Purpose and Education: The Case of Mathematics

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The Case of Mathematics

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Abstract

Why do schools teach mathematics, and why do they teach the mathematics that they do? In this three-part dissertation, I argue that the justifications offered by national education systems are not convincing, and that students are tested on content whose purpose neither they nor their teachers clearly understand. In the first part of the dissertation, I propose a theoretical framework for understanding the content and pedagogy of school mathematics as a set of practices reflecting socio-political values, particularly in relation to labor and citizenship. Beginning with a critical study of history, I trace the origins of modern mathematics education, in the process unearthing common, unexamined assumptions regarding the place and form of mathematics education in contemporary society.

In the second part of the dissertation I use the above theoretical framework to re-examine the literature on mathematical word problems. Word problems have interested research because they operate at the intersection between mathematics, education, and labor. I argue that scholarly discussions of word problems have so far adopted unexamined assumptions regarding the role of history, the structure of everyday life, and the relationship between mathematics and other disciplines. Through the lens of political economy I examine these assumptions and offer new categories and explanation for understanding word problems.

In the final part of the dissertation, I apply my theoretical framework to practice. Using a dialogical approach, I present a group of undergraduate students and pre-service teachers with artifacts and problems that embody some of the defining tensions of mathematics

education. Through twelve weeks of in-depth discussion, fieldwork and exploration, students eventually arrive at a more critical understanding of the social purpose of mathematics and the impact of this purpose on its teaching and learning in various contexts. The results for the students include an expanded vision of the possibilities of mathematics, a radical critique of its place in society, and reports of reduced math anxiety as well as increased curiosity toward mathematics.

A Note to the Reader

The idiosyncratic format of this dissertation, which is intimately related to its content, requires a brief explanation. The three-article organization is due to a rarely-exploited bylaw of the Harvard Graduate School of Education which allows for three manuscripts to make a dissertation so long as those manuscripts share a unifying theme. When I began this work I certainly had a scholarly unifying theme – an investigation into the purpose of mathematics education – and I projected from the beginning that doing the theme justice would require at least three movements: developing a theory, addressing the existing academic discourse, and attempting to combine the theory with practice. The three articles correspond to these movements. In each piece I take up similar dilemmas, sometimes the very same set of phenomena, and examine them from different perspectives and according to different scholarly needs. Variations in prose and formatting style, in turn, correspond to the requirements imposed by the audience and the scholarly publication for which each article was intended.

There are more important reasons for rejecting the traditional dissertation format. While the particular topic of these articles is a re-examination of the purpose of mathematics education, the broader topic is the purpose of education in general. Education, I believe, can no longer be justified through simple idealistic references to practical goals or citizenship. Strict division of labor and social hierarchies do not allow for such justifications. I have expounded the case of mathematics in order to clarify, first and foremost for myself, the work involved in understanding the role of purpose in teaching

and learning. The first article clarifies how difficult it is to speak about purpose and purposefulness in an aspect of human life that is entirely embroiled in the creation and loss of traditions, in the social organism's subconscious attempt at shielding itself from both crisis and stagnation.

To address this topic, the last thing I wanted to do was to take for granted the basic categories of my investigation. In each article I show how educational research tends to rely on half-understood and barely-examined conceptions of such basic categories as "mathematics," "tradition," "education," and "real life." I do not pretend to have a full and final definition for these terms – which is not what is needed here anyway – nor do I mean to suggest that no one else has investigated them as they deserve to be investigated. As far as I know, however, no one has investigated all these categories and their underlying structures at the same time.

While dialectical thought is capable of taking up such an effort and even making headway in the limited space of a dissertation, it would be hopeless if shackled by the constraints of the traditional dissertation format which act, almost strictly, at the service of linear positivist thought. I could not, for example, dedicate a chapter to a lengthy and isolated literature review: my purpose was to cut through the previous literature, rather than merely build on it. Nor could I discuss my methodology in isolation from the content itself, because my method, though derivative, still had to be shaped in relation to my specific topic. Even my criteria for what constitutes educational practice could not strictly follow the pedagogies that have influenced me (in this case, Eleanor Duckworth's and

Paulo Freire's). My conception of teaching and learning in this case was not based on familiarizing students with certain materials, but rather on defamiliarizing aspects of the world that they had so far taken for granted. Last but not least, my prose – fleet-footed but iterative – also had to conform to the particular needs of this research.

I am, of course, not the only doctoral student who has had to struggle against the traditional dissertation format. I was lucky enough to have a committee that cared more about the strength of my arguments than about my adhering to closed-minded notions of what constitutes academic writing.

As my last act of rebellion against the traditional format, I have chosen to forgo an acknowledgments section. Conducting a list of persons to thank for their support, interest and ideas would only do injustice to a very large number of people who cannot be included in that list. It seems that the topic of this dissertation strikes a chord in almost anyone who hears about it. I have never discussed this work without the listeners – academics, teachers, students, people of all ages and professions – offering some sensible and significant question or experience. Many of these have influenced my thinking and writing. There are simply too many people to thank, even if I could remember all of them.

That said, I would like to submit to at least one tradition. I would like to dedicate whatever aspect of this work that is new and mine to my mother and to the memory of

my father. I hope this unfinished work can serve as a partial continuation of the important labor they took up many years ago and, in their turn, left unfinished.

Toward a Political Economy of Mathematics Education

HOUMAN HAROUNI

2015

Abstract:

Why do schools teach the mathematics that they do? In this essay, Houman Harouni argues that the justifications offered by national education systems are not convincing, and that students are tested on content whose purpose neither they nor their teachers clearly understand. He proposes a theoretical framework for understanding the content and pedagogy of school mathematics as a set of practices reflecting socio-political values, particularly in relation to labor and citizenship. Beginning with a critical study of history, Harouni traces the origins of modern math education to the early institutions in which mathematics served a clear utilitarian purpose, in the process unearthing common, unexamined assumptions regarding the place and form of mathematics education in contemporary society.

Introduction: The Why Questions of Mathematics

Nowhere in the discourse on math education can we come across a clear explanation for why schools teach the mathematics that they do. There is certainly plenty of literature on *how* and *what* to teach, but at the base of all these there seems to be a willingness to ignore the fundamental questions of what has become perhaps the most problematic subject matter in schooling. The New Common Core (2010), a set of content standards for American schools, for example, simply cites another study (National Research Council, 2009) as its justification for the importance of mathematics as a subject before moving on to make a vast array of recommendations on content and pedagogy. This latter study in turn passes the burden by citing other texts (e.g. National Research Council, 2001), which do not go beyond repeating the ready slogan that math is necessary to social and economic participation. Similar halls of mirrors are set up everywhere: special hiding places in which assumptions about mandated learning reproduce themselves, along with glossy 'new' curriculums, ad infinitum.[1]

The problem of unjustified school mathematics is not unique to the American and European contexts. In step with the globalization of mandatory schooling, it has reached near-universal status. Studies that purport to show the differences between various national contexts unwittingly achieve the opposite by highlighting the extreme similarity

[1] A decade before the New Common Core, the extremely similar standards proposed by the National Council of Teachers of Mathematic (2000) exhibited the same willingness to pose unjustified benchmarks for learning. Two decades earlier, in England, we observe the same tendency in the Mathematics Counts reports in England (Harouni, 2013).

of national curricula (see, for example, English & Bartolini Bussi, 2008; Leung, Lopez-Real, & Graf, 2006; Wong, Hai, & Lee, 2004). The fact that international evaluations manage to apply themselves to almost every country is assumed to mean that there is a basic set of universal mathematical skills that should be tested. What, however, it indicates is that there is a similar training that occurs across the world regardless of societal backgrounds. These similarities involve content, sequence of topics, and pedagogy. They revolve around certain core ideas regarding what mathematics is and does.

Let's consider the first few months of an average first-grader's math education. Her learning begins with counting—as opposed to, say, geometric reasoning or comparing quantities. She immediately learns to think of numbers as quantities of similar, individual objects (seven apples, ten oranges)—not measures or relationships. This type of reasoning is expressed and taught through a type of fairy tale—the word problem.

> Susan has 12 oranges. Her mother gives her 15 more. How many oranges does she have now?

The oranges have no identity of their own—once you pour them in a pile, you will not be able to tell one from the other. In the context of a word problem, they do not even have an actual group identity: we do not know where they come from or what Susan was doing with a dozen oranges in the first place, let alone why her mother should give her 15 more.

Before the child is able to count even up to 100, she is asked to perform arithmetic operations, and addition is universally the first operation she will learn. It will take

months before her teacher introduces subtraction, and sometimes a year or two before the child will look at multiplication and division. Until high school graduation, pen and paper will remain the dominant instruments of performing mathematics. Rulers, compasses, and protractors make their brief appearances and are quickly set aside in favor of abstract calculation problems. The world, the majority of math textbooks tell us, is full of things that demand immediate manipulation of their numbers.

This model of elementary school math is as resilient as it is prevalent. The basic curriculum has remained largely unchanged since the founding of modern schooling (National Council of Teachers of Mathematics, 1970; Phillips, 2011). It is at the base of most national curricula. At the same time, i.e. from the very first days of public schooling, a debate has raged around math education. There have always been reformers to call the dominant approach outdated and unsound and suggest different approaches. The result is a variety of alternative curriculums that partially challenge the assumptions of dominant school mathematics. The Montessori model, for example, presents numbers as differences in magnitude rather than quantity (Montessori, 1914); Waldorf schools teach that all numbers arise from and add up to a larger *whole*, an essential unity (Aeppli, 1986); the New Math curriculum based its definition of numbers on set theory (Hayden, 1981). The existence of these alternative models deepens the problem regarding the relationship between math and society: if—and this is only an unexamined assumption—math education fulfills a societal need, then an alternative approach indicates a different attitude toward those needs – toward labor and citizenship. What constitutes these differences? The persistence of the debate also indicates that the resilience of the

traditional model is not due to lack of know-how. What, then, is the source of its resilience?

Academic answers to these questions are often either purely utilitarian—studying the efficacy of a method in teaching a certain topic—or idealistic, setting up a vision of citizenship and then proposing pedagogies that seemingly correspond to that vision. The disconnect between these efforts and societal issues eventually leads to a strong suspicion: perhaps school math has *nothing* to do with usefulness; perhaps it is primarily a product of an education system whose main purpose is not learning, but socializing and certifying its students (e.g Lave, 1992; Lundin, 2010b). According to this explanation, the dominant format of math is dominant either because it enjoys the strength of tradition, or because it is a method best suited to keeping students from anti-hegemonic forms of thinking. Reform agendas, it follows, are mechanisms through which schooling maintains itself by venting out and neutralizing all radical critiques of its form and content (Lundin, 2010a). Such arguments, despite their usefulness in creating a strong political stance against the status quo, are purely negative. This is not a shortcoming in itself; however, this negativity relies on a great deal of vagueness. In order to further its critical agenda, it must ignore the kernel of truth that still exists in the idea that math, in one form or another, is still useful. This is why these explanations do not contain any element of redemption, even beyond schooling.

The first step toward a radical reformulation of math education is a genealogical understanding of current assumptions and practices. There is in this essay something of

the resistant student's most common question to his or her math teachers: "Why are we learning this?" Along with those students who, in face of relentless testing and intricate mechanisms of reward and punishment, continue to insist on their right to know, I want to suggest that the common answer "Because it's good for you!" is not good enough at all and should be treated with extreme suspicion. In this essay I will propose a framework for understanding such questions—a historical framework wherein we can begin to place ourselves before the phenomenon of contemporary math education

Addition Is Not Addition: Three Approaches

Let's begin where schools begin: with Susan's twelve oranges. The question of how many oranges she would have if her mother gave her fifteen more is represented as follows:

$$12+15 = ?$$

This format is by far the dominant way of teaching addition in classrooms. It is so familiar as to seem like the basic building block of mathematical thinking, rooted in the most basic social interactions. But what if we imagine something else, which is just as basic:

$$27 = ?$$

There are fundamental differences between these two questions. The former (12+15 = ?) is interested in a particular task; it demands to know what will happen if two numbers are added together. The equal sign in this problem is more or less a command to compute.[2] Coincidentally, this is also the calculator's understanding of the equal sign. By contrast, in the second format (27 = ?) the equal sign is asking a question regarding the *meaning* of number 27. In the first problem there can only be one answer. For 12 plus 15 to become anything other than 27 the world, it would appear, must come apart at the seams. In the second format, however, answers are innumerable, and increase as one's knowledge of arithmetic expands.

Another alternative, based on real-world interactions, challenges the faith one would place in either of the above formats. We might ask,12 of what? Or, 15 of what? What is the result of adding 12 oranges to 15 apples, for example? Adding 12 meters of rope to 15 square centimeters of wood will not give 27 of any unit. If we imagine numbers as referring to real objects in a real context, then the addition sign would rarely (only in very special circumstances) imply a simple accumulation of quantities: it would refer, instead, to a much more complicated process that can only be understood in its proper setting. From this perspective, the equal sign is neither a command to compute nor a question of meaning: It symbolizes an aspect of *labor* performed on *materials*.

I am not merely trying to suggest alternative ways of teaching addition, but rather that there is a qualitative difference between these three formats. Functionally, I have

[2] The vast majority of American elementary students think of the equal sign as a command to calculate, rather than as stating a relationship of equality (Li, Ding, Capraro, & Capraro, 2008).

described the difference by suggesting how each can teach a different attitude toward mathematics, but what I have not described are the contexts that give shape to each approach and the reasons why the first format has come to dominate the teaching of arithmetic everywhere. Such a description is a necessary step in order to free our thinking from its ahistorical foundations.

Three Historical Venues: Shop Floor, Grammar Schools, and Reckoning Schools

By the 16th century, in Western Europe, we discover mathematical learning as taking place in three, very different institutional settings.[3] The first, and the one least covered in history of education, is the institution of apprenticeship. In this setting, craftsmen learned their trade through direct contact and on-the-job training with a master and other apprentices. Many crafts involved what can loosely be thought of as mathematical skills: masons and carpenters, ship-builders and wheelwrights each had the need for a set of numerical or geometrical systems or maneuvers. However, it would be a mistake to think of these mathematical practices as separate from the actual work of these workshops. The carpenter's act of measuring planks, for example, might involve operations similar to what one learns in school today, but, the artisan's math is intertwined with the materials and instruments of his work (Smith, 2004). The ruler he uses defines the meaning of numbers for him. A plank of oak serves a different function and can bear a different weight than a plank of walnut, even if the measurements are precisely the same.

[3] My initial emphasis on Western Europe has an obvious reason: modern education, like the economic and political systems that support it, is essentially a European product. By shedding light on what was European, we might even achieve the happy side effect of emboldening what was not.

Within the institution of apprenticeship, the characteristics of performed labor defined the mode and content of the artisan's mathematical training. Education for artisans did not mean storing up knowledge to use at a later time: every learning rose out of performing or observing a useful function. The little we know about what happened on European shop floors tells us that the apprentice, after a year or so of observing the master and performing simple manual labor, would take up a simple commission that, once complete, would be sold and used (De Munck, 2007). The products of his labor predefined all his actions, including his learning.

The second place for learning math in 16th century Europe was the so-called grammar school.[4] These constitute a gray area in our account, because until late 16th century, most grammar schools did not teach mathematics (Howson, 1982; Struik, 1936). Their primary task, as their name implies, was the teaching of classical languages; even local languages were for centuries a secondary concern and only gained attention as Latin and Greek lost their prominence (Thompson, 1960). Grammar schools served to teach "culture" to the sons and sometimes the daughters of educated commoners: physicians, pastors, lawyers, and town officials. The occasional mathematics taught in grammar schools was closely tied to the type of mathematics taught in the universities of the time, which in turn corresponded to a knowledge of the classics, which included some mathematics (Howson, 1982; National Council of Teachers of Mathematics, 1970). By the age of thirteen or fourteen, students were done with their grammar school education. Those for

[4] Called Latin schools in Prussia and Netherlands, and *schools of the teaching orders* in France and Italy (Jackson, 1906).

whom the basics sufficed would join their families or a master for professional training (apothecaries, copyists, notaries, scrivners, stationers, etc.) Those bent for ecclesiastical or legal careers would enter the university.

As a rule, university math shirked any emphasis on calculation and instead focused on the relationship between numbers in "pure" form (i.e. more "27=?" than "12+15=?"). The textbooks used at the time in Europe began with an introduction to Arabic numerals, which spanned no more than two or three pages, and then moved on to brief definitions of various types of numbers (odd, even, prime, etc. – i.e., patterns). Geometry, which was heavily Euclidean, took up the bulk of the students' learning in mathematics. Arithmetic texts written for use in universities or grammar schools treated the subject quite theoretically. They emphasized definitions rather than application, rarely contained sample problems from daily life, and concentrated on the logical and intuitive relationships between numbers (Jackson, 1906).

However, neither of these two venues—universities and shop floors—resemble the dominant modern form of teaching elementary mathematics. In these two models, there is no intensive teaching of arithmetic operations, nor practical word problems about giving and taking, buying and selling, trading and borrowing. Euclidean geometry barely prepares one for calculating the area and perimeter of shapes, which is the main focus of elementary school geometry (though reduced here to algebraic formulas and stripped of its geometric reasoning).

We find these familiar elements in the third historical venue for teaching mathematics. It may be unfamiliar by name, but easily recognizable by curriculum. In England it was referred to as a reckoning school, and its teacher as a reckonmaster—in Italy as *maestro d'abaco*, in France as *maistre d'algorisme*, and in Germanic territories as *Rechenmeister*. Reckoning schools first appeared in the 14th century in the commercial cities of Italy and later spread along the routes of the Hanseatic League trading confederation (Swetz & Smith, 1987). In 1338, Florence, the most important center for teaching mercantile mathematics, had six reckoning schools; by 1613, with the rise of mercantile economy in Europe, Nuremberg alone boasted some 48 reckonmasters (Swetz & Smith, 1987), while Antwerp had become home to 51 (Meskens, 1996). Students of reckoning schools were the children of merchants and accountants, sent at about the age of 11 or 12 to study commercial arithmetic with a reckonmaster (Jackson, 1906; Swetz & Smith, 1987). In Florence, one Francesco Galigai, in the year 1519, taught a course of instruction for boys between 11 and 15. The course lasted about 2 years, and classes met 6 days a week. His curriculum, typical of its kind (see Goldthwaite, 1972; Howson, 1982; Swetz & Smith, 1987), contained seven consecutive sections, each paid for separately. The seven parts were as follows (Goldthwaite, 1972):

1. Addition, Subtraction, and Multiplication (including memorization of algorithms and fact tables)
2. Division by a single digit
3. Division by a two-digit number
4. Division by three or more digits
5. Fractions (basic operations, used in problem situations)

6. Rule of three

7. Principles of the Florentine monetary system

Textbooks on commercial arithmetic from the time go beyond Galigai's basic curriculum. They dedicate larger sections to monetary systems: topics that include rates of interest, partnership in trade, and currency exchange (Harouni, 2013; Jackson, 1906). The only geometry that makes its way into these texts and classrooms concerns land surveying—the calculation of areas and perimeters.

