^+) dt^+) - \frac{1}{kr} \bar{L}(t^+) dt^+ + O(r^{-2}) dt^\pm + O(r^{-4}) dr, \\
\frac{e^{(-)}}{\ell} &= r dt^+ - \frac{\Delta}{kr} (dt^- - \partial_+ \bar{P}(t^+) dt^+) + O(r^{-2}) dt^\pm + O(r^{-4}) dr, \\
\frac{e^{(3)}}{\ell} &= \frac{dr}{r} + O(r^{-2}) dt^\pm + O(r^{-4}) dr,
\end{aligned} \tag{3.6}$$

where we specified the fluctuating mode $\bar{L}(t^+)$ implied by the asymptotic equations of motion. In terms of the gauge fields this is

$$A = \begin{pmatrix} \frac{dr}{2r} & \frac{\Delta}{kr}(dt^- - \partial_+ \bar{P}(t^+) dt^+) \\ r(dt^- - \partial_+ \bar{P}(t^+) dt^+) & -\frac{dr}{2r} \end{pmatrix} + \begin{pmatrix} O(r^{-2}) & O(r^{-2}) \\ O(r^{-1}) & O(r^{-2}) \end{pmatrix}, \quad (3.7)$$

$$\bar{A} = \begin{pmatrix} -\frac{dr}{2r} + O(r^{-2}) & r dt^+ + O(r^{-2}) \\ \frac{1}{kr} \bar{L}(t^+) dt^+ + O(r^{-2}) & \frac{dr}{2r} + O(r^{-2}) \end{pmatrix}. \quad (3.8)$$

The boundary conditions on the (barred) right moving part are the same as Brown-Henneaux [1], or Coussaert-Henneaux-van Driel [32] in the Chern-Simons formalism. The (unbarred) left moving sector is different both at leading order and at subleading order. There is a crossover right moving Kac-Moody current at leading order. Since Δ is constant, there is also a constraint on the subleading piece of $A_{(+)}$, which prevents the excitation of the left-moving Virasoro modes. The right-moving Virasoro modes are still present in the subleading part of $\bar{A}_{(-)}$. The new boundary conditions can be obtained from the Brown-Henneaux ones by simultaneously applying the change of coordinates $t^- \rightarrow t^- - \bar{P}(t^+)$ and fixing $L(t^-) = \Delta$.

3.3 Conserved charges

The infinitesimal conserved charges associated with the gauge parameters Λ and $\bar{\Lambda}$ defined by

$$\delta A_\mu = D_\mu \Lambda = \partial_\mu \Lambda + [A_\mu, \Lambda], \quad \delta \bar{A}_\mu = \bar{D}_\mu \bar{\Lambda} = \partial_\mu \bar{\Lambda} + [\bar{A}_\mu, \bar{\Lambda}] \quad (3.9)$$

are given by

$$\delta Q_\xi = \frac{k}{2\pi} \int_{2\pi}^0 d\phi \operatorname{Tr} (\delta A_\phi \Lambda - \delta \bar{A}_\phi \bar{\Lambda}). \quad (3.10)$$

Up to an irrelevant local Lorentz transformation, a diffeomorphism is given by $\Lambda = \xi^\mu A_\mu$, $\bar{\Lambda} = \xi^\mu \bar{A}_\mu$. The infinitesimal conserved charge is

$$\delta Q_\xi = -\frac{k}{2\pi} \int_0^{2\pi} d\phi \xi^\mu \operatorname{Tr} (A_\mu \delta A_\phi - \bar{A}_\mu \delta \bar{A}_\phi). \quad (3.11)$$

The boundary conditions require ξ^μ to depend on t^+ only and moreover

$$\delta Q_\xi = \delta \left[\frac{1}{2\pi} \int_0^{2\pi} d\phi (\xi^- \Delta(1 + 2\partial_+ \bar{P}) + \xi^+ (\bar{L}(t^+) - \Delta(\partial_+ \bar{P})^2)) \right] \quad (3.12)$$

which for $\xi^- = \sigma$, $\xi^+ = \epsilon$ agrees with the integrable charges (2.15)-(2.16) in the metric formalism. We have therefore reduced the charges to the ones discussed in Section 2.4.

3.4 From Chern-Simons to constrained WZW

3.4.1 The right \bar{A} sector

The boundary conditions on \bar{A} are identical to those of Brown and Henneaux and the analysis of the resulting boundary theory has been given in [32]. Here we quickly review this work in our notation. The variation of the right moving sector is

$$\delta S_{-k}[\bar{A}] = \frac{k}{2\pi} \int_\Sigma dt d\phi \text{Tr} (\bar{A}_t \delta \bar{A}_\phi). \quad (3.13)$$

The conditions (3.8) imply that on the boundary .

$$\bar{A}_- = O(r^{-2}), \quad (3.14)$$

which enforce $\bar{A}_t = \bar{A}_\phi$ close to the boundary. A good variational principle follows from the complete action

$$S_R[\bar{A}] = S_{-k}[\bar{A}] - \frac{k}{4\pi} \int_\Sigma d^2x \text{Tr} \left(\bar{A}_\phi^2 - \frac{1}{2} (\bar{A}_t - \bar{A}_\phi)^2 \right). \quad (3.15)$$

We chose to add a combination of the constraints to the action in order to obtain an action of second order in time derivatives, see below (3.22). The constraint $\bar{F}_{r\phi} = 0$ is solved by

$$\bar{A}_i = G_R^{-1} \partial_i G_R, \quad i = r, \phi, \quad G_R \sim g_R(t^+, t^-) \begin{pmatrix} \frac{1}{\sqrt{r}} & 0 \\ 0 & \sqrt{r} \end{pmatrix}, \quad (3.16)$$

where the last equation follows from the boundary conditions and $g_R(t^+, t^-)$ is an element of $SL(2, \mathbb{R})$. We find

$$\frac{4\pi}{k} S_R[\bar{A}] = \int \text{Tr} \frac{1}{3} (G_R^{-1} dG_R)^3 + 2 \int_\Sigma dt d\phi \text{Tr} (\bar{A}_+ \bar{A}_-) \quad (3.17)$$

where $\dot{g}_R \equiv g_R^{-1} \partial_t g_R$, $g'_R \equiv g_R^{-1} \partial_\phi g_R$. It is convenient to use the Gauss decomposition

$$G_R(r, t^+, t^-) = \begin{bmatrix} 1 & \hat{X}_R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(\frac{1}{2}\hat{\Phi}_R) & 0 \\ 0 & \exp(-\frac{1}{2}\hat{\Phi}_R) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \hat{Y}_R & 1 \end{bmatrix}. \quad (3.18)$$

One then has

$$\frac{1}{3} \text{Tr}(G_R^{-1} dG_R)^3 = d^3 x \epsilon^{\alpha\beta\gamma} \partial_\alpha \left[e^{-\hat{\Phi}_R} \partial_\beta \hat{X}_R \partial_\gamma \hat{Y}_R \right]. \quad (3.19)$$

