



# A Comparative-Advantage Approach to Government Debt Maturity

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**Internet Appendix for:**  
**A Comparative-Advantage Approach to  
Government Debt Maturity**

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**A Additional Empirical Results**

Our key point in Section II is that T-bills yields appear to embed a particularly large convenience premium, even relative to other Treasuries. However, this claim raises a host of benchmarking issues. Specifically, one might worry that our quantitative results might be an artifact of the smooth Svensson (1994) model that GSW (2007) fit to the Treasury curve. We explore these benchmarking issues here.

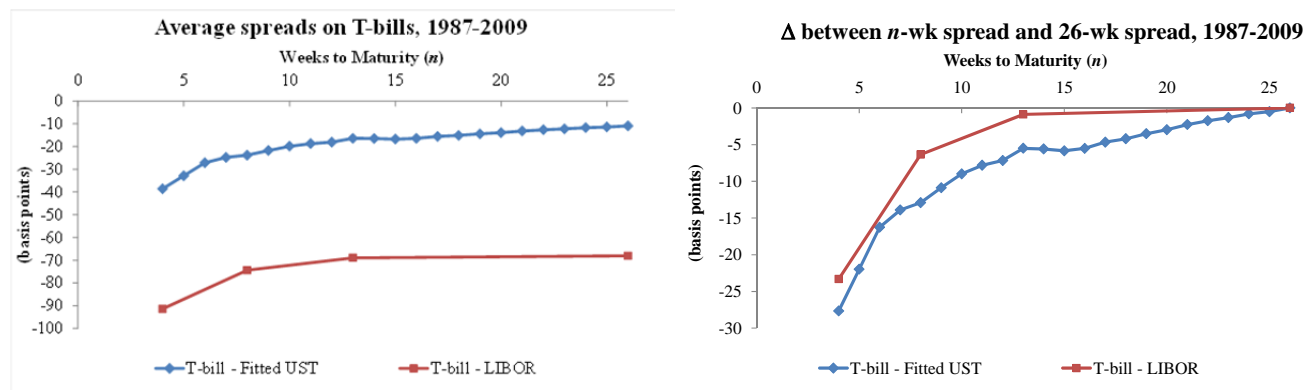
First, we show that similar results obtain when we compare T-bill yields to private money market rates. Second, we show that we obtain very similar results using less parametric models of the Treasury curve which do not impose as much smoothing. Third, we note that once we use high-frequency variation in bills supply to identify the special convenience premium on bills, this benchmarking issue becomes far less important; it is this covariance between T-bill supply and z-spreads that is really our essential point.

*A.1. Comparison of the T-bill curve to other money market curves*

One way to restate the result shown in Figure 1 is to say that at the very short-end of the Treasury-bill curve is often surprisingly steep relative to what one would have expected based on the rest of the Treasury curve. If this is the case, we would also expect the T-bill curve to be steep relative to private money market curves. For the LIBOR curve, we can compute  $y_t^{(n)} - r_t^{(n)}$  for weeks  $n = 4, 8, 13,$  and  $26$  going back continuously to 1987. If anything, using a private money

market curve (as opposed to a fitted Treasury curve) should lead us to underestimate the special steepness at the front-end of the bill curve. Specifically, to the extent that credit spreads curve are normally increasing in maturity,  $n$ , this would lead  $y_t^{(n)} - r_t^{(n)} = -(r_t^{(n)} - y_t^{(n)})$  to *decline* with maturity, whereas the specialness of short-term T-bills suggests that  $y_t^{(n)} - r_t^{(n)}$  should rise (i.e., becoming less negative) with  $n$ .

We show this analysis below for the 1987-2009 period. The results are quite similar if we exclude 2008-2009 and focus only on the 1987-2007 period. The plot on the left compares z-spreads,  $z_t^{(n)} = y_t^{(n)} - \hat{y}_t^{(n)}$ , based on GSW (2007) fitted yields, with the spread between bills and LIBOR,  $y_t^{(n)} - r_t^{(n)}$ . The plot on the right simply contrasts the steepness of these spread curves at the front-end. Specifically, we plot the difference between the  $n$ -week spread and the 26-week (i.e., 6-month) spread. As shown below, whether we use fitted Treasury rate or private money market rates, the data paints a similar picture of the special convenience yield on short-term bills.



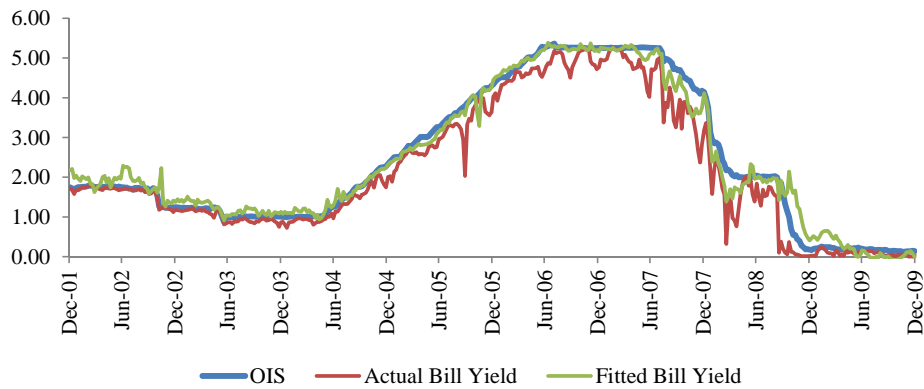
### A.2. Alternative approaches for constructing fitted Treasury yields

Gurknayak, Sack, and Wright (2007) fit the 6-parameter Svensson (1994) model of forward rates to Treasury yields by minimizing a weighted sum of pricing errors for a sample of nominal Treasuries that includes almost all off-the-run notes and bonds with more than 3-months remaining to maturity.<sup>1</sup> Our overall sense, from looking at many of these fitted yield curves is that the GSW (2007) curve tends to fit the short end of the curve remarkably well and that, as a result, our fitted yields are largely capturing what we want them to capture: namely a default-free short-term interest rate that is (largely) free of convenience premium (i.e., demand and supply effects). One way to see

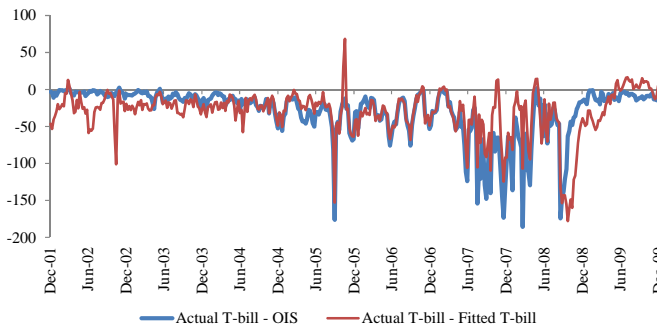
<sup>1</sup> The model is  $f(t) = \beta_0 + \beta_1 \cdot \exp(-t/\tau_1) + \beta_2 \cdot (t/\tau_1)\exp(-t/\tau_1) + \beta_3 \cdot (t/\tau_2)\exp(-t/\tau_2)$ . The 6 parameters have a natural interpretation.  $\beta_0 + \beta_1$  is the current short rate and  $\beta_0$  is the forward rate at very long horizons. The model allows for two humps in the forward curve:  $\beta_2$  and  $\tau_1$  control the magnitude and location of the first hump, while  $\beta_3$  and  $\tau_2$  control the magnitude and location of the second hump.

this is to compare our  $z$ -spreads to the spread between T-bills and the overnight indexed swap (OIS) rate, which we can do starting in 2002. The OIS rate is a good proxy for the default-free short rate that does not contain any moneyness premium.<sup>2</sup> As shown below, our fitted Treasury yields largely tracks the OIS rate. As a result, the time-series of spreads based on these two benchmarks are 69% correlated in levels and 78% in 4-week changes.

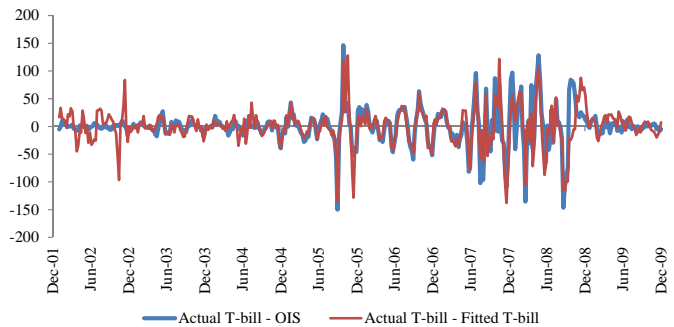
**1-month Yields (%)**



**1-month Spreads (bps)**



**4-week Changes in Spreads (bps)**



We also estimated cubic spline-based models of the forward curve that allow us to explicitly vary the degree of smoothing. As shown below, we find similar results for any estimated forward curve than imposes some minimal degree of smoothing. Formally, we use smoothing cubic splines to

<sup>2</sup> As explained by Sunderam (2013), the OIS rate is unlikely to be affected by default risk since it is based on the expected compounded overnight (Federal funds) rate. And it is largely free of any convenience premium since, like other swap rates, it is not a rate at which a money market investor can invest principal—i.e., a swap involve no exchange of principal and, thus, in not a stable-value store of value in the same way as a T-bill or financial CP.

estimate the forward rate curve following Waggoner (1997). Specifically, we find the cubic spline  $f$  that solves:

$$\min_f \left[ w \cdot \sum_{i=1}^N \left( (P_i - \hat{P}_i(f)) / D_i \right)^2 + (1-w) \cdot \int_0^T (f''(t))^2 dt \right],$$

where  $P_i$  is the price of bond  $i$ ,  $D_i$  is modified duration, and  $\hat{P}_i(f) = \sum_{j=1}^J c_{ij} \exp(-\int_0^{t_j} f(s) ds)$  is the model-implied price.<sup>3</sup> The first term is the sum of squared pricing errors, weighted inversely by modified duration—this is similar to the sum of squared yield fitting errors. The second term is the “roughness penalty” that imposes smoothing on the forward curve (i.e., that penalizes over-fitting). By varying the weight  $w$  between 0 and 1 we are able to put more or less weight on the roughness penalty. For instance, for  $w = 1$ , the procedure finds the cubic spline that best fits the data. As  $w \rightarrow 0$ , the procedure finds the best linear approximation of the forward rate curve. We obtain our estimated zero-coupon yield curve by integrating the estimated forward curve  $\hat{y}(t) = t^{-1} \int_0^t \hat{f}(s) ds$  and compute  $z$ -spreads for Treasury bills using  $z(t) = y(t) - \hat{y}(t)$ .<sup>4</sup>

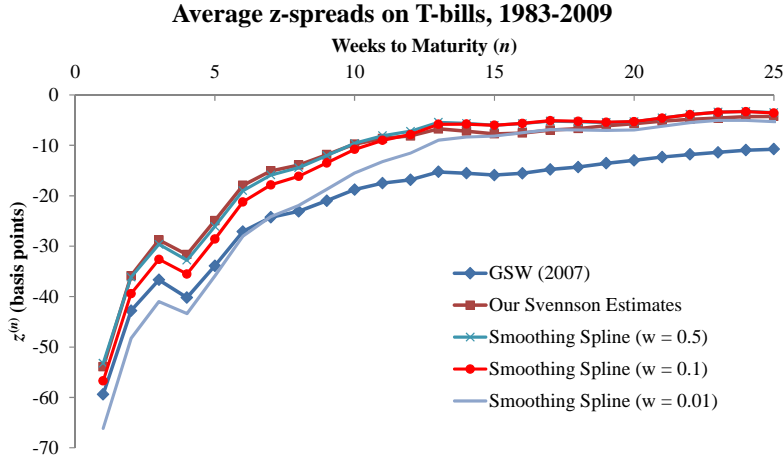
We fit our smoothing cubic spline curves using the same subset of notes and bonds with more than 3-months remaining to maturity as Gurknayak, Sack, and Wright (2007).<sup>5</sup> We then extrapolate these yields to the front-end of the curve to compute  $z$ -spreads for T-bills. We find similar that, on average, short-term bills have surprisingly low yields based on any extrapolation of the rest yield that imposes some minimal degree of smoothing. Specifically, the following figure shows average T-bill  $z$ -spreads by weeks-to-maturity over of 1983-2009 sample using the curve estimated by GSW (2007) as well as five alternative curves (our implementation of the Svensson (1994) model as three smoothing spline models). As shown below, our results do not seem particularly sensitive, either qualitatively or quantitatively, to the specific curve-fitting procedure we use.

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<sup>3</sup> In other words, the forward curve is modeled as a series of piecewise cubic polynomial functions which are required to have continuous first and second derivatives at each of the breakpoints or knots.

<sup>4</sup> We estimate these smoothing cubic splines in Matlab using the `IRFunctionCurve.fitSmoothingSpline` command that is available in Matlab’s Fixed Income Toolbox.

<sup>5</sup> Specifically, Gurknayak, Sack, and Wright (2006) exclude (i) all callable bonds and flower bonds, (ii) all securities with less than 3-months remaining to maturity, (iii) all Treasury bills, (iv) old 20-year bonds beginning in 1996, (v) beginning with securities issued in 1980, the on-the-run and first off-the-run 2, 3, 4, 5, 7, 10, 20, and 30 year notes and bonds, and (vi) a handful of other issues were dropped on an *ad hoc* basis due to persistent pricing errors.