The similarities between reckoning school and most modern curriculums are striking: the same emphasis on calculation, the same sequence of operations, the same computational view of geometry. Reckoning textbooks share even more features with their modern counterparts in schools (see Jackson, 1906). Without exception, they emphasized algorithms for solving operations and demanded a memorization of the most salient arithmetic facts. Salience was a factor of the frequency with which a set of numbers appeared in trade (in England, for example, 12 was an important number, since there were 12 pence to each English shilling). There was little or no attempt to present the underlying principles that make an algorithm work. Here, for example, is the author of *Treviso Arithmetic* (c. 1478), the earliest printed arithmetic book available, introducing his readers to two-digit addition:

> We always begin to add with the lowest order, which is of least value. Therefore, if we wish to add 38 to 59 we write the numbers thus:
>
> 59
> 38

Sum: 97

> We then say, "8 and 9 make 17," writing 7 in the column which was added, and carrying the 1.... This 1 we now add to 3, making 4, and this to 5, making 9, which is written in the column from which it is derived. The two together make 97.

This approach—teaching an algorithm without concern for its mechanism—holds for the vast majority of reckoning books (Jackson, 1906). Alongside the algorithms, the texts also feature a large number of word problems for each topic—many of them, *mutatis mutandis,* could have been written yesterday.

To restate the obvious, in reckoning schools we see many elements of contemporary "traditional" math education. Meanwhile, in certain alternative approaches we observe stronger similarities to the math taught at the other 16th century venues: Waldorf and New Math, with their emphasis on teaching the nature of numerical rather than calculative relationships, for example, have something of university mathematics; Montessori's reliance on math arising from physical experience with objects is closer to the craftsman's approach. Reckoning school math, however, is by far the one most closely and widely replicated in the contemporary context. Perhaps the major difference between the reckoning program and the vast majority of current curriculums is that the more complex monetary practices—dividing profits in a partnership, calculating complex loans and inheritances, for example—have not made it to our classrooms.

The Role of History

How are we to draw connections between these historical forms of teaching mathematics and their modern counterparts? I would like to suggest that there are two distinct ways of drawing connections: a chronological and a theoretical approach. Both are necessary to this discussion; however, it is only the latter, the theoretical approach, which by necessity contains the former, that can give a proper historical sense to our analysis. I will, therefore, begin with highlighting the more significant chronological connections. Let the reader beware, however, that by the end of this dense and hurried section, I must present the shortcomings of this approach and transition to a theoretical perspective.

A step-wise, chronological description of what might link 16th century mathematics to present-day education was offered as early as the turn of the previous century by Jackson (1906). It has since been expanded by other historical overviews of math education (e.g. Howson, 1982; National Council of Teachers of Mathematics, 1970). Here I will present their argument in condensed form. Proceeding chronologically, we draw lines that show the development of modern elementary school from its antecedents: moving step-wise, we try to establish if one practice led to another, if one institution influenced the next, until we arrive at the present. The similarities between reckoning and modern school math cannot be arbitrary. By the late 17th century the merchant classes had gained enough power to impact the curriculum of grammar and parochial schools (Howson, 1982; Swetz & Smith, 1987). This was partly a matter of finances: accountants and merchants were obliged to pay for two types of education—once for literacy to grammar schools and once for numeracy to reckonmasters. They preferred to combine the two for a single fee

(Jackson, 1906). More importantly, the other educated and well-to-do city dwellers also found themselves in closer contact with commerce, bookkeeping, debt and other activities that are part and parcel of middle class life in a money economy. Until the rise of mercantile and capitalist economy in Europe, the higher classes attached a strong stigma to arithmetic due to its connection with trade and banking. Almost all 16th century textbooks began with a sort of apologia: the author had to make a case for the intellectual and spiritual value of arithmetic (Davis, 1960). However, by the late 17th century the stigma attached to excessive counting and accounting departed on the same wave that wrested the control of social life away from the church and eventually placed it in the hands of the bourgeoisie. By mid-17th century, the demand for commercial arithmetic had turned the subject into a main feature of middle class education in most of Western Europe.

Within a few generations, a new type of schoolteacher emerged out of this process: a compromised combination of the old grammar school teacher and reckonmasters, able to teach basic reckoning, but not versed in more advanced financial applications of mathematics. Since grammar school still had to prepare some children for the university, these teachers were also charged with teaching a smattering of the type of mathematics practiced in higher education. Some, usually rote, learning of Euclid became the most salient way of fulfilling this requirement. In upper-class schools, where education was meant to equip students with knowledge of higher culture, the philosophical approach to math remained more prominent than it did in poor or working-class schools.

Grammar school, which in certain English-speaking areas is still the name used for elementary education, in many ways formed the foundations of public education as we know it today. Its structure and values carried over to the schools of the poor, the peasants, and the working class by teachers and reformers who were themselves products of grammar schooling. Across various eras, society would experiment with slight modifications in the grammar-school framework, hoping to make it better "fit" various contexts: the factory-like Lancasterian system for the poor and working class, for example, and the college "prep" model that shaped some of the private schools for the rich (Bowles & Gintis, 1977; Howson, 1982). By and large, however, grammar schools—carrying their reckoning school implants—were the original blueprint for the schools that were eventually established by the state for public education. As it expanded beyond its original base, however, the grammar school model became involved in a strange dialectic: it imposed itself on classes and entire cultures whose economic interests it did not represent. The grammar school was designed neither to establish students within a working class identity nor to help them organize and recognize themselves as a class capable of changing the social dynamics that kept them as such. One result of this institutional mismatch may have been the elimination of more complex financial mathematics from curriculums meant for working and lower-middle class students who, school organizers would admit (e.g. Klapper, 1934), did not need them.

And so, in the step-wise fashion shown above, history seems to bring us to the present. The above, chronological account is a rather efficient way of presenting the sources of contemporary practice. Like many historical accounts of the formation of modern

schooling, it highlights the outdatedness of current practices by showing that they developed in contexts that no longer exist in modern society. Given on their own, such accounts seem to provide us with a two-fold task: to identify and replace outdated practices, and to make education more relevant to "marginalized" classes and cultures. Here, however, we face an essential problem: the historical forces that keep the old practices in place or that make one group marginal to another are still present in society. By confining history to the past, viewing it as merely a source of habits, we stop to think historically. In a purely chronological analysis, the resilience of practices that seem useless or ritualistic is attributed to institutional footdragging: if only we could reorganize schools according to new and relevant values, then a new, more relevant form of education would inevitably arise as well. This is the perspective underlying any reform agenda that chooses to refer to its target primarily as "traditional" schooling. Where, however, will those new values come from, given that the very lenses through which we judge the world are historically formed? Even lethargy rests on something more than mere laziness and ignorance. Tradition owes its force not just to the past, but also to the present.

At the outset of this essay, I proposed that we ought to explore the history of mathematics education in order to understand why it has taken on its present form. However, to understand an existing practice or to create a new one, it is not enough to simply trace its provenance. History is most powerful precisely where it cannot be traced, where it seems to spin out entirely new practices or where it adopts an old practice in a very different context, thus rendering it new. To grasp the interplay of historical forces, step-wise

historicism is deficient. It takes the connection between events for granted, denying detours, failures, and contorted processes. It also forgets to take account of the epistemology that establishes the steps in the first place.

To analyze our own lenses and to bring out a sense of history that includes lost and forgotten opportunities, we need the incisive blade of theory. In the case of mathematics education we are dealing with an overwhelming economic fact that we hold in common with 16th century Europe: commercial and administrative calculation is *still* the dominant intellectual activity of our societies. We are not merely inheritors of reckoning. We *are* reckoners—and perhaps academics, artisans, politicians, and so on—and the math we teach contains our reckoning attitude. This is the starting point for a theoretical understanding.

We can expand on what we know about reckoning to think of it as a category of mathematics. I will refer to this category as *reckoning* or *commercial-administrative* mathematics. The other historical venues for teaching math can also be said to teach rather specific categories of mathematics, which I will refer to as *philosophical* and *artisanal* mathematics. A fourth type, *social-analytical* math emerges at a later time in history.

There is a problem inherent to this process, however. To speak of a category is not to point at an immutable fact: rather it is primarily a device set up for thinking. Each category contains a dilemma. On the one hand, in actual, historical conditions, the type of

math that emerges from, say, commercial-administrative practice is shaped in relationship to the larger social and institutional settings. It changes from era to era, from place to place. On the other hand, there remain enough similarities across a wide variety of contexts, having to do with the similar place of the merchant and administrator in relation to economic production, that it becomes useful to cluster these similarities under a meaningful title. In my analysis I will try to hold both sides of this contradiction at the same time, pointing out what is universal just as it falls back into particular historical conditions.

Categories of Mathematics

Commercial-Administrative Mathematics

Let us try and *hold* reckoning—that is, extract it from the rapid flood of history—for just a moment, knowing that it will immediately slip out of our hands and back into the current. Because counting and exchange appear as fundamental human activities, there is a tendency to think that *commercial-administrative* math is also simply an outgrowth of human nature. For example, here is Constance Kamii drawing on Piaget: "every culture that builds any mathematics at all ends up building exactly the same mathematics, as this is a system of relationships in which absolutely nothing is arbitrary" (Kamii, 1982). Both Piaget and Kamii ignore that for math to turn into what they describe, there first has to be a need to look for and construct relationships in which "absolutely nothing is arbitrary." The underlying assumptions of such conclusions are false. Counting is not a "natural"

human activity, as hunter-gatherer tribes that develop only a "one-two-many" counting system clearly demonstrate (Gordon, 2004; Pinker, 2007). Moreover, the movement from counting sheep to reducing human labor into abstract quantities and keeping books on everything (shipments of olive oil from Greece, the number of workers building a dam, an entire nation's taxes) is not merely a matter of expanding on the basic principles of counting. The development—its form and degree of proliferation—is neither *natural* nor inevitable, but cultural.

The history of reckoning math moves alongside the history of accumulated labor power. Without the opportunity for products and work-hours to accumulate, there would be nothing to book-keep or trade at a level that would require a special discipline of mathematics. This overruling mindset impacts every aspect of reckoning mathematics. First and foremost, number in reckoning is a placeholder, referring to values drawn from the real world. In other words, number in reckoning is never fully abstract or freestanding; instead, it is, at the last instance, the result of counting things. There is a special character to this counting that makes it different from, say, scientific or technical measurement, which I will explain soon. It is important, for now, to notice that if numbers are closely tied to counting, then it becomes quite problematic to apply them to more abstract concepts. As Russell (1919), for example, put it, you cannot develop the various definitions of infinity out of counting, because we could not possibly count an infinite number of things. Nor can certain fractions or irrational numbers be expressed in terms of counting. It is, furthermore, difficult for the reckoner to imagine numbers as emerging out of a dialectical relationship between objects. He can't, for example, see that

a mountain immediately presents itself as being many meters tall: to him length is an accumulation of single units of length, rather than a concept arising out of the comparison between this mountain and other objects. From a scientific perspective, then, reckoning math must have at some point in history died off, being unable to explore all the mathematical problems it engenders. In reality, due to its economic significance, it continues to dominate the "lay" perception of mathematics.

The second essential aspect of reckoning or commercial-administrative math is that it treats calculations as ultimately representing predictable (i.e. regulated and fixed) social interactions. The tremendous use of word problems that reference everyday activity in reckoning textbooks is both a result and a promoter of this attitude. Along with practical considerations, this attitude gives rise to a host of psychological decisions that are particularly bold when expressed in pedagogy. There is, for example, no mathematical or practical reason for the unquestioned primacy of addition in reckoning school (and contemporary) education. There is no developmental reason that bars teaching subtraction first, since even young children can learn it at the same time as they do addition (Starkey & Gelman, 1982). It is rather the commercial-administrative attitude that is at work here: a certain degree of *accumulation* of resources is necessary before any of the other practical functions served by commerce or administration can take place. It is this sociological need for accumulation, seen as the source of all interactions, which can account for the seminal place of addition. The order of teaching operations in Europe prior to the rise of money economy, for example, was not always the same as it is today. Rabbi ben Ezra (c. 1140) and Fibonacci (c. 1202), for example, began with

multiplication, and then went on to division, addition, and subtraction (Swetz & Smith, 1987). Today, Waldorf schools, which explicitly reject a calculative approach to elementary mathematics (see Aeppli, 1986, pp. 41-57), prefer to teach all four basic operations simultaneously.

The earliest appearance of *commercial-administrative* math in historical record is in clay tablets surviving from ancient Sumer and Babylonia (Friberg, 1999). A very large number of tablets have been recovered from the sites of scribal schools and allow a view into how scribes taught and learned the mathematics of their trade. What is astonishing is the similarity between many of these ancient methods and what we observe in reckoning texts or contemporary curriculums: we find in the tablets the same reliance on readymade algorithms, the same use of word problems, the same tendency to reduce concrete things to a numerical value.

Scribes originally functioned as administrators for the various organs—the temple, the state, or private persons, depending on the era—capable of accumulating the labors of many. They kept records on harvests, treasuries, and building projects. They calculated the amount of clay, straw, and bricks needed for each building, and converted these into the labor-power needed to produce them (Friberg, 1996; Friberg, 2007). Here, for example, is a Babylonian example that anticipates our own math problems. The modern versions usually involve a number of workers finishing a job in a certain number of days and the textbook asking us to calculate how long it would take a different number of workers to finish the same job. The Sumerian version is slightly more complicated,

because it does not shy away from the calculation of wages. There is no confusion about why we are looking at work-hours—we want to know how much we have to pay:

> If a man carried 420 bricks for 180 meters, I would give him 10 liters of barley. Supposing he finished after carrying 300 bricks, how much would I give him? (Friberg, 1996)

How can bricks and distances, barley and men, such disparate things, all interact so seamlessly with each other in a problem? By becoming reducible: each entity is reduced to the labor power that it demands or embodies. It is hard to think of activities outside of commerce or administration that would dissolve the material identity of objects and people so readily. This reductionist tendency is so strong that it seeped into every aspect of the scribes' teaching. Eventually, they arrived at nonsensical practice questions that had the student add up ants, birds, barley and people in a pile. The tablet containing the problem can be represented as follows (Friberg, 2005, p. 5):

649,539		barley-corns
72,171		ears of barley
8,091		ants
891		birds
(+) 99		people
730, 791		

The tablet, clearly a practice problem, in all likelihood corresponded to a type of word problem, examples of which reappear in ancient Egypt and Europe, up to the modern era (Friberg, 2005). Notice how blithely the student is asked to add birds, ants, barley, and people. It was necessary for the young learner to get used to seeing things as operating in

this way. What does it mean to add ants to birds and to people? The teacher can be unconcerned, because in his professional sphere numbers present an interstitial virtual space within which all objects and people lose their material identity and blend into abstract value. So long as the student knows how to perform complex operations, the material aspect of numbers is unimportant. The majority of numbers he will deal with are bereft of all meaning except for two: their quantity as items and their value in exchange—and these two are easily convertible to one another. In modern times this exchange value is reified within the concept of money, to which all commodities are converted.

Commercial-administrative math always stands *beside* the process of production, turning objects and labor into abstract quantities. The reckoner is interested in the interaction between numbers because he needs to predict the outcome of contracts, exchanges, partnerships, or investments—the specifics varying according to the mode of production. In any setting he looks to laws of exchange or administration that were set in the past, and he calculates for the future. How long *will* it take the workers to finish the job? How much *will* we have to pay or feed them? What *will* the interest on this loan be? Labor power is time is product is remuneration. All these things are ultimately interchangeable. The future, as seen by this type of math, is ideally a reliable one—the flowering of the seed of the owner's hope in an investment or of the administrator's promise to safeguard one. Without reliability, investment and exchange cannot proliferate. This is one reason why the types of problems reckoning mathematics uses for training are single-answer questions. The obsession with this single answer demands more and more efficient

mechanisms of arriving at the answer. How these mechanisms work is not nearly as important as the assurance that they *do* work. Furthermore, the units that the word problems refer to are not nearly as important as their ability to hint at pre-determined, simple interactions involving the reduction of people and objects to value.

Ultimately, in modern curricula, when a textbook question talks about apples and oranges, it does not mean apples and oranges, it means money. And yet neither teacher nor student is aware of this underlying meaning. As mentioned above, the actual mercantile and administrative content of math has all but disappeared from modern curricula, replaced by odd and nearly always spurious references to domestic activity (see Dowling, 1998; Lave, 1992). The general form, however, has remained the same. In this diffuse form, the *commercial-administrative* mindset addresses itself to all things without ever revealing its own nature.

Artisanal Mathematics

While commercial-administrative math stands outside the process of creative labor, a different kind of "mathematics" emerges from *within* that process. It is shaped by the interaction between people, instruments and materials. In creative labor the material identity of things is not obliterated, but transmuted. The mathematics of the artisan's workshop is therefore part of this transformation process: the workman's skill meeting the material. I will refer to this type of activity as *artisanal mathematics*, keeping in mind that it is so different from other forms of math that often it would not even make sense to refer to it as such.

It is measuring, not counting, that constitutes the artisan's primary encounter with numbers. It is tempting to think of measurement as the counting of units, but this temptation is *the result of our own early education*. Teachers introduce students to measurement by handing them a ruler and asking them to measure the length of a line, counting the centimeters from zero. Actual measurement, however, is rarely so one-dimensional—because in the real world it is rare for a craftsman to care about only one dimension of an object. Any piece of wood, for example, has a type, an age, a weight, a density, a hardness, and a minimum of three dimensions, and a carpenter takes into account a combination of these and other aspects in every stage of woodwork. Furthermore, the measure of something does not reveal itself primarily as a sum of units, but as a comparison between objects that is then expressed in terms of units. So, to divide the length of something into equal parts, the carpenter does not need to use a measuring tape: he or she can hold a piece of string to the object, separating a section that equals the length, and then divide the string as many times as needed.