The boundary conditions imply the leading behavior

$$\hat{X}_R(r, t^+, t^-) \sim X_R(t^+, t^-), \quad \hat{Y}_R(r, t^+, t^-) \sim \frac{1}{r} Y_R(t^+, t^-), \quad (3.20)$$

$$\hat{\Phi}_R(r, t^+, t^-) \sim -\log r + \Phi_R(t^+, t^-). \quad (3.21)$$

The element $g_R(t^+, t^-)$ defined in (3.16) is then also in the Gauss decomposition with fields $X_R(t^+, t^-)$, $Y_R(t^+, t^-)$, $\Phi_R(t^+, t^-)$. Hence, the right moving action becomes

$$S_R[\Phi_R, X_R, Y_R] = \frac{k}{8\pi} \int_\Sigma dt^+ dt^- (\partial_+ \Phi_R \partial_- \Phi_R + 4e^{-\Phi_R} \partial_+ X_R \partial_- Y_R). \quad (3.22)$$

The tangential components $a = t^+, t^-$ behave as

$$\bar{A}_a \sim \begin{pmatrix} \frac{\bar{a}_{(3)a}}{2} & \bar{a}_{(+)a} r \\ -\frac{\bar{a}_{(-)a}}{r} & -\frac{\bar{a}_{(3)a}}{2} \end{pmatrix}, \quad (3.23)$$

where $\bar{a}_a = g_R^{-1} \partial_a g_R$. The boundary conditions further imply

$$\bar{a}_{(3)a} = 0, \quad \bar{a}_{(\pm)-} = 0, \quad \bar{a}_{(+)+} = 1. \quad (3.24)$$

The boundary conditions (3.24) are equivalent to the constraints

$$\begin{aligned} 0 &= Y_R + \frac{1}{2} \partial_+ \Phi_R, \\ 0 &= \partial_- \Phi_R, \\ 1 &= e^{-\Phi_R} \partial_+ X_R, \\ 0 &= e^{-\Phi_R} \partial_- X_R, \\ 0 &= \partial_- Y_R, \end{aligned} \quad (3.25)$$

which automatically enforce the equations of motion. We see that all the fields are determined by one right-moving scalar field $\Phi_R(t^+)$:

$$\Phi_R = \Phi_R(t^+), \quad \partial_+ X_R(t^+) = e^{\Phi_R(t^+)}, \quad Y_R = -\frac{1}{2}\partial_+ \Phi_R(t^+). \quad (3.26)$$

The field Φ_R is subject only to the constraint $\partial_- \Phi_R = 0$.

The classical Dirac bracket implies that

$$\{\partial_+ \Phi_R(t^+), \partial_+ \Phi_R(t'^+)\} = -\frac{4\pi}{k} \partial_{t^+} \delta(t^+ - t'^+) \quad (3.27)$$

The stress tensor following from the action (3.22) is

$$T^{R-}_{+} = \frac{k}{4} (\partial_+ \Phi_R \partial_+ \Phi_R + 4e^{-\Phi_R} \partial_+ X_R \partial_+ Y_R). \quad (3.28)$$

Using (3.25) this becomes the stress-tensor of a linear dilaton CFT,

$$T^{R-}_{+} = \frac{k}{4} (\partial_+ \Phi_R \partial_+ \Phi_R - 2\partial_+^2 \Phi_R). \quad (3.29)$$

We can then read off the classical central charge

$$c_R = 6k. \quad (3.30)$$

We also note that $T^{R\pm}_{-} = T^{R+}_{+} = 0$. We finally note from (3.8) that in terms of metric components we have the on-shell relation

$$T^{R-}_{+} = \bar{L}(t^+). \quad (3.31)$$

3.4.2 The left A sector

The boundary conditions in the unbarred sector are different from those of Brown and Henneaux. In the usual case we would take $\delta A_+ = 0$ and the unbarred sector completes the barred sector to a nonchiral Liouville theory. However this condition is incompatible with the new boundary condition (3.7). The variation of the action reads as

$$\delta S_k[A] = \frac{1}{2\pi} \int_{\Sigma} dt d\phi (\delta \Delta - \delta(\Delta \partial_+ \bar{P}^2(t^+)) + 2\Delta \delta \partial_+ \bar{P}(t^+)). \quad (3.32)$$

A good variational principle arises from the action¹

$$S_L[A] = S_k[A] - \frac{k}{8\pi} \int_{\Sigma} dt^+ dt^- \text{Tr}(A_-^2 - A_+^2 - 2A_+A_- - 4\alpha[A_+, A_-]\sigma^{(3)}). \quad (3.33)$$

where α is a constant since

$$\delta S_L = -\frac{1}{2\pi} \int dt^+ dt^- \partial_+ \bar{P} \delta \Delta = 0. \quad (3.34)$$

With Brown-Henneaux boundary condition we should take $\alpha = 0$. With the new boundary conditions, we will see later that we should choose $\alpha = 1$. The constraint $F_{r\phi} = 0$ can be solved consistently with the boundary conditions by

$$A = G_L^{-1} \partial_i G_L, \quad i = r, \phi, \quad G_L \sim g_L(t^+, t^-) \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix}, \quad (3.35)$$

where $g_L(t^+, t^-)$ is an element of $SL(2, \mathbb{R})$. The action can be written as

$$S_L[A] = \frac{k}{4\pi} \left(- \int d^3x \frac{1}{3} (G_L^{-1} dG_L)^3 + \frac{1}{2} \int_{\Sigma} dt d\phi \text{Tr}(\dot{g}_L^2 - (g'_L)^2) \right),$$

where $\dot{g}_L \equiv a_t$, $g'_L \equiv a_\phi$ with $a_a = g_L^{-1} \partial_a g_L$. Using the Gauss decomposition

$$g_L = \begin{pmatrix} 1 & 0 \\ Y_L & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{\Phi_L}{2}} & 0 \\ 0 & e^{\frac{\Phi_L}{2}} \end{pmatrix} \begin{pmatrix} 1 & X_L \\ 0 & 1 \end{pmatrix}, \quad (3.36)$$

the action becomes

$$S_L[X_L, Y_L, \Phi_L] = \frac{k}{8\pi} \int_{\Sigma} dt^+ dt^- [\partial_+ \Phi_L \partial_- \Phi_L + 4e^{-\Phi_L} \partial_+ X_L \partial_- Y_L + 4\alpha e^{-\Phi_L} (X_L (\partial_+ Y_L \partial_- \Phi_L - \partial_+ \Phi_L \partial_- Y_L) + \partial_+ X_L \partial_- Y_L - \partial_- X_L \partial_+ Y_L)]. \quad (3.37)$$

For Brown-Henneaux boundary conditions, the constraints are

$$\partial_+ X_L = \partial_+ Y_L = \partial_+ \Phi_L = 0, \quad (3.38)$$

$$X_L = -\frac{1}{2} \partial_- \Phi_L, \quad \partial_- Y_L = e^{\Phi_L}. \quad (3.39)$$

¹One can also add a boundary term proportional to $\int dt^+ dt^- \text{Tr}(A_-^2)$ since its variation is proportional to $\delta \Delta = 0$. The only effect of such term is to shift the current T^L_- defined below by a constant.