A.3. *Inferring the convenience premium on bills from high-frequency variation in supply*

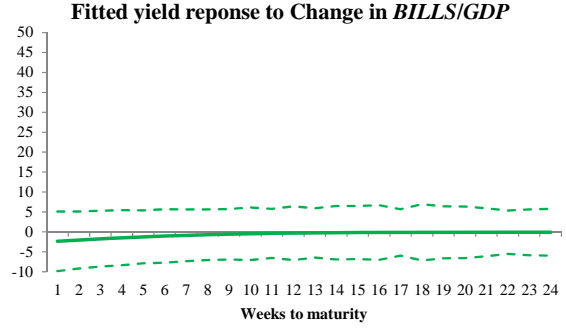
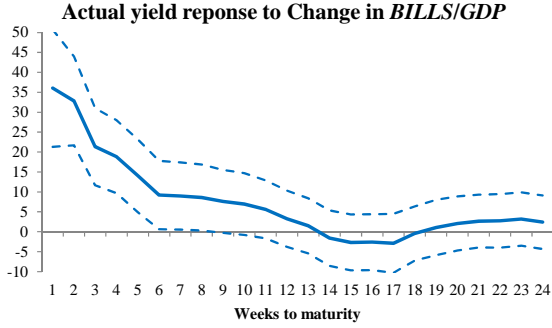
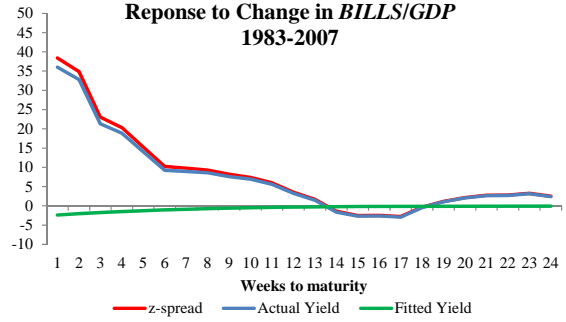
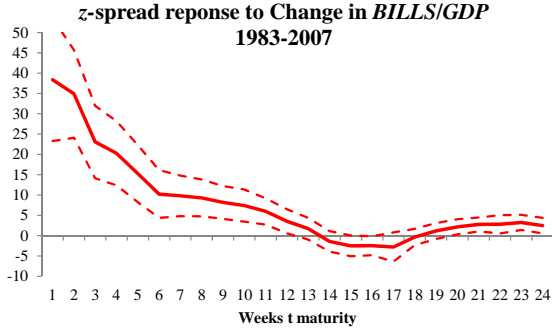
Table A3 presents weekly OLS regressions of the form:

$$\Delta_4 z_t^{(n)} = \Delta_4 y_t^{(n)} - \Delta_4 \hat{y}_t^{(n)} = a^{(n)} + b^{(n)} \cdot \Delta_4 (BILLS / GDP)_t + \varepsilon_t^{(n)} .$$

This is just repeating our baseline regressions, but breaking the z-spread into its underlying components. Here, our use of fitted yields plays almost no role at all. The response of z-spread to changes in bills supply is driven entirely by the response of *actual* yields and not by the response of fitted yields.

Why then look at z-spreads at all? Focusing on z-spreads simply enhances the power of these tests. This is shown in figures below where we decompose the slope coefficients  $b^{(n)}$  from our z-spread regressions into the difference between (i) the coefficients from a regression of changes in *actual* T-bill yields on changes in *BILLS/GDP* and (ii) the coefficients from a regression of changes in *fitted* yields on changes in *BILLS/GDP*. Focusing on z-spreads (as opposed to actual bill yields) simply helps to soak up some of the normal variation in short-rates and gives us more power to isolate the downward-sloping demand for bills.

Reassuringly, the regression-based plots paint a very similar picture to Figure 1: both suggest that the special premium on bills is particular concentrated at the very short end of the yield curve—e.g., in T-bills with less than 3-months to maturity.



We also obtain similar results in Table 1 if we regress spreads between T-bill yields and private money market rates on T-bill supply. Specifically, Table A.3 shows results for spreads on 4-week T-bills relative to commercial paper, bank certificates of deposits, and the OIS rate. Panel A shows results from 1983-2009 and Panel B shows results from 1983-2007. Note that the OIS rate is only available beginning in 2002. The results for bill spreads relative to CP and CD are similar to those for  $z$ -spreads, albeit a bit weaker, because these spreads contain an omitted and time-varying credit risk component. The results for the T-bill vs. OIS spread are very similar to those for  $z$ -spreads.

## B Proofs and Derivations

**Proof of Proposition 1:** The planner's date-1 problem is given by Eq. (11). Differentiating with respect to  $B_{1,2}$  yields the first order condition

$$-\beta(B_{0,1} - \beta B_{1,2}) + \beta(B_{1,2} + B_{0,2}) = 0. \quad (\text{A1})$$

The second order condition is  $\beta(1 + \beta) > 0$ . The solution to (A1) is then

$$B_{1,2} = (B_{0,1} - B_{0,2}) / (1 + \beta), \quad (\text{A2})$$

which implies that

$$\tau_1 = \tau_2 = (B_{0,1} + B_{0,2}\beta) / (1 + \beta). \quad (\text{A3})$$

Consider the problem at  $t = 0$  where  $\tau_0 = G - B_{0,1} - B_{0,2}$ . Substituting (A3) into Eq. (8), yields

$$\min_{\{B_{0,1}, B_{0,2}\}} \left[ \frac{1}{2} \left( (G - B_{0,1} - B_{0,2})^2 + E \left[ \left( \frac{B_{0,1} + \beta B_{0,2}}{1 + \beta} \right)^2 \right] \right) + E \left[ \beta \left( \frac{B_{0,1} + \beta B_{0,2}}{1 + \beta} \right)^2 \right] \right], \quad (\text{A4})$$

which is equivalent to

$$\min_{\{S, D\}} \left[ \frac{1}{2} (G - D)^2 + \frac{1}{2} E \left[ \frac{(SD + (1 - S)D\beta)^2}{1 + \beta} \right] \right] \quad (\text{A5})$$

where we have made the change of variables to  $D = B_{0,1} + B_{0,2}$ , and  $S = B_{0,1} / D$ .

We first differentiate (A5) with respect to  $S$ , yielding

$$D^2 E \left[ \frac{1 - \beta}{1 + \beta} (\beta + S(1 - \beta)) \right] = 0. \quad (\text{A6})$$

We note that  $S^* = 1/2$  is the solution to this the first order condition, since

$$E \left[ \frac{1 - \beta}{1 + \beta} \left( \beta + \frac{1}{2}(1 - \beta) \right) \right] = E \left[ \frac{1}{2}(1 - \beta) \right] = 0. \quad (\text{A7})$$

Noting that

$$E \left[ \frac{\beta(1 - \beta)}{1 + \beta} \right] + \frac{1}{2} E \left[ \frac{(1 - \beta)^2}{1 + \beta} \right] = 0, \quad (\text{A8})$$

and defining  $b = E[(1 - \beta)^2 / (1 + \beta)]$ , we can rewrite (A6) as

$$D^2 b (S - 1/2) = 0. \quad (\text{A9})$$

We now solve for  $D$ , the level of debt. Note that

$$E \left[ \frac{(S^* + (1 - S^*)\beta)^2}{1 + \beta} \right] = E \left[ \frac{(\frac{1}{2} + \frac{1}{2}\beta)^2}{1 + \beta} \right] = \frac{1}{4} E[1 + \beta] = \frac{1}{2}. \quad (\text{A10})$$

Optimal  $D$  thus satisfies

$$\min_D \left[ \frac{1}{2} (G - D)^2 + \frac{1}{4} D^2 \right], \quad (\text{A11})$$

which has first order condition



$$-(G-D)+D/2=0 \Rightarrow D^* = 2G/3. \quad (\text{A12})$$

Using these facts, we can show that

$$\frac{1}{2} E \left[ \frac{(SD+(1-S)D\beta)^2}{1+\beta} \right] = \frac{D^2}{2} (b(S-1/2)^2 + 1/2). \quad (\text{A13})$$

One can confirm that the second order conditions are satisfied at this solution. Moreover, as demonstrated below, the objective is globally convex in  $B_{0,1}$  and  $B_{0,2}$ , so the solution is unique.<sup>6</sup>

*Allowing the government to issue risky securities*

We now show that, in the absence of money demand, these results continue to hold if we allow for arbitrary risky securities whose payouts are possibly contingent on the realization of  $\beta$ . Specifically, we now allow the government to issue face value  $B_R$  of risky securities with payoff  $X_R(\beta)$  at  $t = 2$ . We assume that these securities are fairly priced by households with price  $P_R = E[\beta X_R(\beta)]$ . The government's budget constraint becomes

$$\begin{aligned} t=0: G &= \tau_0 + B_{0,1} + B_{0,2} + B_R P_R \\ t=1: B_{0,1} &= \tau_1 + B_{1,2} P_{1,2} \\ t=2: B_{1,2} + B_{0,2} + B_R X_R(\beta) &= \tau_2 \end{aligned} \quad (\text{A14})$$

As above, we work backwards from  $t = 1$ . The planner's date-1 problem is

$$\min_{B_{1,2}} \left[ \frac{1}{2} (\tau_1^2 + \beta \tau_2^2) \right] = \min_{B_{1,2}} \left[ \frac{1}{2} (B_{0,1} - B_{1,2} \beta)^2 + \frac{1}{2} \beta (B_{1,2} + B_{0,2} + B_R X_R(\beta))^2 \right]. \quad (\text{A15})$$

Taking first order conditions with respect to  $B_{1,2}$  yields

$$B_{1,2} = (B_{0,1} - B_{0,2} - B_R X_R(\beta)) / (1 + \beta), \quad (\text{A16})$$

which implies  $\tau_1 = \tau_2 = (B_{0,1} + B_{0,2} \beta + B_R \beta X_R(\beta)) / (1 + \beta)$ .

Consider the problem at  $t = 0$  where  $\tau_0 = G - B_{0,1} - B_{0,2} - B_R P_R$ :

$$\min_{\{B_{0,1}, B_{0,2}, B_R\}} \left[ \frac{1}{2} (G - B_{0,1} - B_{0,2} - B_R P_R)^2 + \frac{1}{2} E \left[ \frac{(B_{0,1} + B_{0,2} \beta + B_R \beta X_R(\beta))^2}{1 + \beta} \right] \right]. \quad (\text{A17})$$

The first order conditions are

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<sup>6</sup> While the objective may not be globally convex in  $S$  and  $D$ , global convexity in  $B_{0,1}$  and  $B_{0,2}$  shows the solution is unique.

$$\begin{aligned}
& -(G - B_{0,1} - B_{0,2} - B_R P_R) + E \left[ \frac{1}{1 + \beta} (B_{0,1} + B_{0,2} \beta + B_R \beta X_R(\beta)) \right] = 0, \\
& -(G - B_{0,1} - B_{0,2} - B_R P_R) + E \left[ \frac{\beta}{1 + \beta} (B_{0,1} + B_{0,2} \beta + B_R \beta X_R(\beta)) \right] = 0, \\
& -P_R (G - B_{0,1} - B_{0,2} - B_R P_R) + E \left[ \frac{\beta X_R(\beta)}{1 + \beta} (B_{0,1} + B_{0,2} \beta + B_R \beta X_R(\beta)) \right] = 0.
\end{aligned} \tag{A18}$$

Since  $E[\beta]=1$  and  $P_R = E[\beta X_R(\beta)]$ , it is easy to see that  $B_{0,1} = B_{0,2} = G/3$  and  $B_R = 0$  satisfies these three conditions for an arbitrary risky security.

We now show that the objective function is globally convex in its three arguments, showing that this is the unique solution to the planner's problem. Specifically, the Hessian is

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & P_R \\ 1 & 1 & P_R \\ P_R & P_R & P_R^2 \end{bmatrix} + \begin{bmatrix} E[(1 + \beta)^{-1}] & E[(1 + \beta)^{-1} \beta] & E[(1 + \beta)^{-1} \beta X_R] \\ E[(1 + \beta)^{-1} \beta] & E[(1 + \beta)^{-1} \beta^2] & E[(1 + \beta)^{-1} \beta^2 X_R] \\ E[(1 + \beta)^{-1} \beta X_R] & E[(1 + \beta)^{-1} \beta^2 X_R] & E[(1 + \beta)^{-1} \beta^2 X_R^2] \end{bmatrix} \tag{A19}$$

The first matrix is positive semi-definite with eigenvalues of  $2 + P_R^2 > 0$  and 0 (multiplicity 2). Let

$$E^*[Z] = \frac{E[(1 + \beta)^{-1} Z]}{E[(1 + \beta)^{-1}]} \tag{A20}$$

denote the expectation with respect to the  $(1 + \beta)^{-1}$  twisted probability measure and note that the second term can be written as

$$E[(1 + \beta)^{-1}] \cdot E^* \left[ \begin{bmatrix} 1 \\ \beta \\ \beta X_R \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \beta X_R \end{bmatrix}' \right] \tag{A21}$$

which is positive definite, assuming that 1,  $\beta$ , and  $\beta X_R$  are linearly independent. This shows that the objective function is globally convex for an arbitrary  $X_R$  and, hence, that the unique optimum is

$$B_{0,1}^* = B_{0,2}^* = G/3 \text{ and } B_R^* = 0.^7$$

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<sup>7</sup> The matrix is positive semi-definite if these three variables are linearly dependent. Specifically, if  $X_R = c$ , a constant, the security is equivalent to 2-period riskless bonds. In this case, all solutions with  $B_{0,2} + cB_R = G/3$  are equivalent, so while  $B_{0,2} + cB_R$  is determined, neither  $B_{0,2}$  nor  $B_R$  is determined. Similarly, if  $X_R = c/\beta$  so that  $\beta X_R = c$ , the security is equivalent to 1-period riskless from an ultimate tax-perspective. Of course, these are simply different ways of implementing perfect tax-smoothing, so these two indeterminate cases do not alter our substantive conclusion.

**Proof of Proposition 2:** The planner solves

$$\min_{S,D} \left[ \frac{1}{2}(G-D)^2 + \frac{D^2}{2} \left( b(S-1/2)^2 + 1/2 \right) - \gamma f(SD) \right]. \quad (\text{A22})$$

The first order conditions for  $S$  and  $D$  are

$$0 = D^2 b(S-1/2) - D\gamma f'(SD), \quad (\text{A23})$$

and

$$0 = -(G-D) + D \left( b(S-1/2)^2 + 1/2 \right) - S\gamma f'(SD). \quad (\text{A24})$$

The solution takes the form

$$\begin{aligned} S^* &= \frac{1}{2} + \frac{\gamma f'(S^*D^*)}{D^*b} \\ D^* &= \frac{2}{3}G + \frac{\gamma f'(S^*D^*)}{3}. \end{aligned} \quad (\text{A25})$$

Note that the Hessian evaluated at the solution in (A25) is

$$\mathbf{H} = \begin{bmatrix} D^2b - D^2\gamma f''(SD) & \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) \\ \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) & b(S-1/2)^2 + \frac{3}{2} - \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right)^2 \gamma f''(SD) \end{bmatrix}, \quad (\text{A26})$$

with determinant  $\det(\mathbf{H}) = 3bD^2/2 - \gamma f''(SD)(3/2 + b/4) > 0$ , so this is a minimum. Furthermore, so long as  $f''(\cdot) \leq 0$ , the objective is globally convex in  $B_{0,1}$  and  $B_{0,2}$  and the solution is unique.

