Deformations of Local Systems and Eisenstein Series

DEFORMATIONS OF LOCAL SYSTEMS AND EISENSTEIN SERIES

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To our teacher J. Bernstein

Introduction

0.1.   The goal of this paper is to realize a suggestion made by V. Drinfeld. To explain it let us recall the general framework of the geometric Langlands correspondence.

Let G be a reductive group and X a (smooth and complete) curve. Let Bun_G denote the moduli stack of G-torsors on X; let $D^b(\mathrm{Bun}_G)$ be the appropriately defined derived category of constructible sheaves. Let $\check{G}$ denote the Langlands dual group of G.

The nature of $\check{G}$ depends on the sheaf-theoretic context we work in. The following are the main options. If we live over a ground field k of characteristic 0, we can work with holonomic D-modules on schemes over k, and $\check{G}$ will be an algebraic group over k. If the ground field is $\mathbb{C}$, we can work with sheaves of k'-vector spaces (here k' is another field of characteristic 0) in the analytic topology; in this case $\check{G}$ is an algebraic group over k'. Finally, over any ground field, we can work with ℓ-adic sheaves (where ℓ is different from the ground field); in this case $\check{G}$ is an algebraic group over $\overline{\mathbb{Q}_\ell}$.

For the duration of this introduction we can work in either of the above sheaf-theoretic contexts.

Let $E_{\check{G}}$ be a $\check{G}$-local system on X, thought of as a tensor functor $V \mapsto V_{E_{\check{G}}}$ from the category $\mathrm{Rep}(\check{G})$ of finite-dimensional $\check{G}$-representations to that of local systems (=lisse sheaves) on X.

In this case one introduces the notion of *Hecke eigensheaf*, which is an object $\mathcal{S}(E_{\check{G}}) \in D^b(\mathrm{Bun}_G)$, satisfying

$$(0.1) \qquad \mathrm{H}^V(\mathcal{S}(E_{\check{G}})) \simeq \mathcal{S}(E_{\check{G}}) \boxtimes V_{E_{\check{G}}},$$

where $\mathrm{H}^V : D^b(\mathrm{Bun}_G) \to D^b(\mathrm{Bun}_G \times X)$ are the Hecke functors, defined for each $V \in \mathrm{Rep}(\check{G})$; the isomorphisms (0.1) are required to satisfy certain compatibility conditions, that we will not list here.

A basic (but in general unconfirmed, and perhaps even imprecise) expectation is that for every $E_{\check{G}}$ there corresponds a non-zero Hecke eigensheaf $\mathcal{S}(E_{\check{G}})$. This is a weak form of the geometric Langlands conjecture. A stronger form of the conjecture, which only makes sense in the context of D-modules, says that the assignment $E_{\check{G}} \mapsto \mathcal{S}(E_{\check{G}})$ should work in families. In other words, if $E_{\check{G},Y}$ is a Y-family of $\check{G}$-local systems, where Y is a scheme over k, then to it there should correspond a Y-family $\mathcal{S}(E_{\check{G},Y})$. The necessity to use D-modules here, as opposed to any other sheaf-theoretic context, is that it is only in this case that we have a reasonable notion of Y-families of objects of $D^b(\mathcal{X})$ on a scheme (or stack) $\mathcal{X}$.

The strongest (and boldest) form of the geometric Langlands conjecture says that the above assignment should give rise to an equivalence between the category $D^b(\mathrm{Bun}_G)$ and the appropriately defined derived category of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}$, classifying $\check{G}$-local systems on X.

Let us, however, consider the following intermediate case. Let $E_{\check{G}}$ be a fixed local system, and let $E_{\check{G},Y}$ be its *formal* deformation. We note that the notion of a Y-family of sheaves makes sense in any of the above sheaf-theoretic contexts, when Y is a *formal* scheme. Suppose we have found $\mathcal{S}(E_{\check{G}})$ which is a Hecke eigensheaf with respect to $E_{\check{G}}$.

Can we extend $\mathcal{S}(E_{\check{G}})$ to a Y-family $\mathcal{S}(E_{\check{G},Y})$ of eigensheaves? This is the question that Drinfeld asked on several occasions.

0.2. In addition to posing this question, Drinfeld emphasized the following general principle that should lead to a solution.

Let $\mathcal{M}$ be an object of an abelian category $\mathcal{C}$. (We will take $\mathcal{M}$ to be $\mathcal{S}(E_{\check{G}})$ as an object of the category of perverse sheaves on Bun_G, when $\mathcal{S}(E_{\check{G}})$ is known to exist and be perverse.)

Let Y be a formal scheme, or more generally a formal DG-scheme, which means by definition that $Y = \mathrm{Spec}(A^\bullet)$, where $A^\bullet$ is a (super)-commutative formal DG-algebra. Assume that $A^\bullet$ is isomorphic, or rather quasi-isomorphic, to the standard (=Chevalley) complex of a DG Lie algebra $L^\bullet$.

Recall that the standard complex equals the symmetric algebra on the topological graded vector space $(L^\bullet)^*[-1]$ with the differential induced by the differential on $L^\bullet$ and the Lie bracket. (In our main example $L^\bullet$ will only have cohomology in degree 1, implying that $A^\bullet$ is acyclic off cohomological degree 0, so that Y is an honest (and not a DG) formal scheme.)

The following principle is known as the Quillen (or Koszul) duality:

A data of deformation of $\mathcal{M}$ over the base Y is equivalent, up to quasi-isomorphism, to a data of action of $L^\bullet$ on $\mathcal{M}$.

Returning to our problem, let us take Y to be $\mathrm{Def}(E_{\check{G}})$– the base of the universal deformation of $E_{\check{G}}$ as a $\check{G}$-local system. In this case, Y is indeed quasi-isomorphic to the standard complex of a DG algebra canonically attached to $E_{\check{G}}$. Namely, let $\check{\mathfrak{g}}_{X,E_{\check{G}}}$ be the sheaf of Lie algebras on X, associated with the adjoint representation of $\check{G}$. A general result of deformation theory says:

The DG formal scheme $\mathrm{Def}(E_{\check{G}})$ corresponds to the Lie algebra $R\Gamma(X, \check{\mathfrak{g}}_{X,E_{\check{G}}})$.

Our understanding is that results of this type were first discussed in a letter by V. Drinfeld to V. Schechtman, and worked out in a series of papers by V. Hinich and V. Schechtman [HiS] and V. Hinich [Hi, Hi1]. In any case, the above assertion is a theorem in the context of local systems of D-modules and sheaves in the classical topology over $\mathbb{C}$. We are not sure of its status for ℓ-adic sheaves, and for that reason we will avoid evoking it in this context.

Summarizing, we obtain that the existence of the the family $\mathcal{S}(E_{\check{G},\mathrm{Def}(E_{\check{G}})})$ of Hecke eigensheaves parametrized by $\mathrm{Def}(E_{\check{G}})$ is equivalent to the existence of the action of the DG Lie algebra $R\Gamma(X, \check{\mathfrak{g}}_{X,E_{\check{G}}})$ on $\mathcal{S}(E_{\check{G}})$.

Let us note now that the existence of such an action is heuristically very natural: we expect the assignment $E_{\check{G}} \to \mathcal{S}(E_{\check{G}})$ to be functorial; in particular we want automorphisms of $E_{\check{G}}$ to act on $\mathcal{S}(E_{\check{G}})$. Therefore, $R\Gamma(X, \check{\mathfrak{g}}_{X,E_{\check{G}}})$, which can be thought of as the Lie algebra of derived endomorphisms of $E_{\check{G}}$, should act by derived endomorphisms of $\mathcal{S}(E_{\check{G}})$, which is what we are looking for.

0.3. Unfortunately, even the assignment $E_{\check{G}} \to \mathcal{S}(E_{\check{G}})$ has been constructed only in few cases.

One such case is when $G = GL_n$ and $E_{\check{G}} = E_n$ is an n-dimensional irreducible local system. In this case the existence of $\mathcal{S}(E_n)$ is known, and the action of $R\Gamma(X, \check{\mathfrak{g}}_{X, E_{\check{G}}}) \simeq R\operatorname{End}(E_n)$ on $\mathcal{S}(E_{\check{G}})$, has been fully investigated by S. Lysenko in [Lys]. In particular, in *loc.cit.* it was shown that the family $\mathcal{S}(E_{n, \operatorname{Def}(E_n)})$ has the properties expected from the most general form of the geometric Langlands conjecture, mentioned above.

The case that we will study in this paper is, in some sense, the opposite one. We will take $\check{G}$ to be arbitrary, but $E_{\check{G}}$ will be assumed "maximally reducible", i.e., $E_{\check{G}}$ is induced from a local system $E_{\check{T}}$ with respect to the Cartan group $\check{T} \subset \check{G}$. In this case the corresponding Hecke eigensheaf was constructed in [BG], under the name "geometric Eisenstein series". In this paper we denote it by $\overline{\operatorname{Eis}}(E_{\check{T}})$. We will review the construction of $\overline{\operatorname{Eis}}(E_{\check{T}})$ later on.

Remarkably, the action of $R\Gamma(X, \check{\mathfrak{g}}_{X, E_{\check{G}}})$ on $\overline{\operatorname{Eis}}(E_{\check{T}})$ has been essentially constructed in [FFKM], for independent reasons. Thus, we do have the object $\mathcal{S}(E_{\check{G}, \operatorname{Def}(E_{\check{G}})})$, and our current goal is to describe it more explicitly.

At the moment, however, an explicit description of $\mathcal{S}(E_{\check{G}, \operatorname{Def}(E_{\check{G}})})$ is beyond what we know how to do. We will be able to address a more modest question, though:

Namely, let $\operatorname{Def}_{\check{B}}(E_{\check{T}})$ be the base of the universal deformation of $E_{\check{T}}$, thought of as a $\check{B}$-local system (here $\check{B}$ is the Borel subgroup of $\check{G}$), such that the induced $\check{T}$-local system under the canonical projection $\check{B} \twoheadrightarrow \check{T}$ is fixed to be $E_{\check{T}}$.

For example, in the case $G = GL_2$, in which case $\check{G}$ is also isomorphic to GL_2, a $\check{T}$-local system can be thought of as a pair of 1-dimensional local systems (E_1, E_2), and we will be looking for 2-dimensional local systems of the form

$$0 \to E_1 \to E \to E_2 \to 0.$$

The (formal) scheme of such local systems is isomorphic to (the completion at the origin of) the vector space $\operatorname{Ext}^1(E_2, E_1)$, if we ignore the DG complications.

The DG formal scheme $\operatorname{Def}_{\check{B}}(E_{\check{T}})$ maps naturally to $\operatorname{Def}(E_{\check{G}})$, so we can restrict and obtain a $\operatorname{Def}_{\check{B}}(E_{\check{T}})$-family of Hecke eigensheaves $\mathcal{S}(E_{\check{G}, \operatorname{Def}_{\check{B}}(E_{\check{T}})})$. In fact, $\operatorname{Def}_{\check{B}}(E_{\check{T}})$ corresponds to a DG Lie subalgebra in $R\Gamma(X, \check{\mathfrak{g}}_{X, E_{\check{G}}})$, namely, $R\Gamma(X, \check{\mathfrak{n}}_{X, E_{\check{T}}})$, where $\check{\mathfrak{n}}$ is the nilpotent radical of $\check{\mathfrak{b}}$, and $\check{\mathfrak{n}}_{X, E_{\check{T}}}$ is the corresponding local system of Lie algebras on X, twisted by $E_{\check{T}}$ using the adjoint action of $\check{T}$ on $\check{\mathfrak{n}}$.

0.4. A concrete question posed by Drinfeld was that of explicit description of $\mathcal{S}(E_{\check{G}, \operatorname{Def}_{\check{B}}(E_{\check{T}})})$. However, more recently (in the fall of 2003) he himself suggested an answer:

Along with the geometric Eisenstein series $\overline{\operatorname{Eis}}(E_{\check{T}})$ there exists a more naive object, that we can call "classical" Eisenstein series; in this paper we denote it by $\operatorname{Eis}_!(E_{\check{T}})$. When we work over the finite ground field and ℓ-adic sheaves, $\operatorname{Eis}_!(E_{\check{T}})$ goes over under the *faisceaux-fonctions* correspondence to the usual Eisenstein series as defined in the theory of automorphic forms.

Drinfeld's conjecture was that the family $\mathcal{S}(E_{\check{G}, \operatorname{Def}_{\check{B}}(E_{\check{T}})})$ is nothing but (a certain completion of) $\operatorname{Eis}_!(E_{\check{T}})$. This statement has an ideological significance also for the classical (i.e., function theoretic vs. sheaf-theoretic) Langlands correspondence:

The classical Eisenstein series correspond not to homomorphisms Galois $\to \check{G}$ *that factor through* $\check{T}$, *but rather to the universal family of homomorphisms* Galois $\to \check{B}$ *with a fixed composition* Galois $\to \check{B} \twoheadrightarrow \check{T}$.

The present paper is devoted to the proof of Drinfeld's conjecture, under a certain simplifying hypothesis on $E_{\check{T}}$.

Namely, we will assume that $E_{\check{T}}$ is *regular*, i.e., that it is as non-degenerate as possible. This means, by definition, that for every co-root $\check{\alpha}$ of G, which is the same as a root of $\check{G}$, the induced 1-dimensional local system $\check{\alpha}(E_{\check{T}})$ is non-trivial. For example, in the case of GL_2 this means that the two local systems E_1 and E_2 are non-isomorphic.

This regularity assumption is equivalent to requiring that the DG formal scheme $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ be an "honest" scheme. In the GL_2 example this manifests itself in the absence of $\mathrm{Hom}(E_2, E_1)$ and $\mathrm{Ext}^2(E_2, E_1)$.

Moreover, we show that in this case both versions of Eisenstein series, namely, $\overline{\mathrm{Eis}}(E_{\check{T}})$ and $\mathrm{Eis}_!(E_{\check{T}})$ are perverse sheaves. This fact and the simplified nature of $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ makes life significantly easier, since we can avoid a lot of complications of homotopy-theoretic nature. However, we are sure that, once properly formulated, Drinfeld's conjecture that $\mathcal{S}(E_{\check{G}, \mathrm{Def}_{\check{B}}(E_{\check{T}})}) \simeq \mathrm{Eis}_!(E_{\check{T}})$, is true for any $E_{\check{T}}$.

Proving the above conjecture amounts to the following:
- Exhibiting the action of the commutative algebra $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$ of functions on $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ on $\mathrm{Eis}_!(E_{\check{T}})$.
- Establishing an isomorphism $\mathbb{C} \overset{L}{\underset{\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}}{\otimes}} \mathrm{Eis}_!(E_{\check{T}}) \simeq \overline{\mathrm{Eis}}(E_{\check{T}})$, compatible with the $R\Gamma(X, \check{\mathfrak{n}}_{X, E_{\check{T}}})$ actions.
- Verifying the Hecke property $\mathrm{H}^V(\mathrm{Eis}_!(E_{\check{T}})) \simeq \mathrm{Eis}_!(\check{T}) \underset{\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}}{\boxtimes} V_{E_{\check{G}, \mathrm{Def}_{\check{B}}(E_{\check{T}})}}$, where $V_{E_{\check{G}, \mathrm{Def}_{\check{B}}(E_{\check{T}})}}$ is the canonical family of $\check{G}$-local systems over $\mathrm{Def}_{\check{B}}(E_{\check{T}})$.

As is to be expected, the verification of these properties is a nice simple exercise when $G = GL_2$, that we will perform in Sections 2 and 7, but not altogether trivial for other groups.

In the next section we will review the definitions of $\overline{\mathrm{Eis}}(E_{\check{T}})$ and $\mathrm{Eis}_!(E_{\check{T}})$, and state each of the above properties precisely as a theorem.

Acknowledgments. This introduction makes it clear how much we owe V. Drinfeld for the existence of this paper. We would like to thank A. Beilinson for developing and generously explaining to us the theory of chiral homology, which is crucial for a manageable description of the deformation base $\mathrm{Def}_{\check{B}}(E_{\check{T}})$. We would also like to thank M. Finkelberg for numerous illuminating discussions about Drinfeld's compactifications and, particularly, on the paper [FFKM].

1. Background and overview

1.1. **Notation and conventions.** The notation in this paper by and large follows that of [BG]. We refer the reader to *loc. cit.* for conventions regarding stacks, derived categories, etc.

We can work in any of the sheaf-theoretic contexts mentioned in the introduction *excluding* the following one: ℓ-adic sheaves on schemes over a ground field of positive characteristic *other than* $\mathbb{F}_q$. [1] To simplify the notation, from now on we shall assume that the field of coefficients of our sheaves is $\mathbb{C}$, and for a scheme or a stack Y we shall denote by $\mathbb{C}_Y$ *the constant sheaf* on Y.

Let G be a reductive group; as in [BG] we will make a simplifying assumption that the derived group of G is simply connected. Let B be a (fixed) Borel subgroup of G and T its

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[1]The latter exclusion is due to the fact that we will be using Kashiwara's conjecture, proved by Drinfeld in [Dr]. The only one place in this paper that uses this result is Theorem 1.5. We have no doubt, however, that Theorem 1.5 remains valid in the context of ℓ-adic sheaves over fields of positive characteristic, see Sect. 10 for an additional remark.

Cartan quotient. We let Λ denote the lattice of characters of T and $\check{\Lambda}$ the dual lattice. By Λ^+ (resp., Λ^{pos}) we will denote the semi-group of dominant weights (resp, the semi-group generated by positive linear combinations of simple roots); by $\check{\Lambda}^+$ and $\check{\Lambda}^{pos}$ we will denote the corresponding objects for the Langlands dual group.

We let I denote the set of vertices of the Dynkin diagram of G. For $i \in I$ we will denote by $\alpha_i \in \Lambda^{pos}$ and $\check{\alpha}_i \in \check{\Lambda}^{pos}$ the corresponding simple root and co-root, respectively.

1.2. **Drinfeld's compactifications.** Let Bun_G (resp., Bun_B, Bun_T) be the moduli stacks of G (resp., B, T) torsors on X. We have the natural maps

$$\mathrm{Bun}_G \xleftarrow{\mathfrak{p}} \mathrm{Bun}_B \xrightarrow{\mathfrak{q}} \mathrm{Bun}_T .$$

The stacks Bun_B and Bun_T are both unions of connected components, numbered by elements of $\check{\Lambda}$. We will sometimes use the notation $\mathfrak{p}^{\check{\mu}}$ and $\mathfrak{q}^{\check{\mu}}$ for the restrictions of $\mathfrak{p}$ and $\mathfrak{q}$ to the connected component $\mathrm{Bun}_B^{\check{\mu}}$.

Let us now recall the definition of the stack $\overline{\mathrm{Bun}}_B$. By definition, it classifies triples $(\mathcal{P}_G, \mathcal{P}_T, \{\kappa^\lambda\})$, where $\mathcal{P}_G$ is a G-torsor, $\mathcal{P}_T$ is a T-torsor, and κ^λ is an injective map of coherent sheaves

$$\kappa^\lambda : \mathcal{L}_{\mathcal{P}_T}^\lambda \to V_{\mathcal{P}_G}^\lambda,$$

defined for each $\lambda \in \Lambda^+$, where $\mathcal{L}_{\mathcal{P}_T}^\lambda$ is the line bundle associated to $\mathcal{P}_T$ and the character $T \xrightarrow{\lambda} \mathbb{G}_m$, V^λ is the corresponding highest weight representation of G, and $V_{\mathcal{P}_G}^\lambda$ is the associated vector bundle. The collection $\{\kappa^\lambda\}$ is required to satisfy the Plücker relations, see [BG], Sect. 1.2.1.

Let $\overline{\mathfrak{p}}$ (resp., $\overline{\mathfrak{q}}$) denote the natural morphism from $\overline{\mathrm{Bun}}_B$ to Bun_G (resp., Bun_T) that sends a triple $(\mathcal{P}_G, \mathcal{P}_T, \{\kappa^\lambda\})$ to $\mathcal{P}_G$ (resp., $\mathcal{P}_T$). It is a basic fact, established, e.g., in [BG], Proposition 1.2.2, that the map $\overline{\mathfrak{p}}$ is representable and proper.

The stack $\overline{\mathrm{Bun}}_B$ also splits into connected components, denoted $\overline{\mathrm{Bun}}_B^{\check{\mu}}$, $\check{\mu} \in \check{\Lambda}$, according to the degree of $\mathcal{P}_T$. We shall denote by $\overline{\mathfrak{p}}^{\check{\mu}}$ (resp., $\overline{\mathfrak{q}}^{\check{\mu}}$) the restriction of $\overline{\mathfrak{p}}$ (resp., $\overline{\mathfrak{q}}$) to the corresponding connected component.

Consider the open substack of $\overline{\mathrm{Bun}}_B$ corresponding to the condition that all the maps κ^λ are (injective) bundle maps. This substack identifies naturally with Bun_B. We will denote by $\jmath$ the corresponding open embedding, so that

$$\mathfrak{p} = \overline{\mathfrak{p}} \circ \jmath \text{ and } \mathfrak{q} = \overline{\mathfrak{q}} \circ \jmath.$$

Thus, $\overline{\mathrm{Bun}}_B$ can be regarded as a relative compactification of Bun_B over Bun_G. It established in [FGV], Proposition 3.3.1, that the morphism $\jmath$ is affine. In fact, in [BFG], Theorem 11.6, a stronger assertion is proved: namely, that Bun_B is a complement to an effective Cartier divisor on $\overline{\mathrm{Bun}}_B$.

1.3. The stack $\overline{\mathrm{Bun}}_B$ admits a natural stratification related to zeroes of the maps κ^λ:

Let $\check{\lambda}$ be an element of $\check{\Lambda}^{pos}$ equal to $\underset{i \in I}{\Sigma} n_i \cdot \check{\alpha}_i$, let $|\check{\lambda}|$ be the integer equal to Σn_i, and let $X^{\check{\lambda}}$ denote the corresponding partially symmetrized power of X, i.e.,

$$X^{\check{\lambda}} = \underset{i \in I}{\Pi} X^{(n_i)}.$$

Points of $X^{\check{\lambda}}$ can be thought of as effective I-coloured divisors on X; each such point has the form $\Sigma \check{\lambda}_k \cdot x_k$ with $x_k \neq x_{k'}$, $\check{\lambda}_k \in \check{\Lambda}^{pos}$ and $\Sigma \check{\lambda}_k = \check{\lambda}$.

For every $\check\lambda$ there corresponds a finite map

$$\overline{\imath}_{\check\lambda} : X^{\check\lambda} \times \overline{\mathrm{Bun}}_B^{\check\mu} \to \overline{\mathrm{Bun}}_B^{\check\mu - \check\lambda}.$$

It is defined as follows. A triple $(\mathcal{P}_G, \mathcal{P}_T, \{\kappa^\lambda\})$ and $D^{\check\lambda} \in X^{\check\lambda}$ gets sent to $(\mathcal{P}'_G, \mathcal{P}'_T, \{\kappa'^\lambda\})$, where $\mathcal{P}'_G = \mathcal{P}_G$, $\mathcal{P}'_T = \mathcal{P}_T(-D^{\check\lambda})$, i.e.,

$$\mathcal{L}^\lambda_{\mathcal{P}'_T} = \mathcal{L}^\lambda_{\mathcal{P}_T}(-\lambda(D^{\check\lambda})),$$

and $\kappa'^{\check\lambda}$ is the composition

$$\mathcal{L}^\lambda_{\mathcal{P}'_T} \hookrightarrow \mathcal{L}^\lambda_{\mathcal{P}_T} \xrightarrow{\kappa^\lambda} V^\lambda_{\mathcal{P}_G}.$$

Let $\imath_{\check\lambda}$ denote the composition

$$\overline{\imath}_{\check\lambda} \circ \jmath : X^{\check\lambda} \times \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B.$$

The maps $\imath_{\check\lambda}$ are locally closed embeddings and their images define a stratification of $\overline{\mathrm{Bun}}_B$. Since the map $\jmath$ is affine and $\overline{\imath}_{\check\lambda}$ finite, the map $\imath_{\check\lambda}$ is affine as well.

1.4. **Eisenstein series.** Let $E_{\check T}$ be a $\check T$-local system on X. Let $\mathcal{S}(E_{\check T})$ be the local system on $\mathrm{Bun}_T \simeq \mathrm{Pic}(X) \underset{\mathbb{Z}}{\otimes} \check\Lambda$, corresponding to $E_{\check T}$ via the geometric class field theory. (Our normalization of $\mathcal{S}(E_{\check T})$ is so that it is a *sheaf* and not a perverse sheaf.) The defining property of $\mathcal{S}(E_{\check T})$ is that for $\check\lambda \in \check\Lambda$, its pull-back under the corresponding map

$$\mathrm{Bun}_T \times X \to \mathrm{Bun}_T$$

is isomorphic to $\mathcal{S}(E_{\check T}) \boxtimes \check\lambda(E_{\check T})$, where $\check\lambda(E_{\check T})$ is the induced 1-dimensional local system on X under $\check T \xrightarrow{\check\lambda} \mathbb{G}_m$.

For $\check\mu \in \check\Lambda$ the geometric (compactified) Eisenstein series $\overline{\mathrm{Eis}}^{\check\mu}(E_{\check T})$ is an object of $D^b(\mathrm{Bun}_G)$ defined as

$$\overline{\mathrm{Eis}}^{\check\mu}(E_{\check T}) := \overline{\mathfrak{p}}_!^{\check\mu} \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}} \otimes (\overline{\mathfrak{q}}^{\check\mu})^*(\mathcal{S}(E_{\check T})) \right).$$

The naive (non-compactified) Eisenstein series $\mathrm{Eis}_!^{\check\mu}(E_{\check T})$ is is defined as

$$\mathrm{Eis}_!^{\check\mu}(E_{\check T}) := \mathfrak{p}_!^{\check\mu} \left(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}} \otimes (\mathfrak{q}^{\check\mu})^*(\mathcal{S}(E_{\check T})) \right),$$

or which is the same,

$$\overline{\mathfrak{p}}_!^{\check\mu} \left(\jmath_!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}) \otimes (\overline{\mathfrak{q}}^{\check\mu})^*(\mathcal{S}(E_{\check T})) \right).$$

Note, however, that since Bun_B (unlike $\overline{\mathrm{Bun}}_B$) is smooth, $\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}$ is the constant sheaf $\mathbb{C}_{\mathrm{Bun}_B^{\check\mu}}$, up to a cohomological shift.

We define $\overline{\mathrm{Eis}}(E_{\check T})$ and $\mathrm{Eis}_!(E_{\check T})$ as $\check\Lambda$-graded objects of $D^b(\mathrm{Bun}_G)$, equal to $\underset{\check\mu \in \check\Lambda}{\oplus} \overline{\mathrm{Eis}}^{\check\mu}(E_{\check T})$ and $\underset{\check\mu \in \check\Lambda}{\oplus} \mathrm{Eis}_!^{\check\mu}(E_{\check T})$, respectively.

Let us assume now that $E_{\check T}$ is regular, i.e., for every root $\check\alpha$ of $\check G$, the 1-dimensional local system $\check\alpha(E_{\check T})$ is non-trivial. We will prove the following:

Theorem 1.5. *The complexes* $\mathrm{Eis}_!^{\check\mu}(E_{\check T})$ *and* $\overline{\mathrm{Eis}}^{\check\mu}(E_{\check T})$ *are perverse sheaves.*

This result was conjectured in [BG]; the idea of the proof belongs to V. Drinfeld, and is based on the validity of Kashiwara's conjecture proved by him earlier.

Another fact, which we will use only marginally, is that for $E_{\check{T}}$ regular, for every open substack $U \subset \mathrm{Bun}_G$ *of finite type*, the restrictions of $\overline{\mathrm{Eis}}^{\check{\mu}}(E_{\check{T}})|_U$ are non-zero for only finitely many $\check{\mu}$'s. In particular, the direct sum $\overline{\mathrm{Eis}}(E_{\check{T}})$ makes sense as an object of $\mathrm{Perv}(\mathrm{Bun}_G)$.

1.6. **The space of deformations.** Before we state the main result of this paper, we need to discuss the formal scheme of deformations $\mathrm{Def}_{\check{B}}(E_{\check{T}})$. From now on we will assume that $E_{\check{T}}$ is regular.

By definition, $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ is a functor on the category of local Artinian $\mathbb{C}$-algebras that assigns to R the set of isomorphism classes of R-flat $\check{B}$-local systems $E_{\check{B},R}$, such that the induced $\check{T}$-local system $E_{\check{T},R}$, by means of $\check{B} \twoheadrightarrow \check{T}$, is identified with $E_{\check{T}} \otimes R$, and such that the reduction modulo the maximal ideal, $E_{\check{B},R/\mathfrak{m}_R}$ is identified with the local system, induced from $E_{\check{T}}$ by means of $\check{T} \hookrightarrow \check{B}$, in a compatible way.

Since $H^0(X, \check{\mathfrak{n}}_{X,E_{\check{T}}}) = 0$, local systems as above have no non-trivial automorphisms; so, by passing to the set of isomorphism classes of objects, we do not lose information. Moreover, since $H^2(X, \check{\mathfrak{n}}_{X,E_{\check{T}}}) = 0$, the deformation theory is unobstructed. Hence, $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ is representable by a smooth formal scheme.