Embedded in the above tensions we find the fundamental difference between the meaning of units in commercial-administrative and in artisanal mathematics. All units of measurement, including currency, are arbitrary social constructs, employed in accordance with the particular situation. But whereas for the artisan, measures represent aspects of the materials he works with, for the merchant all materials, including their measures, are aspects of the value he invests or earns. The artisan, unlike the merchant, can improvise useful units for his work on the fly—using a piece of string, as mentioned, to express a

length. The only time he needs to think in terms of conventional units (inches, grams, etc.) is when he is communicating with strangers. Here is, for example, a village carpenter, Walter Rose (b. 1871), describing his father/teacher's method of measuring trees in the late 19th century. The interaction between purpose, skill, math, and material reads through every step of the process, in which conventional units appear only at the very end:

> For the measurement of trees my father always used a string and the slide-rule.[5] As the trunks of trees taper lengthwise, the middle was taken as the average girth round which the string was to be passed. I helped him many times, holding the string carefully with my fingers at the place where it terminated the circumference as measured, afterwards doubling it and then redoubling it twice, with the result that the folds held in my hand were each an eighth of the total circumference. Then he would direct me to drop one-eighth part – this an allowance for the bark – and double the remaining seven-eighths twice. Each fold of the string was now one-fourth part of the seven-eighths of the circumference. He would take the length of this with his rule. The measurement thus arrived at represented the "girth," or one side of the squared log, supposing the content of the log to have been square instead of round.
>
> On that basis he would then ascertain the cubic content of the log by the use of his slide-rule … (Rose, 1946)

[5] A slide rule is a ruler with a sliding central strip, marked with logarithmic scales and used for making rapid calculations, esp. multiplication and division. We might as well think of it as a primitive calculator. I am intentionally using an example that involves calculation and algorithms.

The artisanal model of learning (i.e. apprenticeship)—watching an experienced worker closely, emulating complex skills that take into account many aspects of the work at the same time, recreating artifacts, project-based assessment—is a necessary extension of the complex relationship people have with materials. Therefore, textbooks, worksheets, word problems and explanations which, if not engaging, can at least convey the basic ideas of reckoning, are completely inadequate for the learning that creative labor requires. They cannot convey the complexity of things as needed. Even merchants who do more than basic accounting are partly trained on the job. That artisanal learning did not help shape public schooling is due to the rise of industrial capitalism, which rapidly eroded the influence of the artisan class. By the late 17th century in Europe, apprenticeships were proving expensive and outdated (De Munck, 2007). Technological progress simplified the skills needed by the majority of workers on the shop floor, and increasingly the combination of simple wage labor and machinery replaced the work of trained artisans. Today, aspects of the apprenticeship mode of learning survive in engineering, art and technical schools. In places where, under the influence of other academic subjects, training moves away from dealing directly with tools and materials, the student will have to learn the real skills of her trade *on the job*, working with specific tools under the supervision of more experienced workers. The bond between the content of artisanal work and the pedagogy used to teach it remains strong: one demands the other.

In modern elementary education, we rarely come across any instances of artisanal mathematics. Therefore it is harder to imagine how a learning model based on an artisanal attitude would differ from standard classroom mathematics. We do find at least

one well-documented attempt in the works of Maria Montessori (1870-1952). Montessori came upon an artisanal approach to teaching in a roundabout way—not through an attempt at training technicians, but through close observation of very young children interacting with their environment. Early on in her work, she discovered that her kindergarten students were happiest and most focused when working independently with materials. She concluded that curriculum and content should become embedded in tangible things—so called "didactic materials"—that embody certain relationships for the child to explore (Lillard, 2005; Montessori, 1914). This emphasis on materials brought Montessori to certain key elements of artisanal learning. The complexity of the materials meant that children learned better when watching someone else perform a task than when listening to explanations or copying down problems. Montessori decided that for her mode of learning to be effective, the room had to be full of people doing work. There was no way for all this work to be led by the teacher. Therefore, she conceived of classrooms that housed children of various ages, just as shop floors held workers of varying abilities. However, this artisanal attitude, strong in the Montessori pedagogy for the early grades, gives way under the pressure of the common notion of mathematics. The reason is simple: there is no real production process in Montessori schools. By mid elementary years, Montessori materials turn into means of expressing relationships that are not creative, but calculative. Thus we arrive at rather absurd sets of materials: for example, blocks that are expected to represent quadratic equations. Montessori's lack of a strong theory for the purpose of math eventually brings her back to the dominant model.[6]

[6] Davydov's method (Schmittau, 2010), partially re-animated in the U.S. as the 'Measure Up' curriculum, is a much more thoughtful and theoretically sound *artisanal* approach. For that very reason, it would be beyond the scope of this paper to discuss its implications.

Philosophical Mathematics

From our study of 16th century Europe we can conceive of a third type of mathematics – the one corresponding to the math practiced in universities. This type of math stands neither inside nor beside the process of creative labor. Its product is neither an object nor an interaction in the world, but an order in the mind. However, we should resist the temptation to brand this type of math as abstract and impractical, as this math, too, only reaches its development once it becomes part of a larger social *practice.* We can think of it as a *philosophical* type of mathematics—using philosophy as a blanket term to cover also priestly and academic activities. It is exemplified in Euclid, in the astronomical and astrological discourses of Muslim scholars and in the discipline of pure math that is the practice of academic mathematicians. As in the other categories, a large number of different practices fall together – differences that are not merely practical but also cultural. Nonetheless, some important similarities remain.

Philosophical mathematics loves patterns. It draws them out because they hint at meaning, and meaning is the priest and philosopher's sustenance. When philosophical math turns to numbers, it turns to their meaning, the patterns and the logical connections they contain. Sixteenth-century books of arithmetic that were meant for grammar schools first discuss the common number *patterns* (odds, evens, naturals, etc.) before arriving at operations. Books written prior to the dominance of money economics and commercial arithmetic often introduce all four operations at the same time (Al-Biruni's *Instructions,* for example, or Isidore's *Etymologies*). What implicitly steals the show in every page of

philosophical arithmetic (as in the question "27=?") is the concept of "equality," the *meaning* of a number; a number in philosophical math is not so much the carrier of value or signifier of relative quantity as it is a repository of self-referential relationships.

In philosophy or theology rigor is not defined by how well a student predicts the outcome of a practical situation. Scholarly arithmetic texts of the 16th century generally contained abstract or "impractical" puzzles, rather than word-problems. Even when a word problem did involve the use of money, it had nothing to do with a business situation; such problems were uninterested in predicting the result of an interaction and did not try to practice the student's hand at a specific, existing algorithm. For example:

> Three men together have a certain amount of silver, but each one is ignorant of the amount he has. The first and second together have 50 coins, the second and third, 70 coins, the third and first, 60. It is required to know how much each one has. (Jackson, 1906)

There is nothing business-like in this problem (c. 1540), despite the reference to money and merchants. What the student performs here will not help a merchant conduct his work—it is definitely of no help to the three men in the problem.

The more we look at philosophical mathematics, the more we understand the regular complaint of contemporary academic mathematicians that the subject taught in elementary and secondary classrooms is far removed from "mathematics" (e.g. Lockhart,

2009). But what they see as the discipline of mathematics is also far removed from many other perspectives. We observe this inability to recognize one's own perspective in the debate that surrounded the New Math curriculum in the United States. In the 1960's mathematicians were invited to design a curriculum that updated math education and corresponded to contemporary "mathematical" practices—which here meant neither engineering nor modern money mechanics, but pure academic math (Phillips, 2011). The result was a curriculum that seemed even more alienated from everyday life than the traditional curriculum. For this reason and others, its days in the elementary school were numbered. It gave way before the most simple of all attacks: that kids raised on New Math were not quickly proficient in (reckoning) calculations: "Johnny can't count!" was the battle cry of the so-called "back-to-basics" movement (Hayden, 1981).

New Math, with its emphasis on patterns, classifications and analysis, is not the only philosophical approach to math education. The Waldorf curriculum, for example, is a very different attempt from a philosophical perspective, relying on a spiritualist agenda (thus its emphasis, for example, on all numbers adding up to one, and thus promoting the idea of universal oneness). Just as commercial-administrative math is formed in relation to its context, philosophical math, too, is by no means a single, over-determined form of practice. For example, as far as modern philosophical developments are concerned, the dry, academic curriculum of the New Math, corresponding to the academic worldview of its founders, can be seen as philosophically reactionary: its philosophical core was bolstered by willful ignorance of all radical philosophy, from Nietzsche to Marx. In other words, nearly a century after it was shown by radical philosophy that logic itself is

formed in relation to society and human psychology, New Math attempted to teach mathematical logic that is nearly empty of sociological and psychological connections.

But mathematics is not formed only in relation to the larger context, but also in relation to its own various forms. In a strong money economy, as soon as philosophical mathematics leaves its specialized cloisters and addresses itself to the general public, it is fated to meet commercial-administrative mathematics in a dialectical battle. On the one hand, commerce and administration, which rely heavily on mathematics, want philosophical math to submit to and reinforce their agenda. In certain epochs, this agenda is bolstered by the massive economic might that commercial or administrative institutions yield in society and, therefore, over academic and philosophical institutions as well. On the other hand, the philosophical mathematician wants to reassert the independent identity of his or her own discipline, and in that attempt is equipped with a stronger, more scientific version of mathematics. The synthesis can take a variety of forms, depending on the battleground. In public education, however, the result is always disappointing to proponents of reckoning as well as philosophical math, specifically because as adults, they do not intend to change their own practice or view of the purpose of math, but only the way in which the future generation is trained in it. When Socrates, in book 7 of the *Republic*, browbeats Glaucon into accepting that arithmetic is an essential subject for training the rulers of a utopian city, he immediately has to qualify his statement by separating the two forms of mathematical practice:

> Then it would be appropriate, Glaucon, to prescribe this subject in our

> legislation and to persuade those who are going to take part in what is most important in the city to go in for calculation and take it up, *not as laymen do*, but staying with it until they reach the point at which *they see the nature of the numbers by means of understanding itself*; not like tradesmen and retailers, caring about it for the sake of *buying and selling*, but for the sake of war and for ease in *turning the soul* itself around from becoming to *truth and being*. (Plato, 2004, p. 220)

Emphasis is mine. Thus Plato, never shy about his disdain for working and trading classes, attempts to prevent the above-mentioned dialectic by clearly delineating a space for each type of math: Philosophical for aristocrats, reckoning (and possibly artisanal math) for lay people. That elementary school mathematics has managed to retain its basic shape for so long, unaffected by all scientific and philosophical developments, has partly depended on a similar solution. "Basic" math in public schools in the U.S. and many other nations is almost exclusively a watered-down commercial-administrative approach, while the more philosophical approaches are reserved for more advanced students or alternative, private institutions. The now commonplace observation that algebra seems to act as an intellectual barrier against many, particularly working class students, entering the more advanced topics in mathematics (see Moses & Cobb, 2001) is unwittingly hinting at the mismatch between these two types of math. Historically, algebra itself emerged in the Middle East as a reordering of the mathematical practices of merchants (itself possibly relying on earlier, artisanal practices of land-surveyors), placed on scientific footing by philosophical mathematicians (Hoyrup, 1987). It contains the

conflict.

Social-Analytical Mathematics

For the purpose of understanding current trends in math education, we must consider one more category of mathematics – one that forms precisely in the meeting of philosophical and commercial-administrative practices. I will refer to this category as *social-analytical* mathematics. Exemplified in the disciplines of economics and social statistics, this category can only arise once commercial-administrative math is already highly developed, in widespread use, and subject to analysis by competing groups. The earliest, extremely rudimentary recording of this type of practice comes from certain Greek city-states in the 4th century B.C., where state accounts were posted in public for scrutiny (Cuomo, 2001). The accounts themselves are products of commercial-administrative mathematics. Their public posting as well as the way they were read by their intended audience, however, are the result of a social setting in which one accumulator of resources finds himself accountable to another section of society. The most obvious form of such accountability—taxes—can initiate a primitive form of social mathematics, when the ruling class tries to analyze the population to determine the safe margin of taxation or control. In its advanced form, however, social-analytical mathematics requires a situation where competing interests can view and analyze the data *at the same time*. This effort takes the form of an argument that requires new tools for reasoning and representation. Thus, economics and social statistics, which enable such scrutiny, both expanded into

disciplines against the background of the class struggles that shaped 19th century Europe. The scientific basis for both had existed, sometimes for centuries prior (for instance, 9th-century mathematician Al-Kindi had used frequency analysis to decipher encrypted messages) but there had been no reason to use them as tools of social analysis.[7]

Thus social-analytical mathematics contains a special tension. On the one hand it can be used as an administrative tool, either by further reducing people to abstract units in order to predict their behavior or by providing justification for the status quo, or it can work against administration, acting as a tool of critique that simultaneously helps give definition to phenomena that previously appeared too diffuse and scattered to have clear meaning. In the latter sense, Marxist and socialist economics mark a break with prior forms of mathematical practice, turning all disciplines of commercial-administrative mathematics on their heads by placing workers in the role of the analyzer. In this intellectual tradition, math no longer serves only to convert labor power and nature into exchange value. Instead, one uses math to inquire after the life of the worker, his individual or group interest in all commodities and interactions. Unlike a reckoning mindset, where significance is defined in exchange value, here the person using mathematics can refuse such a reduction. Where such a reduction is encountered, one tries to subvert the process by arriving instead at the human beings who created the value in the first place. For wageworkers, here might finally be a math that can be said to concern "citizenship," because it concerns the asymmetric interests that define life in the marketplace.

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[7] I base my analysis on a critical reading of (Porter, 1986) and (Desrosières, 1998), among others.

Drawing on the above promise of social-analytical mathematics, many progressive educators have argued for math education employing statistics and analysis to help students "read the world," as Frankenstein (1983), drawing on Freire, has put it. Frankenstein grouped these efforts under a global movement called "critical math education," which has resulted in a body of curricula that problematize the data on racial, economical, and gender inequality, among other social problems (e.g. Gutstein & Peterson, 2005; Lesser & Blake, 2007; Scovsmose, 1994). In a 2009 essay on word problems, Frankenstein offers the clearest articulation of a social-analytical approach to math education to date. In her approach every mathematical statement is to be seen as a codified social interaction meant for critical analysis. Numbers are to be used by teachers and students to help describe the world, while also showing how numerical descriptions distort or hide reality. The purpose of calculation for critical math pedagogy, in turn, is no longer to compute answers, but to understand and verify the logic of an argument, restate and explain information, and to reveal the unstated data.

Embedded in Frankenstein's proposal we find the tensions that underlie social-analytical mathematics. We should notice that she treats the basic materials of math, both in numerical and technical terms, as having already been provided by another source—one that is essentially suspect. In fact, Frankenstein relies on an unnamed and undescribed form of training that is supposed to equip students with the basic technical skills that enables them to analyze data, which is also, more often than not, gathered elsewhere. This outsourcing of basic technical training is endemic to "critical math education": in

my review, the vast majority of the curriculums published by practitioners of this approach address middle and secondary education. There is almost no "critical" theory of elementary mathematics.

Frankenstein's above-mentioned "source" can partly be described as commercial-administrative mathematics, providing both the methods and raw materials of analysis. This does not, however, cover all grounds. Much of what critical math tries to analyze in fact comes from social-analytical math itself: from statistical or economic analysis performed on social phenomena. Critical math's justified suspicion toward its own raw materials is the result of the observation that social-analytic math itself can easily turn back into an instrument of commerce and administration, rather than a critical tool. Statistics on racial inequality in educational achievement, as Gould (1981), for example, demonstrated, can be used to critique the education system or to uphold both the existing definition of achievement and its related racial injustices. Within math education the threat of a critical perspective turning into yet another justification for the very thing that is critiqued is quite subtle, and it is very often ignored by proponents of social-analytical mathematics. Consider the following example from Gutstein (2006, p. 247), who provides his students with the price tags for a B-2 bomber and a college education, and asks them the following question:

> Last June, about 250 students graduated from Simón Bolivar high school. Could the cost of one B-2 bomber give those graduates a free ride to the [University of Wisconsin-Madison] for four years?

On the surface, the question is a critique of policy that allocates money to bombers instead of education. At its depth, however, it reproduces the logic that reduces all objects and decisions to their exchange value. In one stroke of the pen, the B-2 bomber and a college education are co-defined by their comparative monetary value. One turns into the other like a large bill into change. In completing the problem, students do not ask where the bomber comes from, what purpose it serves, or what its dissolution might mean, nor why a college education in the United States costs as much as it does. If an arms-industry lobbyist points out that the construction of the bomber provides jobs for so many workers, and when sold to Saudi Arabia it provides so much revenue, and in supporting American interest upholds the strength of U.S. trade relations, the same monetary logic ends up supporting the necessity of the bomber. In this sense, the well-meaning question still trades in the logic that it purports to attack. The logic of exchange value cannot easily be subverted by its own tools. Critical math pedagogy's reframing of the purpose of numbers and calculation is a practical framework against these types of mistakes. But it does not go far enough, because it does not contain a theory for critiquing its own instruments. It lacks the theoretical grounding that could help it address the relationship between differing approaches to mathematics.

School Mathematics – A Paradigm

All categories of mathematics are formed in relation to the institutional setting in which they are practiced; and schools, as institutions, impose their own conditions. A high degree of scholasticization generally tends to separate at least aspects of knowledge from

direct use in practice; it reshapes knowledge as "signs" of mastery or enculturation, rather than instruments or thought and labor. This can apply just as much to commercial-administrative settings, like the Babylonian scribal school, as it can to philosophical ones, where, as in Medieval European universities, math turns into a sign of initiation into classical literature (Schrader, 1967), rather than a philosophical pursuit per se.

This tendency has reached its apotheosis in modern public education, where even the least explicit links to practice have disappeared. The false economy of social capital sought in grades and certifications transmutes math as it does every other school subject. Children in schools learn what they learn in great part in order to satisfy school requirements, to gain certification or the approval of teachers. As mentioned before, some theorists of education suggest that school math no longer has any relationship with labor at all (Dowling, 1998), that it is a self-referential discipline born out of the special properties of schooling (Lave, 1992), and the only reason to learn school math is to be able to do more of it, later (Lundin, 2010a; Lundin, 2010b).

Such arguments regarding the separation between labor and learning in schools, powerful as they are, ignore an essential aspect of what constitutes the school curriculum. Schools do not only teach *know-how*, they also teach *attitudes* toward the world and toward labor in particular (Anyon, 1980). The predominance of a commercial-administrative attitude in elementary education speaks to a particular mindset. Implicit in using the dominant model of mathematics is the *overuse* of the intellectual muscles associated with commerce and administration. This type of math is the carrier of the mercantile and

administrative relation to labor and life, the tendency to reduce real-world relationships to economic exchanges. School mathematics reflects, and in turn generates, the calculating attitude that results from immersion in the relations of a money economy. The alienation inherent in commercial-administrative math becomes particularly severe when its real function is masked—for example, when money, in story problems, is replaced with apples and oranges.