We now derive the comparative statics. Consider the impact of  $\gamma$  on  $S^*$  and  $D^*$ :

$$\begin{aligned} \begin{bmatrix} \partial S^* / \partial \gamma \\ \partial D^* / \partial \gamma \end{bmatrix} &= \begin{bmatrix} D^2b - D^2\gamma f''(SD) & \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) \\ \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) & b(S-1/2)^2 + \frac{3}{2} - \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right)^2 \gamma f''(SD) \end{bmatrix}^{-1} \begin{bmatrix} Df'(SD) \\ Sf'(SD) \end{bmatrix} \\ &= \frac{2f'(SD)}{6b - \gamma(6-b)f''(SD)} \begin{bmatrix} \frac{3D - \gamma f'(SD)}{D^2} \\ b \end{bmatrix}. \end{aligned} \quad (\text{A27})$$

Since  $D^* = 2G/3 + \gamma f'(S^*D^*)/3$ , we have  $3D^* > \gamma f'(S^*D^*)$  since  $G > 0$ . Therefore, we have  $\partial S^* / \partial \gamma > 0$  and  $\partial D^* / \partial \gamma > 0$ .

We next examine the impact of  $b$  on  $S^*$  and  $D^*$ :

$$\begin{aligned}
\begin{bmatrix} \partial S^* / \partial b \\ \partial D^* / \partial b \end{bmatrix} &= \begin{bmatrix} D^2 b - D^2 \gamma f''(SD) & \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) \\ \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) & b(S-1/2)^2 + \frac{3}{2} - \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right)^2 \gamma f''(SD) \end{bmatrix}^{-1} \begin{bmatrix} D^2(S-1/2) \\ D(S-1/2)^2 \end{bmatrix} \\
&= \frac{2f'(SD)}{6b - \gamma(6-b)f''(SD)} \begin{bmatrix} \frac{\gamma}{b^2 D^2} (f''(SD)\gamma^2 f'(SD) + f''(SD)bD\gamma/2 - 3bD) \\ -\gamma^2 f''(SD) \\ b \end{bmatrix}.
\end{aligned} \tag{A28}$$

Thus,  $\partial S^* / \partial b < 0$  and  $\partial D^* / \partial b \geq 0$ .

Last, the impact of  $G$  is given by:

$$\begin{aligned}
\begin{bmatrix} \partial S^* / \partial G \\ \partial B^* / \partial G \end{bmatrix} &= \begin{bmatrix} D^2 b - D^2 \gamma f''(SD) & \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) \\ \gamma f'(SD) - D \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right) \gamma f''(SD) & b(S-1/2)^2 + \frac{3}{2} - \left( \frac{1}{2} + \frac{\gamma f'(SD)}{Db} \right)^2 \gamma f''(SD) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{2}{6b - \gamma(6-b)f''(SD)} \begin{bmatrix} \frac{\gamma}{b^2 D^2} (f''(SD)bD - 2bf'(SD) + 2\gamma f'(SD)f''(SD)) \\ 2b - 2f''(SD)\gamma \end{bmatrix}.
\end{aligned} \tag{A29}$$

Thus,  $\partial S^* / \partial G < 0$  and  $\partial D^* / \partial G > 0$ .

**Proof of Proposition 3:** For simplicity, assume that all government debt is sold to foreign investors, including both short-term and long-term debt. Since all debt is sold to foreign investors, domestic household consumption is given by:

$$\begin{aligned}
C_0 &= 1 - \tau_0 - (1/2)\tau_0^2 \\
C_1 &= 1 - \tau_1 - (1/2)\tau_1^2 \\
C_2 &= 1 - \tau_2 - (1/2)\tau_2^2.
\end{aligned} \tag{A30}$$

(Compare (A30) with equation (7) in the text.) Substituting the government's budget constraint in (6) into (A30) we obtain

$$\begin{aligned}
C_0 &= 1 - (1/2)\tau_0^2 - G + (B_{0,1}P_{0,1} + B_{0,2}P_{0,2}) \\
C_1 &= 1 - (1/2)\tau_1^2 + (B_{1,2}P_{1,2} - B_{0,1}) \\
C_2 &= 1 - (1/2)\tau_2^2 - (B_{1,2} + B_{0,2}).
\end{aligned} \tag{A31}$$

As before, domestic consumption is impacted by distortionary tax costs, but relative to (8) there are additional terms which reflect *net* foreign borrowing in each period.

Assuming that  $P_{0,1} = 1 + v'(B_{0,1})$ ,  $P_{0,2} = 1$ , and  $P_{1,2} = \beta$  (i.e., foreign investors have the same preference shock as domestic households), it is easy to see that

$$U = C_0 + E[C_1 + \beta C_2] = 3 - G - \frac{1}{2}[\tau_0^2 + E[\tau_1^2] + E[\beta \tau_2^2]] + B_{0,1}v'(B_{0,1}). \quad (\text{A32})$$

Thus, dropping constants, the nationalistic planner's problem can be rewritten as

$$\min_{S,D} \left[ \frac{1}{2}(G - D - R(DS))^2 + \frac{D^2}{2} \left( b \left( S - \frac{1}{2} \right)^2 + \frac{1}{2} \right) - R(DS) \right], \quad (\text{A33})$$

where  $R(M) = v'(M)M$  denotes seignorage revenue and we assume  $R'(M) > 0$  and  $R''(M) \leq 0$ .

The first order conditions for  $S$  and  $D$  are

$$0 = -(1 + G - D - R(DS))DR'(SD) + D^2b(S - 1/2), \quad (\text{A34})$$

and

$$0 = -(G - D - R(DS)) - (1 + G - D - R(DS))SR'(SD) + D \left( b(S - 1/2)^2 + 1/2 \right). \quad (\text{A35})$$

Solving (A34) and (A35) shows that the solution takes the form

$$S^* = \frac{1}{2} + \frac{R'(D^*S^*)}{b} \frac{(G - R(D^*S^*) + 3)}{R'(D^*S^*)(G - R(D^*S^*) + 3) + 2(G - R(D^*S^*) - R'(D^*S^*))} \quad (\text{A36})$$

$$D^* = \frac{(2 + R'(D^*S^*))(G - R(D^*S^*)) + R'(D^*S^*)}{3 + R'(D^*S^*)}.$$

We now derive the comparative statics. Consider the impact of  $b$  on  $S^*$  and  $D^*$ . We have:

$$\begin{aligned} \begin{bmatrix} \partial S^* / \partial b \\ \partial D^* / \partial b \end{bmatrix} &= \begin{bmatrix} (DR')^2 - (1 + G - D - R)D^2R'' + D^2b & (1 + SR')DR' - (1 + G - D - R)DSR'' + Db(S - 1/2) \\ (1 + SR')DR' - (1 + G - D - R)DSR'' + Db(S - 1/2) & (1 + SR')^2 - (1 + G - D - R)S^2R'' + b(S - 1/2)^2 + 1/2 \end{bmatrix}^{-1} \begin{bmatrix} D^2(S - 1/2) \\ D(S - 1/2)^2 \end{bmatrix} \\ &= \frac{2(S - 1/2)}{6b + 4bR' + (2 + b)(R')^2 - R''(6 + b)(1 + G - D - R)} \begin{bmatrix} -(R' + 2SR' + S(R')^2 - SR''(1 + G - D - R) + 3) \\ D(2R' + (R')^2 - R''(1 + G - D - R)) \end{bmatrix}. \end{aligned}$$

Since  $S^* > 1/2$ ,  $(1 + G - D - R) = (1 + \tau_0) > 0$ ,  $R' > 0$ , and  $R'' < 0$ , we have  $\partial S^* / \partial b < 0$  and  $\partial D^* / \partial b > 0$ .

The impact of  $G$  is given by:

$$\begin{aligned} \begin{bmatrix} \partial S^* / \partial G \\ \partial D^* / \partial G \end{bmatrix} &= \begin{bmatrix} (DR')^2 - (1 + G - D - R)D^2R'' + D^2b & (1 + SR')DR' - (1 + G - D - R)DSR'' + Db(S - 1/2) \\ (1 + SR')DR' - (1 + G - D - R)DSR'' + Db(S - 1/2) & (1 + SR')^2 - (1 + G - D - R)S^2R'' + b(S - 1/2)^2 + 1/2 \end{bmatrix}^{-1} \begin{bmatrix} DR' \\ (1 + SR') \end{bmatrix} \\ &= \frac{1}{6b + 4bR' + (2 + b)(R')^2 - R''(6 + b)(1 + G - D - R)} \begin{bmatrix} (4SR''(1 + G - D - R) - 4(R'/D)) / D \\ 2b + bR' - 2R''(1 + G - D - R) \end{bmatrix}, \end{aligned}$$

where the second line follows by using (A36) to simplify the resulting expression. Thus,

$\partial S^* / \partial G < 0$  and  $\partial D^* / \partial G > 0$ .

**Proof of Propositions 4 and 5:** We solve the second-best problem. The first-best problem can be seen as a special case of the second-best problem which is obtained by setting  $\phi = 1$ . We start with the planner's objective function

$$U_{SOCIAL} = E[g(K) - K] + v(M_0) - \frac{1}{2}[\tau_0^2 + E[\tau_1^2] + E[\beta\tau_2^2]]. \quad (\text{A37})$$

We plug into this the reaction function implicitly defined by equation (21) in the text,  $M_p^*(M_G, \phi)$ :

$$\begin{aligned} U_{SOCIAL} = & p[g(W) - W] + (1-p)[g(W - M_p^*(SD, \phi)) - (W - M_p^*(SD, \phi))] \\ & + v(M_p^*(SD, \phi) + SD) - \frac{1}{2}(G - D)^2 - \frac{D^2}{2} \left( b \left( S - \frac{1}{2} \right)^2 + \frac{1}{2} \right). \end{aligned} \quad (\text{A38})$$

The first order condition for  $S^{***}$  is

$$\begin{aligned} 0 = & -(1-p)(g'(W - M_p^*) - 1)D \frac{\partial M_p^*}{\partial M_G} + v'(SD + M_p^*)D \left( 1 + \frac{\partial M_p^*}{\partial M_G} \right) - D^2 b(S - 1/2) \\ = & (1-p)(\phi - 1)g'(W - M_p^*) \frac{\partial M_p^*}{\partial M_G} D + v'(SD + M_p^*)D - D^2 b(S - 1/2). \end{aligned} \quad (\text{A39})$$

Where the second line follows from the fact that  $v'(M_p^* + SD) = (1-p)(\phi g'(W - M_p^*) - 1)$ .

Rearranging and dividing by  $D$ , we obtain

$$Db(S - 1/2) = (1-p)(\phi - 1)g'(W - M_p^*) \frac{\partial M_p^*}{\partial M_G} + v'(SD + M_p^*), \quad (\text{A40})$$

which is equation (25) in the text. Note that the first-best solution given in equation (21) obtains as a special case of (A40) by setting  $\phi = 1$ . The first order condition for  $D^{***}$  can be written as

$$0 = (1-p)(\phi - 1)g'(W - M_p^*) \frac{\partial M_p^*}{\partial M_G} S + v'(SD + M_p^*)S + (G - D) - D \left( b(S - 1/2)^2 + 1/2 \right). \quad (\text{A41})$$

We later use the fact that the first order conditions for  $S^{***}$  and  $D^{***}$  imply

$$G = D^{***} \left( 3/2 - b(S^{***} - 1/2)/2 \right). \quad (\text{A42})$$

Letting

$$\lambda_p = \frac{\partial M_p^*}{\partial M_G} = - \frac{v''(M_G + M_p^*(M_G, \phi))}{v''(M_G + M_p^*(M_G, \phi)) + (1-p)\phi g''(W - M_p^*(M_G, \phi))} < 0 \quad (\text{A43})$$

(recall that  $-1 < \lambda_p < 0$ ) denote the crowding out effect of short-term government issuance and

$$\Psi = (1-p)(1-\phi)g''(\cdot)\lambda_p^2 + (1-p)(\phi-1)g'(\cdot)\left(\frac{\partial\lambda_p}{\partial M_G} + \frac{\partial\lambda_p}{\partial M_p}\lambda_p\right) + v''(\cdot)(1+\lambda_p). \quad (\text{A44})$$

The Hessian for this problem at the solution defined by (A39) and (A41) is

$$\mathbf{H} = \begin{bmatrix} \Psi D^2 - bD^2 & \Psi SD - Db(S-1/2) \\ \Psi SD - Db(S-1/2) & \Psi S^2 - b(S-1/2)^2 - 3/2 \end{bmatrix} \quad (\text{A45})$$

We assume that  $\Psi < 0$  at  $S = S^{***}$  and  $D = D^{***}$ . This ensures that the second order conditions are satisfied since this implies  $\det(\mathbf{H}) = D^2[(3/2)b - (3/2)\Psi - (1/4)b\Psi] > 0$ . As above, if  $\Psi < 0$ , the objective will be globally concave in  $B_{0,1}$  and  $B_{0,2}$ , ensuring uniqueness.