The tangent space to $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ at the origin is canonically isomorphic to $H^1(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})$. Hence, there exists a *non-canonical* isomorphism between $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ and the completion of $H^1(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})$ at the origin. However, such an isomorphism is indeed very non-canonical, and in order to proceed, we need to describe $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ explicitly in terms of $E_{\check{T}}$. Such a description is provided by Theorem 3.6.

Namely, in Sect. 3 for $\check{\lambda} \in \check{\Lambda}^{pos}$ we introduce a perverse sheaf $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}$ on $X^{\check{\lambda}}$. Set $R_{E_{\check{T}}}^{-\check{\lambda}} = H(X^{\check{\lambda}}, \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}})$. The regularity assumption on $E_{\check{T}}$ implies that the above cohomology is concentrated in degree 0.

The collection $\{\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}\}$ has a natural multiplicative structure with respect to the addition operation on $\check{\Lambda}^{pos}$, making $R_{E_{\check{T}}} := \underset{\check{\lambda}}{\oplus} R_{E_{\check{T}}}^{-\check{\lambda}}$ into a $-\check{\Lambda}^{pos}$-graded commutative algebra, with the 0-graded component isomorphic to $\mathbb{C}$. The completion $\widehat{R}_{E_{\check{T}}}$ of $R_{E_{\check{T}}}$ with respect to the augmentation ideal is isomorphic to $\underset{\check{\lambda}}{\Pi} R_{E_{\check{T}}}^{-\check{\lambda}}$.

We will prove (see Theorem 11) that $\widehat{R}_{E_{\check{T}}}$ is canonically isomorphic to $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$–the (topological) algebra of functions on the formal scheme $\mathrm{Def}_{\check{B}}(E_{\check{T}})$. The $\check{\Lambda}$-grading on $\widehat{R}_{E_{\check{T}}}$ corresponds to the $\check{T}$-action on $\mathrm{Def}_{\check{B}}(E_{\check{T}})$, which comes from the adjoint action of $\check{T}$ on $\check{B}$. We will denote by $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$ the algebra equal to the sum of homogeneous components of $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$; by the above, it is isomorphic to $R_{E_{\check{T}}}$.

Let us comment on how one could guess the above description of $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$. As was mentioned above, $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$ is quasi-isomorphic to the standard complex $\mathbf{C}^\bullet\left(R\Gamma(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})\right)$ of the DG Lie algebra $R\Gamma(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})$.

The machinery of chiral algebras, developed in [BD], implies that $\mathbf{C}^\bullet\left(R\Gamma(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})\right)$ is quasi-isomorphic to the chiral homology of the (super)-commutative chiral algebra on X equal to the standard complex $\mathbf{C}^\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ of the sheaf of Lie algebras $\check{\mathfrak{n}}_{X,E_{\check{T}}}$. By definition, the above chiral homology is computed as the homology of a sheaf associated to $\mathbf{C}^\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ on the *Ran space* of X.

The latter sheaf splits into direct summands corresponding to elements $\check\lambda$, and each direct summand is isomorphic to the direct image image of $\Omega(\check{\mathfrak{n}}_{X,E_{\check T}})^{-\check\lambda}$ under the natural map from $X^{\check\lambda}$ to the Ran space.

1.7. **The main result.** We are now ready to state one of the two main results of this paper:

Theorem 1.8. *Let $E_{\check T}$ be regular.*

(1) *There is a natural action of the $\check\Lambda$-graded algebra $\mathcal{O}_{\mathrm{Def}_{\check B}(E_{\check T})}$ on the $\check\Lambda$-graded object $\mathrm{Eis}_!(E_{\check T}) \in \mathrm{Perv}(E_{\check T})$.*

(2) *The higher $\mathrm{Tor}_i^{\mathcal{O}_{\mathrm{Def}_{\check B}(E_{\check T})}}(\mathbb{C}, \mathrm{Eis}_!(E_{\check T}))$ vanish, and we have a canonical isomorphism of $\check\Lambda$-graded perverse sheaves:*

$$\mathbb{C} \underset{\mathcal{O}_{\mathrm{Def}_{\check B}(E_{\check T})}}{\otimes} \mathrm{Eis}_!(E_{\check T}) \simeq \overline{\mathrm{Eis}}(E_{\check T}).$$

In Sect. 4 we will give an explicit and intrinsic description of the pro-object

$$\widehat{\mathrm{Eis}}_!(E_{\check T}) := \mathrm{Eis}_!(E_{\check T}) \underset{\mathcal{O}_{\mathrm{Def}_{\check B}(E_{\check T})}}{\otimes} \widehat{\mathcal{O}}_{\mathrm{Def}_{\check B}(E_{\check T})},$$

which is our candidate for $\mathcal{S}(E_{\check G, \mathrm{Def}_{\check B}(E_{\check T})})$.

Let us add a few comments on the strategy of the proof of this theorem. Taking into account Corollary 3.7, to define an action as in point (1), we need to construct morphisms

$$(1.1) \qquad\qquad R_{E_{\check T}}^{-\check\lambda} \otimes \mathrm{Eis}_!^{\check\mu+\check\lambda}(E_{\check T}) \to \mathrm{Eis}_!^{\check\mu}(E_{\check T})$$

that are associative in the natural sense. The morphisms (1.1) will be obtained by applying the functor $\overline{\mathfrak{p}}_!^{\check\mu}$ to some canonical morphism of perverse sheaves upstairs.

Namely, in Theorem 4.2 we will show that there exists a canonical map

$$\imath_{\check\lambda!}\left(\Omega(\check{\mathfrak{n}}_X)^{-\check\lambda} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}}\right) \to \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}),$$

where $\Omega(\check{\mathfrak{n}}_X)^{-\check\lambda}$ is the perverse sheaf corresponding to the trivial twist. Tensoring both sides of the above expression by $(\overline{\mathfrak{q}}^{\check\mu})^*(\mathcal{S}(E_{\check T}))$ we obtain a map

$$\imath_{\check\lambda!}\left(\Omega(\check{\mathfrak{n}}_{X,E_{\check T}})^{-\check\lambda} \boxtimes \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}} \otimes (\mathfrak{q}^{\check\mu+\check\lambda})^*(\mathcal{S}(E_{\check T}))\right)\right) \to \jmath_!\left(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}} \otimes (\mathfrak{q}^{\check\mu})^*(\mathcal{S}(E_{\check T}))\right),$$

which gives rise to (1.1).

1.9. **Koszul complex.** To prove point (2) of Theorem 1.8 we proceed as follows. In Sect. 6.4 for each $\check\lambda \in \check\Lambda^{pos}$, we define a certain explicit complex of perverse sheaves on $X^{\check\lambda}$, denoted $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check T}})^{\bullet,-\check\lambda,*}$.

One should think of $\Omega(\check{\mathfrak{n}}_{X,E_{\check T}})^{-\check\lambda}$ and $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check T}})^{\bullet,-\check\lambda,*}$ as Koszul-dual objects in the same way as the universal enveloping algebra $U(\mathfrak{h})$ of a Lie algebra $\mathfrak{h}$ is a Koszul-dual object to the co-standard complex $\mathbf{C}_\bullet(\mathfrak{h})$.

We show (see Theorem 6.6) that the graded perverse sheaf on $\overline{\mathrm{Bun}}_B^{\check\mu}$

$$(1.2) \qquad \mathrm{Kosz}_{\check\mu}(\mathrm{Eis}_!(E_{\check T}))^\bullet := \bigoplus_{\check\lambda \in \check\Lambda^{pos}} \imath_{\check\lambda!}\left(\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check T}})^{\bullet,-\check\lambda,*} \boxtimes \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\lambda}})\right)$$

acquires a natural differential, such that the resulting complex is quasi-isomorphic to $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}$. This implies the assertion of point (2) of the theorem as follows:

The regularity assumption on $E_{\check{T}}$ implies that $H^j(X^{\check{\lambda}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{i,-\check{\lambda},*}) = 0$ for $j \neq 0$. So, we obtain well-defined complexes

$$\mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda},*} := H^0(X^{\check{\lambda}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,-\check{\lambda},*}).$$

Set $\mathfrak{u}_{E_{\check{T}}}^{\bullet,*} := \oplus_{\check{\lambda}} \mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda},*}$. In Sect. 6.4 we show that for any $R_{E_{\check{T}}}$-module $\mathcal{M}$ the tensor product

$$\mathsf{K}(\mathcal{M}) := \mathfrak{u}_{E_{\check{T}}}^{\bullet,*} \otimes \mathcal{M}$$

acquires a natural differential, and the resulting complex is quasi-isomorphic to $\mathbb{C} \overset{L}{\underset{R_{E_{\check{T}}}}{\otimes}} \mathcal{M}$.

Taking the direct image of (1.2) with respect to $\overline{\mathfrak{p}}^{\check{\mu}}$, and summing up over $\check{\mu}$, we obtain

$$\mathbb{C} \overset{L}{\underset{R_{E_{\check{T}}}}{\otimes}} \mathrm{Eis}_!(E_{\check{T}}) \simeq \mathsf{K}(\mathrm{Eis}_!(E_{\check{T}})) \simeq \oplus_{\check{\mu}} \overline{\mathfrak{p}}^{\check{\mu}}_!(\mathrm{Kosz}_{\check{\mu}}(\mathrm{Eis}_!(E_{\check{T}}))^\bullet) \simeq \overline{\mathrm{Eis}}(E_{\check{T}}),$$

as required.

Let us add a comment on the nature of the complexes $\mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda},*}$. In Sect. 6 we show that the collection $\{\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,-\check{\lambda},*}\}$ possesses a natural co-multiplicative structure, thereby endowing $\mathfrak{u}_{E_{\check{T}}}^{\bullet,*}$ with a structure of DG graded co-associative co-algebra (with a trivial differential (!)).

Let $\mathfrak{u}_{E_{\check{T}}}^{\bullet,\check{\lambda}}$ be the dual of $\mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda},*}$, and set $\mathfrak{u}_{E_{\check{T}}}^{\bullet} := \oplus_{\check{\lambda}} \mathfrak{u}_{E_{\check{T}}}^{\bullet,\check{\lambda}}$. We obtain that $\mathfrak{u}_{E_{\check{T}}}^{\bullet}$ is an associative DG algebra (also, with a trivial differential). Although we do not state this explicitly, from Sect. 6.4 one can deduce that $\mathfrak{u}_{E_{\check{T}}}^{\bullet}$ is canonically quasi-isomorphic to the universal enveloping algebra of $R\Gamma(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})$.

By construction, $\mathsf{K}(\mathrm{Eis}_!(E_{\check{T}}))$ is a DG-comodule with respect to $\mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda},*}$, and, hence, a DG-module over $\mathfrak{u}_{E_{\check{T}}}^{\bullet}$. This structure can be viewed as an action of DG Lie algebra $R\Gamma(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})$ on $\overline{\mathrm{Eis}}(E_{\check{T}})$. By definition, this action equals the one arising via the Koszul-Quillen duality on $\mathbb{C} \overset{L}{\underset{R_{E_{\check{T}}}}{\otimes}} \mathrm{Eis}_!(E_{\check{T}})$.

A compatibility result proved in Sect. 6.8 implies that the above $R\Gamma(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})$-action on $\overline{\mathrm{Eis}}(E_{\check{T}})$, coincides with the one given by the construction of [FFKM]. This is equivalent to the fact that $\mathrm{Eis}_!(E_{\check{T}})$ *is* the $\mathrm{Def}_{\check{B}}(E_{\check{T}})$-family corresponding to $\overline{\mathrm{Eis}}(E_{\check{T}})$ with the action of $R\Gamma(X, \check{\mathfrak{n}}_{X,E_{\check{T}}})$ constructed in [FFKM].

1.10. **The Hecke property.** The second main result of the present paper has to do with the verification of the Hecke property of $\mathrm{Eis}_!(E_{\check{T}})$. In order to simplify the exposition, instead of the Hecke functor $\mathrm{H}^V : D^b(\mathrm{Bun}_G) \to D^b(\mathrm{Bun}_G \times X)$, we will consider the local Hecke functor

$$\mathrm{H}^V_x : D^b(\mathrm{Bun}_G) \to D^b(\mathrm{Bun}_G),$$

corresponding to a fixed point $x \in X$. As will be clear from the proof, the case of a moving point (or multiple points, as required for the verification of the additional compatibility condition, which we did not even state explicitly) is analogous.

Thus, to a point $x \in X$ there corresponds a $\check{G}$-torsor, equal to the restriction to the fiber at x of the universal $\check{G}$-local system over $\mathrm{Def}_{\check{B}}(E_{\check{T}})$. For $V \in \mathrm{Rep}(\check{G})$, let $V_{E_{\check{B},\widehat{univ},x}}$ be the corresponding locally free $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$-module. Let $V_{E_{\check{B},univ,x}}$ be the corresponding $\check{\Lambda}$-graded version, which is a locally free module over $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$.

We will prove:

Theorem 1.11. *There exists a canonical isomorphism*

$$\mathrm{H}_x^V\left(\mathrm{Eis}_!(E_{\check{T}})\right) \simeq V_{E_{\check{B},univ,x}} \underset{\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}}{\otimes} \mathrm{Eis}_!(E_{\check{T}}).$$

Let us explain the main ideas involved in the proof of this theorem. First, we need to interpret $V_{E_{\check{B},univ,x}}$ in terms of the isomorphism $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})} \simeq R_{E_{\check{T}}}$. This is also done using chiral homology.

In Sect. 8 for every $\check{\nu} \in \check{\Lambda}$ we construct a perverse sheaf $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T}},x})^{\check{\nu}}$ that lives over an appropriate ind-version $_{\infty\cdot x}X^{\check{\nu}}$ of the space of coloured divisors (essentially, we allow the divisor to be non-effective at x). We set

$$R(V_x)_{E_{\check{T}}}^{\check{\nu}} := H(_{\infty\cdot x}X^{\check{\nu}}, \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T}},x})^{\check{\nu}}), \text{ and } R(V_x)_{E_{\check{T}}} = \underset{\check{\nu}}{\oplus} R(V_x)_{E_{\check{T}}}^{\check{\nu}}.$$

We show that $R(V_x)_{E_{\check{T}}}$ is a locally free $R_{E_{\check{T}}}$-module of rank $\dim(V)$, and that $R(V_x)_{E_{\check{T}}}$ corresponds to $V_{E_{\check{B},univ,x}}$ under the isomorphism $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})} \simeq R_{E_{\check{T}}}$.

Secondly, we need to reinterpret the LHS in Theorem 1.11 locally in terms of $\overline{\mathrm{Bun}}_B^{\check{\mu}}$. To do this, as in [BG], we need to replace $\overline{\mathrm{Bun}}_B^{\check{\mu}}$ by its ind-version $_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}$ that allows the maps κ^λ (see the definition of $\overline{\mathrm{Bun}}_B$) to have poles of an arbitrary order at x. Tautologically, the Hecke functors H_x^V that act on $D^b(\mathrm{Bun}_G)$ lift to functors, denoted $\mathrm{H'}_x^V$, that act on $D^b(_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}})$, in a way compatible with the push-forward functor $\overline{\mathfrak{p}}_!^{\check{\mu}} : D^b(_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}) \to D^b(\mathrm{Bun}_G)$.

Thus, the LHS in Theorem 1.11 is given by

$$\bigoplus_{\check{\nu}} \overline{\mathfrak{p}}_!^{\check{\mu}}\left(\mathrm{H'}_x^V\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \otimes (\overline{\mathfrak{q}}^{\check{\mu}})^*(\mathcal{S}(E_{\check{T}}))\right)\right).$$

As in the case of $\overline{\mathrm{Bun}}_B^{\check{\mu}}$, we have a natural map

$$_{\infty\cdot x}\iota_{\check{\nu}} : {}_{\infty\cdot x}X^{\check{\nu}} \times \mathrm{Bun}_B^{\check{\mu}-\check{\nu}} \to {}_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}.$$

In Theorem 8.8, we show that there exists a canonical map

$$_{\infty\cdot x}\iota_{\check{\nu}!}\left(\Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\nu}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}}\right) \to \mathrm{H'}_x^V\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})\right).$$

Tensoring with the pull-back of $\mathcal{S}(E_{\check{T}})$ under $\overline{\mathfrak{q}}^{\check{\mu}} : {}_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}} \to \mathrm{Bun}_T$, we obtain a map

$$_{\infty\cdot x}\iota_{\check{\nu}!}\left(\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T}},x})^{\check{\nu}} \boxtimes \left(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}} \otimes (\mathfrak{q}^{\check{\mu}-\check{\nu}})^*(\mathcal{S}(E_{\check{T}}))\right)\right) \to$$
$$\to \mathrm{H'}_x^V\left(\jmath_!\left((\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \otimes (\mathfrak{q}^{\check{\mu}})^*(\mathcal{S}(E_{\check{T}}))\right)\right),$$

and applying the direct image under $\overline{\mathfrak{p}}_!^{\check{\mu}}$, we obtain a map in one direction $(\leftarrow)$ in Theorem 1.11.

The proof that the resulting map is an isomorphism follows by considering filtrations defined naturally on the two sides, and showing that the map induced on the associated graded level is an isomorphism.

2. The case of GL_2

In this section we will prove Theorem 1.8 for $G = GL_2$ by an explicit calculation. This will be a prototype of the argument in the general case.

2.1. **The base of deformation.** Let E_1 and E_2 be two non-isomorphic 1-dimensional local systems on X. We regard the pair (E_1, E_2) as a local system $E_{\check{T}}$ with respect to the Cartan subgroup $\check{T} \simeq (\mathbb{G}_m, \mathbb{G}_m)$ of the group $\check{G}$, Langlands dual of GL_2, which is itself isomorphic to GL_2.

By definition, the formal scheme $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ associates to a local Artinian $\mathbb{C}$-algebra R the category of R-flat local systems E_R that fit into the short exact sequence

$$0 \to E_1 \otimes R \to E_R \to E_2 \otimes R \to 0,$$

which modulo the maximal ideal of R are identified with the direct sum $E_1 \oplus E_2$.

Note that since $E_1 \neq E_2$, such short exact sequences admit no automorphisms, so that the above category is (equivalent to) a set.

Hence, we obtain that $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ is naturally isomorphic to the completion at 0 of the $\mathbb{C}$-vector space $\mathrm{Ext}^1(E_2, E_1)$. Let us denote the dual vector space $H^1(X, E_2 \otimes E_1^{-1})$ by W. The corresponding complete commutative algebra $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$ is isomorphic to $\widehat{\mathrm{Sym}}(\mathsf{W})$.

We will also consider the non-completed algebra $\mathrm{Sym}(\mathsf{W})$, endowed with a grading by *negative* integers. We regard $\mathbb{Z}$ as a subgroup of $\mathbb{Z} \oplus \mathbb{Z} \simeq \check{\Lambda}$ via $d \mapsto (d, -d)$.

2.2.   For each $(d_1, d_2) \in \mathbb{Z} \oplus \mathbb{Z}$, let $\mathrm{Eis}_!^{d_1,d_2}(E_{\check{T}}) \in D^b(\mathrm{Bun}_G)$ and $\overline{\mathrm{Eis}}^{d_1,d_2}(E_{\check{T}}) \in D^b(\mathrm{Bun}_G)$ be the corresponding component of the non-compactified and compactified Eisenstein series, respectively, attached to $E_{\check{T}}$.   Both $\mathrm{Eis}_!^{d_1,d_2}(E_{\check{T}})$ and $\overline{\mathrm{Eis}}^{d_1,d_2}(E_{\check{T}})$ are known ([Ga]) to be perverse sheaves; the generalization of this assertion for arbitrary G (due to Drinfeld) will be established in Sect. 10.

Adapting the notation of Theorem 1.8 to the present context, we obtain the following:

Theorem 2.3.

(a) *There exists a grading preserving action*

$$\mathrm{Sym}(\mathsf{W}) \otimes \mathrm{Eis}_!(E_{\check{T}}) \to \mathrm{Eis}_!(E_{\check{T}}).$$

(b) *For each $(d_1, d_2) \in \mathbb{Z} \oplus \mathbb{Z}$, the resulting Koszul complex*

$$\mathsf{K}_{d_1,d_2}(\mathrm{Eis}_!(E_{\check{T}}))^\bullet := \ldots \to \Lambda^d(\mathsf{W}) \otimes \mathrm{Eis}_!^{d_1+d,d_2-d}(E_{\check{T}}) \xrightarrow{\partial_d} \ldots \to$$
$$\to \Lambda^2(\mathsf{W}) \otimes \mathrm{Eis}_!^{d_1+2,d_2-2}(E_{\check{T}}) \xrightarrow{\partial_2} \mathsf{W} \otimes \mathrm{Eis}_!^{d_1+1,d_2-1}(E_{\check{T}}) \xrightarrow{\partial_1} \mathrm{Eis}_!^{d_1,d_2}(E_{\check{T}})$$

is quasi-isomorphic to $\overline{\mathrm{Eis}}^{d_1,d_2}(E_{\check{T}})$.

2.4. **Proof of Theorem 2.3.** Recall the stack $\overline{\mathrm{Bun}}_B^{d_1,d_2}$.   For each non each non-negative integer d let $\overline{\imath}_d^{d_1,d_2}$ denote the finite map

$$X^{(d)} \times \overline{\mathrm{Bun}}_B^{d_1+d,d_2-d} \to \overline{\mathrm{Bun}}_B^{d_1,d_2}.$$

Let $\imath_d^{d_1,d_2}$ denote the composition of $\overline{\imath}^d$ and the open embedding

$$\imath_0^{d_1+d,d_2-d} := \jmath^{d_1+d,d_2-d} : \mathrm{Bun}_B^{d_1+d,d_2-d} \hookrightarrow \overline{\mathrm{Bun}}_B^{d_1+d,d_2-d}.$$

It is known (see [BG], Proposition 6.1.2) that $\imath_d$ is a locally closed embedding, and that these stacks define a stratification of $\overline{\mathrm{Bun}}_B^{d_1,d_2}$. Moreover, the map $\jmath^{d_1+d,d_2-d}$ is known to be affine. Hence, the map $\imath_d^{d_1,d_2}$ is also affine.

Recall that for $G = GL_2$, the stack $\overline{\mathrm{Bun}}_B^{d_1,d_2}$ is smooth; in particular, its intersection cohomology sheaf is constant. Thus, we obtain an exact complex of perverse sheaves on $\overline{\mathrm{Bun}}_B^{d_1,d_2}$:

(2.1) $... \to (\imath_d^{d_1,d_2})_!(\mathrm{IC}_{X^{(d)} \times \mathrm{Bun}_B^{d_1+d,d_2-d}}) \to ...... \to (\imath_2^{d_1,d_2})_!(\mathrm{IC}_{X^{(2)} \times \mathrm{Bun}_B^{d_1+2,d_2-2}}) \to$

$$\to (\imath_1^{d_1,d_2})_!(\mathrm{IC}_{X \times \mathrm{Bun}_B^{d_1+1,d_2-1}}) \to (\imath_0^{d_1,d_2})_!(\mathrm{IC}_{\mathrm{Bun}_B^{d_1,d_2}}) \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{d_1,d_2}}\,.$$

Let $\mathcal{S}(E_{\check{T}})$ be the local system on $\mathrm{Bun}_T \simeq \mathrm{Pic}(X) \times \mathrm{Pic}(X)$, corresponding to $E_{\check{T}}$. We normalize it so that the pull-back of $\mathcal{S}(E_{\check{T}})$ under

$$X^{(d_1')} \times X^{(d_2')} \overset{\mathrm{AJ} \times \mathrm{AJ}}{\to} \mathrm{Pic}(X) \times \mathrm{Pic}(X),$$

where AJ is the Abel-Jacobi map $D \mapsto \mathcal{O}_X(D)$, is $E_1^{(d_1')} \boxtimes E_2^{(d_2')}$.

Recall that by definition,

$$\overline{\mathrm{Eis}}^{d_1,d_2}(E_{\check{T}}) = \overline{\mathfrak{p}}_!^{d_1,d_2}\left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{d_1,d_2}} \otimes \overline{\mathfrak{q}}^{d_1,d_2\,*}(\mathcal{S}(E_{\check{T}}))\right)$$

and

$$\mathrm{Eis}_!^{d_1,d_2}(E_{\check{T}}) = \mathfrak{p}_!^{d_1,d_2}\left(\mathrm{IC}_{\mathrm{Bun}_B^{d_1,d_2}} \otimes \mathfrak{q}^{d_1,d_2\,*}(\mathcal{S}(E_{\check{T}}))\right),$$

where $\overline{\mathfrak{p}}^{d_1,d_2}$ ($\overline{\mathfrak{q}}^{d_1,d_2}$) denote the natural projection from $\overline{\mathrm{Bun}}_B^{d_1,d_2}$ to Bun_G (resp., Bun_T), and $\mathfrak{p}^{d_1,d_2} = \overline{\mathfrak{p}}^{d_1,d_2} \circ \jmath^{d_1,d_2}$ (resp., $\mathfrak{q}^{d_1,d_2} = \overline{\mathfrak{q}}^{d_1,d_2} \circ \jmath^{d_1,d_2}$) is its restriction to $\mathrm{Bun}_B^{d_1,d_2}$.

Tensoring the complex (2.1) with the local system $\overline{\mathfrak{q}}^{d_1,d_2\,*}(\mathcal{S}(E_{\check{T}}))$ we obtain a complex

(2.2) $\mathrm{Kosz}_{d_1,d_2}(\mathrm{Eis}_!(E_{\check{T}}))^\bullet :=$

$$... \to (\imath_d^{d_1,d_2})_!\left((E_2 \otimes E_1^{-1})^{(d)}[d] \boxtimes \left(\mathrm{IC}_{\mathrm{Bun}_B^{d_1+d,d_2-d}} \otimes \mathfrak{q}^{d_1+d,d_2-d\,*}(\mathcal{S}(E_{\check{T}}))\right)\right) \to ...$$

$$... \to (\imath_2^{d_1,d_2})_!\left((E_2 \otimes E_1^{-1})^{(2)}[2] \boxtimes \left(\mathrm{IC}_{\mathrm{Bun}_B^{d_1+2,d_2-2}} \otimes \mathfrak{q}^{d_1+2,d_2-2\,*}(\mathcal{S}(E_{\check{T}}))\right)\right) \to$$

$$\to (\imath_1^{d_1,d_2})_!\left((E_2 \otimes E_1^{-1})[1] \boxtimes \left(\mathrm{IC}_{\mathrm{Bun}_B^{d_1+1,d_2-1}} \otimes \mathfrak{q}^{d_1+1,d_2-1\,*}(\mathcal{S}(E_{\check{T}}))\right)\right) \to$$

$$\to (\imath_0^{d_1,d_2})_!\left(\mathrm{IC}_{\mathrm{Bun}_B^{d_1,d_2}} \otimes \mathfrak{q}^{d_1,d_2\,*}(\mathcal{S}(E_{\check{T}}))\right),$$

which is quasi-isomorphic to $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{d_1,d_2}} \otimes \overline{\mathfrak{q}}^{d_1,d_2\,*}(\mathcal{S}(E_{\check{T}}))$.

Applying $\overline{\mathfrak{p}}_!^{d_1,d_2}$ to $\mathrm{Kosz}_{d_1,d_2}(\mathrm{Eis}_!(E_{\check{T}}))^\bullet$, we obtain a complex

(2.3) $... \to H^d(X, (E_2 \otimes E_1^{-1})^{(d)}) \otimes \mathrm{Eis}_!^{d_1+d,d_2-d}(E_{\check{T}}) \to ... \to$

$$\to H^2(X, (E_2 \otimes E_1^{-1})^{(2)}) \otimes \mathrm{Eis}_!^{d_1+2,d_2-2}(E_{\check{T}}) \to$$

$$\to H^1(X, (E_2 \otimes E_1^{-1})) \otimes \mathrm{Eis}_!^{d_1+1,d_2-1}(E_{\check{T}}) \to \mathrm{Eis}_!^{d_1,d_2}(E_{\check{T}}),$$

which is in turn quasi-isomorphic to $\overline{\mathrm{Eis}}^{d_1,d_2}(E_{\check{T}})$.

Let us recall that $H^d(X, (E_2 \otimes E_1^{-1})^{(d)}) \simeq \Lambda^d\left(H^1(X, E_2 \otimes E_1^{-1})\right) =: \Lambda^d(\mathsf{W})$. Thus, we obtain that the terms of the complex (2.3) coincide with those of the complex $\mathsf{K}_{d_1,d_2}(\mathrm{Eis}_!(E_{\check{T}}))^\bullet$ appearing in Theorem 2.3(b).

