



The Moduli Space of S_1 -Type Zero Loci for $Z/2$ Harmonic Spinors in Dimension 3

Citation

Takahashi, Ryosuke. 2015. The Moduli Space of S_1 -Type Zero Loci for $Z/2$ Harmonic Spinors in Dimension 3. Doctoral dissertation, Harvard University, Graduate School of Arts & Sciences.

Permanent link

<http://nrs.harvard.edu/urn-3:HUL.InstRepos:17463144>

Terms of Use

This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA>

Share Your Story

The Harvard community has made this article openly available.
Please share how this access benefits you. [Submit a story](#).

[Accessibility](#)

The moduli space of S^1 -type zero loci for $\mathbb{Z}/2$ -harmonic spinors in dimension 3

A dissertation presented

by

Ryosuke Takahashi

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

March 2015

© 2015 – Ryosuke Takahashi
All rights reserved.

The moduli space of S^1 -type zero loci for $\mathbb{Z}/2$ -harmonic spinors in dimension 3

Abstract

Let M be a compact oriented 3-dimensional smooth manifold. In this paper, we will construct a moduli space consisting of the following data $\{(\Sigma, \psi)\}$ where Σ is a C^1 -embedding S^1 curve in M , ψ is a $\mathbb{Z}/2$ -harmonic spinor vanishing only on Σ and $\|\psi\|_{L^2_1} = 1$. We will prove that this moduli space can be parametrized by the space $\mathcal{X} = \{ \text{all Riemannian metrics on } M \}$ locally as the kernel of a Fredholm operator.

CONTENTS

Acknowledgements	vi
1. Introduction	1
2. Basic setting and results	4
2.1. Functional spaces	4
2.2. Some analytical properties of Dirac operators on $M - \Sigma$	6
3. Classification of Spin structures with the singular S^1 - curve	11
3.1. Spin structures	11
3.2. Classification of the spin structure on the boundary of a tubular neighborhood	13
3.3. Classification of the real line bundle \mathcal{I}	14
4. The harmonic section defined on the tubular neighborhood with the Euclidean metric	15
4.1. The L^2 and L_1^2 harmonic sections expressed by modified Bessel functions	15
4.2. Regularity properties and the asymptotic behavior of the L^2 -harmonic sections on the tubular neighborhood	24
4.3. Regularity properties and the asymptotic behavior of the L_1^2 -harmonic sections on the tubular neighborhood	26
5. Variational formula and perturbation of curves	31
5.1. Variational formula	31
5.2. Perturbation of Σ : local trivialization	38
5.3. Perturbation of Σ : estimates	40
5.4. Composition of the perturbations: estimates	47
5.5. Variational formula for the perturbed Dirac operators	52
6. The General Σ embedding in M	54
6.1. Asymptotic behavior of the L_1^2 -harmonic section	54
6.2. Modify the propositions in section 5	59

7. Fredholm property	63
7.1. Basic setting	63
7.2. Linearization	67
7.3. Fredholmness of finite Fourier mode case	68
7.4. General cases	84
7.5. Relation between \mathcal{T} and the original equation	87
8. Proof of the main theorem: Part I	87
8.1. Definition of \mathbb{K}_1 and \mathbb{K}_0	88
8.2. Basic setting	88
8.3. Part I of the proof: First order approximation of η_s and \mathfrak{k}_s	90
8.4. Part I of the proof: Iteration of $(\eta_i, (\mathfrak{c}_i^g, \mathfrak{h}_i^g, \mathfrak{e}_i^g, \mathfrak{k}_{i,s}^b, \mathfrak{k}_{i,s}^\perp), \mathfrak{f}_i,)$	103
8.5. Part I of the proof: The set $\pi_1(\mathcal{N})$	116
9. Proof of the main theorem: Part II	117
9.1. Part II of the proof: parametrization of (η_s, ψ_s)	117
9.2. Part II of the proof: C^1 regularity of f	118
9.3. Summary of the proof	120
10. Appendix	121
10.1. Remark of the proof when the metric is not Euclidean around Σ	121
10.2. Upper semi-continuity of $\dim(\text{coker}(p^- _{\ker(p^+)}))$	121
References	122

ACKNOWLEDGEMENTS

Firstly and foremost, I would like to thank my advisor Clifford H. Taubes for lots of inspiring conversations and encouragement during the past five years, for showing me the fearless attitude to attack difficult problems. Without his help, my thesis would not exist. I own my deepest gratitude to him.

Secondly, I also thank Professor Shing-Tung Yau for his help on many aspects. He also suggested me some interesting problems which inspiring me a lot. I also own him a lot for the effort he made on my job applications.

Thirdly, I want to thank Professor Peter Kronheimer for serving on my thesis committee and helping me a lot on my job application. I also want to thank my colleagues Chen-Yu Chi, Si Li, Yu-Shen Lin, Po-Ning Chen, Jie Zhou, Gabriel Bujokas, Atanas Atanasov, Yi Xie, who make my life at Harvard enjoyable.

Thanks also goes to Professor Chang-Shou Lin, Professor Chiun-Chuan Chen at National Taiwan University. They taught me so many important skills on analysis and PDEs and encouraged me for so many years. I want to take this opportunity to express my deep thanks to them.

Finally I want to thank my parents and my love, Josie, who make me brave enough to challenge all the difficulties I faced.

1. INTRODUCTION

In [12], Clifford Taubes proved a generalized version of Uhlenbeck's compactness theorem [13]. Let (M, g) be a 3-dimensional Riemannian manifold. The Uhlenbeck's compactness theorem [14] can be stated in the following way:

Theorem 1.1. *Suppose P is a principle G bundle over M for some compact Lie group G and $\{A_i\}$ be a sequence of connections on P satisfying*

$$(1.1) \quad \|F(A_i)\|_{L^2} \leq C$$

for some constant C which is independent of i . Then there exists a subsequence of $\{A_i\}$ converging (up to gauge transformations) weakly in L^2_1 to a L^2_1 connection.

To state the theorem proved in [12], I need to introduce some notations. Firstly, Clifford Taubes used the fact that $\mathfrak{sl}(2; \mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$ and P can be regarded as one of its $SO(3)$ -reduction associated with $PSL(2; \mathbb{C})$. So he can fix one reduction and denote P by $P \times_{SO(3)} PSL(2; \mathbb{C})$. Therefore he can always decompose a connection $\mathbb{A} = A + ia$ where A is the connection one form on the $SO(3)$ reduction of P and a is a $\mathfrak{su}(2)$ -valued one form.

Secondly, if I denote the group of gauge transformations (the automorphism group of P) by \mathcal{G} , then the lie algebra $\mathfrak{sl}(2; \mathbb{C})$ does not have norms which are invariant under the action of \mathcal{G} . So I should refine the L^2 boundedness condition (1.1) as follows:

Definition 1.2. Let

$$\mathcal{F}(\mathbb{A}) = \inf_{A+ia \in \mathcal{G}_{\mathbb{A}}} \int |F(A) - a \wedge a|^2 + |d_A a|^2 + |d_A * a|^2$$

where $\mathcal{G}_{\mathbb{A}}$ is the \mathcal{G} -orbit of \mathbb{A} .

Now, the generalized Uhlenbeck's compactness theorem proved in [12] can be stated as follows:

Theorem 1.3. *For any sequence of connections $\{\mathbb{A}_i = A_i + ia_i\}$ defined on $P \times_{SO(3)} PSL(2; \mathbb{C})$, which has $\{\mathcal{F}(\mathbb{A}_i)\}$ being bounded, we have*

- *If $\{\|a_i\|_{L^2}\}$ is bounded, then we can find a subsequence of $\{\mathbb{A}_i\}$ which is weakly L^2_1 convergent up to the automorphism of P .*
- *If $\|a_i\|_{L^2} \rightarrow \infty$, we can find a closed, Hausdorff dimension at most 1 subset Σ and a subsequence of $\{\mathbb{A}_i = A_i + ia_i\}$ such that*

1. $\{A_i\}$ converges weakly in $L^2_{1,loc}$ -sense on $M - \Sigma$ up to the automorphism of P and

2. $\{\frac{1}{\|a_i\|_2} a_i\}$ also converges weakly in $L^2_{1,loc}$ -sense on $M - \Sigma$ up to the automorphism of P .

Moreover, the data Σ can be formulated as the zero locus of a \mathbb{Z}_2 -harmonic spinor, say ψ , defined on $M - \Sigma$, see [6]. In [6], Clifford Taubes showed more properties for this data set Σ . Moreover, it is still a conjecture that Σ is a C^1 curve for the "generic" metric g , this conjecture is also mentioned in [1]. So a natural question we can ask is the following: can we find a way to parametrize the data (Σ, ψ) ?

In this paper, I will give a local parametrization of the set of triples of the form (g, Σ, ψ) with g being a Riemannian metric, Σ being a C^1 embedded circle and ψ being a $\mathbb{Z}/2$ harmonic spinor defined on the complement of whose norm extends across as zero as to give a Holder continuous function on M . To say more about this, let

$$\mathcal{A} = \{\Sigma \subset M | \Sigma \text{ is the image of a } C^1 \text{ embedding of the circle}\}.$$

For each $\Sigma \in \mathcal{A}$, define H be the subset of $H^1(M - \Sigma; \mathbb{Z}/2)$ with non-zero monodromy around Σ . Each $e \in H$ corresponds to a real line bundle $\mathcal{I}_{\Sigma, e}$ on $M - \Sigma$. So as Σ

varies, the set H varies continuously to define a finite sheeted covering space of \mathcal{A} . This is denoted by \mathcal{A}_H . Denote by \mathcal{X} the space of Riemannian metrics on M . Each metric $g \in \mathcal{X}$ has a corresponding spinor bundle $\mathcal{S}_g \rightarrow M$. Denote by $\mathcal{S}_{g,\Sigma,e}$ the bundle $\mathcal{S}_g \otimes \mathcal{I}_{\Sigma,e}$; this is a spinor bundle over $M - \Sigma$. This is called the $\mathbb{Z}/2$ spinor bundle. Define \mathcal{Y} to be $\mathcal{X} \times \mathcal{A}_H$.

Let $\mathcal{E} \rightarrow \mathcal{Y}$ denote the infinite dimensional vector bundle defined as follows: Supposing that $y = (g, \Sigma, e) \in \mathcal{X} \times \mathcal{A}_H$, then the fiber of \mathcal{E} over y is the infinite dimensional vector space of L_1^2 sections over $M - \Sigma$ of the $\mathbb{Z}/2$ spinor bundle $\mathcal{S}_{g,\Sigma,e}$. This vector space is denoted by \mathcal{E}_y . Let $D^{(y)}$ denote the Dirac operator defined on \mathcal{E}_y by the metric g . This operator gives a bounded, linear map from \mathcal{E}_y to the space of square integrable sections of $\mathcal{S}_{g,\Sigma,e}$.

With \mathcal{E} understood, the space of interest is the subset \mathfrak{M} in \mathcal{E} whose elements are data sets $(y = (g, \Sigma, e), \psi \in \mathcal{E}_y)$ obeying

- $D^{(y)}\psi = 0$
 - $|\psi|$ extends across Σ as a Holder continuous function on M
- with its zero locus containing Σ .
- $\frac{|\psi|(p)}{\text{dist}(p, \Sigma)^{\frac{1}{2}}} > 0$ near Σ .
 - The L_1^2 norm of ψ is 1.

The set \mathfrak{M} inherits a topology from \mathcal{E} . The goal is to give it some additional structure. To say more about \mathfrak{M} , we can consider the vector bundle \mathcal{F} over \mathcal{Y} whose fiber \mathcal{F}_y is the L^2 sections of \mathcal{S}_y . Then \mathfrak{M} will be contained in the kernel of $D : \mathcal{E} \rightarrow \mathcal{F}$ where $D|_{\mathcal{E}_y} = D^{(y)}$.

I will prove the following:

Theorem 1.4. *Let $(y = (g, \Sigma, e), \psi)$ denote a given element in \mathfrak{M} . There are finite dimensional vector spaces \mathbb{K}_1 and \mathbb{K}_0 , a ball $B \subset \mathbb{K}_1$ centered at the origin, a set $\mathcal{B} \subset \mathcal{X}$ with $\mathcal{B} = \pi_1(\mathcal{N})$ being the projection of \mathcal{N} , a neighborhood of y , from \mathcal{Y} to \mathcal{X} and a C^1 map to be denoted by f from $B \times \mathcal{B}$ to \mathbb{K}_0 such that \mathfrak{M} near (y, ψ) is homeomorphic to $f^{-1}(0)$.*

The vector space \mathbb{K}_1 and \mathbb{K}_0 in this theorem can be generated by the kernel and cokernel of a Fredholm operator respectively. This theorem shows us several facts. First of all, the C^1 -curve component Σ in \mathfrak{M} can only be perturbed in finite dimensional directions. Secondly, when $\dim(\mathbb{K}_0) = 0$, then \mathfrak{M} near (y, ψ) is homeomorphic to $B \times \mathcal{B}$.

2. BASIC SETTING AND RESULTS

2.1. Functional spaces. First of all, we start with some basic setting. Let (M, g) be a compact 3-dimensional Riemannian manifold and $\Sigma \in \mathcal{A}$ be a C^1 circle embedding in M . Moreover, we suppose that g is the product type metric near Σ . Namely, there exists N_R , a small tubular neighborhood of Σ which is parametrized by coordinate $(r, \theta, t) \in [0, R] \times [0, 2\pi] \times [0, 2\pi]$, such that $g|_{N_R} = dr^2 + r^2 d\theta^2 + dt^2$. So we can parametrized the circle Σ by t .

Secondly, let \mathcal{S} be the spinor bundle over M with respect to g and \mathcal{I} be a real line bundle defined on $M - \Sigma$. We suppose that \mathcal{I} cannot be extended to the entire manifold M , which means $\mathcal{I}|_{r=a, t=b} \simeq \mathbb{R} \times [0, 2\pi] / \{(x, 0) \sim (-x, 2\pi) \text{ for all } x \in \mathbb{R}\}$ homeomorphically for all $0 < a < R$ and $t \in [0, 2\pi]$. We also fix a metric $g_{\mathcal{I}}$. So we can define the scale $|v \otimes w| = |v||w|$ for any $(v, w) \in \mathcal{S} \otimes \mathcal{I}$.

Thirdly, the \mathcal{S} itself is equipped with the standard connection $\nabla^{\mathcal{S}}$, see [6]. Meanwhile, the connection 1-form of the real line bundle is zero. So we can define the connection $\nabla^{\mathcal{S} \otimes \mathcal{I}} = \nabla^{\mathcal{I}} \otimes id_{\mathcal{I}}$ on the bundle $\mathcal{S} \otimes \mathcal{I}$.

Now we are ready to define the functional spaces we need in this paper.

Definition 2.1. Let $\mathbf{u} \in C^\infty(M - \Sigma, \mathcal{S} \otimes \mathcal{I})$ be a smooth section of the twisted spinor bundle $\mathcal{S} \otimes \mathcal{I}$. We define the following norms and corresponding space:

1. $\|\mathbf{u}\|_{L^2} = (\int_{M-\Sigma} |\mathbf{u}|^2)^{\frac{1}{2}}$;
2. $\|\mathbf{u}\|_{L_1^2} = (\int_{M-\Sigma} |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2)^{\frac{1}{2}}$;
3. $\|\mathbf{u}\|_{L_{-1}^2} = \sup\{\int_{M-\Sigma} \langle \mathbf{v}, \mathbf{u} \rangle \mid \|\mathbf{v}\|_{L_1^2} \leq 1 \text{ and } \}$.

Moreover, the spaces of sections bounded with respect to these norms will be denoted by

$$L_i^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I}) = \text{closure of } \{\mathbf{u} \in C^\infty(M - \Sigma, \mathcal{S} \otimes \mathcal{I}) \mid \|\mathbf{u}\|_{L_i^2} \leq \infty\}$$

for $i = 1, 0, -1$. In the following paragraphs, we simply use the notation L_i^2 to denote $L_i^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$ and we usually omit the index i when it is zero.

Similarly, we can define the space of compactly supported sections, $L_{i,cpt}^2$ by taking the closure of the set of smooth, compactly supported sections with respect to the norm $\|\cdot\|_{L_i^2}$.

Remark 2.2. We should always remember that the space L_{-1}^2 is the dual space of L_1^2 in our case. In a general open domain Ω on \mathbb{R}^n , the notation $L_{-1}^2(\Omega)$ usually denote the dual space of $L_{1,cpt}^2(\Omega)$. The advantage of taking dual of $L_{1,cpt}^2(\Omega)$ is the following: We can "differentiate" a $L^2(\Omega)$ function formally by coupling it with sections defined on $L_{1,cpt}^2$. This gives us a functional defined on $L_{1,cpt}^2$. Then the compactly supported inputs of this functional allow us doing integration by parts formally without having the boundary terms. Therefore, if we want to extend the domain of this functional to be L_1^2 , we need to prove that there is no contribution from the boundary when we have L_1^2 inputs.

The space L_{-1}^2 has the following property. This is an analogue version of theorem 1 in section 5.9 of [5].

Proposition 2.3. Let $\mathbf{f} \in L^2_{-1}$. Then there exists a pair

$$(\mathbf{f}_0, \mathbf{f}_1) \in L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I}) \times L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I} \otimes T^*M)$$

such that

$$(2.1) \quad \int_{M-\Sigma} \langle \mathbf{v}, \mathbf{f} \rangle = \int_{M-\Sigma} \langle \mathbf{v}, \mathbf{f}_0 \rangle + \langle \nabla \mathbf{v}, \mathbf{f}_1 \rangle$$

for all $\mathbf{v} \in L^2_1$. Furthermore, we have

$$\|\mathbf{f}\|_{L^2_{-1}} = \left(\int_{M-\Sigma} |\mathbf{f}_0|^2 + |\mathbf{f}_1|^2 \right)^{\frac{1}{2}}$$

Proof. Let $T_{\mathbf{f}} : L^2_1 \rightarrow \mathbb{C}$ be a bounded functional sending each \mathbf{v} to $\int_{M-\Sigma} \langle \mathbf{v}, \mathbf{f} \rangle$. By Riesz Representation Theorem, there exists $\mathbf{u} \in L^2_1$ such that

$$(2.2) \quad T_{\mathbf{f}}(\mathbf{v}) = \int_{M-\Sigma} \langle \mathbf{v}, \mathbf{u} \rangle + \langle \nabla \mathbf{v}, \nabla \mathbf{u} \rangle.$$

So we can simply take $\mathbf{f}_0 = \mathbf{v}$ and $\mathbf{f}_1 = \nabla \mathbf{v}$.

To prove the second part, by taking $\mathbf{v} = \mathbf{u}$ in (2.2), we have

$$\|\mathbf{u}\|_{L^2_1}^2 = T_{\mathbf{f}}(\mathbf{u}) \leq \|\mathbf{u}\|_{L^2_1} \|\mathbf{f}\|_{L^2_{-1}}.$$

This inequality implies that $(\int_{M-\Sigma} |\mathbf{f}_0|^2 + |\mathbf{f}_1|^2)^{\frac{1}{2}} = \|\mathbf{u}\|_{L^2_1} \leq \|\mathbf{f}\|_{L^2_{-1}}$.

Meanwhile, from (2.2) we have

$$|T_{\mathbf{f}}(\mathbf{v})| \leq \left(\int_{M-\Sigma} |\mathbf{f}_0|^2 + |\mathbf{f}_1|^2 \right)^{\frac{1}{2}}$$

if $\|\mathbf{v}\|_{L^2} \leq 1$. So by the definition 2.1, we have $\|\mathbf{f}\|_{L^2_{-1}} \leq (\int_{M-\Sigma} |\mathbf{f}_0|^2 + |\mathbf{f}_1|^2)^{\frac{1}{2}}$. \square

2.2. Some analytical properties of Dirac operators on $M - \Sigma$. We will prove the following proposition in this subsection.

Proposition 2.4. Let $D|_{L^2_1} : L^2_1 \rightarrow L^2$ be the Dirac operator. Then we have the following properties:

1. $\ker(D|_{L^2_1})$ is finite dimensional.
2. $\text{range}(D|_{L^2_1})$ is closed.
3. Suppose we write the adjoint of $D|_{L^2_1}$ to be $D|_{L^2}$, then we have

$$L^2 = \text{range}(D|_{L^2_1}) \oplus \ker(D|_{L^2}).$$

Remark 2.5. $\ker(D|_{L^2})$ is not finite dimensional in general.

To prove this proposition, we need the following lemma. This lemma is also very useful in the rest of this article.

Lemma 2.6. *Let $\mathbf{u} \in L^2_1$, then we have*

$$\int_{N_r} |\mathbf{u}|^2 \leq 4\pi^2 r^2 \int_{N_r} |\nabla \mathbf{u}|^2$$

for all $r \leq R$.

Proof. Let $\mathbf{u} \in L^2_1$ and $\{\mathbf{u}_n\}$ be a sequence of smooth sections such that

$$\mathbf{u}_n \rightarrow \mathbf{u}$$

in L^2_1 sense. Since \mathcal{I} is nontrivial along θ direction, we have

$$\begin{aligned} |\mathbf{u}_n(r, s, t)| &\leq \left| \int_0^{2\pi} \partial_\theta |\mathbf{u}_n(r, \theta, t)| d\theta \right| \\ &\leq \int_0^{2\pi} |\nabla_{e_2} \mathbf{u}_n(r, \theta, t)| r d\theta \\ &\leq \sqrt{2\pi} r^{\frac{1}{2}} \left(\int_0^{2\pi} |\nabla_{e_2} \mathbf{u}_n(r, \theta, t)|^2 r d\theta \right)^{\frac{1}{2}} \end{aligned}$$

for any $s, t \in [0, 2\pi]$, $0 < r \leq R$, where $e_2 = \frac{1}{r} \partial_\theta$. So we have

$$\begin{aligned} \int_{N_r} |\mathbf{u}_n|^2 &\leq \int_0^r \int_0^{2\pi} \int_0^{2\pi} |\mathbf{u}_n(r, s, t)|^2 r ds dt dr \\ &\leq 4\pi^2 r^2 \int_{N_r} |\nabla_{e_2} \mathbf{u}_n|^2. \end{aligned}$$

By taking $n \rightarrow \infty$, we prove this lemma. □

Proof. (of Proposition 2.4)

Step 1. First of all, for any $\mathbf{u} \in L_1^2$, we have the Schrodinger-Lichinerowicz formula

$$D^2\mathbf{u} = \Delta\mathbf{u} + \frac{\mathcal{R}}{4}\mathbf{u},$$

in the following sense:

$$(2.3) \quad \int \langle D\zeta, D\mathbf{u} \rangle = \int \langle \nabla\zeta, \nabla\mathbf{u} \rangle + \int \frac{\mathcal{R}}{4} \langle \zeta, \mathbf{u} \rangle$$

for all $\zeta \in L_{1,cpt}^2$. Here \mathcal{R} is the scalar curvature of M . We should prove that (2.3) is true for all $\zeta \in L_1^2$.

By lemma 2.6, we have

$$(2.4) \quad \int_{N_r} |\zeta|^2 \leq 4\pi^2 r^2 \int_{N_r} |\nabla\zeta|^2$$

for all $\zeta \in L_1^2$. Let us denote $(\int_{N_r} |\nabla\zeta|^2)^{\frac{1}{2}} = f_\zeta(r)$. We have $f_\zeta(r) \rightarrow 0$ as $r \rightarrow 0$.

Now we take the family of smooth functions

$$\chi_\delta = \begin{cases} 0 & \text{on } N_{\frac{\delta}{2}} \\ 1 & \text{on } M - N_\delta \end{cases}$$

with $|\nabla(\chi_\delta)| \leq \frac{C}{\delta}$. So by (2.3), we have

$$(2.5) \quad \int \langle D(\chi_\delta\zeta), D\mathbf{u} \rangle = \int \langle \nabla(\chi_\delta\zeta), \nabla\mathbf{u} \rangle + \int \frac{\mathcal{R}}{4} \langle \chi_\delta\zeta, \mathbf{u} \rangle$$

for all $\zeta \in L_1^2$. Clearly the second terms on the right hand side of (2.5) will converges to $\int \frac{\mathcal{R}}{4} \langle \zeta, \mathbf{u} \rangle$ as $\delta \rightarrow 0$ by Cauchy's inequality.

For the left hand side of (2.5), we have

$$\int \langle D(\chi_\delta\zeta), D\mathbf{u} \rangle = \int \chi_\delta \langle D\zeta, D\mathbf{u} \rangle + \epsilon.$$

Because of the inequality (2.4), ϵ can be bounded as follows.

$$|\epsilon| \leq \frac{C}{\delta} \int_{N_\delta} |\langle \zeta, D\mathbf{u} \rangle| \leq \frac{C}{\delta} \left(\int_{N_\delta} |\zeta|^2 \right)^{\frac{1}{2}} \|D\mathbf{u}\|_{L^2} \leq C f_\zeta(\delta) \|D\mathbf{u}\|_{L^2}.$$

So we have

$$\int \langle D(\chi_\delta \zeta), D\mathbf{u} \rangle \rightarrow \int \langle D\zeta, D\mathbf{u} \rangle$$

as $\delta \rightarrow 0$.

Similarly, we have

$$\int \langle \nabla(\chi_\delta \zeta), \nabla \mathbf{u} \rangle \rightarrow \int \langle \nabla \zeta, \nabla \mathbf{u} \rangle$$

as $\delta \rightarrow 0$, too. So we have

$$(2.6) \quad \int \langle D\zeta, D\mathbf{u} \rangle = \int \langle \nabla \zeta, \nabla \mathbf{u} \rangle + \int \frac{\mathcal{R}}{4} \langle \zeta, \mathbf{u} \rangle$$

for all $\zeta \in L_1^2$.

Step 2. We prove $\ker(D|_{L_1^2})$ is finite dimensional in this paragraph. By taking $\zeta = \mathbf{u}$ in (2.6), we have

$$(2.7) \quad \|\mathbf{u}\|_{L_1^2}^2 - C_1 \|\mathbf{u}\|_{L^2}^2 \leq \|D\mathbf{u}\|_{L^2}^2 \leq \|\mathbf{u}\|_{L_1^2}^2 + C_2 \|\mathbf{u}\|_{L^2}^2$$

for some C_1, C_2 depending on the supreme and infimum of the scalar curvature \mathcal{R} .

Now, let $\{\mathbf{u}_n\} \subset \ker(D|_{L_1^2})$ and $\|\mathbf{u}_n\|_{L_1^2} \leq 1$. Then there is a subsequence of $\{\mathbf{u}_n\}$ converging weakly in L_1^2 , which will also converge strongly in L^2 , but the inequality (2.7) shows us that $\|\mathbf{u}\|_{L_1^2}^2 \leq C_1 \|\mathbf{u}\|_{L^2}^2$ for all $\mathbf{u} \in \ker(D|_{L_1^2})$. So this subsequence will actually converge strongly in L_1^2 . Therefore the unit sphere inside the space $\ker(D|_{L_1^2})$ is compact, which means $\ker(D|_{L_1^2})$ is finite dimensional.

Step 3. To prove it has a closed range, we need to show that for any Cauchy sequence $\{D\mathbf{u}_n\}$, it will converge to $D\mathbf{u}$ for some \mathbf{u} . We can suppose that \mathbf{u}_n are orthogonal to $\ker(D|_{L_1^2})$ without loss of generality. Here we claim that the following inequality

$$(2.8) \quad \|D\mathbf{v}\|_{L^2} \geq C_3\|\mathbf{v}\|_{L^2}$$

holds for all \mathbf{v} orthogonal to $\ker(D|_{L_1^2})$. With this inequality in mind, the right hand side of the inequality,

$$\|\mathbf{u}_m - \mathbf{u}_n\|_{L_1^2}^2 \leq \|D(\mathbf{u}_m - \mathbf{u}_n)\|_{L^2}^2 + C_1\|\mathbf{u}_m - \mathbf{u}_n\|_{L^2}^2,$$

provided by inequality (2.7) converges to 0. Therefore $\{\mathbf{u}_n\}$ is a Cauchy sequence.

To prove the inequality (2.8), we just follow the argument of section 4.2 in [6]. only need to show that the spectrum of $D|_{L_1^2}$ is discrete. To prove this statement, we pick up a suitable $\lambda \notin \sigma(D)$ and show that $(D - \lambda)^{-1} : L^2 \rightarrow L^2$ is a compact operator. So let $\{\mathbf{v}_n\}$ be a bounded sequence in L^2 , we have to show that $\{(D - \lambda)^{-1}\mathbf{v}_n\}$ have a converging subsequence. By (2.7) again, we have

$$\|(D - \lambda)^{-1}\mathbf{v}_n\|_{L_1^2}^2 \leq \|\mathbf{v}_n\|_{L^2}^2 + (C_1 + \lambda^2)\|(D - \lambda)^{-1}\mathbf{v}_n\|_{L^2}^2.$$

Meanwhile, because $\lambda \notin \sigma(D)$, so $(D - \lambda)^{-1}$ is a bounded operator. Therefore $\|(D - \lambda)^{-1}\mathbf{v}_n\|_{L^2} \leq C_\lambda\|\mathbf{v}_n\|_{L^2}$ for some constant C_λ depending on λ . So we have

$$\|(D - \lambda)^{-1}\mathbf{v}_n\|_{L_1^2}^2 \leq C_\lambda\|\mathbf{v}_n\|_{L^2}^2$$

which implies $(D - \lambda)^{-1}$ is compact.

Step 4. Now we already have $L^2 = \text{range}(D|_{L_1^2}) \oplus \text{range}(D|_{L_1^2})^\perp$. We only need to show that

$$\text{range}(D|_{L_1^2})^\perp = \ker(D|_{L^2}).$$

To show this is true, by taking any $\mathbf{v} \in \ker(D|_{L^2})$, we have

$$\int \langle \mathbf{u}, D\mathbf{v} \rangle = \int \langle D\mathbf{u}, \mathbf{v} \rangle = 0$$

for all $\mathbf{u} \in L_1^2$. So we can see this fact immediately. \square

So far we prove that $D|_{L_1^2}$ has closed range and finite dimensional kernel. However the cokernel of $D|_{L_1^2}$ which is also the kernel of $D|_{L^2} : L^2 \rightarrow L_{-1}^2$ is infinite dimensional in general. In section 4, we will express the elements in $\ker(D|_{L^2})$ explicitly in terms of Bessel functions directly.

3. CLASSIFICATION OF SPIN STRUCTURES WITH THE SINGULAR S^1 - CURVE

In this section, we classify the spin structure on the singular submanifold. We should start from the basic knowledge of spin structure first, readers can see [6], [9] for the details.

3.1. Spin structures. Let M be a compact Riemannian manifold. Let $\dim(M) = m$. A spin structure of M is a $Spin(m)$ -principle bundle P which is a 2-fold covering of Q , the frame bundle of TM . More precisely, we have following diagram

$$\begin{array}{ccc} P \times Spin(m) & \longrightarrow & P \\ \downarrow \Lambda \times \lambda & & \downarrow \Lambda \searrow \\ Q \times SO(m) & \longrightarrow & Q \longrightarrow M \end{array}$$

commutes.

Now suppose that there is a submanifold $N \subset M$ with $\dim(N) = n$. Moreover, we suppose the normal bundle of N is trivial. Therefore, for any $x \in N$, the cotangent

space $T_x M$ can be decomposed as $T_x N \otimes \nu_x$, where we denote by ν the normal bundle of N . If we fixed an orthonormal sections in $\Gamma(\nu)$ to be $\{v_1, \dots, v_{m-n}\}$, then we can define the frame bundle Q' of N as a subbundle of Q by considering the map $(e_1, \dots, e_n) \mapsto (e_1, \dots, e_n, v_1, \dots, v_{m-n}) \in Q$. Furthermore, we can show that $\Lambda^{-1}(Q') = P'$ is a $Spin(n)$ -structure of N :

Lemma 3.1. *Let M be a m -dimensional Riemannian manifold. Suppose that there is a smooth n -dimensional submanifold $N \subset M$ with trivial normal bundle. Then for any spin structure $P_{Spin(m)}$ defined on M , there is a corresponding spin structure $P'_{Spin(n)}$ defined on N which is a subbundle of $P_{Spin(m)}|_N$. We call P' the **reduced spin structure** on N .*

Proof. Let Q, Q' be the frame bundles over M and N respectively which are defined in the previous paragraph. There is a $SO(n)$ -action on Q' and a inclusion map $i : SO(n) \rightarrow SO(m)$. So we have the following diagram

$$\begin{array}{ccc} \Lambda^{-1}(Q') \times \lambda^{-1} \circ i(SO(n)) & \twoheadrightarrow & \Lambda^{-1}(Q') \\ \downarrow \Lambda \times \lambda & & \downarrow \Lambda \\ Q' \times_i SO(n) & \longrightarrow & Q' \longrightarrow N \end{array}$$

commutes.

It is obvious that $\Lambda^{-1}(Q')$ is a 2-fold covering of Q' . To finish the proof, we only need to show that $\lambda^{-1} \circ i(SO(n))$ is isomorphic to $Spin(n)$. We prove this part as following.