We now examine the comparative statics with respect to  $\phi$ . Differentiate the first order condition for  $S$  with respect to  $\phi$  to obtain:

$$\left( (1-p)g'(\cdot)\lambda_p - (1-p)(\phi-1)g''(\cdot)\lambda_p \frac{\partial M_p^*}{\partial \phi} + (1-p)(\phi-1)g'(\cdot) \left[ \frac{\partial\lambda_p}{\partial \phi} + \frac{\partial\lambda_p}{\partial M_p} \frac{\partial M_p^*}{\partial \phi} \right] + v''(\cdot) \frac{\partial M_p^*}{\partial \phi} \right) D. \quad (\text{A46})$$

Noting that

$$\begin{aligned} \frac{\partial M_p^*}{\partial \phi} &= \frac{(1-p)g'(\cdot)}{v''(\cdot) + (1-p)\phi g''(\cdot)} < 0 \\ \frac{\partial \lambda_p}{\partial \phi} &= \frac{(1-p)v''(\cdot)g''(\cdot)}{[v''(\cdot) + (1-p)\phi g''(\cdot)]^2} > 0 \\ \frac{\partial \lambda_p}{\partial M_p} &= -(1-p)\phi \frac{v'''(\cdot)g''(\cdot) + v''(\cdot)g'''(\cdot)}{[v''(\cdot) + (1-p)\phi g''(\cdot)]^2}, \end{aligned} \quad (\text{A47})$$

which imply  $(1-p)g'(\cdot)\lambda_p + v''(\cdot)(\partial M_p^* / \partial \phi) = 0$ , the expression in (A46) simplifies to

$$\begin{aligned} & -(1-p)(\phi-1)g''(\cdot)\lambda_p \frac{\partial M_p^*}{\partial \phi} + (1-p)(\phi-1)g'(\cdot) \left[ \frac{\partial\lambda_p}{\partial \phi} + \frac{\partial\lambda_p}{\partial M_p} \frac{\partial M_p^*}{\partial \phi} \right] \\ &= \frac{(\phi-1)g'(\cdot)(1-p)^2}{[v''(\cdot) + (1-p)\phi g''(\cdot)]^2} \left( 2v''(\cdot)g''(\cdot) - (1-p)\phi g'(\cdot) \frac{v'''(\cdot)g''(\cdot) + v''(\cdot)g'''(\cdot)}{v''(\cdot) + (1-p)\phi g''(\cdot)} \right) D^{***}. \end{aligned} \quad (\text{A48})$$

The expression in (A48) will be negative when  $\phi < 1$  so long as  $g'''(\cdot)$  and  $v'''(\cdot)$  are not too large which we assume is the case.<sup>8,9</sup>

Combining all of this we have

$$\begin{aligned} \begin{bmatrix} \partial S^{***} / \partial \phi \\ \partial D^{***} / \partial \phi \end{bmatrix} &= \begin{pmatrix} (1-\phi)g'(\cdot)(1-p)^2 \\ [v''(\cdot) + (1-p)\phi g''(\cdot)]^2 \end{pmatrix} \begin{pmatrix} 2v''(\cdot)g''(\cdot) - (1-p)\phi g'(\cdot) \frac{v'''(\cdot)g''(\cdot) + v''(\cdot)g'''(\cdot)}{v''(\cdot) + (1-p)\phi g''(\cdot)} \\ \end{pmatrix} \times \\ &\begin{bmatrix} \Psi D^2 - bD^2 & \Psi SD - Db(S-1/2) \\ \Psi SD - Db(S-1/2) & \Psi S^2 - b(S-1/2)^2 - 3/2 \end{bmatrix}^{-1} \begin{bmatrix} D \\ S \end{bmatrix} \\ &= \frac{1}{\det(\mathbf{H})} \begin{pmatrix} (1-\phi)g'(\cdot)(1-p)^2 \\ [v''(\cdot) + (1-p)\phi g''(\cdot)]^2 \end{pmatrix} \begin{pmatrix} 2v''(\cdot)g''(\cdot) - (1-p)\phi g'(\cdot) \frac{v'''(\cdot)g''(\cdot) + v''(\cdot)g'''(\cdot)}{v''(\cdot) + (1-p)\phi g''(\cdot)} \\ \end{pmatrix} \begin{bmatrix} -G \\ -bD^{***2} / 2 \end{bmatrix}, \end{aligned} \quad (\text{A49})$$

where we have made use of the fact that  $G = D^{***} \left( 3/2 - b(S^{***} - 1/2) / 2 \right)$  from (A42). Thus,

$\partial S^{***} / \partial \phi < 0$  and  $\partial D^{***} / \partial \phi < 0$  so long as  $\phi < 1$  and  $g'''(\cdot)$  and  $v'''(\cdot)$  are not too large (this also implies that  $M_G^{***} = D^{***} S^{***}$  is decreasing in  $\phi$ ).

Finally, note that

$$\begin{aligned} \frac{\delta}{\delta \phi} [M_P^{***}(\phi, M_G^{***}(\phi))] &= \frac{\partial M_P^{***}}{\partial \phi} + \frac{\partial M_P^{***}}{\partial M_G} \frac{\partial M_G^{***}}{\partial \phi} > \frac{\partial M_P^{***}}{\partial \phi} \\ \frac{\delta}{\delta \phi} [M_P^{***}(\phi, M_G^{***}(\phi)) + M_G^{***}(\phi)] &= \frac{\partial M_P^{***}}{\partial \phi} + \frac{\partial M_G^{***}}{\partial \phi} \left( 1 + \frac{\partial M_P^{***}}{\partial M_G} \right) < \frac{\partial M_P^{***}}{\partial \phi} < 0. \end{aligned} \quad (\text{A50})$$

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<sup>8</sup> If  $g'''(\cdot)$  and  $v'''(\cdot)$  were too large, then  $\delta \lambda_p / \delta \phi = (\partial \lambda_p / \partial \phi) + (\partial \lambda_p / \partial M_p)(\partial M_p^* / \partial \phi)$ , the total derivative of  $\lambda_p$  with respect to  $\phi$ , would be a large negative number. In this case, as  $\phi$  declined,  $|\lambda_p|$  would decline significantly (since  $\lambda_p < 0$ ,  $\lambda_p$  would rise), greatly reducing the crowding-out benefit from issuing short-term government debt. If this force were strong enough, it could outweigh the direct effect,  $-(1-p)(\phi-1)g''(\cdot)\lambda_p(\partial M_p^* / \partial \phi) < 0$ , which reflects the fact that  $M_p^*$  rises as  $\phi$  falls, exacerbating the under-investment problem in the bad state. However, note that  $\partial \lambda_p / \partial \phi > 0$  which reflects the fact that, holding  $M_p$  and  $M_G$  fixed, private money creation becomes *more* not less sensitive to the money premium as  $\phi$  declines because firms more severely underweight its costs. Thus, if  $\partial \lambda_p / \partial M_p$  is not too large (i.e. the functions are well approximated locally by quadratics, so  $g'''(\cdot)$  and  $v'''(\cdot)$  are small), then we will have  $\delta \lambda_p / \delta \phi > 0$ , implying that  $\partial S^{***} / \partial \phi < 0$  and  $\partial D^{***} / \partial \phi < 0$ .

<sup>9</sup> The second order conditions for  $S^{***}$  and  $D^{***}$  also depend on  $g'''(\cdot)$  and  $v'''(\cdot)$  through  $\Psi$  which we assume is negative. Specifically, one can show that  $(\partial \lambda_p / \partial M_G) + (\partial \lambda_p / \partial M_p)\lambda_p$  is increasing in  $v'''(\cdot)$  and decreasing in  $g'''(\cdot)$ . Therefore,  $g'''(\cdot)$  cannot be too large if the second order conditions for  $S^{***}$  and  $D^{***}$  are to hold. Specifically, if  $g'''(\cdot)$  is too large, a rise in  $M_G$  would significantly raise  $|\lambda_p|$ , implying an increasing as opposed to diminishing crowding out benefit from issuing more short-term debt.



Thus, the increase in private money following a decline in  $\phi$  is smaller when the government recognizes the “crowding out” benefit of short-term bills. However, the total increase in public plus private short-term debt is greater than in the absence of such a policy because each dollar of additional short-term government debt crowds out less than one dollar of short-term private debt. Finally,  $\delta[M_p^{***}(\phi, M_G^{***}(\phi))]/\delta\phi < 0$ , so long as  $g'''(\cdot)$  and  $v'''(\cdot)$  are not too large and  $\phi$  is not too small (e.g. if  $g'''(\cdot) = v'''(\cdot) = 0$  and  $\phi > 1/2$ ). Since the first best solution obtains when  $\phi = 1$ , the second-best solution involves a larger quantity of government bills and more private money creation.

**Proof of Proposition 6:** Let  $M_p^*(M_G, \theta_p, \phi)$  denote the solution to equation (27) repeated here:

$$v'(M_p^* + M_G) = \theta_p + (1-p)(\phi g'(W - M_p^*) - 1). \quad (\text{A51})$$

It follows that

$$\begin{aligned} \frac{\partial M_p^*}{\partial M_G} &\equiv \lambda_p = -\frac{v''(M_p^* + M_G)}{v''(M_p^* + M_G) + (1-p)\phi g''(W - M_p^*)} < 0 \\ \frac{\partial M_p^*}{\partial \theta_p} &\equiv \eta_p = \frac{1}{v''(M_p^* + M_G) + (1-p)\phi g''(W - M_p^*)} < 0 \\ \frac{\partial M_p^*}{\partial \phi} &= \frac{(1-p)g'(W - M_p^*)}{v''(M_p^* + M_G) + (1-p)\phi g''(W - M_p^*)} < 0, \end{aligned} \quad (\text{A52})$$

with  $-1 < \lambda_p < 0$ . To get the second best solution, we rewrite the planner’s objective function in equation (28) as

$$\begin{aligned} U_{SOCIAL} &= p[g(W) - W] + (1-p)[g(W - M_p^*(SD, \theta_p, \phi)) - (W - M_p^*(SD, \theta_p, \phi))] \\ &\quad + v(M_p^*(SD, \theta_p, \phi) + SD) - \frac{1}{2}(G - D)^2 - \frac{D^2}{2} \left( b \left( S - \frac{1}{2} \right)^2 + \frac{1}{2} \right) - \frac{\Upsilon}{2} \theta_p^2. \end{aligned} \quad (\text{A53})$$

The planner now has three control variables:  $S$ ,  $\underline{D}$ , and  $\theta_p$ .

We first compute optimal taxes. The planner’s first order condition for  $\theta_p$  is

$$\begin{aligned} 0 &= \frac{\partial M_p^*}{\partial \theta_p} [v'(M_p^*(SD, \theta_p, \phi) + SD) - (1-p)g'(W - M_p^*(SD, \theta_p, \phi) - 1)] - \Upsilon \theta_p \\ &= \frac{\partial M_p^*}{\partial \theta_p} [\theta_p + (1-p)(\phi - 1)g'(W - M_p^*)] - \Upsilon \theta_p. \end{aligned} \quad (\text{A54})$$

where the second line uses (A51). This implies that

$$\theta_p^{***} = \frac{|\partial M_p^* / \partial \theta_p|}{|\partial M_p^* / \partial \theta_p| + \Upsilon} (1-p)(1-\phi)g'(W - M_p^*) \leq (1-p)(1-\phi)g'(W - M_p^*). \quad (\text{A55})$$

Thus, we have  $\theta_p^{***} > 0$  if  $\phi < 1$  and  $\Upsilon$  is finite. When  $\Upsilon = 0$ , we obtain

$$\theta_p^{***} = (1-p)(1-\phi)g'(W - M_p^*), \quad (\text{A56})$$

which implies that

$$v'(M_p^* + M_G) = \theta_p^{***} + (1-p)(\phi g'(W - M_p^*) - 1) = (1-p)(g'(W - M_p^*) - 1). \quad (\text{A57})$$

This is the same as the condition defining the first-best level of optimal private money  $M_p^{**}$ .

However, with positive deadweight costs, the optimal tax is only a fraction of the tax that makes the banks fully internalize the fire-sale externality (i.e.,  $(1-p)(1-\phi)g'(W - M_p^*)$ ). This fraction is higher if  $|\partial M_p^* / \partial \theta_p|$  is larger or if the deadweight costs are smaller.

We next compute optimal government debt maturity. The first order condition for  $S$  is the same as before:

$$0 = \left[ -(1-p)(g'(W - M_p^*) - 1) \frac{\partial M_p^*}{\partial M_G} + v'(SD + M_p^*) \left( 1 + \frac{\partial M_p^*}{\partial M_G} \right) \right] D - D^2 b(S - 1/2) \quad (\text{A58})$$

Rearranging and dividing by  $D$ , we obtain equation (29) in the text:

$$\begin{aligned} Db(S - 1/2) &= v'(M_p^* + SD) + [v'(M_p^* + SD) - (1-p)(g'(W - M_p^*) - 1)] \frac{\partial M_p^*}{\partial M_G} \\ &= v'(M_p^* + SD) + [\theta_p^* + (1-p)(\phi - 1)g'(W - M_p^*)] \frac{\partial M_p^*}{\partial M_G} \\ &= v'(M_p^* + SD) + \frac{\Upsilon}{|\partial M_p^* / \partial \theta_p| + \Upsilon} [(\phi - 1)(1-p)(g'(W - M_p^*) - 1)] \frac{\partial M_p^*}{\partial M_G}. \end{aligned} \quad (\text{A59})$$

This manipulation uses the market clearing condition (A51) for  $M_p^*$  to substitute out for  $v'$  and then uses the optimal expression for  $\theta_p$  from (A55). The second term above, which reflects the crowding-out benefits of issuing additional short-term government debt, is positive so long as  $\phi < 1$  and  $\Upsilon > 0$ .

We now turn to the comparative statics calculations for the planner's problem with two tools.

The first order conditions for  $S$ ,  $D$ , and  $\theta_p$  can be written as:

$$\begin{aligned} 0 &= [v'(M_p^* + SD)(1 + \lambda_p) - (1 - p)(g'(W - M_p^*) - 1)\lambda_p]D - D^2b(S - 1/2) \\ 0 &= [v'(M_p^* + SD)(1 + \lambda_p) - (1 - p)(g'(W - M_p^*) - 1)\lambda_p]S + (G - D) - D[b(S - 1/2)^2 + 1/2] \quad (\text{A60}) \\ 0 &= \eta_p[v'(M_p^* + SD) - (1 - p)(g'(W - M_p^*) - 1)] - \Upsilon\theta_p \end{aligned}$$

The Hessian for this problem takes the form

$$\mathbf{H} = \begin{bmatrix} \Psi D^2 - bD^2 & \Psi SD - bD(S - 1/2) & \Phi D \\ \Psi SD - bD(S - 1/2) & \Psi S^2 - b(S - 1/2)^2 - 3/2 & \Phi S \\ \Phi D & \Phi S & \Xi - \Upsilon \end{bmatrix}, \quad (\text{A61})$$

where

$$\begin{aligned} \Psi &= v''(\cdot)(1 + \lambda_p)^2 + (1 - p)g''(\cdot)(\lambda_p)^2 = (1 - p)v''(\cdot)g''(\cdot) \frac{v''(\cdot) + (1 - p)\phi^2 g''(\cdot)}{(v''(\cdot) + (1 - p)\phi g''(\cdot))^2} < 0 \\ \Xi &= (\eta_p)^2[v''(\cdot) + (1 - p)g''(\cdot)] = \frac{v''(\cdot) + (1 - p)g''(\cdot)}{(v''(\cdot) + (1 - p)\phi g''(\cdot))^2} < 0 \quad (\text{A62}) \\ \Phi &= \eta_p \lambda_p [v''(\cdot) + (1 - p)g''(\cdot)] = -v''(\cdot) \frac{v''(\cdot) + (1 - p)g''(\cdot)}{(v''(\cdot) + (1 - p)\phi g''(\cdot))^2} < 0. \end{aligned}$$

These quantities are calculated under the simplifying assumption that  $v'''(\cdot) = g'''(\cdot) = 0$ , so that  $\partial \lambda_p / \partial M_p = \partial \lambda_p / \partial M_G = \partial \eta_p / \partial M_p = \partial \eta_p / \partial M_G = 0$ . However, the inequalities in (A62) will hold so long as the relevant third derivatives are not too large in magnitude. We assume that  $\det(\mathbf{H}) < 0$  by the second order condition for the planner's problem.