In particular, for each pair (d_1',d_2') we obtain a map

$$\partial_1 : \mathsf{W} \otimes \mathrm{Eis}_!^{d_1',d_2'}(E_{\check{T}}) \to \mathrm{Eis}_!^{d_1'-1,d_2'+1}(E_{\check{T}}).$$

Therefore, to finish the proof of the theorem, it suffices to show that for each d the differential in (2.3)

$$(2.4) \quad H^d(X, (E_2 \otimes E_1^{-1})^{(d)}) \otimes \mathrm{Eis}_!^{d_1+d,d_2-d}(E_{\check{T}}) \to$$
$$H^{d-1}(X, (E_2 \otimes E_1^{-1})^{(d-1)}) \otimes \mathrm{Eis}_!^{d_1+d-1,d_2-+1}(E_{\check{T}})$$

equals

$$\Lambda^d(\mathsf{W}) \otimes \mathrm{Eis}_!^{d_1+d,d_2-d}(E_{\check{T}}) \hookrightarrow \Lambda^{d-1}(\mathsf{W}) \otimes \mathsf{W} \otimes \mathrm{Eis}_!^{d_1+d,d_2-d}(E_{\check{T}}) \overset{\mathrm{id} \otimes \partial_1}{\longrightarrow}$$
$$\longrightarrow \Lambda^{d-1}(\mathsf{W}) \otimes \mathrm{Eis}_!^{d_1+d-1,d_2+1}(E_{\check{T}}).$$

Note that we have a commutative diagram of stacks

$$
\begin{array}{ccc}
X^{(d-1)} \times X \times \mathrm{Bun}_B^{d_1+d,d_2-d} & \xrightarrow{\ \mathrm{id} \times \iota_1^{d_1+d-1,d_2-d+1}\ } & X^{(d-1)} \times \overline{\mathrm{Bun}}_B^{d_1+d-1,d_2-d+1} \\
{\scriptstyle \mathrm{sym}_{d-1,1}} \downarrow & & \downarrow {\scriptstyle \bar{\iota}_{d-1}^{d_1,d_2}} \\
X^{(d)} \times \mathrm{Bun}_B^{d_1+d,d_2-d} & \xrightarrow{\ \iota_d^{d_1,d_2}\ } & \overline{\mathrm{Bun}}_B^{d_1,d_2},
\end{array}
$$

where $\mathrm{sym}_{d-1,1}$ is the natural map $X^{(d-1)} \times X \to X^{(d)}$.

Hence, the map

$$(\iota_d^{d_1,d_2})_! \left(\mathrm{IC}_{X^{(d)}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{d_1+d,d_2-d}} \right) \to (\iota_{d-1}^{d_1,d_2})_! \left(\mathrm{IC}_{X^{(d-1)}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{d_1+d-1,d_2-d+1}} \right),$$

appearing in (2.1) equals the composition

$$(\iota_d^{d_1,d_2})_! \left(\mathrm{IC}_{X^{(d)}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{d_1+d,d_2-d}} \right) \to (\iota_d^{d_1,d_2})_! \left(\mathrm{sym}_{d-1,1\,!}(\mathrm{IC}_{X^{(d-1)} \times X}) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{d_1+d,d_2-d}} \right) \simeq$$
$$\simeq (\bar{\iota}_{d-1}^{d_1,d_2})_! \left(\mathrm{IC}_{X^{(d-1)}} \boxtimes (\iota_1^{d_1+d-1,d_2-d+1})_!(\mathrm{IC}_{X \times \mathrm{Bun}_B^{d_1+d,d_2-d}}) \right) \to$$
$$\to (\bar{\iota}_{d-1}^{d_1,d_2})_! \left(\mathrm{IC}_{X^{(d-1)}} \boxtimes (\iota_0^{d_1+d-1,d_2-d+1})_!(\mathrm{IC}_{\mathrm{Bun}_B^{d_1+d-1,d_2-d+1}}) \right) \simeq$$
$$\simeq (\iota_{d-1}^{d_1,d_2})_! \left(\mathrm{IC}_{X^{(d-1)}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{d_1+d-1,d_2-d+1}} \right),$$

implying the desired equality after taking the direct image with respect to $\bar{\mathfrak{p}}^{d_1,d_2}$.

3. DEFORMING LOCAL SYSTEMS

The goal of this section is to describe explicitly the topological algebra $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$.

3.1.   Let $\check{\mathfrak{n}}_X$ be the constant sheaf of Lie algebras over X with fiber $\check{\mathfrak{n}}$. Let $\check{\mathfrak{n}}_{X,E_{\check{T}}}$ denote its twist by means of $E_{\check{T}}$ with respect to the adjoint action of $\check{T}$ on $\check{\mathfrak{n}}$. It is naturally graded by elements of $\check{\Lambda}^{pos}$, where the latter is a sub-semigroup of $\check{\Lambda}$ equal to the positive span of simple co-roots. We will consider the standard complex $\mathbf{C}_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ as a sheaf of co-commutative DG co-algebras on X, also endowed with a grading by means of $\check{\Lambda}^{pos}$.

Recall that to $\check{\lambda} \in \check{\Lambda}^{pos} - 0$ we have attached the corresponding partially symmetrized power of X, denoted $X^{\check{\lambda}}$. We are going to associate to each such $\check{\lambda}$ a certain perverse sheaf $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}$ on $X^{\check{\lambda}}$. (In the non-twisted case, i.e., for $E_{\check{T}}$ trivial, we will denote it simply by $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}$.) This will be based on the following general construction.

Let A be a sheaf of co-commutative DG co-algebras on X, and let n be a non-negative integer. By the procedure of [BD], Sect. 3.4, to A one can associate a sheaf, denoted, $A_{X^{(n)}}$ on $X^{(n)}$, whose fiber at $D = \Sigma\, m_k \cdot x_k \,|\, x_{k_1} \neq x_{k_2}$ is $\bigotimes_k A_{x_k}$.

Suppose now that A is $\check{\Lambda}^{pos}$-graded, and let $\check{\lambda}$ be an element of $\check{\Lambda}^{pos}$. We have a natural map $X^{\check{\lambda}} \to X^{(|\check{\lambda}|)}$. Then the *-pull-back of $A_{X^{(n)}}$ to $X^{\check{\lambda}}$ admits a sub-sheaf, denoted $A_{X^{\check{\lambda}}}$, whose fiber at $D = \check{\lambda}_k \cdot x_k \,|\, x_{k_1} \neq x_{k_2}$ is the subspace

$$\bigotimes_k (A_{x_k})^{\check{\lambda}_k} \subset \bigotimes_k A_{x_k}.$$

We apply this to $A = \mathbf{C}_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ and obtain a complex of sheaves that we denote by $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$. Let us describe it even more explicitly:

Recall that $X^{\check{\lambda}}$ admits a stratification, numbered by partitions

$$\mathfrak{P}^\lambda : \check{\lambda} = \underset{k}{\Sigma}\, m_k \cdot \check{\lambda}_k \,|\, m_k \in \mathbb{Z}^{>0}, \check{\lambda}_k \in \check{\Lambda}^{pos} - 0, \check{\lambda}_k \neq \check{\lambda}_{k'}$$

The corresponding stratum in $X^{\check{\lambda}}$ is isomorphic to $\left(\underset{k}{\Pi}\, X^{(n_k)} \right)_{disj}$, where the subscript "$disj$" denotes the complement to the diagonal divisor in the above product. Its dimension is $|\mathfrak{P}(\check{\lambda})| = \underset{k}{\Sigma}\, m_k$. Let $j^{\mathfrak{P}(\check{\lambda})}$ denote the corresponding locally closed embedding.

Let $\mathrm{Cous}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\mathfrak{P}(\check{\lambda})}$ denote the complex of (both, sheaves, and (shifted) perverse) sheaves

$$j_!^{\mathfrak{P}(\check{\lambda})} \left(\underset{k}{\boxtimes} \left(\Lambda^\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_k} \right)^{(m_k)} \right).$$

The direct sum

$$\mathrm{Cous}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}} := \bigoplus_{\mathfrak{P}(\check{\lambda})} \mathrm{Cous}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\mathfrak{P}(\check{\lambda})},$$

viewed as a graded perverse sheaf, admits a natural differential and the resulting total complex is, by definition, quasi-isomorphic to $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$.

Proposition 3.2. $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ *is acyclic off cohomolgical degree* 0 *in the perverse t-structure.*

3.3. In order to prove Proposition 3.2 we need to introduce some notation that will be also useful in the sequel. Let $\check{\lambda}_1, \check{\lambda}_2$ be two elements of $\check{\Lambda}^{pos}$. Note we have a natural addition map

$$X^{\check{\lambda}_1} \times X^{\check{\lambda}_2} \to X^{\check{\lambda}_1 + \check{\lambda}_2}.$$

This map is finite, and it induces an exact functor

$$\mathrm{Perv}(X^{\check{\lambda}_1}) \times \mathrm{Perv}(X^{\check{\lambda}_2}) \to \mathrm{Perv}(X^{\check{\lambda}_1 + \check{\lambda}_2})$$

that we will denote by $\mathcal{T}_1, \mathcal{T}_2 \mapsto \mathcal{T}_1 \star \mathcal{T}_2$.

The natural increasing filtration on $\mathbf{C}_\bullet(\check{\mathfrak{n}}_{X,E_{\check{T}}})$ defines a filtration on $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$. The associated graded can be obtained by the same procedure, where instead of $\check{\mathfrak{n}}_{X,E_{\check{T}}}$ we use the abelian Lie algebra structure on the same sheaf. Hence,

$$\mathrm{gr}^j \left(\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}} \right) = \bigoplus_{\check{\lambda} = \Sigma\, n_k \cdot \check{\alpha}_k,\, \check{\alpha}_k \in \check{\Delta}^+,\, \Sigma\, n_k = j} \underset{k}{\star} \left(\Lambda^{(n_k)}(E_{\check{T}}^{\check{\alpha}_k}) \right) [j],$$

where in the above formula for a local system F we denote by $\Lambda^{(n)}(\mathsf{F})$ its external exterior power.

This description of the associated graded readily implies that $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ is a perverse sheaf. In addition, we obtain the following:

Corollary 3.4. *Assume that $E_{\check{T}}$ is regular. Then for every $\check{\lambda} \in \check{\Lambda}^{pos}$ the cohomology*

$$H^\bullet\left(X^{\check{\lambda}}, \Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}\right)$$

is concentrated in degree 0.

Proof. It is enough to prove that each $H^\bullet\left(X^{\check{\lambda}}, \mathrm{gr}^j(\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}})\right)$ is concentrated in degree 0. The latter follows from the fact that

$$H^\bullet\left(X^{(n)}, \Lambda^{(n)}(E_{\check{T}}^{\check{\alpha}})[n]\right) \simeq \mathrm{Sym}^n\left(H^\bullet(X, E_{\check{T}}^{\check{\alpha}})[1]\right) \simeq \mathrm{Sym}^n\left(H^1(X, E_{\check{T}}^{\check{\alpha}})\right).$$

$\square$

3.5. Let $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}$ denote the Verdier dual of $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$. From Corollary 3.4 it follows that $H^\bullet\left(X^{\check{\lambda}}, \Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}\right)$ is also concentrated in degree 0.

From the construction of $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ it follows that we have natural maps

(3.1) $$\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_1+\check{\lambda}_2} \to \Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_1} \star \Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_2},$$

that are co-associative and co-commutative in the natural sense. Hence, we obtain the maps

$$\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}_1} \star \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}_2} \to \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}_1-\check{\lambda}_2}$$

that are associative and commutative.

In addition the perverse sheaves $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}$ possess the following factorization property. For $\check{\lambda} = \check{\lambda}_1 + \check{\lambda}_2$ as above, let $\left(X^{\check{\lambda}_1} \times X^{\check{\lambda}_2}\right)_{disj} \subset X^{\check{\lambda}_1} \times X^{\check{\lambda}_2}$ be the open subset corresponding to the condition that the two divisors have a disjoint support. We have a natural isomorphism:

(3.2) $$\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}|_{\left(X^{\check{\lambda}_1} \times X^{\check{\lambda}_2}\right)_{disj}} \simeq \left(\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}_1} \boxtimes \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}_2}\right)|_{\left(X^{\check{\lambda}_1} \times X^{\check{\lambda}_2}\right)_{disj}},$$

and similarly for $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$.

Set

$$R_{E_{\check{T}}}^{-\check{\lambda}} := H(X^{\check{\lambda}}, \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}) \text{ and } R_{E_{\check{T}}} := \bigoplus_{\check{\lambda} \in \check{\Lambda}^{pos}} R_{E_{\check{T}}}^{-\check{\lambda}}, \ \widehat{R}_{E_{\check{T}}} := \prod_{\check{\lambda} \in \check{\Lambda}^{pos}} R_{E_{\check{T}}}^{-\check{\lambda}}.$$

We obtain that $R_{E_{\check{T}}}$ is a commutative $-\check{\Lambda}^{pos}$-graded algebra, and $\widehat{R}_{E_{\check{T}}}$ is isomorphic to the completion of $R_{E_{\check{T}}}$ at the natural augmentation ideal.

Note that the computation in Sect. 3.3 implies that $R_{E_{\check{T}}}$ admits a natural decreasing filtration and

$$\mathrm{gr}(R_{E_{\check{T}}}) \simeq \mathrm{Sym}\left(\bigoplus_{\check{\alpha} \in \check{\Delta}^+} H^1(X, E_{\check{T}}^{\check{\alpha}})^*\right) \simeq \mathrm{Sym}\left(H^1(X, \mathfrak{n}_{X,E_{\check{T}}})^*\right).$$

It is easy to see that the above filtration is given by powers of the augmentation ideal.

We have:

Theorem 3.6. *We have a canonical isomorphism of topological algebras*

$$\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})} \simeq \widehat{R}_{E_{\check{T}}}.$$

The proof will be given in Sect. 11.

Let $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$ denote the algebra equal to the direct sum of homogeneous components of $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$ with respect to the natural $\check{T}$-action. The above theorem implies:

Corollary 3.7. $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})} \simeq R_{E_{\check{T}}}$.

Let us end this section with the following remark. In the context of D-modules, we can look at the *scheme*, and not just the formal scheme, of $\check{B}$-local systems $E_{\check{B}}$, such that the induced $\check{T}$-local system is identified with $E_{\check{T}}$.

It is easy to see that the regularity assumption on $E_{\check{T}}$ implies that this scheme indeed exists; moreover, it is isomorphic to $\mathrm{Spec}(\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})})$. In other words, the algebra of functions on this scheme is isomorphic to $R_{E_{\check{T}}}$.

4. STRUCTURE OF THE EXTENSION BY ZERO

4.1.   For $\check{\lambda} \in \check{\Lambda}^{pos}$ recall the maps

$$\bar{\imath}_{\check{\lambda}} : X^{\check{\lambda}} \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B \quad \text{and} \quad \imath_{\check{\lambda}} : X^{\check{\lambda}} \times \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B.$$

In what follows we will use the following notation. For $\mathcal{T} \in D^b(X^{\check{\lambda}})$ and $\mathcal{S} \in D^b(\overline{\mathrm{Bun}}_B)$, we will denote by $\mathcal{T} \star \mathcal{S} \in D^b(\overline{\mathrm{Bun}}_B)$ the object $(\bar{\imath}_{\check{\lambda}})_!(\mathcal{T} \boxtimes \mathcal{S})$. This operation is clearly associative with respect to $\star : D^b(X^{\check{\lambda}_1}) \times D^b(X^{\check{\lambda}_2}) \to D^b(X^{\check{\lambda}_1 + \check{\lambda}_2})$.

The main result of this section is the following:

Theorem 4.2.

(1) *The 0-th perverse cohomology of* $\imath_{\check{\lambda}}^! \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \right)$ *is canonically isomorphic to the product* $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$. *In particular, by adjunction we obtain a map*

$$(4.1) \qquad \Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}) \to \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}),$$

(2) *For two elements* $\check{\lambda}_1, \check{\lambda}_2 \in \check{\Lambda}^{pos}$ *the diagram*

$$
\begin{array}{ccc}
\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}_1} \star \Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}_2} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}_1+\check{\lambda}_2}}) & \longrightarrow & \Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}_1-\check{\lambda}_2} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}_1+\check{\lambda}_2}}) \\
\downarrow & & \downarrow \\
\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}_1} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}_1}}) & \longrightarrow & \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}).
\end{array}
$$

is commutative.

We shall now explain how this theorem implies point (1) of Theorem 1.8, i.e., that $\mathrm{Eis}_!(E_{\check{T}})$ carries an action of $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$. Taking into account Corollary 3.7, we need to exhibit the maps

$$(4.2) \qquad H(X^{\check{\lambda}}, \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}) \otimes \mathrm{Eis}_!^{\check{\mu}+\check{\lambda}}(E_{\check{T}}) \to \mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}}), \ \check{\lambda} \in \check{\Lambda}^{pos}$$

that are associative in the natural sense.

Let us tensor both sides of (4.1) with $(\overline{\mathfrak{q}}^{\check{\mu}})^*(\mathcal{S}(E_{\check{T}}))$. We obtain a map

$$\imath_{\check{\lambda}!} \left(\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}} \boxtimes \left(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \otimes (\mathfrak{q}^{\check{\mu}+\check{\lambda}})^*(\mathcal{S}(E_{\check{T}})) \right) \right) \to \jmath_! \left(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}} \otimes (\mathfrak{q}^{\check{\mu}})^*(\mathcal{S}(E_{\check{T}})) \right),$$

and taking the direct image with respect to $\overline{\mathfrak{p}}^{\check{\mu}} : \overline{\mathrm{Bun}}_B^{\check{\mu}} \to \mathrm{Bun}_G$, we obtain the map of (4.2), as required. The associativity of the action follows from the commutativity of the diagram in Theorem 4.2.

Let us consider the pro-object in $\mathrm{Perv}(\mathrm{Bun}_G)$ defined as

$$\widehat{\mathrm{Eis}}_!(E_{\check{T}}) := \mathrm{Eis}_!(E_{\check{T}}) \underset{\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}}{\otimes} \widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})},$$

and let us describe it in intrinsic terms.

For $\check{\lambda} \in \check{\Lambda}$, let $\jmath_!^{\tau(\check{\lambda})}(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})$ denote the quotient of $\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})$ by the image of the maps

$$\Omega(\check{\mathfrak{n}}_X)^{\check{\lambda}'} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}'}}) \to \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}),$$

given by (4.1) for $\check{\lambda}' > \check{\lambda}$. Set

$$\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})^{\tau(\check{\lambda})} := \overline{\mathfrak{p}}_!^{\check{\mu}} \left(\jmath_!^{\tau(\check{\lambda})}(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \otimes (\overline{\mathfrak{q}}^{\check{\mu}})^*(\mathcal{S}(E_{\check{T}})) \right).$$

As in Theorem 1.5, each $\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})^{\tau(\check{\lambda})}$ is perverse. Moreover, for every open substack $U \subset \mathrm{Bun}_G$ of finite type there are only finitely many $\check{\mu}$, for which the restriction $\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})^{\tau(\check{\lambda})}|_U$ is non-zero. Hence, the direct sum $\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})^{\tau(\check{\lambda})} := \underset{\check{\mu}}{\oplus} \mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})^{\tau(\check{\lambda})}$ makes sense as an object of $\mathrm{Perv}(\mathrm{Bun}_G)$. We have:

$$\widehat{\mathrm{Eis}}_!(E_{\check{T}}) \simeq \text{''}\varprojlim_{\check{\lambda}}\text{''}\ \mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}}))^{\tau(\check{\lambda})}.$$

4.3. The rest of the present section is be devoted to the proof of Theorem 4.2.

For a local system $E_{\check{T}}$ as above, let $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ denote the following *sheaf* on $X^{\check{\lambda}}$ for $\check{\lambda} \in \check{\Lambda}^{pos}$. Its fiber at a point $\Sigma\,\check{\lambda}_k \cdot x_k$ with x_k's distinct is the tensor product

$$\bigotimes_k U(\check{\mathfrak{n}}_{E_{\check{T},x}})^{\check{\lambda}_k},$$

where the superscript $\check{\lambda}_k$ refers to the corresponding weight component in $U(\check{\mathfrak{n}})$ and the subscript $E_{\check{T},x}$ to the twist by the fiber of $E_{\check{T}}$ at x. These fibers glue to a sheaf by means of the co-multiplication map on $U(\check{\mathfrak{n}})$. When $E_{\check{T}}$ is trivial, we will denote the corresponding sheaf simply by $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}}$.

One of the ingredients in the proof Theorem 4.2 is the following result, which essentially follows from [BFGM]:

Proposition 4.4. *There exists a canonical isomorphism in* $D^b(X^{\check{\lambda}} \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}})$:

$$\imath_{\check{\lambda}}^!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}) \simeq \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}.$$

We should remark that the proof of Proposition 4.4, that we give, uses one piece of unpublished work (see below) that has to do with the identification of the sheaf $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}}$. However, for the proof of Theorem 4.2 we will need only to know the image of $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}}$ in the Grothendieck group, a computation which is fully carried out in [BFGM].

Proof. Let us recall the construction of the Zastava spaces, $\overline{\mathcal{Z}}^{\check{\lambda}}$ for $\check{\lambda} \in \check{\Lambda}^{pos}$. Let B^- (resp., N^-) be the negative Borel subgroup of G (resp., its unipotent radical).

By definition, $\overline{\mathcal{Z}}^{\check{\lambda}}$ is the open subscheme of $\overline{\mathrm{Bun}}_B^{-\check{\lambda}} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{N^-}$, corresponding to the condition that the reduction to N^- and the generalized reduction to B on the given G-bundle are transversal at the generic point of the curve. The stack $\overline{\mathcal{Z}}^{\check{\lambda}}$ is naturally fibered over $X^{\check{\lambda}}$ by means of a projection denoted $\pi^{\check{\lambda}}$, and with a section of this projection, denoted $\mathfrak{s}^{\check{\lambda}}$ (see [BFGM], Sect. 2).

A basic feature of the Zastava spaces is the following factorization property with respect to the projection $\pi^{\check\lambda}$ (see [BFGM], Proposition 2.4):

$$(4.3) \qquad \overline{\mathcal{Z}}^{\check\lambda_1+\check\lambda_2} \underset{X^{\check\lambda_1+\check\lambda_2}}{\times} \left(X^{\check\lambda_1} \times X^{\check\lambda_2} \right)_{disj} \simeq \left(\overline{\mathcal{Z}}^{\check\lambda_1} \times \overline{\mathcal{Z}}^{\check\lambda_2} \right) \underset{X^{\check\lambda_1} \times X^{\check\lambda_2}}{\times} \left(X^{\check\lambda_1} \times X^{\check\lambda_2} \right)_{disj}.$$

Let $\mathcal{Z}^{\check\lambda} \subset \overline{\mathcal{Z}}^{\check\lambda}$ denote the open subscheme, corresponding to the condition that the B-structure is non-degenerate; let $j^{\mathcal{Z}^{\check\lambda}}$ denote its open embedding. Both this subscheme and the section $\mathfrak{s}^{\check\lambda}$ are compatible with the isomorphisms (4.3) above.

It was shown in *loc. cit.* that $\overline{\mathcal{Z}}^{\check\lambda}$ is locally in the smooth topology isomorphic to $\overline{\mathrm{Bun}}_B$, in such a way that $\mathrm{Bun}_B \subset \overline{\mathrm{Bun}}_B$ corresponds to $\mathcal{Z}^{\check\lambda}$, and the locally closed subvariety $X^{\check\lambda} \times \mathrm{Bun}_B \subset \overline{\mathrm{Bun}}_B$ corresponds to a subscheme $\mathfrak{s}^{\check\lambda}(X^{\check\lambda})$ that projects isomorphically onto $X^{\check\lambda}$.

We claim that to prove Proposition 4.4, it is sufficient to establish the isomorphism

$$(4.4) \qquad\qquad\qquad \mathfrak{s}^{!}_{\check\lambda}(\overline{\mathcal{Z}}^{\check\lambda}) \simeq \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda}.$$

Indeed, the local isomorphism between the triples

$$(X^{\check\lambda} \times \mathrm{Bun}_B) \overset{\imath_{\check\lambda}}{\hookrightarrow} \overline{\mathrm{Bun}}_B \overset{\jmath}{\hookleftarrow} \mathrm{Bun}_B \ \text{ and } \ X^{\check\lambda} \overset{\mathfrak{s}^{\check\lambda}}{\hookrightarrow} \overline{\mathcal{Z}}^{\check\lambda} \overset{\jmath^{\mathcal{Z}^{\check\lambda}}}{\hookleftarrow} \mathcal{Z}^{\check\lambda}$$

implies that $\imath^{!}_{\check\lambda}(\mathrm{IC}_{\overline{\mathrm{Bun}}^{\check\mu}_B})$ has the desired shape, except for a possible twist by local systems along the $\mathrm{Bun}^{\check\mu+\check\lambda}_B$ multiple. The fact that no such twist occurs can be seen using the action of the Hecke stack, as in [BG], Sects. 5.2 and 6.2.

In [BFGM], Proposition 5.2, it was also shown that there is a canonical isomorphism

$$\mathfrak{s}^{!}_{\check\lambda}(\mathrm{IC}_{\overline{\mathcal{Z}}^{\check\lambda}}) \simeq \pi^{\check\lambda}_{!}(\mathrm{IC}_{\overline{\mathcal{Z}}^{\check\lambda}}),$$

and that the expression appearing in the above formula is a *sheaf* isomorphic to the top $(=2|\check\lambda|)$ cohomology in the *usual t-structure* of $\pi^{\check\lambda}_{!}(\mathbb{C}_{\mathcal{Z}^{\check\lambda}})$. Thus, we have to show that the latter is isomorphic as a sheaf to $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda}$.

For that we note that $\mathcal{Z}^{\check\lambda}$ is naturally a subscheme in the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,X^{\check\lambda}}$, equal to the intersection of the corresponding semi-infinite orbits. The identification between $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda}$ and the top cohomology of $\pi^{\check\lambda}_{!}(\mathbb{C}_{\mathcal{Z}^{\check\lambda}})$ at the level of fibers follows from the realization of $U(\check{\mathfrak{n}})$ is the top cohomology of the intersection of semi-infinite orbits, see [BFGM], Theorem 5.9.

In order to see that this identification glues to an isomorphism of sheaves one needs to express the co-product on $U(\mathfrak{n})$ in terms of $\pi^{\check\lambda}_{!}(\mathbb{C}_{\mathcal{Z}^{\check\lambda}})$. This relationship has been recently established in [Kam]

$$\square$$

To state a corollary of the above proposition that will be used in the proof of Theorem 4.2, let us consider the following version of the Grothendieck group of perverse sheaves on $\overline{\mathrm{Bun}}^{\check\mu}_B$. We start with the usual Grothendieck group of the Artinian category of perverse sheaves on $\overline{\mathrm{Bun}}_B$ of *finite length*, and we complete it with respect to the topology, where the system of neighbourhoods of zero is given by classes of perverse sheaves, supported on closures of $\imath_{\check\lambda}(X^{\check\lambda} \times \mathrm{Bun}_B)$, $\check\lambda \in \check\Lambda^{pos}$. In particular, the class of each $(\imath_{\check\lambda})_{!}(\mathcal{T})$, $\mathcal{T} \in D^b(X^{\check\lambda} \times \mathrm{Bun}^{\check\mu+\check\lambda}_B)$ is a well-defined element of this group.

Corollary 4.5. *There is an equality* $[\jmath_{!}(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})] = \underset{\check\lambda \in \check\Lambda^{pos}}{\Sigma} [\Omega(\check{\mathfrak{n}}_X)^{-\check\lambda} \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check\mu+\check\lambda}_B}].$

Proof. From Proposition 4.4 it follows that for $\mathfrak{T} \in D^b(X^{\check{\lambda}})$,

$$[\mathfrak{T} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}}] = \underset{\check{\lambda}' \in \check{\Lambda}^{pos}}{\Sigma} [\mathfrak{T} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}'} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}+\check{\lambda}'}})].$$

Moreover, by Sect. 6.4, for $0 \neq \check{\lambda} \in \check{\Lambda}^{pos}$,

$$\underset{\substack{\check{\lambda}_1, \check{\lambda}_2 \in \check{\Lambda}^{pos} \\ \check{\lambda}_1 + \check{\lambda}_2 = \check{\lambda}}}{\Sigma} [\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}_1} \star \Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}_2}] = 0.$$

This implies the assertion of the corollary. $\qquad\square$

Corollary 4.6. *In the Grothendieck group of perverse sheaves on $X^{\check{\lambda}} \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$ we have the following equality:*

$$[h^0 \left(\imath_{\check{\lambda}}^! \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \right) \right)] = [\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}].$$

Proof. Let $\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\lambda}}$ by the open substack of $\overline{\mathrm{Bun}}_B^{\check{\mu}}$, obtained by removing the closed substack equal to the union $\imath_{\check{\lambda}'}(X^{\check{\lambda}'} \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}'})$ for $\check{\lambda}' - \check{\lambda} \in \check{\Lambda}^{pos} - 0$.

Arguing as in the proof of Proposition 4.4, we can replace the original question about $\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\lambda}}$ for one about the Zastava space $\overline{\mathcal{Z}}^{\check{\lambda}}$. Using the factorization property (4.3), and arguing by induction on $|\check{\lambda}|$, can assume that the desired equality holds in the Grothendieck group of perverse sheaves over the open substack $(X^{\check{\lambda}} - \Delta(X)) \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$, where $\Delta(X) \subset X^{\check{\lambda}}$ denotes the main diagonal.