The cases $n < 3$ are easy to check, so we suppose that $n \geq 3$. Since $\pi_1(Spin(n)) = 0$, there is a natural lifting from $Spin(n)$ to $Spin(m)$ such that the following diagram

$$\begin{array}{ccc} Spin(n) & \twoheadrightarrow & Spin(m) \\ \downarrow \lambda & & \downarrow \lambda \\ SO(n) & \xrightarrow{i} & SO(m) \end{array}$$

commutes. We need to show that l is injective. Suppose not, we will have $a, b \in Spin(n)$ such that $l(a) = l(b)$. We can choose a curve $\gamma : [0, 1] \rightarrow Spin(n)$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Then $l \circ \gamma$ will be a trivial loop in $Spin(m)$, which maps to a trivial loop in $i^*(SO(n)) \subset SO(m)$. This is a contradiction. Therefore we have that $l : Spin(n) \rightarrow \lambda^{-1} \circ i(SO(n)) \subset Spin(m)$ is an isomorphism. □

Remark 3.2. There is an intuitive way to understand this lemma. Recall that if M is a product manifold $A \times \mathbb{R}^k$, then any spin structure P of M can be reduced to a spin structure on A . In our case, if we rescale the metric near N along the normal direction, we can construct a manifold $N \times \mathbb{R}^{m-n}$ by taking the limit of this rescaling. Because the spin structure on N is invariant under the scaling, we prove this lemma immediately.

It is well-known that if the second Stiefel-Whitney class $W_2(Q) = 0$, then there exist the spin structures and vice versa. Moreover, the spin structures can be classified by $H^1(M; \mathbb{Z}_2)$ by considering the exact sequence of cohomology groups.

3.2. Classification of the spin structure on the boundary of a tubular neighborhood. Let $m = 3$ and $\Sigma \subset M$ be a C^1 circle embedding in M . We consider the tubular neighborhood $N \simeq \Sigma \times \mathbb{D}$ (\mathbb{D} is the 2 dimensional closed disc) and denote the boundary of \bar{N} by B . Now $B \subset M$ is a submanifold and $dim(B) = 2$. We can parametrize B by $(\theta, \varphi) \mapsto B$ with $\theta, \varphi \in [0, 2\pi]$. We define $\gamma_1 = Im\{\theta = 0\}$ and $\gamma_2 = Im\{\varphi = 0\}$ where γ_1 can bound an embedded disc inside B .

Let P be the spin structure on M , which is a $Spin(3)$ -principle bundle on M , we have the corresponding spin structure P' on B by using lemma 3.1. Since the spin structure on B can be classified by $H^1(B; \mathbb{Z}_2) = H^1(S^1 \times S^1; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, there are exactly 4 spin structures on B .

Similarly, since γ_1 and γ_2 are submanifolds of B , we can apply lemma 3.1 again and get spin structures P''_1 and P''_2 on γ_1 and γ_2 respectively. There are only 2 spin

structures on S^1 :

1. $S^1 \times Spin(1)$;
2. $[0, 2\pi] \times Spin(1)/\{(0, 1) \sim (2\pi, -1)\}$.

Therefore the spin structures defined on B can also be classified by the spin structures on γ_1 and γ_2 .

Now we want to specify the spin structure P' of B . The key point is that this bundle is defined on entire M , so the spin structure P''_1 can be extended inside the disk \mathbb{D} . Now we prove the following lemma given by [15].

Lemma 3.3. *Let \mathbb{D} be a 2-dimensional closed disc equipped with a Riemannian metric g . Suppose there is a spin structure P defined on \mathbb{D} and P' be the spin structure on the S^1 boundary of \mathbb{D} by lemma 3.1. Then P' must be $[0, 2\pi] \times Spin(1)/\{(0, 1) \sim (2\pi, -1)\}$.*

Proof. We parametrize $\partial\mathbb{D} = S^1$ by $\{\theta \in [0, 2\pi]\}$. Let $n(\theta)$ be the inner normal vector defined on S^1 and $v(\theta)$ be the tangent vector on S^1 . Following the notations defined in lemma 3.1, we have $v \in \Gamma(Q')$ and $(v, n) \in \Gamma(Q)$. Since (v, n) is a nontrivial loop in Q , the lifting curve, $(v, n)' \in P$ is not a loop. However, the lifting of v is a loop if P' is trivial. \square

Therefore, by this lemma, $P''_1 = [0, 2\pi] \times Spin(1)/\{(0, 1) \sim (2\pi, -1)\}$.

We write down the following conclusion to close this subsection.

Corollary 3.4. *There are only 2 possible reduced spin structures P' defined on B . Moreover, it is totally determined by the reduced spin structure P''_2 defined on γ_2*

3.3. Classification of the real line bundle \mathcal{I} . Now we should consider the real line bundle \mathcal{I} over $M - \Sigma$. Since the vector bundles can be totally determined by the transition functions $\{U_{\alpha\beta}, g_{\alpha\beta}\}$ which is one-one corresponding to the elements in the sheaf cohomology $H^1(M; G)$. Therefore if we consider the restriction $\mathcal{I}|_B$ over B ,

there are only 4 possibilities ($H^1(B, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$). Furthermore, if we restricted the line bundle on γ_1 and γ_2 , either it will be a trivial line bundle or it will be a Mobius strip. This observation will classify the line bundle $\mathcal{I}|_B$. Moreover, if $\mathcal{I}|_{\gamma_1}$ is trivial, then \mathcal{I} can be extended to Σ for sure. So we suppose that $\mathcal{I}|_{\gamma_1}$ is the nontrivial one.

Now, if we consider the corresponding spinor bundle $\mathcal{S} = P \times_{\kappa} \Delta_3$ on ∂B and then tensor with \mathcal{I} , since both of them are nontrivial along γ_1 , the bundle $\mathcal{S} \otimes \mathcal{I}$ will be trivial along γ_1 . Hence, we can identify $\mathcal{S} \otimes \mathcal{I}$ to $P' \times_{\kappa} \Delta_3$ where P' is the spin structure defined on Σ which cannot extend to M .

By using this observation, we can fix a element $e \in H \subset H^1(M - \Sigma; \mathbb{Z}/2)$ in the rest of this article, which determines one of those two type of spin structures defined above.

4. THE HARMONIC SECTION DEFINED ON THE TUBULAR NEIGHBORHOOD WITH THE EUCLIDEAN METRIC

4.1. The L^2 and L_1^2 harmonic sections expressed by modified Bessel functions. Let us consider the space $N = \mathbb{R}^2 \times \mathbf{S}^1$. Denote $E = \mathcal{S} \otimes \mathcal{I}$ the total space of twisted spinor bundle over N . The Dirac operator on N can be written as

$$D = e_1 \cdot \frac{\partial}{\partial t} + e_2 \cdot \frac{\partial}{\partial z} + e_3 \cdot \frac{\partial}{\partial \bar{z}}$$

where

$$e_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

and $z = x + iy$.

Using the cylindrical coordinate, $r := \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})$, we can write down the Fourier expansion of \mathbf{u} as

$$\mathbf{u}(t, r, \theta) = \sum_{l,k} e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} U_{k,l}^+ \\ e^{i(k+\frac{1}{2})\theta} U_{k,l}^- \end{pmatrix}$$

for any C^∞ -section \mathbf{u} of E . Here k runs over all integers and l can be either in \mathbb{Z} or $\mathbb{Z} + \frac{1}{2}$ according to the spin structure we chose. The Dirac operator can be written in terms of θ, r by changing the coordinate, we have

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = e^{i\theta} \left(\frac{\partial}{\partial r} - \frac{i\partial}{r\partial\theta} \right); \\ \frac{\partial}{\partial \bar{z}} &= \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} = e^{-i\theta} \left(\frac{\partial}{\partial r} + \frac{i\partial}{r\partial\theta} \right). \end{aligned}$$

Suppose \mathbf{u} is a harmonic section. Then we have

$$D\mathbf{u} = \sum_{l,k} e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} (lU^+ + \frac{d}{dr}U^- + \frac{(k+\frac{1}{2})}{r}U^-)_{k,l} \\ e^{i(k+\frac{1}{2})\theta} (-lU^- - \frac{d}{dr}U^+ + \frac{(k-\frac{1}{2})}{r}U^+)_{k,l} \end{pmatrix} = 0$$

which gives us a system of equations:

$$\frac{d}{dr} \begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l} = \begin{pmatrix} \frac{(k-\frac{1}{2})}{r} & -l \\ -l & -\frac{(k+\frac{1}{2})}{r} \end{pmatrix} \begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l}.$$

For $l \neq 0$, this equation has standard solutions of the form

$$\begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l} = \begin{pmatrix} u_{k,l}^+ l^{-k+\frac{1}{2}} I_{k-\frac{1}{2}}(lr) - u_{k,l}^- l^{k+\frac{1}{2}} I_{-k+\frac{1}{2}}(lr) \\ -u_{k,l}^+ l^{-k+\frac{1}{2}} I_{k+\frac{1}{2}}(lr) + u_{k,l}^- l^{k+\frac{1}{2}} I_{-k-\frac{1}{2}}(lr) \end{pmatrix}$$

where I_p is the modified Bessel function which can be written as

$$I_p(r) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+p+1)} \left(\frac{r}{2}\right)^{2m+p}.$$

For the properties of Bessel functions, readers can see [4] for more detail.

For $l = 0$ we have

$$\begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l} = \begin{pmatrix} u_{k,0}^+ r^{k-\frac{1}{2}} \\ u_{k,0}^- r^{-k-\frac{1}{2}} \end{pmatrix}.$$

Clearly we have $I_p(r) = O(r^p)$. To normalized the leading coefficient of $I_p(lr)$, We define $\mathfrak{J}_{p,l}(r) = l^{-p} I_p(lr)$.

Now we apply these results to the sections of $\mathcal{S} \otimes \mathcal{I}$ over N . Fix an $R > 0$, we define $N_R := N \cap \{r < R\}$. Suppose $\mathbf{u} \in L^2(N; \mathcal{S} \otimes \mathcal{I})$ and $D\mathbf{u}|_{N_R} = 0$, then

$$\begin{aligned} \mathbf{u} = & \sum_{k \geq 0; l \neq 0} u_{k,l}^+ e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} \mathfrak{J}_{k-\frac{1}{2},l}(r) \\ -e^{i(k+\frac{1}{2})\theta} l \mathfrak{J}_{k+\frac{1}{2},l}(r) \end{pmatrix} + \sum_{k \leq 0; l \neq 0} u_{k,l}^- e^{ilt} \begin{pmatrix} -e^{i(k-\frac{1}{2})\theta} l \mathfrak{J}_{-k+\frac{1}{2},l}(r) \\ e^{i(k+\frac{1}{2})\theta} \mathfrak{J}_{-k-\frac{1}{2},l}(r) \end{pmatrix} \\ & + \sum_{k \geq 0} \begin{pmatrix} u_{k,0}^+ e^{i(k-\frac{1}{2})\theta} r^{k-\frac{1}{2}} \\ 0 \end{pmatrix} + \sum_{k \leq 0} \begin{pmatrix} 0 \\ u_{k,0}^- e^{i(k+\frac{1}{2})\theta} r^{-k-\frac{1}{2}} \end{pmatrix} \end{aligned}$$

which has the leading term of order $\frac{1}{\sqrt{r}}$, i.e.

$$\begin{aligned} \mathbf{u} = & \begin{pmatrix} u_{0,0}^+ e^{-i\frac{1}{2}\theta} r^{-\frac{1}{2}} \\ u_{0,0}^- e^{i\frac{1}{2}\theta} r^{-\frac{1}{2}} \end{pmatrix} + \sum_{l \neq 0} e^{ilt} [u_{0,l}^+ \begin{pmatrix} e^{-i\frac{1}{2}\theta} \mathfrak{J}_{-\frac{1}{2},l}(r) \\ -l e^{i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \end{pmatrix} + u_{0,l}^- \begin{pmatrix} -l e^{-i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \\ e^{i\frac{1}{2}\theta} \mathfrak{J}_{-\frac{1}{2},l}(r) \end{pmatrix}] \\ & + \text{higher order terms.} \end{aligned}$$

The Bessel functions $I_{\frac{1}{2}}(x)$ and $I_{-\frac{1}{2}}(x)$ can be explicitly written as $\sqrt{\frac{2}{\pi x}} \sinh(x)$ and $\sqrt{\frac{2}{\pi x}} \cosh(x)$. So we can change the basis of the leading term in terms of $\{\frac{e^{lr}}{\sqrt{r}}, \frac{e^{-lr}}{\sqrt{r}}\}$. Let us using this expression and denote by \hat{u}^+ and \hat{u}^- the coefficients of the leading

term, then we have

$$\mathbf{u} = \sum_l e^{ilt} [\hat{u}_{0,l}^+ \begin{pmatrix} \frac{e^{|l|r}}{\sqrt{z}} \\ -\text{sign}(l) \frac{e^{|l|r}}{\sqrt{\bar{z}}} \end{pmatrix} + \hat{u}_{0,l}^- \begin{pmatrix} \frac{e^{-|l|r}}{\sqrt{z}} \\ \text{sign}(l) \frac{e^{-|l|r}}{\sqrt{\bar{z}}} \end{pmatrix}] + \text{higher order terms}$$

where $\hat{u}_{0,l}^+ = (u_{0,l}^+ - \text{sign}(l)u_{0,l}^-)$ and $\hat{u}_{0,l}^- = (u_{0,l}^+ + \text{sign}(l)u_{0,l}^-)$.

We defined the following space.

Definition 4.1. For any $R > 0$ given. Let \mathcal{K}_R be a subspace of $L^2(N_R; \mathcal{S} \otimes \mathcal{I})$ defined by

$$\mathcal{K}_R = \{\mathbf{u} \in L^2(N_R; \mathcal{S} \otimes \mathcal{I}) \mid D\mathbf{u} = 0 \text{ and } \hat{u}_l^- = 0 \text{ for all } |l| > \frac{1}{2R}\}.$$

Definition 4.2. Let $\mathbf{u} \in L^2(N_R)$, then there is the corresponding Fourier coefficients $\{u_{k,l}^\pm\}$. We define the following terminologies:

- We call $\{(\hat{u}_{0,l}^+ + \hat{u}_{0,l}^-, -\text{sign}(l)\hat{u}_{0,l}^+ + \text{sign}(l)\hat{u}_{0,l}^-)\} \in l^2 \times l^2$ to be the leading coefficients of \mathbf{u} .
- Define $\{(u_l^+, u_l^-)\} \in l^2 \times l^2$ to be

$$(u_l^+, u_l^-) = (\hat{u}_{0,l}^+, -\text{sign}(l)\hat{u}_{0,l}^+) \text{ for } |l| > \frac{1}{2R}$$

$$(u_l^+, u_l^-) = (\hat{u}_{0,l}^+, -\text{sign}(l)\hat{u}_{0,l}^+) + (\hat{u}_{0,l}^-, \text{sign}(l)\hat{u}_{0,l}^-) \text{ for } |l| \leq \frac{1}{2R}.$$

We call $\{(u_l^+, u_l^-)\}$ the \mathcal{K}_R -leading coefficients of \mathbf{u} .

- We call $(\sum_l (\hat{u}_{0,l}^+ + \hat{u}_{0,l}^-)e^{ilt}, \sum_l (-\text{sign}(l)\hat{u}_{0,l}^+ + \text{sign}(l)\hat{u}_{0,l}^-)e^{ilt})$ to be the leading term of \mathbf{u} .
- Define $u^+(t) = \sum_l u_l^+ e^{ilt}$ and $u^-(t) = \sum_l u_l^- e^{ilt}$ where $\{u_l^\pm\}$ is the \mathcal{K}_R -leading coefficients of \mathbf{u} . We call $u^\pm(t)$ to be the \mathcal{K}_R -leading term of \mathbf{u} .
- We call $(u^+(t) \frac{1}{\sqrt{z}}, u^-(t) \frac{1}{\sqrt{\bar{z}}})$ the \mathcal{K}_R -dominant term of \mathbf{u} , where $u^\pm(t)$ is the \mathcal{K}_R

-leading term of \mathbf{u} .

Moreover, we can see that if $\mathbf{u} \in \mathcal{K}_R$, then the \mathcal{K}_R -leading coefficients(term) will be the leading coefficients(term) of \mathbf{u} .

Now if we consider $\mathbf{v} \in L_1^2(N_R; \mathcal{S} \otimes \mathcal{I})$ and $D\mathbf{v} = 0$, we will have

$$\begin{aligned} \mathbf{v} = & \sum_{k \geq 1; l \neq 0} v_{k,l}^+ e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} \mathfrak{J}_{k-\frac{1}{2},l}(r) \\ -e^{i(k+\frac{1}{2})\theta} l \mathfrak{J}_{k+\frac{1}{2},l}(r) \end{pmatrix} + \sum_{k \leq -1; l \neq 0} v_{k,l}^- e^{ilt} \begin{pmatrix} -e^{i(k-\frac{1}{2})\theta} l \mathfrak{J}_{-k+\frac{1}{2},l}(r) \\ e^{i(k+\frac{1}{2})\theta} \mathfrak{J}_{-k-\frac{1}{2},l}(r) \end{pmatrix} \\ & + \sum_{k \geq 1} \begin{pmatrix} v_{k,0}^+ e^{i(k-\frac{1}{2})\theta} r^{k-\frac{1}{2}} \\ 0 \end{pmatrix} + \sum_{k \leq -1} \begin{pmatrix} 0 \\ v_{k,0}^- e^{i(k+\frac{1}{2})\theta} r^{-k-\frac{1}{2}} \end{pmatrix} \end{aligned}$$

So we can write

$$\mathbf{v} = \begin{pmatrix} v_{-1,0}^+ e^{i\frac{1}{2}\theta} r^{\frac{1}{2}} \\ v_{1,0}^- e^{-i\frac{1}{2}\theta} r^{\frac{1}{2}} \end{pmatrix} + \sum_{l \neq 0} e^{ilt} \begin{pmatrix} v_{-1,l}^+ e^{i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \\ v_{1,l}^- e^{-i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \end{pmatrix} + \text{higher order terms.}$$

So

$$\mathbf{v} = \begin{pmatrix} v_{-1,0}^+ e^{i\frac{1}{2}\theta} r^{\frac{1}{2}} \\ v_{1,0}^- e^{-i\frac{1}{2}\theta} r^{\frac{1}{2}} \end{pmatrix} + \sum_{l \neq 0} e^{ilt} \begin{pmatrix} v_{-1,l}^+ e^{i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \\ v_{1,l}^- e^{-i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \end{pmatrix} + \text{higher order terms.}$$

Again, we should define the leading coefficients and the leading term of \mathbf{v} .

Definition 4.3. Let $\mathbf{v} \in L_1^2(N_R)$.

- We call the Fourier coefficients, $\{(v_{-1,l}^+, v_{1,l}^-)\}$, denoted by $\{v_l^\pm\} \in (\mathbb{C}^2)^\mathbb{Z}$, to be the leading coefficients of \mathbf{v} .
- We define $v^\pm(t)$, where $v^+(t) = \sum_l v_l^+ e^{ilt}$ and $v^-(t) = \sum_l v_l^- e^{ilt}$, to be the leading term of \mathbf{v} .
- We call $(v^+(t)\sqrt{z}, v^-(t)\sqrt{\bar{z}})$ to be the dominant term of \mathbf{v} , where $v^\pm(t)$ is the leading term of \mathbf{v} .

In the rest of this article, we will always use the letters of Fraktur script, $\mathbf{u}, \mathbf{v}, \mathbf{h}, \mathbf{c}$, etc., to denote the sections defined on $L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$ or $L^2_1(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$. If they satisfy the Dirac equation on N_R for some $R > 0$, then their corresponding (\mathcal{K}_R -)leading coefficients will be denoted by the letters of normal script $\{u_l^\pm\}, \{v_l^\pm\}, \{h_l^\pm\}, \{c_l^\pm\}$, etc. which are in $l^2 \times l^2$. Finally, the corresponding (\mathcal{K}_R -)leading terms will be denoted by $u^\pm = \sum u_l^\pm e^{ilt}, v^\pm, h^\pm, c^\pm$ which are in $L^2(S^1) \times L^2(S^1)$. Therefore we have the L^2 -norm for u^\pm will be the same as $(\|\{u_l^+\}\|_{l^2}^2 + \|\{u_l^-\}\|_{l^2}^2)^{\frac{1}{2}}$.

Now we should prove that the \mathcal{K}_R -leading coefficients of $\mathbf{u} \in \mathcal{K}_R$ and $\mathbf{v} \in L^2_2$ in a smaller tubular neighborhood have the following regularity estimate.

Proposition 4.4. We have the following two properties.

a. Let $\mathbf{u} \in \mathcal{K}_R$, then we can decompose

$$\mathbf{u} = \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix} + \mathbf{u}_{\mathfrak{R}}$$

for some $\mathbf{u}_{\mathfrak{R}} \in L^2_1(N_{\frac{2R}{3}}; \mathcal{S} \otimes \mathcal{I})$ where $u^\pm(t) = \sum u_l^\pm e^{ilt}$ and

$$(4.1) \quad \|\mathbf{u}_{\mathfrak{R}}\|_{L^2_1(N_{\frac{2R}{3}})} \leq CR^{-1} \|\mathbf{u}\|_{L^2(N_R)}$$

for some constant C only depending on the curvature of M . In the following paragraph, we call $(\mathbf{u} - \mathbf{u}_{\mathfrak{R}})$ the \mathcal{K}_R -dominant term of \mathbf{u} (we already define this in definition 4.2) and call $\mathbf{u}_{\mathfrak{R}}$ the remainder term of \mathbf{u} .

b. Let $\mathbf{v} \in L^2_1(N_R; \mathcal{S} \otimes \mathcal{I})$ and $D\mathbf{v} = 0$, then we can decompose

$$\mathbf{v} = \begin{pmatrix} v^+(t) \sqrt{z} \\ v^-(t) \sqrt{\bar{z}} \end{pmatrix} + \mathbf{v}_{\mathfrak{R}}$$

for some $\mathbf{v}_{\mathfrak{R}} \in L^2_2(N_{\frac{2R}{3}}; \mathcal{S} \otimes \mathcal{I})$ where $v^\pm(t) = \sum v_l^\pm e^{ilt}$ and

$$(4.2) \quad \|\mathbf{v}_{\mathfrak{R}}\|_{L^2_2(N_{\frac{2R}{3}})} \leq CR^{-2} \|\mathbf{v}\|_{L^2(N_R)}$$

for some constant C only depending on the curvature of M . Similarly, in the following paragraph, we call $(\mathbf{v} - \mathbf{v}_{\mathfrak{R}})$ the dominant term of \mathbf{v} (we already define this in definition 4.3) and call $\mathbf{v}_{\mathfrak{R}}$ the remainder term of \mathbf{v} .

Proof. (proof of part a). To prove this part, we claim the following two statements:

Firstly, we have $D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix} \in L^2(N_R)$ and

$$(4.3) \quad \left\| D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix} \right\|_{L^2(N_R)}^2 \leq CR^{-2} \|\mathbf{u}\|_{L^2(N_R)}^2$$

for some $C > 0$. Secondly

$$(4.4) \quad \left\| \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix} \right\|_{L^2(N_R)}^2 \leq C \|\mathbf{u}\|_{L^2(N_R)}^2$$

We will prove these claims in the corollary 4.6.

Now, we fix a $K > 0$ and define

$$\mathbf{u}_{\mathfrak{R},K} = \sum_{k \neq 0} \sum_{|l| \leq K} e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} U_{k,l}^+ \\ e^{i(k+\frac{1}{2})\theta} U_{k,l}^- \end{pmatrix} - \sum_{l \neq 0; |l| \leq K} e^{ilt} \begin{pmatrix} u_l^+ \frac{1}{\sqrt{z}} \\ u_l^- \frac{1}{\sqrt{\bar{z}}} \end{pmatrix}.$$

We can easily see that $|\mathbf{u}_{\mathfrak{R},K}| \leq C_K \sqrt{r}$ and $|\nabla \mathbf{u}_{\mathfrak{R},K}| \leq C_K \frac{1}{\sqrt{r}}$, which means there will be no boundary term when we do the integration by part for the Schrodinger-Lichnerowicz formula. Now we let χ be a positive smooth function supported on N_R where

$$\chi = \begin{cases} 1 & \text{on } N_{\frac{2R}{3}} \\ 0 & \text{on } M - N_R \end{cases}$$

and $|\nabla \chi| < C \frac{1}{R}$.

By applying Schrodinger-Lichinerowicz formula on $\chi \mathbf{u}_{\mathfrak{R},K}$ and using claims (4.3), (4.4) above, we have

$$\begin{aligned}
(4.5) \quad \|\mathbf{u}_{\mathfrak{R},K}\|_{L^2_1(N_{\frac{2R}{3}})}^2 &\leq \|D\mathbf{u}_{\mathfrak{R},K}\|_{L^2(N_R)}^2 + C\frac{1}{R^2}\|\mathbf{u}_{\mathfrak{R},K}\|_{L^2(N_R)}^2 \\
&\leq \|D\begin{pmatrix} u^+(t)\frac{1}{\sqrt{z}} \\ u^-(t)\frac{1}{\sqrt{\bar{z}}} \end{pmatrix}\|_{L^2(N_R)}^2 + C\frac{1}{R^2}\|\mathbf{u}_{\mathfrak{R},K}\|_{L^2(N_R)}^2 \\
&\leq CR^{-2}\|\mathbf{u}\|_{L^2(N_R)}^2
\end{aligned}$$

for some $C > 0$.

By taking $K \rightarrow \infty$ in (4.5), we have

$$\|\mathbf{u}_{\mathfrak{R}}\|_{L^2_1(N_{\frac{2R}{3}})}^2 \leq CR^{-2}\|\mathbf{u}\|_{L^2(N_R)}^2.$$

(proof of part b). Similar to the proof of part a, we have the following two claims which will be proved in corollary 4.8.

$$(4.6) \quad \|D\begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix}\|_{L^2(N_R)}^2 \leq CR^{-2}\|\mathbf{v}\|_{L^2(N_R)}^2$$

and

$$(4.7) \quad \left\| \begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix} \right\|_{L^2(N_R)}^2 \leq C\|\mathbf{v}\|_{L^2(N_R)}^2.$$

Now we fix a $K > 0$, define

$$\mathbf{v}_{\mathfrak{R},K} = \sum_{k \neq 0} \sum_{|l| \leq K} e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} V_{k,l}^+ \\ e^{i(k+\frac{1}{2})\theta} V_{k,l}^- \end{pmatrix} - \sum_{l \neq 0; |l| \leq K} e^{ilt} \begin{pmatrix} v_l^+ \sqrt{z} \\ v_l^- \sqrt{\bar{z}} \end{pmatrix}.$$

We have $|\mathbf{u}_{\mathfrak{R},K}| \leq C_K \sqrt{r^3}$ and $|\nabla \mathbf{u}_{\mathfrak{R},K}| \leq C_K \sqrt{r}$ and $|\nabla \nabla \mathbf{u}_{\mathfrak{R},K}| \leq C_K \frac{1}{\sqrt{r}}$. So by applying Schrodinger-Lichinerowicz formula on $\chi \mathbf{v}_{\mathfrak{R},K}$, we have

$$\begin{aligned}
(4.8) \quad \|\mathbf{v}_{\mathfrak{R},K}\|_{L_1^2(N_{\frac{2R}{3}})}^2 &\leq \|D\mathbf{v}_{\mathfrak{R},K}\|_{L^2}^2 + C \frac{1}{R^2} \|\mathbf{v}_{\mathfrak{R},K}\|_{L^2(N_R)}^2 \\
&\leq \|D \begin{pmatrix} v^+(t) \frac{1}{\sqrt{z}} \\ v^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix}\|_{L^2(N_R)}^2 + C \frac{1}{R^2} \|\mathbf{v}_{\mathfrak{R},K}\|_{L^2(N_R)}^2 \\
&\leq CR^{-2} \|\mathbf{v}\|_{L^2(N_R)}^2
\end{aligned}$$

for some $C > 0$. By taking the limit $K \rightarrow \infty$, we have

$$\|\mathbf{v}_{\mathfrak{R}}\|_{L_1^2(N_{\frac{2R}{3}})}^2 \leq CR^{-2} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

Notice that $[\nabla_i, D] = 0$, so we can use the same argument on $\nabla_i \mathbf{v}$ now. Here we need the following claim which is also proved in corollary 4.8.

$$(4.9) \quad \|D[\nabla \begin{pmatrix} v^+(t) \sqrt{z} \\ v^-(t) \sqrt{\bar{z}} \end{pmatrix}]\|_{L^2(N_R)}^2 \leq CR^{-4} \|\mathbf{v}\|_{L^2(N_R)}^2$$

and

$$(4.10) \quad \left\| \begin{pmatrix} v^+(t) \sqrt{z} \\ v^-(t) \sqrt{\bar{z}} \end{pmatrix} \right\|_{L_1^2(N_R)}^2 \leq CR^{-2} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

So we have

$$\begin{aligned}
(4.11) \quad \|\mathbf{v}_{\mathfrak{R},K}\|_{L_2^2(N_{\frac{2R}{3}})}^2 &\leq \|D[\nabla \mathbf{v}_{\mathfrak{R},K}]\|_{L^2(N_R)}^2 + C \frac{1}{R^2} \|\mathbf{v}_{\mathfrak{R},K}\|_{L_1^2(N_R)}^2 \\
&\leq \|D[\nabla \begin{pmatrix} v^+(t) \frac{1}{\sqrt{z}} \\ v^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix}]\|_{L^2(N_R)}^2 + C \frac{1}{R^2} \|\mathbf{v}_{\mathfrak{R},K}\|_{L_1^2(N_R)}^2 \\
&\leq CR^{-4} \|\mathbf{v}\|_{L^2(N_R)}^2
\end{aligned}$$

for some $C > 0$. By taking the limit $K \rightarrow \infty$, we prove this proposition. \square

In the next section, we will derive some regularity properties about the leading coefficients.

4.2. Regularity properties and the asymptotic behavior of the L^2 -harmonic sections on the tubular neighborhood. In this subsection, we will derive some regularity theorem for those $\mathbf{u} \in L^2(N_R; \mathcal{S} \otimes \mathcal{I})$ and $D\mathbf{u} = 0$. These estimates are similar to the doubling estimates appearing in [11]. Recall that, by standard regularity theorem, \mathbf{u} is a smooth section on any compact subset of N_R . We write

$$\mathbf{u} = \sum_{l,k} e^{ilt} \begin{pmatrix} e^{(k-\frac{1}{2})\theta} U_{k,l}^+ \\ e^{(k+\frac{1}{2})\theta} U_{k,l}^- \end{pmatrix}$$

where

$$\begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,l} = \begin{pmatrix} u_{k,l}^+ \mathfrak{J}_{k-\frac{1}{2},l}(r) - u_{k,l}^- l \mathfrak{J}_{-k+\frac{1}{2},l}(r) \\ -u_{k,l}^+ l \mathfrak{J}_{k+\frac{1}{2},l}(r) + u_{k,l}^- \mathfrak{J}_{-k-\frac{1}{2},l}(r) \end{pmatrix}$$

for $l \neq 0$ and

$$\begin{pmatrix} U^+ \\ U^- \end{pmatrix}_{k,0} = \begin{pmatrix} u_{k,0}^+ r^{k-\frac{1}{2}} \\ u_{k,0}^- r^{-k-\frac{1}{2}} \end{pmatrix}.$$

Since $\mathbf{u} \in L^2$, so we have

$$u_{k,l}^+ = 0 \text{ for } k \leq -1;$$

$$u_{k,l}^- = 0 \text{ for } k \geq 1.$$

Moreover, let

$$E_{k,l} = \left\{ e^{ilt} \begin{pmatrix} u_{k,l}^+ e^{i\frac{1}{2}\theta} \mathfrak{J}_{k-\frac{1}{2},l}(r) - u_{k,l}^- e^{i\frac{1}{2}\theta} l \mathfrak{J}_{-k+\frac{1}{2},l}(r) \\ -u_{k,l}^+ e^{-i\frac{1}{2}\theta} l \mathfrak{J}_{k+\frac{1}{2},l}(r) + u_{k,l}^- e^{-i\frac{1}{2}\theta} \mathfrak{J}_{-k-\frac{1}{2},l}(r) \end{pmatrix} \in L^2 \right\},$$

then $E_{k,l} \perp E_{k',l'}$ for any two $(k,l) \neq (k',l')$.

By using these observation, we can prove the following proposition.

Proposition 4.5. Let $\mathbf{u} \in L^2(N_R; \mathcal{S} \otimes \mathcal{I}) \cap \ker(D)$ with the corresponding Fourier coefficients $\{u_{k,l}^\pm\}$. Then the \mathcal{K}_R -leading coefficients $\{u_l^\pm\}$ is in l_k^2 for any $k \in \mathbb{N}$. Moreover, we have

$$(4.12) \quad \|(l^k u_l^\pm)_{l \in \mathbb{Z}}\|_2^2 \leq 3 \frac{(2k+1)!}{R^{2k+1}} \|\mathbf{u}\|_{L^2}^2.$$

Proof. First of all, let $P_{k,l} : L^2 \cap \ker(D) \rightarrow E_{k,l}$ to be the orthonormal projection.

We have

$$P_{0,l}(\mathbf{u}) = e^{ilt} \begin{pmatrix} \hat{u}_{0,l}^+ \frac{e^{|l|r}}{\sqrt{z}} + \hat{u}_{0,l}^- \frac{e^{-|l|r}}{\sqrt{z}} \\ -\text{sign}(l) \hat{u}_{0,l}^+ \frac{e^{|l|r}}{\sqrt{z}} + \text{sign}(l) \hat{u}_{0,l}^- \frac{e^{-|l|r}}{\sqrt{z}} \end{pmatrix}$$

for any l .