The comparative statics with respect to  $\Upsilon$  follow from

$$\begin{aligned}
\begin{bmatrix} \partial S^{***} / \partial \Upsilon \\ \partial D^{***} / \partial \Upsilon \\ \partial \theta_p^{***} / \partial \Upsilon \end{bmatrix} &= - \begin{bmatrix} \Psi D^2 - bD^2 & \Psi SD - bD(S-1/2) & \Phi D \\ \Psi SD - bD(S-1/2) & \Psi S^2 - b(S-1/2)^2 - 3/2 & \Phi S \\ \Phi D & \Phi S & \Xi - \Upsilon \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -\theta_p \end{bmatrix} \\
&= \frac{\theta_p}{\det(\mathbf{H})} \begin{bmatrix} \Phi G \\ \frac{1}{2} \Phi b D^2 \\ \frac{1}{4} D^2 (6b - (6+b)\Psi) \end{bmatrix}, \tag{A63}
\end{aligned}$$

where we have used the fact that  $G = D^{***} (3/2 - b(S^{***} - 1/2)/2)$  by (A60) to simplify the resulting expression. Thus, we have  $\partial S^{***} / \partial \Upsilon > 0$ ,  $\partial D^{***} / \partial \Upsilon > 0$ , and  $\partial \theta_p^{***} / \partial \Upsilon < 0$ .

The comparative statics with respect to  $b$  follow from

$$\begin{aligned}
\begin{bmatrix} \partial S^{***} / \partial b \\ \partial D^{***} / \partial b \\ \partial \theta_p^{***} / \partial b \end{bmatrix} &= - \begin{bmatrix} \Psi D^2 - bD^2 & \Psi SD - bD(S-1/2) & \Phi D \\ \Psi SD - bD(S-1/2) & \Psi S^2 - b(S-1/2)^2 - 3/2 & \Phi S \\ \Phi D & \Phi S & \Xi - \Upsilon \end{bmatrix}^{-1} \begin{bmatrix} -D^2(S-1/2) \\ -D(S-1/2)^2 \\ 0 \end{bmatrix} \\
&= \frac{-1}{\det(\mathbf{H})} \begin{bmatrix} D^2(S-1/2) \left( \frac{S}{2} (\Psi \Upsilon - (\Psi \Xi - \Phi^2)) + \frac{3}{2} (\Xi - \Upsilon) \right) \\ \frac{1}{2} D^3(S-1/2) ((\Psi \Xi - \Phi^2) - \Psi \Upsilon) \\ -\frac{3}{2} \Phi D^3(S-1/2) \end{bmatrix}. \tag{A64}
\end{aligned}$$

Making the natural regularity assumption that  $\Psi \Xi > \Phi^2$  so the social returns to limiting private money creation are concave in  $M_G$  and  $\theta_p$  (a sufficient condition is that  $(1-p)\phi g''(\cdot) < v''(\cdot)$ ) and noting that  $S^{***} > 1/2$ , we have  $\partial S^{***} / \partial b < 0$ ,  $\partial D^{***} / \partial b > 0$ , and  $\partial \theta_p^{***} / \partial b > 0$ .

Turning to the comparative statics with respect to  $\phi$ , note that  $\partial \lambda_p / \partial \phi > 0$  and  $\partial \eta_p / \partial \phi > 0$  (i.e., the efficacy of both crowding out and regulation rise as  $\phi$  falls). Next let

$$\begin{aligned}
\Theta &= [v'(\cdot) - (1-p)(g'(\cdot) - 1)] \frac{\partial \lambda_p}{\partial \phi} + [v''(\cdot)(1 + \lambda_p) + (1-p)g''(\cdot)\lambda_p] \frac{\partial M_p^*}{\partial \phi} \\
&= [v'(\cdot) - (1-p)(g'(\cdot) - 1) - (1-p)(1-\phi)g'(\cdot)] \frac{(1-p)g''(\cdot)v''(\cdot)}{(v''(\cdot) + (1-p)\phi g''(\cdot))^2} < 0 \tag{A65}
\end{aligned}$$

(the term in square brackets is negative for  $\phi < 1$ ) which reflects the lower social returns to crowding out when the externality becomes less severe (i.e., when  $\phi$  rises). Similarly, let

$$\Gamma = [v'(\cdot) - (1-p)(g'(\cdot) - 1)] \frac{\partial \eta_p}{\partial \phi} + \eta_p [v''(\cdot) + (1-p)g''(\cdot)] \frac{\partial M_p^*}{\partial \phi} < 0, \quad (\text{A66})$$

(the term in square brackets is negative for  $\phi < 1$ ), which reflects the lower social returns to direct regulation when  $\phi$  rises. We then have

$$\begin{aligned} \begin{bmatrix} \partial S^{***} / \partial \phi \\ \partial D^{***} / \partial \phi \\ \partial \theta_p^{***} / \partial \phi \end{bmatrix} &= - \begin{bmatrix} \Psi D^2 - bD^2 & \Psi SD - bD(S-1/2) & \Phi D \\ \Psi SD - bD(S-1/2) & \Psi S^2 - b(S-1/2)^2 - 3/2 & \Phi S \\ \Phi D & \Phi S & \Xi - \Upsilon \end{bmatrix}^{-1} \begin{bmatrix} \Theta D \\ \Theta S \\ \Gamma \end{bmatrix} \\ &= \frac{-1}{\det(\mathbf{H})} \begin{bmatrix} ((\Gamma\Phi - \Theta\Xi) + \Theta\Upsilon)G \\ \frac{1}{2}((\Gamma\Phi - \Theta\Xi) + \Theta\Upsilon)bD^2 \\ (\Theta\Phi - \Gamma\Psi)\frac{1}{4}D^2(6+b) + \frac{3}{2}bD^2\Gamma \end{bmatrix}. \end{aligned} \quad (\text{A67})$$

Since crowding out and regulation are substitutes from the perspective of limiting private money creation (i.e.,  $\Phi < 0$ ), both  $(\Gamma\Phi - \Theta\Xi)$  and  $(\Theta\Phi - \Gamma\Psi)$  are ambiguous because they are the difference of two positive terms. However, if either (i)  $|\Phi|$  is small relative to both  $|\Xi|$  and  $|\Psi|$  or (ii) both  $b$  and  $\Upsilon$  are sufficiently large, then we have  $\partial S^{***} / \partial \phi < 0$ ,  $\partial D^{***} / \partial \phi < 0$ , and  $\partial \theta_p^{***} / \partial \phi < 0$ .

**Proof of Proposition 7:** We now extend the model by adding an additional period (i.e., the dates of the model are now  $t = 0, 1, 2, 3$ ) and by allowing short-term debt to generate monetary services at the interim dates ( $t = 1$  and  $t = 2$ ) in addition to the initial date. (For simplicity, we do not allow for private money creation.) This extension serves two purposes. First, it shows that our results are not driven by the simplifying assumption that households only enjoy money services at time 0. Secondly, the extension allows us to investigate how the hedging opportunities afforded by multiples maturities alter the tax-smoothing costs faced by the government.

The government finances a one-time expenditure  $G$  at date 0 by issuing short-term (1-period) bonds  $B_{0,1}$ , medium-term (2-period) bonds  $B_{0,2}$ , and long-term (3-period) bonds  $B_{0,3}$  to households and by levying distortionary taxes,  $\tau_0$ . At time 1, the government must repay any maturing debt by

levying taxes and issuing new short- and long-term bonds. At time 2, the government repays maturing debt by levying taxes and issuing new short-term bonds. All debt maturing at time 3 must be repaid by levying taxes. Thus, the sequence of government budget constraints is given by:

$$\begin{aligned}
t=0: G &= \tau_0 + B_{0,1}P_{0,1} + B_{0,2}P_{0,2} + B_{0,3}P_{0,3} \\
t=1: B_{0,1} &= \tau_1 + B_{1,2}P_{1,2} + B_{1,3}P_{1,3} \\
t=2: B_{0,2} + B_{1,2} &= \tau_2 + B_{2,3}P_{2,3} \\
t=3: B_{0,3} + B_{1,3} + B_{2,3} &= \tau_3,
\end{aligned} \tag{A68}$$

where  $P_{0,1}$ ,  $P_{0,2}$ , and  $P_{0,3}$  denote the prices of short-, medium-, and long-term bonds issued at date 0,  $P_{1,2}$  and  $P_{1,3}$  denote the uncertain prices of short- and long-term bonds issued at date 1, and  $P_{2,3}$  is uncertain price of short-term bonds issued at date 2.

There are now two uncertain interest rates. At time 1, households learn  $\beta_1$  which pins down the short rate between periods 1 and 2. At time 2, households learn  $\beta_2$  which determines the short rate between periods 2 and 3. However, at time 1, households also update their expectations of  $\beta_2$  based on the realization of  $\beta_1$ . Specifically, households learn  $\delta_1 = E[\beta_2 | \beta_1]$  at time 1, but  $\beta_2 = \delta_1 \varepsilon_2$  is only realized at time 2 where  $E[\varepsilon_2 | \beta_1, \delta_1] = 1$ . Note that  $1/\delta_1$  is simply the gross forward short-term interest rate at time 1. Thus, there are now effectively three interest rate shocks: the realization of  $\beta_1$  at time 1, the “news” about  $\beta_2$  at time 1, and the ultimate realization of  $\beta_2$  at time 2. If all shifts in the yield curve are parallel, then  $\delta_1 \equiv \beta_1$  so, in that case, there are only two non-degenerate shocks. Without loss of generality, we assume the term structure is initially flat:  $E[\beta_1] = E[\beta_1 \beta_2] = 1$ . However, we allow  $\beta_1$  and  $\delta_1$  to be correlated. Since  $1 = E[\beta_1 \delta_1] = Cov[\beta_1, \delta_1] + E[\beta_1]E[\delta_1]$ , we have  $Cov[\beta_1, \delta_1] = 1 - E[\delta_1]$ .

By the above assumptions, the prices of 1-, 2-, and 3-period bonds are all equal to 1 at time 0:  $P_{0,1} = P_{0,2} = P_{0,3} = 1$ . At time 1, the price of 1-period bonds maturing at time 2 is  $P_{1,2} = \beta_1$  and the price of 2-period bonds maturing at time 3 is  $P_{1,3} = \beta_1 \delta_1$ . Finally, at time 2, the price of 1-period bonds maturing at time 3 is  $P_{2,3} = \beta_2 = \delta_1 \varepsilon_2$ .

Household enjoy monetary services at time 0, 1, and 2 based on the total stock of outstanding 1-period bonds at each date. For simplicity, we work with linear money utility in this extension, so households obtain money utility  $v(M_t) = \gamma \cdot M_t$  at time  $t$  where  $M_t$  is the total stock of outstanding 1-period bonds at  $t$ . Thus, we have  $M_0 = B_{0,1}$ ,  $M_1 = B_{0,2} + B_{1,2}$ , and  $M_2 = B_{0,3} + B_{1,3} + B_{2,3}$ .

*Basic intuition:* Before proceeding with the proof, we first provide some intuition for Proposition 7. To begin, consider the perfect smoothing case. As before, the government needs to finance an expenditure of 1. One option is to set  $B_{0,1} = B_{0,2} = B_{0,3} = 1/4$ , which corresponds to a weighted average debt maturity of 2, which allows taxes to be perfectly smoothed ( $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 1/4$ ). If there is no motive to create monetary services, this is indeed the optimal debt structure.

Now, suppose that the government wishes to provide monetary services at time 0 by increasing the supply of short-term bonds. Assume for simplicity that households do not derive monetary services at time 1 and 2.<sup>10</sup> A simple version of the barbell strategy would be to set  $B_{0,1} = 3/8$ ,  $B_{0,2} = 0$ , and  $B_{0,3} = 3/8$ , levying taxes  $\tau_0 = 1/4$ . Note that this keeps the average debt maturity unchanged at 2.

At time 1, the government must pay off debt of 3/8 through a combination of taxes, new short-term debt, and new two-period debt. First, suppose that the term structure remains flat at time 1, so that  $\beta_1 = E[\beta_2 | \beta_1] = 1$ . The desire to smooth taxes will lead the government to raise taxes of 1/4 at time 1, leaving it with 1/8 to finance using new issues. In order to smooth taxes going forward, the government should raise 1/4 in new short-term debt, using 1/8 to pay off the maturing short-term debt and the other 1/8 to *repurchase* long-term bonds due at time 3. This operation leaves the government with 1/4 of debt maturing at both time 2 and time 3 which it repays by levying taxes of 1/4 at each date. In other words, given an initial barbell maturity structure and the desire to smooth taxes, the government issues additional short-term debt at time 1 and some of the proceeds are used to repurchase long-term bonds maturing at date 3. The issuance and repayment schedule associated with such a barbell strategy is summarized in the table below:

	$t = 0$	$t = 1$	$t = 2$	$t = 3$
$G$	-1.000			
Taxes	0.250	0.250	0.250	0.250
Issuance of debt due $t = 1$	0.375	-0.375	-	-
Issuance of debt due $t = 2$	0.000	0.250	-0.250	-
Issuance of debt due $t = 3$	0.375	-0.125	0.000	0.250

Of course,  $\beta_1$  and  $E[\beta_2 | \beta_1]$  are not fixed, but instead vary randomly, which creates uncertainty about future taxes and, in turn, generates tax-smoothing costs. How do taxes and debt

<sup>10</sup> As shown below, very similar results obtain if households enjoy monetary services based on the stock of short-term debt outstanding at the interim dates ( $t = 1$  and 2) in addition to the initial date. Allowing for monetary services at these interim dates has little impact on the optimal maturity structure at time 0 because the government always has the option to re-optimize its debt maturity structure at times 1 and 2.

issuance vary as a function of the interest rate shocks? Given the initial reliance on short-term debt, the government is naturally exposed to short rates at time 1: the government needs to borrow less and can tax less when  $\beta_1$  is high (short rates are low) and it can roll-over its existing short-term debt on more favorable terms. However, holding fixed  $\beta_1$ , the price of the extra long-term bonds the government needs to retire (with the proceeds of new short-term bonds) is increasing in  $E[\beta_2 | \beta_1]$ . Thus, under the barbell strategy both short-term issuance and taxes at time 1 are increasing in  $E[\beta_2 | \beta_1]$ . If the realizations of  $\beta_1$  and  $E[\beta_2 | \beta_1]$  are highly correlated—e.g., if there are only level shifts in the yield curve—then this barbell strategy means that the government has offsetting exposures to the  $\beta_1$  and  $E[\beta_2 | \beta_1]$  shocks, better allowing it to smooth taxes. Intuitively, the government creates a “short” position in long-term bonds at time 1. When time 1 movements in short rates and forward rates are highly correlated, this “short” position in long bonds hedges the government’s natural “long” position in short-term bonds.