The left-moving chiral boson is then expressed as $\frac{k}{4}((\partial_- \Phi_L)^2 - 2\partial_-^2 \Phi) = L(t^-)$ and the left action with $\alpha = 0$ is similar to the right action (3.22) with right fields replaced by left fields. Note the Gauss decomposition (3.18)-(3.36) was instrumental in describing the left and right sector with similar variables.

Under the new boundary conditions, using

$$A_a \sim \begin{bmatrix} \frac{a_{(3)a}}{2} & \frac{a_{(+)a}}{r_{a_{(3)a}}} \\ -a_{(-)a}r & -\frac{\Delta}{2} \end{bmatrix}, \quad a = +, - \quad (3.40)$$

we see that (3.7) implies

$$\begin{aligned} a_{(3)a} &= 0, & a_{(-)-} &= -1, & a_{(-)+} &= \partial_+ \bar{P}, \\ a_{(+)-} &= \frac{\Delta}{k}, & a_{(++)} &= -\frac{\Delta}{k} a_{(-)+}. \end{aligned} \quad (3.41)$$

In terms of the Gauss decomposition this becomes

$$\partial_+ X_L = (X_L^2 - \frac{\Delta}{k}) \partial_+ \bar{P}, \quad (3.42)$$

$$\partial_- X_L = \frac{\Delta}{k} - X_L^2, \quad (3.43)$$

$$\partial_- \Phi_L = -2X_L, \quad (3.44)$$

$$\partial_+ \Phi_L = 2\partial_+ \bar{P} X_L, \quad (3.45)$$

$$\partial_+ Y_L = -e^{\Phi_L} \partial_+ \bar{P}, \quad (3.46)$$

$$\partial_- Y_L = e^{\Phi_L}. \quad (3.47)$$

These equations determine Φ_L , X_L and Y_L in terms of Δ and $\partial_+ \bar{P}(t^+)$. The solutions to the constraints is

$$\begin{aligned} X_L &= \sqrt{\frac{\Delta}{k}} \tanh \left(\sqrt{\frac{\Delta}{k}} (t^- - \bar{P}) \right), \\ e^{\Phi_L} &= e^{\phi_0} \cosh^{-2} \left(\sqrt{\frac{\Delta}{k}} (t^- - \bar{P}) \right), \\ Y_L &= y_0 + e^{\phi_0} \sqrt{\frac{k}{\Delta}} \tanh \left(\sqrt{\frac{\Delta}{k}} (t^- - \bar{P}) \right). \end{aligned} \quad (3.48)$$

This solution is periodic in $\phi = (t^+ - t^-)/2$ when $\Delta = -k/4$ (which corresponds to the AdS vacuum). It is periodic in ϕ for $\Delta > 0$ only when $\bar{P} = -t^+ + f(t^+)$ where f is a periodic function.

Using the constraint conditions and choosing $\alpha = 1$, one can write the action as

$$S_L[X_L, Y_L, \Phi_L] = \frac{k}{8\pi} \int_{\Sigma} dt^+ dt^- [-(\partial_+ \Phi_L \partial_- \Phi_L + 4e^{-\Phi_L} \partial_+ X_L \partial_- Y_L) + 8 \frac{\Delta}{k} \frac{\partial_+ X_L}{\frac{\Delta}{k} - X_L^2}]. \quad (3.49)$$

One can check that the equations of motion from (3.49) are fully compatible with the constraints. (The second line in the action does not contribute to the equations of motion.) Therefore, it is correct to use the action (3.49). The action of the right moving conformal transformations on these fields is nontrivial because they depend on t^+ . Under the reparameterization $\delta t^+ = \xi(t^+)$, Φ_R, Y_R, X_R transform as scalars, and $\partial_+ \bar{P}$ transform as a weight one operator

$$\delta_{\xi}(2\Delta \partial_+ \bar{P}(t^+)) = -\partial_+ \xi (2\Delta \partial_+ \bar{P}(t^+)) - \xi \partial_+ (2\Delta \partial_+ \bar{P}(t^+)). \quad (3.50)$$

One can construct the stress-tensor as the Noether current

$$\begin{aligned} T^{L-}{}_{+}(t^+) &= -\frac{k}{4} ((\partial_+ \Phi_L)^2 + 4e^{-\Phi_L} \partial_+ X_L \partial_+ Y_L) \\ &= \frac{1}{k_{KM}} (2\Delta \partial_+ \bar{P})^2, \end{aligned} \quad (3.51)$$

together with $T^{L+}{}_{+}(t^+) = 0$. Summing up the contributions from the left and right sectors, we obtain that there is no correction to the central charge obtained from the right sector $c = 6k$ (although at the quantum level there would be a shift from $c = 6k + 1$ to $c = 6k + 2$).

Now we wish to construct the Noether current associated to the crossover symmetry (2.14) under which $\delta_{\sigma} t^- = \sigma(t^+)$. The fields transform as

$$\delta_{\sigma} \Phi_L = -\sigma \partial_- \Phi_L, \quad \delta_{\sigma} Y_L = -\sigma \partial_- Y_L, \quad \delta_{\sigma} X_L = -\sigma \partial_- X_L, \quad (3.52)$$

or equivalently

$$\delta_{\sigma}(2\Delta \partial_+ \bar{P}) = 2\Delta \partial_+ \sigma. \quad (3.53)$$

This is generated by

$$\begin{aligned} T^{L-}{}_{-}(\sigma) &= -\frac{k}{2} ((\partial_+ \Phi_L \partial_- \Phi_L + 4e^{-\Phi_L} \partial_+ X_L \partial_- Y_L) \\ &= 2\Delta \partial_+ \bar{P}(t^+), \end{aligned} \quad (3.54)$$

whose commutators form a Kac-Moody algebra at level k_{KM} as anticipated. The zero mode of the left translation is generated by

$$\begin{aligned} T^{L+}_-(\sigma) &= -\frac{k}{4} \left((\partial_- \Phi_L \partial_- \Phi_L + 4e^{-\Phi_L} \partial_- X_L \partial_- Y_L - 8 \frac{\Delta}{k} \frac{\partial_- X_L}{\frac{\Delta}{k} - X_L^2}) \right) \\ &= \Delta. \end{aligned} \quad (3.55)$$

The combined charge

$$\mathcal{Q}_\xi \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi \left((T^{R+}_+ + T^{L+}_+) \epsilon + (T^{L+}_- + T^{L-}_-) \sigma \right) \quad (3.56)$$

generates the diffeomorphism $\xi^+ = \epsilon$, $\xi^- = \sigma$, agreeing with the charges computed in the gravity calculation (2.15) and (2.16).