Now recall that $(u_l^+, u_l^-) = (\hat{u}_{0,l}^+, -\text{sign}(l) \hat{u}_{0,l}^+)$ for $|l| > \frac{1}{2R}$ and $(u_l^+, u_l^-) = (\hat{u}_{0,l}^+, -\text{sign}(l) \hat{u}_{0,l}^+) + (\hat{u}_{0,l}^-, \text{sign}(l) \hat{u}_{0,l}^-)$ for $|l| \leq \frac{1}{2R}$.

We can compute directly to get

$$\begin{aligned} \|\mathbf{u}\|_{L^2(N_R)}^2 &\geq \sum_l |P_{0,l}(\mathbf{u})|^2 \\ &\geq \sum_l |\hat{u}_{0,l}^+|^2 \int_0^R e^{2|l|r} dr + \sum_l |\hat{u}_{0,l}^-|^2 \int_0^R e^{-2|l|r} dr \\ &\geq \sum_l |u_{0,l}^+|^2 \left(\int_0^R e^{2|l|r} dr \right) \\ &\geq \sum_k \sum_l |u_{0,l}^+|^2 \frac{(2l)^{2k} R^{2k+1}}{(2k+1)!}. \end{aligned}$$

Meanwhile, the second line of this inequality also tells us that

$$\begin{aligned} \|\mathbf{u}\|_{L^2(N_R)}^2 &\geq \sum |\hat{u}_{0,l}^-|^2 \int_0^R e^{-2|l|r} dr \\ &\geq \sum_{|l| \leq \frac{1}{2R}} e^{-1} |\hat{u}_{0,l}^-|^2 R \\ &\geq \sum_{|l| \leq \frac{1}{2R}} e^{-1} |\hat{u}_{0,l}^-|^2 |l|^{2k} R^{2k+1}. \end{aligned}$$

So we prove (4.12). □

By using this proposition, we can prove (4.3) and (4.4) in the following way.

Corollary 4.6. *Suppose that $\begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix}$ is the \mathcal{K}_R -dominant term of a L^2 -harmonic section \mathbf{u} as we showed in proposition 4.4, then*

$$\|D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix}\|_{L^2(N_R)}^2 \leq CR^{-2} \|\mathbf{u}\|_{L^2(N_R)}^2$$

and

$$\left\| \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix} \right\|_{L^2(N_R)} \leq C \|\mathbf{u}\|_{L^2(N_R)}$$

for some constant $C > 0$.

Proof. We can compute directly that

$$D \begin{pmatrix} u^+(t) \frac{1}{\sqrt{z}} \\ u^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix} = \begin{pmatrix} \dot{u}^+(t) \frac{1}{\sqrt{z}} \\ \dot{u}^-(t) \frac{1}{\sqrt{\bar{z}}} \end{pmatrix}.$$

Then by proposition 4.5, we can prove this corollary immediately. □

4.3. Regularity properties and the asymptotic behavior of the L_1^2 -harmonic sections on the tubular neighborhood. Suppose that $\mathbf{v} \in L_1^2(N_R; \mathcal{S} \otimes \mathcal{I})$ and

$D\mathbf{v} = 0$, then we can write

$$\mathbf{v} = \sum_{l,k} e^{ilt} \begin{pmatrix} e^{(k-\frac{1}{2})\theta} V_{k,l}^+ \\ e^{(k+\frac{1}{2})\theta} V_{k,l}^- \end{pmatrix}$$

where

$$\begin{pmatrix} V^+ \\ V^- \end{pmatrix}_{k,l} = \begin{pmatrix} v_{k,l}^+ \mathfrak{J}_{k-\frac{1}{2},l}(r) - v_{k,l}^- l \mathfrak{J}_{-k+\frac{1}{2},l}(r) \\ -v_{k,l}^+ l \mathfrak{J}_{k+\frac{1}{2},l}(r) + v_{k,l}^- \mathfrak{J}_{-k-\frac{1}{2},l}(r) \end{pmatrix}$$

for $l \neq 0$ and

$$\begin{pmatrix} V^+ \\ V^- \end{pmatrix}_{k,0} = \begin{pmatrix} v_{k,0}^+ r^{k-\frac{1}{2}} \\ v_{k,0}^- r^{-k-\frac{1}{2}} \end{pmatrix}.$$

Since $\mathbf{v} \in L_1^2$, so we have

$$v_{k,l}^+ = 0 \text{ for } k \leq 0;$$

$$v_{k,l}^- = 0 \text{ for } k \geq 0.$$

Proposition 4.7. Let $\mathbf{v} \in L_1^2(N_R; \mathcal{S} \otimes \mathcal{I}) \cap \ker(D)$ with the corresponding coefficient $\{v_{k,l}^\pm\}$. Then the leading coefficients $\{(v_l^\pm)\}$ defined in definition 4.3 is in l_k^2 for any $k \in \mathbb{N} \cup \{0\}$. Moreover, we have

$$(4.13) \quad \|(l^k v_l^\pm)_{l \in \mathbb{Z}}\|_2^2 \leq \frac{(2k+3)!}{R^{2k+3}} \|\mathbf{v}\|_{L^2}^2$$

Proof. We use the notations defined in proposition 4.5.

$$P_{-1,l} = \begin{pmatrix} -v_{-1,l}^- l \mathfrak{J}_{\frac{3}{2},l}(r) \\ v_{-1,l}^- \mathfrak{J}_{\frac{1}{2},l}(r) \end{pmatrix}, \quad P_{1,l} = \begin{pmatrix} v_{1,l}^+ \mathfrak{J}_{\frac{1}{2},l}(r) \\ -v_{1,l}^+ l \mathfrak{J}_{\frac{3}{2},l}(r) \end{pmatrix}.$$

for $l \neq 0$ and

$$P_{-1,0} = \begin{pmatrix} 0 \\ v_{-1,0}^- r^{\frac{1}{2}} \end{pmatrix}, \quad P_{1,0} = \begin{pmatrix} v_{1,0}^+ r^{\frac{1}{2}} \\ 0 \end{pmatrix}.$$

Since $\mathfrak{J}_{\frac{1}{2},l} = \frac{\sinh(lr)}{l\sqrt{r}}$, we have

$$\begin{aligned} \|\mathbf{v}\|_{L^2}^2 &\geq \sum_{l \neq 0} (|v_{1,l}^+|^2 + |v_{-1,l}^-|^2) \int_0^R \frac{\sinh^2(lr)}{l^2} dr + (|v_{1,0}^+|^2 + |v_{-1,0}^-|^2) \int_0^R r^2 dr \\ &\geq \sum_l (|v_{1,l}^+|^2 + |v_{-1,l}^-|^2) \sum_{k=0}^{\infty} \frac{l^{2k} R^{2k+3}}{(k+3)!} \\ &= \sum_l |v_l^\pm|^2 \sum_{k=0}^{\infty} \frac{l^{2k} R^{2k+3}}{(2k+3)!}. \end{aligned}$$

Therefore we prove this proposition. □

We also have the following corollary which is similar to the corollary 4.6. We omit the proof for this corollary.

Corollary 4.8. *Suppose $\begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix}$ is the dominant term of a L_1^2 -harmonic section \mathbf{v} as we showed in proposition 4.4, then we have*

a.

$$\|D \begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix}\|_{L^2(N_R)}^2 \leq CR^{-2} \|\mathbf{v}\|_{L^2(N_R)}^2$$

and

$$\left\| \begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix} \right\|_{L^2(N_R)}^2 \leq C \|\mathbf{v}\|_{L^2(N_R)}^2$$

for some constant $C > 0$.

b.

$$\|D[\nabla \begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix}]\|_{L^2(N_R)}^2 \leq CR^{-4}\|\mathbf{v}\|_{L^2(N_R)}^2$$

and

$$\|\begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix}\|_{L_1^2(N_R)}^2 \leq CR^{-1}\|\mathbf{v}\|_{L^2(N_R)}^2$$

for some constant $C > 0$.

Finally, we can prove the following theorem by using proposition 4.7 now.

Theorem 4.9. *For any $\mathbf{v} \in L_1^2(N_R) \cap \ker(D)$, we have*

$$\|\mathbf{v}\|_{L^2(N_r)}^2 \leq r^3 \frac{C}{R^3} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

Moreover, we can also prove that

$$\|\mathbf{v}_t\|_{L^2(N_r)}^2 \leq r^3 \frac{C}{R^5} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

for some constant $C > 0$ and all $r \leq \frac{R}{2}$.

Proof. To prove the first statement, we use the lemma 2.6 to get

$$\|\mathbf{v}\|_{L^2(N_r)}^2 \leq Cr^2 \|\nabla \mathbf{v}\|_{L^2(N_r)}^2$$

for all $\mathbf{v} \in L_1^2(N_R)$ and $r < R$.

By the lemma 2.6, proposition 4.7 and proposition 4.4 **b**, we have

$$\begin{aligned} \int_{N_r} |\mathbf{v}|^2 &\leq Cr^2 \int_{N_r} |\nabla \mathbf{v}|^2 \leq 2Cr^2 \int_{N_r} \left| \nabla \begin{pmatrix} v^+(t)\sqrt{z} \\ v^-(t)\sqrt{\bar{z}} \end{pmatrix} \right|^2 + |\nabla \mathbf{v}_{\mathfrak{R}}|^2 \\ &\leq 2C \frac{r^3}{R^3} \|\mathbf{v}\|_{L^2(N_R)}^2 + 2Cr^4 \|\mathbf{v}_{\mathfrak{R}}\|_{L^2_2(N_R)}^2 \\ &\leq 4C \frac{r^3}{R^3} \|\mathbf{v}\|_{L^2(N_R)}^2 \end{aligned}$$

for some $C > 0$.

To prove the second statement, we notice that by applying lemma 2.6 on \mathbf{v}_t ,

$$\int_{N_r} |\mathbf{v}_t|^2 \leq r^2 \int_{N_r} |\nabla \mathbf{v}_t|^2 \leq 2r^2 \int_{N_r} \left| \nabla \begin{pmatrix} v_t^+(t)\sqrt{z} \\ v_t^-(t)\sqrt{\bar{z}} \end{pmatrix} \right|^2 + |\nabla(\mathbf{v}_{\mathfrak{R}})_t|^2.$$

By using the proposition 4.7, we have

$$r^2 \int_{N_r} \left| \nabla \begin{pmatrix} v_t^+(t)\sqrt{z} \\ v_t^-(t)\sqrt{\bar{z}} \end{pmatrix} \right|^2 \leq 2 \frac{r^3}{R^5} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

So we have

$$\int_{N_r} |\mathbf{v}_t|^2 \leq 2 \frac{r^3}{R^5} \|\mathbf{v}\|_{L^2(N_R)}^2 + 2r^2 \|\mathbf{v}_{\mathfrak{R}}\|_{L^2_2(N_r)}^2.$$

Then by the first statement proved above and proposition 4.4 **b**,

$$\|\mathbf{v}_{\mathfrak{R}}\|_{L^2_2(N_r)}^2 \leq \frac{C}{R^4} \|\mathbf{v}\|_{L^2(N_{2r})}^2 \leq C \frac{r^3}{R^7} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

So we prove the second statement. □

Remark 4.10. By using this theorem, proposition 2.3 and lemma 2.6, we can prove that for any $\mathbf{v} \in L^2_1(N_R) \cap \ker(D)$, we have

$$\|\mathbf{v}\|_{L^2_{-1}(N_r)}^2 \leq r^5 \frac{C}{R^3} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

Moreover, we can also prove that

$$\|\mathbf{v}_t\|_{L^2_{-1}(N_r)}^2 \leq r^5 \frac{C}{R^5} \|\mathbf{v}\|_{L^2(N_R)}^2.$$

for some constant $C > 0$.

5. VARIATIONAL FORMULA AND PERTURBATION OF CURVES

We introduce some tools needed for the proof of the main theorem here.

5.1. Variational formula. We should review the following fact about the Sobolev inequality and introduce a modified Poincare inequality first.

Let $\mathbf{u} \in L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$. We have $|\mathbf{u}| \in L^2(M - \Sigma; \mathbb{R})$. Since Σ is a measure zero subset of M , we can extend $|\mathbf{u}|$ as a L^2 section on M . Moreover, suppose \mathbf{u} is in $L^2_1(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$, we will have $|\mathbf{u}| \in L^2_1(M; \mathbb{R})$.

Now, by Sobolev inequality, we have

$$(5.1) \quad \|\mathbf{u}\|_{L^6(M; \mathbb{R})} \leq C \|\mathbf{u}\|_{L^2_1(M; \mathbb{R})}$$

for some constant $C > 0$. Another important tool we need is a modified type of Poincare inequality.

Lemma 5.1. *Let $\mathbf{u} \in L^2_1$ and $\mathbf{u} \perp \ker(D)$, then we have*

$$(5.2) \quad \|\mathbf{u}\|_{L^2} \leq C \|D\mathbf{u}\|_{L^2}$$

for some C depending only on the volume of M .

Proof. The lemma can be showed immediately by proving the Dirac operator has empty radius spectrum and empty continuous spectrum and has nonnegative 1st eigenvalue. See [6] for the proof. \square

Now, we define the following functional:

Definition 5.2. Let $\mathbf{f} \in L^2_{-1}$, we define the functional

$$E_{\mathbf{f}}(\mathbf{u}) = \int_{M-\Sigma} |D\mathbf{u}|^2 + \langle \mathbf{u}, \mathbf{f} \rangle$$

for all $\mathbf{u} \in L^2_1$.

Since D is self-adjoint, the Euler-Lagrange equation of $E_{\mathbf{f}}$ will be

$$(5.3) \quad D^2\mathbf{u} = \mathbf{f}$$

We can prove the following proposition:

Proposition 5.3. For any $\mathbf{f} \in L^2_{-1}$, the corresponding functional $E_{\mathbf{f}}$ is bounded from below and for any $\mathbf{u} \in L^2_1$, we have

$$(5.4) \quad E_{\mathbf{f}}(\mathbf{u}) \geq \alpha \|D\mathbf{u}\|_{L^2}^2 - \beta$$

for some $\alpha > 0, \beta \in \mathbb{R}$ (This property is usually called coercive). Moreover, if we consider the admissible set of $E_{\mathbf{f}}$ to be all sections in $L^2_1 \cap \ker(D)^\perp$, then $E_{\mathbf{f}}(\mathbf{u})$ has a unique minimizer.

Proof. We separate the proof into 3 parts.

Step 1. First of all, we define the following smooth functions on M :

$$\chi_1 = \begin{cases} 1 & \text{on } M - N_{\frac{R}{2}} \\ 0 & \text{on } N_{\frac{R}{3}} \end{cases}$$

and $\chi_2 = 1 - \chi_1$.

Then we claim the following statement: There exist $\delta, K_0 > 0$ such that

$$(5.5) \quad E_{\mathbf{f}}(\mathbf{u}) \geq \delta \int_{M-\Sigma} |D(\chi_1\mathbf{u})|^2 + \int_{M-\Sigma} \langle \chi_1\mathbf{u}, \mathbf{f} \rangle - K_0.$$

Assuming this claim is true, then by proposition 2.3, we have

$$\int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f} \rangle = \int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f}_0 \rangle + \langle \nabla(\chi_1 \mathbf{u}), \mathbf{f}_1 \rangle$$

for some $(\mathbf{f}_0, \mathbf{f}_1) \in L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I}) \times L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I} \otimes T^*M)$. So by Cauchy's inequality, we have

$$\begin{aligned} E_{\mathbf{f}}(\mathbf{u}) &\geq \delta \int_{M-\Sigma} |D(\chi_1 \mathbf{u})|^2 + \int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f} \rangle - K_0 \\ &\geq \delta \|D(\chi_1 \mathbf{u})\|_{L^2}^2 - \varepsilon \|\nabla(\chi_1 \mathbf{u})\|_{L^2}^2 - \frac{1}{4\varepsilon} \|\mathbf{f}_1\|_{L^2}^2 + \int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f}_0 \rangle - K_0. \end{aligned}$$

Meanwhile, since $\chi_1 \mathbf{u} = 0$ on $N_{\frac{R}{3}}$, we have

$$\|\nabla(\chi_1 \mathbf{u})\|_{L^2}^2 \leq \|D(\chi_1 \mathbf{u})\|_{L^2}^2 + \sup |\mathcal{R}| \|\mathbf{u}\|_{L^2}^2$$

by Schrodinger-Lichnerowicz formula. Therefore for any $\varepsilon \leq \frac{\delta}{2}$ we have

$$\begin{aligned} E_{\mathbf{f}}(\mathbf{u}) &\geq \frac{\delta}{2} \|D(\chi_1 \mathbf{u})\|_{L^2}^2 - \varepsilon \|\mathcal{R}\|_{L^\infty}^2 \|\chi_1 \mathbf{u}\|_{L^2}^2 - \frac{1}{4\varepsilon} \|\mathbf{f}_1\|_{L^2}^2 + \int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f}_0 \rangle - K_0 \\ &\geq \frac{\delta}{2} \|D(\chi_1 \mathbf{u})\|_{L^2}^2 - (1 + \|\mathcal{R}\|_{L^\infty}^2) \varepsilon \|\chi_1 \mathbf{u}\|_{L^2}^2 - \frac{1}{4\varepsilon} \|\mathbf{f}\|_{L^2_{-1}}^2 - K_0. \end{aligned}$$

Now since $\chi_1 \mathbf{u} = 0$ on $\{r \leq \frac{R}{3}\}$, by regularity theory of the elliptic operator, we have

$$\|\chi_1 \mathbf{u}\|_{L^2} \leq C \|D(\chi_1 \mathbf{u})\|_{L^2}.$$

So by taking ε small enough, we have

$$(5.6) \quad E_{\mathbf{f}}(\mathbf{u}) \geq \frac{\delta}{4} \|D(\chi_1 \mathbf{u})\|^2 - \frac{1}{4\varepsilon} \|\mathbf{f}\|_{L^2_{-1}}^2 - K_0$$

Therefore, by setting $\alpha = \frac{\delta}{4}$ and $\beta = \frac{1}{4\varepsilon} \|\mathbf{f}\|_{L^2_{-1}}^2 + K_0$, we can prove that $E_{\mathbf{f}}$ is bounded from below and coercive.

Step 2. We prove the claim (5.5) now. By using Schrodinger-Lichinerowicz formula, we have

$$(5.7) \quad E_{\mathfrak{f}}(\mathbf{u}) \geq \int_{M-\Sigma} |\nabla \mathbf{u}|^2 - 2 \int_{M-\Sigma} |\mathcal{R}||\mathbf{u}|^2 + \gamma \int_{M-\Sigma} \mathcal{R}|\mathbf{u}|^2 + \int_{M-\Sigma} \langle (1 - \chi_1)\mathbf{u}, \mathfrak{f} \rangle + \langle \chi_1 \mathbf{u}, \mathfrak{f} \rangle$$

for any $\gamma \leq 1$, where \mathcal{R} is the scalar curvature on M . Since we have product metric on N_R , so $\mathcal{R} = 0$ on N_R . Therefore there is a constant $C > 0$ such that

$$\int_{M-\Sigma} |\mathcal{R}||\mathbf{u}|^2 \leq C \int_{M-\Sigma} |\chi_1 \mathbf{u}|^2.$$

Now for any $\varepsilon > 0$, by using Holder inequality and Sobolev inequality, we have

$$\begin{aligned} C \int_{M-\Sigma} |\chi_1 \mathbf{u}|^2 &\leq C \left(\int_{M-\Sigma} |\chi_1 \mathbf{u}|^6 \right)^{\frac{1}{3}} (Vol(M))^{\frac{2}{3}} \\ &\leq \varepsilon \left(\int_{M-\Sigma} |\chi_1 \mathbf{u}|^6 \right)^{\frac{1}{3}} + C_\varepsilon (Vol(M))^{\frac{2}{3}} \\ &\leq \varepsilon \|\chi_1 \mathbf{u}\|_{L_1^2}^2 - K_\varepsilon \end{aligned}$$

for some constant $K_\varepsilon > 0$ depending on ε .

Meanwhile, we have

$$\int_{M-\Sigma} |\nabla \mathbf{u}|^2 \geq \frac{1}{2} \int_{M-\Sigma} |\nabla(\chi_1 \mathbf{u})|^2 - C \int_{M-\Sigma} |\sigma(\chi_1) \mathbf{u}|^2.$$

Since $\sigma(\chi_1)$ is supported on $M - N_{\frac{R}{2}}$, by proposition 2.6, we have

$$\int_{M-\Sigma} |\sigma(\chi_1) \mathbf{u}|^2 \leq C \int_{N_R} |\nabla \mathbf{u}|^2.$$

Therefore we have

$$(5.8) \quad \int_{M-\Sigma} |\nabla \mathbf{u}|^2 \geq \frac{1}{2} \int_{M-\Sigma} |\nabla(\chi_1 \mathbf{u})|^2 - C \int_{N_R} |\nabla \mathbf{u}|^2.$$

Similarly, since $(1 - \chi_1)$ is also supported on $M - N_{\frac{R}{2}}$, we have

$$\|(1 - \chi_1)\mathbf{u}\|_{L^2} \leq C \int_{N_R} |\nabla \mathbf{u}|^2$$

which implies

$$(5.9) \quad \left| \int_{M-\Sigma} \langle (1 - \chi_1)\mathbf{u}, \mathbf{f} \rangle \right| \leq \varepsilon \|(1 - \chi_1)\mathbf{u}\|_{L^2} + C_\varepsilon \|\mathbf{f}\|_{L^2_{-1}} \leq C\varepsilon \int_{N_R} |\nabla \mathbf{u}|^2 + K'_\varepsilon$$

for some constant K'_ε depending on ε .

Therefore by (5.8) and (5.9) there exists $\delta > 0$ small enough such that

$$\int_{M-\Sigma} |\nabla \mathbf{u}|^2 + \int_{M-\Sigma} \langle (1 - \chi_1)\mathbf{u}, \mathbf{f} \rangle \geq 3\delta \int_{M-\Sigma} |\nabla(\chi_1 \mathbf{u})|^2 - K'_\varepsilon.$$

Now, by taking $\varepsilon \leq \delta$ and $\gamma = 2\delta$, we can estimate (5.7) as follows

$$\begin{aligned} E_{\mathbf{f}}(\mathbf{u}) &\geq 3\delta \int_{M-\Sigma} |\nabla(\chi_1 \mathbf{u})|^2 + 2\delta \int_{M-\Sigma} \mathcal{R}|\mathbf{u}|^2 - \varepsilon \|\chi_1 \mathbf{u}\|_{L^2}^2 - K_\varepsilon - K'_\varepsilon + \int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f} \rangle \\ &\geq 2\delta \int_{M-\Sigma} |D(\chi_1 \mathbf{u})|^2 - 2\varepsilon \|\chi_1 \mathbf{u}\|_{L^2}^2 - K_\varepsilon - K'_\varepsilon - \frac{1}{4\varepsilon} \|\mathbf{f}\|_{L^2_{-1}}^2 + \int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f} \rangle. \end{aligned}$$

Again, by regularity theory of Dirac operator

$$\|\chi_1 \mathbf{u}\|_{L^2} \leq C \|D(\chi_1 \mathbf{u})\|_{L^2},$$

we can take ε small such that

$$\varepsilon \|\chi_1 \mathbf{u}\|_{L^2}^2 \leq \frac{\delta}{2} \|D(\chi_1 \mathbf{u})\|_{L^2}^2.$$

So we have

$$E_{\mathbf{f}}(\mathbf{u}) \geq \delta \int_{M-\Sigma} |D(\chi_1 \mathbf{u})|^2 + \int_{M-\Sigma} \langle \chi_1 \mathbf{u}, \mathbf{f} \rangle - K_\varepsilon - K'_\varepsilon - \frac{1}{4\varepsilon} \|\mathbf{f}\|_{L^2_{-1}}^2.$$

Let $K_\varepsilon + K'_\varepsilon + \frac{1}{4\varepsilon} \|\mathbf{f}\|_{L^2_{-1}}^2 = K_0$, then we proved our claim.

Step 3. Now we should prove that E_f has a unique minimizer in $L_1^2 \cap \ker(D)^\perp$.

Suppose we have a sequence $\{\mathbf{u}_n\} \subset L_1^2 \cap \ker(D)^\perp$ such that

$$\lim_{n \rightarrow \infty} E_f(\mathbf{u}_n) = \inf_{\mathbf{u} \in L_1^2} E_f(\mathbf{u}).$$

Let us call $\inf_{\mathbf{u} \in L_1^2} E_f(\mathbf{u}) = m$. Then there exists $n_0 \in \mathbb{N}$ such that

$$E_f(\mathbf{u}_n) \leq m + 1$$

for all $n > n_0$. So

$$\alpha \|D\mathbf{u}_n\|_{L^2}^2 - \beta \leq E_f(\mathbf{u}_n) \leq m + 1$$

for all $n > n_0$. This inequality implies the sequence $\{\|D\mathbf{u}_n\|_{L^2}\}_{n > n_0}$ is bounded. By lemma 5.1, we have $\{\|\mathbf{u}_n\|_{L_1^2}\}$ is bounded. So a subsequence of $\{\mathbf{u}_n\}$ has a weak limit, say \mathbf{u} , which is a minimizer of E_f .

Finally, we prove the uniqueness. Suppose we have $\mathbf{u}_a, \mathbf{u}_b$ are two minimizers in $L_1^2 \cap \ker(D)^\perp$, then

$$\begin{aligned} E_f\left(\frac{\mathbf{u}_a + \mathbf{u}_b}{2}\right) &= \int \frac{1}{4} (|D\mathbf{u}_a + D\mathbf{u}_b|^2) + \frac{1}{2} \langle \mathbf{u}_a, \mathbf{f} \rangle + \frac{1}{2} \langle \mathbf{u}_b, \mathbf{f} \rangle \\ &\leq \int \frac{1}{2} |D\mathbf{u}_a|^2 + \frac{1}{2} |D\mathbf{u}_b|^2 + \frac{1}{2} \langle \mathbf{u}_a, \mathbf{f} \rangle + \frac{1}{2} \langle \mathbf{u}_b, \mathbf{f} \rangle \\ &= m \end{aligned}$$

by Cauchy's inequality. The equality holds if and only if $D\mathbf{u}_a = D\mathbf{u}_b$, which implies $\mathbf{u}_a = \mathbf{u}_b$ by lemma 5.1.

□

With this proposition in mind, we have the following proposition.

Proposition 5.4. Suppose that $\mathbf{f} \in L_{-1}^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$ and $\mathbf{f}|_{N_r} = 0$ for some $r > 0$.

Then there exists a $\mathbf{h} \in L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$ such that $D\mathbf{h} = \mathbf{f}$ and

- a. $\|\mathfrak{h}\|_{L^2} \leq C\|\mathfrak{f}\|_{L^2_{-1}}$ for some universal constant $C > 0$;
- b. $\mathfrak{h}|_{N_r} \in \mathcal{K}_r$;
- c. The \mathcal{K}_r -leading term of \mathfrak{h} , h^\pm , will satisfies

$$r\|h^\pm\|_{L^2}^2, r^3\|(h^\pm)_t\|_{L^2}^2, r^5\|(h^\pm)_{tt}\|_{L^2}^2 \leq C\|\mathfrak{f}\|_{L^2_{-1}}^2$$

for some universal constant $C > 0$.

Proof. First of all, we define $\mathbf{u}_l \in L^2(M - \Sigma; \mathcal{S} \otimes \mathcal{I})$ to be

$$\mathbf{u}_l = e^{ilt} \begin{pmatrix} \frac{e^{-|l|r}}{\sqrt{z}} \\ \text{sign}(l) \frac{e^{-|l|r}}{\sqrt{\bar{z}}} \end{pmatrix}$$

on N_R and $D\mathbf{u}_l = 0$. Then we have

$$\|\mathbf{u}_l\|_{L^2} \leq \frac{2C}{|l|^{\frac{1}{2}}}.$$

Meanwhile, by using proposition 5.3, there exists a $\hat{\mathfrak{h}} \in L^2_1$ such that $D^2\hat{\mathfrak{h}} = \mathfrak{f}$. Taking $\tilde{\mathfrak{h}} = D\hat{\mathfrak{h}}$, we have $D\tilde{\mathfrak{h}} = \mathfrak{f}$. Now, since $\tilde{\mathfrak{h}} \in \text{range}(D)$, it will perpendicular to $\ker(D|_{L^2})$ by proposition 2.4. So it will perpendicular to \mathbf{u}_l . Suppose that the Fourier coefficients of $\tilde{\mathfrak{h}}$ are $h^\pm_{k,l}$.

Therefore we can define

$$\hat{\mathbf{u}}_l \equiv \frac{\hat{h}^-_{0,l}}{|\hat{h}^-_{0,l}|} \mathbf{u}_l$$

where $\hat{h}^+_{0,l} = (h^+_{0,l} - \text{sign}(l)h^-_{0,l})$ and $\hat{h}^-_{0,l} = (h^+_{0,l} + \text{sign}(l)h^-_{0,l})$. So we also have

$$\|\hat{\mathbf{u}}_l\|_{L^2} \leq 2C.$$

Meanwhile, we have

$$\int_{M-\Sigma} \langle \tilde{\mathfrak{h}}, \hat{\mathbf{u}}_l \rangle = 0 = |\hat{h}^-_{0,l}| \int_0^r e^{-2|l|r} dr + \int_{M-N_r} \langle \tilde{\mathfrak{h}}, \hat{\mathbf{u}}_l \rangle$$

So we have

$$|\hat{h}_{0,l}^-| \leq \frac{4C|l|^{\frac{1}{2}}}{1 - e^{-2|l|r}} \|P_l(\tilde{\mathfrak{h}})\|_{L^2}$$

where P_l is the orthogonal projection from L^2 to $\text{span}\{\mathbf{u}_l\}$. Suppose \mathfrak{h} is a L^2 section such that $D\mathfrak{h} = 0$ and

$$\mathfrak{h} = \sum_{|l| > \frac{1}{2R}} \hat{u}_{0,l}^- \mathbf{u}_l$$

on N_r . Then we have

$$\begin{aligned} \|\mathfrak{h}\|_{L^2} &\leq C \|r^{\frac{1}{2}} \mathfrak{h}|_{\Sigma}\|_{L^2_{-1/2}} = \sum_{|l| > \frac{1}{2r}} \frac{|\hat{h}_{0,l}^-|^2}{|l|} \leq \sum_{|l| > \frac{1}{2r}} \frac{4C}{(1 - e^{-2|l|r})^2} \|P_l(\tilde{\mathfrak{h}})\|_{L^2}^2 \\ &\leq \frac{4C}{(1 - e^{-2})^2} \sum_l \|P_l(\tilde{\mathfrak{h}})\|_{L^2}^2 \leq C \|\tilde{\mathfrak{h}}\|_{L^2}^2. \end{aligned}$$

Now we define $\mathfrak{h} = \tilde{\mathfrak{h}} - \mathfrak{h}$, which satisfies $D\mathfrak{h} = 0$ and $\mathfrak{h} \in \mathcal{K}_r$. Moreover, we have

$$\|\mathfrak{h}\|_{L^2} \leq C \|\tilde{\mathfrak{h}}\|_{L^2}.$$

Notice that by lemma 5.1, we have by Cauchy inequality

$$\|\hat{\mathfrak{h}}\|_{L^2}^2 \leq C \|\tilde{\mathfrak{h}}\|_{L^2}^2 \leq C \|\hat{\mathfrak{h}}\|_{L^2_1} \|\mathfrak{f}\|_{L^2_{-1}} \leq \varepsilon \|\hat{\mathfrak{h}}\|_{L^2_1} + \frac{C}{4\varepsilon} \|\mathfrak{f}\|_{L^2_{-1}}$$

So by choosing ε small enough, we have

$$\|\tilde{\mathfrak{h}}\|_{L^2} \leq C \|\mathfrak{f}\|_{L^2_{-1}}.$$

So we prove **a** and **b**. For **c**, we can get it immediately by using proposition 4.5.

Therefore we finish our proof. □

5.2. Perturbation of Σ : local trivialization. In this subsection, we define some notations and explain the local trivialization of \mathcal{E} . First of all, let N_R be the tubular

neighborhood of $\Sigma \in \mathcal{A}$. There exists a neighborhood of Σ in \mathcal{A} , say \mathcal{V}_Σ , such that $\Sigma' \subset N_{\frac{R}{2}}$ for all $\Sigma' \in \mathcal{V}_\Sigma$. Therefore, we can parametrize the elements in \mathcal{V}_Σ by $\{\eta : S^1 \rightarrow \mathbb{C} \mid \eta \in C^1 \text{ and } \|\eta\|_{C^1} \leq C_R\}$ for some C_R depending on R . We map η to $\{(\eta(t), t)\} = \Sigma' \subset N_R$.

Here we choose a variable $\mathfrak{r} < \frac{R}{4}$. This variable will also be used in the rest of this paper. Also, we fix a $T > 1$ which will be specified in the following subsections. Finally, we define a smooth, real valued function $\chi : M \rightarrow [0, 1]$ such that

$$\chi^{(\mathfrak{r})} = \begin{cases} 1 & \text{on } N_{\frac{\mathfrak{r}}{T}} \\ 0 & \text{on } M - N_{\mathfrak{r}} \end{cases}$$

(We will omit the label (\mathfrak{r}) later, but keep in mind that this function depends on \mathfrak{r}).

Now, for each (η, \mathfrak{r}) , we define the following map

$$(5.10) \quad \begin{aligned} \phi^{(\mathfrak{r})} : M - \Sigma &\rightarrow M - \Sigma'; \\ (z, t) &\mapsto (z + \chi^{(\mathfrak{r})}(z)\eta(t), t) \end{aligned}$$

with $\Sigma' = \{(\eta(t), t)\}$. This map is a diffeomorphism if $\|\eta\|_{C^1} \leq C_{\mathfrak{r}}$ for some constant $C_{\mathfrak{r}}$ depending on \mathfrak{r} .