The key insight from the multi-period extension is that if interest rate shocks are primarily driven by parallel shifts in the yield curve, then a barbell strategy enables the government to hedge out most of the interest rate exposure created by its initial reliance on short-term debt. In the limiting case in which  $E[\beta_2 | \beta_1] = \beta_1$ , the central tradeoff between tax smoothing and the production of monetary services disappears, because the government can perfectly immunize itself against interest rate shocks. In other words, the barbell strategy allows for a decoupling of money creation—which is accomplished by issuing more short-term debt—and tax smoothing, which, roughly speaking, is accomplished by keeping duration fixed at the right value.

At the other extreme, if  $E[\beta_2 | \beta_1]$  and  $\beta_1$  are independently distributed, then the barbell strategy no longer provides an effective hedge. Thus, if the government wants to create more in the way of monetary services, it must do so by shortening the weighted average maturity of its debt and by accepting some loss of tax smoothing. More generally, the effectiveness of this kind of barbell strategy is increasing in the correlation between the absolute value of  $E[\beta_2 | \beta_1]$  and  $\beta_1$ .

*The government’s time-2 problem:* To solve the model, we work backwards from time 2, taking time-1 and time-2 issuance as given. Using (A68), the expressions for prices, and the expression for  $M_2$ , the government’s problem at time 2 is:



$$\begin{aligned} & \min_{B_{2,3}} \left[ \frac{1}{2} (\tau_2^2 + \beta_2 \tau_3^2) - \gamma \cdot M_2 \right] \\ & = \min_{B_{2,3}} \left[ \frac{1}{2} (B_{0,2} + B_{1,2} - B_{2,3} \beta_2)^2 + \frac{1}{2} \beta_2 (B_{0,3} + B_{1,3} + B_{2,3})^2 - \gamma \cdot (B_{0,3} + B_{1,3} + B_{2,3}) \right]. \end{aligned} \quad (\text{A69})$$

The solution to (A69) is then

$$B_{2,3}^* = \frac{(B_{0,2} + B_{1,2}) - (B_{0,3} + B_{1,3})}{1 + \beta_2} + \frac{\gamma}{\beta_2(1 + \beta_2)}, \quad (\text{A70})$$

which is the natural generalization of (A2). The first term in (A70) implements perfect tax smoothing between times 2 and 3 and the second component is the optimal deviation from perfect smoothing: the government tilts toward short-term debt because households derive monetary services from short-term debt at time 2 (i.e.,  $\gamma > 0$ ). This solution implies that

$$\begin{aligned} \tau_2^* &= \frac{(B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3})}{1 + \beta_2} - \frac{\gamma}{(1 + \beta_2)} \\ \tau_3^* &= \frac{(B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3})}{1 + \beta_2} + \frac{\gamma}{\beta_2(1 + \beta_2)}. \end{aligned} \quad (\text{A71})$$

The present value of taxes is  $\tau_2^* + \beta_2 \tau_3^* = (B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3})$  independently of  $\gamma$ . Thus, the government responds to  $\gamma > 0$  by taxing a bit less at  $t = 2$  and a bit more at  $t = 3$  in order to create additional money services at  $t = 2$ . Algebra shows that the minimized objective function is given by

$$\begin{aligned} V_2(B_{0,2} + B_{1,2}, B_{0,3} + B_{1,3}, \beta_2) &= \min_{B_{2,3}} \left[ \frac{1}{2} (\tau_2^2 + \beta_2 \tau_3^2) - \gamma \cdot M_2 \right] \\ &= \frac{1}{2} \frac{((B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3}))^2}{1 + \beta_2} - \gamma \cdot \frac{(B_{0,2} + B_{1,2}) + \beta_2 (B_{0,3} + B_{1,3})}{1 + \beta_2} - \frac{\gamma^2}{2} \frac{1}{\beta_2(1 + \beta_2)}. \end{aligned} \quad (\text{A72})$$

We can omit the final term when we move backwards to time 1 since this term is independent of the government's prior debt maturity choices.

*The government's time-1 problem:* Now consider the government's problem at time 1 taking time 0 issuance as given. Recall that at time 1 agents learn  $\beta_1$  and  $\delta_1 = E[\beta_2 | \beta_1]$ , but that  $\beta_2 = \delta_1 \varepsilon_2$  is still uncertain as of time 1 since  $\varepsilon_2$  is not realized until time 2. The government's time 1 problem is:

$$\begin{aligned}
& \min_{B_{1,2}, B_{1,3}} \left[ \frac{1}{2} \tau_2^2 - \gamma \cdot M_1 + \beta_1 \cdot E \left[ V_2(B_{0,2} + B_{1,2}, B_{0,3} + B_{1,3}, \beta_2) \mid \beta_1, \delta_1 \right] \right] \\
& = \min_{B_{1,2}, B_{1,3}} \left[ \frac{1}{2} (B_{0,1} - B_{1,2}\beta_1 - B_{1,3}\beta_1\delta_1)^2 - \gamma \cdot (B_{0,2} + B_{1,2}) \right. \\
& \quad \left. + \beta_1 E \left[ \frac{1}{2} \frac{((B_{0,2} + B_{1,2}) + \delta_1 \varepsilon_2 (B_{0,3} + B_{1,3}))^2}{1 + \delta_1 \varepsilon_2} - \gamma \cdot \frac{(B_{0,2} + B_{1,2}) + \delta_1 \varepsilon_2 (B_{0,3} + B_{1,3})}{1 + \delta_1 \varepsilon_2} \mid \delta_1 \right] \right]. \tag{A73}
\end{aligned}$$

The first order conditions for  $B_{1,2}$  and  $B_{1,3}$  are

$$\begin{aligned}
0 & = -\beta_1 (B_{0,1} - B_{1,2}\beta_1 - B_{1,3}\beta_1\delta_1) - \gamma + \beta_1 E \left[ \frac{((B_{0,2} + B_{1,2}) + \delta_1 \varepsilon_2 (B_{0,3} + B_{1,3}))}{1 + \delta_1 \varepsilon_2} - \gamma \cdot \frac{1}{1 + \delta_1 \varepsilon_2} \mid \delta_1 \right] \\
0 & = -\beta_1 \delta_1 (B_{0,1} - B_{1,2}\beta_1 - B_{1,3}\beta_1\delta_1) + \beta_1 E \left[ \delta_1 \varepsilon_2 \frac{((B_{0,2} + B_{1,2}) + \delta_1 \varepsilon_2 (B_{0,3} + B_{1,3}))}{1 + \delta_1 \varepsilon_2} - \gamma \cdot \frac{\delta_1 \varepsilon_2}{1 + \delta_1 \varepsilon_2} \mid \delta_1 \right]. \tag{A74}
\end{aligned}$$

We can rewrite this system as

$$\begin{bmatrix} \beta_1 + w_1 & \beta_1 \delta_1 + x_1 \\ \beta_1 \delta_1 + x_1 & \beta_1 \delta_1^2 + y_1 \end{bmatrix} \begin{bmatrix} B_{1,2} \\ B_{1,3} \end{bmatrix} = \begin{bmatrix} \gamma(1/\beta_1 + w_1) + B_{0,1} - w_1 B_{0,2} - x_1 B_{0,3} \\ \gamma x_1 + \delta_1 B_{0,1} - x_1 B_{0,2} - y_1 B_{0,3} \end{bmatrix}, \tag{A75}$$

where

$$w_1 \equiv E \left[ \frac{1}{1 + \delta_1 \varepsilon_2} \mid \delta_1 \right], \quad x_1 \equiv E \left[ \frac{\delta_1 \varepsilon_2}{1 + \delta_1 \varepsilon_2} \mid \delta_1 \right], \quad \text{and } y_1 \equiv E \left[ \frac{(\delta_1 \varepsilon_2)^2}{1 + \delta_1 \varepsilon_2} \mid \delta_1 \right], \tag{A76}$$

are random variables that are functions of the realized value of  $\delta_1$ .

It is straightforward to verify that the perfect tax-smoothing ‘‘consol’’ bond solution extends to the 4-period model when  $\gamma = 0$ . This in turn implies that  $B_{1,2}^* = B_{1,3}^* = 0$  for all realizations of  $\beta_1$  and  $\delta_1$  when  $\gamma = 0$  and  $B_{0,1} = B_{0,2} = B_{0,3}$  which implies

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - w_1 - x_1 \\ \delta_1 - x_1 - y_1 \end{bmatrix} \Rightarrow x_1 = 1 - w_1 \quad \text{and} \quad y_1 = w_1 - (1 - \delta_1). \tag{A77}$$

Thus, optimal issuance is given by

$$\begin{aligned}
\begin{bmatrix} B_{1,2}^* \\ B_{1,3}^* \end{bmatrix} & = \begin{bmatrix} \beta_1 + w_1 & \beta_1 \delta_1 + 1 - w_1 \\ \beta_1 \delta_1 + 1 - w_1 & \beta_1 \delta_1^2 + w_1 + (1 - \delta_1) \end{bmatrix}^{-1} \begin{bmatrix} \gamma(1/\beta_1 + w_1) + B_{0,1} - w_1 B_{0,2} - (1 - w_1) B_{0,3} \\ \gamma(1 - w_1) + \delta_1 B_{0,1} - (1 - w_1) B_{0,2} - (w_1 - (1 - \delta_1)) B_{0,3} \end{bmatrix} \\
& = \frac{1}{1 + \beta_1 + \beta_1 \delta_1} \begin{bmatrix} (B_{0,1} - B_{0,2}) - \beta_1 \delta_1 (B_{0,2} - B_{0,3}) \\ (B_{0,1} - B_{0,3}) + \beta_1 (B_{0,2} - B_{0,3}) \end{bmatrix} \\
& \quad + \frac{\gamma}{\beta_1 (1 + \beta_1 + \beta_1 \delta_1) (w_1 + \delta_1 w_1 - 1)} \begin{bmatrix} w_1 (1 + \beta_1 (1 + \beta_1 \delta_1) (1 + \delta_1)) + (\beta_1 (\delta_1^2 - 1) + \delta_1 (1 - \beta_1^2) - 1) \\ w_1 (1 - \beta_1^2 (1 + \delta_1)) + (\beta_1^2 - \beta_1 \delta_1 - 1) \end{bmatrix}, \tag{A78}
\end{aligned}$$

and optimal time 1 taxes  $\tau_1^* = B_{0,1} - B_{1,2}^* \beta_1 - B_{1,3}^* \beta_1 \delta_1$  are

$$\tau_1^* = \frac{B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}}{1 + \beta_1 + \beta_1 \delta_1} - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1}. \quad (\text{A79})$$

In both (A78) and (A79), the second term reflects the government's optimal response to money demand whereas the first term reflects a pure tax-smoothing motive (e.g., the first term in (A79) is the constant tax rate such that the present value of taxes equals the present value of future debt obligations). These tax-smoothing and money-creation motives decouple neatly in this model because the government has the option to re-optimize at time 2.<sup>11</sup> Finally, tedious algebra shows that the minimized time-1 objective function takes the form:

$$\begin{aligned} V_1(B_{0,1}, B_{0,2}, B_{0,3}, \beta_1, \delta_1) &= \min_{B_{1,2}, B_{1,3}} \left[ \frac{1}{2} \tau_1^2 - \gamma \cdot M_1 + \beta_1 \cdot E \left[ V_2(B_{0,2} + B_{1,2}, B_{0,3} + B_{1,3}, \beta_2) \mid \beta_1, \delta_1 \right] \right] \\ &= \frac{1}{2} \frac{(B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3})^2}{1 + \beta_1 + \beta_1 \delta_1} - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}) \end{aligned} \quad (\text{A80})$$

For simplicity, we have omitted a term from (A80) that does not depend on initial maturity choices.

*The government's time-0 problem:* Let  $D = B_{0,1} + B_{0,2} + B_{0,3}$  denote the total amount of debt issued at time 0 and let  $S = B_{0,1}/D$  and  $L = B_{0,3}/D$  denote the fraction of debt that is short- and long-term. Straightforward manipulations allow us to successively rewrite the time 0 problem as:

$$\begin{aligned} & \min_{B_{0,1}, B_{0,2}, B_{0,3}} \left[ \frac{1}{2} \tau_1^2 - \gamma \cdot M_0 + E \left[ V_1(B_{0,1}, B_{0,2}, B_{0,3}, \beta_1, \delta_1) \right] \right] \\ &= \min_{B_{0,1}, B_{0,2}, B_{0,3}} \left[ \frac{1}{2} (G - B_{0,1} - B_{0,2} - B_{0,3})^2 - \gamma \cdot B_{0,1} \right. \\ & \quad \left. + E \left[ \frac{1}{2} \frac{(B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3})^2}{1 + \beta_1 + \beta_1 \delta_1} - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} (B_{0,1} + \beta_1 B_{0,2} + \beta_1 \delta_1 B_{0,3}) \right] \right] \quad (\text{A81}) \\ &= \min_{B_{0,1}, B_{0,2}, B_{0,3}} \left[ \frac{1}{2} (G - D)^2 - \gamma D \left( (1 + A_1) S + A_2 (1 - S - L) + A_3 L \right) \right. \\ & \quad \left. + \frac{1}{2} D^2 \left( \frac{1}{3} + b \left( S - \frac{1}{3} \right)^2 - 2c \left( S - \frac{1}{3} \right) \left( L - \frac{1}{3} \right) + d \left( L - \frac{1}{3} \right)^2 \right) \right], \end{aligned}$$

where

<sup>11</sup> In addition, the linear specification for money utility implies that the marginal monetary services from each additional unit of short-term debt do not depend on the existing stock of short-term debt. With non-linear money utility the tax-smoothing and money demand motives would interact in indirect ways because the marginal money benefit would depend on past issuance and, hence, on past interest rate shocks.

$$b \equiv E \left[ \frac{(\beta_1 - 1)^2}{1 + \beta_1 + \beta_1 \delta_1} \right], \quad c \equiv E \left[ \frac{\beta_1 (\delta_1 - 1)(\beta_1 - 1)}{1 + \beta_1 + \beta_1 \delta_1} \right], \quad \text{and} \quad d \equiv E \left[ \frac{\beta_1^2 (\delta_1 - 1)^2}{1 + \beta_1 + \beta_1 \delta_1} \right] \quad (\text{A82})$$

reflect the dispersion of  $\beta_1$ , the co-movement between  $\beta_1$  and  $\delta_1$ , and the dispersion of  $\delta_1$ , respectively. (Note that equation (A81) is the natural generalization of equation (A22).) For instance,  $c > 0$  means that spot ( $\beta_1$ ) and forward short rates ( $\delta_1$ ) tend to move together at time 1. Naturally, higher level of  $b$  and  $d$  reflect greater uncertainty about spot and forward short rates. Furthermore

$$A_1 \equiv E \left[ \frac{1 + \beta_1}{1 + \beta_1 + \beta_1 \delta_1} \right], \quad A_2 \equiv E \left[ \frac{\beta_1 (1 + \beta_1)}{1 + \beta_1 + \beta_1 \delta_1} \right], \quad \text{and} \quad A_3 \equiv E \left[ \frac{\beta_1 \delta_1 (1 + \beta_1)}{1 + \beta_1 + \beta_1 \delta_1} \right] \quad (\text{A83})$$

reflect the expected present value of future monetary services associated with issuing an additional unit of short-, medium-, and long-term debt at time 0, respectively.