The mathematics that dominates elementary education is an amputated version of the math taught in 16th century reckoning schools, having lost the original emphasis on anything other than the most basic commercial situations. Though it reflects the attitude embodied in reckoning, it is geared toward a different purpose than training merchants or even accountants. Elementary math today could better be described as *consumer* mathematics. This downgrade (from merchant/administrator to consumer) is a function of the social downgrade in the role of schools: from schools for the upper middle classes to schools for the lower classes, and then for the population at large. It was not—and still is not—conceivable that working class children might need to learn actual financial mathematics for the administration of capital and labor. School math, therefore, has been gradually and deliberately reduced to the most basic aspects of reckoning—just enough for shopping or for working as a petty bureaucrat, a soldier, or a cashier. It must be mentioned, however, that consumer math, by virtue of the passive social position that it corresponds to, can by no means develop an expansive form of mathematics. Its underlying structure is borrowed. School mathematics at best appears as a shanty built on top of vast, ruined foundations meant for an arena. In this process, it has lost whatever

social power mathematics possessed. Remnants of philosophical math (in the form of calculus, for example) have suffered a similar sea-change as they are cut off from their more expansive purpose in science: how many calculus students know that what they are learning was designed to accommodate physics?

Of course, today's teachers are not reckonmasters. In addition to commercial-administrative math, we find strong traces of other types of mathematics in the classroom, particularly in private, alternative, and elite schools. It would be impossible to critique every one of these approaches. However, one thing is clear: none of them can be complete in themselves, because at this point in history it is impossible to chart a direct link between education and action, education and purpose. So long as it is deployed in the isolation of the classroom, math should primarily be analyzed in terms of the social attitudes it promotes—terms that are ideological rather than utilitarian, and their critique, therefore, is a critique of ideology.

Conclusions and Implications

If we accept that mathematics education, down to its most elementary aspects, is a historical process reflecting economic values and political attitudes, then the implications for theory and practice are enormous. The theoretical framework I have outlined in this essay can be loosely summarized in four tenets:

1. The economic purpose of math defines its most basic characteristics.
2. The economic characteristics of math impact how it can be taught.
3. The institutional setting within which math is taught also modifies the

character of its practice.

4. All of the above aspects impact one another in relation to the socio-economic forces that shape them.

This framework addresses all those involved in math education—including teachers, parents, theorists, proponents of deschooling, and, perhaps most intimately, students. It presents a challenge to individuals and communities to define their own view of mathematics, and to not take the discipline for granted at any level. The result of my own work in this field points to various areas where accepting such a challenge would lead to new directions in research and practice.

First, this framework challenges the idealistic discourse that underlies nearly all discussions regarding school mathematics. As I have demonstrated, rhetoric that presents math learning as an absolute good, as necessary to work and citizenship, masks a deeper discussion regarding the role of labor and politics in society. One cannot take any ideas regarding the 'usefulness' of math education for granted. What is needed instead is a precise, dialectical approach that clarifies what shapes curriculum, for whom, and to what end.

Second, this framework challenges the notion that there is a single "basic" math that constitutes the foundation of all other mathematical practices. On the contrary, an educator's worldview and place in society defines how he or she conceives of such concepts as numbers, precision, and context. While further study may better clarify the connections between worldview and math education, for now it is sufficient to observe

that all school examinations, without exception, do not test students' "basic knowledge of math." They impose specific notions of what is and is not valid knowledge.

Third, teacher education can not only concern itself with *what* teachers teach and *how* they teach it. The *what* and *how* questions of math cannot be answered by sole recourse to an objective reservoir of knowledge. Any program that views teachers as more than mere functionaries will have to involve an exploration of the *why* questions of mathematics—with the understanding that such an exploration may lead teachers to rebel against the confines and assumptions of their own position.

Finally, this framework provides a critical basis from which we can engage the role of mathematics education in the lives of individuals and societies. The statement attributed to Foucault holds true in regard to math education: "We know what we do; frequently we know why we do it; but what we don't know is what what we do does." Nonetheless, any impact that math might have on individuals and society depends on two processes: the one that forms the content of curriculum and the one that delivers it to people. Once these structures are more transparent, and only then, we can finally begin to discuss what math education is making of us as human beings.

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Oranges are Money:

Reframing the Discussion on Word Problems

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2015

Abstract:

Many studies have attempted to describe the apparently significant place of word problems at the intersection of mathematics, education and the real world. In this essay the author suggests that the discussion of word problems needs to be reframed within a historical and dialectical conception of "mathematics," "the real world" and pedagogical institutions. This paper suggests that, historically, various socio-economic needs and attitudes have given rise to different forms of mathematics and consequently different forms of word problems. Contemporary schooling, isolated from all specific forms of practice other than teaching and learning, nonetheless draws on these different approaches, both reproducing and altering the social attitudes that are embedded in them. The author argues that the literature on word problems takes for granted certain unexamined conceptions of history, everyday experience and mathematics that severely curtail our understanding of the topic. These assumptions can be effectively re-examined in light of political economy.

Introduction: Hidden assumptions

The practice of questioning the uses of word problems is nearly as old as modern schooling. The earliest attempts took the form of complaints about a perceived lack of connection between word problems and real life situations (see Verschaffel, Greer & De Corte, 2000). This has remained a fruitful line of inquiry: what makes mathematical word problems interesting for theory is an implicit promise that through their analysis we might arrive at an understanding of the problematic relationship between school mathematics and the rest of the world. In the past three decades, research has arrived at considerably more sophisticated theorizations. For example, in a meticulously argued essay, Lave (1992) suggests that word problems do not merely disconnect from reality but also serve to impose mathematics on everyday tasks that either do not involve mathematics or involve a different type of mathematical thinking. Using literary theory, Gerofsky (2004) recasts word problems as more than simply fictitious, but as constituting a literary genre with its own requirements and traditions. Dowling (1998), by analyzing the way word problems function within the larger institution of schooling, concludes that beyond any pedagogic function, these problems serve to create various mythologies around the usefulness and relevance of mathematics and, by extension, schooling in society. Various empirical researchers have studied the way students navigate the relationship between problems and reality and concluded that students situate the problems within the very real confines of their own experience of schooling (*e.g.*, Inoue, 2009; Schoenfeld, 1991).

To further the investigation of word problems, we must ask what foundations underlie the way in which we frame our questions. This line of inquiry is encouraged by a relatively recent book that brings together and summarizes the literature on word problems (Verschaffel, Greer, Van Dooren & Mukhopadhyay, 2009). The editors argue that this literature shares three stated assumptions:

1. that those involved in mathematics education should not use word problems or any other form of pedagogy "mindlessly, 'because that is how it has always been done'", but "rather need to reflect deeply on what they are doing and why";

2. that "attention should be given in teaching mathematics to make connections with children's lived experience";

3. and that "in contrast to typical … teaching in schools … an understanding of the very idea that mathematics can be used to model aspects of reality, and that this process is complex, and has many limitations and dangers, is essential to effective and responsible citizenship." (p. xxv)

In this article, I argue that each of these three stated assumptions envelops an essential and hidden assumption. The above statements, respectively, take for granted 1) a particular relationship between thought and history, 2) an unexamined description and even valorization of "lived experience" and everyday life, and, 3) a non-dialectical conception of "mathematics" wherein mathematics stands as a cloistered and coherent discipline, unaffected by the multifaceted reality whose aspects it models. Examining these hidden assumptions, I believe, can uncover new possibilities for research and theory.

In Verschaffel *et al*.'s first identified assumption, history is acknowledged as a factor impacting the present, and its impact is to be identified and critically understood by educators. However, here, and in the literature in general, the role of history seems to be limited to the force of tradition. It is of course true that educators need to "reflect deeply on what they are doing and why," instead of taking traditions for granted; but history does not only hand us what we do, it also forms the lens with which we reflect on what we do. Just when we think we are no longer acting with blind traditionalism, when we are at our most inventive, we may be most at the mercy of unrecognized, historically-formed assumptions. It is perhaps a lack of attention to this role of history that has allowed the literature on word problems, with the exception of Gerofsky's work, to be ahistorical: Lave (1992, pp. 74-75), for example, ignores historical evidence to incorrectly assume that the disconnect between word problems and real life activity are only recent products of modern schooling; and Dowling (1998) does not discuss whether extremely similar word problems throughout history also served to mythologize mathematics education. Why, for example, would Babylonian scribes need to pretend that their mathematics was applicable to all areas of life, as do modern mathematics textbooks?

Gerofsky's work, on the other hand, contains a detailed historical study of word problems and their functions throughout history. Drawing on evidence from Ancient Babylonia, Renaissance Europe and contemporary schooling, Gerofsky (2004) concludes that "the format of word problems has survived and thrived for 4,000 years while its purposes have changed from riddles to exemplars of mathematical generality to practical, applied

problems" (p. 131). The statement is admittedly an over-simplification of Gerofsky's complex research; nonetheless it illustrates my main criticism of the literature's view of history as a linear process leading to the present. All three purposes that Gerofsky names for word problems are, in fact, present *simultaneously* in each of the historical eras she examines (see Harouni, 2015; Hoyrup, 1994; Jackson, 1906). Each purpose, as I will show, is connected to a particular labor context and, therefore, to a particular type of economic institution. The question here is: why would certain institutions come to dominate an epoch or our current view of it? An explanation of such phenomena, I believe, is possible through an understanding of history that incorporates political economy: that is, the processes through which societal divisions interact with all aspects of culture.

The second stated assumption, which concerns the role of students' "lived experience," is connected to the above assumption about history. The literature on word problems tends to take concepts such as "practice", "everyday" and the "real world" without clarifying the tremendous complexity that these terms mask. In nearly all the literature I have mentioned, students' "everyday life" appears to be beyond criticism, a place of good, practical, and useful sensibility, in which the fictionalized settings of word problems can appear to be an irrelevant and even damaging outsider. But the everyday also produces waste, alienation, oppression, and organized ignorance (see Lefebvre, 1971). We can see the importance of this problem more clearly if we notice that even in Frankenstein's Freirian approach to teaching mathematics, in which word problems are reshaped to reflect socio-economic conflicts (Frankenstein, 2009), students seem always to reflect on

an *external* reality. A major step in Freire's pedagogy, i.e. reflecting on *one's own* complicity in relations of domination (Freire, 2000, p. 115) is not described.

By proposing a sociological investigation of mathematics education, Dowling (1998) and other researchers (for example, Gellert & Jablonka, 2009) have taken major steps toward remedying the above problem. By studying the institutional setting of mathematics education, Dowling reframes the school as a problematic part of students' lived experience, rather than a place isolated from reality. Dowling also acknowledges class divisions in society and how these divisions may shape the discourse within and surrounding mathematics education. In this way, he arrives at a conception of word problems as working beyond their immediate pedagogical purpose, as part of a discourse on the utility of mathematics, education, and schooling; a discourse that further enables schools to reproduce social inequalities. This analysis can be deepened to take into account the underlying structures of societal division, that is, the division of labor that occurs within specific modes of production. In this approach, everyday life is reestablished as a site of economic and political activities, the most important of which are production, alienation of labor, exchange and consumption, within which mathematics and mathematics education operate as flexible instruments.

The third stated assumption of the literature concerns mathematics as a discipline that can be used (not unproblematically) to model the world. The literature's overall conception of mathematics is undialectical. Dialectical thought, by not taking its own categories and relationships for granted, *i.e.* by acknowledging thought as always mediated by the

systems (economical, political, cultural, *etc*.) that produce it, encourages a constant re-examination of reality and one's relationship to it. By assuming that "mathematics" refers to a unified, coherent body of knowledge, the literature creates the illusion that mathematics is shielded from all influences that do not come from within the (vaguely defined) discipline of mathematics itself. This is essentially a metaphysical (Hegel, 2010, pp. 64-78) conception of mathematics. My purpose here is not to offer a new, dictionary-ready definition for "mathematics": that would be to fall back on simple, metaphysical thought. Rather, I propose that we see mathematics as a vague category that refers to a host of practices mediated by historically situated activities ranging from accounting, to engineering, to social statistics, to philosophical work, and many others, last but not least of which is modern schooling. By extension, word problems also lose their simple definition as a single genre or a single pedagogical device and become part of a more complex conception.

Many elements of this type of thinking are already present in the literature. My proposal is to push these ideas toward their ultimate conclusion, so that they take into account the entirety of the socio-political systems that underlie them. This article contributes one step in that direction.

Categories of mathematics and their word problems

Hoyrup has shown that the institutional setting in which mathematics is practiced and/or taught impacts its character (Hoyrup, 1994). To Hoyrup's thesis I add a second: that, in

divergent contexts, similar relationships to labor and society lead to certain similarities in mathematical knowledge, practice and pedagogy (Harouni, 2015). Sixteenth century European mercantile mathematics, for example, shares many idiosyncrasies with Babylonian scribal mathematics, similarities that can be explained in terms of the comparable administrative stance of scribes and merchants toward the world. Word problems are not an exception to this double-sided thesis. Even when used in the isolated school context, these problems have to refer to a view of mathematics and the world, and these views are inevitably shaped by politics, culture and economy.

From a study of these similarities and differences I arrive at the idea that there is not one mathematics, but various *interrelated* categories of mathematics and, consequently, word problems (Harouni, 2015). These categories must *not* be taken taxonomically, as if they designate species of a larger genus. My categories are tools meant to help our thinking as we reincorporate mathematics into political economy. They change with context, culture, and scientific development. Within certain situations one category can even turn into another, as when statistics, which I primarily categorize as a *social-analytical* form of mathematics, is turned into a purely *commercial-administrative* instrument.

Finally, I have opted for including categories that can help us better understand our current situation. So, even if there may be enough evidence to speak of, for example, a category of *architectural mathematics*, doing so would not be helpful in understanding the dilemmas of modern schooling. A related category, which I refer to as *artisanal mathematics*, on the other hand, despite its very broad embrace, sheds a strong new light

on our situation. I ask the reader to be aware of the subjective aspect of my descriptions. In my work with teachers and educators, I encourage them to reconfigure the categories and propose, using historical evidence, other categories that they feel relate more clearly to their own contexts. The strength of my own divisions is that they relate closely to divisions of labor, which at this point in history have universal relevance.

Commercial-administrative mathematics

The similarities between the mathematical texts recovered from Babylonian scribal schools and what was taught in Renaissance-era books on commercial arithmetic are startling (Harouni, 2015; Hoyrup, 1994), and include an enthusiastic penchant for word problems. We find in both places the same emphasis on problems designed for ready-made algorithms, the same computational problems in which objects and people are reduced to numerical relationships, and even the same proclivity for allowing mathematics to overrun rather non-mathematical situations.

In practice, scribe and merchant share a characteristic that dominates all their other professional attitudes. They stand outside the process of creative labor, measuring the relationship between owners, producers, and products. The basic building block of this relationship is labor power, exerted by the producer, embodied in the product, appropriated by the owner. In modern economy, this abstract labor is often reified as money; in scribal schools we find it expressed both directly, as labor power itself (Friberg, 1996), and indirectly as wages. The reduction of all things to a value, to

measures of abstract labor, is the root of what I refer to as *commercial-administrative mathematics*.

Word problems, particularly the computational type common in modern schooling, dominate mathematical learning most, if not exclusively, in situations where commercial and administrative sectors dictate the agenda. When researchers speak about the relationship between problems and reality, they generally leave out the subjective aspect that co-defines objective reality. Nesher (1980), for example, says that word problems are unlike "real world" problems, because a real problem does not hint at all the relevant data as well as the method required for solving it. In fact, "real" mercantile or administrative problems concern social interactions that are highly regulated, predictable, and self-contained. The primary way in which most word problems distort reality corresponds to the way in which an administrator tends to understand and act on reality.

The person posing the average computational word problem is uninterested in the particular fictional set-up of the problem, and yet he or she is interested once these particulars are turned into abstract, exchangeable values. When a problem asks how many oranges Sarah has after her mother gave her so many more oranges, we do not care about Sarah, her mother, or oranges. The movement of value from one side of an equation to another is all. The commercial-administrative mindset consistently turns its gaze on other sectors of production and consumption (including the family) and picks out new computational situations. Lave (1992), drawing a strictly commercial example (her only historical reference) from *Treviso Arithmetic*, a fifteenth-century Italian text for

merchants, argues that prior to the era of schooling, word problems referred to their immediate practical context. Her theory, however, would not explain why only a few pages later in *Treviso* we come across this example:

> I have bought 9 yards and 2/3 of cloth, 2 yards and 3/4 wide, wishing to make a garment. I wish to line it with cloth 1 yard and 1/8 wide. Required is the amount of lining needed. (Swetz & Smith, 1987, p. 133)

Which tailor, to ask a question after Lave's argument, would ever calculate the lining for a single garment in this way? The mathematical answer to the question, 23 and 17/27 yards, is, in practical terms, absurd: the tailor cannot and need not measure out 17/27 of a yard of anything.

Why should the author of *Treviso*, who need not prove the usefulness of mathematics to his audience of merchants, have felt the need to resort to such an example? The reason, I argue, has to do with where he stood: he did not think of the problem as a tailor, but as an *administrator* of tailors. The question might be absurd for one garment; it is quite sensible if we are dealing with two hundred garments and need to plan for materials. This explanation satisfies Dowling's discomfort with most of the examples he uses to argue that word problems are impractical and merely a function of the general tendency of mathematics to subsume real-world interactions.

One of Dowling's examples (1998, p. 8), involves a boy who wants to know how much tape he needs to buy in order to cover the top of a circular lampshade. In real life, of course, no one buys tape just to cover a single, small object. This seemingly impractical word problem, however, can be quite practical in the context of a factory making hundreds of lampshades. A type of mythology is, as Dowling correctly concludes, created in the problem's willingness to talk about handicraft as necessarily mathematical, but this mythologization is not a function of mathematics as such or even of scholastacized mathematics. Its base is in an administrative attitude toward all forms of creative labor.

Both Lave and Dowling, among other researchers, express a concern that word problems "mathematize the everyday" and thus devalue everyday experiences. To demonstrate her point, Lave relies on studies that show shoppers facing problems that do not correspond to the mathematical attitude represented in word problems. In slightly higher spheres of a money economy than supermarket consumption, however, there is less discord between "school math" and the "everyday." The administrator uses this kind of calculation on an everyday basis; the average consumer, clerk, and many other sectors of society do not. The confusion in regard to most (though not all, as I will show) school word problems stems from the application of a commercial-administrative attitude to *consumer* situations, while denying or hiding the original administrative purpose entirely.

Philosophical mathematics

How mathematical activity turns into an ecclesiastical or philosophical tool is beyond the scope of this paper. It will suffice to say that here the role of coordinated numerical and

geometric systems is neither to tally up abstract labor nor to help create objects. This type of mathematics is interested in patterns, meaning, and magic, which are what its practitioners offer in order to justify their social positions. This mathematics is priestly, academic, or philosophical. I refer to it as *philosophical mathematics*, keeping in mind that it can contain all or some of these three attitudes.