Recall that the fiber of \mathcal{E} over $(g, \Sigma', e) \in \mathcal{X} \times \mathcal{A}_H$ is the space $L_1^2(M - \Sigma'; \mathcal{S}_{g, \Sigma', e})$, which can be identified to $L_1^2(M - \Sigma; \mathcal{S}_{\phi^{(\mathfrak{r})} * g, \Sigma, e}) \simeq L_1^2(M - \Sigma; \mathcal{S}_{g, \Sigma, e})$. Therefore, for an element $(g, \Sigma) \in \mathcal{X} \times \mathcal{A}_H$, there exists $\mathcal{N} \subset \mathcal{X} \times \mathcal{A}_H$, a neighborhood of (g, Σ) , such that the bundle $\mathcal{E}|_{\mathcal{N}} \simeq \pi_1(\mathcal{N}) \times \mathbb{B}_\varepsilon \times L_1^2$ where $L_1^2 \simeq L_1^2(M - \Sigma; \mathcal{S}_{g, \Sigma, e})$ and $\mathbb{B}_\varepsilon = \{\eta : S^1 \rightarrow \mathbb{C} \mid \eta \in C^1 \text{ and } \|\eta\|_{C^1} \leq \varepsilon\}$ for some small $\varepsilon > 0$.

By the same token, we have the local trivialization of \mathcal{F} near (g, Σ, e) to be $\pi_1(\mathcal{N}) \times \mathbb{B}_\varepsilon \times L^2$. The Dirac operator $D : \mathcal{E} \rightarrow \mathcal{F}$ will be a family of first order differential operator mapping from $\mathbb{B}_\varepsilon \times L_1^2$ to $\mathbb{B}_\varepsilon \times L^2$. Therefore, the tangent space of \mathfrak{M} will be contained in $\mathbb{V} \times L_1^2$ where $\mathbb{V} = \{\eta : S^1 \rightarrow \mathbb{C} \mid \eta \in C^1\}$. By proposition 2.4, we know

the perturbation along L_1^2 is finite dimensional. We will prove that the perturbation along \mathbb{V} is also finite dimensional in section 7.

5.3. Perturbation of Σ : estimates. In this subsection, I will say more about the estimates we get when the curve Σ be perturbed. Recall that we assume the product metric defined on N_R , which is $g_{N_R} = dr^2 + r^2 d\theta + dt^2$. In the following subsections, we choose a positive constant $\mathfrak{r} < \frac{R}{4}$. The precise value of \mathfrak{r} can be assumed to be decrease between each successive appearance. Also, we fix a $T > 1$ which will be specified in the following subsections.

Suppose there exists a pair (χ, η) where $\eta \in C^\infty(S^1; \mathbb{C})$ and

$$\chi = \begin{cases} 1 & \text{on } N_{\frac{\mathfrak{r}}{T}} \\ 0 & \text{on } M - N_{\mathfrak{r}} \end{cases} .$$

For any pair (χ, η) , we can define the following corresponding one parameter family of diffeomorphism

$$(5.11) \quad \begin{aligned} \phi_s : M - \Sigma &\rightarrow M - \Sigma_s; \\ (z, t) &\mapsto (z + s\chi(z)\eta(t), t) \end{aligned}$$

with $s \leq c_0$ for some small c_0 and $\Sigma_s = \{(s\eta(t), t)\}$. Now we fix a $s \leq c_0$ and use (u, τ) to denote the coordinate on $\phi_s(N_R)$ in the following paragraphs.

If we write down the relationship of ∂_t , ∂_z and $\partial_{\bar{z}}$ and the pull-back tangent vectors $(\phi_s)^*(\partial_\tau)$, $(\phi_s)^*(\partial_u)$ and $(\phi_s)^*(\partial_{\bar{u}})$,

$$\begin{cases} \partial_t = (\phi_s)^* \left(\frac{\partial \tau}{\partial t} \partial_\tau + \frac{\partial u}{\partial t} \partial_u + \frac{\partial \bar{u}}{\partial t} \partial_{\bar{u}} \right) \\ \partial_z = (\phi_s)^* \left(\frac{\partial u}{\partial z} \partial_u + \frac{\partial \bar{u}}{\partial z} \partial_{\bar{u}} \right) \\ \partial_{\bar{z}} = (\phi_s)^* \left(\frac{\partial u}{\partial \bar{z}} \partial_u + \frac{\partial \bar{u}}{\partial \bar{z}} \partial_{\bar{u}} \right) \end{cases} ,$$

we will have

$$\begin{pmatrix} (\phi_s)^*(\partial_\tau) \\ (\phi_s)^*(\partial_u) \\ (\phi_s)^*(\partial_{\bar{u}}) \end{pmatrix} = \mathcal{M} \begin{pmatrix} (\partial_t) \\ (\partial_z) \\ (\partial_{\bar{z}}) \end{pmatrix}$$

where

$$\mathcal{M} = \frac{1}{1 + s(\chi_z\eta + \chi_{\bar{z}}\bar{\eta})} \begin{pmatrix} 1 + s(\chi_z\eta + \chi_{\bar{z}}\bar{\eta}) & 0 & 0 \\ -s\chi\dot{\eta} - s^2\chi\chi_{\bar{z}}(\dot{\eta}\bar{\eta} - \dot{\eta}\eta) & 1 + s\chi_{\bar{z}}\bar{\eta} & -s\chi_{\bar{z}}\eta \\ -s\chi\dot{\eta} - s^2\chi\chi_z(\dot{\eta}\bar{\eta} - \dot{\eta}\eta) & -s\chi_z\bar{\eta} & 1 + s\chi_z\eta \end{pmatrix}.$$

Since we are not going to change our metric and spinor bundle over M here, so the spin representation $\kappa : TM \rightarrow Cl(TM)$ will always send $\partial_\tau, \partial_u, \partial_{\bar{u}}$ to $e_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ respectively. Therefore, the Dirac operator D_s defined on $\phi_s(N_R)$ will be

$$\begin{aligned} D_s &= e_1 \cdot \partial_\tau + e_2 \cdot \partial_u + e_3 \cdot \partial_{\bar{u}} \\ &+ \frac{1}{2} \sum_{i=1}^3 e_i \sum_{k,l} \omega_{kl}(e_i) e_k e_l \end{aligned}$$

In the following sections, we will identify all these perturbed curves to Σ by using the pull-back operator $(\phi_s)^*$. So we have to write down explicitly

$$\begin{aligned} D_{s\chi\eta} &= (\phi_s)^* \circ D_s = e_1 \cdot (\phi_s)^*(\partial_\tau) + e_2 \cdot (\phi_s)^*(\partial_u) + e_3 \cdot (\phi_s)^*(\partial_{\bar{u}}) \\ &+ \frac{1}{2} \sum_{i=1}^3 e_i \sum_{k,l} (\phi_s)^*(\omega_{kl}(e_i)) e_k e_l. \end{aligned}$$

We can see that, after some standard computation,

$$(5.12) \quad (\phi_s)^*(\omega_{kl}(e_i)) = \mathcal{M}(\omega(e_i))\mathcal{M}^{-1} + (d\mathcal{M})\mathcal{M}^{-1} = (d\mathcal{M})\mathcal{M}^{-1}.$$

Here we write down precisely the $O(s)$ order term of $\sum_{i=1}^3 e_i \sum_{k,l} (\phi_s)^*(\omega_{kl}(e_i)) e_k e_l$, which can denote by

$$\begin{aligned} & - [(d\mathcal{M})_{11}(e_1)Id + (d\mathcal{M})_{11}(e_2)e_2 + (d\mathcal{M})_{11}(e_3)e_3] \\ & + [-(d\mathcal{M})_{12}(e_1)e_2 - (d\mathcal{M})_{13}(e_1)e_3 + (d\mathcal{M})_{23}(e_1)e_1e_2e_3 + (d\mathcal{M})_{32}(e_1)e_1e_3e_2] \\ & = D(s(\chi_z\eta + \chi_{\bar{z}}\bar{\eta})Id) + D\left(\begin{pmatrix} 0 & i\chi\dot{\eta} \\ -i\chi\dot{\eta} & 0 \end{pmatrix}\right) = \mathcal{F}_s. \end{aligned}$$

So the term $\frac{1}{2} \sum_{i=1}^3 e_i \sum_{k,l} (\phi_s)^*(\omega_{kl}(e_i)) e_k e_l$ can be expressed as

$$(5.13) \quad \frac{1}{2} \sum_{i=1}^3 e_i \sum_{k,l} (\phi_s)^*(\omega_{kl}(e_i)) e_k e_l = \mathcal{F}_s + \mathcal{A}_s$$

where \mathcal{F}_s is the $O(s)$ -zero order differential operator described as above and \mathcal{A}_s is the remainder which is an $O(s^2)$ -order operator.

Meanwhile, suppose that we have the following assumptions: There exist κ_0 such that

$$(5.14) \quad \|\eta\|_{L^2(S^1)} \leq \kappa_0 \mathfrak{r}^2,$$

$$(5.15) \quad \|\eta_t\|_{L^2(S^1)} \leq \kappa_0 \mathfrak{r},$$

$$(5.16) \quad \|\eta_{tt}\|_{L^2(S^1)} \leq \kappa_0,$$

We can see that these inequalities will imply that there exists $\kappa_1 = O(\kappa_0)$ such that

$$(5.17) \quad \max\{|\chi_z|\eta|, |\chi_{\bar{z}}|\eta|, |\eta_t|\} \leq \gamma_T \kappa_1 \mathfrak{r}^{\frac{1}{2}}$$

$$(5.18) \quad \|\chi_z \eta_t\|_{L^2}, \|\chi_{\bar{z}} \eta_t\|_{L^2} \leq \gamma_T \kappa_1$$

$$(5.19) \quad \|\chi_{zz} \eta\|_{L^2}, \|\chi_{z\bar{z}} \eta\|_{L^2}, \|\chi_{\bar{z}\bar{z}} \eta\|_{L^2} \leq \gamma_T^2 \kappa_1.$$

where we use the notion $\gamma_T = (\frac{T}{T-1})$.

Here we prove these implications. Firstly, notice that by Sobolev inequality, we have η is continuous. So

$$\begin{aligned}
|\eta|^2(t) &\leq \frac{1}{2\pi} \int_0^{2\pi} |\eta|^2 + \int_0^{2\pi} \partial_t(|\eta|^2) \\
&\leq \frac{1}{2\pi} \|\eta\|_{L^2}^2 + 2\|\eta\|_{L^2} \|\eta_t\|_{L^2} \\
&\leq \frac{1}{2\pi} \kappa_0^2 \mathbf{r}^4 + 2\kappa_0^2 \mathbf{r}^3 \\
&\leq \left(\frac{1}{2\pi} + 2\right) \kappa_0^2 \mathbf{r}^3.
\end{aligned}$$

Meanwhile, we have $|\chi_z|, |\chi_{\bar{z}}| \leq C\gamma_T \frac{1}{\mathbf{r}}$. Therefore

$$|(\chi_i)_z| |\eta_i|, |(\chi_i)_{\bar{z}}| |\eta_i| \leq C\kappa_0 \mathbf{r}^{\frac{1}{2}}.$$

This implies (5.17). The inequality (5.18) can be proved by the fact $|\chi_z|, |\chi_{\bar{z}}| \leq C\gamma_T \frac{1}{\mathbf{r}}$ and (5.15) and (5.19) can be proved by the fact $|\chi_{zz}|, |\chi_{\bar{z}z}|, |\chi_{\bar{z}\bar{z}}| \leq C\gamma_T^2 \frac{1}{\mathbf{r}^2}$ and (5.14).

Under these assumptions, for any s , we have

$$\left| \frac{1}{1 + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})} - 1 \right| \leq 2\gamma_T \kappa_1 s.$$

So we can write $\frac{1}{1 + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})} = 1 + \varrho_{s\chi\eta}$ for some $|\varrho_{s\chi\eta}| \leq 2\gamma_T \kappa_1 s$.

For the perturbed Dirac operator $D_{s\chi\eta}$, we have the following proposition.

Proposition 5.5. There exists $\kappa_1 = O(\kappa_0)$ depending on κ_0 with the following significance. The perturbed Dirac operator $D_{s\chi\eta}$ satisfies (5.14) - (5.19) can be written as follows:

$$(5.20) \quad D_{s\chi\eta} = (1 + \varrho_{s\chi\eta})D + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})(e_1 \partial_t) + \Theta_s + \mathcal{R}_s + \mathcal{A}_s + \mathcal{F}_s$$

where

- $\Theta_s = [e_1(s\chi\eta\partial_z + s\chi\bar{\eta}\partial_{\bar{z}}) + e_2(s\chi_{\bar{z}}\bar{\eta}\partial_z - s\chi_z\eta\partial_{\bar{z}}) + e_3(-s\chi_{\bar{z}}\eta\partial_z + s\chi_z\eta\partial_{\bar{z}})]$ is a

first order differential operator.

- $\mathcal{R}_s : L_1^2 \rightarrow L^2$ is a $O(s^2)$ -first order differential operator supported on $N_{\mathfrak{r}} - N_{\frac{\mathfrak{r}}{T}}$ with its operator norm $\|\mathcal{R}_s\| \leq \gamma_T^2 \kappa_1^2 s^2$.
- \mathcal{A}_s is a $O(s)$ -zero order differential operator supported on $N_{\mathfrak{r}} - N_{\frac{\mathfrak{r}}{T}}$. Moreover, let us denote by $\vec{n} = \partial_r$ the vector field defined on N_R , then

$$(5.21) \quad \int_{\{r=r_0\}} |\mathcal{A}_s|^2 i_{\vec{n}} dVol(M) \leq \gamma_T^2 \kappa_1^4 \mathfrak{r} s^4$$

for all $r_0 \leq \mathfrak{r}$.

- \mathcal{F}_s is a $O(s)$ -zero order differential operator where

$$(5.22) \quad \mathcal{F}_s = D(s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})) Id + D\left(\begin{pmatrix} 0 & si\chi\dot{\eta} \\ -si\chi\dot{\bar{\eta}} & 0 \end{pmatrix}\right).$$

Proof. By using the conventions defined above, we have

$$(5.23) \quad \mathcal{M} = (1 + \varrho_{s\chi\eta}) \left[\begin{pmatrix} 1 + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -s\chi\dot{\eta} & s\chi_{\bar{z}}\bar{\eta} & -s\chi_{\bar{z}}\eta \\ -s\chi\dot{\bar{\eta}} & -s\chi_z\bar{\eta} & s\chi_z\eta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -s^2\chi\chi_{\bar{z}}(\dot{\eta}\bar{\eta} - \dot{\bar{\eta}}\eta) & 0 & 0 \\ -s^2\chi\chi_z(\eta\dot{\bar{\eta}} - \dot{\eta}\bar{\eta}) & 0 & 0 \end{pmatrix} \right].$$

Therefore by (5.12) we can write

$$(5.24) \quad \begin{aligned} D_{s\chi\eta} = & (1 + \varrho_{s\chi\eta})(D + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta}))(e_1 \partial_t) \\ & + (1 + \varrho_{s\chi\eta})[e_1(s\chi\dot{\eta}\partial_z + s\chi\dot{\bar{\eta}}\partial_{\bar{z}}) + e_2(s\chi_{\bar{z}}\bar{\eta}\partial_z - s\chi_z\bar{\eta}\partial_{\bar{z}}) \\ & + e_3(-s\chi_{\bar{z}}\eta\partial_z + s\chi_z\eta\partial_{\bar{z}})] \\ & + \hat{\mathcal{R}}_s + \mathcal{A}_s + \mathcal{F}_s \end{aligned}$$

where we define

$$(5.25) \quad \hat{\mathcal{R}}_s = \frac{-1}{1 + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})} [e_2(s^2 \chi \chi_{\bar{z}} (\dot{\eta} \bar{\eta} - \dot{\bar{\eta}} \eta) \partial_t) + e_3(s^2 \chi \chi_z (\eta \dot{\bar{\eta}} - \dot{\eta} \bar{\eta}) \partial_t)]$$

to be a first order differential operator with $\|\hat{\mathcal{R}}_s\| \leq \gamma_T^2 \kappa_1^2 s^2$.

Finally, we denote

$$\begin{aligned} \Theta_s = & [e_1(s\chi \dot{\eta} \partial_z + s\chi \dot{\bar{\eta}} \partial_{\bar{z}}) \\ & + e_2(s\chi_{\bar{z}} \bar{\eta} \partial_z - s\chi_z \bar{\eta} \partial_{\bar{z}}) \\ & + e_3(-s\chi_{\bar{z}} \eta \partial_z + s\chi_z \eta \partial_{\bar{z}})] \end{aligned}$$

and

$$\begin{aligned} \delta \mathcal{R}_s^{(1)} = & \varrho_{s\chi\eta} [e_1(s\chi \dot{\eta} \partial_z + s\chi \dot{\bar{\eta}} \partial_{\bar{z}}) \\ & + e_2(s\chi_{\bar{z}} \bar{\eta} \partial_z - s\chi_z \bar{\eta} \partial_{\bar{z}}) \\ & + e_3(-s\chi_{\bar{z}} \eta \partial_z + s\chi_z \eta \partial_{\bar{z}})] \end{aligned}$$

where $\delta \mathcal{R}_s^{(1)}$ is a $O(s^2)$ -first order differential operator. We should also notice that $(1 + \varrho_{s\chi\eta})(s\chi_z \eta + \chi_{\bar{z}} \bar{\eta})(e_1 \partial_t) = s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})(e_1 \partial_t) + \delta \mathcal{R}_s^{(2)}$ where $\delta \mathcal{R}_s^{(2)}$ is also a $O(s^2)$ -first order differential operator. So we can rewrite

$$D_{s\chi\eta} = (1 + \varrho_{s\chi\eta})D + s(\chi_z \eta + \chi_{\bar{z}} \bar{\eta})(e_1 \partial_t) + \Theta_s + \mathcal{R}_s + \mathcal{A}_s + \mathcal{F}_s$$

where $\mathcal{R}_s = \hat{\mathcal{R}}_s + \delta \mathcal{R}_s^{(1)} + \delta \mathcal{R}_s^{(2)}$.

To prove the estimate (5.21) of \mathcal{A}_s , we should notice that the term $(d\mathcal{M})\mathcal{M}^{-1}$ involves at most the second derivative of χ and η , which can be estimated by (5.16), (5.18) and (5.19). So we get (5.21) immediately. \square

By using the conventions of this proposition, we can prove the following proposition.

Proposition 5.6. Let $\psi \in L_1^2$ be a harmonic section. Then

$$\|\mathcal{R}_s(\psi)\|_{L^2} \leq C\gamma_T^{\frac{3}{2}}\kappa_1^2\mathfrak{r}^2s^2$$

for some constant C depending on the $\|\psi\|_{L_1^2}$. In fact, this estimate is true for any $\psi \in L_1^2$ which can be expressed as $\psi = \sqrt{r}v(t, \theta, r)$ with v being a C^1 -bounded section.

Proof. By proposition 4.4 **b.**, we have $\psi = \sqrt{r}v(t, \theta, r)$ with v being a C^1 -bounded section. We write down the definition of \mathcal{R}_s

$$\begin{aligned} \mathcal{R}_s &= \frac{-1}{1 + s(\chi_z\eta + \chi_{\bar{z}}\bar{\eta})} [e_2(s^2\chi\chi_{\bar{z}}(\dot{\eta}\bar{\eta} - \dot{\eta}\eta)\partial_t) + e_3(s^2\chi\chi_z(\eta\dot{\eta} - \dot{\eta}\bar{\eta})\partial_t)] \\ &\quad + \varrho_{s\chi\eta} [e_1(s\chi\dot{\eta}\partial_z + s\chi\dot{\eta}\partial_{\bar{z}}) + e_2(s\chi_{\bar{z}}\bar{\eta}\partial_z - s\chi_z\bar{\eta}\partial_{\bar{z}}) + e_3(-s\chi_{\bar{z}}\eta\partial_z + s\chi_z\eta\partial_{\bar{z}})] \\ &\quad + \varrho_{s\chi\eta}(s\chi_z\eta + \chi_{\bar{z}}\bar{\eta})(e_1\partial_t). \end{aligned}$$

Then by using (5.14), (5.15), (5.16), (5.17), (5.18) and (5.19) we will notice that every term of \mathcal{R}_s can be written as the type $s^2\alpha\beta\partial_i$ where $\|\alpha\|_{L^\infty} \leq \gamma_T\kappa_1\mathfrak{r}^{\frac{1}{2}}$ and

$$\int_{r=r_0} |\beta|^2 i_{\bar{n}} dVol(M) \leq \gamma_T\kappa_1^2\mathfrak{r}^2.$$

So we have

$$\|\mathcal{R}_s(\psi)\|_{L^2} \leq s^2\|v\|_{C^1}\gamma_T^{\frac{3}{2}}\kappa_1^2\mathfrak{r}^2.$$

\square

5.4. Composition of the perturbations: estimates. In this subsection, we discuss the composition of perturbations and its corresponding Dirac operator. These computations will be used in the following subsections.

Let $\mathfrak{r} < \frac{R}{4}$, $T > P > 1$ be fixed for a moment. We assume that there is a sequence $\{(\chi_i, \eta_i)\}$ satisfying the following two properties:

1. χ_i is a smooth function satisfying

$$\chi_i = \begin{cases} 1 & \text{on } N_{\frac{\mathfrak{r}}{T^{i+1}}} \\ 0 & \text{on } M - N_{\frac{\mathfrak{r}}{T^i}} \end{cases}.$$

for all $i \in \mathbb{N} \cup \{0\}$.

2. There exists $\kappa_2 > 0$ such that

$$(5.26) \quad \|\eta_i\|_{L^2(S^1)} \leq \kappa_2 \frac{\mathfrak{r}^2}{T^{2i}},$$

$$(5.27) \quad \|(\eta_i)_t\|_{L^2(S^1)} \leq \kappa_2 \frac{\mathfrak{r}}{T^i},$$

$$(5.28) \quad \|(\eta_i)_{tt}\|_{L^2(S^1)} \leq \kappa_2,$$

for all $i \in \mathbb{N}$.

Similar to the argument of (5.16), (5.18) and (5.19), we have the following implications

$$(5.29) \quad \max\{|\langle \chi_i \rangle_z| |\eta_i|, |\langle \chi_i \rangle_{\bar{z}}| |\eta_i|, |(\eta_i)_t|\} \leq \gamma_T \kappa_3 \frac{\mathfrak{r}^{\frac{1}{2}}}{T^{\frac{i}{2}}},$$

$$(5.30) \quad \|(\chi_i)_z \eta_i\|_{L^2}, \|(\chi_i)_{\bar{z}} \eta_i\|_{L^2} \leq \gamma_T \kappa_3,$$

$$(5.31) \quad \|(\chi_i)_{zz} \eta_i\|_{L^2}, \|(\chi_i)_{z\bar{z}} \eta_i\|_{L^2}, \|(\chi_i)_{\bar{z}\bar{z}} \eta_i\|_{L^2} \leq \gamma_T^2 \kappa_3$$

for some $\kappa_3 = O(\kappa_2)$.

Furthermore, we denote $\eta^i = \sum_{j=1}^i \chi_j \eta_j$.

As we have showed in previous subsection, we can define the following family of diffeomorphisms

$$(5.32) \quad \begin{aligned} \phi_s^i : M - \Sigma &\rightarrow M - \Sigma_s; \\ (z, t) &\mapsto (z + s(\eta^i(t)(t)), t) \end{aligned}$$

with $s \leq c_0$ for some small c_0 and $\Sigma_s = \{(s(\eta^i(t)), t)\}$. Now fix a s , we use (u, τ) to denote the coordinate on $\phi_s^i(N_R)$.

The Dirac operator $D_{s\eta^i}$ define on $M - \Sigma$ will be

$$\begin{aligned} D_{s(\eta^i)} &= (\phi_s)^i \circ D_s = e_1 \cdot (\phi_s)^i(\partial_\tau) + e_2 \cdot (\phi_s)^i(\partial_u) + e_3 \cdot (\phi_s)^i(\partial_{\bar{u}}) \\ &\quad + \frac{1}{2} \sum_{i=1}^3 e_i \sum_{k<l} (\phi_s)^i(\omega_{kl}(e_i)) e_k e_l. \end{aligned}$$

In this subsection, we prove the following proposition.

Proposition 5.7. There exists a $\kappa_3 = O(\kappa_2)$ depending on κ_2 with the following significance. The perturbed Dirac operator $D_{s(\eta^i)}$ which satisfies the hypothesis (5.26) - (5.31) can be written as follows:

$$(5.33) \quad \begin{aligned} D_{s\eta^{i+1}} &= (1 + \varrho^{i+1})D_{s\eta^i} + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) e_1 \partial_t \\ &\quad + \Theta_s^{i+1} + \mathcal{R}_s^{i+1} + \hat{\mathcal{A}}_s^{i+1} + \mathcal{F}_s^{i+1} \end{aligned}$$

where

- Θ_s^{i+1} , the $(\chi, \eta) = (\chi_{i+1}, \eta_{i+1})$ version of Θ_s^0 , is a first order differential operator of order $O(s)$.
- $\mathcal{R}_s^{i+1} : L_1^2 \rightarrow L^2$ is a $O(s^2)$ -first order differential operator supported on

$N_{\frac{\mathfrak{r}}{T^i}} - N_{\frac{\mathfrak{r}}{T^{i+1}}}$ with its operator norm bounded in the following way:

$$\|\mathcal{R}_s^{i+1}\| \leq \gamma_T^2 \kappa_3^2 s^2.$$

- $\hat{\mathcal{A}}_s^{i+1}$ is a $O(s)$ -zero order differential operator. Moreover, let us denote by $\vec{n} = \partial_r$ the vector field defined on N_R , then

$$(5.34) \quad \int_{\{r=r_0\}} |\hat{\mathcal{A}}_s^{i+1}|^2 i_{\vec{n}} dVol(M) \leq \gamma_T^4 \kappa_3^4 \left(\frac{(i+1)\mathfrak{r}}{T^{i+1}}\right) s^4.$$

for all $r_0 \leq \frac{\mathfrak{r}}{T^i}$.

- \mathcal{F}_s^{i+1} is a $O(s)$ -zero order differential operator where

$$(5.35) \quad \mathcal{F}_s^{i+1} = D(s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) Id) + D\left(\begin{array}{cc} 0 & si\chi_{i+1}\dot{\eta}_{i+1} \\ -si\chi_{i+1}\dot{\bar{\eta}}_{i+1} & 0 \end{array}\right).$$

Proof. We can also define the matrix \mathcal{M}^i to be

$$\begin{pmatrix} (\phi_s^i)^*(\partial_\tau) \\ (\phi_s^i)^*(\partial_u) \\ (\phi_s^i)^*(\partial_{\bar{u}}) \end{pmatrix} = \mathcal{M}^i \begin{pmatrix} (\partial_t) \\ (\partial_z) \\ (\partial_{\bar{z}}) \end{pmatrix}.$$

Notice that the support of $(\chi_i)_z$ and $(\chi_j)_{\bar{z}}$ are disjoint for all $i \neq j$. Therefore we can write \mathcal{M}^{i+1} as follows

$$(5.36) \quad \mathcal{M}^{i+1} = \frac{1}{1 + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1})} \mathcal{M}^i + \mathcal{N}^{i+1}$$

where \mathcal{N}^{i+1} is a (χ_{i+1}, η_{i+1}) version of \mathcal{M} :

$$\mathcal{N}^{i+1} = \frac{1}{1 + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1})} \cdot \begin{pmatrix} s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) & 0 & 0 \\ -s\chi_{i+1} \dot{\eta}_{i+1} \\ -s^2 \chi_{i+1} (\chi_{i+1})_{\bar{z}} (\dot{\eta}_{i+1} \bar{\eta}_{i+1} - \dot{\bar{\eta}}_{i+1} \eta_{i+1}) & s(\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1} & -s(\chi_{i+1})_{\bar{z}} \eta_{i+1} \\ -s\chi_{i+1} \dot{\bar{\eta}}_{i+1} \\ -s^2 \chi_{i+1} (\chi_{i+1})_z (\eta_{i+1} \dot{\bar{\eta}}_{i+1} - \dot{\eta}_{i+1} \bar{\eta}_{i+1}) & -s(\chi_{i+1})_z \bar{\eta}_{i+1} & s(\chi_{i+1})_z \eta_{i+1} \end{pmatrix}.$$

Let us define $\frac{1}{1+s((\chi_i)_z \eta_i + (\chi_i)_{\bar{z}} \bar{\eta}_i)} = 1 + \varrho^i$. Define Θ_s^{i+1} and \mathcal{R}_s^{i+1} to be the (χ_{i+1}, η_{i+1}) version of Θ_s and \mathcal{R}_s . Then we have

$$\begin{aligned} D_{s\eta^{i+1}} &= (1 + \varrho^{i+1})(D_{s\eta^i} - \mathbb{A}_s^i) + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) e_1 \partial_t \\ &\quad + \Theta_s^{i+1} + \mathcal{R}_s^{i+1} + \mathbb{A}_s^{i+1} \\ &= (1 + \varrho^{i+1}) D_{s\eta^i} + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) e_1 \partial_t \\ &\quad + \Theta_s^{i+1} + \mathcal{R}_s^{i+1} + [\mathbb{A}_s^{i+1} - (1 + \varrho^{i+1}) \mathbb{A}_s^i] \end{aligned}$$

where $\mathbb{A}_s^{i+1} = \sum_{j=1}^3 e_j \sum_{k<l} (\omega_{kl}^{(i+1)}(e_j)) e_k e_l$ with $\omega^{(i+1)}$ being the pull back of Levi-Civita connection $(\phi_s^{i+1})^*(\omega)$. Using these conventions, we have

$$|\mathbb{A}_s^{i+1} - (1 + \varrho^{i+1}) \mathbb{A}_s^i| = |(d\mathcal{M}^{i+1})(\mathcal{M}^{i+1})^{-1} - (d\mathcal{M}^i)(\mathcal{M}^i)^{-1} - \varrho^{i+1} (d\mathcal{M}^i)(\mathcal{M}^i)^{-1}|.$$

Now by using (5.35) and (5.36), we can see that \mathcal{F}_s^{i+1} is the $O(s)$ order term of the $(d\mathcal{M}^{i+1})(\mathcal{M}^{i+1})^{-1} - (d\mathcal{M}^i)(\mathcal{M}^i)^{-1}$. Therefore by using (5.26), (5.27) and (5.29), we

have

$$\int_{r=r_0} | (d\mathcal{M}^{i+1})(\mathcal{M}^{i+1})^{-1} - (d\mathcal{M}^i)(\mathcal{M}^i)^{-1} - \mathcal{F}_s^{i+1}|^2 i_{\bar{n}} dVol(M) \leq C \gamma_T^4 \left(\frac{\mathfrak{r}}{T^{i+1}}\right) \kappa_0^4 s^4$$

for some universal constant C . Therefore we can choose $\kappa_3 = O(\kappa_2)$ large enough such that the right hand side of this equation is smaller than $\frac{1}{4} \gamma_T^4 \left(\frac{\mathfrak{r}}{T^{i+1}}\right) \kappa_3^4 s^4$.

Meanwhile, we have

$$(5.37) \quad \int_{r=r_0} | (d\mathcal{M}^{j+1})(\mathcal{M}^{j+1})^{-1} - (d\mathcal{M}^j)(\mathcal{M}^j)^{-1}|^2 i_{\bar{n}} dVol(M) \leq C \gamma_T^2 \kappa_3^2 s^2$$

for all j , so we have

$$\int_{r=r_0} | (d\mathcal{M}^j)(\mathcal{M}^j)^{-1}|^2 i_{\bar{n}} dVol(M) \leq \gamma_T^2 (i+1) \kappa_3^2 s^2.$$

Now recall that $|\varrho^{i+1}| \leq \gamma_T s \kappa_3 \left(\frac{\mathfrak{r}}{T^{i+1}}\right)^{\frac{1}{2}}$. So we have

$$(5.38) \quad \int_{r=r_0} |\varrho^{i+1} (d\mathcal{M}^j)(\mathcal{M}^j)^{-1}|^2 i_{\bar{n}} dVol(M) \leq \gamma_T^4 (i+1) \left(\frac{\mathfrak{r}}{T^{i+1}}\right) \kappa_3^4 s^4.$$

Therefore by taking

$$\hat{\mathcal{A}}_s^{i+1} = \mathbb{A}_s^{i+1} - (1 + \varrho^{i+1}) \mathbb{A}_s^i - \mathcal{F}_s^{i+1},$$

we prove this proposition. □

Similarly, we have a i -th version of proposition 5.6 as follows

Proposition 5.8. Let $\psi \in L_1^2$ be a harmonic section. Then

$$\|\mathcal{R}_s^{i+1}(\psi)\|_{L^2} \leq C \gamma_T^{\frac{3}{2}} \kappa_3^2 \left(\frac{\mathfrak{r}}{T^{i+1}}\right)^2 s^2.$$

for some constant C depending on the $\|\psi\|_{L_1^2}$. In fact, this estimate is true for any $\psi \in L_1^2$ which can be expressed as $\psi = \sqrt{r}v(t, \theta, r)$ where v is a C^1 -bounded section.