Taking into account Corollary 4.5, we have to show that there does not exist $\check{\lambda}' \in \check{\Lambda}^{pos}$ with $0 \neq \check{\lambda}' \neq \check{\lambda}$ and a perverse sheaf $\mathfrak{T}$ on $X^{\check{\lambda}'}$, appearing as subquotient of $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}'}$, and a non-trivial extension of perverse sheaves on $\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\lambda}}$:

$$0 \to (\mathfrak{T} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}'}})|_{\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\lambda}}} \to \mathfrak{T}' \to (\Delta_!(\mathbb{C}_X)[1] \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}) \to 0.$$

Obviously, an extension as above does not exist unless $\check{\lambda} - \check{\lambda}' \in \Lambda^{pos}$. In the latter case, it would be given by a morphism from $\Delta_!(\mathbb{C}_X[1]) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$ to

$$h^1 \left(\imath_{\check{\lambda}}^! \left(\overline{\imath}_{\check{\lambda}'!}(\mathfrak{T} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}'}}) \right) \right).$$

By Proposition 4.4, the latter expression is isomorphic to

$$h^1 \left((\mathfrak{T} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}-\check{\lambda}'}) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \right).$$

Note that $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}-\check{\lambda}'}$ is concentrated in the perverse cohomological degrees ≥ 1. The assertion of the corollary follows now from the fact that $\Delta^!(\mathfrak{T} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}-\check{\lambda}'})$ is concentrated in the cohomological degrees ≥ 2.

$\qquad\square$

4.7.   Our present goal is to construct a map

$$\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}) \to \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}),$$

or, equivalently, a map

(4.5) $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \to h^0 \left(\imath_{\check{\lambda}}^! \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \right) \right).$

Let $\overset{\circ}{X}{}^{\check{\lambda}}$ be the open subset of $X^{\check{\lambda}}$, corresponding to the *full* partition. I.e., it corresponds to coloured divisors of the form $\Sigma\,\check{\lambda}_k \cdot x_k$ with x_k's pairwise distinct and each $\check{\lambda}_k$ being a simple coroot. Let $j^{\check{\lambda}}$ denote the open embedding $\overset{\circ}{X}{}^{\check{\lambda}} \hookrightarrow X^{\check{\lambda}}$.

First, we claim that here exists a map (in fact, an isomorphism)

$$(4.6) \qquad j^{\check{\lambda}*}\left(\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}\right) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \simeq (j^{\check{\lambda}} \times \mathrm{id})^*\left(h^0\left(\imath_{\check{\lambda}}^! \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})\right)\right)$$

over $\overset{\circ}{X}{}^{\check{\lambda}} \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$. In other words, we claim that the isomorphism stated in Theorem 4.2 holds over the open substack $\overset{\circ}{X}{}^{\check{\lambda}} \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$.

Proof. As in the proof of Proposition 4.4, the assertion reduces to one about the Zastava space. Namely, we have to show that the restriction of

$$h^0\left((\mathfrak{s}^{\check{\lambda}})^! \circ \jmath_!^{\mathcal{Z}^{\check{\lambda}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}})\right)$$

to $\overset{\circ}{X}{}^{\check{\lambda}}$ is isomorphic to the restriction of $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}$.

Write $\check{\lambda} = \underset{i \in I}{\Sigma}\, n_i \cdot \check{\alpha}_i$. Then

$$\overset{\circ}{X}{}^{\check{\lambda}} \simeq \left(\underset{i \in I}{\Pi}\, X^{(n_i)}\right)_{disj},$$

and the restriction of $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}$ to it isomorphic to

$$\underset{i \in I}{\boxtimes}\, \Lambda^{(n_i)}((\check{\mathfrak{n}}^*)_X^{-\check{\alpha}_i}[1]),$$

where $(\check{\mathfrak{n}}^*)_X^{-\check{\alpha}_i}$ denotes the constant sheaf on X with fiber $(\check{\mathfrak{n}}^*)^{-\check{\alpha}_i}$, and $\Lambda^{(n_i)}(\cdot)$ the external exterior power of a local system.

By the same argument as in [BFGM], Proposition 5.2, $(\mathfrak{s}^{\check{\lambda}})^! \circ \jmath_!^{\mathcal{Z}^{\check{\lambda}}}(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}}) \simeq (\pi^{\check{\lambda}} \circ \jmath^{\mathcal{Z}^{\check{\lambda}}})_!(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}})$. So, we have to calculate the 0-th perverse cohomology of the direct image with compact supports of the constant sheaf on $\mathcal{Z}^{\check{\lambda}}$, cohomologically shifted by $[2|\check{\lambda}|]$.

However, by the factorization property given by (4.3),

$$\mathcal{Z}^{\check{\lambda}} \underset{X^{\check{\lambda}}}{\times} \overset{\circ}{X}{}^{\check{\lambda}} \simeq \underset{i \in I}{\Pi}\, \left(\mathcal{Z}^{\check{\alpha}_i}\right)^{\times n_i}/\Sigma_{n_i},$$

where Σ_{n_i} is the corresponding symmetric group. Moreover, each $\mathcal{Z}^{\check{\alpha}_i}$ is isomorphic to the product $X \times \mathbb{G}_m$, and

$$h^0\left((\pi^{\check{\alpha}_i} \circ \jmath^{\mathcal{Z}^{\check{\alpha}_i}})_!(\mathrm{IC}_{\mathcal{Z}^{\check{\alpha}_i}})\right) \simeq \mathbb{C}_X[1] \otimes H^1(\mathbb{G}_m, \mathbb{C}).$$

This makes the required isomorphism manifest once we identify each of the lines $(\check{\mathfrak{n}}^*)^{-\check{\alpha}_i}$ with $H^1(\mathbb{G}_m, \mathbb{C})$. There exists a natural identification like this, when we realize $U(\check{\mathfrak{n}})^{\check{\lambda}}$ as the top cohomology of the corresponding intersection in the affine Grassmannian.

$\qquad\quad\square$

Thus, our task is to show that the isomorphism (4.6) extends to a map (and an isomorphism) over the entire $X^{\check{\lambda}} \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$.

Lemma 4.8. *The canonical map*

$$\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \to j_*^{\check{\lambda}} \circ j^{\check{\lambda}*}\left(\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}\right)$$

is injective.

Proof. Using (3.2) we can apply induction on $|\check{\lambda}|$. The assertion clearly holds for $|\check{\lambda}| = 1$, i.e., when $\check{\lambda}$ is a simple root. Thus, we can assume that $|\check{\lambda}| \geq 2$ and that the injectivity assertion holds over $X^{\check{\lambda}} - \Delta(X)$.

We have to show that $\Delta^!(\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}})$ has no cohomologies in degrees ≤ 0. This amounts to the fact that the complex $\mathbf{C}^\bullet(\check{\mathfrak{n}})^{-\check{\lambda}}$ has no cohomologies in degrees ≤ 1. But this is evidently so: the kernel of the co-bracket $\check{\mathfrak{n}}^* \to \Lambda^2(\check{\mathfrak{n}}^*)$ is spanned by the duals of the simple roots. $\square$

The main geometric ingredient in the construction of the map (4.5) is the following:

Proposition 4.9. *The map of perverse sheaves on* $X^{\check{\lambda}} \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$

$$h^0\left(\imath_{\check{\lambda}}^! \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})\right) \to (j^{\check{\lambda}} \times \mathrm{id})_* \circ (j^{\check{\lambda}} \times \mathrm{id})^*\left(h^0\left(\imath_{\check{\lambda}}^! \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})\right)\right)$$

is injective.

Let us assume this proposition, and proceed with the construction of the map (4.5). [2]

4.10. By Proposition 4.9, we obtain that if the map (4.6) extends to a map as in (4.5), then it does so uniquely. Furthermore, by Lemma 4.8, the latter map is automatically injective. We claim that in this case, it is surjective as well, implying the isomorphism statement of Theorem 4.2. Indeed, the equivalence of injectivity and surjectivity properties of the map in question follows immediately from Corollary 4.6.

The commutativity of the diagram of Theorem 4.2 also follows from Lemma 4.8.

Thus, let us assume by induction that the map (4.6) has been shown to extend to a map (4.5) for all parameters $\check{\lambda}'$ with $|\check{\lambda}'| < |\check{\lambda}|$. Since the the question of extension is local, we can pass to the Zastava space $\overline{\mathcal{Z}}^{\check{\lambda}}$ as in the proof of Proposition 4.4.

Using the factorization property (4.3), and by the induction hypothesis, we can assume that the map (4.5) has been extended over $(X^{\check{\lambda}} - \Delta(X)) \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$. Let us distinguish two cases.

Case 1, when $\check{\lambda}$ is not a root of $\check{\mathfrak{g}}$. In this case, by Corollary 4.6 and Sect. 3.3, we obtain that neither $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$ nor $h^0\left(\imath_{\check{\lambda}}^!\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})\right)\right)$ has sub-quotients supported on $\Delta(X) \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$. Hence, both sides of (4.5) are the minimal extensions of their respective restrictions to $(X^{\check{\lambda}} - \Delta^!(X)) \times \mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$, and the extension assertion follows by the functoriality of the minimal extension operation.

Case 2, when $\check{\lambda}$ is a root of $\check{\mathfrak{g}}$. Consider the sum of the images of $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$ and $h^0\left(\imath_{\check{\lambda}}^!\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})\right)\right)$ in

$$j_*^{\check{\lambda}} \circ j^{\check{\lambda}*}\left(\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}}\right) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \simeq (j^{\check{\lambda}} \times \mathrm{id})_* \circ (j^{\check{\lambda}} \times \mathrm{id})^*\left(h^0\left(\imath_{\check{\lambda}}^! \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})\right)\right).$$

Let us denote this perverse sheaf by $\mathcal{T}$. We claim that $\Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$ maps isomorphically to $\mathcal{T}$. Indeed, if it did not, we would have a non-trivial extension

$$0 \to \Omega(\check{\mathfrak{n}}_X)^{-\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \to \mathcal{T} \to \Delta_!(\mathbb{C}_X[1]) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \to 0.$$

However, we have:

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[2]Added in Jan. 2008: a simpler proof of this proposition has been found that does not rely on [FFKM] and makes Sect. 5 redundant. Namely, one can argue by induction on $|\check{\lambda}|$ analyzing $\Delta^! \circ \pi_!^{\check{\lambda}} \circ \jmath_!(\mathrm{IC}_{\mathcal{Z}^{\check{\lambda}}})$.

Lemma 4.11. $\mathrm{Ext}^1_{X^{\check\lambda}}(\mathbb{C}_{\Delta(X)}[1], \Omega(\check{\mathfrak{n}}_X)^{-\check\lambda}) = 0$ *for $\check\lambda$ being a root of $\check{\mathfrak{g}}$.*

Proof. By adjunction, it suffices to show that $h^1\left(\Delta^!(\Omega(\check{\mathfrak{n}}_X)^{-\check\lambda})\right) = 0$. This amounts to the calculation of the component of weight $-\check\lambda$ in $H^2(\check{\mathfrak{n}}, \mathbb{C})$. It is known that the weights that appear in $H^2(\check{\mathfrak{n}}, \mathbb{C})$ are of the form $-\check\lambda = -\check\alpha_i - s_i(\check\alpha_j)$ for pairs of simple roots α_i and α_j. However, a weight $\check\lambda$ of the above form is never a root. $\square$

Thus, we obtain that $h^0\left(\imath^!_{\check\lambda}\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})\right)\right)$ is a sub-perverse sheaf of $\Omega(\check{\mathfrak{n}}_X)^{-\check\lambda} \boxtimes \mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\lambda}_B}$. The fact that this inclusion is an isomorphism follows from Corollary 4.6.

5. The FFKM construction and proof of Proposition 4.9

5.1.   The goal of this section is to prove Proposition 4.9, which is, in a way, the main technical point of this paper. We will first make a reduction to the case when $\check\lambda$ is a root of $\check{\mathfrak{g}}$, and in the latter case we will deduce the required assertion from a construction of [FFKM].

5.2.   We argue by induction on $|\check\lambda|$, and we claim that we can assume that the morphism

$$(5.1) \qquad h^0\left(\imath^!_{\check\lambda} \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})\right) \to (j^{\check\lambda} \times \mathrm{id})_* \circ (j^{\check\lambda} \times \mathrm{id})^*\left(h^0\left(\imath^!_{\check\lambda} \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})\right)\right)$$

is injective over the open substack $(X^{\check\lambda} - \Delta(X)) \times \mathrm{Bun}^{\check\mu+\check\lambda}_B$.

Indeed, the injectivity statement is local, so we can replace $\overline{\mathrm{Bun}}^{\check\mu,\leq\check\lambda}_B$ by the Zastava space $\overline{\mathcal{Z}}^{\check\lambda}$, and apply the factorization property (4.3).

This reduces the assertion of the proposition to the following one:

Proposition 5.3. *Assume that $|\check\lambda| > 1$. Then the perverse sheaf $h^0\left(\imath^!_{\check\lambda} \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})\right)$ does not have sub-objects supported on $\Delta(X) \times \mathrm{Bun}^{\check\mu+\check\lambda}_B$.*

The rest of this section is essentially devoted to the proof of this proposition. Let us assume first that $\check\lambda$ is not a root of $\check{\mathfrak{g}}$. Then by Corollary 4.6 and Sect. 3.3, the Jordan-Hölder series of the perverse sheaf $h^0\left(\imath^!_{\check\lambda} \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})\right)$ does not contain terms that are supported on the closed substack $\Delta(X) \times \mathrm{Bun}^{\check\mu+\check\lambda}_B$, implying, in particular, that it does not have such sub-objects.

5.4.   Hence, it remains to analyze the case when $\check\lambda$ is a root $\check\alpha$, but not a simple root. We will argue by contradiction, assuming that $h^0\left(\imath^!_{\check\alpha} \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})\right)$ admits $\Delta_!(\mathbb{C}_X)[1] \boxtimes \mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\alpha}_B}$ as a sub-object.

Consider the quotient

$$(5.2) \qquad\qquad h^0\left(\imath^!_{\check\alpha} \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu}_B})\right) \Big/ \left(\Delta_!(\mathbb{C}_X)[1] \boxtimes \mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\alpha}_B}\right).$$

By Corollary 4.6, it is isomorphic to the intermediate extension of its own restriction to the open substack $(X^{\check\lambda} - \Delta(X)) \times \mathrm{Bun}^{\check\mu+\check\lambda}_B$. By the induction hypothesis and the above factorization argument, we can assume that Theorem 4.2 holds over $(X^{\check\lambda} - \Delta(X)) \times \mathrm{Bun}^{\check\mu+\check\lambda}_B$. Hence, the above quotient perverse sheaf is isomorphic to

$$\left(\ker\left(\Omega(\check{\mathfrak{n}}_X)^{-\check\alpha} \to \Delta_!((\check{\mathfrak{n}}^*)^{-\check\alpha}_X[1])\right)\right) \boxtimes \mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\alpha}_B},$$

since the first multiple in the above formula equals the intermediate extension of the restriction of $\Omega(\check{\mathfrak{n}}_X)^{-\check\alpha}$ to $X^{\check\lambda} - \Delta(X)$.

Let $\check{\beta}$ and $\check{\gamma}$ be two roots such that $\check{\alpha} = [\check{\beta}, \check{\gamma}]$. By Sect. 3.3, the perverse sheaf of (5.2) admits a further quotient, isomorphic to

$$(5.3) \qquad \mathcal{F}'_1 := \left(\left((\check{\mathfrak{n}}^*)_X^{-\check{\beta}}[1] \right) \star \left((\check{\mathfrak{n}}^*)_X^{-\check{\gamma}}[1] \right) \right) \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}} .$$

Let $\mathcal{F}'$ denote the corresponding quotient of $\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}$. This perverse sheaf has a 3-step filtration with $\mathcal{F}'_1$ being the above perverse sheaf on $X^{\check{\alpha}} \times \mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}$, thought of as a closed substack of $\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}$, and

$$\mathcal{F}'_2 := \ker(\mathcal{F}' \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}),$$

so that $\mathcal{F}'_3/\mathcal{F}'_2 \simeq \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}$ and

$$\mathcal{F}'_2/\mathcal{F}'_1 \simeq \left(\ker(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}} \right) \Big/ \left((\iota_{\check{\alpha}})_! h^0 \left(i_{\check{\alpha}}^! \circ \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \right) \right).$$

By Corollary 4.5 and Proposition 4.4, the only terms in the Jordan-Hölder series of $\jmath_!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})|_{\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}}$ that can have a non-trivial Ext^1 to the perverse sheaf (5.3) are

$$(5.4) \qquad \left((\check{\mathfrak{n}}^*)_X^{-\check{\beta}}[1] \right) \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\gamma}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}} \text{ and } \left((\check{\mathfrak{n}}^*)_X^{-\check{\gamma}}[1] \right) \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\beta}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}} .$$

Moreover, no other terms in the Jordan-Hölder series of $\jmath_!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})|_{\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}}$ apart from $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}$ admit a non-trivial Ext^1 to either of the perverse sheaves appearing in (5.4).

Thus, we obtain that the perverse sheaf $\mathcal{F}'$ on $\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\beta}}$ admits a quotient that we shall denote by $\mathcal{F}$, endowed with a 3-step filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = \mathcal{F},$$

such that

$$\mathcal{F}_1 \simeq \mathcal{F}'_1 \simeq \left(\mathbb{C}_X^{\check{\beta}}[1] \star \mathbb{C}_X^{\check{\gamma}}[1] \right) \star \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}},$$

$$\mathcal{F}_2/\mathcal{F}_1 \simeq \mathbb{C}_X^{\check{\beta}}[1] \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\gamma}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}} \bigoplus \mathbb{C}_X^{\check{\gamma}}[1] \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\beta}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}$$

and $\mathcal{F}_3/\mathcal{F}_2 \simeq \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}$.

The corresponding elements in

$$\mathrm{Ext}^1_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}} \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}, \mathbb{C}_X^{\check{\beta}}[1] \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\gamma}}} \right), \text{ and } \mathrm{Ext}^1_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}} \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}, \left(\mathbb{C}_X^{\check{\gamma}}[1] \right) \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\beta}}} \right)$$

are non-zero, and correspond, therefore, to nonzero multiples of the maps

$$\mathbb{C}_X \to \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\beta}} \text{ and } \mathbb{C}_X \to \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\gamma}},$$

given by $\check{\mathfrak{n}}^{\check{\beta}} \to U(\check{\mathfrak{n}})^{\check{\beta}}$ and $\check{\mathfrak{n}}^{\check{\gamma}} \to U(\check{\mathfrak{n}})^{\check{\gamma}}$, respectively.

The extension

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_2/\mathcal{F}_1 \to 0$$

is non-trivial, since otherwise we would obtain that there exist a non-trivial element in

$$\mathrm{Ext}^1_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\alpha}}} \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}, \left(\mathbb{C}_X^{\check{\beta}}[1] \star \mathbb{C}_X^{\check{\gamma}}[1] \right) \star \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}} \right),$$

which is impossible by Proposition 4.4.

We are going to show now, using results of [FFKM] that a perverse sheaf $\mathcal{F}$ with a filtration having such properties does not exist.

5.5. Recall the *sheaf* $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}}$ on $X^{\check{\lambda}}$. The multiplication operation on $U(\check{\mathfrak{n}})$ defines a map

$$(5.5) \qquad \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}_1} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}_2} \to \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}_1 + \check{\lambda}_2}.$$

The result of [FFKM], Sect. 2 that we will use can be summarized as follows:

Theorem 5.6.

(1) *For every $\check{\lambda} \in \check{\Lambda}^{pos}$ there exists an isomorphism $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \simeq \imath_{\check{\lambda}}^{!}(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})$.*

(2) *The above isomorphism extends to a unique morphism $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}} \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}} \to \bar{\imath}_{\check{\lambda}}^{!}(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})$, or, equivalently (by adjunction), to a morphism*

$$\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}} \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}.$$

(3) *For $\check{\lambda} = \check{\lambda}_1 + \check{\lambda}_2$ the diagram*

$$
\begin{array}{ccc}
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}_1} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}_2} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}} & \longrightarrow & \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}} \\
\downarrow & & \downarrow \\
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}_1} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}_1}} & \longrightarrow & \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}},
\end{array}
$$

commutes.

Let us add several remarks. First, comparing point (1) of Theorem 5.6 and that of Proposition 4.4, we obtain that there are *a priori* two isomorphisms

$$\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}} \rightrightarrows \imath_{\check{\lambda}}^{!}(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}).$$

At this stage it is not clear why these two maps coincide.

The existence of the map stated in point (2) of the theorem is proved in [FFKM] by purity considerations. The statement about uniqueness of this extension (which is omitted in *loc. cit.*) follows by analyzing cohomological degrees of various subquotients.

5.7. We shall analyze the commutative diagram of Theorem 5.6 in the following particular case.

Let us first take $\check{\lambda}_1 = \check{\beta}$ and $\check{\lambda}_2 = \check{\gamma}$, and identify $\check{\mathfrak{n}}_X^{\check{\beta}}$ and $\check{\mathfrak{n}}_X^{\check{\gamma}}$ with $\mathbb{C}_X$ (up to a scalar). We obtain extensions

$$0 \to \mathbb{C}_X[1]^{\check{\gamma}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\gamma}}} |_{\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\alpha}}} \to \widetilde{\mathcal{F}}_{2,\beta} \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}} |_{\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\alpha}}} \to 0$$

and

$$(5.6) \qquad 0 \to \mathbb{C}_X[1]^{\check{\beta}} \star \mathbb{C}_X[1]^{\check{\gamma}} \star \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}} \to \widetilde{\mathcal{F}}_{1,\beta} \to \mathbb{C}_X[1]^{\check{\gamma}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\gamma}}} |_{\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\alpha}}} \to 0,$$

whose cup product is an element in

$$(5.7) \qquad \mathrm{Ext}^2_{\overline{\mathrm{Bun}}_B^{\check{\mu}, \leq \check{\alpha}}} \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}, \mathbb{C}_X[1]^{\check{\beta}} \star \mathbb{C}_X[1]^{\check{\gamma}} \star \mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\alpha}}} \right),$$

corresponding to the map

$$\mathbb{C}_{X \times X} \to \check{\mathfrak{n}}_X^{\check{\beta}} \star \check{\mathfrak{n}}_X^{\check{\gamma}} \to \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\alpha}}.$$

Interchanging the roles of $\check{\beta}$ and $\check{\gamma}$ we obtain another element in the above Ext^2 group. The difference of these two elements is non-zero for any choice of non-zero scalars as above, since the roots $\check{\beta}$ and $\check{\gamma}$ do not commute.

Going back to the perverse sheaf $\mathcal{F}$ of Sect. 5.4 we obtain that the corresponding element in

$$\operatorname{Ext}^1_{\overline{\operatorname{Bun}}_B^{\check{\mu},\leq\check{\alpha}}}\left(\left((\check{\mathfrak{n}}^*)_X^{-\check{\beta}}[1]\right)\star\operatorname{IC}_{\overline{\operatorname{Bun}}_B^{\check{\mu}+\check{\gamma}}},\mathcal{F}_1\right)$$

is non-zero, and hence equals, up to a scalar to the one of (5.6). A similar assertion holds for $\check{\beta}$ replaced by $\check{\gamma}$.

Let us now interpret what the existence of a sheaf $\mathcal{F}$ with a 3-step extension having the properties specified above would mean:

It implies that that the difference of the two resulting elements in (5.7) is zero, contradicting the assertion made earlier.

6. The Koszul complex

6.1. In the previous sections we have endowed $\operatorname{Eis}_!(E_{\check{T}})$ with an action of $\mathcal{O}_{\operatorname{Def}_{\check{B}}(E_{\check{T}})}$. The goal of this section is to show that

$$\mathbb{C}\overset{L}{\underset{\mathcal{O}_{\operatorname{Def}_{\check{B}}(E_{\check{T}})}}{\otimes}}\operatorname{Eis}_!(E_{\check{T}})\simeq\overline{\operatorname{Eis}}(E_{\check{T}}),$$

thereby proving point (2) of Theorem 1.8.

In order to do this we shall first construct a certain Koszul complex, by means of which one can compute $\mathbb{C}\overset{L}{\underset{\mathcal{O}_{\operatorname{Def}_{\check{B}}(E_{\check{T}})}}{\otimes}}\mathcal{M}$ for any $\mathcal{O}_{\operatorname{Def}_{\check{B}}(E_{\check{T}})}$-module $\mathcal{M}$.

6.2. Recall again the *sheaf* $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}\in D^b(X^{\check{\lambda}})$, introduced in Sect. 4.3. We are now going to represent it by an explicit complex of perverse sheaves.

For a positive integer m consider the direct sum

$$\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m,\check{\lambda}}:=\bigoplus_{\substack{\check{\lambda}_1,\ldots,\check{\lambda}_m\in\check{\Lambda}^{pos}\\\check{\lambda}_j\neq0,\Sigma\check{\lambda}_j=\check{\lambda}}}\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_1}\star\ldots\star\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_m},$$

where $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\mu}}$'s are as in Sect. 3.1. Note that we have natural maps

(6.1) $$\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}'}\star\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m,\check{\lambda}''}\to\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m+1,\check{\lambda}'+\check{\lambda}''}.$$

We define a differential $\mathfrak{d}_{\mathfrak{U}}^{m,\check{\lambda}}:\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m,\check{\lambda}}\to\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m+1,\check{\lambda}}$ as follows. By induction, assume that

$$\mathfrak{d}_{\mathfrak{U}}^{m-1,\check{\lambda}-\check{\lambda}_1}:\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_2}\star\ldots\star\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_m}\to\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m,\check{\lambda}-\check{\lambda}_1}$$

has been defined. We let $\mathfrak{d}_{\mathfrak{U}}^{m,\check{\lambda}}$ be the sum of

$$\operatorname{id}_{\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_1}}\star\mathfrak{d}_{\mathfrak{U}}^{m-1,\check{\lambda}-\check{\lambda}_1}$$

and

$$\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_1}\star\left(\underset{j=2,\ldots,m}{\star}\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_j}\right)\to$$

$$\to\bigoplus_{\substack{\check{\lambda}_1'+\check{\lambda}_1''=\check{\lambda}_1\\\check{\lambda}_1',\check{\lambda}_1''\neq0}}\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_1'}\star\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_1''}\star\left(\underset{j=2,\ldots,m}{\star}\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_j}\right),$$

coming from (3.1). It is straightforward to check that $\mathfrak{d}_{\mathfrak{U}}^{m,\check{\lambda}}\circ\mathfrak{d}_{\mathfrak{U}}^{m-1,\check{\lambda}}=0$, so we can form a complex of perverse sheaves that we will denote by $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$.

Lemma 6.3. *We have an isomorphism in $D^b(X^{\check{\lambda}})$:*

$$\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}} \simeq \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}.$$

Proof. Let us compute the fiber of $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$ at a point $\Sigma\,\check{\lambda}_k \cdot x_k$ of $X^{\check{\lambda}}$. It is easy to see that this fiber will be isomorphic, as a complex, to the product of the corresponding complexes for the x_k's; so we can assume that our point is $\check{\lambda} \cdot x$.

In this case the resulting complex is quasi-isomorphic to the complex associated to the bi-complex, whose m-th column is

$$\bigoplus_{\substack{\check{\lambda}_1,\ldots,\check{\lambda}_m\in\check{\Lambda}^{pos}\\ \check{\lambda}_j\neq 0,\Sigma\check{\lambda}_j=\check{\lambda}}} \left(\Lambda^{\bullet}(\check{\mathfrak{n}}_{E_{\check{T},x}})\right)^{\check{\lambda}_1} \otimes \ldots \otimes \left(\Lambda^{\bullet}(\check{\mathfrak{n}}_{E_{\check{T},x}})\right)^{\check{\lambda}_m}.$$

The vertical differential in this bi-complex comes from the Lie algebra structure on $\check{\mathfrak{n}}$, and the horizontal one is defined as in the case of $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m,\check{\lambda}}$ via the co-multiplication on $\Lambda^{\bullet}(\check{\mathfrak{n}}_{E_{\check{T},x}})$.

The 0-th term of this complex is isomorphic to

$$\bigoplus_{\substack{m,\check{\lambda}_1,\ldots,\check{\lambda}_m\in\check{\Lambda}^{pos}\\ \check{\lambda}_j\neq 0,\Sigma\check{\lambda}_j=\check{\lambda}}} \check{\mathfrak{n}}_{E_{\check{T},x}}^{\check{\lambda}_1} \otimes \ldots \otimes \check{\mathfrak{n}}_{E_{\check{T},x}}^{\check{\lambda}_m},$$

and it maps to $U(\check{\mathfrak{n}}_{E_{\check{T},x}})^{\check{\lambda}}$ via the product operation in this algebra.

This map is easily seen to induce a quasi-isomorphism from the above complex to $U(\check{\mathfrak{n}}_{E_{\check{T},x}})^{\check{\lambda}}$. In addition, it is straightforward to check that the above fiber-wise calculation identifies $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$ with the cohomology of $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$.