The first order conditions for  $D$ ,  $S$ , and  $L$  are

$$\begin{aligned} 0 &= -(G - D) - \gamma \left( (1 + A_1)S + A_2(1 - S - L) + A_3L \right) + D \left( \frac{1}{3} + b \left( S - \frac{1}{3} \right)^2 - 2c \left( S - \frac{1}{3} \right) \left( L - \frac{1}{3} \right) + d \left( L - \frac{1}{3} \right)^2 \right). \\ 0 &= -\gamma D(1 + A_1 - A_2) + D^2 \left( b \left( S - \frac{1}{3} \right) - c \left( L - \frac{1}{3} \right) \right) \\ 0 &= -\gamma D(A_3 - A_2) + D^2 \left( -c \left( S - \frac{1}{3} \right) + d \left( L - \frac{1}{3} \right) \right) \end{aligned} \quad (\text{A84})$$

Under the assumption that the second order condition for this problem is satisfied, it is straightforward to show that  $bd > c^2$ . Assuming that  $b > 0$  and  $d > 0$ , the solution takes the form

$$\begin{aligned} D^* &= \frac{3}{4} \left( G + \gamma \left( \frac{1}{3}(1 + A_1) + \frac{1}{3}A_2 + \frac{1}{3}A_3 \right) \right) \\ S^* &= \frac{1}{3} + \gamma \frac{d + c(A_3 - A_2) + d(A_1 - A_2)}{D^*(bd - c^2)} \\ L^* &= \frac{1}{3} + \gamma \frac{c + b(A_3 - A_2) + c(A_1 - A_2)}{D^*(bd - c^2)}. \end{aligned} \quad (\text{A85})$$

The average duration of debt issued at time 0 is

$$\begin{aligned} \overline{DUR} &= 1 \cdot S^* + 2 \cdot (1 - S^* - L^*) + 3 \cdot L^* \\ &= 2 + \frac{\gamma}{D^*} \frac{(c - d)(1 + A_1 - A_2) + (b - c)(A_3 - A_2)}{bd - c^2}. \end{aligned} \quad (\text{A86})$$

Thus, in the absence of monetary services ( $\gamma = 0$ ) we again obtain the perfect tax-smoothing outcome ( $\tau_0^* = \tau_1^* = \tau_2^* = \tau_3^* = G/4$ ) which is implemented by issuing a ‘‘consol’’ bond:

$D^* = (3/4)G$  and  $S^* = L^* = 1/3$ . However, with positive monetary services ( $\gamma > 0$ ) the government issues more short-term debt in order to satisfy household money demand:  $S^* > 1/3$ .

It is easy to show that  $A_3 < A_2 < A_1$  so long as  $Cov[\beta_1, \delta_1] > 0$ . However, in general,  $A_1 \approx A_2 \approx A_3$  for almost any plausible parameterization of the two interest rate shocks. As a result, allowing for interim monetary services has a modest effect on the choice of  $D$ , but has little if any effect on the choice of  $S$  and  $L$ —i.e., on the optimal maturity structure of the debt. To simplify the analysis, we apply the approximation that  $A_1 \approx A_2 \approx A_3$ ,<sup>12</sup> and obtain:

$$\begin{aligned} S^* &= \frac{1}{3} + \gamma \frac{1}{D^*(b - c^2/d)} \\ L^* &= \frac{1}{3} + \gamma \frac{c/d}{D^*(b - c^2/d)}. \\ \overline{DUR} &\approx 2 + \frac{\gamma(c-d)}{D^*bd - c^2}. \end{aligned} \tag{A87}$$

Equation (A87) shows that  $\partial S^* / \partial b < 0$ ,  $\partial S^* / \partial d < 0$ ,  $\partial S^* / \partial c \propto c$ ,  $\partial L^* / \partial b < 0$ ,  $\partial L^* / \partial d < 0$ , and  $\partial L^* / \partial c > 0$ . Thus, the government issues more short-term debt when uncertainty about spot or forward rates is lower (i.e.,  $b$  or  $d$  is lower). However, a larger absolute correlation between these two shocks enables the government to better hedge its interest rate exposure and ultimately take on more roll-over risk (i.e., a larger value of  $|c|$  is associated with a higher value of  $S$ ). In the natural case where  $c > 0$ , this is accomplished via a “barbell” strategy in which the government issues lots of short- and long-term debt at time 0, but little if any medium-term debt. Indeed the government may even choose to *lend* on an intermediate-dated basis (i.e., we may have  $1 - S - L < 0$ ).

The intuition is that this barbell strategy enables the government to hedge the roll-over risk that is created by deviating from the “consol” solution by issuing larger amounts short-term debt at time 0. To better see the intuition, note that time 1 taxes are

$$\tau_1^* = D \frac{S + \beta_1(1 - S - L) + \beta_1\delta_1 L}{1 + \beta_1 + \beta_1\delta_1} - \gamma \cdot \frac{1 + \beta_1}{1 + \beta_1 + \beta_1\delta_1}. \tag{A88}$$

so that

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<sup>12</sup> The expressions given in (A87) obtain exactly under the assumption that households derive no monetary services from short-term debt at time 1 and 2—i.e., they only derive utility from monetary services at time 0.

$$\begin{aligned}\frac{\partial \tau_1^*}{\partial \beta_1} &= D \frac{\delta_1(L-S) - 2(S - (1-L)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} - \gamma \cdot \frac{\delta_1}{(1 + \beta_1 + \beta_1 \delta_1)^2} \\ \frac{\partial \tau_1^*}{\partial \delta_1} &= D \beta_1 \frac{(L-S) + 2\beta_1(L - (1-S)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} + \gamma \cdot \frac{\beta_1(1 + \beta_1)}{(1 + \beta_1 + \beta_1 \delta_1)^2}.\end{aligned}\tag{A89}$$

In an attempt to satisfy money demand at time 0, they government will choose  $S > L$  and  $S > (1-L)/2$  which implies that  $\partial \tau_1^* / \partial \beta_1 < 0$ . In other words, the need to roll-over short-term debt means that taxes will be high when the short-term interest rate is high at time 1 (i.e., when  $\beta_1$  is low). Thus, choosing a high value of  $S$  naturally exposes the government budget and hence taxes to  $\beta_1$  shocks. What about the government's exposure to  $\delta_1$  shocks? Equation (A89) shows that by pursuing a barbell strategy in which  $L > (1-S)/2$  the government can reduce the exposure of taxes to  $\delta_1$  or even create an *offsetting* exposure such that  $\partial \tau_1^* / \partial \delta_1 > 0$ . When the correlation between  $\beta_1$  and  $\delta_1$  is high, this barbell strategy allows the government to hedge the exposure of time 1 taxes to interest rate shocks. This hedging strategy lowers the tax-smoothing costs associated with issuing additional short-term debt at time 0 which explains why  $S^*$  is increasing in  $|c|$ .

How is this strategy implemented in terms of time 1 issuance? We have

$$\begin{aligned}\begin{bmatrix} B_{1,2}^* \\ B_{1,3}^* \end{bmatrix} &= \frac{D}{1 + \beta_1 + \beta_1 \delta_1} \begin{bmatrix} 2(S - (1-L)/2) + 2\beta_1 \delta_1(L - (1-S)/2) \\ (S-L) - 2\beta_1(L - (1-S)/2) \end{bmatrix} \\ &+ \frac{\gamma}{\beta_1(1 + \beta_1 + \beta_1 \delta_1)(w_1 + \delta_1 w_1 - 1)} \begin{bmatrix} w_1(1 + \beta_1(1 + \beta_1 \delta_1)(1 + \delta_1)) + (\beta_1(\delta_1^2 - 1) + \delta_1(1 - \beta_1^2) - 1) \\ w_1(1 - \beta_1^2(1 + \delta_1)) + (\beta_1^2 - \beta_1 \delta_1 - 1) \end{bmatrix}.\end{aligned}\tag{A93}$$

Ignoring the terms that depend on  $\gamma$ , we have

$$\begin{aligned}\begin{bmatrix} \partial B_{1,2}^* / \partial \beta_1 \\ \partial B_{1,3}^* / \partial \beta_1 \end{bmatrix} &= D \frac{-\delta_1(S-L) - 2(S - (1-L)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} \partial B_{1,2}^* / \partial \delta_1 \\ \partial B_{1,3}^* / \partial \delta_1 \end{bmatrix} &= \beta_1 D \frac{-(S-L) + 2\beta_1(L - (1-S)/2)}{(1 + \beta_1 + \beta_1 \delta_1)^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.\end{aligned}\tag{A94}$$

Thus, the government must issue more of both maturities when  $\beta_1$  is low (i.e., when short rates are high at time 1). How does issuance vary with shocks to  $\delta_1$ ? Since  $S > L$ , there is the natural tendency for issuance to fall with  $\delta_1$ —i.e., issuance is also higher when  $\delta_1$  is lower (forward rates are higher) at time 1. However, to the extent that  $L > (1-S)/2$ —i.e., if the government is using a barbell strategy—issuance can actually increase in  $\delta_1$ . Intuitively, the government needs to issue more to repurchase long-term bonds when long-term bond prices are high (i.e.,  $\delta_1$  is high). Thus, by pursuing

a barbell strategy at time 0, the government can partially hedge its time 1 issuance and hence time taxes against interest rate shocks.

*Numerical example:* Here we explore a simple numerical example. We assume that:

- $\Pr(\beta_1 = 1 - \sigma_\beta) = \Pr(\beta_1 = 1 + \sigma_\beta) = 1/2$ , so that  $\sigma_\beta$  is the standard deviation of  $\beta_1$ .
- $\Pr(\delta_1 = \bar{\delta} - \sigma_\delta) = \Pr(\delta_1 = \bar{\delta} + \sigma_\delta) = 1/2$ , so that  $\sigma_\delta$  is the standard deviation of  $\delta_1$ .
- $\Pr(\varepsilon_2 = 1 - \sigma_\varepsilon) = \Pr(\varepsilon_2 = 1 + \sigma_\varepsilon) = 1/2$ , so that  $\sigma_\varepsilon$  is the standard deviation of  $\varepsilon_2$ .
- $\rho$  is the correlation between  $\beta_1$  and  $\delta_1$ —i.e.,  $\Pr(\delta_1 = \bar{\delta} + \sigma_\delta | \beta_1 = 1 + \sigma_\beta) = (1 + \rho)/2$ . Note that  $\bar{\delta} = 1 - \rho\sigma_\beta\sigma_\delta$  since  $E[\beta_1\delta_1] = 1$  and that an increase in  $\rho$  also raises  $c$  in (A82).

Given these assumptions it is straightforward to calculate the model parameters given in (A82) and (A83) and then to compute the optimal debt structure given in (A85) or (A87) which imposes the approximation that  $A_1 \approx A_2 \approx A_3$ . In the following table, we compute the optimal values of  $S$ ,  $L$ , and  $\overline{DUR}$  varying the level of money demand,  $\gamma$ , and the correlation between the shocks to  $\beta_1$  and  $\delta_1$ ,  $\rho$ .

Table A1 below computes the optimal values of  $S^*$ ,  $L^*$ , and  $DUR$  for various values of  $\gamma$  and  $\rho$  using equations (A88) based on the parameters defined in (A82) and (A83). Computations based on (A87) yield nearly identical results. The computations in Table A1 assume that  $G = 1$  and that  $\sigma_\beta = \sigma_\delta = \sigma_\varepsilon = 30\%$ . The table shows that  $S^*$  is increasing in both  $\gamma$  and  $\rho$ ;  $L^*$  is increasing in both  $\gamma$  and  $\rho$  for  $\rho > 0$  ( $L^*$  is decreasing in  $\gamma$  for  $\rho = 0$  which translates into a tiny *negative* value for  $c$ ); and  $DUR$  is increasing in  $\rho$  for  $\gamma > 0$ .

**Table A1. Numerical Example.** This table computes the optimal values of  $S^*$ ,  $L^*$ , and  $DUR$  for various values of  $\gamma$  and  $\rho$  using equations (A88) based on the parameters defined in (A82) and (A83). Computations based on (A87) yield nearly identical results. The computations in Table A1 assume that  $G = 1$  and that  $\sigma_\beta = \sigma_\delta = \sigma_\epsilon = 30\%$ . The table shows that  $S^*$  is increasing in both  $\gamma$  and  $\rho$ ;  $L^*$  is increasing in both  $\gamma$  and  $\rho$  for  $\rho > 0$  ( $L^*$  is decreasing in  $\gamma$  for  $\rho = 0$  which translates into a tiny *negative* value for  $c$ ; and  $DUR$  is increasing in  $\rho$  for  $\gamma > 0$ ).