3.5 From chiral bosons to Liouville

3.5.1 Brown-Henneaux boundary conditions

For the case of Brown-Henneaux boundary conditions, our analysis determined all the on shell fields in terms of two free bosons $Y_L(t^-)$ and $X_R(t^+)$. The other fields are given by

$$e^{-\Phi_L} \partial_- Y_L = 1, \quad X_L = -\frac{1}{2} \partial_- \Phi_L, \quad (3.57)$$

$$e^{-\Phi_R} \partial_+ X_R = 1, \quad Y_R = -\frac{1}{2} \partial_+ \Phi_R, \quad (3.58)$$

where $(X, Y, \Phi)_{L/R}$ parameterize the group element $g_{L/R}$ by

$$g_L = \begin{bmatrix} 1 & 0 \\ Y_L & 1 \end{bmatrix} \begin{bmatrix} \exp(-\frac{1}{2}\Phi_L) & 0 \\ 0 & \exp(\frac{1}{2}\Phi_L) \end{bmatrix} \begin{bmatrix} 1 & X_L \\ 0 & 1 \end{bmatrix}, \quad (3.59)$$

$$g_R = \begin{bmatrix} 1 & X_R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(\frac{1}{2}\Phi_R) & 0 \\ 0 & \exp(-\frac{1}{2}\Phi_R) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_R & 1 \end{bmatrix}. \quad (3.60)$$

The Liouville field Φ comes from the Gauss decomposition of $g \equiv g_L^{-1} g_R$,

$$g = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(\frac{1}{2}\Phi) & 0 \\ 0 & \exp(-\frac{1}{2}\Phi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}. \quad (3.61)$$

Therefore

$$e^{-\frac{1}{2}\Phi} = e^{-\frac{1}{2}(\Phi_L + \Phi_R)}(1 - X_R Y_L). \quad (3.62)$$

Plugging the solution (3.58) into the above definition, we get the relation

$$e^\Phi = \frac{\partial_+ X_R(t^+) \partial_- Y_L(t^-)}{(1 - X_R(t^+) Y_L(t^-))^2}, \quad (3.63)$$

known as a Bäcklund transformation. Φ satisfies the Liouville field equation

$$\partial_+ \partial_- \Phi - 2e^\Phi = 0. \quad (3.64)$$

Hence the two free fields are equivalent to the Liouville scalar.

3.5.2 New boundary conditions

Using the field Φ defined in (3.62), we still find that

$$e^\Phi = \frac{\partial_- Y_L \partial_+ X_R}{(1 - Y_L X_R)^2} \quad (3.65)$$

but Φ and $h \equiv \partial_+ \bar{P}$ are solutions of the following constrained system

$$S = \frac{k}{8\pi} \int dt^+ dt^- \left(\partial_+ \Phi \partial_- \Phi + 4e^\Phi + h((\partial_- \Phi)^2 - 2\partial_-^2 \Phi - \frac{4\Delta}{k}) \right), \quad (3.66)$$

$$\partial_- h = 0. \quad (3.67)$$

The right-moving current \bar{T}^-_+ that generates the diffeomorphism $t^+ \rightarrow t^+ + \epsilon(t^+)$ is

$$\begin{aligned} \bar{T}^-_+ &= \frac{k}{4} ((\partial_+ \Phi)^2 - 2\partial_+^2 \Phi + 2h(\partial_+ \Phi \partial_- \Phi - 2\partial_+ \partial_- \Phi) - 2\partial_+ h \partial_- \Phi), \\ \partial_- \bar{T}^-_+ &= 0. \end{aligned} \quad (3.68)$$

The right-moving current that generates the diffeomorphism $t^- \rightarrow t^- + \sigma(t^-)$ is

$$\bar{T}^-_- = \frac{k}{2} h ((\partial_- \Phi)^2 - 2\partial_-^2 \Phi) = 2\Delta h \quad (3.69)$$

$$\bar{T}^+_- = \frac{k}{4} ((\partial_- \Phi)^2 - 2\partial_-^2 \Phi) = \Delta \quad (3.70)$$

The on shell results agree with the previous subsection, and reduce to the standard Liouville stress tensors at $h = 0$. The currents generate the symmetry transformations via the Dirac brackets

$$\{2\Delta h(t^+), 2\Delta h(s^+)\} = \pi k_{KM} \partial_{t^+} \delta(t^+ - s^+), \quad (3.71)$$

$$\{2\Delta h(t^+), \Phi(s^+, s^-)\} = -2\pi \delta(t^+ - s^+) \partial_{s^-} \Phi(s^+, s^-), \quad (3.72)$$

$$\{\bar{T}^+_{-}(t^+), h(s^+)\} = -2\pi \partial_{s^+} (\delta(t^+ - s^+) h(s^+, s^-)), \quad (3.73)$$

$$\{\bar{T}^+_{-}(t^+), \Phi(s^+, s^-)\} = -2\pi (\delta(t^+ - s^+) \partial_{s^+} \Phi(s^+, s^-) - \partial_{t^+} \delta(t^+ - s^+)),$$

$$\begin{aligned} \{\bar{T}^+_{-}(t^+), \bar{T}^+_{-}(s^+)\} &= 2\pi (\partial_{t^+} - \partial_{s^+}) (\delta(t^+ - s^+) \bar{T}^+_{-}(s^+, s^-)) \\ &\quad - \frac{\pi c_R}{6} \partial_{t^+}^3 \delta(t^+ - s^+). \end{aligned} \quad (3.74)$$

The first line is just the $U(1)$ Kac-Moody algebra with level $\tilde{k}_{KM} = -4\Delta$. The last is the Virasoro algebra with central charge c_R .

In the companion paper [31] we derive the action (3.66) (in the notation $\Phi = 2\rho$) by imposing a chiral gauge condition on the Polyakov-Liouville action for two-dimensional gravity. The equations of motion agree and applying the canonical formalism to this chiral Liouville theory gives the Dirac brackets (3.71)-(3.74), with the identification $\bar{T}^+_{-} = j^-_{\epsilon}$, $\bar{T}^-_{-} = j^-_{\sigma}$, and $c = c_R = 6k$. Therefore the two-dimensional chiral Liouville theory and AdS₃ Chern-Simons gravity with the new boundary conditions are semiclassically equivalent.