While there can be no scholasticized teaching of commercial-administrative mathematics without some recourse to word problems (*i.e.*, to codified examples of exchange), philosophical mathematics can be entirely free of that need. Euclid presents his geometry without any reference to other areas of activity. Biruni's eleventh-century *Instructions*, for centuries a primer for Iranians approaching mathematics as another aspect of their spiritual education, does not contain a single word problem. When a text with primarily philosophical concerns does use word problems, the questions are better characterized as puzzles. Books written in sixteenth-century Europe for grammar schools, for example, contain far fewer word problems than those written for merchant schools by *reckonmasters* (Jackson, 1906). In grammar school texts, the word problems rely on artificial set-ups; but in most cases there is no attempt to pretend at practicality. Here is, for example, a problem from Frisius (c. 1540), meant for grammar school students:

> A man having a certain number of gold coins bought for each as many pounds of pepper as equaled half of the whole number of coins. Then upon selling the pepper he received for each 25 pounds as many gold coins as he had at the beginning. Finally, he had 20 times as many coins as he had at first. The number

of coins and the quantity of pepper are required.

The problem, despite its mercantile set-up, has nothing to do with trade or any codified social interaction. Unlike the tailor problem from *Treviso*, you could not have it make sense by turning it into an administrative calculation. Puzzles of this sort are deployed in philosophical mathematics, not least because of their relationship to magic tricks: they give rise to a sensation of wonder and send the student searching for relationships.

This is not, however, to imply that this category and its problems, despite their relative intellectual autonomy, stand entirely outside the relationships of production. Hoyrup has shown that the first of such puzzles appear in Babylonia as exceptions among much more abundant examples of scribal school administrative word problems. Their appearance corresponds to the rise of scribes as a semi-autonomous professional class. At this point, scribes began to seek meaning and signs of mastery within, but also beyond, their explicit economic function (Hoyrup, 1994). It is not a coincidence, I believe, that the scribes' philosophical puzzles appear at the same time that magical and occult practices also become part of a young scribe's training (Hoyrup, 1994, p. 6). The shared origins of these two categories of mathematics points to the possibility of their interconnectedness: a philosophical approach that owes its existence to a division of labor that places the approach entirely outside creative labor, also has a tendency to subsume the material identity of objects. The character of this subsumption, nonetheless, is in most cases distinct from its commercial-administrative cognate.

Artisanal mathematics

To call the abstracting systems of formal relationships that emerge *within* certain processes of creative labor "mathematics" is merely a theoretical device. In nearly all cases (engineering, architecture, carpentry, *etc*.), the mathematical systems are inseparable from the larger practice, tied to its tools, materials and products. It is more or less a modern academic misunderstanding to imagine that the engineer studies "mathematics", and that he studies precisely the same mathematics as the future accountant. In practice, the engineer needs a certain type of "mathematics". This type of mathematics does not annihilate the identity of materials, but places measurements of their relevant aspects in relationship to tools and the final product. Number, in this case, is primarily an expression of relative magnitude, mediated by instruments and the final product of the work, rather than exchange value or patterns. The idea taught in schools that measuring is basically the counting of units is the result of a commercial-administrative mindset applied to what is essentially (due to the importance of actual instruments and materials) an *artisanal* practice.

The impact of *artisanal mathematics* on modern schooling has been minimal (Harouni, 2015). It does, however, survive in academic programs where technological learning is still tied to some level of practice, such as in various forms of on-the-job training and apprenticeship. Despite the false opposition that much of the literature tends to draw between word problems and "practice," artisanal learning is not antagonistic to word problems. Rather, it produces a distinct kind of word problem that has not been considered in any of the literature. In my own engineering training, I was often given a problem (*e.g*., to draw the circuitry of an elevator that privileges a particular floor) and

asked to *design* a solution. The fictional client with a fictional piece of land is a regular presence in architecture schools. The set up is imaginary, verbal; the solution is tangible, specific, and yet open-ended.

The absence of this type of problem in both school mathematics and the literature on word problems is a sign of the supremacy of commercial-administrative and philosophical socio-economic attitudes in modern times. One of the few efforts to go against this trend occurred in Soviet Russia, where Vygotskian psychology, under the influence of Marxist philosophy, emphasized the role of instruments and creative labor in shaping cognition. In the Davydov curriculum, for example (see Schmittau, 2010, p. 256), where measuring objects is the primary means of acquiring number sense, the teacher might ask the children to pretend to be members of an ancient people who do not know how to count. He then asks them to measure the length of an object and communicate it to another group. Students must invent objects and forms of measurement from what they can find in their environment (twigs, tokens, *etc*.). This is a word problem, but it refers to a creative activity, the product being an instrument for measuring and communicating length. Notice, however, how quickly the process has become self-referential: the product of measurement is an instrument of measurement. The Davydov curriculum could not offer true artisanal questions, because the Soviet school, like its Western counterparts, was isolated from actual creative labor.

Social-analytical mathematics

Under historical conditions where one or more social groups begin a critical examination of the commercial-administrative activities of itself or another group, a new type of mathematics emerges. This type, exemplified by social statistics and economics, is very closely tied to commercial-administrative mathematics; so closely that, once deprived of its critical agenda, it can itself become an administrative instrument. Used as a critical tool, however, it can reverse the reductive tendency of commercial-administrative activity: its aim becomes to take numbers and try to retrace them to the social experience to which they refer. In the process, *social-analytical mathematics* highlights imbalances that we could not have articulated without its aid. Rigor, which in commercial-administrative mathematics was defined by the ability to best predict an outcome, here refers to the doggedness with which one reconnects numbers to real-world situations.

Many educators have proposed that this type of mathematics, *i.e.*, mathematics that can help people "read the world," is the most appropriate for school education (*e.g.*, Frankenstein, 1989; Gutstein & Peterson, 2013). A large array of curricula have been proposed according to this ideal. Frankenstein (2009) offers one of the most sophisticated conceptions of this mathematical attitude, particularly in relationship to word problems. Here numbers are seen as "describing the world", which, as she points out, also includes "hiding" certain relationships, thus mystifying them. The role of calculation, instead of to predict results, becomes to discover, interpret and evaluate relationships. All these mathematical instruments, finally, are deployed with an understanding that they must constantly refer back to the world, forming and reevaluating connections.

This perspective leads to a new understanding of word problems as well. We can understand them as instances of Freirian "codifications": texts that contain a complex of relevant themes or contradictions, presented not in order to be solved, but in order to be decoded (Freire, 2000). For a reader, to decode a codification means to reformulate the relationships contained in it by applying his or her own consciousness, thereby generating a discussion that deepens his or her own, as well as a group's, understanding of a social theme. Nearly all of Frankenstein's (2009) examples fulfill these requirements, although they do not ask students to evaluate their own participation in the social problems that they are investigating.

Without a strong theory that distinguishes social-analytical mathematics from its commercial-administrative origins, however, the social-analytical word problem is in constant threat of returning to an uncritical stance. Gutstein (2006, pp. 238-240), for example, in a lesson on the impact of real-estate development on a low-income neighborhood, presents his students with a local newspaper article on the topic and then asks them a series of questions, which they are to answer and discuss. For instance:

> If a family needs [a yearly] income of $47,000 to buy a $125,000 house, how much is needed to buy a $350,000 house? (p. 239)

This problem, treated in separation or even in relationship to a newspaper article is still a commercial-administrative one. Notice how the ultimate aim is a calculation; how easily we can replace the situation with another; how the problem takes the business transactions that it refers to for granted (*i.e.*, that families must have an income, that they

buy houses, that they borrow money to buy houses, *etc*.) It becomes a social-analytical problem only once it is used to engender discussion. Even then, the dialectic between the two types of mathematics is not at all simple. The moment students cease to go further in their critique, they risk falling back into a commercial-administrative mindset. In this sense, the above example is doomed from the beginning: its original relationship to the world is administrative, not analytical. Bankers do not calculate loans using simple proportion, as the question implies. Therefore, the students (unless they are trained to critically tear apart anything that the teacher offers them) cannot use the problem as an artifact to discuss loans and bankers. The question is not whether the word problem is real or not: the question is how it opens to reality.

Conclusions for school mathematics

How can we understand word problems as they are used in modern classrooms? The vast majority of school word problems are of the commercial-administrative type, applied to a vast array of situations; but this answer does not convey the complexity of the issue. Schools do not teach commerce or administration. The content of school mathematics has undergone a transformation, commensurate with changes in society and the aims of schooling. Overtly, today's school word problems do not refer to administration or production, but consumption. The chiefly domestic setting of these word problems, to which an administrative mindset is applied, is a result of the interactions between school and society, between consumption and administration. The everyday experience of students is one of consumption; the overall social attitude that oversees their everyday

experience (including their schooling) is commercial-administrative. This attitude is so dominant that even the critics of traditional schooling usually fail to imagine an alternative that escapes its limitations.

Let me return to the hidden and stated assumptions of the literature. Gerofsky (2004) is correct to suggest that if word problems are to be studied at all, whether by students or researchers, the study must not treat the problems as neutral conveyors of mathematical knowledge, but rather must take them critically, to explore their ambiguities and limitations (p. 142). In their essence, these ambiguities and limits correspond to antagonisms and restraints within society itself; and society works hard to hide the true nature of its internal contradictions. Bar an awareness of the role of these social forces, the more minutely we analyze a word problem, the more we are in danger of being seduced into ignoring its larger implications. Even critical perspectives are not immune to this danger. Lave (1992) offers the astute observation that school word problems often turn "problems of sense" (*i.e.*, ones that can be addressed by people within their daily activities) into "problems of scale" (*i.e.*, ones where the sheer magnitude of the activity places the situation beyond the immediate reach of students). But in concrete terms, what are these activities and what defines their scale and people's access to them? Are they anything other than relationships within the system of production, consumption and administration? Only a properly historical perspective can force a word problem to finally speak its own name.

We cannot expect educators and students to arrive at a critical understanding of word problems by relating mathematical modeling to their own "lived experience." The commercial-administrative attitude that sees the world as an endless series of calculations is the result of life within a money economy. Those who worry that school word problems trivialize and reduce the complexity of everyday life have put the cart before the horse: everyday consumer life and its progenitor, alienated and administrated labor, are the *primary* trivializers and reducers of life. Dreams of inexhaustible wealth, of ever-greater production and consumption, operate within the average student as within the simple, average word problem. The main task of a critical mathematics education may be to constantly struggle against its own mathematical logic, which on the one hand is encouraged by society to dominate all aspects of life, and on the other hand can reveal the mechanisms of social domination. In this light, word problems that (perhaps by being overtly absurd) force students to come up against the material, psychological and moral limits of commercial-administrative logic may be more to the point than word problems that unwittingly encourage students to increasingly channel their own moral and emotional experience through that logic.

Finally, as mathematics, too, is demythologized so that it is understood as changing with context, at least three possibilities open up for a reframing of word problems. The first is the possibility to examine the content and meaning of word problems in relationship to their larger purpose, a step I have somewhat demonstrated in this article. Second, we can begin to pose word problems from relatively coherent mathematical vantage points that could not be clearly imagined if we were to remain stuck in common sense definitions of

mathematics. Not only can we think of artisanal word problems, we can keep an eye on our own socially-acquired tendency to let artisanal activity be subsumed by other categories of mathematics. Third, the possibility of interaction between categories, in turn, presents us with entirely new possibilities for word problems, where types of mathematics that until now had remained within separate sectors of the economy can mix to form new categories. Word problems that, like artisanal problems, take into account the limitations posed by material conditions, if combined with the speculative aspect of commercial-administrative problems, can help set a proper foundation for a reworking of both categories within an environmentalist approach to administration: an approach wherein the administrator must constantly reconsider the impact of each act of exchange on the social and physical environment in which the act takes place.

All these possibilities, however, are at this point purely theoretical. There is no way to predict how researchers, educators and students, within particular situations, will re-examine or reinforce their existing relationships to mathematics, education and society. The structural elements, whose power I have outlined in this essay, by necessity will constantly re-introduce themselves into any attempt at new forms of practice. It is therefore only in practice that the extent and limitations of these possibilities can be determined.

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A “Why” Approach to Mathematics Teacher Education

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Abstract

The mathematical training of pre-service teachers has, for the most part, focused on the content and method of teaching. Most teachers, like their future students, do not know how to grapple with the common classroom question regarding *why* math is taught and *why* it is taught the way that it is. In this article the author describes an experiment in working with pre-service teachers to raise and address such questions. Using a dialogical approach, the teacher-researcher presents his students with artifacts and problems that embody some of the defining tensions of mathematics education. Through in-depth discussion, fieldwork and exploration, students eventually arrive at a more critical understanding of the social purpose of mathematics and the impact of this purpose on its teaching and learning in various contexts. The results include an expanded vision of the possibilities of mathematics, a radical critique of its place in society, and reports of reduced math anxiety as well as increased curiosity toward mathematics.

Week 8.

"I'm going to try and say it again, but it won't come out right," says Shannon, a 19-year-old pre-service teacher, after a break in the class discussion. "There is 'one', and it's like an adjective, 'one apple', 'one kid'… But, what Plato is saying, there's also 'oneness', and that doesn't have anything to do with things*… oneness is the concept of 'one', you can't cut it up like you do with an apple and then call each part 'one half of an apple'. Oneness is its own thing."*

The class is reading through the section in Plato's Republic *that deals with the teaching of arithmetic* (Plato, 2004, pp. 255-7)*. Shannon's explanation is the closest the group has come, after forty-five minutes of discussion, to articulating Plato's distinction between two definitions of the number one – in Shannon's articulation, it's the distinction between 'one' as something that describes real things and one as a concept unto itself, the pure number. The class is at the same time grappling with another issue from the text: Plato believes that each conception of numbers belongs to the education of a particular sector of society.*

Jim has focused on this aspect of the dialogue and returns to it: "He [Plato] wants the important people in the city to study arithmetic, but" Jim stops midsentence and after a few seconds gives up on articulating his thought and goes back to squinting at the page in front of him.

"Go on," I say.

Jim reads directly from the text: "It would be appropriate… to persuade those who are going to take part in what is most important to the city to go in for calculation and take it up, but not as laymen do… not like tradesman and retailers, caring about it for the sake of buying and selling, but for the sake of war and for ease in turning the soul itself around from becoming to truth and being" (Plato, 2004, p. 220).

Aaron, who is usually quiet in the discussions, makes the connection that Jim seems to be hinting at: "Learning about 'one' is for regular people. Learning 'oneness' is for people who run the city."

Jim seems to feel the importance of this point more than the others in the class. It takes until the very end of the discussion for him to finally voice what is bothering him. "I think," he says, "in school they only teach us about 'one', but never about 'oneness'…"

Immediately Janine and Dana disagree with him. Their own experiences were different. Jim, however, holds fast to his opinion. In his journal entry for the week he writes: "I keep returning to the question of oneness. I feel like in my [teacher training] courses I'm told what to teach, but no one ever talks about why we are teaching things. Why would you teach more about 'one' than 'oneness'?"

1. Introduction: Purpose and contradictions

The "why" of a subject matter or a curriculum – why it is taught and why in a particular way – can never be adequately taught as a lecture. The questions always involve arguments the strength of whose reasons, if not subjective, are at least dependent on one's relationship to society. There are, of course, explanations that can help our thinking. We can, for example, draw genealogies of current practices in order to understand the forces that shape them (e.g. Harouni, 2015b; Hoyrup, 1994; Swetz & Smith, 1987). We can also study the institutions that surround a subject matter, try to understand who and what frames and formulates educational discourses, and search for and study alternative practices (e.g. Freudenthal, 2002; Schmittau, 2010). These are useful measures and can partially be expressed in coherent texts that a prospective teacher or researcher can read. What, however, is the reader's own genealogy, his or her own place within institutions, discourses and practice – these require a concurrent investigation that will constantly face the investigator's own limitations, not least among which is his or her formal education. Math is the math that we were taught – and it was usually taught with no questions asked. The process of better understanding the purpose of pedagogy is inevitably about deepening one's own questions.

In this paper I will describe an attempt at teacher training that reaches beyond proscribing *what* to teach and *how* to teach in order to provoke the *why* questions of mathematics education. The attempt is fundamentally different from offering potential teachers explanations of a curriculum or a pedagogy that they are to deploy in their future

classrooms. My concern is with those questions for which there are no entirely objective explanations – that is, with pedagogical problems that spin out of contradictions at the core of the individual's relationship with mathematics, society, and schooling.

In my investigations into the theory and practice of math education, I have tried to unearth the hidden assumptions of school mathematics (2015b). The theoretical aspect of the project has involved an archeology of math education, clarifying the relationships between math, society and schooling from a historical perspective. The contradictions in math education that the theory identifies are rooted in more basic contradictions in society at large: division of labor, the divide between work and schooling, and the tension in a rather uniform approach to teaching math despite math's manifold practices. The result is a theory that points out, among other factors, the confusions that arise as age-old practices are taken up by schooling in the modern era. The purpose of the theory is to help the reader reject his or her role as a mere consumer of math education and move toward an informed vista for critique and creativity. What I will describe in this essay is part of the practical application of that theory – an intervention in teacher education.

The element of practice is an essential part of my theoretical work. No theory can adequately identify all the contradictions that its readers have experienced, nor can it perfectly anticipate its own impact, the way in which it is taken up and transformed into new questions. Therefore the themes and contradictions that form the basis of a training program, such as the one I will describe, always partly emerge out of the experience of

the program itself. They are the result of co-investigations conducted by the teacher and students.

In the above discussion on Plato, I can identify, with the experience of the course behind me, at least four contradictions addressed by the students. Each of these themes or contradictions is interconnected with the others. I will briefly name them in order to use them as an organizing principle of this paper.

The first theme or contradiction involves the student's own relationship to schooling. Although modern schools claim to prepare their students for every profession and social position, in reality students must go on to live in a society defined by strict division of labor and, therefore, trainings. How, the students ask, do schools define their purpose, and what roles are allotted to students? There is, usually, a strict divide between those who set the curriculum and those to whom the curriculum is applied. And yet even the briefest open investigation of a curriculum – here, Plato's – gives rise to a sense of unease among the students. Why have I been subjected to this particular vision of math and education, Jim, my student, asks himself in his journal entry.

The second contradiction, tied to the first, arises out of the students noticing that different societal sectors – in Plato's case, the aristocratic philosopher and the lay buyer and seller – can have different conceptions of mathematical needs and even mathematics itself. What is the relationship between school math and math as a part of labor? Jim's frustration with his own training as a future elementary school teacher is legitimate, not

only as a challenge to educational hierarchies, but as a form of concern for one's own place within society. The opening pages of mathematics standards and curriculums are chockfull of truisms about the usefulness of mathematics in labor and citizenship; but the individual mathematical topics in these texts are presented without clear and specific justification (Heymann, 2003; Lave, 1988; Noyes, 2007). The lack of justifications serves to further obscure a basic feature of school mathematics: in schools, math is more than itself, because it is not just a skill, but also a sign in a false economy of diplomas and certifications. For that very reason, having lost its real cultural and economic context and, therefore, meaningful expression, school math is in a sense something less than math – it is a shadow, a simulation of skill-earning played out on academic records. Can this contradiction between school math and practical math ever be overcome?