$\qquad\square$

The complexes $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$ possess the following structures. For $\check{\lambda} = \check{\lambda}_1 + \check{\lambda}_2$ there is a multiplication map

(6.2) $$\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}_1} \star \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}_2} \to \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}},$$

inducing the map (5.5). In addition, they have a factorization property similar to that of (3.2):

(6.3) $$\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}\Big|_{\left(X^{\check{\lambda}_1}\times X^{\check{\lambda}_2}\right)_{disj}} \simeq \left(\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}_1} \boxtimes \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}_2}\right)\Big|_{\left(X^{\check{\lambda}_1}\times X^{\check{\lambda}_2}\right)_{disj}},$$

also compatible with the corresponding isomorphism for $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$.

6.4.   Consider now the direct sum

$$\mathrm{Kosz}(E_{\check{T}})^{\bullet,\check{\lambda},*} := \bigoplus_{\check{\lambda}_1+\check{\lambda}_2=\check{\lambda}} \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}_1} \star \Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_2}.$$

We can view it as a "module" over the $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$'s via (6.2):

(6.4) $$\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\mu}} \star \mathrm{Kosz}(E_{\check{T}})^{\bullet,\check{\lambda},*} \to \mathrm{Kosz}(E_{\check{T}})^{\bullet,\check{\lambda}+\check{\mu},*}.$$

It is also a "co-module" over the $\Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}}$'s via the maps (3.1):

(6.5) $$\mathrm{Kosz}(E_{\check{T}})^{\bullet,\check{\lambda},*} \to \bigoplus_{\check{\lambda}=\check{\lambda}_1+\check{\lambda}_2} \mathrm{Kosz}(E_{\check{T}})^{\bullet,\check{\lambda}_1,*} \star \Upsilon(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\lambda}_2}.$$

Now, by the very construction of the complexes $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$, we can endow $\mathrm{Kosz}(E_{\check{T}})^{\bullet,\check{\lambda},*}$ with a differential, which makes it into an acyclic complex for all $\check{\lambda} \neq 0$. This differential is compatible with the maps (6.4) and (6.5).

Let $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,-\check{\lambda},*}$ be the complex obtained from $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$ by applying Verdier duality term-wise. Similarly, set

$$\mathrm{Kosz}(E_{\check{T}})^{\bullet,-\check{\lambda}} := \bigoplus_{\check{\lambda}_1+\check{\lambda}_2=\check{\lambda}} \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,-\check{\lambda}_1,*} \star \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}_2}.$$

This is a complex possessing the structures dual to those of (6.4) and (6.5).

Let us assume now that $E_{\check{T}}$ is regular. Note that by Corollary 3.4, the terms of $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m,\check{\lambda}}$ are such that $H^j(X^{\check{\lambda}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{m,\check{\lambda}}) = 0$ unless $j = 0$. Applying the functor $H^0(X^{\check{\lambda}}, ?)$ term-wise to $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,\check{\lambda}}$, we obtain a $\mathbb{Z}^{\geq 0}$-graded vector space that we shall denote by $\mathfrak{u}_{E_{\check{T}}}^{\bullet,\check{\lambda}}$. The direct sum

$$\mathfrak{u}_{E_{\check{T}}}^{\bullet} := \bigoplus_{\check{\lambda}\in\check{\Lambda}^{pos}} \mathfrak{u}_{E_{\check{T}}}^{\bullet,\check{\lambda}}$$

is a $\mathbb{Z}$-graded associative algebra. Dually, we set

$$\mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda},*} = H(X^{\check{\lambda}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet,-\check{\lambda},*}) \text{ and } \mathfrak{u}_{E_{\check{T}}}^{\bullet,*} := \bigoplus_{\check{\lambda}\in\check{\Lambda}^{pos}} \mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda},*},$$

the latter being a $\mathbb{Z}^{\leq 0}$-graded co-associative co-algebra.

In addition, for $\check{\lambda} \in \check{\Lambda}^{pos}$, we define the complex $\mathsf{K}(E_{\check{T}})^{\bullet,-\check{\lambda}}$ by applying $H(X^{\check{\lambda}}, ?)$ term-wise to $\mathrm{Kosz}(E_{\check{T}})^{\bullet,-\check{\lambda}}$. In other words,

$$\mathsf{K}(E_{\check{T}})^{\bullet,-\check{\lambda}} = \bigoplus_{\check{\lambda}_1+\check{\lambda}_2=\check{\lambda}} \mathfrak{u}_{E_{\check{T}}}^{\bullet,-\check{\lambda}_1,*} \otimes R_{E_{\check{T}}}^{-\check{\lambda}_2}.$$

Set also $\mathsf{K}(E_{\check{T}})^{\bullet} = \bigoplus_{\check{\lambda}\in\check{\Lambda}} \mathsf{K}(E_{\check{T}})^{\bullet,-\check{\lambda}}$. This is a DG module over $R_{E_{\check{T}}}$. By the acyclicity of $\mathrm{Kosz}(E_{\check{T}})^{\bullet,-\check{\lambda}}$, this DG module is quasi-isomorphic to $\mathbb{C}$. Moreover, by construction, when we disregard the differential, it is free and isomorphic to $\mathfrak{u}_{E_{\check{T}}}^{\bullet,*} \otimes R_{E_{\check{T}}}$.

Let $\mathcal{M}$ is a module over $R_{E_{\check{T}}} \simeq \mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$. We obtain that $\mathbb{C} \overset{L}{\underset{\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}}{\otimes}} \mathcal{M}$ can be computed by means of the complex

$$(6.6) \qquad\qquad \mathfrak{u}_{E_{\check{T}}}^{\bullet,*} \otimes \mathcal{M} \simeq \mathsf{K}(E_{\check{T}})^{\bullet} \underset{R_{E_{\check{T}}}}{\otimes} \mathcal{M},$$

where the differential is obtained as a composition

$$\mathfrak{u}_{E_{\check{T}}}^{\bullet,*} \otimes \mathcal{M} \to \mathsf{K}(E_{\check{T}})^{\bullet} \otimes \mathcal{M} \simeq \mathfrak{u}_{E_{\check{T}}}^{\bullet,*} \otimes R_{E_{\check{T}}} \otimes \mathcal{M} \to \mathfrak{u}_{E_{\check{T}}}^{\bullet,*} \otimes \mathcal{M},$$

where the first arrow is given by the differential on $\mathsf{K}(E_{\check{T}})^{\bullet}$, and the last arrow is given by the action of $R_{E_{\check{T}}}$ on $\mathcal{M}$.

6.5. Consider now the direct sum

$$\mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}^{\bullet} := \bigoplus_{\check{\lambda}'\in\check{\Lambda}^{pos}} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda}',*} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check{\mu}+\check{\lambda}'}})$$

as a graded perverse sheaf on $\overline{\mathrm{Bun}}_B^{\check{\mu}}$.

It acquires a differential by means of

$$\bigoplus_{\check\lambda'\in\check\Lambda^{pos}} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda',*} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\lambda'}}) \to \bigoplus_{\check\lambda'=\check\lambda_1'+\check\lambda_2'} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda_1',*} \star \Omega(\check{\mathfrak{n}}_X)^{-\check\lambda_2'} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\lambda'}}) \to$$

$$\to \bigoplus_{\check\lambda_1'\in\check\Lambda^{pos}} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda_1',*} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\lambda_1'}}),$$

where the first arrow is given by the differential on $\mathrm{Kosz}(E_{\check T})^{\bullet,-\check\lambda'}$ (for $E_{\check T}$ being trivial) and the second arrow is given by (4.1).

Theorem 6.6. *The map* $\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}}) \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}$ *defines a quasi-isomorphism*

$$\mathrm{Kosz}^{\bullet}_{\overline{\mathrm{Bun}}_B^{\check\mu}} \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}.$$

Before proving this theorem, let us show how it implies the assertion of point (2) of Theorem 1.8.

We have to show that

$$(6.7) \qquad\qquad \mathbb{C} \overset{L}{\underset{\mathcal{O}_{\mathrm{Def}_{\check B}(E_{\check T})}}{\otimes}} \mathrm{Eis}_!(E_{\check T}) \simeq \overline{\mathrm{Eis}}(E_{\check T}).$$

Let us tensor the complex $\mathrm{Kosz}^{\bullet}_{\overline{\mathrm{Bun}}_B^{\check\mu}}$ by $(\overline{\mathsf{q}}^{\check\mu})^*(\mathcal{S}(E_{\check T}))$. We obtain a complex

$$\mathrm{Kosz}_{\check\mu}(\mathrm{Eis}_!(E_{\check T}))^{\bullet} := \bigoplus_{\check\lambda'\in\check\Lambda^{pos}} \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check T}})^{\bullet,-\check\lambda',*} \star \jmath_! \left(\mathrm{IC}_{\mathrm{Bun}^{\check\mu+\check\lambda'}} \otimes (\mathsf{q}^{\check\mu+\check\lambda'})^*(\mathcal{S}(E_{\check T}))\right).$$

Applying the functor $\overline{\mathsf{p}}_!^{\check\mu}$ to it term-wise, we obtain a complex

$$\mathsf{K}_{\check\mu}(\mathrm{Eis}_!(E_{\check T}))^{\bullet} := \bigoplus_{\check\lambda'\in\check\Lambda^{pos}} \mathfrak{u}_{E_{\check T}}^{\bullet,-\check\lambda',*} \otimes \mathrm{Eis}_!^{\check\mu+\check\lambda'}(E_{\check T}).$$

The differential on $\mathsf{K}_{\check\mu}(\mathrm{Eis}_!(E_{\check T}))^{\bullet}$ is given by the formula for the differential on (6.6), using the action maps (4.2). Hence, by Sect. 6.4, the direct sum

$$(6.8) \qquad\qquad \mathsf{K}(\mathrm{Eis}_!(E_{\check T}))^{\bullet} := \bigoplus_{\check\mu} \mathsf{K}_{\check\mu}(\mathrm{Eis}_!(E_{\check T}))^{\bullet} \simeq \mathfrak{u}_{E_{\check T}}^{\bullet,*} \otimes \mathrm{Eis}_!(E_{\check T})$$

is quasi-isomorphic to the LHS of (6.7).

Now, by Theorem 6.6, the complex $\mathrm{Kosz}_{\check\mu}(\mathrm{Eis}_!(E_{\check T}))^{\bullet}$ is quasi-isomorphic to the perverse sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}} \otimes (\overline{\mathsf{q}}^{\check\mu})^*(\mathcal{S}(E_{\check T}))$. We obtain that $\mathsf{K}_{\check\mu}(\mathrm{Eis}_!(E_{\check T}))^{\bullet}$ is quasi-isomorphic to $\overline{\mathrm{Eis}}_{\check\mu}(E_{\check T})$.

Therefore, $\mathsf{K}(\mathrm{Eis}_!(E_{\check T}))^{\bullet}$ is quasi-isomorphic to the RHS of (6.7), as required.

6.7. **Proof of Theorem 6.6.** We proceed by induction on $|\check\lambda|$ by showing that the map in question is a quasi-isomorphism over the open substack $\overline{\mathrm{Bun}}_B^{\check\mu,\leq\check\lambda}$. The base of the induction, i.e., $\check\lambda = 0$ evidently holds. Thus, we assume that the quasi-isomorphism in question is valid for all $\check\lambda'$ with $|\check\lambda'| < |\check\lambda|$.

Thus, it is sufficient to show that the map

$$(6.9) \qquad\qquad \imath_{\check\lambda}^* \left(\mathrm{Kosz}^{\bullet}_{\overline{\mathrm{Bun}}_B^{\check\mu}}\right) \to \imath_{\check\lambda}^* \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}\right)$$

is a quasi-isomorphism.

Note now that by Lemma 6.3, the LHS of (6.9) is quasi-isomorphic to

$$(6.10) \qquad\qquad \mathfrak{U}(\check{\mathfrak{n}}_X)^{-\check\lambda,*} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\lambda}}.$$

By Proposition 4.4, the RHS of (6.9) is also isomorphic to the expression in (6.10). Thus, to prove the theorem, we need to show that the resulting endomorphism of (6.10) is an isomorphism. [3]

Consider the canonical filtration on $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda},*}$, corresponding to the perverse t-structure. The associated graded is isomorphic to

$$\bigoplus_{\check{\lambda}=m_1\cdot\check{\beta}_1+...+m_k\cdot\check{\beta}_k} ((\check{\mathfrak{n}}^*)_X^{-\check{\beta}_1})^{(n_1)} \star ... \star ((\check{\mathfrak{n}}^*)_X^{-\check{\beta}_k})^{(n_k)}[2(n_1+...+n_k)],$$

where $\check{\beta}_1,...,\check{\beta}_k$ are *not necessarily simple* roots of $\check{\mathfrak{g}}$. Each of the summands, except the one corresponding to $k=1$ (in which case $\check{\lambda}$ is itself a root), is a cohomologically shifted perverse sheaf, which is the intermediate restriction of its extension to $X^{\check{\lambda}}-\Delta(X)$.

However, we claim that by the induction hypothesis we can assume that (6.9) is an isomorphism over $(X^{\check{\lambda}}-\Delta(X))\times\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}$. Indeed, since the assertion is local, we can replace $\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\lambda}}$ by the Zastava space, and then apply the factorization principle, (4.3).

Hence, the map (6.9) induces an isomorphism on the associated graded pieces of (6.10), except, possibly, on $(\check{\mathfrak{n}}^*)_X^{-\check{\lambda}}[2]\boxtimes\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}+\check{\lambda}}}$, where $(\check{\mathfrak{n}}^*)_X^{-\check{\lambda}}[2]$ is the last quotient of $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda},*}$ (and which can only occur if $\check{\lambda}$ is a root).

Suppose, by contradiction, that this map was not an isomorphism, i.e., equal to zero. We would obtain that

$$\mathrm{Cone}\left(\mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check{\mu}}} \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}\right) \simeq \mathrm{Cone}\left((\imath_{\check{\lambda}})_!((\check{\mathfrak{n}}^*)_X^{-\check{\lambda}}[2]) \to (\imath_{\check{\lambda}})_!((\check{\mathfrak{n}}^*)_X^{-\check{\lambda}}[2])\right),$$

and therefore

$$h^0\left(\mathrm{Cone}\left(\mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check{\mu}}} \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}\right)\right)|_{\overline{\mathrm{Bun}}_B^{\check{\mu},\leq\check{\lambda}}} \neq 0.$$

But this is a contradiction, since $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}$ is an irreducible perverse sheaf, and the complex $\mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}$ is concentrated in non-positive perverse cohomolgical degrees.

6.8. A comparison of two isomorphisms.

Let us denote by $\mathrm{Kosz}^{\bullet,*}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}$ the complex obtained from $\mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}$ by applying term-wise Verdier duality. Its terms are given by

$$\bigoplus_{\check{\lambda}\in\check{\Lambda}^{pos}} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check{\lambda}} \star \jmath_*(\mathrm{IC}_{\mathrm{Bun}^{\check{\mu}+\check{\lambda}}}),$$

and the differential by that on $\mathrm{Kosz}(E_{\check{T}})^{\bullet,\check{\lambda},*}$ and maps dual to those of (4.1).

As this complex is quasi-isomorphic to $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}$, it can be used to calculate $\imath_{\check{\lambda}}^!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})$. From Lemma 6.3, we obtain a quasi-isomorphism

$$(6.11) \qquad \imath_{\check{\lambda}}^!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}) \simeq \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check{\lambda}} \boxtimes \mathrm{IC}_{\mathrm{Bun}^{\check{\mu}+\check{\lambda}}}.$$

Note that Proposition 4.4 and Theorem 5.6(1) give two more isomorphisms between the same objects.

Conjecture 6.9. *The isomorphisms of Proposition 4.4 and Theorem 5.6(1) coincide.*

In the remainder of this section we will prove that the isomorphisms of (6.11) and Theorem 5.6(1) coincide.

---

[3] We do not claim at this stage that this is the identity automorphism.

6.10.   The starting point is the following observation:

Lemma 6.11. *Suppose that we have a system of automorphisms $\phi^{\check\lambda}$ of the sheaves $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda}$ with the following two properties:*

- $\phi^{\check\lambda} = \mathrm{id}$ *for $\check\lambda$ being a simple root $\check\alpha_i$.*
- *The system $\{\phi^{\check\lambda}\}$ is compatible with the maps (5.5).*

Then $\phi^{\check\lambda} = \mathrm{id}$ for all $\check\lambda$.

The lemma follows from the fact that the simple root spaces generate $U(\check{\mathfrak{n}})$. We apply it in our situation for $\phi^{\check\lambda}$ being the discrepancy of the maps (6.11) and Theorem 5.6(1). The fact that $\phi^{\check\alpha_i} = \mathrm{id}$ follows from the construction of both maps. Thus, it remains to check the compatibility with the product operation (5.5).

First, let us observe that the map

$$(6.12) \qquad \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\lambda}}) \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}},$$

corresponding by adjunction to (6.11), is represented by

$$\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check\lambda} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu+\check\lambda}}) \to \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check\lambda} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}} \hookrightarrow \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check\mu}}^{\bullet,*},$$

where the image of the last arrow belongs to the kernel of the differential, because $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu'}} \subset \jmath_*(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu'}})$ lies in the kernel of the maps dual to those of (4.1). By construction, the map (6.12) extends to a map

$$(6.13) \qquad \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}} \simeq \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check\lambda} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}} \to \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check\mu}}^{\bullet,*} \simeq \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}},$$

whose existence is asserted in Theorem 5.6(2).

Using Theorem 5.6(3), it remains to establish the commutativity of the following diagram in $D^b(\overline{\mathrm{Bun}}_B^{\check\mu})$, whose arrows are induced by (6.13):

$$
\begin{array}{ccc}
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda_1} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda_2} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda_1+\check\lambda_2}} & \longrightarrow & \mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}} \\
\downarrow & & \downarrow \\
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda_1} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda_1}} & \longrightarrow & \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}} .
\end{array}
$$

Consider the map

$$(6.14) \qquad \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check\lambda} \star \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}}^{\bullet,*} \to \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check\mu}}^{\bullet,*},$$

induced by (6.2). The compatibility of the latter with the differential on the $\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check\lambda}$'s implies that (6.14) is a map of complexes.

The commutativity of the above diagram follows now from the next statement:

Lemma 6.12. *The diagram*

$$
\begin{array}{ccc}
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\check\lambda} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}} & \longrightarrow & \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}} \\
\sim \downarrow & & \sim \downarrow \\
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check\lambda} \star \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}}^{\bullet,*} & \longrightarrow & \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check\mu}}^{\bullet,*}
\end{array}
$$

is commutative in the derived category.

Proof. This follows from the fact that the composition

$$\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}} \to \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,\check{\lambda}} \star \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}}^{\bullet,*} \xrightarrow{(6.14)} \mathrm{Kosz}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}}^{\bullet,*}$$

equals the map of (6.13). □

7. The Hecke property in the case of GL_2

In this section we will prove Theorem 1.11 for GL_2 by a direct calculation for $V \in \mathrm{Rep}(\check{G})$ being the standard 2-dimensional representation of $\check{G} = GL_2$. We retain the notation of Sect. 2.

7.1. Let $E_{\check{B},\widehat{univ}}$ denote the tautological $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$-family of 2-dimensional local systems on X. Let $E_{\check{B},\widehat{univ},x}$ be its fiber at x; this is a locally free $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$-module of rank 2. Let $E_{\check{B},univ,x}$ be the corresponding $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})} \simeq \mathrm{Sym}(\mathsf{W})$-module, i.e., the direct sum of homogeneous components of $E_{\check{B},\widehat{univ},x}$ with respect to the natural $\check{T}$-action. We have a short exact sequence

$$0 \to \mathrm{Sym}(\mathsf{W}) \otimes E_{1,x} \to E_{\check{B},univ,x} \to \mathrm{Sym}(\mathsf{W}) \otimes E_{2,x} \to 0.$$

We need to prove that for any $V \in \mathrm{Rep}(\check{G})$ being the standard 2-dimensional representation of GL_2 there exists a canonical isomorphism of $\check{\Lambda}$-graded perverse sheaves on Bun_G:

$$(7.1) \qquad \mathrm{H}_x^{\mathrm{basic}}(\mathrm{Eis}_!(E_{\check{T}})) \simeq E_{\check{B},univ,x} \underset{\mathrm{Sym}(\mathsf{W})}{\otimes} \mathrm{Eis}_!(E_{\check{T}}),$$

where $\mathrm{H}_x^{\mathrm{basic}}$ is the Hecke functor corresponding to the standard 2-dimensional representation of GL_2.

7.2. Let us first describe the fiber $E_{\check{B},univ,x}$ explicitly.

Set $\mathsf{W}' := H^1(X-x, E_2 \otimes E_1^{-1})$. The cokernel W'/W is canonically isomorphic to $E_{2,x} \otimes E_{1,x}^{-1}$. Consider the map

$$(7.2) \qquad \mathrm{Sym}(\mathsf{W}) \otimes \mathsf{W} \to (\mathrm{Sym}(\mathsf{W}) \otimes \mathsf{W}') \bigoplus \mathrm{Sym}(\mathsf{W}),$$

where the first component corresponds to the embedding of W into W', and the second component is given by the multiplication map.

Lemma 7.3. *The* $\mathrm{Sym}(\mathsf{W})$*-module* $E_{\check{B},univ,x}$ *is canonically isomorphic cokernel of the map of* (7.2), *tensored by* $E_{1,x}$.

Corollary 7.4. *Let* $\mathcal{F}$ *be an object of some abelian category endowed with an action of* $\mathrm{Sym}(\mathsf{W})$. *Then*

$$\mathcal{F} \underset{\mathrm{Sym}(\mathsf{W})}{\overset{L}{\otimes}} E_{\check{B},univ,x}$$

is canonically quasi-isomorphic to the complex

$$\mathsf{W} \otimes \mathcal{F} \to (\mathsf{W}' \otimes \mathcal{F}) \bigoplus \mathcal{F},$$

tensored by $E_{1,x}$.

Thus, we obtain that the existence of the isomorphism (7.1) is equivalent to the following assertion:

Proposition 7.5. *The object* $\mathrm{H}_x^{\mathrm{basic}}(\mathrm{Eis}_!^{d_1,d_2}(E_{\check{T}})) \otimes E_{1,x}^{-1}$ *is canonically quasi-isomorphic to the complex*

$$H^1(X, E_2 \otimes E_1^{-1}) \otimes \mathrm{Eis}_!^{d_1,d_2-1}(E_{\check{T}}) \to H^1(X-x, E_2 \otimes E_1^{-1}) \otimes \mathrm{Eis}_!^{d_1,d_2-1}(E_{\check{T}}) \bigoplus \mathrm{Eis}_!^{d_1-1,d_2}(E_{\check{T}}).$$

7.6. **Proof of Proposition 7.5.** Let $\mathcal{H}_x^{\mathrm{basic}}$ denote the stack classifying triples

$$(\mathcal{M}, \mathcal{M}', \beta : \mathcal{M} \hookrightarrow \mathcal{M}'),$$

where $(\mathcal{M}, \mathcal{M}') \in \mathrm{Bun}_G = \mathrm{Bun}_2$, and β is an embedding such that the quotient $\mathcal{M}'/\mathcal{M}$ has length 1 and is supported at x. Let $\overleftarrow{h}, \overrightarrow{h}$ denote the two projections from $\mathcal{H}_x^{\mathrm{basic}}$ to Bun_G that remember $\mathcal{M}$ and $\mathcal{M}'$, respectively.

Consider the Cartesian product

$$\mathcal{H}_x'{}^{\,\mathrm{basic}} := \mathcal{H}_x^{\mathrm{basic}} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B^{d_1, d_2},$$

where $\mathcal{H}_x^{\mathrm{basic}}$ maps to Bun_G via $\overrightarrow{h}$. This stack classifies quintuples

$$(\mathcal{L}' \overset{\kappa'}{\hookrightarrow} \mathcal{M}'; \ \mathcal{M} \overset{\beta}{\hookrightarrow} \mathcal{M}'),$$

where $(\mathcal{M}, \mathcal{M}', \beta)$ are as above, $\mathcal{L}'$ is a line bundle on X, and κ' is an embedding of $\mathcal{L}'$ into $\mathcal{M}'$ as a coherent sub-sheaf.

Let us denote by $\overrightarrow{h}'$ the natural projection $\mathcal{H}_x'{}^{\,\mathrm{basic}} \to \overline{\mathrm{Bun}}_B^{d_1, d_2}$ that remembers the data of $\mathcal{L}' \overset{\kappa'}{\hookrightarrow} \mathcal{M}'$. Let $\overleftarrow{h}'$ denote the map $\mathcal{H}_x'{}^{\,\mathrm{basic}} \to \overline{\mathrm{Bun}}_B^{d_1, d_2 + 1}$ that sends a quintuple as above to $(\mathcal{L}, \kappa, \mathcal{M})$, where $\mathcal{L} := \mathcal{L}'(-x)$ and κ is the (unique and well-defined) embedding of $\mathcal{L}$ into $\mathcal{M}$, such that $\beta \circ \kappa$ equals

$$\mathcal{L} \hookrightarrow \mathcal{L}' \overset{\kappa'}{\hookrightarrow} \mathcal{M}'.$$

By construction and base change,

$$\mathrm{H}_x(\mathrm{Eis}_!^{d_1, d_2}(E_{\check{T}})) \simeq \overline{\mathfrak{p}}_!^{d_1 - 1, d_2} \circ \overrightarrow{h}'_! \circ \overleftarrow{h}'^* \left((\iota_0^{d_1, d_2})_! (\mathrm{IC}_{\mathrm{Bun}_B^{d_1, d_2}}) \otimes (\overline{\mathfrak{q}}^{d_1, d_2})^* (\mathcal{S}(E_{\check{T}})) \right),$$

which, in turn, is isomorphic to

$$\overline{\mathfrak{p}}_!^{d_1 - 1, d_2} \left(\overrightarrow{h}'_! \circ \overleftarrow{h}'^* \left((\iota_0^{d_1, d_2})_! (\mathrm{IC}_{\mathrm{Bun}_B^{d_1, d_2}}) \right) \otimes (\overline{\mathfrak{q}}^{d_1 - 1, d_2})^* (\mathcal{S}(E_{\check{T}})) \right) \otimes E_{1, x}[1].$$

Thus, to prove Proposition 7.5, it is sufficient to show that

$$(7.3) \quad \overrightarrow{h}'_! \circ \overleftarrow{h}'^* \left((\iota_0^{d_1, d_2})_! (\mathrm{IC}_{\mathrm{Bun}_B^{d_1, d_2}}) \right) \simeq$$

$$\simeq \mathrm{Co\text{-}Ker} \left((\overline{\imath}_1^{d_1, d_2 - 1})_! \left(\mathrm{IC}_X \boxtimes (\iota_0^{d_1, d_2 - 1})_! (\mathrm{IC}_{\mathrm{Bun}_B^{d_1, d_2 - 1}}) \right) \to \right.$$

$$\left. \to (\overline{\imath}_1^{d_1, d_2 - 1})_! \left(j_{x *} (\mathrm{IC}_{X - x}) \boxtimes (\iota_0^{d_1, d_2 - 1})_! (\mathrm{IC}_{\mathrm{Bun}_B^{d_1 - 1, d_2}}) \right) \bigoplus (\iota_0^{d_1 - 1, d_2})_! \left(\mathrm{IC}_{\mathrm{Bun}_B^{d_1 - 1, d_2}} \right) \right),$$

where j_x denotes the open embedding $X - x \hookrightarrow X$.

To establish the required isomorphism, note that both the LHS and the RHS are extensions by 0 from the open substack

$$\iota_0^{d_1 - 1, d_2}(\mathrm{Bun}_B^{d_1 - 1, d_2}) \cup \iota_1^{d_1, d_2 - 1}(\mathrm{Bun}_B^{d_1, d_2 - 1}) \subset \overline{\mathrm{Bun}}_B^{d_1 - 1, d_2}.$$

Over this open subset, $\iota_1^{d_1, d_2 - 1}(X \times \mathrm{Bun}_B^{d_1, d_2 - 1})$ is a smooth divisor, which itself contains the divisor, corresponding to the point $x \in X$. The map $\overleftarrow{h}'$ is an isomorphism away from $x \times \mathrm{Bun}_B^{d_1, d_2 - 1}$, and over this codimension-2 closed substack it is a fibration with typical fiber $\mathbb{P}^1$.

Therefore, our situation admits the following local model. Let $f : \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$ be the blow-up of the affine plane at the origin. Let $\imath^1$ be the embedding of a fixed line $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$, and let $\imath^0$ be the embedding of its complement; we will denote by $\widetilde{\imath}^0$ the embedding of the complement of the proper transform of $\mathbb{A}^1$ into $\widetilde{\mathbb{A}}^2$. Finally, let j denote the embedding $\mathbb{A}^1 - 0 \hookrightarrow \mathbb{A}^1$. We have:

Lemma 7.7.

$$f_! \left(\widetilde{\imath}_!^0 (\mathrm{IC}_{\widetilde{\mathbb{A}}^2 - \mathbb{A}^1}) \right) \simeq \mathrm{Co\text{-}Ker}\left(\imath_!^1 (\mathrm{IC}_{\mathbb{A}^1}) \to \imath_!^1 (j_*(\mathrm{IC}_{\mathbb{A}^1 - 0})) \bigoplus \imath_!^0 (\mathrm{IC}_{\mathbb{A}^2 - \mathbb{A}^1}) \right).$$

The proof is a straightforward verification.