4 Bulk Kac-Moody representations

In this section we describe the lowest-weight Kac-Moody representations in terms of a weakly interacting gas of bulk particles. We work in the global AdS₃ coordinates

$$\frac{ds^2}{\ell^2} = -\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2. \quad (4.1)$$

In these coordinates, with $u = \tau + \phi$, $v = \tau - \phi$, the $\overline{SL(2, \mathcal{R})}_R$ generators are described by the vector fields

$$\begin{aligned} \bar{\mathcal{L}}_0 &= i\partial_u, \\ \bar{\mathcal{L}}_{-1} &= ie^{-iu} \left[\frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v + \frac{i}{2} \partial_\rho \right], \end{aligned}$$

$$\bar{\mathcal{L}}_1 = ie^{iu} \left[\frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v - \frac{i}{2} \partial_\rho \right], \quad (4.2)$$

normalized so that

$$[\bar{\mathcal{L}}_0, \bar{\mathcal{L}}_{\pm 1}] = \mp \bar{\mathcal{L}}_{\pm 1}, \quad [\bar{\mathcal{L}}_1, \bar{\mathcal{L}}_{-1}] = 2\bar{\mathcal{L}}_0. \quad (4.3)$$

The quadratic Casimir of $SL(2, \mathcal{R})_R$ on scalar fields is

$$\bar{\mathcal{L}}^2 = \frac{1}{2}(\bar{\mathcal{L}}_1 \bar{\mathcal{L}}_{-1} + \bar{\mathcal{L}}_{-1} \bar{\mathcal{L}}_1) - \bar{\mathcal{L}}_0^2 = -\frac{1}{4} \left[\partial_\rho^2 + 2 \frac{\cosh 2\rho}{\sinh 2\rho} \partial_\rho + \frac{1}{\sinh^2 \rho} \partial_\phi^2 - \frac{1}{\cosh^2 \rho} \partial_\tau^2 \right], \quad (4.4)$$

which is the laplacian times $-\ell^2/4$. Therefore a scalar field of mass m has

$$\bar{\mathcal{L}}^2 = -m^2 \ell^2 / 4, \quad (4.5)$$

and the conformal algebra can be used to classify the solutions of the wave equation. We also have a $U(1)$ isometry

$$\bar{\mathcal{P}}_0 = i\partial_v, \quad (4.6)$$

sometimes called \mathcal{L}_0 . In the usual manner consider states (i.e. scalar field wavefunctions) with weights (h, p) under $\bar{\mathcal{L}}_0, \bar{\mathcal{P}}_0$ so that

$$\bar{\mathcal{L}}_0 |\psi\rangle = h |\psi\rangle, \quad \bar{\mathcal{P}}_0 |\psi\rangle = p |\psi\rangle. \quad (4.7)$$

It follows that

$$|\psi\rangle = e^{-ihu - ipv} F(\rho). \quad (4.8)$$

Periodicity requires

$$h - p \in \mathbf{Z}. \quad (4.9)$$

Now suppose that $|\psi\rangle$ is a $SL(2, \mathcal{R})_R$ primary state in the sense that $\bar{\mathcal{L}}_1 |\psi\rangle = 0$. This implies that F satisfies

$$2h \frac{\cosh 2\rho}{\sinh 2\rho} F - \frac{2p}{\sinh 2\rho} F + \partial_\rho F = 0, \quad (4.10)$$

which is solved by

$$F(p, h) = \text{const} \frac{(\sinh \rho)^{p-h}}{(\cosh \rho)^{p+h}}. \quad (4.11)$$

Regularity at the origin requires

$$p \geq h. \quad (4.12)$$

Demanding that $|\psi\rangle$ obey the mass m wave equation and be normalizable at large ρ with Dirichlet boundary conditions imposes the additional constraint

$$h = \frac{1}{2} \left(1 + \sqrt{m^2 \ell^2 + 1} \right). \quad (4.13)$$

Starting from these primary states labelled by $p \geq h$ we can generate all other normalizable solutions with Dirichlet boundary conditions at infinity by acting with $\bar{\mathcal{L}}_{-1}$. These descendants all have the same quadratic Casimirs, but higher integer-spaced eigenvalues of $\bar{\mathcal{L}}_0$. They correspond in an obvious way to $\overline{SL(2, \mathcal{R})}_R$ descendants of the primary operators.

Recall the usual procedure with Brown-Henneaux boundary conditions in AdS_3 is to also use the $SL(2, \mathcal{R})_L$ isometries (which look like (4.2) with a $u \leftrightarrow v$ exchange). All solutions then arise from left-right primaries and their descendants. The states above correspond to acting on the primary with $p = h$ with \mathcal{L}_{-1}^{p-h} :

$$F(p, h) = \mathcal{L}_{-1}^{p-h} F(h, h). \quad (4.14)$$

With the usual Brown-Henneaux boundary conditions, the left-right primary wavefunction is (4.11) with $p = \bar{h} = h$. The full set of Virasoro descendants are then all states of the form

$$\left((\bar{\mathcal{L}}_{-\bar{k}})^{n_{\bar{k}}} \dots (\bar{\mathcal{L}}_{-2})^{n_2} (\bar{\mathcal{L}}_{-1})^{n_1} \right) \left((\mathcal{L}_{-k})^{n_k} \dots (\mathcal{L}_{-2})^{n_2} (\mathcal{L}_{-1})^{n_1} \right) F(h, h) \quad (4.15)$$

This state has

$$\mathcal{L}_0 = h + \sum_{j=1}^k j n_j, \quad \bar{\mathcal{L}}_0 = h + \sum_{\bar{j}=1}^{\bar{k}} \bar{j} n_{\bar{j}}. \quad (4.16)$$

Using the commutation relations, it is not hard to see that we can always arrange to move the higher moded operators to the left. The particle interpretation of this is as follows. The original primary is a minimal energy single scalar particle sitting in the middle of AdS_3 . The action of powers of $\bar{\mathcal{L}}_{-1}$ and \mathcal{L}_{-1} serve to translate its position around to all possible locations in AdS_3 . $\bar{\mathcal{L}}_{-2}$ and \mathcal{L}_{-2} create left and right boundary gravitons in their minimal energy states, and the higher Virasoro generators create excited boundary

gravitons. Hence the left-right Virasoro module corresponds to multi-particle states consisting of a single scalar particle accompanied by a gas of boundary gravitons. The standard character is

$$tr[q^{\mathcal{L}_0}\bar{q}^{\bar{\mathcal{L}}_0}] = \frac{q^{h-\frac{1}{24}}\bar{q}^{h-\frac{1}{24}}}{\eta(q)\eta(\bar{q})}. \quad (4.17)$$

However with the new chiral boundary conditions \mathcal{L}_{-n} is not part of the asymptotic symmetry group (except for $\mathcal{L}_0 = \bar{\mathcal{P}}_0$) so these solutions do not all obey the new boundary conditions. Using the commutation relations the general state obeying the new boundary conditions can be written in the form

$$((\bar{\mathcal{P}}_{-\bar{l}})^{q_{\bar{l}}}\dots(\bar{\mathcal{P}}_{-2})^{q_2}(\bar{\mathcal{P}}_{-1})^{q_1}) ((\bar{\mathcal{L}}_{-k})^{n_k}\dots(\bar{\mathcal{L}}_{-2})^{n_2}(\bar{\mathcal{L}}_{-1})^{n_1}) F(h, p) \quad (4.18)$$