A third tension arises out of the conflict between the commonly held opinion that math is a single, coherent discipline, and the fact that the practice of math, as well as its teaching and learning, can assume wildly different forms. In Plato's example, we are confronted with two historical views of the definition of numbers. On the one hand numbers can be seen as entirely abstract ideas, and on the other hand they can be definitely tied to material objects. Neither view can be easily dismissed. In my historical research I have shown how, until the modern era, each definition was adopted by a separate economic sector (Harouni, 2015b). Even today, the accountant and the engineer, for example, employ numbers according to quite distinct working definitions (even if pressed to present a theory of numbers, they would defer to the professional mathematician's formulation). To pretend that we know perfectly well what it is that we talk about when

we talk about "mathematics," as if it were a single, perfectly defined discipline, we would be turning math into myth. This mythologization, however, is the rule when it comes to the discourse surrounding math education.

A fourth conflict concerns math as a social and, therefore, political instrument. In the discussion of Plato's *Republic*, this theme is present only in its embryonic form, as a suspicion that perhaps Plato wants the general populace *not* to have access to a certain type of mathematical thinking. There are many aspects to this tension, but one in particular is important both to understanding math education and to the class discussions in my course. Just as math has the potential to illuminate relationships, it also has an essential, abstracting function that obscures reality. Salient evidence of this conflict is the politicians' use of statistics to obscure the realities of a situation – realities that could not have been expressed without the use of numbers in the first place (Frankenstein, 2009). The problem, however, runs much deeper, stemming from those activities that engendered math as their most potent intellectual instrument: commerce and administration. Math as a commercial-administrative tool upholds the entire field of economic relations, the reduction of living people's life-force to abstract value – wages, work hours and exchange rates. In the same breath that we speak of the uses of mathematics, we must address its abuses. One is not external to the other.

Schooling as an institution is so well-established that it can deal with these four basic contradictions quite off-handedly. Teacher training programs for K-12 teachers can engender a self-referential logic that delivers a seemingly reasonable justification for

school mathematics. A rather common approach is to add a “math in context” course to the training curriculum – usually a piecemeal approach to describing the relationship between various topics in mathematics and real-world activities. This approach inevitably takes some version of school math for granted. It starts from the textbook and offers justifications for each topic in terms of the world. But because the world is a dynamic place while textbooks are static orderings of knowledge, often brought together under a half-understood definition of “practice,” such courses have neither the ability nor the intention to deal with contradictions as deep as the four that I have named so far.

We find a different solution in books that, addressing teachers, try to initiate them into the disagreements surrounding math education (e.g. Freudenthal, 2002; Heymann, 2003; Noyes, 2007). It would be impossible to pass judgment on all these efforts; however, I can point out one significant shortcoming endemic to the effort itself. Because these books are designed for teachers, who are supposed to enter the classroom and act as confident agents, the authors try to eventually arrive at a place of productive harmony. The tendency is pervasive in the most influential studies of math education (e.g. the works of Hans Freudenthal or Alan Bishop). Though the authors acknowledge that math curriculum is subject to the value systems that form it, including the teacher’s own, they end up presenting curriculum design as a sort of balancing act between various values and priorities. But can a teacher ever strike such a balance on her own? Of course not: she does not even have access to all the relevant information. To strike that balance, one must conclude, is the task of experts, national commissions, university professors, and curriculum designers. The teacher must once again retreat to her socially accepted

position as a mere functionary, accepting the greater wisdom that is handed down to her. But how is one to know who to trust? Even the national commissions who claim to shoulder the burden of balancing the curriculum are not sincere. Curriculum design and standard setting are, by definition, forms of tilting knowledge toward a particular perspective. Values, spoken or unspoken, shape the curriculum. The fact that national commissions such as the National Council of Teachers of Mathematics (NCTM) in the US, and Mathematics Counts Commission in Great Britain, after consulting experts, stakeholders, etc. arrive at a curriculum that barely differs from what was already in place is not a sign of the efficacy of the traditional scope-and-sequence but of the unwillingness of these commissions to take a stand and invite disagreement (for a discussion see Harouni, 2013b). They rest secure within the structures they help fortify.

We then arrive at a third, more candid approach: to speak to teachers from a specific, declared political and cultural position, shedding off the cloak of disinterestedness. This is the case, for example, with what is referred to as "critical math education" (Frankenstein, 2009). In this Freirian approach, the teacher studies the political conflicts that help shape one's pedagogy. The method is more honest, but so far it has not matched its convictions with the required rigor. The conflicts and contradictions of math education involve mathematics itself. Unless a pedagogic approach already contains a strong theory of mathematics that allows the teacher to distinguish the contradictions of math education, it will have no choice but to take math itself for granted. This is precisely what plagues critical mathematics, which has a great deal to say about how math should be applied to social issues, but has almost nothing to say about how politics would impact

the very logic of arithmetic, algebra, geometry and so on. This is also why critical math education is practically silent on elementary education, offering no perspective on basic mathematical learning. The implications of this lack of theory are dire for teacher-training: the teacher, who is supposed to take a critical position vis-à-vis society, is to accept the fundamentals of her subject uncritically.

In brief, engaging the conflicts and contradictions of one's own practice is by no means a matter of simply being aware of such conflicts. It means actively entering them, gauging them against one's own experience and values, understanding them in light history and practice, and seeing them in the context of communities and society at large. To any conception of teacher knowledge I add this necessary compliment: the ability to act as a cultural critic at least within one's own practice (Harouni, 2013a). This is different from learning *about* theory (though that might be a component); primarily it means to enter that place of reasoned uncertainty in which theory is forged.

As I will show, if this is to receive some aid from a teacher trainer, both the methods and results will vary from what we have come to expect from teacher training courses. Instead of trying to justify math education, the course would try to strip it of all its apparent justifications. This means to act in conflict with many accepted notions, including the unquestioned faith placed in the usefulness of mathematics. So, even though at the end of our semester my students reported considerable improvements in their comfort with mathematics, their understanding of teaching and learning, and their curiosity about math – among other positive indicators – many also spoke of feeling

tensions that they did not feel before. The intent was not to train happy employees, but educators who are able to strive for autonomy.

2. Context and methodology

The course I will describe took place in a New England university with a strong teacher-preparation program. Fifteen of of my 22 students were either pre-service teachers or future school counselors. As with the majority of pre-service educators in the U.S., they had entered college with even more aversion toward mathematics than the average college student (Gresham, 2007). In the beginning of the course eighteen students described themselves as experiencing mild to very high levels of math anxiety – the fear to perform or even think about mathematics. The university marketed the course, titled "Problem solving: A critical approach to mathematics," primarily to students who were not proficient enough in math to be able to pass their teacher certification exams. The students who were not in teacher-training programs took the course either out of personal curiosity or because they thought it would be an easy out for their one and only college math requirement. We met as a class for 12 weeks, over 22 sessions, each 75 minutes long (an administrative decision made by the university: my own preference would have been for weekly, 3-hour meetings).

My teaching-research method was an extension of what Eleanor Duckworth has called Critical Exploration in the Classroom – an application of Jean Piaget and Barbel

Inhelder's clinical interviewing method to teaching and learning (Duckworth, 2006). The teacher presents materials or a problem that he or she expects will give rise to a range of contradictions in the students' minds. As they encounter the materials, the teacher-researcher tries to understand their thinking by asking them questions and giving them secondary problems or materials that complicate their reasoning. As students wrestle with their own reasoning, and as a trusting environment is established in which all participants feel free to give voice to their confusions, they come to ask more probing questions and think more rigorously. From the process the teacher-researcher gleans new data for designing new materials and investigations. He or she learns not only about a discipline or about the development of concepts in students, but about the possible interaction between the world and people's consciousness.

The pedagogy, as Duckworth (in personal communication) has pointed out, closely resembles Paulo Freire's teaching methodology. Freire referred to his didactic materials as "codifications," which in the process of "decodification" by a group provoke the discussion of certain themes and contradictions, filtered through the students' experience and language (Freire, 2000). An essential aspect of Freire's method is that the codifications should contain contradictions that relate to the socio-political structures that surround the students. The result of the decodification process – that is, the discussion – is itself a new codification worth exploring (I would regularly hand my students a transcript of their previous talks, allowing them to better grasp their own ideas). While investigating texts and artifacts allows students to voice the contradictions they

experience, the secondary investigation of their voiced ideas helps them grasp their experience and understand its underlying individual and communal structures.

The data gathered from the course was substantial. It included pre- and post-surveys, transcripts of class discussions, weekly student journal entries and fieldwork reports (no less than 5 double-spaced pages from each student, each week), one-on-one interviews, and final papers. After nearly every session a trained teacher-assistant and I shared and discussed our observations, deciding on the best way to proceed forward. In other words, the curriculum was not set in advance. This report, then, does not describe a curriculum so much as it does what considerations can create the type of continually evolving curriculum that would address the fundamental questions of math education.

3. Course components: activities, artifacts and readings

Whereas in traditional teacher education the goal of every activity is to familiarize students with some aspect of the topic, in this work perhaps the first goal of each activity is to *defamiliarize* the topic, creating a distance from whose vantage they can view their own basic assumptions. An unprescribed curriculum is key, because it allows the facilitator to flexibly deploy each component to disquiet the particular stasis reached by the group. The three main components of the course – *problem solving, artifact decoding,* and *reading discussions* – serve a range of pedagogic purposes. Where I needed to defamiliarize math as an *action*, we engaged in fieldwork and problem-solving; where we

needed to challenge our *conceptions* of mathematics, we switched toward readings and historical artifacts.

Problem solving refers to hands on activities that engage students in a math-related phenomenon. Each activity would in some way inspire the question: is what I am doing mathematics? One activity (inspired by Duckworth) asked students where on a wall should they place a small mirror in order for two people standing at random locations in the room to see each other in the mirror. For an hour or so students played with mirrors, strings, and flashlights in order to come up with a sure-fire way of choosing the right spot on the wall. Was this mathematics?

Voicing their doubts gradually brought them to one of the main conflicts in math education. Henry, who was by far the most mathematically proficient student in the course, found a geometrical solution that was quite satisfying to him. His teammates eventually figured out a solution using a string and very little formal mathematics. It turned out that the cottage-industry solution was faster and, for our practical purposes, more accurate. Were we dealing with a mathematical problem? Eugenia formulated the confusion in a later session:

Eugenia: When we started doing the mirror thing, I was sure we would eventually get to math. But it turned out most of us really didn't need to do that.

…

Taylor: You can't teach yourself most of math from everyday stuff. You can't just do things and end up learning calculus.

Janine, however, disagreed.

Janine: The whole mirror thing is about angles. You can learn about angles this way… Think of playing pool. Isn't pool all geometry?

Taylor: … [in pool] you're not really thinking, "What kind of an angle am I using, what kind of formula would help me."

I will not relate the rest of the discussion. The point for now is that the debate regarding whether everyday life can lead to mathematical learning or whether math, and nearly all school math problems, tend to impose math on everyday activities that do not require them is central to any theory of math education (Dowling, 1998; Lave, 1992). The students had arrived at this question through engaging their own thoughts about one of our problem-solving activities.

Problem-solving was also a way of challenging the students' assumptions about teaching and learning. Each activity minimized the role of the teacher as speaker while demanding tremendous energies from the teacher as facilitator and problem-poser. The teacher might appear like he or she is not "teaching" or is even avoiding responsibilities; and yet the teacher is by no means inert or inactive. Once we had worked through a problem, the

students were to find someone outside the course and "teach" their volunteer learner by posing to him or her the same problem that we had worked on in class. They were to do so without revealing the solutions they had developed for themselves .In weekly fieldwork reports they described the experience, how they felt about restraining their urge to lecture, the role of the materials, and the process of watching someone learn. As they watched someone else come to his or her own conclusions, they came to further question their own assumptions about the teacher's role in the classroom.

Artifact decoding, the second component of the course, refers to the investigation of texts that are dense with contradictions. Their purpose is to spark conversations that would not normally arise, but do so without hitting the students over the head with an already formulated problem. Artifacts, or *codifications*, as Freire says, "should be simple in their complexity and offer various decoding possibilities in order to avoid the brainwashing tendencies of propaganda" (Freire, 2000, p. 115). The student formulates a problem out of her own sense of the materials. So, for example, when I give the class a set of extremely similar word problems drawn from disparate eras ranging from ancient Babylonia to modern times, I have consciously coded a number of contradictions (*themes*, in Freire's phrasing) into these artifacts and expect some of them to emerge from the decodification – i.e. the collective reading of the artifact. What I cannot expect is what students bring to the table: would it be channeled through indifference or fascination for history? Do they feel frustrated or motivated by word problems? What experiences inform or hinder their thinking? And so on. Later, by giving them a type of word problem that they have never seen in their school careers and that differs in

significant ways from the previous examples, I can challenge some of the conclusions they have drawn from the discussion.

Finally, the course *readings* addressed three broad categories: history of mathematics (e.g. Cuomo, 2001; Jackson, 1906; Swetz & Smith, 1987), pedagogy in general (e.g. Duckworth, 2006; Harouni, 2013a; Kohl, 1991), and theories of mathematics education (Dowling, 1998; Frankenstein, 1983; Lave, 1992). While I chose most of the readings before the course began, the order of their assignment depended on classroom discussions. A few readings were added based on topics that became problematic for this particular group of students. I assigned Herbert Kohl's essay on student resistance to learning because of an important discussion about students who remained silent in our sessions.

I would ignore a major aspect of the experience if I did not explain that all three kinds of readings were, generally speaking, quite difficult for most of the students. They were not used to reading theory, philosophy or primary historical texts. For the first seven weeks I did not lecture at all on the readings but allowed plenty of time for discussions. This was painful at first, because students had to grapple with texts that they did not clearly understand, and there was no authoritative voice to shield them from their own confusion. By encouraging students to speak about their observations and confusions, we eventually reached a shared belief that even the more difficult texts could be penetrated in group-discussions. This is how Henry characterized the experience in his journal for week 8:

> Never have I been asked simply "what do you see" when given a reading… It took me a while to see beyond the words, to figure out how to take a text that I know nothing about and make meaningful connections. This was such a difficult process because at first it is hard to understand why we are doing this; what am I supposed to see? We're so programmed to respond with answers that we believe to be satisfactory to the teacher that it takes practice to rid ourselves of that mindset and think freely.

Dana, in her final paper, discussed the role of the communal sharing of confusions in the development of her individual reading habits:

> It took me until I took this course to understand that I actually *hate* reading things that I don't immediately understand. It was an important experience for me to hear all the confusion my classmates had about the readings that I found difficult as well, and then to see how by sharing our confusions we could relax and really begin to understand… And I thought: this is exactly the experience children have with any reading.

In the final evaluation, almost all students indicated that the course helped them feel "much more willing to spend time analyzing a text they did not initially understand."

4. The learning experience

Time is a crucial dimension in this type of study, and so it would be intuitive to describe the learning experience chronologically. When dealing with a topic that involves multiple interrelated themes, however, the collective understanding of students expands and contracts in overlapping circles. A confusion that develops in one corner of the classroom might manifest itself only weeks later, and initially as an inarticulate non sequitur. It would be more useful, therefore, to describe the experience thematically, organized along the four major contradictions of math education that I described in the first section of the essay. To reiterate, the themes are, 1) the tensions inherent in the teacher-student relationship in the mathematics classroom; 2) the distance between math as practice and math as a school subject; 3) the contradiction between the perception of math as a single discipline and the fundamentally different approaches to the practice of mathematics; and 4) the dissonance between the potential of mathematics to illuminate the world and its tendency to abstract and reduce reality. I will preserve a roughly chronological sense by presenting these themes in the order in which each reached its zenith in classroom discussions.

4.1. Teachers and students: Who decides what and how to learn

There is a fundamental problem in exploring the teacher-student relationship in the context of a classroom in which the teacher remains an authority. The professor lecturing for hours about student autonomy or student-centered teaching is a familiar, comical image to anyone who has studied education. Even if I could eliminate grading in my

classroom – which I did for individual assignments but could not do for the final transcript report – students would still depend on me to approve their credits. More importantly, by entering the classroom we all bring in all the hard-earned behaviors we have developed through repeated interactions with authority. The only solution once we are so deeply placed within an institution is to create a critical distance from which students can at least consider those relationships.

Most of what we did in the first four weeks of the course centered on disturbing the students' notion of our respective roles in the classroom. I would barely speak, breaking my silence only to try and better understand their ideas: I would ask what they noticed, what puzzled them, or what they thought of something that a classmate had said earlier. I never offered a justification for my choice of materials and homework, and I would reassign the same reading or assignment until it caused a backlash. Once the backlash was there, we finally had something to talk about – a text that coded both their and my frustrations with the teacher-student dynamics. Because my "unjustified" behavior spanned both our regular interactions and our mathematical activities, questioning either one would immediately lead to a discussion of the other element.

The first of such reactions occurred when, in week 3, Sheila demanded that I explain what I really wanted from students during our discussions of a particularly difficult reading (chapter 2 of Freire's *Pedagogy of the Oppressed*):

Sheila: How am I supposed to know if I'm on the right track? I give you an explanation and you either nod or ask me questions that mostly just make me doubt myself... Are you just trying to confuse me and get me frustrated?

Houman: Tell me where those two options come from.

Other students immediately seconded Sheila's frustration. It was a full 10 minutes before someone offered a different line of explanation.

Madison: I think you [Houman] are trying to get us to think. You could obviously tell us the answers, but instead you want us to think for ourselves. I think. I could be totally off!

Taylor [*indicating Houman*]: Still no answers! [*Students laugh*]

Sheila: You take everything and just sit there and ponder on it…

Houman: I'd like to hear a different perspective. Maybe you have something different or harsher to say.

Dana: I think you are trying to get us to think about the way people think. It's more psychology than math!

Aaron eventually offered a more personal explanation.

Aaron: When you don't answer a question, I have to think and be more sure of my own thoughts. I work harder, I perfect my method for myself. I think you're training us to have real confidence, instead of just mirroring your confidence.

The interaction does not challenge my authority, but it definitely problematizes it. In their journals that week, many students began to discuss the role of the teacher in the classroom, including their own role as future teachers. The next week, the journals showed more concrete thoughts about the goal of the classroom. Dana wrote: "We are learning not only material but we are learning how we learn material. We are seeing the process not only of ourselves but of the overall roles of a student and a teacher."