8. The Hecke property

8.1. Let $E_{\check{B}, \widehat{univ}}$ the canonical $\check{B}$-local system over X over the formal scheme $\mathrm{Def}_{\check{B}}(E_{\check{T}})$. For a point $x \in X$ and $V \in \mathrm{Rep}(\check{G})$, let $V_{E_{\check{B}, \widehat{univ}, x}}$ be the fiber at x of the local system associated with $E_{\check{B}, \widehat{univ}}$ and V. This is a locally free $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$-module of rank equal to $\dim(V)$.

As in the case of GL_2, the first step is to describe $V_{E_{\check{B}, \widehat{univ}, x}}$ explicitly as an $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$-module, in terms of the isomorphism of Theorem 3.6.

Let $\check{\eta} \in \check{\Lambda}$ be a large enough weight, so that $\check{\eta} - \check{\nu} \in \check{\Lambda}^{pos}$ whenever $V(\check{\nu}) \neq 0$. (If G is not of the adjoint type, we will assume that $Z(G)$ acts on V by a single character.) For $\check{\lambda} \in \check{\Lambda}^{pos}$ we consider the following complex on $X^{\check{\lambda}}$:

We consider the stratification of $X^{\check{\lambda}}$, numbered by triples: $(\check{\lambda}_1, \check{\lambda}_2 \,|\, \check{\lambda}_1 + \check{\lambda}_2 = \check{\lambda}, \mathfrak{P}(\check{\lambda}_1))$, which each stratum corresponds to the configuration

$$\check{\lambda}_1^k \cdot x_k, \check{\lambda}_2 \cdot x, \ x_k \neq x_{k'} \neq x, \Sigma \check{\lambda}_1^k = \check{\lambda}_1.$$

On each such stratum we put the locally-constant complex, denoted $\mathrm{Cous}(\check{\mathfrak{n}}_{E, \check{T}}^*, V)^{\check{\lambda}_1, \check{\lambda}_2, \mathfrak{P}(\check{\lambda}_1)}$, whose !-stalk at the above point is

$$\bigotimes_k \left(\check{\Lambda}^\bullet (\check{\mathfrak{n}}_{E_{\check{T}, x_k}}^*) \right)^{-\check{\lambda}_1^k} \bigotimes \left(\check{\Lambda}^\bullet (\check{\mathfrak{n}}_{E_{\check{T}, x}}^*) \otimes V_{E_{\check{T}, x}} \right)^{\check{\eta} - \check{\lambda}_2},$$

with the standard Chevalley differential.

Let $j^{\mathfrak{P}(\check{\lambda}_1), \check{\lambda}_2}$ denote the embedding of the corresponding stratum into $X^{\check{\lambda}}$. The direct sum of complexes

$$\bigoplus_{\check{\lambda}_1, \check{\lambda}_2, \mathfrak{P}(\check{\lambda}_1)} j_*^{\mathfrak{P}(\check{\lambda}_1), \check{\lambda}_2} \left(\mathrm{Cous}(\check{\mathfrak{n}}_{E, \check{T}}^*, V)^{\check{\lambda}_1, \check{\lambda}_2, \mathfrak{P}(\check{\lambda}_1)} \right)$$

acquires a natural differential. We shall denote the complex, associated to the resulting bi-complex by $\Omega(\check{\mathfrak{n}}_{X, E_{\check{T}}}, V_{E_{\check{T}, x}})^{\check{\eta} - \check{\lambda}}$. When $E_{\check{T}}$ is trivial (i.e., when there is no twisting), we shall denote this complex simply by $\Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\eta} - \check{\lambda}}$.

Proposition 8.2. *The complex* $\Omega(\check{\mathfrak{n}}_{X, E_{\check{T}}}, V_{E_{\check{T}, x}})^{\check{\eta} - \check{\lambda}}$ *is a perverse sheaf. If $E_{\check{T}}$ is regular, then the cohomology* $H(X^{\check{\lambda}}, \Omega(\check{\mathfrak{n}}_{X, E_{\check{T}}}, V_{E_{\check{T}, x}})^{\check{\eta} - \check{\lambda}})$ *is concentrated in degree* 0.

Proof. When we regard V as a $\check{B}$-module, it carries a canonical filtration, parametrized by the partially order set $\check{\Lambda}$ (with the order relation $\check{\lambda}' \geq \check{\lambda}''$ if $\check{\lambda}' - \check{\lambda}'' \in \check{\Lambda}^{pos}$), such that $\mathrm{gr}^{\check{\nu}}(V) \simeq V(\check{\nu})$ (here and in the sequel $V(\check{\nu})$ denotes the $\check{\nu}$ weight space of V).

This filtration induces a filtration on $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\eta}-\check{\lambda}}$, such that

$$\mathrm{gr}^{\check{\nu}}\left(\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\eta}-\check{\lambda}}\right) \simeq \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\eta}-\check{\lambda}-\check{\nu}} \otimes V(\check{\nu}),$$

where $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\eta}-\check{\lambda}-\check{\nu}}$, which is by definition a perverse sheaf on $X^{\check{\lambda}-(\check{\eta}-\check{\nu})}$, is viewed as a perverse sheaf on $X^{\check{\lambda}}$ via the map

$$X^{\check{\lambda}-(\check{\eta}-\check{\nu})} \to X^{\check{\lambda}-(\check{\eta}-\check{\nu})} \times X^{\check{\eta}-\check{\nu}} \to X^{\check{\lambda}},$$

where the first arrow corresponds to the point $(\check{\eta} - \check{\nu}) \cdot x \in X^{\check{\eta}-\check{\nu}}$.

This proves both points of the proposition in view of Proposition 3.2 and Corollary 3.4.

$\square$

8.3.   Let us make the following observation. Let us replace the element $\check{\eta}$ by another element $\check{\eta}'$; with no restriction of generality we can assume that $\check{\eta}' - \check{\eta} \in \check{\Lambda}^{pos}$. Let $\check{\nu} = \check{\eta} - \check{\lambda} = \check{\eta}' - \check{\lambda}'$ with $\check{\lambda}, \check{\lambda}' \in \check{\Lambda}^{pos}$. Then we have the perverse sheaf $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\eta}-\check{\lambda}}$ on $X^{\check{\lambda}}$ and the perverse sheaf $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\eta}'-\check{\lambda}'}$ on $X^{\check{\lambda}'}$. However, it is easy to see that the latter is canonically isomorphic to the direct image of the former under the closed embedding

$$X^{\check{\lambda}} \hookrightarrow X^{\check{\lambda}'},$$

corresponding to adding the coloured divisor $(\check{\lambda}' - \check{\lambda}) \cdot x$.

For $\check{\nu} \in \check{\Lambda}$, let $_{\infty \cdot x}X^{\check{\nu}}$ denote the ind-scheme $\varinjlim\limits_{\check{\lambda} \in \check{\Lambda}^{pos}} X^{\check{\lambda}}$, which we think of as classifying divisors of the form $\check{\lambda}' \cdot x - \Sigma \check{\lambda}_k \cdot x_k$, where $\check{\lambda}_k \in \check{\Lambda}^{pos}$ for $x_k \neq x$, and $\check{\lambda}' \in \check{\Lambda}$ arbitrary, but so that $\check{\lambda}' - \Sigma \check{\lambda}_k = \check{\nu}$.

We obtain that for $\check{\nu} \in \check{\Lambda}$ we have a well-defined perverse sheaf $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\nu}}$ on $_{\infty \cdot x}X^{\check{\nu}}$, which equals $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\eta}-\check{\lambda}}$ on $X^{\check{\lambda}}$ for $\check{\eta}$ and $\check{\lambda}$ large enough with $\check{\eta} - \check{\lambda} = \check{\nu}$. [4] Letting V be the trivial representation, we recover $\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}}$ as a perverse sheaf on $X^{\check{\lambda}} \subset {}_{\infty \cdot x}X^{-\check{\lambda}}$.

For each $\check{\lambda} \in \check{\Lambda}^{pos}$ We have natural addition maps

$$_{\infty \cdot x}X^{\check{\nu}_1} \times {}_{\infty \cdot x}X^{\check{\nu}_2} \to {}_{\infty \cdot x}X^{\check{\nu}_1+\check{\nu}_2},$$

and the corresponding functors

$$\star : D^b({}_{\infty \cdot x}X^{\check{\nu}_1}) \times D^b({}_{\infty \cdot x}X^{\check{\nu}_2}) \to D^b({}_{\infty \cdot x}X^{\check{\nu}_1+\check{\nu}_2}),.$$

By construction, there exists a canonical map

$$(8.1) \qquad \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V^1_{E_{\check{T},x}})^{\check{\nu}_1} \star \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V^2_{E_{\check{T},x}})^{\check{\nu}_2} \to \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, (V^1 \otimes V^2)_{E_{\check{T},x}})^{\check{\nu}_1+\check{\nu}_2},$$

which is associative in the natural sense. Letting V_2 be the trivial representation, we obtain the map

$$(8.2) \qquad \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\nu}} \star \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{-\check{\lambda}} \to \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\nu}-\check{\lambda}}.$$

In particular, assuming that $E_{\check{T}}$ is regular, set

$$R(V_x)^{\check{\nu}}_{E_{\check{T}}} := H\left({}_{\infty \cdot x}X^{\check{\nu}}, \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\nu}}\right) \text{ and } R(V_x)_{E_{\check{T}}} := \bigoplus_{\check{\nu}} R(V_x)^{\check{\nu}}_{E_{\check{T}}}.$$

We obtain that $R(V_x)_{E_{\check{T}}}$ is a ($\check{\Lambda}$-graded) module over the ($\check{\Lambda}$-graded) commutative algebra $R_{E_{\check{T}}}$.

This module is finitely generated and projective. Indeed, the filtration, introduced in the proof of Proposition 8.2, induces a filtration on the above module, with the associated graded being the free module on the vector space V.

---

[4]This construction has the advantage that it makes sense whether or not $Z(G)$ acts on V by a single character.

Set also

$$\widehat{R}(V_x)_{E_{\check{T}}} = \prod_{\check{\nu}} R(V_x)_{E_{\check{T}}}^{\check{\nu}} \simeq R(V_x)_{E_{\check{T}}} \underset{R_{E_{\check{T}}}}{\otimes} \widehat{R}_{E_{\check{T}}}.$$

Lemma 8.4. *Under the isomorphism $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})} \simeq \widehat{R}_{E_{\check{T}}}$, the module $V_{E_{\check{B},\widehat{univ},x}}$ corresponds to* $\widehat{R}(V_x)_{E_{\check{T}}}$.

The proof will be given in Sect. 11.

Let $V_{E_{\check{B},univ,x}}$ be the $\check{\Lambda}$-graded version of $V_{E_{\check{B},\widehat{univ},x}}$. We obtain:

Corollary 8.5. *Under the isomorphism $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})} \simeq R_{E_{\check{T}}}$, the $\mathcal{O}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$-module $V_{E_{\check{B},univ,x}}$ corresponds to to the $R_{E_{\check{T}}}$-module $R(V_x)_{E_{\check{T}}}$.*

8.6. Let $_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}$ denote the ind-version of $\overline{\mathrm{Bun}}_B^{\check{\mu}}$, where the maps κ^λ are allowed to have poles of arbitrary order at x, see [BG], Sect. 4.1.1. The stack $\overline{\mathrm{Bun}}_B^{\check{\mu}}$ is a closed substack of $_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}$; hence perverse sheaves (or objects of the derived category) on the former can be thought of as corresponding objects on the latter. We will denote by $_{\infty\cdot x}\overline{\mathrm{Bun}}_B$ the union of the $_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}$'s over $\check{\mu} \in \check{\Lambda}$.

For $\check{\nu} \in \check{\Lambda}$ we have a natural map

$$_{\infty\cdot x}\overline{\imath}_{\check{\nu}} : \, _{\infty\cdot x}X^{\check{\nu}} \times \, _{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}-\check{\nu}} \to \, _{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}},$$

defined in the same way as $\overline{\imath}_{\check{\lambda}}$. Let $_{\infty\cdot x}\imath_{\check{\nu}}$ denote the restriction of $_{\infty\cdot x}\overline{\imath}_{\check{\nu}}$ to the locally closed substack

$$_{\infty\cdot x}X^{\check{\nu}} \times \mathrm{Bun}_B^{\check{\mu}-\check{\nu}} \subset \, _{\infty\cdot x}X^{\check{\nu}} \times \, _{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}-\check{\nu}}.$$

The images of the maps $_{\infty\cdot x}\imath_{\check{\nu}}$ for $\check{\nu} \in \check{\Lambda}$ define a stratification of $_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}$.

Using the maps $_{\infty\cdot x}\overline{\imath}_{\check{\nu}}$ we define the functors

$$(8.3) \qquad\qquad \star : D^b(_{\infty\cdot x}X^{\check{\nu}}) \times D^b(_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\nu}}) \to D^b(_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}}).$$

Let $\mathcal{H}_x$ be the Hecke stack for G, and let $\mathcal{H}'_x$ be its version over $_{\infty\cdot x}\overline{\mathrm{Bun}}_B$ (see [BG], 4.1.2), so that we have a commutative diagram with both squares Cartesian:

$$
\begin{array}{ccccc}
_{\infty\cdot x}\overline{\mathrm{Bun}}_B & \xleftarrow{\overleftarrow{h}'} & \mathcal{H}'_x & \xrightarrow{\overrightarrow{h}'} & _{\infty\cdot x}\overline{\mathrm{Bun}}_B \\
{\scriptstyle\overline{\mathsf{p}}}\downarrow & & {\scriptstyle\overline{\mathsf{p}}'}\downarrow & & {\scriptstyle\overline{\mathsf{p}}}\downarrow \\
\mathrm{Bun}_G & \xleftarrow{\overleftarrow{h}} & \mathcal{H}_x & \xrightarrow{\overrightarrow{h}} & \mathrm{Bun}_G .
\end{array}
$$

Thus, for each $V \in \mathrm{Rep}(\check{G})$ we can associate the perverse sheaves $\mathcal{V}$ and $\mathcal{V}'$ on $\mathcal{H}_x$ and $\mathcal{H}'_x$, respectively, and the Hecke functors

$$\mathrm{H}_x^V : D^b(\mathrm{Bun}_G) \to D^b(\mathrm{Bun}_G), \text{ given by } \mathcal{T} \mapsto \overleftarrow{h}_!(\mathcal{V} \otimes \overrightarrow{h}^*(\mathcal{T}))$$

and

$$\mathrm{H}'_x^V : D^b(_{\infty\cdot x}\overline{\mathrm{Bun}}_B) \to D^b(_{\infty\cdot x}\overline{\mathrm{Bun}}_B), \text{ given by } \mathcal{T}' \mapsto \overleftarrow{h}'_!(\mathcal{V}' \otimes \overrightarrow{h}'^*(\mathcal{T}')).$$

Note that the convolution functors (8.3) introduced above and the Hecke functors H'_x^V naturally commute.

8.7.   The main geometric ingredient, from which we shall deduce Theorem 1.11 is a description of the object

$$\text{(8.4)} \qquad \qquad \text{H}'^{V}_{x}\left(\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu}})\right).$$

Before stating the corresponding theorem, let us make several observations.

First, let us recall from [BG], Theorem 3.3.2, that there is a canonical isomorphism:

$$\text{(8.5)} \qquad \qquad \text{H}'^{V}_{x}(\text{IC}_{\overline{\text{Bun}}_B^{\check\mu}}) \simeq \underset{\check\nu}{\oplus}\, V(\check\nu) \otimes \delta_x^{\check\nu} \star \text{IC}_{\overline{\text{Bun}}_B^{\check\mu-\check\nu}},$$

where $\delta_x^{\check\nu}$ denotes the sky-scraper at the point $\check\nu \cdot x \in {}_{\infty\cdot x}X^{\check\nu}$. Here the weight spaces $V(\check\nu)$ are realized as cohomologies of the corresponding spherical sheaves on the affine Grassmannian along semi-infinite orbits (see [BG], proof of Theorem 3.3.2).

Combining this with Corollary 4.5, we obtain that the object (8.4) is a perverse sheaf.

Moreover, the fiber product

$$\mathcal{H}'_x \underset{\text{Bun}_G,\, \overrightarrow{h}}{\times} \text{Bun}_B^{\check\mu}$$

is naturally stratified, according to the order of zero/pole at x of the corresponding generalized B-structure under the projection $\overleftarrow{h}'$, see [BG], Sect. 3.3.4. Hence, we obtain a filtration on (8.4), parametrized by $\check\Lambda$ with

$$\text{gr}^{\check\nu}\left(\text{H}'^{V}_{x}\left(\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu}})\right)\right) \simeq V(\check\nu)\otimes\delta_x^{\check\nu}\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu+\check\nu}}).$$

We will prove:

Theorem 8.8.

(A) *We have an isomorphism of perverse sheaves:*

$$\text{(8.6)} \qquad \Omega(\check{\mathfrak{n}}_X, V_x)^{\check\nu} \boxtimes \text{IC}_{\text{Bun}_B^{\check\mu-\check\nu}} \simeq h^0\left({}_{\infty\cdot x}i_{\check\nu}^{!}\left(\text{H}'^{V}_{x}\left(\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu}})\right)\right)\right).$$

In particular, by adjunction we obtain a canonical map of perverse sheaves

$$\text{(8.7)} \qquad \Omega(\check{\mathfrak{n}}_X, V_x)^{\check\nu}\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu-\check\nu}}) \to \text{H}'^{V}_{x}\left(\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu}})\right).$$

(B) *For $\check\lambda \in \check\Lambda^{pos}$ the following diagram is commutative:*

$$
\begin{array}{ccc}
\Omega(\check{\mathfrak{n}}_X, V_x)^{\check\nu}\star\Omega(\check{\mathfrak{n}}_X)^{-\check\lambda}\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu-\check\nu+\check\lambda}}) & \xrightarrow{\text{(4.1)}} & \Omega(\check{\mathfrak{n}}_X, V_x)^{\check\nu}\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu-\check\nu}}) \\
{\scriptstyle(8.2)}\downarrow & & {\scriptstyle(8.7)}\downarrow \\
\Omega(\check{\mathfrak{n}}_X, V_x)^{\check\nu-\check\lambda}\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu-\check\nu+\check\lambda}}) & \xrightarrow{\text{(8.7)}} & \text{H}'^{V}_{x}\left(\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu}})\right).
\end{array}
$$

(C) *The map (8.7) is compatible with filtrations, and resulting map on the associated graded level makes the following diagram commutative:*

$$
\begin{array}{ccc}
\text{gr}^{\check\nu'}\left(\Omega(\check{\mathfrak{n}}_X, V_x)^{\check\nu}\right)\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu-\check\nu}}) & \longrightarrow & \text{gr}^{\check\nu'}\left(\text{H}'^{V}_{x}\left(\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu}})\right)\right) \\
\sim\downarrow & & \sim\downarrow \\
V(\check\nu')\otimes\delta_x^{\check\nu'}\star\Omega(\check{\mathfrak{n}}_X)^{\check\nu-\check\nu'}\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu-\check\nu}}) & \xrightarrow{\text{(4.1)}} & V(\check\nu')\otimes\delta_x^{\check\nu'}\star\jmath_{!}(\text{IC}_{\text{Bun}_B^{\check\mu-\check\nu'}}).
\end{array}
$$

The proof will be given in Sect. 9.

8.9. Let us show how Theorem 8.8 implies Theorem 1.11. Tensoring both sides of (8.7) by $(\overline{\mathfrak{q}}^{\check{\mu}})^*(\mathcal{S}(E_{\check{T}}))$, we obtain a map

$$(\infty\cdot x\overline{i}_{\check{\nu}})_! \left(\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\nu}} \boxtimes \jmath_! \left(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}} \otimes (\mathfrak{q}^{\check{\mu}-\check{\nu}})^*(\mathcal{S}(E_{\check{T}})) \right) \right) \to$$
$$\to \mathrm{H}'_x^V \left(\jmath_! \left(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}} \otimes (\mathfrak{q}^{\check{\mu}})^*(\mathcal{S}(E_{\check{T}})) \right) \right).$$

Taking the direct image of both sides of the above expression with respect to the natural morphism $\overline{\mathfrak{p}}^{\check{\mu}} : {}_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check{\mu}} \to \mathrm{Bun}_G$, we obtain a map

$$R(V_x)_{E_{\check{T}}}^{\check{\nu}} \otimes \mathrm{Eis}_!^{\check{\mu}-\check{\nu}}(E_{\check{T}}) \simeq H\left({}_{\infty\cdot x}X^{\check{\nu}}, \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}}, V_{E_{\check{T},x}})^{\check{\nu}} \right) \otimes \mathrm{Eis}_!^{\check{\mu}-\check{\nu}}(E_{\check{T}}) \to \mathrm{H}_x^V(\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})).$$

Summing up over all $\check{\nu}$ and $\check{\mu}$, we obtain a map

$$(8.8) \qquad\qquad R(V_x)_{E_{\check{T}}} \otimes \mathrm{Eis}_!(E_{\check{T}}) \to \mathrm{H}_x^V(\mathrm{Eis}_!(E_{\check{T}})).$$

Point (B) of Theorem 8.8 implies that the latter map respects the action of the algebra $R(V_x)_{E_{\check{T}}}$, i.e., we obtain a map

$$(8.9) \qquad\qquad R(V_x)_{E_{\check{T}}} \underset{R_{E_{\check{T}}}}{\otimes} \mathrm{Eis}_!(E_{\check{T}}) \to \mathrm{H}_x^V(\mathrm{Eis}_!(E_{\check{T}})).$$

We claim that the above map is an isomorphism, implying Theorem 1.11.

Indeed, the filtration on $\mathcal{H}'_x \underset{\mathrm{Bun}_G, \overrightarrow{h}}{\times} \mathrm{Bun}_B^{\check{\mu}}$ defines a filtration on each $\mathrm{H}_x^V(\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}}))$ with

$$\mathrm{gr}^{\check{\nu}}\left(\mathrm{H}_x^V(\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})) \right) \simeq V(\check{\nu}) \otimes \mathrm{Eis}_!^{\check{\mu}-\check{\nu}}(E_{\check{T}}).$$

By point (C) of Theorem 8.8 we obtain that the map (8.8) respects the filtrations, and the corresponding map on the associated graded level can be identified with

$$V \otimes R_{E_{\check{T}}} \otimes \mathrm{Eis}_!(E_{\check{T}}) \to V \otimes \mathrm{Eis}_!(E_{\check{T}}),$$

given by the $R_{E_{\check{T}}}$-action on $\mathrm{Eis}_!(E_{\check{T}})$.

Hence, the map (8.9) also respects the filtrations, and on the associated graded level induces the identity isomorphism of $V \otimes \mathrm{Eis}_!(E_{\check{T}})$, implying that (8.9) is itself an isomorphism.

8.10. Recall from Theorem 5.6 that for each $\check{\lambda} \in \check{\Lambda}^{pos}$ we have a canonical map in $D^b(\overline{\mathrm{Bun}}_B^{\check{\mu}})$:

$$(8.10) \qquad\qquad \mathfrak{U}(\mathfrak{n}_X)^{\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}} \to \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}},$$

and the Verdier dual map

$$(8.11) \qquad\qquad \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}} \to \mathfrak{U}(\mathfrak{n}_X)^{*,-\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}+\check{\lambda}}}.$$

In this subsection we will study how these maps are compatible with the isomorphisms of (8.5). Namely, we will establish the following:

Theorem 8.11. *For $V \in \mathrm{Rep}(\check{G})$ and $\check{\lambda}$ as above the following diagram is commutative*

$$\begin{array}{ccc}
\mathrm{H}'^{\,V}_x(\mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}}_B}) & \xrightarrow{(8.11)} & \mathrm{H}'^{\,V}_x\left(\mathfrak{U}(\mathfrak{n}_X)^{*,-\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}+\check{\lambda}}_B}\right) \\
 & & \sim\downarrow \\
(8.5)\downarrow & & \mathfrak{U}(\mathfrak{n}_X)^{*,-\check{\lambda}} \star \mathrm{H}'^{\,V}_x(\mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}+\check{\lambda}}_B}) \\
 & & (8.5)\downarrow \\
\underset{\check{\nu}}{\oplus} V(\check{\nu}) \otimes \delta^{\check{\nu}}_x \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}-\check{\nu}}_B} & \longrightarrow & \underset{\check{\nu}'}{\oplus} V(\check{\nu}') \otimes \delta^{\check{\nu}'}_x \star \mathfrak{U}(\mathfrak{n}_X)^{*,-\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}-\check{\nu}'+\check{\lambda}}_B},
\end{array}$$

where the lower horizontal arrow is comprised of the maps

$$V(\check{\nu}) \otimes \delta^{\check{\nu}}_x \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}-\check{\nu}}_B} \xrightarrow{(8.11)} V(\check{\nu}) \otimes \delta^{\check{\nu}}_x \star \mathfrak{U}(\mathfrak{n}_X)^{*,-\check{\lambda}} \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}-\check{\nu}+\check{\lambda}}_B}$$

and the map

$$V(\check{\nu}) \otimes \delta^{\check{\nu}}_x \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}-\check{\nu}}_B} \to V(\check{\nu}+\check{\lambda}) \otimes \mathfrak{U}(\mathfrak{n}_X)^{*,-\check{\lambda}} \star \delta^{\check{\nu}+\check{\lambda}}_x \star \mathrm{IC}_{\overline{\mathrm{Bun}}^{\check{\mu}-\check{\nu}}_B},$$

given by the identification of the !-stalk of $\mathfrak{U}(\mathfrak{n}_X)^{*,-\check{\lambda}}$ at $\check{\lambda} \cdot x \in X^{\check{\lambda}}$ with $(U(\check{\mathfrak{n}})^{\check{\lambda}})^*$ and the map

$$V(\check{\nu}) \to V(\check{\nu}+\check{\lambda}) \otimes (U(\check{\mathfrak{n}})^{\check{\lambda}})^*,$$

given by the action of $U(\check{\mathfrak{n}})$ on V.

The rest of this section will be devoted to the proof of this theorem. We will need some preliminaries. Consider the graded perverse sheaf on $_{\infty \cdot x}X^{\check{\nu}}$

$$\mathrm{Kosz}(V_x)^{\bullet,\check{\nu}} := \bigoplus_{\check{\lambda} \in \check{\Lambda}^{pos}} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda},*} \star \Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\nu}+\check{\lambda}}.$$

The maps (8.2) define on $\mathrm{Kosz}(V_x)^{\bullet,\check{\nu}}$ a differential, by the same formula as in the case of $\mathrm{Kosz}^{\bullet,\check{\nu}}_{\overline{\mathrm{Bun}}^{\check{\mu}}_B}$. The canonical projection

$$\Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\nu}} \to \mathrm{gr}^{\check{\nu}}(\Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\nu}}) \simeq \delta^{\check{\nu}}_x \otimes V(\check{\nu})$$

defines a map

$$(8.12) \qquad\qquad \mathrm{Kosz}(V_x)^{\bullet,\check{\nu}} \to \delta^{\check{\nu}}_x \otimes V(\check{\nu}).$$

Lemma 8.12.

(1) *The map (8.12) is a quasi-isomorphism.*

(2) *For $\check{\lambda} \in \check{\Lambda}^{pos}$, the diagram*

$$\begin{array}{ccc}
\mathrm{Kosz}(V_x)^{\bullet,\check{\nu}} & \longrightarrow & \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda},*} \star \mathrm{Kosz}(V_x)^{\bullet,\check{\nu}+\check{\lambda}} \\
\downarrow & & \downarrow \\
\delta^{\check{\nu}}_x \otimes V(\check{\nu}) & \longrightarrow & \mathfrak{U}(\check{\mathfrak{n}}_X)^{-\check{\lambda},*} \star \delta^{\check{\nu}+\check{\lambda}}_x \otimes V(\check{\nu}+\check{\lambda})
\end{array}$$

commutes in the derived category, where the top horizontal arrow is induced by (6.4), and the bottom horizontal arrow is as Theorem 8.11.

Recall the complex

$$\mathrm{Kosz}^{\bullet}_{\overline{\mathrm{Bun}}^{\check{\mu}}_B} := \bigoplus_{\check{\lambda} \in \check{\Lambda}^{pos}} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda},*} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check{\mu}+\check{\lambda}}_B}).$$

According to Theorem 6.6, it is quasi-isomorphic to $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}$. Moreover, by Lemma 6.12, the map (8.11) is represented by the map

$$(8.13) \qquad \mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu}} \to \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda,*} \star \mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}},$$

induced by the map dual to (6.2).