This has

$$\bar{\mathcal{L}}_0 = h + \sum_{j=1}^k j n_j + \sum_{\bar{j}=1}^{\bar{l}} \bar{j} q_{\bar{j}}, \quad \bar{\mathcal{P}}_0 = p. \quad (4.19)$$

We see that instead of right and left moving gravitons we have right moving gravitons and photons. The number of states is roughly the same: for every state in the non-chiral theory we can get one in the chiral theory by replacing $p = h + n_{\bar{1}}$ and $q_{\bar{l}} = n_{\bar{l}+1}$. The chiral character is

$$tr[\bar{q}^{\bar{\mathcal{L}}_0}\bar{\mu}^{\bar{\mathcal{P}}_0}] = \frac{\bar{q}^{h-\frac{1}{12}}\bar{\mu}^h}{\eta^2(\bar{q})(1-\bar{\mu})}. \quad (4.20)$$

5 String theory and warped AdS_3

String theory is an excellent source of fully consistent theories of Einstein gravity with matter on AdS_3 . Usually these are taken to obey standard Brown-Henneaux boundary conditions. In this section we consider the string solutions studied in [24] for which the new boundary conditions may be relevant. These solutions generically contain a warped AdS_3 factor, but reduce to AdS_3 in an appropriate limit. We will see that a natural set of boundary conditions for the generic case reduce to our new boundary conditions in the AdS_3 limit.

The universal massless NS-NS sector of string theory compactified to six dimensions is governed by the low energy effective lagrangian

$$L = \frac{1}{16\pi G_6} \sqrt{-g} e^{-2\phi} (R + 4(\partial\phi)^2 - \frac{1}{12} H^2). \quad (5.1)$$

The six dimensional warped $\text{AdS}_3 \times \text{S}^3$ black string solution discussed in [24] is

$$\begin{aligned} \frac{4ds_6^2}{\ell^2} &= -\frac{\rho^2 - (2\pi^2 T_L T_R)^2}{\pi^2 T_L^2} (dt^+)^2 + \frac{d\rho^2}{\rho^2 - (2\pi^2 T_L T_R)^2} + 4d\Omega_3^2 \\ &\quad + 4\pi^2 T_L^2 \Xi^2 + 8\pi T_L \frac{\sinh \alpha}{\cosh^2 \alpha} \Xi (d\chi + \cos \theta d\phi), \\ B &= -\frac{\ell^2}{4} \left(\cos \theta d\phi \wedge d\chi + 2\rho \Xi \wedge dt^+ + 4\pi T_L \frac{\sinh \alpha}{\cosh^2 \alpha} (d\chi + \cos \theta d\phi) \wedge \Xi \right), \\ e^{-2\phi} &= \cosh^2 \alpha, \end{aligned} \quad (5.2)$$

where

$$\sinh \alpha = \lambda \pi T_L, \quad \Xi = d\tau^- - \frac{\rho dt^+}{2\pi^2 T_L^2}, \quad (5.3)$$

$$4d\Omega_3^2 = (d\chi + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2. \quad (5.4)$$

We take $(\pi \lambda T_L)^2 < 1$. The coordinate $\psi = \chi + 2\pi T_L \sinh \alpha \tau^-$ is identified with period 4π while $t^- = \tau^- + \frac{\lambda}{2} \chi$ is unidentified. Equivalently $\chi \sim \chi + \frac{4\pi}{1 - \sinh^2 \alpha}$. Using the coordinates $(t^+, \tau^-, \rho, \theta, \phi, \chi)$, we impose the following boundary conditions on the string metric and B field,

$$\begin{aligned} \frac{g_{\mu\nu} + B_{\mu\nu}}{\ell^2} &= \begin{pmatrix} \rho \partial_+ \bar{P}(t^+) & -\rho & 0 & 0 & -\frac{\tanh \alpha}{\pi T_L \cosh \alpha} \cos \theta \rho & -\frac{\tanh \alpha}{\pi T_L \cosh \alpha} \rho \\ 0 & \pi^2 T_L^2 & 0 & 0 & \frac{2\pi T_L \tanh \alpha}{\cosh \alpha} \cos \theta & \frac{2\pi T_L \tanh \alpha}{\cosh \alpha} \\ 0 & 0 & \frac{1}{4\rho^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{\cos \theta}{2} & \frac{1}{4} \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathcal{O}(\rho^0) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^0) & \mathcal{O}(\rho^0) & \mathcal{O}(\rho^0) \\ \mathcal{O}(\rho^0) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) \\ \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-3}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-2}) \\ \mathcal{O}(\rho^0) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) \\ \mathcal{O}(\rho^0) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) \\ \mathcal{O}(\rho^0) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-2}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-1}) \end{pmatrix} \end{aligned} \quad (5.5)$$

where all components are also allowed to depend upon (θ, ϕ, ψ) . Note that we impose $(g + B)_{+-} = O(\rho^{-1})$ while $(g + B)_{-+} = O(\rho^0)$. The asymptotic symmetry group is generated by the following combination of asymptotic vectors and large gauge transformations,

$$\begin{aligned}
(\xi_m, \Lambda_m^\xi) &= e^{imt^+} \left(\partial_+ - \partial_- - im\rho\partial_\rho - \frac{m^2}{2\rho}\partial_- + O(\rho^{-2})\partial_\pm + O(\rho^{-1})\partial_r, \right. \\
&\quad \left. l^2\pi^2 T_L^2 d\tau^- - \frac{il^2 m}{4\rho} d\rho \right), \\
(\eta_m, \Lambda_m^\eta) &= e^{imt^+} \left(\frac{\sqrt{2G_6}}{\pi^2 T_L \ell^2} \partial_-, \sqrt{2G_6} T_L d\tau^- \right), \\
(\eta_m^a, \Lambda_m^\eta) &= e^{imt^+} \left(k^a, \Lambda^{k^a} \right), \\
(\bar{\eta}_m, \Lambda_m^{\bar{\eta}}) &= e^{imt^+} \left(\partial_\chi, \frac{l^2}{4} d\chi \right),
\end{aligned} \tag{5.6}$$

where $\partial_+ = \partial_{t^+}$, $\partial_- = \partial_{\tau^-}$ and k^a are $SU(2)_L$ generators

$$k^3 = -i\partial_\phi, \quad k^\pm = e^{\mp i\phi} (\partial_\theta \mp i(\cot\theta\partial_\phi - \csc\theta\partial_\chi)), \tag{5.7}$$

and

$$\begin{aligned}
\Lambda^{k^3} &= i\frac{l^2}{4} d\phi, \\
\Lambda^{k^\pm} &= e^{\mp i\phi} \frac{l^2}{4} \left(-d\theta \mp i(\cot\theta d\phi + \csc\theta d\chi) \right).
\end{aligned} \tag{5.8}$$

It may be shown that the boundary conditions are preserved by the action of the asymptotic symmetry group. The subleading term $\sim \frac{1}{\rho}\partial_-$ in ξ_m ensures that $\delta g_{\rho-} = O(\rho^{-2})$.