Their relationship with math became problematic in light of school relationships. In week 5, Janine mentioned in class that math for her was really school math, and her

relationship to it had always been defined through the person of the teacher. Other students connected my deemphasizing of "right answers" to the over-emphasis on right answers in math education and wondered about the reasons. Who is asking the questions? Why do most math problems tend to have only a single correct answer? Later, in week 9, Jim, already involved in his exploration of the meaning of "oneness," described the relationship between what he saw as my teaching style and mathematics in the following way:

> It's crazy to even think about education and teaching that I've seen or been subject to when I really think of Houman's teaching style. The number "one" brings me new questions every second and that's the way math should be for students. Houman makes us think about one thing, not just one question but the **one thing** in every aspect. [Emphasis mine]

In Jim's complicated phrasing (he plays with the concept of oneness in more than one way) my classroom approach appears as a particular experience that challenges the entire experience of the student with math education. This is what he might mean when he says that I force him to think of "one thing, not just one question": by refusing to filter his thought process through my own questions, I force him to consider math or a specific mathematical topic (such as the number one) as a *whole*. It's the whole, with its infinite aspects, that bursts out in "new questions every second." To see social phenomena as pregnant with questions is the essence of a critical attitude; an attitude that requires an actual experience of one's own questioning mind.

Notice, however, in Jim's comment the idealistic tendency to want to simply replace one system for another, more improved one. "That's the way math should be for students," he says. My pedagogy – or rather Jim's perception of it – creeps in as a ready replacement for what Jim sees as traditional pedagogy. This is the danger that requires a recurrent process of self-examination both for me as the teacher and for Jim as student – a process that is by no means complete once the course has reached its arbitrarily-set finish line.

By week 5 we had established enough critical distance between ourselves and the process of teaching and learning to be able to attend to the core question of this theme: who sets the agenda for schooling and how does that impact math education? To do so, however, we needed to know more about math itself. The exploration at this point linked tightly with the second theme of the course: the dissonance between math as a school subject and math as a "real world" activity.

4.2.1. School math and the "real world" (Part 1): Usefulness as a problem

> For if numbering be so common (as you grant it to be) that no man can do anything alone, and much less talk or bargain with other, but he shall still have to do with number: this proveth not number to be contemptible and vile, but rather right excellent and of high reputation, since it is the ground of all men's affairs, so that without it no tale can be told, no

> communication without it can be long continued, no bargaining without it can duly be ended, nor no business that man hath justly completed.
>
> Robert Recorde (1543)

The passage is from Recorde's opening dialogue to his widely read 16th century introduction to arithmetic, *The ground of arts*. In the dialogue a "master" and a "scholar" debate the usefulness of mathematics, with the master, of course, demolishing the student's flabby arguments against learning mathematics. I gave this dialogue as a codification in the 6th week of the course. At first, almost everyone seemed to go along with the master's reasoning. "This," said Taylor, "confirms what I have been thinking since the beginning of the course: math is in everything. It's useful in everything you do."

This rather common perception of the uses of mathematics was subsequently applied to contemporary school mathematics:

> Eugenia: Students usually think of math as something that can't help them in their everyday life. But when it's broken down and students are told that numbers specifically can help them with their jobs and so forth, I feel as though it may resonate [with them] in a deeper way.

In its initial function, the codification has solicited just the type of thinly reasoned generalization that nearly every curriculum or list of standards dishes out in its opening

paragraphs. The argument is quite old: Plato, again in Book 7 of the *Republic,* refers to arithmetic in the same way: "that common thing, the one that every type of craft, thought, and knowledge uses, and that is among the first things everyone has to learn." (p. 216).

The praise for math as a fundamentally useful art takes a more convoluted form in contemporary educational discourse. NCTM's rather influential (if providing justification for business as usual can be called influence) *Principals and standards for mathematics* (2000) is just one example:

> In this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence opens doors to productive futures. A lack of mathematical competence keeps those doors closed. NCTM challenges the assumption that mathematics is only for the select few. On the contrary, everyone needs to understand mathematics. (p. 5)

"In this changing world," it could just as well be argued, computers decrease the everyday need for mathematics and confine its uses to even more highly specialized arenas; but it would not do for NCTM to argue against its own *raison d'être*. We should mind the tricky language of the statement: it is never clear whether it is actual mathematical knowledge or the ability to perform mathematics in school or on college entrance exams that will open the doors of opportunity.

Now that a version of these commonplace arguments was on the table and spoken with conviction, I could complicate it with another codification regarding the disconnect

between math and the real world. As it turns out, I did not have to resort to a second artifact, because the Recorde text is itself dense with contradictions. With extended discussion a second puzzle came to the foreground:

Raven:	I wonder who he is arguing against.
Houman:	I'm not sure I understand…
Taylor:	He is arguing against the "scholar," no?
Raven:	But the dialogue isn't *real*. It's not the master that's talking, it's the author [Robert Recorde]. He is trying to convince someone else that people should learn math. It means there were people who thought math *shouldn't* be taught. Who were they?
Houman:	You want to know who…
Raven:	They had to be important enough for him to start the book with them.

Raven's significant comment (in fact, NCTM is also positioning itself against unnamed opponents) came at the very end of that session. It did, nonetheless, make some impression. Once the universal need for math education is transformed from an assumption to a point of argument, we can begin to investigate what we mean by the usefulness of mathematics. To aid the process, I assigned for the next week a reading from Frank Swetz' book (1987) on the rise of arithmetic education in 15th and 16th century Europe. Swetz links the proliferation of mathematical training to an obvious fact:

the rise of money economy in Western Europe. The math taught in 15th and 16th century commercial schools closely resembles the math taught in our schools today. Many students noted this similarity in their journals. Henry wondered in writing about the aim of his own mathematical training:

> … the Swetz reading made me reflect on whether I'm learning for "mercantile pursuits" or whether I'm learning to better myself as an individual.

This is a subtly worded question. By "mercantile pursuits," the rest of his journal entry makes clear, he means both learning about commerce and learning for a certification that would allow him to, potentially, make more money in the future. He goes on to say that school never taught him how to balance a checkbook. Although our curriculum seems similar to that of a 15th century mercantile school, we have somehow lost nearly all the actual commercial content. What remains, he says, is the pursuit of academic degrees.

Struck for the first time by the existing connection between math and commerce, some students began making comparisons between 15th century commercial mathematics and math education in other contexts. Cara quoted in her journal from another artifact we had studied, an 11th century Iranian astronomy primer by Biruni that contains an introduction to mathematics (Biruni, 1984). This is the passage she refers to:

> "What is one? It is that which takes on oneness and can be named. In its totality, it does not suffer reduction or increase, and it is not disturbed by multiplication or division [by itself]. It is the root of the powers and properties of all numbers." (p. 33, translation is mine)

Cara links Biruni's more philosophical concerns with those of merchants:

> I wonder if when Italians [in the 15th century] were teaching about multiplication and division that they also taught how powerful "one" out of all the numbers is...

The very asking of the question speaks of at least a nascent understanding of a link between the institutional setting in which math is practiced and its teaching and learning. She can see that Biruni's approach to numbers would not have served any purpose in a mercantile setting. She goes on:

> What does oneness mean? The only definition I could find online had to do with a connection with God, the great Divine. And this definition is irrelevant because, as stated in the reading of "Capitalism and Arithmetic" [by Swetz] people exploring mathematics [in the 15th century European mercantile settings were] trying to stay clear of religious meanings.

Four other students point out the dissonance between Biruni and commercial mathematics. Madison, for example, quite thoughtfully expresses the role of the larger institution on a teacher's definition of numbers. She points out Recorde's definition of numbers as expressed quantities or values and contrasts them with Biruni's definition, and speculates that Biruni's status as a philosopher and astronomer may have caused the difference:

> Biruni seemed to take a somewhat philosophical view on numbers that I had never quite heard before…. [he refers to] the need for humans to have contact with something greater than themselves and how this void has an unconscious impact on our being and oneness with ourselves. I suppose this measurement or observation of stars could potentially be one of our first mathematical ventures as a society.

Historical evidence upholds the idea that mercantile and philosophical approaches to math tend to result in two distinct types of math education (Harouni, 2015b; Jackson, 1906). Evident in my students' comments are seeds of a critique of contemporary math education: to what end has it been designed? How can variations in approach be explained? I must emphasize the striking subtlety of the language used by Henry, Cara and Madison in the above quotations. The ideas stretch the language that expresses them – all three students use wordplays; they attempt to create a more complex, personal relationship with specific words and concepts. This is a consequence of the students having had a chance to formulate their own questions. Their ideas, part of an inner

dialogue, are not merely put together and delivered to a teacher; they are, one could say, possessed, spirited.

When in the following week I introduced a simple image of a group of barrel-makers as an artifact (Figure 1), the class quickly remembered the connection between institutional setting and math education and now arrived at a puzzle about the role of math in artisanal activity. Obviously people making barrels did not learn their math from mercantile schools – so where did they learn it, and did it have its own characteristics?

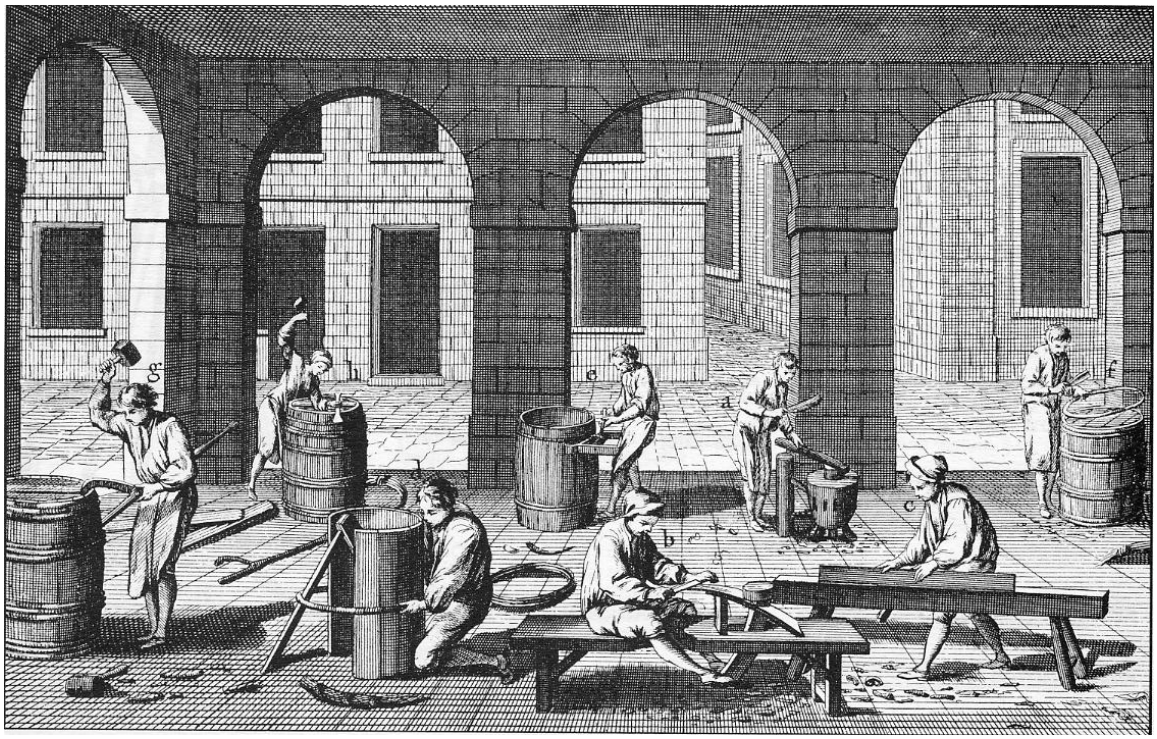

Figure 1

Taylor: [Carpentry math] seems like the ultimate working class math, just like from the beginning of time… wasn't Jesus, I

think, maybe, or his dad was a carpenter, right? … I think a lot of times people get put down for working with their hands when, in fact, watching these guys and women over here moving that whole entire building,

[*Taylor points out the window, to where a three-story historical church is being moved to an empty lot forty yards away*]

that just takes a lot more smarts, I think, than solving problems on a board. I don't know.

Eugenia: It's a lot of discipline, too. Cuz if you screw up, you're *screwin' up*. Yeah.

Raven: I also think that there are certain ways—if you practice or you think about it, there are certain ways that you can build a thing with the same materials. Depending on how you build it, it will be stronger or able to hold more. I think that kind of relates to math, as well.

The class discussed the ramifications of this type of need for mathematics, contrasting it, again, with both academic and mercantile approaches. Once again, they returned to the question of education:

Eugenia: Like where did they learn the math, if they're like the working class?

Raven: Where were they being taught?

At this point, the group began to speak about the biases with which schools evaluate skills. I will soon show how this conversation grew into a major topic for the class. However, at this point I will suspend my description of the exploration in order to show how similar concepts emerged from a different investigation.

4.2.2. School math and the "real world" (Part 2): Word problems

Word problems have been a major focus of theoretical investigations into the sociology of mathematics education (Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009). The obvious explanation for this interest is that the fictional setting of a word problem hints at a relationship between math and other social activities, and an investigation of these hints can be revealing of how math educators see the role of their subject in the world outside their classrooms. Another reason for this type of investigation is the historical nature of word problems: as old as mathematics, they crop up throughout history, in nearly identical forms (Gerofsky, 2004). What the pattern of their appearance indicates is subject to debate: Hoyrup (1994) believes they belong to situations where a subscientific branch of math is taught in highly scholasticized settings: in Babylonian scribal schools or in European mercantile schools, for example. Gerofsky (2004) believes they appeared as a form of shoring up mathematical ideas before there was an algebraic language. Lave (1992) and Dowling (1998) see most word problems as attempts by mathematicians to impose their discipline on the students' everyday experience. In my analysis word problems are reflections of different economic attitudes – the differences

accounting for various types of problems (Harouni, 2015a). The point here is not to support one perspective over the others but to show how ready the ground is for theoretical cultivation.

645,539	Barley-corns
72,171	Ears of barley
8,019	Ants
891	Birds
99	People
730,719	Diverse things

Figure 2

In week 9, I introduced the above artifact (Figure 2) – a translation of an Old Babylonian tablet, recovered from a scribal school (Friberg, 2005). The tablet undoubtedly refers to a word problem posed by a teacher to his student. According to Friberg (p. 5), it could have sounded something like this:

> There were 645,539 barley corns, 9 barley-corns on each ear of barley, 9 ears of barley eaten by each ant, 9 ants swallowed by each bird, and 9 birds caught by each man. How many were there altogether?

The artifact encodes, as the discussion will show, all the four major themes that I earlier defined for mathematics education.

Knowing nothing about scribal culture or the background of the artifact beyond what is in Figure 2, most students did not at first connect the artifact with a word problem. Many hypotheses were floated. Some students speculated that the artifact must have been a list:

Aaron: It looks like a sort of census, something for keeping track of things.

Henry: At first I thought it might be an early form of keeping track of trade, but…

Janine: No, no. What about the ants?

Henry: Yeah! The ants!

Aaron may have noticed the administrative feel of the problem (theme: math and the "real world"). If it *is* a list, however, we are in strange territory:

Sheila: But why would you count human beings [together] with things then? If it is for trade, then why are they counting humans?

Alice: Slavery?

What sort of administrator would add humans to ants and barley? Why? The discussion that followed revolved around the idea of grouping diverse things and adding them regardless of their identity (theme: math abstracting the world). Eventually, someone pointed out the pattern connecting the numbers, and so the class arrived at the idea that this is some kind of school exercise. With more discussion, the group proposed a word problem that could have solicited the artifact as a response. The questions, at this point, revolved around the pedagogic approach of the Babylonian problem-poser. Why create

such a list and have the student add these numbers? What would be the purpose (theme: math as a school subject)?

I then gave the students a list of four other Babylonian word problems. These resemble modern word problems closely, with the obvious difference that the Babylonian problems refer to aspects of Babylonian life: calculating the manpower to dig a trench, for example, or the weighing of a very large stone. With these new artifacts in mind, the discussion returned to the original artifact.

Eugenia: Well, I mean we can't really solve this, but this might be a great example of what the Babylonians—like this probably was a good word problem for them because this had to do with everything that they were dealing with at the time, so maybe this—I don't know.

Janine: I keep telling you, you're forgetting about the ants! What do ants have to do with anything? Anyway, it's a pretty abstract problem, it seems to me.

Taylor: I think that these were examples of like a classroom word problem as opposed to real life examples. Sure, they probably do relate to real life back in the Babylonian times, but I still feel like these were examples of math as opposed to representation of how much it cost to dig a trench.

Raven: These were hypothetical problems and not real problems that someone actually did to solve for something.

Taylor: Yeah, exactly. Mm-hmm. These are like their Jack and Jill problems! Their version of it back in the day, a little more advanced.

Cara: Like the 15th century problems?

At this point we have arrived at the theme of differences and similarities of math education in various contexts.

For the rest of the day we discussed scribal schools and looked at other word problem examples, the themes repeating and now consciously reemerging in the discussion as communally held questions. In the next session students suggested that they use the elementary school word problems they were designing for their other teacher-training courses and discuss their relationship to different types of mathematics and different readings. At home they read Jean Lave's dense chapter, *Word problems: A microcosm of theories of learning*, which in its sophisticated argumentation both justified and challenged their own earlier effort at understanding the role of word problems.

4.3. Mathematics as more than a single discipline

The discussions in the previous section already involve the tension between the perception of math as a single, continuous discipline and its relationship to particular

social contexts. The themes, as I have said, are interwoven. I would like to focus on how parts of our conversation around this third theme unfolded in the course of a discussion in the 10th week of study. By this point in the course, the class had been exposed to a wide range of primary and secondary sources on math in different eras, economic contexts, and cultures. In this particular discussion they used their understanding of these sources to complicate each other's reasoning. No hypothesis managed to win the day. The takeaway, instead, was a series of good questions.

12+15 = ?

27 = ?

12 > 15

Figure 3

We began the discussion by looking at the simple artifact above (Figure 3) for the second time in the course of the semester. Whereas in the first discussion the focus had been on the problematic last statement (12>15), this time they focused on the first two items. The consensus in the group was that 12+15=? is the dominant format of teaching arithmetic in schools. Janine, however, disagreed. In her school the second format had been just as familiar.

"Raise your hand, please," I said, "if your experience in elementary school was similar to Janine's."

Three students other than Janine raised their hands. I knew that Janine had gone to a private Montessori school, but I did not know about the others' backgrounds. I decided to take a risk.

"Raise your hand now," I said, "if you went to a private elementary school."

The same hands that had gone up the first time went up this time, too. What I had done was not good social science – the correlation between school type and math pedagogy in this small sample did not prove anything. I knew from my own research, however, that the second approach to arithmetic (27=?) is, indeed, more common in Montessori, Waldorf, and other alternative forms of elementary education.

There was a long pause. "That's strange," someone finally said.