Next, we shall reinterpret the isomorphisms of Theorem 8.8 and (8.5). Consider the graded perverse sheaf on $_{\infty\cdot x}\overline{\mathrm{Bun}}_B^{\check\mu}$

$$\mathrm{Kosz}(V_x)^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu}} := \bigoplus_{\check\nu\in\check\Lambda,\check\lambda\in\check\Lambda^{pos}} \Omega(\check{\mathfrak{n}}_X,V_x)^{\check\nu} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda,*} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu-\check\nu+\check\lambda}}).$$

It also acquires a natural differential differential. The projection on the direct summand corresponding to $\check\lambda=0$ and the map (8.7) define a map of complexes:

$$(8.14) \qquad \mathrm{Kosz}(V_x)^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu}} \to \mathrm{H}'\,{}^V_x\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu}})\right).$$

Proposition 8.13. *The map* (8.14) *is a quasi-isomorphism.*

Proof. Considering the filtrations on both sides as in Theorem 8.8(C) and passing to the associated graded we obtain a map, whose $\check\nu'$-component is

$$V(\check\nu')\otimes\delta_x^{\check\nu'} \star \left(\underset{\check\lambda,\check\nu}{\oplus}\ \Omega(\check{\mathfrak{n}}_X)^{\check\nu-\check\nu'} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda,*} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu-\check\nu+\check\lambda}})\right) \to V(\check\nu')\otimes\delta_x^{\check\nu'} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}^{\check\mu-\check\nu'}}).$$

Regrouping the terms, the RHS of the above expression is the direct sum over $\check\eta$ of

$$V(\check\nu')\otimes\delta_x^{\check\nu'} \star \left(\underset{\check\lambda}{\oplus}\ \Omega(\check{\mathfrak{n}}_X)^{\check\eta+\check\lambda} \star \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda,*} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu-\check\eta-\check\nu'}})\right).$$

However, it follows from the acyclicity of $\mathrm{Kosz}(E_{\check T})^{\bullet,-\check\eta}$ that the latter expression is acyclic unless $\check\eta=0$, and quasi-isomorphic to $V(\check\nu')\otimes\delta_x^{\check\nu'} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check\mu-\check\nu'}})$ in the latter case, as required. $\square$

Let us apply the functor $\mathrm{H}'\,{}^V_x$ term-wise to the complex $\mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu}}$. By Proposition 8.13, the result is quasi-isomorphic to the complex

$$(8.15) \qquad \bigoplus_{\check\lambda} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda,*} \star \mathrm{Kosz}(V_x)^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}}$$

with a natural differential. Using (8.12) we obtain a map from the expression in (8.15) to

$$\bigoplus_{\check\nu'} V(\check\nu')\otimes\delta_x^{\check\nu'} \star \mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu-\check\nu'}}.$$

The next assertion follows from Theorem 8.8:

Lemma 8.14. *The following diagram commutes in the derived category:*

$$
\begin{array}{ccc}
\displaystyle\bigoplus_{\check\lambda} \mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check\lambda,*} \star \mathrm{Kosz}(V_x)^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu+\check\lambda}} & \longrightarrow & \displaystyle\bigoplus_{\check\nu'} V(\check\nu')\otimes\delta_x^{\check\nu'} \star \mathrm{Kosz}^\bullet_{\overline{\mathrm{Bun}}_B^{\check\mu-\check\nu'}} \\[2ex]
\sim\Big\downarrow & & \sim\Big\downarrow \\[2ex]
\mathrm{H}'\,{}^V_x(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu}}) & \longrightarrow & \displaystyle\bigoplus_{\check\nu'} V(\check\nu')\otimes\delta_x^{\check\nu'} \star \mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check\mu-\check\nu'}}.
\end{array}
$$

Now, using Lemma 8.12, and Lemma 8.14, we can prove Theorem 8.11. Indeed, the diagram, whose commutativity we have to prove, is equivalent to the following one, which is manifestly commutative:

$$
\mathrm{H'}_{x}^{\,V}\left(\mathrm{Kosz}^{\bullet}_{\overline{\mathrm{Bun}}_{B}^{\check{\mu}}}\right)\xrightarrow{\;(8.13)\;}\mathrm{H'}_{x}^{\,V}\left(\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda},*}\star\mathrm{Kosz}^{\bullet}_{\overline{\mathrm{Bun}}_{B}^{\check{\mu}+\check{\lambda}}}\right)
$$

$$
\sim\downarrow
$$

$$
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda},*}\star\mathrm{H'}_{x}^{\,V}\left(\mathrm{Kosz}^{\bullet}_{\overline{\mathrm{Bun}}_{B}^{\check{\mu}+\check{\lambda}}}\right)
$$

$$
\sim\downarrow\qquad\qquad\qquad\qquad\qquad\downarrow
$$

$$
\bigoplus_{\check{\lambda}'}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda}',*}\star\mathrm{Kosz}(V_x)^{\bullet}_{\overline{\mathrm{Bun}}_{B}^{\check{\mu}+\check{\lambda}'}}\longrightarrow\bigoplus_{\check{\lambda}''}\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda},*}\star\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda}'',*}\star\mathrm{Kosz}(V_x)^{\bullet}_{\overline{\mathrm{Bun}}_{B}^{\check{\mu}+\check{\lambda}+\check{\lambda}''}},
$$

where the bottom arrow is given by

$$
\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda}',*}\to\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda},*}\star\mathfrak{U}(\check{\mathfrak{n}}_X)^{\bullet,-\check{\lambda}'',*}
$$

for $\check{\lambda}'=\check{\lambda}+\check{\lambda}''$.

9. Proof of Theorem 8.8

9.1.   The strategy of the proof of Theorem 8.8 will be largely parallel to that of Theorem 4.2. Without restricting the generality, we can assume that the representation V is irreducible, i.e., $V=V^{\check{\eta}}$ for some highest weight $\check{\eta}\in\check{\Lambda}^{+}$. Then the support of $\mathrm{H'}_{x}^{\,V}\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B}^{\check{\mu}}})\right)$ is contained in the closed substack of $_{\infty\cdot x}\overline{\mathrm{Bun}}_{B}^{\check{\mu}}$ equal to the image of

$$
\overline{\mathrm{Bun}}_{B}^{\check{\mu}-\check{\eta}}\simeq(\check{\eta}\cdot x)\times\overline{\mathrm{Bun}}_{B}^{\check{\mu}-\check{\eta}}\overset{\infty\cdot x\bar{i}_{\check{\eta}}}{\hookrightarrow}{}_{\infty\cdot x}\overline{\mathrm{Bun}}_{B}^{\check{\mu}}.
$$

The perverse sheaf $\Omega(\check{\mathfrak{n}}_X,V_x^{\check{\eta}})^{\check{\nu}}$ is non-zero only for $\check{\nu}\leq\check{\eta}$ and is supported on the subscheme

$$
X^{\check{\eta}-\check{\nu}}\times(\check{\eta}\cdot x)\subset{}_{\infty\cdot x}X^{\check{\nu}}.
$$

From now on, we will work on $\overline{\mathrm{Bun}}_{B}^{\check{\mu}-\check{\eta}}$ and $X^{\check{\eta}-\check{\nu}}$ rather than on $_{\infty\cdot x}\overline{\mathrm{Bun}}_{B}^{\check{\mu}}$ and $_{\infty\cdot x}X^{\check{\nu}}$.

Consider the open subset $(X\overset{\circ}{-}x)^{\check{\eta}-\check{\nu}}\subset X^{\check{\eta}-\check{\nu}}$. Let us denote its open embedding by $j_x^{\check{\eta}-\check{\nu}}$. We have:

$$
j_x^{\check{\eta}-\check{\nu}*}\left(\Omega(\check{\mathfrak{n}}_X,V_x^{\check{\eta}})^{\check{\nu}}\right)\simeq j_x^{\check{\eta}-\check{\nu}*}\left(\Omega(\check{\mathfrak{n}}_X)^{\check{\nu}-\check{\eta}}\right)\otimes V^{\check{\eta}}(\check{\eta}).
$$

First, we claim that the isomorphism stated in (8.6) holds over

$$
(X\overset{\circ}{-}x)^{\check{\eta}-\check{\nu}}\times\mathrm{Bun}_{B}^{\check{\mu}-\check{\nu}}\subset X^{\check{\eta}-\check{\nu}}\times\mathrm{Bun}_{B}^{\check{\mu}-\check{\nu}},
$$

i.e.,

$$
(9.1)\quad j_x^{\check{\eta}-\check{\nu}*}\left(\Omega(\check{\mathfrak{n}}_X,V_x^{\check{\eta}})^{\check{\nu}}\right)\boxtimes\mathrm{IC}_{\mathrm{Bun}_{B}^{\check{\mu}-\check{\nu}}}\simeq(j_x^{\check{\eta}-\check{\nu}}\times\mathrm{id})^{*}\left(h^{0}\left({}_{\infty\cdot x}i_{\check{\nu}}^{!}\left(\mathrm{H'}_{x}^{\,V^{\check{\eta}}}\left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B}^{\check{\mu}}})\right)\right)\right)\right).
$$

This follows from the definition of the functor $\mathrm{H'}_{x}^{\,V}$ and (4.6).

We have the following assertion, parallel to Lemma 4.8:

Lemma 9.2. *The canonical map*

$$
\Omega(\check{\mathfrak{n}}_X,V_x^{\check{\eta}})^{\check{\nu}}\to j_x^{\check{\eta}-\check{\nu}}{}_{*}\circ j_x^{\check{\eta}-\check{\nu}*}\left(\Omega(\check{\mathfrak{n}}_X,V_x^{\check{\eta}})^{\check{\nu}}\right)
$$

is injective.

Proof. Let $i_x^{\check{\eta}-\check{\nu}}$ denote the embedding of the point $(\check{\eta}-\check{\nu})\cdot x$ into $X^{\check{\eta}-\check{\nu}}$. By induction on $|\check{\eta}-\check{\nu}|$ we have to show that

$$i_x^{\check{\eta}-\check{\nu}!}\left(\Omega(\check{\mathfrak{n}}_X, V_x^{\check{\eta}})^{\check{\nu}}\right)$$

has no cohomologies in degrees ≤ 0, unless $\check{\nu} = \check{\eta}$.

By the definition of $\Omega(\check{\mathfrak{n}}_X, V_x^{\check{\eta}})^{\check{\nu}}$, the above !-stalk is quasi-isomorphic to the weight $\check{\nu}$-component in the cohomology $H^\bullet(\check{\mathfrak{n}}, V^{\check{\eta}})$. Hence, the 0-th cohomology, which corresponds to the highest weight in $V^{\check{\eta}}$, is of weight $\check{\eta}$, and not $\check{\nu}$. $\qquad\square$

Another assertion that we shall need is parallel to Proposition 4.9:

Proposition 9.3. *The canonical map*

$$h^0\left({}_{\infty\cdot x}i_{\check{\nu}}^!\left(\mathrm{H}'\,{}_x^{V^{\check{\eta}}}\left(\jmath_!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})\right)\right)\right) \to (j^{\check{\lambda}}\times\mathrm{id})_*\circ(j^{\check{\lambda}}\times\mathrm{id})^*\left(h^0\left({}_{\infty\cdot x}i_{\check{\nu}}^!\left(\mathrm{H}'\,{}_x^{V^{\check{\eta}}}\left(\jmath_!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})\right)\right)\right)\right)$$

is injective.

Let us assume this proposition and proceed with the proof of Theorem 8.8.

As in the proof of Theorem 4.2, from Lemma 9.2 and Proposition 9.3, we obtain that if the isomorphism (9.1) extends to a map in one direction

$$(9.2) \qquad \Omega(\check{\mathfrak{n}}_X, V_x^{\check{\eta}})^{\check{\nu}}\times\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}-\check{\nu}}} \to h^0\left({}_{\infty\cdot x}i_{\check{\nu}}^!\left(\mathrm{H}'\,{}_x^V\left(\jmath_!(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}}})\right)\right)\right),$$

then it does so uniquely. Moreover, this map will automatically be an isomorphism and point (B) of Theorem 8.8 will hold.

Thus, our present goal will be to establish the required extension property.

9.4. We will distinguish two cases. One is when $\check{\eta}-\check{\nu}$ is a multiple of a simple coroot, and another when it is not. Let us first treat the latter case. We will argue by induction on $|\check{\eta}-\check{\nu}|$, so we can assume that the extension (9.2) exists for all $\check{\nu}'$ with $\check{\nu}' > \check{\nu}$.

Since the assertion about extension is local, as in the proof of Theorem 4.2, we can pass from $\overline{\mathrm{Bun}}_B^{\check{\mu}}$ to a suitable version of the Zastava space, and using factorization, we can assume that the required isomorphism holds over the open substack

$$(9.3) \qquad \left(X^{\check{\eta}-\check{\nu}}-((\check{\eta}-\check{\nu})\cdot x)\right)\times\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}.$$

As in the proof of Theorem 4.2, to establish the required extension property, it suffices to prove the following:

Lemma 9.5. *Assume that $|\check{\eta}-\check{\nu}|$ is not a multiple of a simple coroot. Then*

$$\mathrm{Ext}^1_{X^{\check{\eta}-\check{\nu}}}\left(i_x^{\check{\eta}-\check{\nu}}{}_!(\mathbb{C}), \Omega(\check{\mathfrak{n}}_X, V_x^{\check{\eta}})^{\check{\nu}}\right) = 0.$$

Proof. The Ext^1 of the lemma is isomorphic to the weight $\check{\nu}$ component in $H^1(\check{\mathfrak{n}}, V^{\check{\eta}})$. However, the 1-st cohomology in question consists of weights of the form

$$s_i(\check{\eta}+\check{\rho})-\check{\rho} = \check{\eta}-\check{\alpha}_i\cdot(\langle\alpha_i,\check{\eta}\rangle+1),$$

for simple reflections s_i. The difference between such a weight and $\check{\eta}$ is a multiple of $\check{\alpha}_i$. $\qquad\square$

9.6.   To complete the proof of Theorem 8.8 it remains to do three things: (1) prove the extension property for $\check{\nu} = \check{\eta} - n \cdot \check{\alpha}_i$, (2) prove Proposition 9.3, and (3) establish the compatibility of point (C) of the theorem. We will do this by reducing to the case of groups of semi-simple rank one.

Thus, let us assume for a moment that G is of semi-simple rank 1, and show that Theorem 8.8 holds in this case. First, it is easy to see that we can assume that $G = GL_2$, as the statement of the theorem is stable under isogenies.

Next, we claim that the explicit computation we did in Sect. 7.6 for V being the standard representation $V^{(1,0)}$ amounts to the statement of Theorem 8.8 in this case. Indeed, the perverse sheaf $\Omega(\check{\mathfrak{n}}_X, V_x^{\check{\eta}})^{\check{\nu}}$ is non-zero only for $\check{\nu}$ being $(1,0)$ or $(0,1)$, and is the sky-scraper in the former case and the sheaf $j_{x*}(\mathrm{IC}_{X-x})$ in the latter case.

Hence, the isomorphism assertion of Theorem 8.8(A) follows from (7.3). The assertion of Theorem 8.8(C) is also manifest.

Now, we claim that Theorem 8.8 holds for an arbitrary representation V of $\check{G} \simeq GL_2$. Indeed, it is easy to see from (8.1) (and this is valid for any group G) that if Theorem 8.8 holds for representations V^1 and V^2, then it also holds for their tensor product.

In addition, by Lemma 9.2 and Proposition 9.3, we obtain that if Theorem 8.8 holds for some representation V, then it also holds for any of its direct summands.

To prove Theorem 8.8 for a representation V of GL_2 it suffices to observe that V is isomorphic to a direct summand of a tensor power of the standard representation, up to some power of the determinant.

9.7.   Next, we shall study how the Hecke functors H'^V_x, that act on the derived category on $_{\infty \cdot x}\overline{\mathrm{Bun}}^{\check{\mu}}_B$, are related to similar functors for Levi subgroups of G.

Let P be a parabolic in G, and let M be the corresponding Levi quotient; let $B(M)$ denote the Borel subgroup in M. If we choose P so that it contains B, then $B(M)$ is the projection of B to M. Let $\overline{\mathrm{Bun}}_{B(M)}$ and $_{\infty \cdot x}\overline{\mathrm{Bun}}_{B(M)}$ be the corresponding stacks for the group M.

Note that we have a natural isomorphism:
$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{B(M)} \simeq \mathrm{Bun}_B,$$
which extends to a locally closed embedding
$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} {}_{\infty \cdot x}\overline{\mathrm{Bun}}_{B(M)} \overset{\iota_M}{\hookrightarrow} {}_{\infty \cdot x}\overline{\mathrm{Bun}}_B.$$

Recall also that we have a diagram of affine Grassmannians:
$$\mathrm{Gr}_G \overset{\mathfrak{p}_{\mathrm{Gr}}}{\hookleftarrow} \mathrm{Gr}_P \overset{\mathfrak{q}_{\mathrm{Gr}}}{\twoheadrightarrow} \mathrm{Gr}_M,$$
and the restriction functor $\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{M})$ corresponds to the following operation on spherical perverse sheaves:
$$\mathcal{V} \in \mathrm{Perv}(\mathrm{Gr}_G) \mapsto \mathfrak{q}_{\mathrm{Gr}!} \circ \mathfrak{q}_{\mathrm{Gr}}^*(\mathcal{V}) \in \mathrm{Perv}(\mathrm{Gr}_M),$$
up to a cohomological shift.

For $U \in \mathrm{Rep}(\check{M})$, let $\mathrm{H}'^U_{M,x}$ denote the corresponding Hecke functor acting on the derived category on $_{\infty \cdot x}\overline{\mathrm{Bun}}_{B(M)}$. Let us denote by $\mathrm{H}'^U_{P,x}$ the Hecke functor acting on the derived category on the stack $\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} {}_{\infty \cdot x}\overline{\mathrm{Bun}}_{B(M)}$. These functors are compatible via the pull-back functor corresponding to the projection
$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} {}_{\infty \cdot x}\overline{\mathrm{Bun}}_{B(M)} \to {}_{\infty \cdot x}\overline{\mathrm{Bun}}_{B(M)}.$$

For $\mathcal{F} \in D^b({}_{\infty \cdot x}\overline{\mathrm{Bun}}_B)$ we have a functorial isomorphism

$$(9.4) \qquad \imath_M^* \left((\mathrm{H'}{}_x^V (\mathcal{F})) \simeq \mathrm{H'}{}_{P,x}^U (\imath_M^*(\mathcal{F})) \right),$$

where $U = \mathrm{Res}_M^{\check{G}}(V)$.

Let us observe that when we apply (9.4) to $\mathcal{F} = \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}})$, the resulting isomorphism is compatible with the filtrations.

9.8. Let us now establish the extension property of the isomorphism (9.1) to a morphism (9.2) when $\check{\eta} - \check{\nu}$ is a multiple of a simple coroot, say $\check{\alpha}_i$. We take P to be the minimal parabolic, corresponding to the chosen vertex of the Dynkin diagram; the corresponding Levi M is of semi-simple rank one.

Note that the map

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \overline{\mathrm{Bun}}_{B(M)}^{\check{\eta}} \to \overline{\mathrm{Bun}}_B^{\check{\eta}},$$

induced by $\imath_M$, is an open embedding (which is true for any parabolic), and its image contains the stratum $X^{\check{\eta}-\check{\nu}} \times \mathrm{Bun}_B^{\check{\mu}-\check{\nu}}$.

The required assertion follows now from (9.4) and the fact that Theorem 8.8 holds for M.

9.9. Let us now prove Proposition 9.3. We argue by induction on $|\check{\eta} - \check{\nu}|$. The assertion is evidently true for $\check{\nu} = \check{\eta}$, and we assume that it holds for all $\check{\nu}' > \check{\nu}$. As in the proof of Proposition 4.9, a factorization argument reduces the assertion of the proposition to the fact that there are no non-zero morphisms

$$(9.5) \qquad \delta_x^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}}) \to \mathrm{H'}{}_x^{V^{\check{\eta}}} \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \right).$$

Let us suppose by contradiction that a morphism like this existed. Let us recall the filtration on $\mathrm{H'}{}_x^{V^{\check{\eta}}} \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \right)$ introduced in Sect. 8.7. The only subquotient that admits a map from $\delta_x^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}})$ is

$$\mathrm{gr}^{\check{\nu}} \left(\mathrm{H'}{}_x^{V^{\check{\eta}}} \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}}) \right) \right) \simeq V^{\check{\eta}}(\check{\nu}) \otimes \delta_x^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}}).$$

Hence, a map in (9.5) defines an element $\mathbf{v} \in V^{\check{\eta}}(\check{\nu})$. Let $\check{\alpha}_i$ be a simple root of $\check{\mathfrak{g}}$, for which $\mathbf{v}$ is *not* a highest weight vector. (Such $i \in I$ exists, for otherwise $\mathbf{v}$ would have been a highest weight vector for $\check{\mathfrak{g}}$, which is impossible, since $\check{\eta} \neq \check{\nu}$.)

Let us consider the restriction of the map (9.5) under $\imath_M$ for M being the Levi of the minimal parabolic, corresponding to the vertex i. We obtain a map

$$(9.6) \qquad \delta_x^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B(M)}^{\check{\mu}-\check{\nu}}}) \to \mathrm{H'}{}_{M,x}^U \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B(M)}^{\check{\mu}}}) \right),$$

whose projection to the subquotient

$$\mathrm{gr}^{\check{\nu}} \left(\mathrm{H'}{}_{M,x}^U \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B(M)}^{\check{\mu}}}) \right) \right) \simeq U(\check{\nu}) \otimes \delta_x^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B(M)}^{\check{\mu}-\check{\nu}}}),$$

corresponds to the same vector $\mathbf{v} \in U := \mathrm{Res}_M^{\check{G}}(V^{\check{\eta}})$.

But this is a contradiction: we know that in the rank one situation the only non-trivial maps as in (9.6) correspond to *highest weight vectors* in $U(\check{\nu})$.

9.10. Finally, let us prove point (C) of Theorem 8.8. By induction, we can assume that the assertion holds for all $\check{\nu}_1 < \check{\nu}$. Let us note that the penultimate terms of the filtration on both sides of (8.7) are isomorphic to the intermediate extensions of their own restrictions to the open substack (9.3).

Using the Zastava model and factorization, this implies that the map (8.7) is compatible with filtrations, and that the induced maps on $\mathrm{gr}^{\check{\nu}'}$ for $\check{\nu}' < \check{\nu}$ are as required. The map on the last term of the filtration

$$V(\check{\nu}) \otimes \delta_x^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}}) \to V(\check{\nu}') \otimes \delta_x^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}-\check{\nu}}})$$

corresponds, therefore, to an endomorphism $\phi(\check{\nu})$ of $V(\check{\nu})$.

We have to show that $\phi(\check{\nu})$ equals the identity. For that it is enough to show that for every simple root $\check{\alpha}_i$ of $\check{\mathfrak{g}}$ the diagram

(9.7)
$$\begin{array}{ccc}
\check{\mathfrak{n}}^{\check{\alpha}_i} \otimes V(\check{\nu}) & \longrightarrow & V(\check{\nu} + \check{\alpha}_i) \\
{\scriptstyle \mathrm{id} \otimes \phi(\check{\nu})} \downarrow & & {\scriptstyle \mathrm{id}} \downarrow \\
\check{\mathfrak{n}}^{\check{\alpha}_i} \otimes V(\check{\nu}) & \longrightarrow & V(\check{\nu} + \check{\alpha}_i)
\end{array}$$

commutes.

Let P and M be as above. Note that the substack

$$\left(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} {}_{\infty \cdot x}\overline{\mathrm{Bun}}_{B(M)} \right) \underset{\infty \cdot x \overline{\mathrm{Bun}}_B}{\times} \left(X^{\check{\eta}-\check{\nu}} \times \mathrm{Bun}_B^{\check{\mu}-\check{\nu}} \right)$$

identifies with

$$X^{(n_i)} \times \mathrm{Bun}_B^{\check{\mu}-\check{\nu}} \subset X^{\check{\eta}-\check{\nu}} \times \mathrm{Bun}_B^{\check{\mu}-\check{\nu}},$$

where $n_i = \langle \alpha_i, \check{\eta} - \check{\nu} \rangle$. The 0-th perverse cohomology of the *-restriction on $\Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\nu}}$ under

$$X^{(n_i)} \hookrightarrow X^{\check{\eta}-\check{\nu}}$$

identifies with the corresponding perverse sheaf $\Omega(\check{\mathfrak{n}}_X^{\check{\alpha}_i}, U_x)^{\check{\nu}}$ on $X^{(n_i)}$. It can be also thought of as the maximal quotient of $\Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\nu}}$ supported on the above subscheme, and it equals the quotient of $\Omega(\check{\mathfrak{n}}_X, V_x)^{\check{\nu}}$ corresponding to the terms of the canonical filtration with $\check{\nu}' \in \check{\eta} - \check{\alpha}_i \cdot \mathbb{Z}^{\geq 0}$.

Let us apply the functor $\imath_M^*$ to the morphism (8.7). We obtain a map

(9.8)
$$\Omega(\check{\mathfrak{n}}_X^{\check{\alpha}_i}, U_x)^{\check{\nu}} \star \jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B(M)}^{\check{\mu}-\check{\nu}}}) \to \mathrm{H}'{}_{M,x}^U \left(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_{B(M)}^{\check{\mu}}}) \right).$$

The restriction of this map to

$$(X^{(n_i)} - n_i \cdot x) \times \mathrm{Bun}_{B(M)}^{\check{\mu}-\check{\nu}} \subset X^{(n_i)} \times \mathrm{Bun}_{B(M)}^{\check{\mu}-\check{\nu}}$$

equals the map (8.7) for the group M, by the induction hypothesis. Hence, by Proposition 9.3 (applied to M), the map (9.8) equals (8.7), at least when restricted to the direct summand, corresponding to the direct summand of $U = \mathrm{Res}_M^{\check{G}}(V^{\check{\eta}})$, denoted $U^{\neq \check{\nu}}$, complementary to the isotypic component of highest weight $\check{\nu}$.

This implies that the map $\phi : V(\check{\nu}) \simeq U(\check{\nu}) \to U(\check{\nu}) \simeq V(\check{\nu})$ equals the identity map, when restricted to $U^{\neq \check{\nu}}$. This implies the commutativity of (9.7).

10. Regular Eisenstein series are perverse

10.1. The goal of this section is to prove Theorem 1.5. Thus, we suppose that $E_{\check{T}}$ is a $\check{T}$-local system, which is regular, i.e., $\alpha(E_{\check{T}})$ is non-trivial for all $\alpha \in \Delta^+$.

It is easy to see that we can replace the initial group G by an isogenous one with a connected center, and for which $[G, G]$ is still simply connected. In this case, the assumption that $E_{\check{T}}$ is regular implies that it is *strongly regular*, i.e., that $E_{\check{T}}^w \neq E_{\check{T}}$ for any non-trivial element w of the Weyl group.

In this case, the following strengthening of Theorem 1.5 will hold:

Theorem 10.2. *Assume that $E_{\check{T}}$ is strongly regular. Then:*

(1) *The complex $\overline{\mathrm{Eis}}^{\check{\mu}}(E_{\check{T}})$ is an irreducible perverse sheaf.*

(2) *For $\check{\mu}' \neq \check{\mu}$, the perverse sheaves $\overline{\mathrm{Eis}}^{\check{\mu}}(E_{\check{T}})$ and $\overline{\mathrm{Eis}}^{\check{\mu}'}(E_{\check{T}})$ are non-isomorphic.*

(3) *The complex $\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})$ is also a perverse sheaf.*

The main step in the proof is the following:

Proposition 10.3. $R^0 \mathrm{Hom}(\overline{\mathrm{Eis}}^{\check{\mu}}(E_{\check{T}}), \overline{\mathrm{Eis}}^{\check{\mu}'}(E_{\check{T}}))$ *is 1-dimensional for $\check{\mu} = \check{\mu}'$ and vanishes for $\check{\mu} \neq \check{\mu}'$.*

Let us first explain how Proposition 10.3 implies Theorem 10.2. Indeed, by Kashiwara's conjecture (=Drinfeld's theorem, see [Dr]), the complex $\overline{\mathrm{Eis}}^{\check{\mu}}(E_{\check{T}})$ is semi-simple and satisfies Hard Lefschetz. Hence, the first assertion of the corollary implies point (1) of the theorem. Point (2) of the theorem follows from the second assertion of the corollary.

Finally, to see that $\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})$ is perverse, recall that by Sect. 4, the complex

$$ \jmath_! \left(\mathrm{IC}_{\mathrm{Bun}_B^{\check{\mu}}} \otimes (\mathfrak{q}^{\check{\mu}})^* (\mathcal{S}(E_{\check{T}})) \right) $$

on $\overline{\mathrm{Bun}}_B^{\check{\mu}}$ admits a filtration by complexes of the form

$$ (\bar{\imath}_{\check{\mu}'-\check{\mu}})_! \left(\Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\mu}-\check{\mu}'} \boxtimes \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\check{\mu}'}} \otimes (\mathfrak{q}^{\check{\mu}'})^* (\mathcal{S}(E_{\check{T}})) \right) \right) . $$

Hence, $\mathrm{Eis}_!^{\check{\mu}}(E_{\check{T}})$ admits a filtration by complexes of the form

$$ \overline{\mathrm{Eis}}^{\check{\mu}'}(E_{\check{T}}) \otimes H(X^{\check{\mu}'-\check{\mu}}, \Omega(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\mu}-\check{\mu}'}), $$

which are all perverse sheaves by point (1) of Theorem 10.2 and Corollary 3.4.