5.5. Variational formula for the perturbed Dirac operators. In subsection 5.1, we prove that there exists a unique minimizer for $E_{\mathfrak{f}}$ in $L^2 \cap \ker(D)$. The argument in subsection 5.1 also works not only for D but also for the perturbed Dirac operator $D_{s\chi\eta}$, $D_{s\eta^j}$ appearing in subsection 5.3 and 5.4. However, using the variational method to find the solution $D_{s\eta^j}\mathbf{u}_s = \mathfrak{f}$ wouldn't give us enough information about \mathbf{u}_s changing by varying s . Therefore, we should say more about this in this subsection.

Proposition 5.9. For any $j > 0$ fixed. Suppose $\mathfrak{f} \in L^2_{-1}$ and $\mathbf{u}_0 \in L^2$ be a harmonic section satisfies

$$D\mathbf{u}_0 = \mathfrak{f},$$

then there exist $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}^s$ and $c_0 > 0$ such that

$$D_{s\eta^j}\mathbf{u} = \mathfrak{f}$$

and $\|\mathbf{u}^s\|_{L^2} \leq C(\|\mathbf{u}_0\|_{L^2} + \|\mathfrak{f}\|_{L^2_{-1}})s$ for $s \in [0, c_0]$ and C being a universal constant C . Furthermore, the existence of \mathbf{u}_0 can be given by proposition 5.4.

Proof. Suppose $D_{s\eta^j}$ is a perturbed Dirac operator and $\mathfrak{f} \in L^2_{-1}$. We want to solve $\mathbf{u} \in L^2$ satisfies

$$D_{s\eta^j}\mathbf{u} = \mathfrak{f}.$$

We solve this equation iteratively. Firstly, we know that the perturbed Dirac operator $D_{s\eta^j}$ can be written as $D + \delta_s^j$ where $\delta_s^j : L^2 \rightarrow L^2_{-1}$ is a first order differential operator with its operator norm $\|\delta_s^j\| \leq Cs$ for some $C > 0$. Meanwhile, by proposition 5.4, there exists $\mathbf{u}_0 \in L^2$ such that

$$D\mathbf{u}_0 = \mathfrak{f}.$$

So we have

$$D_{s\eta^j} \mathbf{u}_0 = \mathbf{f} - \delta_s(\mathbf{u}_0).$$

Since $\|\mathbf{u}_0\|_{L^2} \leq C\|\mathbf{f}\|_{L^2_{-1}}$, we have $\|\delta_s^j(\mathbf{u}_0)\|_{L^2_{-1}} \leq Cs\|\mathbf{f}\|_{L^2_{-1}}$. By taking s small enough, we have $\|\delta_s^j(\mathbf{u}_0)\|_{L^2_{-1}} \leq \frac{1}{2}\|\mathbf{f}\|_{L^2_{-1}}$.

Now we solve $\mathbf{v}_1 \in L^2$ such that

$$D\mathbf{v}_1 = \delta_s^j(\mathbf{u}_0)$$

by using proposition 5.3. Then we have $D_{s\eta^j}(\mathbf{u}_0 + \mathbf{v}_1) = \mathbf{f} + \delta_s^j(\mathbf{v}_1)$ where $\|\delta_s^j(\mathbf{v}_1)\|_{L^2_{-1}} \leq \frac{1}{2}\|\delta_s^j(\mathbf{u}_0)\| \leq \frac{1}{4}\|\mathbf{f}\|_{L^2_{-1}}$.

If we call $\delta_s^j(\mathbf{u}_0) = \mathfrak{z}_0$, $-\delta_s^j(\mathbf{v}_1) = \mathfrak{z}_1$ and $\mathbf{u}_0 + \mathbf{v}_1 = \mathbf{u}_1$, we can see the patten of induction. Suppose that we have

$$D_{s\eta^j} \mathbf{u}_i = \mathbf{f} - \mathfrak{z}_i$$

with $\|\mathfrak{z}_i\|_{L^2_{-1}} \leq \frac{1}{2^{i+1}}\|\mathbf{f}\|_{L^2_{-1}}$, then we can solve $\mathbf{v}_{i+1} \in L^2$ which satisfies

$$D\mathbf{v}_{i+1} = \mathfrak{z}_i$$

by proposition 5.3. Therefore we have

$$D_{s\eta^j}(\mathbf{u}_i + \mathbf{v}_{i+1}) = \mathbf{f} + \delta_s^j(\mathbf{v}_{i+1}).$$

where $\|\delta_s^j(\mathbf{v}_{i+1})\|_{L^2_{-1}} \leq \frac{1}{2}\|\mathfrak{z}_i\| \leq \frac{1}{2^{i+2}}\|\mathbf{f}\|_{L^2_{-1}}$. By taking $\mathbf{u}_i + \mathbf{v}_{i+1} = \mathbf{u}_{i+1}$ and $-\delta_s^j(\mathbf{v}_{i+1}) = \mathfrak{z}_{i+1}$, we finish our argument of induction.

Finally, we take the limit $i \rightarrow \infty$, then we have $\mathbf{u}_{i+1} \rightarrow \mathbf{u}$ in L^2 -sense which will satisfy

$$D_{s\eta^j}(\mathbf{u}) = \mathbf{f}.$$

Moreover, since $\mathbf{u} - \mathbf{u}_0 = \sum_{i=1}^{\infty} \mathbf{v}_i$ and $\mathbf{v}_i = \pm \delta_s^j(\mathbf{v}_{i-1})$, we have $\sum_{i=1}^{\infty} \mathbf{v}_i$ is a $O(s)$ -order L^2 section. We call $\sum_{i=1}^{\infty} \mathbf{v}_i = \mathbf{u}^s$.

Therefore, we construct a solution $\mathbf{u}_0 + \mathbf{u}^s$ satisfies

$$(5.39) \quad D_{s\eta^j}(\mathbf{u}_0 + \mathbf{u}^s) = \mathfrak{f}.$$

□

Here we have the following remark for this proposition.

Remark 5.10. In our proof, since we can always write $\delta_s^j = \sum_{i=1}^{\infty} s^i \delta_i^j$ where the operation norm of δ_i^j is bounded uniformly. So \mathbf{u} can be written as $\sum_{i=0}^{\infty} s^i \mathbf{u}^{(i)}$, where $\|\sum_{i=m}^{\infty} s^i \mathbf{u}^{(i)}\|_2 \rightarrow 0$ as $m \rightarrow \infty$.

6. THE GENERAL Σ EMBEDDING IN M

Now we try to derive the same results as we did in previous section, but this time we don't need to assume that Σ has a product type tubular neighborhood.

6.1. Asymptotic behavior of the L_1^2 -harmonic section. Let g be a smooth metric and $\Sigma \subset M$ be a C^1 curve embedded in M . We can define the tubular neighborhood of Σ by sending the elements in the normal bundle $\{v \in \nu_{\Sigma} \mid |v| \leq R\}$ to M by the exponential map. We can parametrize this neighborhood by cylindrical coordinate (r, θ, t) and $g = dr^2 + r^2 d\theta^2 + dt^2 + O(r^2)$. In the following section, we will use D_{prod} to denote the Dirac operator of product metric.

The argument in section 5 can be modified for the general metric if we can prove the following lemma

Lemma 6.1. *For any $R > 0$ fixed. Let $\mathbf{v} \in L_1^2(N_R; \mathcal{S} \otimes \mathcal{I})$ such that $D(\mathbf{v}) = 0$, then there exists $\mathbf{v}^* \in L_1^2(N_R; \mathcal{S} \otimes \mathcal{I})$ such that $D_{prod} \mathbf{v}^* = 0$ and*

$$(6.1) \quad \mathbf{v} = \mathbf{v}^* + \mathbf{v}_{\mathfrak{R},0}^*$$

satisfies the following estimate:

$$(6.2) \quad \|\mathbf{v}_{\mathfrak{A},0}^*\|_{L_1^2(N_r)} \leq O(r^{\frac{3}{2}}).$$

Proof. We divide our proof into two parts.

Step 1. Here we set up the strategy of the proof. First of all, it is clearly that we can write $D = D_{prod} + O(r^2)\mathcal{L}_1 + O(r)\mathcal{L}_2$ where \mathcal{L}_1 is a bounded first order differential operator and \mathcal{L}_2 is a zero order operator which is composed by some Clifford multiplication.

Secondly, the argument in Lemma 2.6 still works for the elements in $L_1^2(N_r; \mathcal{S} \otimes \mathcal{I})$. So by using the equation

$$D_{prod}\mathbf{v} = O(r^2)\mathcal{L}_1(\mathbf{v}) + O(r)\mathcal{L}_0(\mathbf{v}),$$

we have $\|O(r^2)\mathcal{L}_1(\mathbf{v}) + O(r)\mathcal{L}_0(\mathbf{v})\|_{L^2(N_a)} \leq O(a^2)$ for all $a < R$. Let us call this term f . So we have

$$D_{prod}\mathbf{v} = f$$

for some f satisfies $\|f\|_{L^2(N_a)} \leq O(a^2)$ for all $a < R$.

Here we need to use the regularity theorem in [3].

Theorem 6.2. *Let R be the Atiyah-Patodi-Singer boundary condition for the spinor bundle \mathcal{S} on the manifold X with boundary Y , then we have for any $\mathbf{v} \in L_1^2(X; \mathcal{S})$ we have*

$$\|\mathbf{v}\|_{L_1^2(X)} \leq C(\|R(\mathbf{v}|_Y)\|_{L_{\frac{3}{2}}^2(Y)} + \|\mathbf{v}\|_{L^2(X)} + \|D_{prod}\mathbf{v}\|_{L^2(X)})$$

for some constant C .

In our case, Σ can be regarded as a degenerated boundary. We take $X = M - N_r$ and $Y = \partial N_r$, then we have

$$\|\mathbf{v}\|_{L^2_1(M-N_r)} \leq C(\|R(\mathbf{v}|_Y)\|_{L^2_{\frac{1}{2}}(\partial N_r)} + \|\mathbf{v}\|_{L^2(M-N_r)} + \|D_{prod}\mathbf{v}\|_{L^2(M-N_r)}).$$

Now, if we take r goes to 0, the boundary term $\|R(\mathbf{v}|_Y)\|_{L^2_{\frac{1}{2}}(\partial N_r)}$ will vanish by lemma 2.6. So we have

$$(6.3) \quad \|\mathbf{v}\|_{L^2_1(M-\Sigma)} \leq C(\|\mathbf{v}\|_{L^2(M-\Sigma)} + \|D_{prod}\mathbf{v}\|_{L^2(M-\Sigma)}).$$

Therefore, if we can prove that there exists a $\mathbf{v}^* \in L^2_1$ such that $D_{prod}\mathbf{v}^* = 0$ and

$$\left[\int_{\{r=r_0\}} |\mathbf{v} - \mathbf{v}^*|^2 i_{\bar{n}} dVol(M) \right]^{\frac{1}{2}} = o(r_0^{\frac{1}{2}})$$

then we can prove the lemma by using (6.3).

Step 2. Now we prove the existence of $\mathbf{v}^* \in L^2_1$. To prove this part, we write down the Fourier expression of \mathbf{v} on N_R as we have done in section 3.

$$\mathbf{v}(t, r, \theta) = \sum_{l,k} e^{ilt} \begin{pmatrix} e^{i(k-\frac{1}{2})\theta} V_{k,l}^+ \\ e^{i(k+\frac{1}{2})\theta} V_{k,l}^- \end{pmatrix}.$$

The equation $D\mathbf{v} = 0$ will give us the equation

$$\begin{aligned} \frac{d}{dr} V_{k,l}^- + \alpha V_{k,l}^+ + \frac{l}{(1+O(r^2))} V^+ + P_{k,l}^+(f) &= 0; \\ \frac{d}{dr} V_{k,l}^+ - \beta V_{k,l}^- + \frac{l}{(1+O(r^2))} V^- + P_{k,l}^-(f) &= 0 \end{aligned}$$

where P^+ is the projection mapping to the first component of the Fourier expansion. α, β have the form $\frac{(k+\frac{1}{2})+O(r^2)}{r(1+O(r^2))}$.

Now we can find the two nonzero functions $\rho_k^\pm(r)$ by solving the following ODE:

$$\begin{aligned}\frac{d}{dr}\rho_{k,l}^+ &= \alpha\rho_{k,l}^+; \\ \frac{d}{dr}\rho_{k,l}^- &= -\beta\rho_{k,l}^-.\end{aligned}$$

So we have $C_1r^{(k+\frac{1}{2})} < \rho_{k,l}^+ < C_2r^{(k+\frac{1}{2})}$ and $C_1r^{-(k+\frac{1}{2})} < \rho_{k,l}^- < C_2r^{-(k+\frac{1}{2})}$ for some $C_2 > C_1 > 0$.

Therefore we have

$$(6.4) \quad \frac{d}{dr}(\rho_{k,l}^+V_{k,l}^-) = -\rho_{k,l}^+\frac{l}{(1+O(r^2))}V_{k,l}^+ - \rho_{k,l}^+P_{k,l}^+(\mathbf{v});$$

$$(6.5) \quad \frac{d}{dr}(\rho_{k,l}^-V_{k,l}^+) = -\rho_{k,l}^-\frac{l}{(1+O(r^2))}V_{k,l}^- - \rho_{k,l}^-P_{k,l}^-(\mathbf{v});$$

for all k, l .

Suppose $k \geq 0$, the integral of (6.4) shows that

$$\begin{aligned}(6.6) \quad |\rho_{k,l}^+V_{k,l}^-(b) - \rho_{k,l}^+V_{k,l}^-(a)| &\leq \int_a^b \rho_{k,l}^+(|V_{k,l}^+| + |P_{k,l}^+(\mathbf{v})|) \leq (b^{2k+2} - a^{2k+2})^{\frac{1}{2}} \left(\int_a^b O(1) \right)^{\frac{1}{2}} \\ &\leq C(b^{2k+2} - a^{2k+2})^{\frac{1}{2}}(b-a)^{\frac{1}{2}}.\end{aligned}$$

By using this inequality we have

$$\lim_{r \rightarrow 0} \rho_{k,l}^+V_{k,l}^-(r) = c$$

for some $c \in \mathbb{C}$. $|V_{k,l}^+| > \frac{|c|}{2}r^{-k-\frac{1}{2}} \geq \frac{|c|}{2}r^{-\frac{1}{2}}$ which contradicts the lemma 2.6 if $c \neq 0$.

So we have $\lim_{r \rightarrow 0} \rho_{k,l}^+V_{k,l}^-(r) = 0$.

Now, by taking $a \rightarrow 0$ in (6.6), we have

$$C_1b^{k+\frac{1}{2}}|V_{k,l}^-(b)| \leq |\rho_{k,l}^+V_{k,l}^-(b)| \leq b^{k+\frac{3}{2}}.$$

So we have

$$(6.7) \quad |V_{k,l}^-|(r) \leq n_{k,l}O(r)$$

for all $k \geq 0$ with some number $\sum_{k,l} |n_{k,l}|^2 < \infty$. Similarly, by using the same argument, we can also prove that

$$(6.8) \quad |V_{k,l}^+|(r) \leq n_{k,l}O(r).$$

for all $k \leq 0$.

For the case $k = -1$, by (6.7) we have $\lim_{r \rightarrow 0} \rho_{k,l}^+ V_{k,l}^- = c$ for some $c \in \mathbb{C}$. So we have

$$V_{-1,l}^-(r) = v_{-1,l} r^{\frac{1}{2}} + o(r^{\frac{1}{2}}).$$

Similarly, we have

$$V_{1,l}^+(r) = v_{1,l} r^{\frac{1}{2}} + o(r^{\frac{1}{2}}).$$

For the case $k < -1$, if we have

$$\limsup_{r \rightarrow 0} |\rho_{k,l}^+ V_{k,l}^-|(r) = c < \infty$$

then we have $|V_{k,l}^-|(r) \leq cr^{-k-\frac{1}{2}} \leq cr^{\frac{3}{2}}$. On the other hand, if we have

$$\limsup_{r \rightarrow 0} |\rho_{k,l}^+ V_{k,l}^-|(r) = \infty,$$

then we have $k < -2$ by (6.6) and (6.8). Moreover, (6.6) implies that

$$|\rho_{k,l}^+ V_{k,l}^-(b) - \rho_{k,l}^+ V_{k,l}^-(a)| \leq C \rho_{k,l}^+(a) a^2.$$

So

$$\left| \frac{\rho_{k,l}^+(b)}{a\rho_{k,l}^+(a)} V_{k,l}^-(b) - a^{-2} V_{k,l}^-(a) \right| \leq n_{k,l} O(1).$$

Therefore we have

$$\limsup_{a \rightarrow 0} |a^2 V_{k,l}^-(a)| \leq n_{k,l} O(1)$$

which implies

$$|V_{k,l}^-(r)| \leq n_{k,l} O(r^2).$$

So we can conclude that

$$|V_{k,l}^-(r)| \leq n_{k,l} O(r^{\frac{3}{2}})$$

for all $k < -1$. We finish our proof. □

Remark 6.3. By the same token, we can also show that the element in $\ker(D|_{L^2})$ has similar decomposition. To be more precisely, for any $\mathbf{u} \in \ker(D|_{L^2})$, there is a decomposition $\mathbf{u} = \mathbf{u}^* + \mathbf{u}_{\mathfrak{N},0}^*$ such that $D_{prod}(\mathbf{u}^*) = 0$ and $|\mathbf{u}_{\mathfrak{N},0}^*| = o(\frac{1}{\sqrt{r}})$.

6.2. Modify the propositions in section 5. . Now we modify the results we get in section 5 without the assumption of having Euclidean tubular neighborhood.

First of all, we should set up several notations. Let N_R to be the tubular neighborhood of Σ , we should now use D_{prod} to denote the Dirac operator with respect to Euclidean metric on N_R . We define $D^{(n)} = \chi_n D_{prod} + (1 - \chi_n) D$, where χ_n is defined in section 5.4. So we have

$$D^{(n)} = D_{prod} \text{ on } N_{\frac{\epsilon}{T^{n+1}}}.$$

Moreover, we have the following proposition. Here we take $\partial_1 = \partial_r$, $\partial_2 = \partial_\theta$ and $\partial_3 = \partial_t$.

Proposition 6.4. Let $[D^{(n)} - D] = \delta^{(n)}$, we have

$$\delta^{(n)} = \delta_1^{(n)} + \delta_0^{(n)}$$

where

- $\delta_1^{(n)}$ is a first order differential operator supported on $N_{\frac{r}{T^n}}$ such that

$$\delta_1^{(n)} = \sum a_i \partial_i \text{ with } |a_1| \leq O(r^2) \text{ and } |a_2|, |a_3| \leq O(r).$$

- $\delta_0^{(n)}$ is a zero order differential operator supported on $N_{\frac{r}{T^n}}$ such that

$$|\delta_0^{(n)}| = O(r).$$

Meanwhile, we can write a new version of propositions in section 5 as follows.

We follow the setting in section 5. Suppose (η_1, χ_1) satisfies (5.14), (5.15), (5.16).

We also define

$$\phi_s(t, z, \bar{z}) = (t, z + s\eta_1(t), \bar{z} + s\bar{\eta}_1(t))$$

and

$$D_{s\eta^i} = \sum_{i=1}^3 e_i \cdot (\phi_s)^*(e_i) + \sum_{i=1}^3 e_i \sum_{j,k=1}^3 (\phi_s)^*(w_{jk}) e_j e_k.$$

Then we have the following proposition

Proposition 6.5. The perturbed Dirac operator can be written as

$$D_{s\eta_1} = (1 + \rho^1)D^{(1)} + s((\chi_1)_z \eta_1 + (\chi_1)_{\bar{z}} \bar{\eta}_1)(e_1 \partial_t) + \Theta_s + \mathcal{R}_s + \mathcal{A}_s + \mathcal{F}_s + \delta^{(1)}.$$

where

- $\Theta_s = [e_1(s\chi\dot{\eta}\partial_z + s\chi\dot{\eta}\partial_{\bar{z}}) + e_2(s\chi_{\bar{z}}\bar{\eta}\partial_z - s\chi_z\bar{\eta}\partial_{\bar{z}}) + e_3(-s\chi_{\bar{z}}\eta\partial_z + s\chi_z\eta\partial_{\bar{z}})]$ is a first order differential operator.
- $\mathcal{R}_s : L_1^2 \rightarrow L^2$ is a $O(s^2)$ -first order differential operator supported on $N_{\mathfrak{r}} - N_{\frac{\mathfrak{r}}{T}}$ with its operator norm from $\|\mathcal{R}_s\| \leq \gamma_T^2 \kappa_1^2 s^2$. Moreover, for any $\psi \in L_1^2 \cap \ker(D)$ we have

$$\|\mathcal{R}_s(\psi)\|_{L^2} \leq C\gamma_T^{\frac{3}{2}}\kappa_1^2\mathfrak{r}^2s^2$$

for some constant C depending on $\|\psi\|_{L_1^2}$.

- \mathcal{A}_s is a $O(s)$ -zero order differential operator supported on $N_{\mathfrak{r}} - N_{\frac{\mathfrak{r}}{T}}$. Moreover, let us denote $\vec{n} = \partial_r$ be the vector field defined on N_R , then

$$(6.9) \quad \int_{\{r=r_0\}} |\mathcal{A}_s|^2 i_{\vec{n}} dVol(M) \leq \gamma_T^4 \kappa_1^4 \mathfrak{r} s^4$$

for all $r_0 \leq \mathfrak{r}$.

- \mathcal{F}_s is a $O(s)$ -zero order differential operator where

$$(6.10) \quad \mathcal{F}_s = D(s(\chi_z\eta + \chi_{\bar{z}}\bar{\eta}))Id + D\left(\begin{pmatrix} 0 & si\chi\dot{\eta} \\ -si\chi\dot{\eta} & 0 \end{pmatrix}\right).$$

- $\delta^{(1)}$ can be written as $\delta^{(1)} = \delta_0^{(1)} + \delta_1^{(1)}$ where $\delta_1^{(1)}$ is a first order operator with

$$\delta_1^{(1)} = \sum a_i \partial_i \text{ with } |a_i| \leq O(r^2) \text{ and } |a_2|, |a_3| \leq O(r)$$

and $\delta_0^{(1)}$ is a zero order operator with

$$|\delta_0^{(1)}| = O(r).$$

Moreover, $\delta^{(1)}$ is supported on N_R .

Similarly, we have a new version of proposition 5.7. Suppose that we have a sequence of pairs, $\{(\chi_i, \eta_i)\}$, which is defined in subsection 5.3. Moreover, we suppose η_i satisfies (5.26),(5.27) and (5.28) and we write $\eta^i = \sum_{j=0}^i s^{j+1} \chi_j \eta_j$. Then we have

Proposition 6.6. There exists $\kappa_3 = O(\kappa_2)$ depending on κ_2 with the following significance. The perturbed Dirac operator $D_{s(\eta^i)}$ which satisfies the hypothesis (5.26), (5.27), (5.28), (5.29), (5.30), (5.31) can be written as follows:

(6.11)

$$\begin{aligned} D_{s\eta^{i+1}} &= (1 + \varrho^{i+1})D_{s\eta^i}^{(i+1)} + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1})e_1 \partial_t \\ &\quad + \Theta_s^{i+1} + \mathcal{R}_s^{i+1} + \hat{\mathcal{A}}_s^{i+1} + \mathcal{F}_s^{i+1} + \delta^{(i+1)} \end{aligned}$$

where

- Θ_s^{i+1} , the $(\chi, \eta) = (\chi_{i+1}, \eta_{i+1})$ version of Θ_s^0 , is a first order differential operator of order $O(s)$.
- $\mathcal{R}_s^{i+1} : L_1^2 \rightarrow L^2$ is a $O(s^2)$ -first order differential operator supported on $N_{\frac{\mathfrak{r}}{T^i}} - N_{\frac{\mathfrak{r}}{T^{i+1}}}$ with its operator norm

$$\|\mathcal{R}_s^{i+1}\| \leq \gamma_T^2 \kappa_3^2 s^2.$$

- $\hat{\mathcal{A}}_s^{i+1}$ is a $O(s)$ -zero order differential operator. Moreover, let us denote $\vec{n} = \partial_r$ be the vector field defined on N_R , then

$$(6.12) \quad \int_{\{r=r_0\}} |\hat{\mathcal{A}}_s^{i+1}|^2 i_{\vec{n}} dVol(M) \leq \gamma_T^4 \kappa_3^4 \left(\frac{(i+1)\mathfrak{r}}{T^{i+1}}\right) s^4.$$

for all $r_0 \leq \frac{\mathfrak{r}}{T^i}$.

- \mathcal{F}_s^{i+1} is a $O(s)$ -zero order differential operator where

(6.13)

$$\mathcal{F}_s^{i+1} = D(s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1})Id) + D\left(\begin{pmatrix} 0 & si\chi_{i+1}\dot{\eta}_{i+1} \\ -si\chi_{i+1}\dot{\bar{\eta}}_{i+1} & 0 \end{pmatrix}\right).$$

- $\delta^{(i+1)}$ can be written as $\delta^{(i+1)} = \delta_0^{(i+1)} + \delta_1^{(i+1)}$ where $\delta_1^{(i+1)}$ is a first order operator with

$$\delta_1^{(i+1)} = \sum a_i \partial_i \text{ with } |a_i| \leq O(r^2) \text{ and } |a_2|, |a_3| \leq O(r)$$

and $\delta_0^{(i+1)}$ is a zero order operator with

$$|\delta_0^{(i+1)}| = O(r).$$

Moreover, $\delta^{(i+1)}$ is supported on $N_{\frac{R}{T^i}}$.

7. FREDHOLM PROPERTY

7.1. Basic setting. In this section, we develop an important theorem which indicates that the the perturbation along \mathbb{V} is finite dimensional as I mentioned in section 5.2. The operator, \mathcal{T}_{d^+, d^-} , we construct in this section can be regarded as the linear approximation of the moduli space \mathfrak{M} we defined in our main theorem. We roughly sketch the idea of construction this operator in the following subsections first.

First of all, we consider the idea which comes from [2]. Let N be a tubular neighborhood of Σ equipped with the Euclidean metric. By the computation in subsection 4.1, we know that for any \mathbf{u} in the $\ker(D|_{L^2(N; \mathcal{S} \otimes \mathcal{I})})$ can be written as

$$\mathbf{u} = \sum_l e^{ilt} \left[\hat{u}_{0,l}^+ \begin{pmatrix} \frac{e^{|l|r}}{\sqrt{z}} \\ -\text{sign}(l) \frac{e^{|l|r}}{\sqrt{\bar{z}}} \end{pmatrix} + \hat{u}_{0,l}^- \begin{pmatrix} \frac{e^{-|l|r}}{\sqrt{z}} \\ \text{sign}(l) \frac{e^{-|l|r}}{\sqrt{\bar{z}}} \end{pmatrix} \right] \\ + \text{higher order terms.}$$

We define B mapping from $\ker(D|_{L^2(N; \mathcal{S} \otimes \mathcal{I})})$ to $L^2(S^1; \mathbb{C}^2)$ by $B(\mathbf{u}) = (\sum_l \hat{u}_{0,l}^+ e^{i|l|t} + \sum_l \hat{u}_{0,l}^- e^{-i|l|t}, -\sum_l \text{sign}(l) \hat{u}_{0,l}^+ e^{i|l|t} + \sum_l \text{sign}(l) \hat{u}_{0,l}^- e^{-i|l|t})$.

Secondly, we define the following spaces

$$\begin{aligned} Exp^+ &= \{(\sum_l u_l e^{i|l|t}, \sum_l -\text{sign}(l)u_l e^{i|l|t}) | (u_l) \in l_2\} \text{ and} \\ Exp^- &= \{(\sum_l u_l e^{-i|l|t}, \sum_l \text{sign}(l)u_l e^{-i|l|t}) | (u_l) \in l_2\}, \end{aligned}$$

then we have the corresponding projections $\pi^\pm : L^2(S^1; \mathbb{C}^2) \rightarrow Exp^\pm$.

Proposition 7.1. Define the maps $p^\pm = \pi^\pm \circ B$ in the following diagram.

$$\begin{array}{ccc} & & Exp^+ \\ & \nearrow p^+ & \uparrow \pi^+ \\ ker(D|_{L^2(M-\Sigma; \mathcal{S} \otimes \mathcal{I})}) & \xrightarrow{B} & L^2(S^1; \mathbb{C}^2) \\ & \searrow p^- & \downarrow \pi^- \\ & & Exp^- \end{array}$$

We will have

- a. $p^-|_{ker(p^+)} : ker(p^+) \rightarrow Exp^-$ is a Fredholm operator.
- b. $p^+|_{ker(p^-)} : ker(p^-) \rightarrow Exp^+$ is a compact operator.

Proof. a. First of all, for any $r > 0$ small enough we have

$$\int_{M-N_r} |Du|^2 = \int_{M-N_r} |\nabla \mathbf{u}|^2 + \int_{\partial N_r} \langle \mathbf{u}, \partial_r \mathbf{u} \rangle i_{\partial_r} dVol + \int_{M-N_r} \langle \mathcal{B} \mathbf{u}, \mathbf{u} \rangle$$

by Schrodinger-Lichinerowicz formula. Now by taking the limit $r \rightarrow 0$ and $\mathbf{u} \in ker(p^+)$, we have

$$0 = \int_{M-\Sigma} |\nabla \mathbf{u}|^2 - \sum_l |l| |\hat{u}_{0,l}^-|^2 + \int_{M-\Sigma} \mathcal{B} |\mathbf{u}|^2.$$

So

$$\|\mathbf{u}\|_{L^2_1(M-\Sigma)}^2 \leq \sum_l |l| |\hat{u}_{0,l}^-|^2 + C \|\mathbf{u}\|_{L^2(M-\Sigma)}^2$$

for some constant $C = \sup |\mathcal{R}|$. If $\mathbf{u} \in \ker(p^-|_{\ker(p^+)})$, then $\sum_l |l| |\hat{u}_{0,l}^-|^2 = 0$, which implies that

$$\|\mathbf{u}\|_{L^2_1(M-\Sigma)}^2 \leq C \|\mathbf{u}\|_{L^2(M-\Sigma)}^2.$$

So the kernel of $p^-|_{\ker(p^+)}$ will be finite dimensional.

To prove that $p^-|_{\ker(p^+)}$ has finite dimensional cokernel, we can prove the following statement instead: There exists $n > 0$ with the following significance. For any $\sum_l \hat{u}_{0,l}^- e^{ilt} \in \text{Exp}^-$ such that $\hat{u}_{0,l}^- = 0$ for all l satisfying $|l| \leq n$, there exists $\mathbf{u} \in \ker(D|_{L^2})$ such that $\|B(\mathbf{u}) - (\hat{u}_{0,l}^-)\|_{L^2} \leq \frac{1}{2} \|(\hat{u}_{0,l}^-)\|_{L^2}$.

Suppose this claim is true. We let $\mathbb{W} := \{\sum_l e^{ilt} \hat{u}_{0,l}^- | \hat{u}_{0,l}^- = 0 \text{ for all } |l| > n\}$. We prove that $\text{range}(p^-|_{\ker(p^+)}) + \mathbb{W} = \text{Exp}^-$ as the follows. Suppose not, there exists $v \in L^2(S^1; \mathbb{C})$ such that $v \notin \text{range}(p^-|_{\ker(p^+)}) + \mathbb{W}$. Then we can assume that $v \perp (\text{range}(p^-|_{\ker(p^+)}) + \mathbb{W})$. So by using the claim in previous paragraph, for any $\sum_l e^{ilt} \hat{u}_{0,l}^- \in \text{Exp}^-$ with $\|\sum_l e^{ilt} \hat{u}_{0,l}^-\|_{L^2} = 1$, we have

$$\begin{aligned} \langle v, \sum_l e^{ilt} \hat{u}_{0,l}^- \rangle &= \langle v, \sum_{|l| \leq n} e^{ilt} \hat{u}_{0,l}^- \rangle + \langle v, \sum_{|l| > n} e^{ilt} \hat{u}_{0,l}^- \rangle \\ &= \langle v, \sum_{|l| \leq n} e^{ilt} \hat{u}_{0,l}^- \rangle + \langle v, B(\mathbf{u}) \rangle + X \\ &= X \end{aligned}$$

where $|X| \leq \frac{1}{2} \|v\|_{L^2}$, which is a contradiction. Therefore we have

$$\dim(\text{coker}(p^-|_{\ker(p^+)})) \leq 2n + 1.$$

Here we suppose $\|(\hat{u}_{0,l}^-)\|_{L^2} = 1$ without loss of generality. To prove this claim, we can consider the following section

$$\begin{aligned} \mathbf{u}_0 &= \chi \sum_{|l| \geq n} e^{ilt} u_{0,l}^- \left[\begin{pmatrix} e^{-i\frac{1}{2}\theta} \mathfrak{J}_{-\frac{1}{2},l}(r) \\ -le^{i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \end{pmatrix} + \text{sign}(l) \begin{pmatrix} -le^{-i\frac{1}{2}\theta} \mathfrak{J}_{\frac{1}{2},l}(r) \\ e^{i\frac{1}{2}\theta} \mathfrak{J}_{-\frac{1}{2},l}(r) \end{pmatrix} \right] \\ &= \chi \sum_{|l| \geq n} e^{ilt} \hat{u}_{0,l}^- \begin{pmatrix} \frac{e^{-|l|r}}{\sqrt{z}} \\ \text{sign}(l) \frac{e^{-|l|r}}{\sqrt{z}} \end{pmatrix} \end{aligned}$$

(χ is the smooth function defined in subsection 5.3). So by this setting, we have

$$\|D(\mathbf{u}_0)\|_{L^2} \leq C \frac{e^{-nR}}{R}.$$

By proposition 5.4, there exists \mathbf{u}^* such that $D(\mathbf{u}^*) = D(\mathbf{u}_0)$. Moreover, we have $\|B(\mathbf{u}^*)\|_{L^2} \leq C \frac{e^{-nR}}{R}$. So by taking $\mathbf{u} = \mathbf{u}_0 - \mathbf{u}^*$, we finish the proof of this claim.

b. By the similar argument, we have

$$\sum_l |l| |\hat{u}_{0,l}^+|^2 \leq C \|\mathbf{u}\|_{L^2(M-\Sigma)}^2$$

So any bounded sequence $\{\mathbf{u}^{(n)}\}$ such that $\{p^+(\mathbf{u}^{(n)}) = (\hat{u}_{0,l}^{(n)+})\}$ converges, we have

$$\sum_l |\hat{u}_{0,l}^+|^2 + \sum_l |l| |\hat{u}_{0,l}^+|^2 \leq C.$$

This implies that there exists a convergent subsequence of $\{\mathbf{u}^{(n)}\}$ which converges to some \mathbf{u} and $\lim_{n \rightarrow \infty} p^+(\mathbf{u}^{(n)}) = p^+(\mathbf{u})$. Therefore $p^+|_{\ker(p^-)}$ is compact. \square

We should remember that under a small perturbation of the metric and Σ , the dimension of cokernel of $p^+|_{\ker(p^-)}$ will be a upper semi-continuous function. I will leave this proof in appendix 10.2.