A subtlety is that the asymptotic generators are not uniquely fixed by the boundary conditions (5.5). They are constrained with the supplementary conditions

$$\delta_{(\xi, \Lambda)}(g + B)_{\phi+} = O(\rho^{-1}), \quad \delta_{(\xi, \Lambda)}(g + B)_{\chi+} = O(\rho^{-1}), \tag{5.9}$$

for $(\xi, \Lambda) = (\xi_m, \Lambda_m^\xi), (\eta_m, \Lambda_m^\eta), (\bar{\eta}_m, \Lambda_m^{\bar{\eta}})$, while

$$\delta_{(\eta_m^a, \Lambda_m^\eta)}(g + B)_{+\phi} = O(\rho^{-1}), \quad \delta_{(\eta_m^a, \Lambda_m^\eta)}(g + B)_{+\chi} = O(\rho^{-1}). \tag{5.10}$$

These mixed boundary conditions can be motivated by the string construction of vertex operators [24].

The asymptotic charges and asymptotic symmetry algebra can be obtained using the formalism of [36, 37, 40, 41]. In particular, using the 4 form $k_{\xi,\Lambda}^{\mu\nu}\epsilon_{\mu\nu\alpha_1\dots\alpha_4}$ derived from the Lagrangian (5.1), the conserved charges are given by

$$\oint Q_{\xi,\Lambda} = \int d\sigma d\chi d\phi d\theta (k_{\xi,\Lambda}^{rt^+} + k_{\xi,\Lambda}^{r\tau^-} + \frac{\lambda}{2} k_{\xi,\Lambda}^{r\chi}) \quad (5.11)$$

where $\sigma = (t^+ - t^-)/2$ is taken to be of period 2π . We then obtain the right moving Virasoro central charge

$$c_{R,\lambda} = \frac{3\pi^2 \ell^4}{G_6} \quad (5.12)$$

a $U(1)$ Kac-Moody level

$$k_{U(1)} = -4 \quad (5.13)$$

an $SU(2)$ Kac-Moody level

$$k_{SU(2)} = \frac{\pi^2 \ell^4}{2G_6} \quad (5.14)$$

and $\overline{U(1)}$ Kac-Moody level

$$\bar{k}_{U(1)} = \frac{\pi^2 \ell^4}{2G_6}. \quad (5.15)$$

Accounting for normalization conventions ($2G_6 = \pi^2$) this agrees with the results of the world sheet computations of the space-time central charges given in [24]. We note that [24] also gives a construction of the world sheet vertex operators for the boundary photons and gravitons. In three-dimensional language (after reduction on the three-sphere), the level (5.13) is negative for spacelike warped AdS_3 black holes ($T_L^2 > 0$) and becomes positive for timelike warped spacetimes ($T_L^2 < 0$) after setting a real normalization in (5.6) (because $\eta \rightarrow i\eta$ leads to $k_{U(1)} \rightarrow -k_{U(1)}$). These qualitative features of the $U(1)$ Kac-Moody level are similar to the ones derived earlier in another class of warped geometries [13, 15].

The boundary conditions (5.9)-(5.10) are crucial in order to derive the large gauge transformations in (5.6). These large gauge transformations then lead to the correct values of the Kac-Moody levels (5.14)-(5.15). This is an

example where matter fields directly contribute to central extensions in the asymptotic symmetry algebra. We expect that the boundary conditions (5.5) (with possible mild modifications) will lead to finite, integrable and conserved charges associated with the generators (5.6). This however remains to be fully checked.

To compare with the preceding sections, we should take the warping factor $\lambda = 0$ and look at the 3d Einstein frame metric $g_{\mu\nu}$ defined by

$$ds_{6(s)}^2 = e^{4\phi} ds_{3(E)}^2 + \ell^2 d\Omega_3^2 \quad (5.16)$$

The boundary conditions (5.5) reduce to

$$\frac{g_{\mu\nu}}{\ell^2} \sim \begin{pmatrix} \rho\partial_+\bar{P}(t^+) & -\frac{\ell}{2} & 0 \\ \pi^2\bar{T}_L^2 & 0 & 0 \\ & \frac{1}{4\rho^2} & \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\rho^0) & \mathcal{O}(\rho^0) & \mathcal{O}(\rho^{-2}) \\ & \mathcal{O}(\rho^{-1}) & \mathcal{O}(\rho^{-2}) \\ & & \mathcal{O}(\rho^{-3}) \end{pmatrix}.$$

These reduce to the our new boundary conditions (2.1), using the following relations

$$\rho = r^2, \quad \pi^2 T_L^2 = \frac{4G_3\Delta}{l}, \quad G_6 = 2\pi^2 l^3 G_3. \quad (5.17)$$

The asymptotic symmetry generators and the central charges then reduce to the ones derived in Section 2.4.

Acknowledgements

We are grateful to A. Castro, T. Hartman, M. Henneaux, F. Larsen, J. Maldacena for useful conversations, and especially to S. Detournay for his early collaboration. This work was supported by NSF grant 1205550. W.S. would like to thank Hamburg University and DESY for their hospitality during her visit. W.S. is supported in part by the Harvard Society of Fellows, G.C. is a Research Associate of the Fonds de la Recherche Scientifique F.R.S.-FNRS (Belgium) and A.S. is a Fellow at the Radcliffe Institute for Advanced Study. G.C. also acknowledges the University of Amsterdam and funds from a NWO Vici grant for support during the earliest stages of this project.

Appendices

A. Conventions

We use r , $t^\pm = t \pm \phi$ and we take the orientation $\varepsilon^{t\phi r} = \varepsilon^{t\phi} = -1$ so that $\varepsilon^{+-r} = \varepsilon^{+-} = +1$, $d^2x = dt \wedge d\phi = +\frac{1}{2}dt^- \wedge dt^+$ and $\int_{\mathcal{M}} d^3x \partial_r = \int_{\partial\mathcal{M}} d^2x$. We define $\epsilon^{\mu\nu\alpha} = \varepsilon^{\mu\nu\alpha} / \sqrt{-g}$. Surface charges on the boundary circle S spanned by the circle ϕ can be obtained by

$$\int_S k_\xi = \int_{2\pi}^0 \frac{1}{2} k_\xi^{\mu\nu} \varepsilon_{\mu\nu\phi} d\phi = - \int_0^{2\pi} d\phi k_\xi^{r+} - \int_0^{2\pi} d\phi k_\xi^{r-}, \quad (0.18)$$

where $k_\xi = \frac{1}{2} k_\xi^{\mu\nu} \varepsilon_{\mu\nu\alpha} dx^\alpha$ is the surface charge 1-form.