Remark. Let us assume that the validity of the following (very plausible) extension of Kashiwara's conjecture:

Conjecture 10.4. *Let $f : Y' \to Y''$ be a proper map between schemes over an algebraically closed field, and let $\mathcal{F}$ be an irreducible ℓ-adic perverse sheaf on Y', which can be obtained by the 6 operations from a 1-dimensional local system on some curve X. Then $f_!(\mathcal{F})$ is semi-simple and satisfies Hard Lefschetz.*

Then the above argument deducing Theorem 1.5 from Proposition 10.3 would be valid in the context of ℓ-adic sheaves over an arbitrary ground field.

10.5. From now on our goal is to prove Proposition 10.3. We shall deduce is from another proposition, which amounts to a computation of the constant term of Eisenstein series in the geometric context:

Proposition 10.6. $R^i \operatorname{Hom}(\operatorname{Eis}_!^{\check{\mu}}(E_{\check{T}}), \overline{\operatorname{Eis}}^{\check{\mu}'}(E_{\check{T}})) = 0$ *if* $i < 0$. *For* $i = 0$ *it vanishes if* $\check{\mu} \neq \check{\mu}'$ *and is* 1-*dimensional for* $\mu = \mu'$.

This proposition implies Proposition 10.3 as follows:

The perverse sheaf $\operatorname{IC}_{\overline{\operatorname{Bun}}_B^{\check{\mu}}} \otimes \overline{\mathfrak{q}}^*(\mathcal{S}(E_{\check{T}}))$ admits a filtration in the derived category by complexes of the form

$$(\imath_{\check{\mu}'-\check{\mu}})_! \left(\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet, \check{\mu}'-\check{\mu}, *} \boxtimes \left(\operatorname{IC}_{\operatorname{Bun}_B^{\check{\mu}'}} \otimes (\mathfrak{q}^{\check{\mu}'})^*(\mathcal{S}_E) \right) \right).$$

Hence, $\overline{\operatorname{Eis}}^{\check{\mu}}(E_{\check{T}})$ admits a filtration by complexes of the form

$$H \left(X^{\check{\mu}'-\check{\mu}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet, \check{\mu}'-\check{\mu}, *} \right) \otimes \operatorname{Eis}_!^{\check{\mu}'}(E_{\check{T}}).$$

However, since $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\check{\mu}'-\check{\mu}}$ is a *sheaf*, its cohomology is concentrated in non-negative degrees. Hence, $H \left(X^{\check{\mu}'-\check{\mu}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\check{T}}})^{\bullet, \check{\mu}'-\check{\mu}, *} \right)$ is concentrated in non-positive degrees. Moreover, the fact that $E_{\check{T}}$ is regular implies that the above cohomology is concentrated in strictly negative degrees unless $\check{\mu} = \check{\mu}'$.

Therefore, $R^i \operatorname{Hom}(\operatorname{Eis}_!^{\check{\mu}}(E_{\check{T}}), \overline{\operatorname{Eis}}^{\check{\mu}'}(E_{\check{T}})) = 0$ for $i < 0$, as guaranteed by Proposition 10.6, implies that

$$R^0 \operatorname{Hom}(\overline{\operatorname{Eis}}^{\check{\mu}}(E_{\check{T}}), \overline{\operatorname{Eis}}^{\check{\mu}'}(E_{\check{T}})) \simeq R^0 \operatorname{Hom}(\operatorname{Eis}_!^{\check{\mu}}(E_{\check{T}}), \overline{\operatorname{Eis}}^{\check{\mu}'}(E_{\check{T}})),$$

and we deduce the assertion of Proposition 10.3.

10.7. **Proof of Proposition 10.6.** Let

$$\operatorname{CT}^{\check{\mu}} : D(\operatorname{Bun}_G) \to D(\operatorname{Bun}_T^{\check{\mu}})$$

be the constant term functor, the right adjoint to the functor $\operatorname{Eis}_!^{\check{\mu}}$ that sends

$$\mathcal{F} \in D^b(\operatorname{Bun}_T^{\check{\mu}}) \mapsto \mathfrak{p}_!^{\check{\mu}} \circ \mathfrak{q}^{\check{\mu}*}(\mathcal{F})[\dim(\operatorname{Bun}_B^{\check{\mu}})] \in D^b(\operatorname{Bun}_G).$$

The functor $\operatorname{CT}^{\check{\mu}}$ is thus given by

$$\mathcal{F}' \in D^b(\operatorname{Bun}_G) \mapsto \mathfrak{q}_*^{\check{\mu}} \circ \mathfrak{p}^{\check{\mu}!}(\mathcal{F}')[-\dim(\operatorname{Bun}_B^{\check{\mu}})] \in D^b(\operatorname{Bun}_T^{\check{\mu}}),$$

which is well-defined, since $\operatorname{Bun}_B^{\check{\mu}}$ is of finite type and the morphism $\mathfrak{q}^{\check{\mu}}$ is a generalized affine fibration.

Thus, we have to compute $R \operatorname{Hom} \left(\mathcal{S}^{\check{\mu}}(E_{\check{T}}), \operatorname{CT}^{\check{\mu}}(\overline{\operatorname{Eis}}^{\check{\mu}'}(E_{\check{T}})) \right)$. Consider the diagram

$$
\begin{array}{ccccc}
\operatorname{Bun}_B^{\check{\mu}} \underset{\operatorname{Bun}_G}{\times} \overline{\operatorname{Bun}}_B^{\check{\mu}'} & \xrightarrow{\mathfrak{p}'^{\check{\mu}}} & \overline{\operatorname{Bun}}_B^{\check{\mu}'} & \xrightarrow{\overline{\mathfrak{q}}^{\check{\mu}'}} & \operatorname{Bun}_T^{\check{\mu}'} \\
\overline{\mathfrak{p}}'^{\check{\mu}'} \downarrow & & \overline{\mathfrak{p}}^{\check{\mu}'} \downarrow & & \\
\operatorname{Bun}_B^{\check{\mu}} & \xrightarrow{\mathfrak{p}^{\check{\mu}}} & \operatorname{Bun}_G & & \\
\mathfrak{q}^{\check{\mu}} \downarrow & & & & \\
\operatorname{Bun}_T^{\check{\mu}} . & & & &
\end{array}
$$

By base change, and using the fact that $\overline{\mathfrak{p}}^{\breve{\mu}'}$ is proper, $\mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))$ is the direct image onto $\mathrm{Bun}_T^{\breve{\mu}}$ from $\mathrm{Bun}_B^{\breve{\mu}} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B^{\breve{\mu}'}$, of the complex

$$(10.1) \qquad \mathfrak{p}'^{\breve{\mu}!} \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B^{\breve{\mu}'}} \otimes (\overline{\mathfrak{q}}^{\breve{\mu}'})^*(\mathcal{S}(E_{\breve{T}})) \right) [-\dim(\mathrm{Bun}_B^{\breve{\mu}})].$$

The stack $\mathrm{Bun}_B^{\breve{\mu}} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B^{\breve{\mu}'}$ is the union of locally closed substacks $(\mathrm{Bun}_B^{\breve{\mu}} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B^{\breve{\mu}'})^w$ numbered by elements w of the Weyl group that measure the relative position of the two flags at the generic point of the curve. We obtain a filtration on $\mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))$ as an object of the derived category, whose subquotients we will denote by $\mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))^w$.

Proposition 10.8. *The complex* $\mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))^w$ *on* $\mathrm{Bun}_T^{\breve{\mu}}$ *is an extension in the derived category of complexes isomorphic to* $\mathcal{S}_{E_{\breve{T}}^w}$, *where* $E_{\breve{T}}^w$ *is the w-twist of* $E_{\breve{T}}$.

This proposition will be proved in the next subsection. Assuming it, let us finish the proof of Proposition 10.6. First, we obtain that

$$R\,\mathrm{Hom}\left(\mathcal{S}^{\breve{\mu}}(E_{\breve{T}}), \mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))^w \right) = 0$$

for any $1 \neq w \in W$. Indeed, since $E_{\breve{T}}$ is strongly regular, the local systems $\mathcal{S}_{E_{\breve{T}}^w}$ and $\mathcal{S}_{E_{\breve{T}}}$ are non-isomorphic, and hence $R\,\mathrm{Hom}$ between them over Bun_T, which is essentially an abelian variety, vanishes.

Thus, it remains to analyze $\mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))^1$. We have;

$$(\mathrm{Bun}_B^{\breve{\mu}} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B^{\breve{\mu}'})^1 \simeq X^{\breve{\mu}'-\breve{\mu}} \times \mathrm{Bun}_B^{\breve{\mu}},$$

and the map $\mathfrak{p}'^{\breve{\mu}}$ identifies with $\imath_{\breve{\mu}'-\breve{\mu}}$. In terms of this identification, the !-restriction of the complex in (10.1) to $(\mathrm{Bun}_B^{\breve{\mu}} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_B^{\breve{\mu}'})^1$ becomes

$$\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\breve{T}}})^{\breve{\mu}'-\breve{\mu}} \boxtimes \mathrm{IC}_{\mathrm{Bun}_B^{\breve{\mu}}}[-\dim(\mathrm{Bun}_B^{\breve{\mu}})].$$

Since $\mathrm{Bun}_B^{\breve{\mu}} \to \mathrm{Bun}_T^{\breve{\mu}}$ is a generalized affine fibration, we obtain:

$$\mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))^1 \simeq \mathcal{S}^{\breve{\mu}}(E_{\breve{T}}) \otimes H\left(X^{\breve{\mu}'-\breve{\mu}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\breve{T}}})^{\breve{\mu}'-\breve{\mu}} \right).$$

However, since $\mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\breve{T}}})^{\breve{\mu}'-\breve{\mu}}$ is a sheaf, the cohomology $H\left(X^{\breve{\mu}'-\breve{\mu}}, \mathfrak{U}(\check{\mathfrak{n}}_{X,E_{\breve{T}}})^{\breve{\mu}'-\breve{\mu}} \right)$ is concentrated in non-negative degrees (and strictly positive if $\breve{\mu} \neq \breve{\mu}'$ since $E_{\breve{T}}$ was assumed regular).

This implies that

$$R^i\,\mathrm{Hom}\left(\mathcal{S}^{\breve{\mu}}(E_{\breve{T}}), \mathrm{CT}^{\breve{\mu}}(\overline{\mathrm{Eis}}^{\breve{\mu}'}(E_{\breve{T}}))^1 \right) = 0$$

for $i < 0$ and for $i = 0$ and $\breve{\mu} \neq \breve{\mu}'$.

10.9. **Proof of Proposition 10.8.** Let $\mathrm{Eis}_*^{\breve{\mu}'}(E_{\breve{T}})$ denote $\mathfrak{p}_*^{\breve{\mu}} \left(\mathrm{IC}_{\overline{\mathrm{Bun}}_B} \otimes (\mathfrak{q}^{\breve{\mu}})^*(\mathcal{S}(E_{\breve{T}})) \right)$. I.e., we replace the functor $\mathfrak{p}_!^{\breve{\mu}}$ in the definition of $\mathrm{Eis}_!^{\breve{\mu}'}(E_{\breve{T}})$ by $\mathfrak{p}_*^{\breve{\mu}}$.

By Corollary 4.5 and Verdier duality, it is enough to calculate $\mathrm{CT}^{\breve{\mu}}(\mathrm{Eis}_*^{\breve{\mu}'}(E_{\breve{T}}))^w$. This will be parallel to the calculation of the w-term in the expression for the constant term of Eisenstein series in the classical theory of automorphic forms. Recall that in this theory the sought-for expression is known explicitly, and equals (the function corresponding to) $\mathcal{S}(E_{\breve{T}}^w)$ multiplied by an appropriate ratio of L-functions. Unfortunately, in the present geometric context we will

not be able to obtain an explicit formula for $CT^{\check\mu}(Eis_*^{\check\mu'}(E_{\check T}))^w$; we will only show that it is isomorphic to $S(E_{\check T}^w)$ times *some* complex of vector spaces.

Let Fl denote the flag variety G/B, and let Fl_w (resp., $\overline{Fl}_w$) denote the Schubert cell, corresponding to w (resp., its closure). Let us denote the corresponding locally closed substack in $(Bun_B \underset{Bun_G}{\times} Bun_B)^w$ (resp., $(Bun_B^{\check\mu} \underset{Bun_G}{\times} Bun_B^{\check\mu'})^w$) by $\mathcal{Z}_w$ (resp., $\mathcal{Z}_w^{\check\mu,\check\mu'}$). For $w = w_0$ this stack is closely related to the Zastava spaces $\mathcal{Z}^{\check\mu}$ introduced in Sect. 4.3.

By definition, the stack $\mathcal{Z}_w$ classifies the data of pairs $(\mathcal{F}_B, \sigma)$, where $\mathcal{F}_B$ is a B-torsor on X, and σ is a section of $\mathcal{F}_B \overset{B}{\times} \overline{Fl}_w$, such that, over the generic point of the curve, σ hits $\mathcal{F}_B \overset{B}{\times} Fl_w$. Let us call the locus of X where σ does *not* hit $\mathcal{F}_B \overset{B}{\times} Fl_w$ the "locus of degeneration" of $(\mathcal{F}_B, \sigma)$. We shall now construct a map $\pi^{\check\mu,\check\mu',w} : \mathcal{Z}_w^{\check\mu,\check\mu'} \to X^{w(\check\mu')-\check\mu}$, such that for $(\mathcal{F}_B, \sigma)$ as above, the support of the divisor $\pi^{\check\mu,\check\mu',w}(\mathcal{F}_B, \sigma)$ is its locus of degeneration.

For each irreducible G-module V^λ, let $V^{\lambda,\geq w}$ be the canonical B-stable subspace, spanned by vectors with weights $\geq w(\lambda)$. Let $V^{\lambda,> w} \subset V^{\lambda,\geq w}$ be the codimension-1 subspace, spanned by vectors with weights $> w(\lambda)$. Then, a point of Fl, which gives a line $\mathfrak{l}^\lambda$ in each V^λ, belongs to $\overline{Fl}_w$ if and only if $\mathfrak{l}^\lambda \in V^{\lambda,\geq w}$ for each $\lambda \in \Lambda^+$, and it belongs to Fl_w if the image of $\mathfrak{l}^\lambda$ in $V^{\lambda,\geq w}/V^{\lambda,> w}$ is non-zero for every λ.

Similarly, a data of $(\mathcal{F}_B, \sigma)$, where σ is a section of $\mathcal{F}_B \overset{B}{\times} Fl$ defines a sub-bundle for every $\lambda \in \Lambda^+$

$$\mathcal{L}^\lambda \hookrightarrow V_{\mathcal{F}_B}^\lambda,$$

and such a point belongs to $\mathcal{Z}_w$ if and only if $\mathcal{L}^\lambda$ belongs to $V_{\mathcal{F}_B}^{\lambda,\geq w}$ and the composed map of line bundles

$$(10.2) \qquad \mathcal{L}^\lambda \to V_{\mathcal{F}_B}^{\lambda,\geq w} \to \left(V^{\lambda,\geq w}/V^{\lambda,> w}\right)_{\mathcal{F}_B}$$

is non-zero. Moreover, the locus of degeneration of $(\mathcal{F}_B, \sigma)$ is where the maps (10.2) have zeroes.

If $(\mathcal{F}_B, \sigma)$ belongs to the component $\mathcal{Z}_w^{\check\mu,\check\mu'}$ of $\mathcal{Z}_w$, the degree of the line bundle $\mathcal{L}^\lambda$ is by definition $\langle\lambda,\check\mu\rangle$ and that of $\left(V^{\lambda,\geq w}/V^{\lambda,> w}\right)_{\mathcal{F}_B}$ equals $\langle\lambda, w(\check\mu')\rangle$. The divisors of zeroes of the maps (10.2) for all $\lambda \in \Lambda^+$ are encoded by a point of $X^{w(\check\mu')-\check\mu}$. This point is, by definition, the sought-for $\pi^{\check\mu,\check\mu',w}(\mathcal{F}_B, \sigma)$.

Let us denote by $\mathfrak{q}^{\check\mu}$ (resp., $\mathfrak{q}^{\check\mu'}$) the natural map from $\mathcal{Z}_w^{\check\mu,\check\mu'}$ to $Bun_T^{\check\mu}$ (resp., $Bun_T^{\check\mu'}$). Note that $\mathfrak{q}^{\check\mu'}$ equals the composition

$$\mathcal{Z}_w^{\check\mu,\check\mu'} \overset{\mathfrak{q}^{\check\mu} \times \pi^{\check\mu,\check\mu',w}}{\longrightarrow} Bun_T^{\check\mu} \times X^{w(\check\mu')-\check\mu} \overset{id \times AJ}{\longrightarrow} Bun_T^{\check\mu} \times Bun_T^{w(\check\mu')-\check\mu} \to Bun_T^{w(\check\mu')} \overset{w^{-1}}{\to} Bun_T^{\check\mu'}.$$

Applying the Verdier duality and the projection formula, we obtain that in order to show that $CT^{\check\mu}(Eis_*^{\check\mu'}(E_{\check T}))^w$ has the desired form, it is sufficient to prove the following:

Proposition 10.10. *The direct image with compact supports of the constant sheaf along the map*

$$\mathfrak{q}^{\check\mu} \times \pi^{\check\mu,\check\mu',w} : \mathcal{Z}_w^{\check\mu,\check\mu'} \to Bun_T^{\check\mu} \times X^{w(\check\mu')-\check\mu}$$

is an extension of complexes, each of which is a pull-back of a complex on the second multiple (i.e., $X^{w(\check\mu')-\check\mu}$).

10.11. The rest of this section is devoted to the proof of Proposition 10.10. Fix a point $x \in X$, and consider the open substack

$$\operatorname{Bun}_T^{w(\check{\mu})} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \subset \operatorname{Bun}_T^{w(\check{\mu})} \times X^{w(\check{\mu}') - \check{\mu}},$$

and let us analyze the restriction to it of the complex $(\mathfrak{q}^{\check{\mu}} \times \pi^{\check{\mu}, \check{\mu}', w})_! (\mathbb{C}_{\mathcal{Z}_w^{\check{\mu}, \check{\mu}'}})$.

Let us choose a subgroup $N' \subset N$, normalized by T, in such a way that its action on Fl_w is simply-transitive. Let $B' = N' \cdot T$ be the corresponding subgroup of B. Consider the sheaves of groups $\operatorname{Maps}(X, B') \subset \operatorname{Maps}(X, B)$.

By conjugating $B(\widehat{\mathcal{O}}_x)$ inside $B(\widehat{\mathcal{K}}_x)$ by an element of $T(\widehat{\mathcal{K}}_x)$, [5] we can modify both these sheaves at the point x, and obtain sheaves of groups $\widetilde{\operatorname{Maps}}(X, B') \subset \widetilde{\operatorname{Maps}}(X, B)$ in such a way that the forgetful map

$$\operatorname{Bun}_B^{\check{\mu}} \underset{\left(\widetilde{\operatorname{Maps}}(X,B)\text{-tors}\right)}{\times} \left(\widetilde{\operatorname{Maps}}(X, B')\text{-tors}\right) \to \operatorname{Bun}_B^{\check{\mu}}$$

is a smooth generalized affine fibration.

Let us denote by $\widetilde{\mathcal{Z}}_w^{\check{\mu}, \check{\mu}'}$ the Cartesian product

$$(X - x)^{w(\check{\mu}') - \check{\mu}} \underset{X^{w(\check{\mu}') - \check{\mu}}}{\times} \mathcal{Z}_w^{\check{\mu}, \check{\mu}'} \underset{\operatorname{Bun}_B^{\check{\mu}}}{\times} \left(\operatorname{Bun}_B^{\check{\mu}} \underset{\left(\widetilde{\operatorname{Maps}}(X,B)\text{-tors}\right)}{\times} \left(\widetilde{\operatorname{Maps}}(X, B')\text{-tors}\right) \right),$$

and by $\widetilde{\mathfrak{q}}^{\check{\mu}}$ (resp, $\widetilde{\pi}^{\check{\mu}, \check{\mu}', w}$) its map to $\operatorname{Bun}_T^{\check{\mu}}$ (resp., $(X - x)^{w(\check{\mu}') - \check{\mu}}$). It suffices to analyze the object $(\widetilde{\mathfrak{q}}^{\check{\mu}} \times \widetilde{\pi}^{\check{\mu}, \check{\mu}', w})_! (\mathbb{C}_{\widetilde{\mathcal{Z}}_w^{\check{\mu}, \check{\mu}'}})$.

We will show that after a suitable base change, in the map

$$\widetilde{\mathfrak{q}}^{\check{\mu}} \times \widetilde{\pi}^{\check{\mu}, \check{\mu}', w} : \widetilde{\mathcal{Z}}_w^{\check{\mu}, \check{\mu}'} \to \left(\operatorname{Bun}_T^{\check{\mu}} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \right),$$

$\operatorname{Bun}_T^{\check{\mu}}$ splits off a direct factor.

More precisely, we shall exhibit a smooth morphism with connected fibers (in fact, a principal bundle with respect to a connected group-scheme)

$$\left(\operatorname{Bun}_T^{\check{\mu}} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \right)' \to \left(\operatorname{Bun}_T^{\check{\mu}} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \right)$$

and a stack $\mathcal{W}_w^{\check{\mu}, \check{\mu}'}$ over $(X - x)^{w(\check{\mu}') - \check{\mu}}$ such that there exists a Cartesian diagram

$$\begin{array}{ccc}
\left(\operatorname{Bun}_T^{\check{\mu}} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \right)' \underset{\left(\operatorname{Bun}_T^{\check{\mu}} \times (X-x)^{w(\check{\mu}') - \check{\mu}} \right)}{\times} \widetilde{\mathcal{Z}}_w^{\check{\mu}, \check{\mu}'} & \longrightarrow & \mathcal{W}_w^{\check{\mu}, \check{\mu}'} \\
\downarrow & & \downarrow \\
\left(\operatorname{Bun}_T^{\check{\mu}} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \right)' & \longrightarrow & (X - x)^{w(\check{\mu}') - \check{\mu}}.
\end{array}$$

Namely, we take $\left(\operatorname{Bun}_T^{\check{\mu}} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \right)'$ to be the stack, whose fiber over a point

$$(\mathcal{F}_T, D) \in \left(\operatorname{Bun}_T^{\check{\mu}} \times (X - x)^{w(\check{\mu}') - \check{\mu}} \right)$$

is the scheme of (sufficiently high) level structures of the T-torsor $\mathcal{F}_T$ over the support of D.

The sought-for scheme $\mathcal{W}_w^{\check{\mu}, \check{\mu}'}$ is constructed as follows. It classifies triples $(\mathcal{F}_{N'}, D, \sigma')$, where:

- $\mathcal{F}_{N'}$ is a principal N'-bundle on $X - x$,

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[5] Here $\widehat{\mathcal{O}}_x$ (resp., $\widehat{\mathcal{K}}_x$) is the local ring (resp., field) corresponding to the point $x \in X$.

- D is a point of $(X-x)^{w(\check\mu')-\check\mu}$, and
- σ' is a section of $\mathcal{F}_{N'} \overset{N'}{\times} \overline{\mathrm{Fl}}_w$, such that the resulting divisor (obtained as in (10.2)) equals D.

In other words, $\mathcal{W}_w^{\check\mu,\check\mu'}$ is the scheme, classifying maps from $X - x$ to the stack $\overline{\mathrm{Fl}}_w/N'$ of a given degree (cf. [BFG], Sect. 2.16).

11. Proof of Theorem 3.6

We will present a short argument, pointed out to us by A. Beilinson, that proves the theorem and simultaneously makes the assertion of Lemma 8.4 manifest.

11.1. Let us construct the map

$$(11.1) \qquad \widehat{\mathcal{O}}_{\mathrm{Def}_{\check B}(E_{\check T})} \to \widehat{R}_{E_{\check T}},$$

i.e., a map

$$(11.2) \qquad \mathrm{Spf}(\widehat{R}_{E_{\check T}}) \to \mathrm{Def}_{\check B}(E_{\check T}).$$

We can represent the topological algebra $\widehat{R}_{E_{\check T}}$ as $\varprojlim_{n\in\mathbb{Z}^{\geq 0}} R_{E_{\check T}}^n$, where

$$R_{E_{\check T}}^n := \underset{\check\lambda\in\check\Lambda^{pos},|\check\lambda|=n}{\oplus} \widehat{R}_{E_{\check T}}^{-\check\lambda}.$$

Thus, we need to exhibit a compatible family of $R_{E_{\check T}}^n$-points of the functor $\mathrm{Def}_{\check B}(E_{\check T})$. This means that for each n we need to exhibit a tensor functor

$$\mathsf{F}_n : \{\check B\text{-mod}\} \to \{\text{local systems of } R_{E_{\check T}}^n\text{-modules over } X\},$$

with identifications

$$\mathsf{F}_n(V) \underset{R_{E_{\check T}}^n}{\otimes} \mathbb{C} \simeq V_{E_{\check T}} \text{ for } V \in \check B\text{-mod}$$

and

$$\mathsf{F}_n(V') \simeq R_{E_{\check T}}^n \otimes V'_{E_{\check T}} \text{ for } V' \in \check T\text{-mod} \subset \check B\text{-mod}.$$

11.2. The sought-for functor F_n is constructed as follows. First, for $V \in \check B$-mod and $\check\nu \in \check\Lambda$, let $\Omega(\check{\mathfrak{n}}_{X,E_{\check T}}, V_{E_{\check T}})$ be the relative version of $\Omega(\check{\mathfrak{n}}_{X,E_{\check T}}, V_{E_{\check T},x})$, which is a perverse sheaf on the corresponding ind-scheme $_\infty(X \times X^{\check\nu})$. Let $R(V)_{E_{\check T}}^{\check\nu}$ be its direct image under the natural projection

$$_\infty(X \times X^{\check\nu}) \to X.$$

We have the natural maps $\widehat{R}_{E_{\check T}}^{-\check\lambda} \otimes R(V)_{E_{\check T}}^{\check\nu} \to R(V)_{E_{\check T}}^{\check\nu-\check\lambda}$. Consider $R(V)_{E_{\check T}} := \underset{\check\nu}{\oplus} R(V)_{E_{\check T}}^{\check\nu}$ as a $\check\Lambda$-graded local system on X; it is acted on by $R_{E_{\check T}}$. Set

$$R(V)_{E_{\check T}}^n := R(V)_{E_{\check T}} \underset{R_{E_{\check T}}}{\otimes} R_{E_{\check T}}^n;$$

this is an $R_{E_{\check T}}^n$-local system on X and $V \mapsto R(V)_{E_{\check T}}^n$ provides the desired functor F_n.

The functors F_n are evidently compatible for different n, and hence we obtain the morphism in (11.2). Note that by definition, for $V \in \check B$-mod, the pull-back of $V_{E_{\check B,\widehat{univ},x}}$ under this morphism identifies, by construction, with $\widehat{R}(V_x)_{E_{\check T}}$, i.e., Lemma 8.4 holds. Thus, it remains to show that the above map (11.2) is an isomorphism.

11.3. The regularity assumption on $E_{\check{T}}$ implies that the deformation theory of $E_{\check{B}}$ is unobstructed. Hence, $\widehat{\mathcal{O}}_{\mathrm{Def}_{\check{B}}(E_{\check{T}})}$ is (non-canonically) isomorphic to the completion of a polynomial algebra. Likewise, by Sect. 3.5, the algebra $R_{E_{\check{T}}}$ is regular, and hence $\widehat{R}_{E_{\check{T}}}$ is also (non-canonically) isomorphic to the completion of a polynomial algebra. Therefore, in order to show that the map (11.2) is an isomorphism, it is sufficient to do so at the level of tangent spaces at the maximal ideal.

From Sect. 3.5, it follows that $\widehat{R}_{E_{\check{T}}}/\mathfrak{m}^2_{\widehat{R}_{E_{\check{T}}}}$ identifies with

$$\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta^+} H^1(X, \mathfrak{n}^{\check{\alpha}}_{X,E_{\check{T}}})^* \simeq \mathbb{C} \oplus H^1(X, \mathfrak{n}_{X,E_{\check{T}}})^*,$$

where $\mathfrak{m}_{\widehat{R}_{E_{\check{T}}}}$ denotes the maximal ideal in $\widehat{R}_{E_{\check{T}}}$. I.e., the tangent space to $\mathrm{Spf}(\widehat{R}_{E_{\check{T}}})$ at its closed point identifies canonically with $H^1(X, \mathfrak{n}_{X,E_{\check{T}}})$.

In addition, by standard deformation theory, the tangent space to $\mathrm{Def}_{\check{B}}(E_{\check{T}})$ at its closed point also identifies with $H^1(X, \mathfrak{n}_{X,E_{\check{T}}})$. Moreover, it is easy to see from the construction of the map (11.2) that it induces the identity map on $H^1(X, \mathfrak{n}_{X,E_{\check{T}}})$, implying our assertion.

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