With this proposition in mind, we can derive the linearization for our moduli space in the following subsection.

7.2. **Linearization.** Here we derive the linearization of \mathfrak{M} along \mathbb{V} (defined in section 5.2). Suppose that we have a L^2_1 -harmonic spinor

$$\psi = \begin{pmatrix} d^+(t)\sqrt{z} \\ d^-(t)\sqrt{\bar{z}} \end{pmatrix} + \text{higher order terms.}$$

Now we perturb the metric on a subset which is supported away from Σ , then we have a corresponding family of Dirac operators $D^{(s)}$. So

$$D^{(s)}\psi = s\mathfrak{f} + o(s).$$

for some $\mathfrak{f} \in L^2$ and \mathfrak{f} is compactly supported away from Σ . By proposition 5.4, there exists $\mathfrak{h} \in L^2$ such that $D^{(s)}\mathfrak{h} = \mathfrak{f}$. So we have

$$D^{(s)}(\psi - s\mathfrak{h}) = o(s).$$

So $\psi - s\mathfrak{h}$ is a first order approximation of harmonic spinor. However, this section is just in L^2 , which means that it has the leading order term of the order $O(\frac{1}{\sqrt{r}})$.

Now, we consider the perturbation of Σ in a tubular neighborhood, say $\Sigma_s = \{(t, s\eta(t))\}$ for some C^1 function $\eta : S^1 \rightarrow \mathbb{C}$. Then we can write down the following Taylor expansion

$$\psi(t, z - s\eta(t)) - s\mathfrak{h}(z - s\eta) = \psi(z) - \left[\begin{pmatrix} \frac{d^+\eta}{2\sqrt{z}} \\ \frac{d^-\bar{\eta}}{2\sqrt{\bar{z}}} \end{pmatrix} + \begin{pmatrix} \frac{h^+}{\sqrt{z}} \\ \frac{h^-}{\sqrt{\bar{z}}} \end{pmatrix} \right]s + O(s^2).$$

Our goal is to make this section in L^2_1 , so we need to kill the $O(\frac{1}{\sqrt{r}})$ terms. To achieve this goal, the perturbation η must satisfies

$$d^+\eta = -2h^+,$$

$$d^-\bar{\eta} = -2h^-.$$

However, this equation is not solvable. It even doesn't have finite dimensional cokernel. So we need to add an auxiliary part to this equation. Since the choice of \mathfrak{h} is not unique, by using the proposition 7.1, we can rewrite our linearization equation as follows

$$\begin{aligned}d^+ \eta + c^+ &= -2h^+, \\d^- \bar{\eta} + c^- &= -2h^-\end{aligned}$$

where $(c^\pm) \in \text{Exp}^-$. Our goal in the next section is to prove that operator is Fredholm.

7.3. Fredholmness of finite Fourier mode case. The linearization in previous subsection can be written as follows

$$\begin{aligned}d^+ \eta + c^+ &= -2h^+, \\d^- \bar{\eta} + c^- &= -2h^-.\end{aligned}$$

with the following constraint:

$$(7.1) \quad |d^+|^2 + |d^-|^2 \neq 0.$$

This constraint comes from the assumption that $\frac{|\psi(p)|}{\text{dist}(p, \Sigma)^{\frac{1}{2}}} > 0$ for all p near Σ . Now, this equation implies

$$(7.2) \quad \bar{d}^- c^+ - d^+ \bar{c}^- = -2\bar{d}^- h^+ + 2d^+ \bar{h}^-.$$

Moreover, there is the following relationship between c^+ and c^- : if we write $c^+ = \sum p_l e^{ilt}$, then we have $c^- = \sum \text{sign}(l) p_l e^{ilt}$. Namely, the c^- is determined by c^+ . Therefore, we can define the following operator

$$\mathcal{T}_{d^+, d^-}(c^+) = \bar{d}^- c^+ - d^+ \bar{c}^-$$

In this section, we will use the following notation.

Definition 7.2. Let $g = \sum_l g_l e^{ilt} \in L^2$, we write $g^{aps} = \sum_l \text{sign}(l) g_l e^{ilt}$.

So we can rewrite our operator in the following way:

$$\mathcal{T}_{d^+, d^-}(c) = \bar{d}^- c - d^+ \overline{c^{aps}}$$

with $\mathcal{T}_{d^+, d^-} : L^2 \rightarrow L^2$. Here we should explain the meaning of L^2 . We can easily see that, \mathcal{T}_{d^+, d^-} is not a \mathbb{C} -linear operator, since the conjugate term $\overline{c^{aps}}$ involved. However, it is still a \mathbb{R} -linear operator. Therefore we define our index under the real vector space, even though the underlying space has some complex structure.

So in our case, we should define the inner product to be

$$(f, g) := \text{Re} \left(\int_{S^1} f \cdot \bar{g} dt \right)$$

for all $f, g \in C^\infty(S^1; \mathbb{C})$. We can see that, under this definition, the L^2 -bounded space will coincident with the one equipped with the usual inner product over \mathbb{C} .

We will prove the following property:

Proposition 7.3. \mathcal{T}_{d^+, d^-} is a Fredholm operator and $\text{index}(\mathcal{T}_{d^+, d^-}) = 0$ when both d^+ and d^- have only finite many Fourier modes:

$$d^+ = \sum_{-M}^M d_l^+ e^{ilt}; \quad d^- = \sum_{-M}^M d_l^- e^{ilt}$$

for some $M \in \mathbb{N}$.

In this section, we will assume that d^\pm have only finite many Fourier modes and prove the proposition 7.3. Then we will prove the general case in the next subsection.

Before we prove this property, we should define some notations.

Definition 7.4. Let $a = (x, y) \in \mathbb{C} \times \mathbb{C}$, we define the **spouse** of a , denoted by \hat{a} , to be $(\bar{y}, -\bar{x}) \in \mathbb{C} \times \mathbb{C}$. We can easily see that $\hat{\hat{a}} = -a$.

Similarly, for any p -tuple of complex pairs, we have the following definition.

Definition 7.5. Suppose we have $A = (a_1, a_2, \dots, a_{p-1}, a_p) \in (\mathbb{C} \times \mathbb{C})^p$ for some $p \in \mathbb{N}$, we define the **spouse** of A , denoted by \hat{A} , to be $(\hat{a}_p, \hat{a}_{p-1}, \dots, \hat{a}_2, \hat{a}_1) \in (\mathbb{C} \times \mathbb{C})^p$.

Now we write our proof of proposition 7.3 in the following 8 steps.

Step 1. In this and the next step, we will prove that \mathcal{T}_{d^+, d^-} has finite dimensional kernel. Firstly, we notice that the n -th Fourier coefficient of $(\bar{d}^- c - d^+ \overline{c^{aps}})$ can be expressed as

$$(\bar{d}^- c - d^+ \overline{c^{aps}})_n = \sum_{l=-M}^M \bar{d}_{-l}^- p_{n-l} + \text{sign}(l-n) d_l^+ \bar{p}_{l-n}.$$

If we take $n > M$, then we have $\text{sign}(l-n) = -1$ for all $l = -M, \dots, M$. So we have

$$(7.3) \quad (\bar{d}^- c - d^+ \overline{c^{aps}})_n = \sum_{l=-M}^M \bar{d}_{-l}^- p_{n-l} - d_l^+ \bar{p}_{l-n}$$

for $n > M$.

Similarly

$$(\bar{d}^- c - d^+ \overline{c^{aps}})_n = \sum_{l=-M}^M \bar{d}_{-l}^- p_{n-l} + d_l^+ \bar{p}_{l-n}$$

for $n < -M$.

If we take $n = -n'$ and then take the conjugation on the both side of this equation, we will have the following equation:

$$(7.4) \quad (d^- \bar{c} - \bar{d}^+ c^{aps})_{n'} = \sum_{l=-M}^M \bar{d}_{-l}^+ p_{n'-l} + d_l^- \bar{p}_{l-n'}.$$

for $n' > M$.

To show that the kernel of \mathcal{T}_{d^+,d^-} is finite dimensional, here is the idea: we claim that every element in $\ker(\mathcal{T}_{d^+,d^-})$ can be determined by their Fourier coefficients from $-2M$ to $2M$. Therefore the dimension of $\ker(\mathcal{T}_{d^+,d^-})$ cannot exceed $4M + 2$. To prove this statement, suppose there are two elements c_1 and c_2 in $\ker(\mathcal{T}_{d^+,d^-})$ who have same Fourier coefficients from $-2M$ to $2M$. Then $c_1 - c_2$ is also in $\ker(\mathcal{T}_{d^+,d^-})$. Therefore, our claim is true iff any $c \in \ker(\mathcal{T}_{d^+,d^-})$ which has zero Fourier coefficients from $-2M$ to $2M$ is identically zero.

Step 2. Now we prove this claim. Suppose that $c \in \ker(\mathcal{T}_{d^+,d^-})$ has zero Fourier coefficients from $-2M$ to $2M$. Because $c \in \ker(\mathcal{T}_{d^+,d^-})$, we have

$$\begin{aligned} \sum_{l=-M}^M \bar{d}_{-l}^- p_{n-l} - d_l^+ \bar{p}_{l-n} &= 0 \\ \sum_{l=-M}^M \bar{d}_{-l}^+ p_{n-l} + d_l^- \bar{p}_{l-n} &= 0 \end{aligned}$$

for $n > M$, we can rewrite this equation by pairing $(p_j, \bar{p}_{-j}) := v_j$ and $(\bar{d}_{-j}^-, -d_j^+) := d_j$ for all $j \in \mathbb{Z}$. Now we have

$$\begin{aligned} \sum_{l=-M}^M \langle d_l, \bar{v}_{n-l} \rangle &= 0 \\ \sum_{l=-M}^M \langle \hat{d}_{-l}, \bar{v}_{n-l} \rangle &= 0 \end{aligned}$$

with the bracket $\langle \cdot, \cdot \rangle$ denoting the usual inner product over \mathbb{C} . Here we can use the following convention: Let $U = (u_i), W = (w_i) \in (\mathbb{C} \times \mathbb{C})^{\mathbb{Z}}$, we define a new bracket $\langle\langle \cdot, \cdot \rangle\rangle$ to be

$$\langle\langle U, W \rangle\rangle_n = \sum_{i \in \mathbb{Z}} \langle u_i, w_{n-i} \rangle.$$

So our equation can be written as

$$\langle\langle D, \bar{V} \rangle\rangle_n = 0$$

$$\langle\langle \hat{D}, \bar{V} \rangle\rangle_n = 0$$

where $D = (d_l)$ and $V = (v_l)$ and $n > M$.

Now we apply the following squeezing lemma.

Lemma 7.6. *Given $A = (a_j)_{j=1,2,\dots,p} \in (\mathbb{C} \times \mathbb{C})^p$. If $V = (v_j)_{j \in \mathbb{Z}} \in (\mathbb{C} \times \mathbb{C})^{\mathbb{Z}}$ satisfies*

$$\langle\langle A, \bar{V} \rangle\rangle_m = 0; \langle\langle \hat{A}, \bar{V} \rangle\rangle_m = 0$$

for all $m > 0$. Then there is $B = (0, \dots, 0, b_1, \dots, b_q) \in (\mathbb{C} \times \mathbb{C})^p$ with and $\det \begin{pmatrix} b_q \\ \hat{b}_1 \end{pmatrix} \neq 0$ such that

$$\langle\langle B, \bar{V} \rangle\rangle_m = 0; \langle\langle B^*, \bar{V} \rangle\rangle_m = 0,$$

where $B^* = (0, \dots, 0, \hat{b}_q, \dots, \hat{b}_1)$, for all $m > 0$.

Proof. If $\det \begin{pmatrix} a_p \\ \hat{a}_1 \end{pmatrix} \neq 0$, then we can just take $A = B$. The lemma holds trivially.

Suppose now $\det \begin{pmatrix} a_p \\ \hat{a}_1 \end{pmatrix} = 0$. Then we have $\alpha a_p = \hat{a}_1$ for some $\alpha \in \mathbb{C} - \{0\}$. So we have

$$\langle\langle \hat{A}, \bar{V} \rangle\rangle_m - \alpha \langle\langle A, \bar{V} \rangle\rangle_m = \langle\langle \hat{A} - \alpha A, \bar{V} \rangle\rangle_m = 0.$$

We also have

$$\langle\langle A, \bar{V} \rangle\rangle_m + \bar{\alpha} \langle\langle \hat{D}, \bar{V} \rangle\rangle_m = \langle\langle A + \bar{\alpha} \hat{A}, \bar{V} \rangle\rangle_m = 0.$$

Denote

$$B'_1 = (\hat{A} - \alpha A) = (\hat{a}_p - \alpha a_1, \hat{a}_{p-1} - \alpha a_2, \dots, \hat{a}_2 - \alpha a_{p-1}, 0).$$

Notice that: Since $\alpha a_p = \hat{a}_1$, we have $\hat{a}_p - \alpha a_1 = \hat{a}_p + |\alpha|^2 \hat{a}_p = (1 + |\alpha|^2) \hat{a}_p \neq 0$. This implies $B'_1 \neq 0$.

Now let $B_1 = (0, \hat{a}_p - \alpha a_1, \hat{a}_{p-1} - \alpha a_2, \dots, \hat{a}_2 - \alpha a_{p-1})$. We can easily verify that

$$\langle\langle \hat{A} - \alpha A, \bar{V} \rangle\rangle_{m+1} = \langle\langle B_1, \bar{V} \rangle\rangle_m = 0.$$

for all $m > 0$.

Since $(\hat{A} - \alpha A) = (A + \bar{\alpha} \hat{A})^\wedge$, the second equation gives us

$$\langle\langle A - \bar{\alpha} \hat{A}, \bar{V} \rangle\rangle_m = \langle\langle B_1^*, \bar{V} \rangle\rangle_m = 0.$$

for all $m > 0$.

Now by repeating this process inductively, we prove this lemma. □

Back to our problem, we have the equations

$$\langle\langle D, \bar{V} \rangle\rangle_n = 0$$

$$\langle\langle \hat{D}, \bar{V} \rangle\rangle_n = 0$$

for $n > M$. Now we can apply lemma 7.6 on $A = (d_{-M}, d_{-(M-1)}, \dots, d_M)$, $m = n - M$.

So there exists $B \in (\mathbb{C} \times \mathbb{C})^p$ such that $\det \begin{pmatrix} b_q \\ \hat{b}_1 \end{pmatrix} \neq 0$ and

$$\langle\langle B, \bar{V} \rangle\rangle_n = 0$$

$$\langle\langle B^*, \bar{V} \rangle\rangle_n = 0$$

for all $n > M$. Together with the condition $v_l = 0$ for $l = 0, 1, \dots, 2M$, we have

$$\langle\langle B, \bar{V} \rangle\rangle_{M+1} = \langle b_q, v_{2M+1} \rangle = b_q^+ p_{(2M+1)} + b_q^- \bar{p}_{-(2M+1)} = 0$$

$$\langle\langle B^*, \bar{V} \rangle\rangle_{M+1} = \langle \hat{b}_1, v_{2M+1} \rangle = \hat{b}_1^- p_{(2M+1)} + \hat{b}_1^+ \bar{p}_{-(2M+1)} = 0,$$

which implies $v_{2M+1} = 0$ because $\det \begin{pmatrix} b_q \\ \hat{b}_1 \end{pmatrix} \neq 0$. Now we can solve v_k inductively:

Suppose v_1, v_2, \dots, v_{M+k} are all zero for some $k > M + 1$. Then the equation tells us that

$$\langle\langle B, \bar{V} \rangle\rangle_{k+1} = \langle b_q, v_{M+k+1} \rangle = b_q^+ p_{(M+k+1)} + b_q^- \bar{p}_{-(M+k+1)} = 0$$

$$\langle\langle B^*, \bar{V} \rangle\rangle_{k+1} = \langle \hat{b}_1, v_{M+k+1} \rangle = \hat{b}_1^- p_{(M+k+1)} + \hat{b}_1^+ \bar{p}_{-(M+k+1)} = 0.$$

So we have $v_{M+k+1} = 0$. Therefore we have $v_l = 0$ for all l which implies $c \equiv 0$.

Step 3. We still have several parts to prove. To show that \mathcal{T}_{d^+, d^-} is a Fredholm operator, we can either prove \mathcal{T}_{d^+, d^-} has finite dimensional cokernel, or we can prove the following properties instead:

1. $\ker(\mathcal{T}_{d^+, d^-}^*)$ is finite dimensional,
2. $\text{range}(\mathcal{T}_{d^+, d^-})$ is closed,
3. $\text{range}(\mathcal{T}_{d^+, d^-}^*)$ is closed.

We prove these properties step by step.

Step 4. Here we prove $\ker(\mathcal{T}_{d^+, d^-}^*)$ is finite dimensional. Here \mathcal{T}_{d^+, d^-}^* is the adjoint operator of \mathcal{T}_{d^+, d^-} . We can get the following computation by definition: Let $c =$

$$\sum p_l e^{ilt}, k = \sum q_l e^{ilt} \in L^2$$

$$\begin{aligned}
(\mathcal{T}_{d^+, d^-}(c), k) &= \operatorname{Re} \left(\int_{\mathbb{S}^1} \mathcal{T}_{d^+, d^-}(c) \cdot \bar{k} dt \right). \\
&= \frac{1}{2} \left(\int_{\mathbb{S}^1} \mathcal{T}_{d^+, d^-}(c) \cdot \bar{k} dt + \int_{\mathbb{S}^1} k \cdot \overline{\mathcal{T}_{d^+, d^-}(c)} dt \right) \\
&= \sum_{n \in \mathbb{Z}} \sum_{l=-M}^M (\bar{d}_{-l} p_{n-l} + \operatorname{sign}(l-n) d_l^+ \bar{p}_{l-n}) \bar{q}_n \\
&\quad + \sum_{n \in \mathbb{Z}} \sum_{l=-M}^M q_n (d_{-l}^- \bar{p}_{n-l} - \operatorname{sign}(l-n) \bar{d}_l^+ p_{l-n}) \\
&= \sum_{n \in \mathbb{Z}} \left(\sum_{l=-M}^M d_{-l}^- q_{n+l} + \operatorname{sign}(n) d_l^+ \bar{q}_{-n+l} \right) \bar{p}_n \\
&\quad + \sum_{n \in \mathbb{Z}} p_n \left(\sum_{l=-M}^M (\bar{d}_{-l}^- q_{n+l} + \operatorname{sign}(n) d_l^+ \bar{q}_{-n+l}) \right) \\
&= (c, \mathcal{T}_{d^+, d^-}^*(k)).
\end{aligned}$$

We get the last equality by taking

$$\mathcal{T}_{d^+, d^-}^*(k) = \sum_{n \in \mathbb{Z}} \left(\sum_{i=-M}^M d_{-i}^- q_{n+i} + \operatorname{sign}(n) d_i^+ \bar{q}_{-n+i} \right) e^{int}.$$

Now we can apply the argument in step 1 and 2 on \mathcal{T}_{d^+, d^-}^* , then we will get $\dim(\ker(\mathcal{T}_{d^+, d^-}^*)) < \infty$.

Step 5. Properties 2 and 3 in step 3 are similar. Here we only prove property 2. Readers can prove the property 3 by applying the same argument again.

Before we prove the property 2, we need the following lemma.

Lemma 7.7. *Let $P_k : L^2 \rightarrow L^2$ is the projection defined by*

$$P_k : \sum f_n e^{int} \mapsto \sum_{|n| \leq k} f_n e^{int}.$$

Then we have

$$\mathcal{T}_{d^+, d^-}|_{(I-P_{2M})(L^2)} : (I - P_{2M})(L^2) \rightarrow (I - P_M)(L^2)$$

is injective.

Proof. Let $f \in (I - P_{2M})(L^2)$, clearly $T(f) \in (I - P_M)(L^2)$, so we should prove this map is one to one.

Suppose $f = \sum f_k e^{ikt} \in (I - P_M)(L^2)$, by solving the equation given by lemma 7.6, we have

$$\begin{aligned} \langle \langle B, \bar{V} \rangle \rangle_{M+1} &= \langle b_q, v_{2M+1} \rangle = b_q^+ p_{(2M+1)} + b_q^- \bar{p}_{-(2M+1)} = f_{M+1} \\ \langle \langle B^*, \bar{V} \rangle \rangle_{M+1} &= \langle \hat{b}_1, v_{2M+1} \rangle = \hat{b}_1^- p_{(2M+1)} + \bar{b}_1^+ \bar{p}_{-(2M+1)} = \bar{f}_{-(M+1)}. \end{aligned}$$

So we can solve $v_{(2M+1)} = (p_{2M+1}, \bar{p}_{-(2M+1)})$, which is unique.

Now suppose $v_{(2M+1)}, \dots, v_{M+k}$ are uniquely determined (where $k > M + 1$), we consider the equation

$$\begin{aligned} \langle \langle B, \bar{V} \rangle \rangle_{k+1} &= \langle b_q, v_{M+k+1} \rangle \\ &= b_q^+ p_{(M+k+1)} + b_q^- \bar{p}_{-(M+k+1)} + F_k(v_{(2M+1)}, \dots, v_{M+k}) = f_{k+1} \\ \langle \langle B^*, \bar{V} \rangle \rangle_{k+1} &= \langle \hat{b}_1, v_{M+k+1} \rangle \\ &= \hat{b}_1^- p_{(M+k+1)} + \bar{b}_1^+ \bar{p}_{-(M+k+1)} + G_k(v_{(2M+1)}, \dots, v_{M+k}) = \bar{f}_{-(k+1)}. \end{aligned}$$

where $F_k(v_{(2M+1)}, \dots, v_{M+k}) = f_{k+1}$ and $G_k(v_{(2M+1)}, \dots, v_{M+k})$ are determined by $\{v_{(2M+1)}, \dots, v_{M+k}\}$.

So we can solve $v_{(M+k+1)}$ uniquely.

Therefore, the map $\mathcal{T}_{d^+, d^-}|_{(I-P_{2M})(L^2)}$ is an injective map from $(I - P_{2M})(L^2)$ to $(I - P_M)(L^2)$. \square

If we decompose $L^2 = P_{2M}(L^2) \oplus (I - P_{2M})(L^2)$, we have $\mathcal{T}_{d^+,d^-}(P_{2M}(L^2)) \subset P_{3M}(L^2)$ and $\mathcal{T}_{d^+,d^-}((I - P_{2M})(L^2)) \subset (I - P_M)(L^2)$.

Step 6. We will prove the property 2 in Step 3 in the following 2 steps. Suppose now we have $c^k \in L^2$, $k \in \mathbb{N}$ such that $\{\mathcal{T}_{d^+,d^-}(c^k)\}$ converges to some $f \in L^2$. Let $\{v_p^k\}$ be the corresponding pairing l^2 -coefficients of c^k . Here we can assume that $c^k \perp \ker(\mathcal{T}_{d^+,d^-})$ without loss of generality. We have to show that there exist c such that $\mathcal{T}_{d^+,d^-}(c) = f$.

First of all, suppose that c^k is bounded by some constant K . We choose a subsequence, which denote by c^k again, such that $\{v_p^k\}_{k \in \mathbb{N}}$ converge for all $p \leq 3J$ with some $J > M$. Let us say

$$v_p^k \rightarrow v_p$$

for $p \leq 3J$ and here we choose J large enough such that $v_p \neq 0$. Now by lemma 7.7, we have a unique solution c such that

$$\mathcal{T}_{d^+,d^-}(c) = f$$

where the corresponding l^2 -coefficients of c are v_p for $p \leq 3M$. So we only need to show that c is in L^2 .

Now for any $r \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{i \leq r} \|v_i\|_{l^2}^2 &\leq \sum_{i \leq r} \|v_i^k - v_i\|_{l^2}^2 + \sum_{i \leq r} \|v_i^k\|_{l^2}^2 \\ &\leq \sum_{i \leq r} \|v_i^k - v_i\|_{l^2}^2 + K \end{aligned}$$

and we the first term converges to 0 as $k \rightarrow \infty$. Therefore we have

$$\sum_{i \leq r} \|v_i\|_{l^2}^2 \leq 1 + K$$

for any $r > 0$. So $c \in L^2$.

Step 7. Suppose that c^k is unbounded, say $\|c_k\|_{L^2} \rightarrow \infty$ (by taking subsequence if it is necessary). we can take $\check{c}^k = \frac{c_k}{\|c_k\|_{L^2}}$ which satisfies $\mathcal{T}_{d^+, d^-}(\check{c}^k) \rightarrow 0$. We should prove that this case will lead a contradiction. This is the part that condition (7.1) involved.

To begin with, we should define the following notations.

Definition 7.8. We define the number $\tau = \inf\{\sqrt{|d^+|^2 + |d^-|^2}\}$. We also define the following sets as

1. $\Omega_1 = \{|d^+| = |d^-|\} \subset S^1$,
2. $\Omega_{1,\varepsilon} = \{||d^-| - |d^+|| \leq \varepsilon\tau\}$,
3. $\Omega_\varepsilon^+ = \{|d^+| > |d^-| + \varepsilon\tau\}$,
4. $\Omega_\varepsilon^- = \{|d^-| > |d^+| + \varepsilon\tau\}$.

So we have $S^1 = \Omega_{1,\varepsilon} \cup \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$.

Now we fix an $\varepsilon \leq \frac{1}{6}$ which will be specified later. We define $\chi_{1,\varepsilon}$ to be a nonnegative real valued function defined on S^1 which has value 1 in $\Omega_{1,\frac{\varepsilon}{2}}$ and 0 in $\Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$. Also, define $\chi_{2,\varepsilon}$ to be 1 in Ω_ε^+ and 0 on $\{|d^+| \leq |d^-| + \frac{\varepsilon}{2}\tau\}$. Define $\chi_{3,\varepsilon}$ to be 1 in Ω_ε^- and 0 on $\{|d^-| \geq |d^+| + \frac{\varepsilon}{2}\tau\}$. Moreover, suppose that

$$\chi_{1,\varepsilon} + \chi_{2,\varepsilon} + \chi_{3,\varepsilon} \equiv 1.$$

Step 8. In this Step, we will modify our statement in step 7 by some observation and define some notations which will be used later. First of all, for any $L \in \mathbb{N}$ let $P_L : L^2 \rightarrow L^2$ be a projection which maps $\sum_{l \in \mathbb{Z}} q_l e^{ilt}$ to $\sum_{|l| \leq L} q_l e^{ilt}$. Now suppose we have a sequence $\{c^k\}$ with all have L^2 norms equalling 1 and $\mathcal{T}_{d^+, d^-}(c^k)$ converges to 0 in L^2 sense. For any $i \in \mathbb{Z}$ fixed, suppose that the limit sup of $\{|c_i^k|\}$ is nonzero, than we can using the argument in step 6 by taking $J > i$ to achieve a contradiction.

By using this observation, for any $L \in \mathbb{N}$ given, we can add the additional assumption into our statement: $P_L(c^k) = 0$ for any k . This number L will be specified later which is determined by ε and $\chi_{i,\varepsilon}$. Here we should restate our statement as following:

Lemma 7.9. *There exists $L \in \mathbb{N}$ depending only on d^\pm , such that for any sequence $\{c^k\}_{k \in \mathbb{N}} \subset L^2$ satisfying*

$$\|c^k\|_{L^2} = 1, \quad P_L(c^k) = 0 \text{ for all } k \in \mathbb{N},$$

we have $\inf_{k \in \mathbb{N}} \{\|\mathcal{T}_{d^+,d^-}(c^k)\|_{L^2}\} > C_0$, where C_0 depending only on the C^1 -norm of d^\pm and τ .

We still have several constants to define. We consider the function $Q = \frac{d^+}{d^-}$ defined on $\Omega_{1,\varepsilon}$. Extend this function as a C^1 function defined on S^1 . Then we can approximate it by its first N_1 Fourier modes, S , such that the L^2 -norm and L^∞ -norm of $|Q - S|$ are $O(\varepsilon)$.

Since $\chi_{1,\varepsilon} + \chi_{2,\varepsilon} + \chi_{3,\varepsilon} \equiv 1$, we have

$$1 = \|c^k\|_{L^2} \leq \|\chi_{1,\varepsilon}c^k\|_{L^2} + \|\chi_{2,\varepsilon}c^k\|_{L^2} + \|\chi_{3,\varepsilon}c^k\|_{L^2}.$$

Therefore, there exists $i \in \{1, 2, 3\}$ such that $\|\chi_{i,\varepsilon}c^k\|_{L^2} \geq \frac{1}{3}$ infinite many times. We take this subsequence and renumber them consecutively from 1. Since $\chi_{1,\varepsilon}$ is a smooth function, we approximate $\chi_{1,\varepsilon}$ by its first N_2 Fourier mode, denoted by $\zeta_{1,\varepsilon}$, such that $\|\chi_{1,\varepsilon} - \zeta_{1,\varepsilon}\|_{L^2} \leq \varepsilon < \frac{1}{6}$ and $\sup |\chi_{1,\varepsilon} - \zeta_{1,\varepsilon}| \leq \varepsilon$, so by Cauchy's inequality, we have $\|\zeta_{i,\varepsilon}c^k\|_{L^2} \geq \frac{1}{6}$. Now we shall start our proof of lemma 7.9 case by case.

Proof. Case 1. If $i = 1$, we have

$$\zeta_{1,\varepsilon}\mathcal{T}_{d^+,d^-}(c^k) = \zeta_{1,\varepsilon}f^k.$$

where $\limsup \|\zeta_{1,\varepsilon} f^k\|_2 \leq \varepsilon$ Now we can write

$$\zeta_{1,\varepsilon} \mathcal{T}_{d^+,d^-}(c^k) = \mathcal{T}_{d^+,d^-}(\zeta_{1,\varepsilon} c^k) + (\zeta_{1,\varepsilon} \mathcal{T}_{d^+,d^-}(c^k) - \mathcal{T}_{d^+,d^-}(\zeta_{1,\varepsilon} c^k)).$$

We can write the second term as $[\mathcal{T}_{d^+,d^-}, \zeta_{1,\varepsilon}](c^k)$. Let $\zeta_{1,\varepsilon} = \sum_{l \in \mathbb{Z}} \varsigma_l e^{ilt}$, then we can get

$$\begin{aligned} [\mathcal{T}_{d^+,d^-}, \zeta_{1,\varepsilon}](c^k) &= \zeta_{1,\varepsilon}((c^k)^{aps}) - (\zeta_{1,\varepsilon} c^k)^{aps} \\ &= \sum_{n \in \mathbb{Z}} [(\sum_{|j| \leq N_1} \varsigma_j \text{sign}(n-j) p_{n-j}^k) - (\sum_{|j| \leq N_1} \text{sign}(n) \varsigma_j p_{n-j})] e^{int} \\ &= \sum_{|n| \leq N_1} \pm 2 (\sum_{|j| \leq N_1} \varsigma_j p_{n-j}) e^{int}. \end{aligned}$$

So this term will be 0 by taking $L > 2N_1$.