		$S^* = \text{Initial Short-Term Share}$						
		$\rho$						
		0.0%	33.0%	66.0%	80.0%	90.0%	95.0%	99.0%
$\gamma$	0.00%	33.33%	33.33%	33.33%	33.33%	33.33%	33.33%	33.33%
	0.01%	33.76%	33.79%	34.00%	34.34%	35.16%	36.82%	50.10%
	0.02%	34.19%	34.25%	34.67%	35.34%	36.99%	40.30%	66.86%
	0.03%	34.61%	34.71%	35.34%	36.34%	38.81%	43.78%	83.62%
	0.04%	35.04%	35.17%	36.01%	37.34%	40.64%	47.27%	100.38%
	0.05%	35.47%	35.63%	36.68%	38.34%	42.46%	50.75%	117.13%
	0.06%	35.90%	36.09%	37.35%	39.35%	44.29%	54.23%	133.88%
	0.07%	36.32%	36.55%	38.02%	40.35%	46.11%	57.71%	150.62%
	0.08%	36.75%	37.01%	38.69%	41.35%	47.94%	61.19%	167.37%
	0.09%	37.18%	37.47%	39.36%	42.35%	49.76%	64.67%	184.10%
0.10%	37.60%	37.93%	40.03%	43.35%	51.58%	68.15%	200.84%	

		$L^* = \text{Initial Long-term Share}$						
		$\rho$						
		0.0%	33.0%	66.0%	80.0%	90.0%	95.0%	99.0%
$\gamma$	0.00%	33.33%	33.33%	33.33%	33.33%	33.33%	33.33%	33.33%
	0.01%	33.32%	33.46%	33.76%	34.14%	35.06%	36.89%	51.49%
	0.02%	33.30%	33.58%	34.18%	34.95%	36.78%	40.44%	69.64%
	0.03%	33.28%	33.71%	34.60%	35.75%	38.51%	43.99%	87.78%
	0.04%	33.26%	33.83%	35.02%	36.56%	40.23%	47.54%	105.92%
	0.05%	33.25%	33.96%	35.44%	37.37%	41.96%	51.09%	124.06%
	0.06%	33.23%	34.09%	35.86%	38.17%	43.68%	54.64%	142.20%
	0.07%	33.21%	34.21%	36.28%	38.98%	45.41%	58.19%	160.33%
	0.08%	33.19%	34.34%	36.70%	39.79%	47.13%	61.74%	178.45%
	0.09%	33.18%	34.46%	37.13%	40.59%	48.85%	65.29%	196.58%
0.10%	33.16%	34.59%	37.55%	41.40%	50.58%	68.84%	214.70%	

		$DUR = \text{Average Duration}$						
		$\rho$						
		0.0%	33.0%	66.0%	80.0%	90.0%	95.0%	99.0%
$\gamma$	0.00%	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
	0.01%	1.9956	1.9967	1.9975	1.9980	1.9990	2.0007	2.0139
	0.02%	1.9911	1.9933	1.9950	1.9961	1.9980	2.0014	2.0277
	0.03%	1.9867	1.9900	1.9925	1.9941	1.9970	2.0021	2.0416
	0.04%	1.9822	1.9866	1.9901	1.9922	1.9960	2.0028	2.0555
	0.05%	1.9778	1.9833	1.9876	1.9902	1.9950	2.0035	2.0693
	0.06%	1.9733	1.9800	1.9851	1.9883	1.9940	2.0041	2.0832
	0.07%	1.9689	1.9766	1.9826	1.9863	1.9929	2.0048	2.0970
	0.08%	1.9644	1.9733	1.9801	1.9844	1.9919	2.0055	2.1109
	0.09%	1.9600	1.9699	1.9777	1.9825	1.9909	2.0062	2.1247
0.10%	1.9556	1.9666	1.9752	1.9805	1.9899	2.0069	2.1386	



**Table A.2**  
**The Money Premium on T-bills and the Supply of Short-term Treasuries, 1992-2009**

The table reports weekly regressions of  $z$ -spreads on the supply on Treasury bills scaled by GDP and the supply of Treasury notes and bonds. The  $n$ -week  $z$ -spread  $z_t^{(n)} = y_t^{(n)} - \hat{y}_t^{(n)}$  is the difference between the actual yield on an  $n$ -week Treasury bill and the  $n$ -week fitted yield, based on the fitted Treasury yield curve in Gurkaynak, Sack and Wright (2007). Each day they estimate a 6-parameter model of the instantaneous forward curve. Zero coupon yields are derived by integrating along the estimated forward curve. Yield curves are estimated using a set of sample securities that includes nearly all off-the-run Treasury notes and bonds with a remaining maturity of more than 3 months. We estimate this specification in both levels and in 4-week differences:

$$z_t^{(n)} = a^{(n)} + b^{(n)} \cdot (BILLS / GDP)_t + c^{(n)} \cdot (NONBILLS / GDP)_t + \varepsilon_t^{(n)} \quad \text{and}$$

$$\Delta_4 z_t^{(n)} = a^{(n)} + b^{(n)} \cdot \Delta_4 (BILLS / GDP)_t + c^{(n)} \cdot \Delta_4 (NONBILLS / GDP)_t + \Delta_4 \varepsilon_t^{(n)} .$$

To compute the ratio of Treasury bills and notes and bond to GDP at the end of each week, we use detailed data on the size and timing of Treasury auctions from <http://www.treasurydirect.gov/>. The first twelve columns report OLS estimates in levels and changes for  $n = 2, 4,$  and 10-week bills. The final six columns report instrumental variables (IV) estimates which exploit seasonal variation in Treasury supply driven by the Federal tax calendar. In the first stage, we regress  $\Delta_4 BILLS / GDP$  (and  $\Delta_4 NONBILLS / GDP$ ) on a set of week-of-year dummies; in the second stage, we regress changes in  $z$ -spreads on fitted values from the first stage. The units of the dependent variable are basis points and units of the independent variables are percentage points. Newey-West (1987)  $t$ -statistics, allowing for serial correlation up to 12 weeks in the levels regressions and up to 8 weeks in the changes regressions, are in brackets. For the bivariate regressions, we report the  $t$ -statistic for the hypothesis that  $b^{(n)} - c^{(n)} = 0$ , labeled  $t_{\{b^{(n)} = c^{(n)}\}}$  below. Panel A shows results for 1992-2009; Panel B shows results for 1992-2007. The  $F$ -statistic in the first-stage regression of  $\Delta_4 BILLS / GDP$  on the week-of-year dummies is  $F = 25.73$  for 1992-2007 and  $F = 20.05$  for 1992-2009. The  $F$ -statistic in the first-stage regression of  $\Delta_4 NONBILLS / GDP$  on the week-of-year dummies is  $F = 10.03$  for 1992-2007 and  $F = 3.75$  for 1992-2009.

	Z-spread Levels						Z-spread Changes											
	OLS						OLS						IV					
	2-week $z$		4-week $z$		10-week $z$		2-week $z$		4-week $z$		10-week $z$		2-week $z$		4-week $z$		10-week $z$	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
Panel A: 1992-2009 ( $N = 939$ )																		
$b^{(n)}$	3.92	3.22	2.42	2.49	1.46	-0.05	15.89	17.22	7.00	7.37	-0.31	-1.02	34.74	48.70	15.52	21.15	6.02	6.27
$[t]$	[1.86]	[1.02]	[1.27]	[0.83]	[0.90]	[-0.02]	[1.45]	[1.58]	[1.06]	[1.11]	[0.11]	[-0.34]	[5.41]	[6.62]	[3.67]	[4.21]	[2.57]	[2.47]
$c^{(n)}$		0.46		-0.05		1.00		-7.71		-2.13		4.13		-51.67		-20.84		-0.93
$[t]$		[0.53]		[-0.06]		[1.60]		[-1.23]		[-0.47]		[1.54]		[-3.58]		[-2.05]		[-0.20]
$[t_{\{b^{(n)} = c^{(n)}\}}]$		[0.70]		[0.68]		[-0.37]		[1.92]		[1.16]		[-1.15]		[5.44]		[3.15]		[1.20]
$R^2$	0.04	0.04	0.02	0.02	0.02	0.07	0.03	0.03	0.01	0.01	0.00	0.01						
Panel B: 1992-2007 ( $N = 834$ )																		
$b^{(n)}$	8.59	15.83	5.55	13.39	5.16	7.73	38.82	43.27	21.62	23.53	6.89	6.11	37.29	54.53	18.07	25.04	8.23	8.19
$[t]$	[5.21]	[5.78]	[4.04]	[6.21]	[5.36]	[6.42]	[7.81]	[7.94]	[5.84]	[6.00]	[3.11]	[2.67]	[5.45]	[7.30]	[4.10]	[4.96]	[3.49]	[3.26]
$c^{(n)}$		-2.31		-2.50		-0.82		-18.51		-7.94		3.25		-51.84		-20.96		0.11
$[t]$		[-4.08]		[-5.40]		[-3.58]		[-2.40]		[-1.45]		[1.39]		[-4.59]		[-2.46]		[0.03]
$[t_{\{b^{(n)} = c^{(n)}\}}]$		[5.63]		[6.24]		[6.31]		[5.60]		[4.09]		[0.78]		[6.89]		[3.92]		[1.55]
$R^2$	0.09	0.14	0.07	0.16	0.19	0.23	0.09	0.10	0.06	0.06	0.03	0.03						

**Table A.3**  
**The Money Premium on T-bills and the Supply of Short-term Treasuries, 1983-2009**

The table reports weekly regressions of spreads between 4-week T-bills yields and 4-week private money market rates,  $y_t^{(4)} - r_t^{(4)}$  on the supply on Treasury bills scaled by GDP and the supply of Treasury notes and bonds. We use three different proxies for  $r_t^{(4)}$ : the rate on 4-week commercial paper (CP) from the Federal Reserve's H-15 release, the rate on 4-week bank certificates of deposit (CD) from Federal Reserve's H-15 release, and the 4-week OIS rate from Bloomberg. The OIS rate is only available beginning in 2002. We estimate this specification in both levels and in 4-week differences:

$$(y_t^{(4)} - r_t^{(4)}) = a + b \cdot (BILLS / GDP)_t + c \cdot (NONBILLS / GDP)_t + \varepsilon_t \quad \text{and}$$

$$\Delta_4(y_t^{(4)} - r_t^{(4)}) = a + b \cdot \Delta_4(BILLS / GDP)_t + c \cdot \Delta_4(NONBILLS / GDP)_t + \Delta_4\varepsilon_t .$$

To compute the ratio of Treasury bills and notes and bond to GDP at the end of each week, we use detailed data on the size and timing of Treasury auctions from <http://www.treasurydirect.gov/>. The first six columns report OLS estimates in levels, the next six columns report OLS estimates in 4-week differences, the final six columns report instrumental variables (IV) estimates of these difference specifications which exploit seasonal variation in Treasury supply driven by the Federal tax calendar. In the first stage, we regress  $\Delta_4BILLS/GDP$  (and  $\Delta_4NONBILLS/GDP$ ) on a set of week-of-year dummies; in the second stage, we regress changes in z-spreads on fitted values from the first stage. The units of the dependent variable are basis points and units of the independent variables are percentage points. Newey-West (1987)  $t$ -statistics, allowing for serial correlation up to 12 weeks in the levels regressions and up to 8 weeks in the changes regressions, are in brackets. For the bivariate regressions, we report the  $t$ -statistic for the hypothesis that  $b^{(n)} - c^{(n)} = 0$ , labeled  $t_{\{b^{(n)} = c^{(n)}\}}$  below. Panel A shows results for 1983-2009; Panel B shows results for 1983-2007. The  $F$ -statistic in the first-stage regression of  $\Delta_4BILLS/GDP$  on the week-of-year dummies is  $F = 13.94$  for 1983-2007 and  $F = 13.12$  for 1983-2009. The  $F$ -statistic in the first-stage regression of  $\Delta_4NONBILLS/GDP$  on the week-of-year dummies is  $F = 13.00$  for 1983-2007 and  $F = 6.93$  for 1983-2009.

	Levels						Changes											
	CP		CD		OIS		CP		CD		OIS		IV					
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
	Panel A: 1983-2009																	
$b^{(n)}$	6.10	9.21	6.03	7.21	4.34	7.79	4.72	8.90	0.39	3.84	8.02	8.60	-2.52	27.43	-3.88	27.17	15.61	26.15
$[t]$	[2.82]	[3.62]	[2.01]	[1.87]	[3.21]	[2.37]	[0.82]	[1.55]	[0.04]	[0.40]	[1.82]	[1.92]	[0.31]	[2.94]	[0.41]	[2.54]	[3.03]	[2.90]
$c^{(n)}$		-2.46		-0.92		-3.54		-19.87		-16.74		-6.15		-67.50		-70.49		-96.75
$[t]$		[-2.71]		[-0.82]		[-1.51]		[-3.27]		[-2.52]		[-1.16]		[-5.18]		[-5.07]		[-2.92]
$N$	1406	1406	1408	1408	422	422	1406	1406	1408	1408	418	418	1406	1406	1408	1408	418	418
$R^2$	0.03	0.06	0.02	0.02	0.09	0.12	0.00	0.02	0.00	0.01	0.02	0.03						
$[t_{\{b=c\}}]$		[3.71]		[1.72]		[2.07]		[3.25]		[1.69]		[1.97]		[4.86]		[4.62]		[3.37]
	Panel B: 1983-2007																	
$b^{(n)}$	3.32	9.68	7.08	11.81	31.06	27.97	9.08	17.10	6.17	13.53	13.80	16.27	-4.93	27.56	-7.90	23.76	16.17	29.09
$[t]$	[1.10]	[2.36]	[2.23]	[2.78]	[6.00]	[4.93]	[1.86]	[3.40]	[1.11]	[2.28]	[2.70]	[2.43]	[-0.58]	[3.04]	[-0.89]	[2.58]	[3.70]	[4.69]
$c^{(n)}$		-2.85		-2.12		-7.17		-27.03		-24.80		-15.40		-63.36		-61.74		-61.11
$[t]$		[-2.77]		[-1.96]		[-3.98]		[-3.76]		[-3.36]		[-1.25]		[-5.34]		[-5.22]		[-3.44]
$N$	1303	1303	1303	1303	317	317	1303	1303	1303	1303	313	313	1303	1303	1303	1303	313	313
$R^2$	0.00	0.04	0.02	0.04	0.37	0.42	0.00	0.03	0.00	0.02	0.04	0.05						
$[t_{\{b=c\}}]$		[2.58]		[2.77]		[6.77]		[4.53]		[3.56]		[1.77]		[4.97]		[4.72]		[4.18]