Soon the group began to speculate about the causes of the imbalance. Taylor thought it was another example of "you get what you pay for." Private school parents pay to "get an extra way," he said, "they don't just learn the straight 12-plus-17 thing." Raven thought the difference had to do with entirely different views of what constitutes math education, but she could not explain why the differences would be defined by socio-economic boundaries. This discussion continued, meandering at times, until Eugenia, always ready to read beyond a text or a discussion, threw out what initially seemed like a non sequitur:

Eugenia: Do you think that certain societies use math in different ways and so they would only need to learn those certain ways? Or do they use it *because* that's what they've learned? I don't know if that makes any sense.

Reading closely and in light of the discussion that followed, we can get a sense of what she was asking: do the uses of math in society shape mathematics education, or does mathematics education shape its uses in society? The question was wonderfully complex, and it exceeds even her ability to voice it. She went on to clarify it, but in the process she moved into a perfectly related but slightly different question – the role of culture.

Eugenia: Take China—I don't know… Because they live in China, do people need to learn a certain way to do math? I think that's my question.

Houman: Then there was a second part that was very interesting. You said—

Eugenia: Yeah. Would they need to use the math that we use in the United States? Because it's US math and it might not pertain to Chinese society, would they need to learn it? I don't know. That's really confusing.

This question immediately interested the rest of the group. On the one hand, they knew that math has taken different shapes in different contexts. On the other hand, they also believed that math is a single discipline, responding to basic human needs. The second idea is usually augmented by the belief, commonly held by many scholars, that math constitutes a perfectly logical system that in its perfection must be also universal. "Every culture that builds any mathematics at all," says Kamii (1982), for example, "ends up building exactly the same mathematics, as this is a system of relationships in which absolutely nothing is arbitrary." Sheila tried to reconcile all these ideas with logic that resembles Kamii's:

> Sheila: The Egyptians did it differently, I'm pretty sure, than [people did] somewhere in Europe, just because the stuff that they had to use were different. The way that they learned to do it was different; however, it all comes back to the same point. It's just a different manner of getting to that point—[mumbling to herself] not necessarily in everything, though….

We can see that Sheila is not sure about her ideas. Janine tried to solve the problem by moving toward a post-modern explanation, softening the edges of the conflict to such fine grains that things could slide against each other without friction:

Janine: I think there's just a direct correlation between where you live, what you learn and what you do. Think about it. Most of us live in different areas…we just learned that people who went to private schools learned different things than people who went to public schools. I think it's gonna be the same thing for [different] countries and all those things. It just depends on where you live and where you're at in your life. I'm not gonna be the same as you and you're not gonna be the same as me. I feel like my math is not gonna be the same as anyone else's.

But this was not satisfying to anyone, including Janine herself. There were certainly strong similarities as well as differences between various conceptions and pedagogies of math. Raven offered a solution that took many of these arguments into account:

Raven: …If you look back in time, the math that [people] were using would be a lot different depending on where you are, because... it depended on what they had and what they needed, but now everyone has the same needs…

Eugenia [trying to make sense]: Because technology is so universal, everybody can kinda learn the same stuff now? I guess everybody—I don't know.

By "I guess everybody" Eugenia was once again reminding herself that math is certainly not always taught or seen in the same way, not even across the contemporary world. Nonetheless, the point about the divergence and convergence of economic and technological needs made a great impression on the group. Janine now tried to explain the similarities between various approaches as a matter of increased communication:

Janine: I feel it's because we're all progressing…When civilizations first started it, they didn't have contact with each other. No one really knew what was going on, so they had to develop their own [science and technology], but now that everyone's progressing, I feel like—

Aaron: Progressing toward the same thing!

Janine: Yeah. The math is, too, and it's more practical for us to be learning the same things, because we're building the same buildings they are and we're driving the same cars, so it's more practical for us to have the same equations and know the same things that others are—but back then, whatever they

were doing—their boats were way different than the ones we had, so then the math was different—there was a reason for that.

Even this explanation was not convincing. Sheila immediately pointed out that the math that is taught in schools today is extremely similar to the math taught in 15th century Italian mercantile schools. And this type of math, with some variations, is now taught all around the world.

Sheila: I understand what you are saying Janine and I agree with it, but I was also saying that if you take the stuff that was written in the 1400s, why would that have traveled across the world to another area where they could learn the same thing?

I will not recount the rest of the conversation, which remained compelling throughout the session – questions of communication, dominance, and commerce were brought to bear on the problem. I would like to draw the reader's attention to the variety and complexity of ideas that here emerge in a single discussion. Almost no one expresses an idea with absolute certainty, and just as each person is about to end her comment she seems to be confronted in her mind with evidence from past discussions. These students are neither using simple cultural relativism nor are they entirely taken by universalist conceptions of mind and mathematics. In the course of the discussion we can indeed see a tendency to

quickly formulate an answer that can explain away the evidence. However, the back and forth among these students, and the fact that each is willing to change her mind based on new ideas, show that at this point in the course – i.e. as we approach the final sessions – they have strong enough questions and pieces of evidence in mind to be able to consider any new hypothesis more carefully, draw further connections, and look for other pieces of evidence with enthusiasm. Or, at least, they can do so if speaking in a group where any idea offered seriously by a member is taken seriously.

4.4. Math illuminating and abstracting the world

In her essay on mathematical word problems, Marilyn Frankenstein argues that a critical approach to math education while helping students understand what numbers describe in the world should at the same time also help them understand "the meanings that numbers can hide in descriptions of the world" (Frankenstein, 2009, p. 118). Frankenstein does not describe the cause of this tension, which, as her examples clearly show, is inherent to any application of math to social reality: unemployment data in the US or war casualty reports, for example, both indicate and hide the magnitude of the problems. I have proposed that the tendency to reduce reality to manageable numbers is a fundamental aspect of math when used as an administrative instrument (Harouni, 2015b). The reduction is part and parcel of economic systems (the birthplace of math) that abstract human life and labor into a set of exchangeable or manageable values. Understanding this tendency is an important step toward realizing a critical approach to mathematics

education.

Our course did not manage to study this set of conflicts in math education with nearly as much depth or precision as the topic requires. We arrived at them at the very end of the semester, and then, as I will show, only as questions regarding the origins of mathematics. Our experience in this regard can be valuable for future efforts: it might be a good option to begin, earlier in the semester, with an exploration of the origins of mathematics. The early history of math as an administrative instrument can provide strong codifications for this theme.

Our main conversation about the origins of math took place in week 11 with Raven imagining a world without mathematics. The image cut through all preceding conversations and pushed the entire group to try to gain what we can call an anthropological lens on mathematics.

Raven: I was wondering…I understand that [math] is obviously there and people had to develop *something* to help them count. But I'm just thinking, what if no one did? What if we just looked at something, without labeling it with, like, "three"? Obviously when they did stuff, they needed something to count. They had to keep track of it.

Notice the conflict in Raven's own mind. On the one hand, math seems to emerge out of human interactions, almost naturally (it is "obviously there," people "had to develop" it), on the other hand there seems to be a *will* behind that emergence. This means that a lack of such will or even an anti-mathematical will are also possible. Raven, not incidentally, is the same student who in week 6 (see section 4.2.1) had wondered why Robert Recorde in 16th century England had felt the need to try to convince his readers of the usefulness of mathematics.

At this point the students had not read anything about hunter-gatherer tribes that do not develop formal mathematical systems (see Pinker, 2007). Taylor, who until now had been proposing the theory that math is a natural cognitive ability – "you can't look at your fingers without knowing that five fingers is more than four fingers" – followed up Raven's comment with a simple question.

Taylor: Who invented math and when?

Once he asked the question, Taylor seemed entranced by its force. He gazed at me with eyes wide open. Raven used the question to think through her own conflict. First, she reiterated Taylor's nativist explanation, replacing fingers with apples:

Raven: If you had one apple on the table, you'd know that that was one. Even if you didn't know it was one, you'd know it was alone and then if you put another

one on the table, even if you didn't know that it was two, you'd know that it wasn't alone anymore. That's the same everywhere.

But simply seeing a difference in quantities was not enough. At what point does the difference begin to signify something?

Raven: Sure, you have two apples, but then what do they mean for you? If you need to know that—if you live around an apple orchard and you need to count thousands of apples, you'll probably have a different system—whereas if you're just like a farmer, you have four apples. You just need to know that there are four.

Houman: There's a tree full of apples. You just eat from it.

Raven: There are many apples and I would like two for a pie, but if you run an orchard, you need to know how many are gonna fit in a box.

So far, we have arrived at what for this group is a familiar concept. It is the need for administering possessions that can give rise to some of the uses of mathematics. Drawing on what they have understood so far, Taylor can expand on Raven's idea to argue that this use is not uniform across contexts.

Taylor: No. Different theorems, calculus, how to build a building – stuff like that, that's more of a thinking thing. You have to *learn* it and develop that somehow. That's not just pointing at an apple and saying "one." It's not one story of a building that's alone and then you have three stories. Do you know what I mean? It's a different mentality!

Madison: "One" is not always a third of three… You don't just build two stories on top of the first one. You have to know how to *re*calculate everything in a special way. It's a whole discipline.

The discussion at this point drew toward speculations about this difference in "mentality" and how it could have resulted in different schools of mathematics. Is math a natural, universal system, or is it cultural, dependent on its uses in society and its modes of communication? The conversation was merely a better-informed version of the discussion I recounted in the previous section (4.3). We only returned to Raven's fundamental speculation – a world without mathematics, or an anti-mathematizing will – at the very end, and it came through the participation of an outside observer, Ezra Fox, who was attending our sessions for a second time. Fox, somewhat prompted by me, returned us to Raven's original puzzle.

Fox: You're asking about the history, but all the things you were talking about are things that we still have taken from *this* life ... That was one big question that I had in general. It was the history of math. Did money create numbers or did numbers create money? With this thing about the apple orchard, you have to count four apples. Why? What would happen if you didn't have an exact count? What's the worst that would happen?

Raven: Exactly!

Fox: When did a finger become the number one? When did these things turn into symbols of things that they're not? What are the implications for us when we start thinking of real things as something that's not what they actually are? When you can turn an apple into "one" and your finger into "one" and something else into "one," too, everything becomes kind of equal. I think the universality of math is obviously its strength, but I think it's also its weakness.

Fox, it should be said, had read my earlier work on mathematics. He did, however, explain to me after the session that he could share in Raven's vision of a world without

math because in the Jewish tradition, he knew, it is strictly prohibited to count human beings. This is clearly an example of what I referred to as an anti-mathematical will.

It might seem that we have entered fantasyland at this point: a world without counting. The ability to imagine such a world, however, can be helpful in re-examining our assumptions about mathematics in society. A political attitude toward math education has so far been defined as one that promotes political participation in society. This, I believe, is only part of the definition. A larger goal would include a radical re-reading of math itself. Here math suffers a symbolic death in order to live again in a different world, possibly as part of a different, utopian set of economic relations. The course I have discussed did not manage to develop this theme – mainly because I did not present students with adequate codifications. A course like this, in any case, will always conclude on the verge of possibilities that have not been visited. The discussion recounted above is merely a good start to a more in-depth conversation without which any critical discussion of math education would be severely incomplete.

5. Conclusions

5.1. *The students*

A single semester is not nearly enough time to complete what this course sets out to achieve. If taught in isolation, as was the case here, a course like this will have to shoulder the responsibility of helping students reexamine their relationship with teaching

and learning, with math in schools and math in society, with theoretical knowledge, and even with political economy. The takeaways and failures of a course that concentrates on generating questions about all these areas of experience can only be adequately understood over time, as students take these questions into their lives. It is their willingness and ability to explore that needs to be measured. In their final papers for the course, the students spoke about their learning experience. Their writings show that they also see the experience in terms of the possibilities it opens for thinking. Alice writes in her paper:

> Learning about the history of mathematics involved thinking critically about how math is taught in schools. Therefore, by learning about the history… the class was left thinking about the way math is taught today and whether this needs to be changed or not. It left us thinking about the future of education.

For Jim, too, the course was about its questions. As a future teacher he linked it to his own potential for autonomy:

> The traditional teaching method is made for us educators to have children succeed in what authorities think needs to be learned. Children and teachers are provided with what to be taught and what to learn. With all these guidelines and standards, teachers must teach certain things quickly and efficiently. The main point in Houman's teaching method is to get

> people to explore. Once children, teachers, and people explore they discover things that were not made to be discovered.

Aaron, echoing Jim's comment, points out the frustrations the course can engender in a future teacher. These frustrations, for him, *are* the takeaway:

> The fact that I was never given a choice, or explored enough to know that there was a choice for my future [teaching practice] frustrates me… Am I going through life trying to understand what I don't know, or am I simply accepting what I don't know as something that I have to know in order to "benefit my future"?

This frustration, however, can replace a very different type of anxiety. Janine wrote:

> I came to university terrified of the idea of having to take any form of math, because I felt that it would only lead to more failure… Before this course, I thought there was only one specific way that each subject should be taught; common standards, textbooks, and following a strict curriculum… Most importantly [the course] allowed me to improve my perception of myself as someone who is capable of doing mathematics. [It] pushed my critical thinking, taught to my abilities, and showed me that math can be taught in an unconventional yet more beneficial way... I

> learned a lot more about myself as a math teacher in this problem-solving course than I did in any previous education classes that I was enrolled in.

Many of the students seem to share her experience of feeling more capable as teachers, learners and thinkers. Of the 22 students, 15 reported in a post-survey that they had found the course very important to their understanding of mathematics – all but one had found it at least somewhat important in this regard. Sixteen students found the course very important for their developing understanding of teaching and learning. Seventeen said that they were now much more curious about mathematics. Nearly all (20) wrote that the course had made them more willing to spend time analyzing a complex text. Of the students who, in the beginning of the course, had reported experiencing mild to severe math anxiety, all except one wrote that the course had diminished their discomfort with mathematics.

In her paper, Janine goes on to explain how she understands what, in the quote above, she refers to as an "unconventional yet more beneficial way" of teaching mathematics. It is an approach that involves a certain set of beliefs that many of the students, particularly the pre-service teachers, as Dana phrased it in her paper, "had heard about but had not experienced with all its problems." These are, in Dana's words, the belief that "there can always be more than one answer to a question," that "math can be a hands-on activity," and that "the teacher is not the sole authority" and "people can learn in different ways."

For the last four weeks of the course, students worked in pairs on designing and testing a mathematical learning experience. They were both the teachers and the students in the project. Each week, for three weeks, they turned in a report of what they had learned, what puzzles they had encountered, what new materials they had designed for themselves. The range of their subjects is astonishing, showing a much wider understanding of math than the traditional approach would allow. Some, like Jim, took a philosophical approach, studying the meaning of the number one. Others integrated math with science, studying the relationship between shadows and objects, or measuring the strength of magnets, or the formula for sinking and floating. Some built things: Dana studied the geometric set-up that would allow a certain number of vertically placed pencils to bear the largest possible weight; Shannon studied perspective in paintings. Henry (recall his thoughts on school as a "mercantile pursuit") took a social-analytical approach: he conducted and analyzed surveys of students' reasons for attending university. The critical approach of the course had led to new directions in teaching and learning mathematics.

5.2. *The teacher-researcher*

It is conceivable that even if the course could teach me something about pedagogy – I indeed learned to design better codifications, to give time in the beginning of the semester to investigating teacher-student dynamics, to dedicate more time and energy to the final fieldwork projects so that they do not fall flat as they did for some students, to attend more consciously to the anthropological aspects of mathematics – the experience

could nonetheless have no impact on my understanding of my own theory of mathematics education. After all, I came to this course equipped with what I already believed to be a strong theory. I had developed this theory after years of intense research into the history of mathematics education. My ideas were, and are, based on a life-long study of philosophy and political economy. I also have an engineer's and a social scientist's training in mathematics. And what is more, this was not my first time teaching this topic: I had conducted two pilot studies with graduate students at Harvard. Finally, both Duckworth and Freire's open-ended approaches to teaching nonetheless provide the teacher with powerful means of highlighting, through the juxtaposition of materials, almost precisely the contradictions that he or she means to highlight. The impact can in instances be even more authoritarian than a lecture, because whereas in a lecture the students are conscious that everything they hear is mediated by the teacher, in an investigatory approach the student might forget the role of the teacher entirely and think that the contradictions are definitely and immediately present in objective reality itself. Even if the investigative teacher risks a great deal more of himself than does the traditional lecturer, he still has access to mechanisms for parrying away significant challenges to his own assumptions (Harouni, 2013a).

However, a dialectical theory of education, one that sees phenomena and the perception of those phenomena as intimately related, cannot remain entirely insensitive to the student's encounters with materials – no matter how basic or naïve the student's perceptions may be. The theory takes even that naiveté as an added material. The latest iteration of my theory of math education (Harouni, 2015b), which I wrote after teaching

this course, is in significant aspects different from the version I had written prior to teaching the course (Harouni, 2013b). Some of these differences are based on my reflections on the course experience. My students, for example, turned me on to the importance of addressing the cultural and psychological origins of mathematics. Raven and the other students' fantasy about a world without mathematics, though not a new concept for me, encouraged me to adjust my theory to account for both the common perception of mathematics as a "natural" phenomenon and for actual evidence from cultures that do not develop mathematics.

More importantly, my students' initial unwillingness to actively engage the materials, their tendency to treat even an invitation to think critically about mathematics as a chore, signaled to me that I had entirely underestimated the extent to which schooling can turn even the "why" questions of mathematics into moot points. Their resistance to genuinely engage the questions, softened only after I showed *my* willingness to radically address the teacher-student dynamic, was a legitimate response to an education system in which even exploration can turn into a tool for keeping students docile, compliant and invested in the existing structure of schooling. Not only did I need to turn the teacher-student contradiction into a main theme of the course, I also had to make sure that my theory did not take the same contradiction for granted.

In an unfree society, every pedagogy – that is, the positive work of training educators who will then go on to take part in institutions that reproduce a relationship of domination – is at some level a pedagogy of guilt. This is one way to understand, for

example, Freire's rather obsessive and inconsistent preoccupation with ascertaining that the themes that form the content/program of education "come from the people" (Freire, 2000, p. 121), even though, in his pedagogy, the identification and codification of themes is primarily the work of the most educated vanguard of society. What is missing in most investigatory pedagogies is an active co-investigation of the teacher-student dynamic by teachers and students. This investigation is not merely an exploration of the *form* of the dialogue between teachers and students; it must take into account also the content of learning and the way in which it is mediated by the teacher, the institution, and society at large. I mentioned that for the first few weeks of my course I remained disturbingly silent during class discussions, sharing almost none of my own ideas. This tendency only changed once students became willing to attack all my choices. As they came out of their passivity or resistant silence, I could gradually loosen the hold on myself and participate as an active member of the discussions. By the end of the course, I could even give a lecture on some of my ideas, knowing the students would challenge them (which they did). In this way some of the legitimate guilt could be expiated from the pedagogy, and a small measure of freedom could enter our interactions.

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