We choose a $SL(2, \mathbb{R})$ representation such that

$$\text{Tr}(\sigma^{(a)} \sigma^{(b)}) = \frac{1}{2} \eta^{(a)(b)} \quad (0.19)$$

where

$$\eta_{(a)(b)} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ab}, \quad (0.20)$$

which lowers the triad indices. The representation is given by Pauli matrices as

$$\sigma_\alpha^{(-)\beta} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \sigma_\alpha^{(+)\beta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_\alpha^{(3)\beta} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (0.21)$$

The triad and spin connection can be written in spinor notation as

$$e_\alpha^\beta = e_a \sigma_\alpha^{a\beta}, \quad \omega_\alpha^\beta = \frac{1}{2} \epsilon_c^{ab} \omega_{ab} \sigma_\alpha^{c\beta}, \quad (0.22)$$

where $\omega_{ab} = e_a^\nu (\partial_\mu e_{b\nu} - \Gamma_{\mu\nu}^\rho e_{b\rho}) dx^\mu$. Our choice of dreibein leads to

$$\epsilon^{(+)(-)(3)} = -2. \quad (0.23)$$

B. Boundary conditions with Δ varying

In this appendix, we discuss a variant of our boundary conditions where Δ is allowed to fluctuate but $\bar{P}(t^+)$ is periodic. The action which provides a good variational principle is simply S_0 because evaluating (2.5) gives

$$\delta S_0 = \frac{1}{2\pi} \int dt^+ dt^- \Delta \delta(\partial_+ \bar{P}) = 0. \quad (0.24)$$

Note however that the variation vanishes only globally and not locally. One would therefore expect source terms on the right-hand side of boundary field equations in the reduction of the theory if one repeats the analysis done in Section 3.1.

The canonical infinitesimal charges associated with the asymptotic generators (2.14) are given for a generic solution in the phase space by

$$\delta Q_{\xi_R} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \epsilon(t^+) \left(\delta (\bar{L}(t^+) - \Delta(\partial_+ \bar{P}(t^+))^2) - \partial_+ \bar{P}(t^+) \delta \Delta \right), \quad (0.25)$$

$$\delta Q_\eta = \frac{1}{2\pi} \int_0^{2\pi} d\phi \sigma(t^+) (\delta \Delta + \delta \Delta \partial_+ \bar{P}(t^+) + 2\Delta \delta \partial_+ \bar{P}(t^+)). \quad (0.26)$$

The charges are finite and conserved. One wishes to integrate the charges to get globally defined quantities on the phase space. δQ_{ξ_R} and δQ_η are not closed one-forms on phase space and cannot be integrated. However the combination

$$\delta Q_{\xi_R(\epsilon)} - \delta Q_{\eta(\epsilon)} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \epsilon(t^+) \delta [\bar{L}(t^+) - \Delta(1 + \partial_+ \bar{P}(t^+))^2] \quad (0.27)$$

is exact. Setting $\epsilon = e^{int^+}$ and integrating we obtain the globally defined charge

$$\bar{\mathcal{L}}_n \equiv Q_{\xi_R(e^{int^+}) - \eta(e^{int^+})} = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{int^+} (\bar{L}(t^+) - \Delta(1 + \partial_+ \bar{P}(t^+))^2). \quad (0.28)$$

In order to integrate δQ_η when $\Delta \neq 0$, we must use a vector field $\eta_n = \frac{e^{int^+}}{\sqrt{|\Delta|}}$ normalized to a constant which we take to be $\pm 4G\ell$ depending on whether it is spacelike or timelike. We then have

$$\bar{\mathcal{P}}_n \equiv Q_{\eta(|\Delta|^{-1/2} e^{int^+})} = \frac{\Theta(\Delta)}{\pi} \int_0^{2\pi} d\phi e^{int^+} \sqrt{|\Delta|} (1 + \partial_+ \bar{P}(t^+)), \quad (0.29)$$

where $\Theta(\cdot)$ is the sign function.

The Dirac bracket between these generators is given by

$$i\{\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\} = (m-n)\bar{\mathcal{L}}_{m+n} + \frac{c_R}{12}m^3\delta_{m,-n}, \quad (0.30)$$

$$i\{\bar{\mathcal{L}}_m, \bar{\mathcal{P}}_n\} = -n\bar{\mathcal{P}}_{m+n}, \quad (0.31)$$

$$i\{\bar{\mathcal{P}}_m, \bar{\mathcal{P}}_n\} = \frac{\tilde{k}_{KM}}{2}m\delta_{m,-n}. \quad (0.32)$$

The central charge and level of the current algebra are given by

$$c_R = \frac{3\ell}{2G}, \quad \tilde{k}_{KM} = -4\Theta(\Delta). \quad (0.33)$$

Because $\bar{\mathcal{P}}_0$ is a central element of the algebra it is possible to define generators without the awkward Θ function by

$$\bar{\mathcal{T}}_n \equiv \frac{1}{2}\Theta(\bar{\mathcal{P}}_0)\bar{\mathcal{P}}_0\bar{\mathcal{P}}_n = \frac{1}{\pi} \int_0^{2\pi} d\phi e^{int^+} \Delta(1 + \partial_+ \bar{P}(t^+)), \quad (0.34)$$

These obey²

$$[\bar{\mathcal{T}}_m, \bar{\mathcal{T}}_n] = \frac{k_{KM}}{2}m\delta_{m,-n} \quad k_{KM} = -2\bar{\mathcal{T}}_0, \quad (0.35)$$

where $\bar{\mathcal{T}}_0 = -\frac{\ell}{8G}$ in the global vacuum and $\bar{\mathcal{T}}_0 = 2\Delta$ in the black hole sector. These generators correspond to the unnormalized vector fields $\sigma(t^+)\partial_-$. Usually a field-dependent level is inconsistent, but here no problem arises because $\bar{\mathcal{T}}_0$ is a central element of the algebra.

In summary, we defined three distinct phase spaces: $\Delta < 0$, $\Delta = 0$ and $\Delta > 0$ since we had to divide by $\sqrt{|\Delta|}$ in order to define the generators. The sector $\Delta < 0$ contains the AdS_3 vacuum $\Delta = -k/4$ and its Virasoro-Kac-Moody descendants. The Kac-Moody level is $k = c/6$. The sector $\Delta = 0$ contains one class of extremal BTZ black holes and their Virasoro descendants while the Kac-Moody generators are pure gauge. The sector $\Delta > 0$ contains the other class of extremal BTZ black holes, non-extremal BTZ black holes and their Kac-Moody-Virasoro descendants. The Kac-Moody level is negative.

²Note that k_{KM} , unlike \tilde{k}_{KM} , has dimensions of action as expected for a central term in a Dirac Bracket algebra persisting in the classical limit. The units are scaled out of \tilde{k}_{KM} by the field-dependent normalization of the generators.

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