Therefore we have

$$\begin{aligned} \mathcal{T}_{d^+,d^-}(\zeta_{1,\varepsilon} c^k) &= \zeta_{1,\varepsilon} f^k \\ &= \bar{d}^- \zeta_{1,\varepsilon} c^k - d^+ \overline{(\zeta_{1,\varepsilon} c^k)^{aps}} \end{aligned}$$

divided both side by \bar{d}^- , then we have

$$\zeta_{1,\varepsilon} c^k - \frac{d^+}{\bar{d}^-} \overline{(\zeta_{1,\varepsilon} c^k)^{aps}} = \frac{\zeta_{1,\varepsilon} f^k}{\bar{d}^-}.$$

Notice that $|\bar{d}^-| \geq \tau(1 - \varepsilon)$ on $\Omega_{1,\varepsilon}$, so the right hand side still converges to 0 in L^2 sense. Moreover, because $\frac{d^+}{\bar{d}^-} = Q$ on $\Omega_{1,\varepsilon}$ and $|\zeta_{1,\varepsilon}| \leq \varepsilon$ outside $\Omega_{1,\varepsilon}$, so we have

$$\zeta_{1,\varepsilon} c^k - Q \overline{(\zeta_{1,\varepsilon} c^k)^{aps}} = \frac{\zeta_{1,\varepsilon} f^k}{\bar{d}^-} + O_{L^2}(\tau\varepsilon)$$

(This $O_{L^2}(\varepsilon)$ term has its L^2 norm of order $O(\varepsilon)$). Write $Q = S + (Q - S)$ where the L^2 -norm and L^∞ -norm of $Q - S$ are $O(\varepsilon)$. So we have

$$\zeta_{1,\varepsilon}c^k - S\overline{(\zeta_{1,\varepsilon}c^k)^{aps}} = \frac{\zeta_{1,\varepsilon}f^k}{\bar{d}^-} + O_{L^2}(\varepsilon).$$

Finally, let $P^\pm : l^2 \rightarrow l^2$ be the projections map $\sum_l p_l e^{ilt}$ to $\sum_{l>0} p_l e^{ilt}$ and $\sum_{l<0} p_l e^{ilt}$ respectively. Here we denote $\zeta_{1,\varepsilon}c^k = A^k$ and $\overline{(\zeta_{1,\varepsilon}c^k)^{aps}} = B^k$ for a while. We have

$$P^\pm A^k + P^\pm S B^k = \frac{\zeta_{1,\varepsilon}f^k}{\bar{d}^-} + O_{L^2}(\varepsilon).$$

We notice that

$$P^\pm(A^k) = \overline{P^\mp(B^k)},$$

and

$$\begin{aligned} [P^+, S]B^k &= (P^+ S B^k - S P^+ B^k) \\ &= \sum_{n>0} \left(\sum_{|j|\leq N_2} S_j B_{n-j} e^{int} \right) - \sum_{n\geq j} \left(\sum_{|j|\leq N_2} S_j B_{n-j} e^{int} \right) \\ &= \sum_{|n|\leq N_2} \sum_{|j|\leq N_2} |S_j B_{n-j}| e^{int} \end{aligned}$$

the last term will be 0 when we take $L > 2N_1 + 2N_2$.

Therefore we have

$$\begin{aligned} P^+ A^k + S P^+ B^k &= O_{L^2}(\varepsilon) + P^+ \left(\frac{\zeta_{1,\varepsilon}f^k}{\bar{d}^-} \right), \\ \overline{P^+ B^k} - S \overline{P^+ A^k} &= O_{L^2}(\varepsilon) + \overline{P^- \left(\frac{\zeta_{1,\varepsilon}f^k}{\bar{d}^-} \right)} \end{aligned}$$

Since $\|A^k\|_{L^2} \geq \frac{1}{6}$, we can suppose that either $\|P^+A^k\|_{L^2} > \frac{1}{12}$ or $\|P^-A^k\|_{L^2} > \frac{1}{12}$. Suppose that $\|P^+A^k\|_{L^2} > \frac{1}{12}$ then we will have

$$\begin{aligned} P^+A^k + SP^+B^k - S(P^+B^k - \bar{S}P^+A^k) &= (1 + |S|^2)P^+A^k \\ &= O_{L^2}(\varepsilon) + P^+\left(\frac{\zeta_{1,\varepsilon}f^k}{\bar{d}^-}\right) + \bar{S}P^-\left(\frac{\zeta_{1,\varepsilon}f^k}{\bar{d}^-}\right) \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{1}{12} &\leq \|P^+A^k\|_{L^2} \leq \|(1 + |S|^2)P^+A^k\|_{L^2} \leq O(\varepsilon) + \|P^+f^k\|_{L^2} + \frac{1}{\tau}(\|\bar{S}P^-f^k\|_{L^2}) \\ &\leq O(\varepsilon) + \frac{1}{\tau}4\|f^k\|_{L^2} \end{aligned}$$

for ε arbitrary. so we have

$$\|f^k\|_{L^2} \geq \frac{\tau}{13}$$

for all k .

Case 2. If $i = 2$, we have

$$\begin{aligned} \zeta_{2,\varepsilon}\mathcal{T}_{d^+,d^-}(c^k) &= \mathcal{T}_{d^+,d^-}(\zeta_{2,\varepsilon}c^k) = \zeta_{2,\varepsilon}f^k \\ &= \bar{d}^-(\zeta_{2,\varepsilon}c^k) + d^+\overline{(\zeta_{2,\varepsilon}c^k)^-} = \zeta_{2,\varepsilon}f^k \end{aligned}$$

divided both side by d^+ and notice that $|\frac{\bar{d}^-}{d^+}| \leq 1 - \frac{\tau}{2}\varepsilon$ on Ω_ε^+ , so we have

$$\begin{aligned} \left\| \frac{\zeta_{2,\varepsilon}f^k}{d^+} \right\|_{L^2(\Omega_\varepsilon^+)} &= \left\| \frac{\bar{d}^-}{d^+}(\zeta_{2,\varepsilon}c^k) + \overline{(\zeta_{2,\varepsilon}c^k)^-} \right\|_{L^2(\Omega_\varepsilon^+)} \\ &\geq \left\| \overline{(\zeta_{2,\varepsilon}c^k)^-} \right\|_{L^2(\Omega_\varepsilon^+)} - \left(1 - \frac{\tau}{2}\varepsilon\right) \|\zeta_{2,\varepsilon}c^k\|_{L^2(\Omega_\varepsilon^+)} \\ &= \frac{\tau}{2}\varepsilon \|\zeta_{2,\varepsilon}c^k\|_{L^2(\Omega_\varepsilon^+)}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{\tau}{2}\varepsilon\left(\frac{1}{3} - O(\varepsilon)\right) &\leq \frac{\tau}{2}\varepsilon\|\zeta_{2,\varepsilon}c^k\|_{L^2(\Omega_\varepsilon^+)} \\ &\leq \left\|\frac{\zeta_{2,\varepsilon}f^k}{d^+}\right\|_{L^2(\Omega_\varepsilon^+)} \leq \frac{2}{\tau}\|f^k\|_{L^2}. \end{aligned}$$

Then we fix a small ε such that the left end is a positive constant. We get

$$\|f^k\|_{L^2} \geq C\tau^2$$

where C is a constant depending only on C^1 -norm of d^\pm . □

Remark 7.10. We should notice that this lower bounded C_0 can be chosen as a continuous function $C_0(\tau, \|d^+\|_{C^1}, \|d^-\|_{C^1})$. Moreover, if we have a sequence of $\{d^{\pm,(k)}\}$ such that the corresponding $\tau^{(k)}, \|d^{\pm,(k)}\|_{C^1}$ are bounded and do not accumulate at 0, then $\inf\{C_0(\tau^{(k)}, \|d^{+,(k)}\|_{C^1}, \|d^{-,(k)}\|_{C^1})\} > 0$.

So far we have proved that \mathcal{T}_{d^+,d^-} is a Fredholm operator. However, we haven't show that the index is a constant when we change (d^+, d^-) . To prove this part, we consider both (d_1^+, d_1^-) and (d_2^+, d_2^-) have nonzero Fourier coefficients from $-M$ to M .

According to the lemma 7.7, we have $\mathcal{T}_{d_i^+,d_i^-}|_{(I-P_{2M})(L^\infty)} : (I - P_{2M})(L^2) \rightarrow (I - P_M)(L^2)$ is injective for $i = 1, 2$. Therefore, we have the quotient map:

$$\bar{\mathcal{T}}_{d_i^+,d_i^-} : L^2/((I - P_{2M})(L^2)) \rightarrow L^2/\mathcal{T}_{d_i^+,d_i^-}((I - P_{2M})(L^2))$$

with $index(\bar{\mathcal{T}}_{d_i^+,d_i^-}) = index(\mathcal{T}_{d_i^+,d_i^-})$ for $i = 1, 2$. However, both $L^2/((I - P_{2M})(L^2))$ and $L^2/\mathcal{T}_{d_i^+,d_i^-}((I - P_M)(L^2))$ are finite dimensional spaces. So

$$index(\bar{\mathcal{T}}_{td_1^+ + (1-t)d_2^+, td_1^- + (1-t)d_2^-})$$

is a constant for all $t \in [0, 1]$.

7.4. **General cases.** Now we turn to the proof of the general case. We will prove the following theorem

Theorem 7.11. *Let*

$$\mathcal{T}_{d^+, d^-}(c) = \bar{d}^- c - d^+ \overline{c^{aps}}$$

be the operator from L^2 to L^2 , with the following constraint:

$$(7.5) \quad |d^+|^2 + |d^-|^2 \neq 0.$$

Moreover, suppose that

$$\|d^+\|_{C^1}, \|d^-\|_{C^1} < \infty.$$

Then we have \mathcal{T}_{d^+, d^-} is a Fredholm operator and the index will be 0.

Step 1. To prove this theorem, notice that we can approximate the operator \mathcal{T}_{d^+, d^-} by a sequence of Fredholm operators $\{\mathcal{T}_{d^{+, (k)}, d^{-, (k)}}\}_{k \in \mathbb{N}}$, where $d^{\pm, (k)}$ are the first k Fourier modes of d^\pm . Since that the Fredholm operators form an open set inside the $Hom(L^2)$, this is insufficient to say that \mathcal{T}_{d^+, d^-} itself is a Fredholm operator. However, recall that we have the following well-known equivalent statement for the Fredholm operators [8].

Lemma 7.12. *Let X be a Hilbert space and $F \in Hom(X)$. Then F is a Fredholm operator iff there is an inverse $S \in Hom(X)$ such that*

$$SF = FS = I \text{ mod}(Com(X))$$

where $Com(X)$ is the subspace(ideal) consisted by all compact operators mapping from X to itself.

Now since $\mathcal{T}_{d^+, (k), d^-, (k)}$ is a Fredholm operator for all $k \in \mathbb{N}$ by proposition 7.3, there exists a sequence of right inverse $\{S^k\}$ such that

$$\mathcal{T}_{d^+, (k), d^-, (k)} S^k = I \text{ mod}(\text{Com}(X)).$$

Suppose that $\|S^k\|$ is bounded uniformly by a number K . For any $\varepsilon > 0$, there exists a constant $N > 0$ such that $\|\mathcal{T}_{d^+, d^-} - T_{d^+, (k), d^-, (k)}\| \leq \varepsilon$ for all $k \geq N$. So we have

$$\mathcal{T}_{d^+, d^-} S^N = T_{d^+, (N), d^-, (N)} S^N + O(\varepsilon) S^N = I + O(\varepsilon) S^N \text{ mod}(\text{Com}(X)).$$

Since $\|O(\varepsilon) S^N\| \leq O(\varepsilon) K$, we can choose ε small enough such that $\|O(\varepsilon) S^N\| \leq \frac{1}{2}$. Therefore we have $I + O(\varepsilon) S^N$ invertible. Let V be the inverse of $I + O(\varepsilon) S^N$, we have

$$\mathcal{T}_{d^+, d^-} S^N V = I \text{ mod}(\text{Com}(X)).$$

So \mathcal{T}_{d^+, d^-} have the inverse $S^N V$ modular the ideal of compact operators. Therefore it is a Fredholm operator.

Step 2. In step 1, we prove that if there is an uniform bound for $\{\|S^k\|\}$, then theorem 7.11 will be immediately true. To prove this claim, we should know how to construct these inverses S^k . In the following paragraphs, we use \mathcal{T}^k to denote the operator $\mathcal{T}_{d^+, (k), d^-, (k)}$ and \mathcal{T} to denote \mathcal{T}_{d^+, d^-} .

A standard way to construct S^k is to use the decomposition $L^2 = N(\mathcal{T}^k) \oplus N(\mathcal{T}^k)^\perp = R(\mathcal{T}^k) \oplus N(\mathcal{T}^{k*})$. By the standard Fredholm alternative, we have $\mathcal{T} : N(\mathcal{T}^k)^\perp \rightarrow R(\mathcal{T}^k)$ is a bijection. Therefore by open mapping theorem (see [10]), there is a bounded inverse map $\hat{S}^k : R(\mathcal{T}^k) \rightarrow N(\mathcal{T}^k)^\perp$. Now, we define S^k to be $\hat{S}^k \circ P_{R(\mathcal{T}^k)}$.

Here we should imitate this idea to construct S^k . Here we know that $\mathcal{T}^k : (I - P_L)(L^2) \rightarrow \mathcal{T}^k((I - P_L)(L^2)) \subset (I - P_{L-k})(L^2)$ is a bijection, where L is the

number given by lemma 7.9. Moreover, we can prove that $\mathcal{T}^k((I - P_{2k})(L^2))$ is a closed subspace by using the argument in step 6,7,8 in section 7.3. Therefore we have an bounded inverse $\hat{R}^k : \mathcal{T}^k((I - P_L)(L^2)) \rightarrow (I - P_L)(L^2)$. Meanwhile, the remark 7.10 tells us that \hat{R}^k have a uniform bounded norm. Now we set our S^k to be $\hat{R}^k \circ P_{\mathcal{T}^k((I - P_L)(L^2))}$. So $\{\|S^k\|\}$ has a uniform bound.

Step 3. Finally, we shall prove that S^k is actually an inverse of \mathcal{T}^k , modular the ideal of compact operators. To prove this, just recall that both $(I - P_L)(L^2)$ and $\mathcal{T}((I - P_L)(L^2))$ are finite codimensional. We denote $(I - P_L)(L^2) = A$ and $\mathcal{T}((I - P_L)(L^2)) = B$ for a while, so $L^2 = A \oplus A^\perp = B \oplus B^\perp$

$$(\mathcal{T}^k S^k - I)(v) = 0 \text{ for any } v \in B.$$

So for any bounded sequence $\{v^k = (v_1^k, v_2^k) \in B \oplus B^\perp = L^2\}$, we have

$$(\mathcal{T}^k S^k - I)(v^k) = (\mathcal{T}^k S^k - I)(v_2^k)$$

where $\{v_2^k\}$ lies in a finite dimensional space B^\perp . We can get a convergence subsequence of $\{v_2^k\}$ easily. This implies

$$(\mathcal{T}^k S^k - I) = 0 \text{ mod}(Com(X)).$$

Similarly, we have $(S^k \mathcal{T}^k - I) = 0 \text{ mod}(Com(X))$, too. Therefore we finish our proof.

Remember that d^\pm are the leading coefficients of a L_1^2 harmonic section, so by proposition 4.7, it is smooth. Meanwhile, notice that \mathcal{T}_{d^+, d^-} maps from L_k^2 to L_k^2 for any $k \in \mathbb{N}$. We can easily show that all these maps are Fredholm by using the same argument.

Especially, we have

Corollary 7.13. *The map $\mathcal{T}_{d^+,d^-} : L_2^2 \rightarrow L_2^2$ is a Fredholm operator.*

In the following sections, we will assume that the domain of \mathcal{T}_{d^+,d^-} is $L_2^2(S^1; \mathbb{C})$.

7.5. Relation between \mathcal{T} and the original equation. Recall that by the argument in subsection 7.2, we want to solve the equation

$$\begin{aligned} d^+ \eta + c &= -2h^+, \\ d^- \bar{\eta} + c^{aps} &= -2h^- \end{aligned}$$

which will give us the equation $\mathcal{T}_{d^+,d^-}(c) = -2(\bar{d}^- h^+ - d^+ \bar{h}^-)$. Here we define the map \mathcal{J} by $\mathcal{J}(h^+, h^-) = -2(\bar{d}^- h^+ - d^+ \bar{h}^-)$ and the map \mathcal{O} , which maps from $\ker(\mathcal{T})$ to $L^2(S^1; \mathbb{C})$, by

$$\mathcal{O}(c) = -\frac{\bar{d}^+ c + d^- \overline{c^{aps}}}{|d^+|^2 + |d^-|^2}.$$

This map will give us η when $h^\pm = 0$.

Now by using the notations in section 7.1, we can always be decomposed the pair $(u^+, u^-) \in L^2(S^1; \mathbb{C}) \times L^2(S^1; \mathbb{C})$ as $\pi^+(u^+, u^-) + \pi^-(u^+, u^-)$. By using this proposition and the Fredholmness of \mathcal{T}_{d^+,d^-} , we can find a finite dimensional vector space $\mathbb{H}_0 \subset Exp^+$ such that $range(\mathcal{T}_{d^+,d^-}) \oplus \mathcal{J}(\mathbb{H}_0) = L^2$

8. PROOF OF THE MAIN THEOREM: PART I

In this section, we will prove the theorem 1.4 in the version without showing f is C^1 . In the next section, we will prove that f is a C^1 map. The argument in this section assumes that the metric g defined on a tubular neighborhood is Euclidean. The general case is more complicated but with the similar argument, see appendix 10.1 for the detail.

8.1. **Definition of \mathbb{K}_1 and \mathbb{K}_0 .** Here we define the finite dimensional spaces \mathbb{K}_1 and \mathbb{K}_0 . By the discussion in section 5.2, we know that the vector bundle \mathcal{E} and \mathcal{F} can be locally trivialized as $\pi_1(\mathcal{N}) \times \mathbb{B}_\varepsilon \times L_1^2$ and $\pi_1(\mathcal{N}) \times \mathbb{B}_\varepsilon \times L^2$ respectively.

Use \mathbb{H}_1 to denote the space $\mathcal{O}(ker(\mathcal{T}_{d^+,d^-}))$. We define the vector spaces as follows

$$\mathbb{K}_1 = \mathbb{H}_1 \times ker(D|_{L_1^2});$$

$$\mathbb{K}_0 = \mathbb{H}_0.$$

where B is defined in proposition 7.1.

Since \mathcal{T}_{d^+,d^-} is a Fredholm operator, $\mathcal{O}(ker(\mathcal{T}_{d^+,d^-}))$ is finite dimensional. Meanwhile, by proposition 2.4, $ker(D|_{L_1^2})$ is also finite dimensional. So \mathbb{K}_1 is finite dimensional.

In fact, the map \mathcal{O} is injective on $ker(\mathcal{T}_{d^+,d^-})$ since the equation

$$\begin{cases} \bar{d}^+ c + d^- \overline{c^{aps}} = 0, \\ \bar{d}^- c - d^+ \overline{c^{aps}} = 0 \end{cases}$$

implies $c = 0$. So $dim(\mathcal{O}(ker(\mathcal{T}_{d^+,d^-}))) = dim(ker(\mathcal{T}_{d^+,d^-})) = dim(\mathbb{H}_1)$. Meanwhile, by definition of \mathbb{H}_0 , we have $dim(\mathbb{H}_0) = dim(coker(\mathcal{T}_{d^+,d^-}))$.

8.2. **Basic setting.** Before we start our argument, we define the following notations first.

Firstly, in the following paragraphs, we fix $\tau < \frac{R}{4}$, $T > 1$ for a moment. The precise values of τ and T will be specified later. Moreover, let us assume that $\|\mathcal{T}_{d^+,d^-}^{-1}|_{range(\mathcal{T}_{d^+,d^-}^{-1})}\| \leq 1$.

Secondly, we suppose that there exists $c_0 > 0$ which is the upper bound for s . The precise value of c_0 can be assumed to decreased between each successive appearance. Now we define the following notations.

Definition 8.1. For any $A \subset M$, we call a section $\mathbf{u} : [0, c_0] \times A \rightarrow \mathcal{S} \otimes \mathcal{I}$ is in $C^\omega([0, c_0]; L_i^2(A; \mathcal{S} \otimes \mathcal{I}))$ if and only if $\|\mathbf{u}(s, \cdot)\|_{L_i^2(A; \mathcal{S} \otimes \mathcal{I})} < \infty$ for all $s \in [0, c_0]$ and $\mathbf{u}(\cdot, x) : [0, c_0] \rightarrow (\mathcal{S} \otimes \mathcal{I})_x$ varies analytically on $[0, c_0]$ (The remainder of Taylor series will converge to zero in L^2 -sense).

Definition 8.2. For any $i \in \mathbb{N}$ $\kappa > 0$, we define

(8.1)

$$\mathfrak{A}_{i+1}^\kappa = \{\mathbf{f} \in C^\omega([0, c_0]; L_{-1}^2(M - N_R; \mathcal{S} \otimes \mathcal{I})) \mid \|\mathbf{f}(s, \cdot)\|_{L_{-1}^2} \leq \frac{\kappa}{T^{\frac{5i}{2}}}\};$$

(8.2)

$$\mathfrak{B}_{i+1}^\kappa = \{\mathbf{f} \in C^\omega([0, c_0]; L^2(N_{\frac{\mathbf{r}}{T^i}} - N_{\frac{\mathbf{r}}{T^{i+1}}}; \mathcal{S} \otimes \mathcal{I})) \mid \|\mathbf{f}(s, \cdot)\|_{L_{-1}^2} \leq \frac{\kappa}{T^{\frac{5i}{2}}}\};$$

(8.3)

$$\mathfrak{C}_{i+1}^\kappa = \{\mathbf{f} \in C^\omega([0, c_0]; L^2(N_{\frac{\mathbf{r}}{T^i}}; \mathcal{S} \otimes \mathcal{I})) \mid \|\mathbf{f}(s, \cdot)\|_{L^2(N_{r_1} - N_{r_2})}^2 \leq \kappa(r_1^3 - r_2^3) \left(\frac{\mathbf{r}}{T^i}\right)^{\frac{1}{4}}\}$$

for all $r_2 < r_1 \leq \frac{\mathbf{r}}{T^i}$.

Thirdly, suppose that we perturb the metrics g on the region $M - N_R$ analytically with the parameter s . Let us call this family of perturbed metric g^s . We use the notation $D_{pert} = D + T^s$ to denote the Dirac operator perturb by metric. The operator $T^s : L^2 \rightarrow L_{-1}^2$ will be a 1st order differential operator with its operator norm $\|T^s\| \leq Cs$.

Therefore we have

$$D_{pert}\psi = s\mathbf{f}_0$$

for some $\mathbf{f}_0 = T^s(\psi) \in C^\omega([0, c_0]; L^2)$.

To prove theorem 1.4, we need to prove the following claim: There exists $\varepsilon > 0$ with the following significance. For any $\xi \in \mathcal{O}(ker(\mathcal{T}_{d^-, d^-}))$ with $\|\xi\|_{L^2} = \varepsilon$ there

exist $\eta_s \in C^\omega([0, c_0]; C^1(S^1; \mathbb{C}))$ and $\mathfrak{k}_s \in \{\mathbf{u} \in L^2 | B(\mathbf{u}) \in \mathbb{H}_0\}$ such that

$$(8.4) \quad D_{pert, \eta_s}(\psi + s\mathfrak{k}_s) = 0$$

for all $s \in [0, c_0]$ with the constraint $\eta_s = s\xi + \eta_s^\perp$ where $\eta_s \perp \mathcal{O}(ker(\mathcal{T}_{d^+, d^-}))$. Moreover, we have to show these data (η_s, \mathfrak{k}_s) will be homeomorphic to a open set in \mathbb{R}^k with $k = dim(ker(D|_{L_1^2}))$. By using this claim, we can define the map f by $f(g^s, s\xi, \hat{\psi}) = B(s\mathfrak{k}_s)$ for any $\hat{\psi} \in dim(ker(D|_{L_1^2}))$ with $\|\hat{\psi}\|_2$ small. Then, I should prove that this map is C^1 .

So I separate my proof into 3 parts. In this section, I will prove that there exists (η_s, \mathfrak{k}_s) satisfying (8.4). In the next section, I will prove the set of data (η_s, \mathfrak{k}_s) satisfying (8.4) will be homeomorphic to a open set in \mathbb{R}^k with $k = dim(ker(D|_{L_1^2}))$ and f is a C^1 -map.

Remark 8.3. In fact, the ξ we choose in our claim can be a smooth map $\xi : [0, c_0] \rightarrow \mathcal{O}(ker(\mathcal{T}_{d^+, d^-}))$ with $\|\xi\|_{L_2^2} = \varepsilon$ and ψ can be replaced by a smooth family $\psi(s) \in ker(D|_{L_1^2})$. The argument in the rest of this section will still hold under this setting.

8.3. Part I of the proof: First order approximation of η_s and \mathfrak{k}_s . Now we are ready to prove our claim. I separate this proof into 10 steps.

Firstly, we define the following smooth function:

$$\chi^* = \begin{cases} 0 & \text{on } N_{\frac{R}{2}} \\ 1 & \text{on } M - N_R \end{cases}.$$

Step 1. In this and the next step, we will denote by κ_0 a $O(1)$ constant. The precise value of κ_0 can be assumed to increase between each successive appearance.

By using proposition 5.3, there exists $\mathfrak{h}_0 \in L^2$ such that

$$D\mathfrak{h}_0 = \mathfrak{f}_0.$$

So we have

$$D_{pert}(\psi - s\mathfrak{h}_0) = -sT^s(\mathfrak{h}_0).$$

Since T^s is a first order differential operator, we have

$$\|T^s(\mathfrak{h}_0)\|_{L^2_{-1}} \leq Cs\|\mathfrak{h}_0\|_{L^2} \leq Cs\|\mathfrak{f}_0\|_{L^2_{-1}}.$$

This implies

$$sT^s(\mathfrak{h}_0) \in s^2\mathfrak{A}_1^{\frac{\kappa_0}{2}}$$

by taking $\kappa_0 \geq 2C\|\mathfrak{f}_0\|_{L^2_{-1}}$ large enough.

Step 2. In this step we construct the data of perturbation η_0 and prove η_0 will satisfy the condition (5.26), (5.27), (5.28) and (5.29).

Since $\mathfrak{f}_0 = 0$ on N_τ , we have $D\mathfrak{h}_0 = 0$ on N_τ . So by proposition 4.4, we can write

$$\mathfrak{h}_0 = \begin{pmatrix} \frac{h_0^+}{\sqrt{z}} \\ \frac{h_0^-}{\sqrt{\bar{z}}} \end{pmatrix} + \mathfrak{h}_{\mathfrak{A},0}.$$

on N_r . By theorem 7.11, there exists (η_0, c_0) such that

$$\begin{cases} 2h_0^+ + d^+\eta_0 + c_0 = k_0^+ \\ 2h_0^- + d^-\bar{\eta}_0 + c_0^{aps} = k_0^- \end{cases}$$

where $(k_0^+, k_0^-) \in \mathbb{H}_0$. So there is a corresponding \mathfrak{c}_0 which satisfies $D\mathfrak{c}_0 = 0$ on $M - \Sigma$ and

$$\mathfrak{c}_0 = \begin{pmatrix} \frac{c_0}{2\sqrt{z}} \\ \frac{c_0^{aps}}{2\sqrt{\bar{z}}} \end{pmatrix} + \mathfrak{c}_{\mathfrak{A},0}.$$

Since we have \mathfrak{h}_0 satisfies $D\mathfrak{h}_0 = \mathfrak{f}_0$ which is given by proposition 5.3, so

$$\mathfrak{r}\|h_0^\pm\|_{L^2}^2, \mathfrak{r}^3\|(h_0^\pm)_t\|_{L^2}^2, \mathfrak{r}^5\|(h_0^\pm)_{tt}\|_{L^2}^2 \leq C\|\mathfrak{h}_0\|_{L^2(N_{\frac{R}{2}})}^2 \leq C\|\mathfrak{f}_0\|_{L^2_{-1}}^2$$

by part **c** of proposition 5.4. By taking $\kappa_0 \geq 2\frac{C}{\mathfrak{r}^{\frac{5}{2}}}\|\mathfrak{f}_0\|_{L^2_{-1}}$, we have

$$(8.5) \quad \begin{aligned} \|h_0^\pm\|_{L^2}^2 &\leq \frac{\kappa_0}{2}\mathfrak{r}^2, \|(h_0^\pm)_t\|_{L^2}^2 \leq \frac{\kappa_0}{2}\mathfrak{r}, \\ \|(h_0^\pm)_{tt}\|_{L^2}^2 &\leq \frac{\kappa_0}{2}, \|\mathfrak{h}_0\|_{L^2} \leq \kappa_0. \end{aligned}$$

Moreover, since $\mathcal{T}_{d^+, d^-}(c_0) = \bar{d}^- h^+ - d^+ \bar{h}^- \text{ mod}(\mathcal{J}(\mathbb{H}_0))$, we can choose c_0 such that

$$(8.6) \quad \begin{aligned} \|c_0\|_{L^2}^2 &\leq \frac{\kappa_0}{2}\mathfrak{r}^2, \|(c_0)_t\|_{L^2}^2 \leq \frac{\kappa_0}{2}\mathfrak{r}, \\ \|(c_0)_{tt}\|_{L^2}^2 &\leq \frac{\kappa_0}{2}, \|\mathfrak{c}_0\|_{L^2} \leq \kappa_0. \end{aligned}$$

So $\eta_0 = \frac{\bar{d}^+}{(|d^+|^2 + |d^-|^2)}(k_0^+ - 2h_0^+ - c_0) + \frac{d^-}{(|d^+|^2 + |d^-|^2)}\overline{(k_0^- - 2h_0^- - c_0^{aps})}$ will satisfy (5.14), (5.15) and (5.16), so it satisfies (5.17), (5.18) and (5.19).

We should notice that the condition $\kappa_0 \geq 2\frac{C}{\mathfrak{r}^{\frac{5}{2}}}\|\mathfrak{f}_0\|_{L^2_{-1}}$ will give us a constraint for g^s . In the following paragraphs, we should always assume $\|\mathfrak{f}_0\|_{L^2_{-1}} \leq \mathfrak{r}^{\frac{5}{2}}$. This assumption will give us some restriction to define \mathcal{N} in theorem 1.4. We will discuss this part in section 8.5.

By this setting, we will also have

$$(8.7) \quad \|k_0^\pm\|_{L^2}^2 \leq \frac{\kappa_0}{2}\mathfrak{r}^2, \|(k_0^\pm)_t\|_{L^2}^2 \leq \frac{\kappa_0}{2}\mathfrak{r}, \|(k_0^\pm)_{tt}\|_{L^2}^2 \leq \frac{\kappa_0}{2}.$$

Furthermore, since

$$\|T^s(\mathfrak{c}_0)\|_{L^2_{-1}} \leq Cs\|\mathfrak{c}_0\|_{L^2} \leq s\frac{\kappa_0}{2}$$

so we have $sT^s(\mathfrak{c}_0) \in s^2\mathfrak{A}_1^{\frac{\kappa_0}{2}}$.

Finally, notice that we still have some options for the choice the c_0 . We can choose another c_0 by adding an element in $\ker(\mathcal{T})$. So we can choose c_0 such that the

corresponding η_0 satisfies $\eta_0 = \xi + \eta_0^\perp$ where $\eta_0^\perp \perp \mathcal{O}(\ker(\mathcal{T}))$. Moreover, we should assume ξ also satisfy (5.14), (5.15) and (5.16). So (8.6) still holds in this case.

Before we move on to the next step, I would like to make some remarks here.

Remark 8.4. We know that η_0 satisfies (5.14), (5.15) and (5.16). By using the same argument in the proof of (5.17), we have

$$\|\eta_0\|_{C^1} \leq C(\|\eta_0\|_{L^2}^2 + \|\eta_0\|_{L^2}\|\eta_0\|_{L^2}) \leq C\kappa_0^2\mathfrak{r}$$

Meanwhile, we can estimate the following Holder seminorm (I follow the standard way to estimate the Holder norms, reader can see [7] for the detail):

$$[|\eta_t|]_{0, \frac{1}{4}} = \sup_{a \neq b} \frac{|\eta_t|(a) - |\eta_t|(b)}{|a - b|^{\frac{1}{4}}}.$$

When $|a - b| \leq \mathfrak{r}$, we have

$$[|\eta_t|]_{0, \frac{1}{4}} \leq \sup_{a \neq b} \left| \frac{1}{|a - b|^{\frac{1}{4}}} \int_a^b \partial_t |\eta_t|(s) ds \right| \leq \sup \|\eta_{tt}\| |a - b|^{\frac{1}{4}} \leq C\kappa_0 \mathfrak{r}^{\frac{1}{4}};$$

when $|a - b| > \mathfrak{r}$, we have

$$[|\eta_t|]_{0, \frac{1}{4}} \leq C \sup |\eta_t| \frac{1}{\mathfrak{r}^{\frac{1}{4}}} \leq C\kappa_0 \mathfrak{r}^{\frac{1}{4}}.$$

So we have the Holder estimate

$$(8.8) \quad \|\eta_0\|_{C^{1, \frac{1}{4}}} \leq C\kappa_0 \mathfrak{r}^{\frac{1}{4}}.$$

Remark 8.5. We should also notice that the choice of (η_0, k_0^\pm) is unique. More precisely, for any $\xi \in \mathcal{O}(\ker(\mathcal{T}))$, the choice of η_0^\perp will be unique.

Step 3. Now we can fix κ_0 forever. In this and the next steps, we will determine another constant $\kappa_1 = O(\kappa_0)$. The precise value of κ_1 can be assumed to increase between each successive appearance. First of all, since η_0 satisfies (5.17) to (5.19), we should assume κ_1 is the constant appearing in these estimates in the beginning.

On N_R we can define

$$\mathfrak{h}_0^b = \chi_0 \begin{pmatrix} \frac{h_0^+}{\sqrt{z}} \\ \frac{h_0^-}{\sqrt{\bar{z}}} \end{pmatrix}; \quad \mathfrak{c}_0^b = \chi_0 \begin{pmatrix} \frac{c_0}{2\sqrt{z}} \\ \frac{c_0^{aps}}{2\sqrt{\bar{z}}} \end{pmatrix}; \quad \mathfrak{k}_0^b = \chi_0 \begin{pmatrix} -i\frac{k_0^+}{\sqrt{z}} \\ i\frac{k_0^-}{\sqrt{\bar{z}}} \end{pmatrix}.$$

We also define $\mathfrak{h}_0^g = \mathfrak{h}_0 - \mathfrak{h}_0^b$ and $\mathfrak{c}_0^g = \mathfrak{c}_0 - \mathfrak{c}_0^b$.

So we have

$$\begin{aligned} D_{pert}(\psi + s\mathfrak{c}_0 - s\mathfrak{h}_0) &= sT^s(\mathfrak{c}_0 - \mathfrak{h}_0) \\ &= D_{pert}(\psi + s\mathfrak{c}_0^g - s\mathfrak{h}_0^g) + D_{pert}(s\mathfrak{c}_0^b - s\mathfrak{h}_0^b) \\ &= D_{pert}(\psi + s\mathfrak{c}_0^g - s\mathfrak{h}_0^g) + D|_{N_R}(s\mathfrak{c}_0^b - s\mathfrak{h}_0^b). \end{aligned}$$

Notice that

$$\begin{aligned} D|_{N_R}(-s\mathfrak{c}_0^b - s\mathfrak{h}_0^b) &= sD|_{N_R}\left(\chi_0 \begin{pmatrix} \frac{-c_0 - 2h_0^+}{2\sqrt{z}} \\ \frac{-c_0^{aps} - 2h_0^-}{2\sqrt{\bar{z}}} \end{pmatrix}\right) \\ &= s\chi_0 \begin{pmatrix} i\frac{\dot{c}_0 + 2h_0^+}{2\sqrt{z}} \\ -i\frac{\dot{c}_0^{aps} + 2h_0^-}{2\sqrt{\bar{z}}} \end{pmatrix} + s\sigma(\chi_0)\mathfrak{c}_0^b - s\sigma(\chi_0)\mathfrak{h}_0^b \\ &= s\chi_0 \begin{pmatrix} -i\frac{d^+\eta_0 + d^+\dot{\eta}_0}{\sqrt{z}} \\ i\frac{d^-\bar{\eta}_0 + d^-\dot{\bar{\eta}}_0}{\sqrt{\bar{z}}} \end{pmatrix} + s\sigma(\chi_0)\mathfrak{c}_0^b - s\sigma(\chi_0)\mathfrak{h}_0^b \\ &\quad - s\mathfrak{k}_0^b - s\sigma(\chi_0)\mathfrak{k}_0^b - sD|_{N_R}(\mathfrak{k}_0^b) \\ &= s\chi_0 \begin{pmatrix} -i\frac{d^+\eta_0}{\sqrt{z}} \\ i\frac{d^-\bar{\eta}_0}{\sqrt{\bar{z}}} \end{pmatrix} + s\chi_0 \begin{pmatrix} -i\frac{d^+\dot{\eta}_0}{\sqrt{z}} \\ i\frac{d^-\dot{\bar{\eta}}_0}{\sqrt{\bar{z}}} \end{pmatrix} + s\sigma(\chi_0)\mathfrak{c}_0^b - s\sigma(\chi_0)\mathfrak{h}_0^b \\ &\quad - s\sigma(\chi_0)\mathfrak{k}_0^b - sD|_{N_R}(\mathfrak{k}_0^b). \end{aligned}$$

where we have

$$s\sigma(\chi_0)\mathbf{c}_0^b - s\sigma(\chi_0)\mathbf{h}_0^b - s\sigma(\chi_0)\mathbf{k}_0^b = s \begin{pmatrix} \chi_{\bar{z}} \frac{c_0^{aps} - h_0^-}{\sqrt{\bar{z}}} \\ -\chi_z \frac{c_0 - h_0^+}{\sqrt{z}} \end{pmatrix} = s \begin{pmatrix} -\chi_{\bar{z}} \bar{\eta}_0 \frac{d^-}{\sqrt{\bar{z}}} \\ \chi_z \eta_0 \frac{d^+}{\sqrt{z}} \end{pmatrix}$$

We can check that

$$s\chi_0 \begin{pmatrix} -i \frac{d^+ \eta_0}{\sqrt{z}} \\ i \frac{d^- \bar{\eta}_0}{\sqrt{\bar{z}}} \end{pmatrix} + s\sigma(\chi_0)\mathbf{c}_0^b - s\sigma(\chi_0)\mathbf{h}_0^b - s\sigma(\chi_0)\mathbf{k}_0^b = \Theta_s^0(\psi).$$

So

$$(8.9) \quad D|_{N_R}(-s\mathbf{c}_0^b - s\mathbf{h}_0^b) = s\chi_0 \begin{pmatrix} -i \frac{d^+ \eta_0}{\sqrt{z}} \\ i \frac{d^- \bar{\eta}_0}{\sqrt{\bar{z}}} \end{pmatrix} + \Theta_s^0(\psi) - D_{pert}(s\mathbf{k}_0^b).$$

Meanwhile, we define

$$\mathbf{e}_0 = \chi_0 \begin{pmatrix} -i d^- \bar{\eta}_0 \sqrt{\bar{z}} \\ -i d^+ \eta_0 \sqrt{z} \end{pmatrix}$$

which satisfies $D(s\mathbf{e}_0) = \chi_0 \begin{pmatrix} -i \frac{d^+ \eta_0}{\sqrt{z}} \\ i \frac{d^- \bar{\eta}_0}{\sqrt{\bar{z}}} \end{pmatrix} + se_1 \partial_t \mathbf{e}_0 + sD(\chi_0) \left(\frac{\mathbf{e}_0}{\chi_0} \right)$. So we can simplify (8.9) as follows

$$(8.10) \quad D|_{N_R}(-s\mathbf{c}_0^b - s\mathbf{h}_0^b) = D(s\mathbf{e}_0) - se_1 \partial_t \mathbf{e}_0 + \Theta_s^0(\psi) - sD(\chi_0) \left(\frac{\mathbf{e}_0}{\chi_0} \right) - D_{pert}(s\mathbf{k}_0^b).$$

Recall that the Dirac operator $D_{s\chi_0\eta_0}$ can be written as

$$\begin{aligned} D_{s\chi_0\eta_0} &= (1 + \varrho^0)D + s((\chi_0)_z \eta_0 + (\chi_0)_{\bar{z}} \bar{\eta}_0) e_1 \partial_t + \Theta_s^0 + \mathcal{A}_s^0 + \mathcal{F}_s^0 + \mathcal{R}_s^0 \\ &= (1 + \varrho^0)D + s((\chi_0)_z \eta_0 + (\chi_0)_{\bar{z}} \bar{\eta}_0) e_1 \partial_t + \hat{\Theta}_s^0 + \mathcal{W}_s^0 + \mathcal{A}_s^0 + \mathcal{F}_s^0 + \mathcal{R}_s^0 \end{aligned}$$

where

$$\hat{\Theta}_s^0 = e_1(s\chi_0\dot{\eta}_0\partial_z + s\chi_0\dot{\bar{\eta}}_0\partial_{\bar{z}}),$$

$$\mathcal{W}_s^0 = e_2(s(\chi_0)_{\bar{z}}\bar{\eta}_0\partial_z - s(\chi_0)_z\bar{\eta}_0\partial_{\bar{z}}) + e_3(-s(\chi_0)_{\bar{z}}\eta_0\partial_z + s(\chi_0)_z\eta_0\partial_{\bar{z}}).$$

We use the following notations to simplify the upcoming equation:

$$(8.11) \quad \mathcal{W}_s^0(s\mathbf{e}_0) + \varrho^0(e_2\partial_z + e_3\partial_{\bar{z}})(s\mathbf{e}_0) = s^2\mathcal{B}_1;$$

$$(8.12) \quad -\mathcal{W}_s^0(s\mathbf{c}_0^g - s\mathbf{h}_0^g) - \varrho^0(e_2\partial_z + e_3\partial_{\bar{z}})(s\mathbf{c}_0^g - s\mathbf{h}_0^g) = s^2\mathcal{B}_2;$$

$$(8.13) \quad \mathcal{W}_s^0(s\mathbf{e}'_0) + \varrho^0(e_2\partial_z + e_3\partial_{\bar{z}})(s\mathbf{e}'_0) = s^2\mathcal{B}_3;$$

$$(8.14) \quad s((\chi_0)_z\eta_0 + (\chi_0)_{\bar{z}}\bar{\eta}_0)e_1\partial_t\psi = s\mathcal{C}_0;$$

$$(8.15) \quad -sD(\chi_0)\left(\frac{\mathbf{e}_0}{\chi_0}\right) + (\varrho^0 - 1)e_1\partial_t(s\mathbf{e}_0) + s((\chi_0)_z\eta_0 + (\chi_0)_{\bar{z}}\bar{\eta}_0)e_1\partial_t(s\mathbf{e}_0) = s\mathcal{C}_1;$$

$$(8.16) \quad -\varrho^0e_1\partial_t(s\mathbf{c}_0^g - s\mathbf{h}_0^g) - s((\chi_0)_z\eta_0 + (\chi_0)_{\bar{z}}\bar{\eta}_0)e_1\partial_t(s\mathbf{c}_0^g - s\mathbf{h}_0^g) = s\mathcal{C}_2;$$

$$(8.17) \quad \varrho^0e_1\partial_t(s\mathbf{e}'_0) + s((\chi_0)_z\eta_0 + (\chi_0)_{\bar{z}}\bar{\eta}_0)e_1\partial_t(s\mathbf{e}'_0) = s\mathcal{C}_3;$$

$$(8.18) \quad \hat{\Theta}_s^0(s\mathbf{e}_0) = s^2\mathcal{Q}_1;$$

$$(8.19) \quad -\hat{\Theta}_s^0(s\mathbf{c}_0^g - s\mathbf{h}_0^g) = s^2\mathcal{Q}_2$$

$$(8.20) \quad \hat{\Theta}_s^0(s\mathbf{e}'_0) = s^2\mathcal{Q}_3.$$

where

$$\mathbf{e}'_0 = -(\chi_z\eta + \chi_{\bar{z}}\bar{\eta})\psi - \begin{pmatrix} 0 & i\chi\dot{\eta} \\ -i\chi\dot{\bar{\eta}} & 0 \end{pmatrix} \psi$$

which has the property

$$D(s\mathbf{e}'_0) = -\mathcal{F}_s^0(\psi).$$

Now by using the fact that $D\psi = 0$, (8.10) yields

$$\begin{aligned} D|_{N_R}(-s\mathbf{c}_0^b - s\mathbf{h}_0^b) &= D(s\mathbf{e}_0) - se_1\partial_t\mathbf{e}_0 + D_{s\chi_0\eta_0}(\psi) - (\mathcal{A}_s^0 + \mathcal{F}_s^0 + \mathcal{R}_s^0)(\psi) + s\mathcal{C}_1 - s\mathbf{k}_0^b \\ &= D_{s\chi_0\eta_0}(s\mathbf{e}_0) + D_{s\chi_0\eta_0}(\psi) - (\mathcal{A}_s^0 + \mathcal{F}_s^0 + \mathcal{R}_s^0)(\psi + s\mathbf{e}_0) \\ &\quad + s^2\mathcal{B}_1 + s\mathcal{C}_0 + s\mathcal{C}_1 - s^2\mathcal{Q}_1 - D_{pert}(s\mathbf{k}_0^b). \end{aligned}$$

Therefore we have

$$\begin{aligned} D_{pert}(\psi + s\mathbf{c}_0 - s\mathbf{h}_0) &= sT^s(\mathbf{c}_0 - \mathbf{h}_0) \\ &= D_{pert}(\psi + s\mathbf{c}_0^g - s\mathbf{h}_0^g + s\mathbf{e}'_0) + D_{s\chi_0\eta_0}(\psi + s\mathbf{e}_0) - \mathcal{F}_s^0(s\mathbf{e}_0) \\ &\quad - (\mathcal{A}_s^0 + \mathcal{R}_s^0)(\psi + s\mathbf{e}_0) + s^2\mathcal{B}_1 + s(\mathcal{C}_0 + \mathcal{C}_1) - s^2\mathcal{Q}_1 - D_{pert}(s\mathbf{k}_0^b). \end{aligned}$$

So

(8.21)

$$\begin{aligned} D_{s\chi_0\eta_0,pert}(\psi + s\mathbf{c}_0^g - s\mathbf{h}_0^g + s\mathbf{e}_0) &= (\mathcal{A}_s^0 + \mathcal{R}_s^0 + \mathcal{F}_s^0)(s\mathbf{e}_0 + s\mathbf{e}'_0 + s\mathbf{c}_0^g - s\mathbf{h}_0^g) \\ &\quad + (\mathcal{A}_s^0 + \mathcal{R}_s^0)(\psi) + sT^s(\mathbf{c}_0 - \mathbf{h}_0) \\ &\quad - s^2\left(\sum_{i=1}^3 \mathcal{B}_i\right) - s\left(\sum_{i=0}^3 \mathcal{C}_i\right) + s^2\left(\sum_{i=1}^3 \mathcal{Q}_i\right) - D_{pert}(s\mathbf{k}_0^b). \end{aligned} \tag{8.22}$$

Here we show that

$$\mathcal{A}_s^0(\psi) + (\mathcal{A}_s^0 + \mathcal{F}_s^0)(s\mathbf{e}_0 + s\mathbf{e}'_0 + s\mathbf{c}_0^g - s\mathbf{h}_0^g) - s\left(\sum_{i=0}^3 \mathcal{C}_i\right) \in s\mathfrak{E}_1^{\kappa_1}, \tag{8.23}$$

$$\mathcal{R}_s^0(\psi + s\mathbf{e}_0 + s\mathbf{e}'_0 + s\mathbf{c}_0^g - s\mathbf{h}_0^g) - s^2\left(\sum_{i=1}^3 \mathcal{B}_i\right) \in s^2\mathfrak{B}_1^{\kappa_1}, \tag{8.24}$$

$$sT^s(\mathbf{c}_0 - \mathbf{h}_0) \in s^2\mathfrak{A}_1^{\kappa_0}. \tag{8.25}$$

We already show that $sT^s(\mathbf{h}_0) \in \mathfrak{A}_1^{\frac{\kappa_0}{2}}$ in step 1 and $sT^s(\mathbf{c}_0) \in \mathfrak{A}_1^{\frac{\kappa_0}{2}}$ in step 2, so we only need to prove (8.23) and (8.24). By the definition of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, proposition

4.4 and Sobolev embedding, we can see that $s\mathcal{C}_3 \in s\mathfrak{C}_i^{\kappa_1}$ for some κ_1 . Meanwhile, by proposition 5.5, we have $\mathcal{A}_s^0(\psi + s\mathfrak{e}_0 + s\mathfrak{e}'_0 + s\mathfrak{c}_0^g - s\mathfrak{h}_0^g) \in s\mathfrak{C}_1^{\kappa_1}$. So we prove (8.23).

Finally, by proposition 5.5, we have

$$\begin{aligned} \|\mathcal{R}_s^0(s\mathfrak{e}_0 + s\mathfrak{e}'_0 + s\mathfrak{c}_0^g - s\mathfrak{h}_0^g)\|_{L^2} &\leq \gamma_T^2 \kappa_1^2 s^3 \|\psi + s\mathfrak{e}_0 + s\mathfrak{c}_0^g - s\mathfrak{h}_0^g\|_{L^2} \\ &\leq \gamma_T^2 \kappa_1^2 s^3 \mathfrak{r}^{\frac{3}{2}} \leq \kappa_1 \mathfrak{r}^{\frac{3}{2}} s^2 \end{aligned}$$

for any $s \leq \frac{1}{\gamma_T^2 \kappa_1}$. Meanwhile, by proposition 5.6, we have

$$\|\mathcal{R}_s^0(\psi)\|_{L^2} \leq C \gamma_T^3 \kappa_1^3 \mathfrak{r}^2 s^2 \leq \kappa_1 \mathfrak{r}^{\frac{3}{2}} s^2$$

for any $\mathfrak{r} \leq \frac{1}{C^2 \gamma_T^6 \kappa_1^4}$. So we prove (8.24).

Step 4. In this step we prove that there exists $e' \in L_1^2$ such that $D_{s\chi_0\eta_0}(s^2 e'_i) = s^2 \mathcal{Q}_i + s^3 \mathcal{B} + s^2 \mathcal{C}$ for some $\mathcal{B} \in \mathfrak{B}_1^{\kappa_1}$ and $\mathcal{C} \in \mathfrak{C}_1^{\kappa_1}$ where $i = 1, 2, 3$.

Here we have to prove the following lemma.

Lemma 8.6. *Let \mathcal{Q} be either of the type $s^2 \chi_0 \begin{pmatrix} \frac{q^+(t)}{\sqrt{z}} \\ \frac{q^-(t)}{\sqrt{\bar{z}}} \end{pmatrix}$ or of the type $s^2 \chi_0 \begin{pmatrix} \frac{q^+(t)}{\sqrt{\bar{z}}} \\ \frac{q^-(t)}{\sqrt{z}} \end{pmatrix}$ where $\|q^\pm\|_{L^2} \leq \kappa_1 \mathfrak{r}$, $\|(q^\pm)_t\|_{L^2} \leq \kappa_1$. Then there exists a L_1^2 section \mathfrak{e}' which can be written as*

$$\mathfrak{e}' = \sum_{i \geq 2} s^i \chi_0^i \begin{pmatrix} e_i^+(t) \sqrt{\bar{z}} \\ e_i^-(t) \sqrt{z} \end{pmatrix}$$

for the first type and

$$\mathfrak{e}' = \sum_{i \geq 0} s^i \chi_0^i \begin{pmatrix} e_i^+(t) \sqrt{z} \\ e_i^-(t) \sqrt{\bar{z}} \end{pmatrix}$$

for the second such that $D_{s\chi_0\eta_0}(s^2 \mathfrak{e}') = s^2 \mathcal{Q} + s^2 \mathcal{B} + s^2 \mathcal{C}$ for some $\mathcal{B} \in \mathfrak{B}_1^{\kappa_1}$ and $\mathcal{C} \in \mathfrak{C}_1^{\kappa_1}$

for all $s \leq \frac{1}{2\gamma_T^2 \kappa_1 \mathfrak{r}^{\frac{1}{2}}}$. Furthermore, we have $\|\mathfrak{e}'\|_{L_1^2} \leq 2\kappa_1$.

Proof. First of all, let \mathcal{Q} is of the first type. We start with the element

$$\mathbf{e}'_0 = \chi_0 \begin{pmatrix} q^- \sqrt{\bar{z}} \\ q^+ \sqrt{z} \end{pmatrix}.$$

Under a straight-forward direct computation, we have

$$D(s^2 \mathbf{e}'_0) = s^2 \mathcal{Q} + s^2 \mathcal{B} + s^2 \mathcal{C}$$

with $\mathcal{B} \in \mathfrak{B}_1^{s\gamma_T^2 \kappa_1^2}$ and $\mathcal{C} \in \mathfrak{C}_1^{s\kappa_1}$. Recall that by proposition 5.5, we have

$$D_{s\chi_0 \eta_0} = (1 + \varrho^0)D + s((\chi_0)_z \eta_0 + (\chi_0)_{\bar{z}} \bar{\eta}_0) e_1 \partial_t + \Theta_s^0 + \mathcal{A}_s^0 + \mathcal{F}_s^0 + \mathcal{R}_s^0.$$

By the argument proving the results (8.23) and (8.24), we have

$$\begin{aligned} s((\chi_0)_z \eta_0 + (\chi_0)_{\bar{z}} \bar{\eta}_0) e_1 \partial_t (s^2 \mathbf{e}'_0) + (\mathcal{A}_s^0 + \mathcal{F}_s^0)(s^2 \mathbf{e}'_0) &\in s^2 \mathfrak{C}_1^{s\kappa_1^2} \\ \varrho^0 D(s^2 \mathbf{e}'_0) + \mathcal{R}_s^0(s^2 \mathbf{e}'_0) &\in s^2 \mathfrak{B}_1^{s^2 \gamma_T^2 \kappa_1^3}. \end{aligned}$$

Meanwhile, recall that $\Theta_s^0 = [e_1(s\chi\dot{\eta}\partial_z + s\chi\dot{\bar{\eta}}\partial_{\bar{z}}) + e_2(s\chi_{\bar{z}}\bar{\eta}\partial_z - s\chi_z\eta\partial_{\bar{z}}) + e_3(-s\chi_{\bar{z}}\eta\partial_z + s\chi_z\bar{\eta}\partial_{\bar{z}})]$. Here we recall the decomposition

$$\Theta_s^0 = \hat{\Theta}_s^0 + \mathcal{W}_s^0.$$

Notice that \mathcal{W}_s^0 is a $O(s\kappa_1)$ -first order differential operator with its support on $N_{\tau} - N_{\frac{\tau}{2}}$, which implies $\mathcal{W}_s^0(s^2 \mathbf{e}'_0) \in s^2 \mathfrak{B}_1^{s\gamma_T^2 \kappa_1^2}$. So we have

$$\Theta_s^0(s^2 \mathbf{e}'_0) = \hat{\Theta}_s^0(s^2 \mathbf{e}'_0) + s^3 \mathcal{B}$$

for some $\mathcal{B} \in \mathfrak{B}_1^{s\gamma_T^2 \kappa_1^2}$. Moreover, since

$$\Theta_s^0(s^2 \mathbf{e}'_0) = \chi_0 \Theta_s^0(s^2 \frac{\mathbf{e}'_0}{\chi_0}) + \Theta_s^0(\chi_0) s^2 \frac{\mathbf{e}'_0}{\chi_0}$$

and the second term is in $s^2\mathfrak{B}_1^{s\gamma_T^2\kappa_1^2}$, so we have

$$\Theta_s^0(s^2\mathbf{e}'_0) = \chi_0\Theta_s^0\left(\frac{\mathbf{e}'_0}{\chi_0}\right) + s^2\mathcal{B}$$

for some $\mathcal{B} \in \mathfrak{B}_1^{s\gamma_T^2\kappa_1^2}$.

Now we call $\mathcal{Q}_1 = \chi_0\Theta_s^0\left(\frac{\mathbf{e}'_0}{\chi_0}\right)$, which can be simplified as

$$\mathcal{Q}_1 = s\chi_0^2 \begin{pmatrix} \frac{q_1^+(t)}{\sqrt{\bar{z}}} \\ \frac{q_1^-(t)}{\sqrt{z}} \end{pmatrix}$$

where

$$\begin{aligned} q_1^+ &= -i(\chi_0\dot{\eta}_0)q^+ \\ q_1^- &= -i(\chi_0\dot{\eta}_0)q^-. \end{aligned}$$

By using the fact $\|q^\pm\|_{L^2} \leq \kappa_1\mathbf{r}$, $\|(q^\pm)_t\|_{L^2} \leq \kappa_1$, fundamental theorem of calculus and Holder's inequality, we have $\|q^\pm\|_{L^\infty} \leq C\kappa_1\mathbf{r}^{\frac{1}{2}}$. Therefore by using (5.16), (5.17), we have $\|q_1^\pm\|_{L^2} \leq \kappa_1^2\mathbf{r}^{\frac{3}{2}}$, $\|(q_1^\pm)_t\|_{L^2} \leq \kappa_1^2\mathbf{r}^{\frac{1}{2}}$. So we have

$$(8.26) \quad D_{s\chi_0\eta_0}(s^2\mathbf{e}'_0) = s^2\mathcal{Q}_1 + s^2\mathcal{B}_0 + s^2\mathcal{C}_0$$

for some $\mathcal{B}_0 \in \mathfrak{B}_1^{s\gamma_T^2\kappa_1^2\mathbf{r}}$ and $\mathcal{C}_0 \in \mathfrak{C}_1^{s\kappa_1\mathbf{r}^{\frac{1}{2}}}$.

Here we define a $L^2(S^1; \mathbb{C})$ -module \mathbb{V} which is generalized by

$$\left\{ \begin{pmatrix} z^a \bar{z}^b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z^a \bar{z}^b \end{pmatrix} \mid (a, b) \in (\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z} \text{ or } (a, b) \in \mathbb{Z} \times (\mathbb{Z} + \frac{1}{2}), a + b = \frac{1}{2} \right\}.$$

Now, we define a linear map J by the following rule:

$$J \begin{pmatrix} q^+ z^a \bar{z}^b \\ q^- z^b \bar{z}^a \end{pmatrix} = \begin{pmatrix} -i\dot{\eta}_0 q^+ z^b \bar{z}^a \\ i\dot{\eta}_0 q^- z^a \bar{z}^b \end{pmatrix} + \frac{b}{a+1} \begin{pmatrix} -i\dot{\eta}_0 q^+ z^{b-1} \bar{z}^{a+1} \\ i\dot{\eta}_0 q^- z^{a+1} \bar{z}^{b-1} \end{pmatrix}$$

This map is not will defined on the entire \mathbb{V} since it makes no sense when $a = -1$. However, if we start with $x = \begin{pmatrix} q^+ z^a \bar{z}^b \\ q^- z^b \bar{z}^a \end{pmatrix}$ with $(a, b) = (\frac{1}{2}, 0)$ or $(a, b) = (0, \frac{1}{2})$, we can always define $J^n(x)$ for any n . Here we call the term $x = \begin{pmatrix} q^+ z^a \bar{z}^b \\ q^- z^b \bar{z}^a \end{pmatrix}$ is of the type (a, b) . To prove that $J^n(x)$ is well-defined for all n when x is of the type $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$, we should prove that there is no term in $J^n(x)$ which is of the type $(-1, \frac{3}{2})$ or $(\frac{3}{2}, -1)$. We show this fact inductively. When $n = 0$, this statement is obvious true. Suppose there exists a smallest $n \in \mathbb{N}$ such that $J^n(x)$ has a component of the type $(-1, \frac{3}{2})$ or $(\frac{3}{2}, -1)$. For the first case that the component appearing in $J^n(x)$ is of the type $(-1, \frac{3}{2})$, it must generated from a component in $J^{n-1}(x)$ of the type $(\frac{3}{2}, -1)$, which is a contradiction. For the second case that the component appearing in $J^n(x)$ is of the type $(\frac{3}{2}, -1)$, either this component comes from a component in $J^{n-1}(x)$ of the type $(-1, \frac{3}{2})$, which is a contradiction again, or it comes from a component in $J^{n-1}(x)$ of the type $(\frac{5}{2}, -2)$. The later case is also impossible because we start from the term of the type $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ and each time we apply J on it will only change (a, b) by $(\pm 1, \pm 1)$. So there must be a number $m < n - 1$ such that $J^m(x)$ contains a component of the type $(-1, \frac{3}{2})$ or $(\frac{3}{2}, -1)$, which leads a contradiction. Therefore all the component in $J^n(x)$ are not of the type $(-1, \frac{3}{2})$, which means $J^n(x)$ is well-defined for all n .

Now we define \mathbf{e}'_k inductively by

$$\mathbf{e}'_k = s\chi_0^{k+1} J\left(\frac{\mathbf{e}'_{k-1} - \mathbf{e}'_{k-2}}{\chi_0^k}\right) + \mathbf{e}'_{k-1}.$$

By induction hypothesis, suppose that $\mathbf{e}_k \in L_1^2$ such that

$$D_{s\chi_0\eta_0}(s^2\mathbf{e}'_k) = s^2\mathcal{Q}_{k+1} + s^2\mathcal{B}_k + s^2\mathcal{C}_k$$

where $\mathcal{B}_k \in \mathfrak{B}_1^{\sum_{j=0}^k s^{k+1}(k+1)\gamma_T^2\kappa_1^2\tau}$, $\mathcal{C}_k \in \mathfrak{C}_1^{\sum_{j=0}^k s^{k+1}(k+1)\kappa_1\tau^{\frac{1}{2}}}$. The first term on the right, \mathcal{Q}_{k+1} , can be written as

$$\mathcal{Q}_{k+1} = \chi_0^{k+1} \hat{\Theta}_s^0 \left(\frac{\mathbf{e}'_k - \mathbf{e}'_{k-1}}{\chi_0^{k+1}} \right).$$

By taking $s < \frac{1}{\kappa_1\tau^{\frac{1}{2}}}$, we can see that the sequence $\{\mathbf{e}_k\}$ will converge in L_1^2 sense to some \mathbf{e}' . Meanwhile, we can see that

$$D_{s\chi_0\eta_0}(s^2\mathbf{e}'_{k+1}) = \chi_0^{k+2} \hat{\Theta}_s^0 \left(s^2 \frac{(\mathbf{e}'_{k+1} - \mathbf{e}'_k)}{\chi_0^{k+2}} \right) + s^2 \delta\mathcal{B}_{k+1} + s^2 \delta\mathcal{C}_{k+1} + s^2\mathcal{B}_k + s^2\mathcal{C}_k.$$

where $\delta\mathcal{B}_{k+1} \in \mathfrak{B}_1^{s^{k+2}(k+2)\gamma_T^2\kappa_1^2}$ and $\delta\mathcal{C}_{k+1} \in \mathfrak{C}_1^{s^{k+2}(k+2)\kappa_1}$. We define inductively that $\mathcal{B}_{k+1} = \delta\mathcal{B}_{k+1} + \mathcal{B}_k$, $\mathcal{C}_{k+1} = \delta\mathcal{C}_{k+1} + \mathcal{C}_k$ and

$$\chi_0^{k+2} \hat{\Theta}_s^0 \left(\frac{(\mathbf{e}'_{k+1} - \mathbf{e}'_k)}{\chi_0^{k+2}} \right) = \mathcal{Q}_{k+2}.$$

Furthermore, if we take s small enough such that $\sum_{j=0}^{\infty} s^{k+1}(k+1) = \frac{s}{(1-s)^2} \leq \frac{1}{\gamma_T^2\kappa_1}$ e.g. $s \leq \frac{3-\sqrt{5}}{2} \frac{1}{\gamma_T^2\kappa_1}$, then we have $\mathcal{B}_k \rightarrow \mathcal{B} \in \mathfrak{B}_1^{\kappa_1}$ and $\mathcal{C}_k \rightarrow \mathcal{C} \in \mathfrak{C}_1^{\kappa_1}$.

Therefore, by taking $k \rightarrow \infty$, we finish our proof by induction.

To get the L_1^2 -estimate of \mathbf{e}' , we notice that

$$\|\mathbf{e}'_{k+1} - \mathbf{e}'_k\|_{L_1^2} = \|s^{k+1} \chi_0^{k+2} J^k \left(\frac{\mathbf{e}'_0}{\chi_0} \right)\|_{L_1^2} \leq \frac{1}{2^k} \kappa_1$$

by using the fact $\|q_{k+1}^{\pm}\|_{L^2} \leq \frac{\kappa_1^{k+1}\tau}{4}$ and $\|(q_{k+1}^{\pm})_t\| \leq \kappa_1^{k+2}$. So we have $\|\mathbf{e}'\|_{L_1^2} \leq 2\kappa_1$ \square

Now we apply this lemma to the \mathcal{Q}_1 , \mathcal{Q}_2 and \mathcal{Q}_3 in (8.18), (8.19) and (8.20), we can find \mathbf{e}'_1 and \mathbf{e}'_2 such that

$$D_{s\chi_0\eta_0}(s^2\mathbf{e}'_i) = s^2\mathcal{Q}_i + s^2\mathcal{B} + s\mathcal{C} \text{ for } i = 1, 2.$$

For \mathcal{Q}_3 , we notice that

$$s^2 \mathcal{Q}_3 = \hat{\Theta}_s^0 (s\chi_z \eta + s\chi_{\bar{z}} \bar{\eta}) \psi + \hat{\Theta}_s^0 \begin{pmatrix} 0 & i\chi \dot{\eta} \\ -i\chi \dot{\eta} & 0 \end{pmatrix} \psi.$$

the first term is in $s\mathfrak{C}_1^{\kappa_1}$ and the second term is the first type of lemma 8.6. So there exists \mathfrak{e}'_3 such that

$$D_{s\chi_0 \eta_0} (s^2 \mathfrak{e}'_3) = s^2 \mathcal{Q}_3 + s^2 \mathcal{B} + s\mathcal{C}.$$

Finally, we can prove that $D_{pert}(s\mathfrak{k}_0^b) = D_{s\chi_0 \eta_0, pert}(s\mathfrak{k}_{0,s}^b)$ for some $\hat{\mathfrak{k}}_{0,s}^b \in L^2$ by proposition 5.9. Furthermore, we can decompose $\hat{\mathfrak{k}}_{0,s}^b = \mathfrak{k}_{0,s}^b + s\mathfrak{k}_{0,s}^\perp$ where $B(\mathfrak{k}_{0,s}^b) \in \mathbb{H}_0$ and $B(\mathfrak{k}_{0,s}^\perp) \in \mathbb{H}_0^\perp$. Again, by proposition 5.4, we have the following estimates for $B(\mathfrak{k}_{0,s}^\perp)$:

$$(8.27) \quad \|B(\mathfrak{k}_{0,s}^\perp)\|_{L^2}^2 \leq \frac{\kappa_0}{2} \mathfrak{r}^2, \|(B(\mathfrak{k}_{0,s}^\perp))_t\|_{L^2}^2 \leq \frac{\kappa_0}{2} \mathfrak{r}, \|(B(\mathfrak{k}_{0,s}^\perp))_{tt}\|_{L^2}^2 \leq \frac{\kappa_0}{2}.$$

Therefore we can rewrite (8.22) as

$$(8.28) \quad D_{s\chi_0 \eta_0, pert}(\psi - s\mathfrak{c}_0^g - s\mathfrak{h}_0^g - s\mathfrak{e}_0^g + s\mathfrak{k}_{0,s}^b + s\mathfrak{k}_{0,s}^\perp) = s^2 \mathcal{A} + s^2 \mathcal{B} + s\mathcal{C}$$

where $\mathfrak{e}_0^g = \mathfrak{e}_0 + \mathfrak{e}'_0 + s \sum_{i=1}^3 \mathfrak{e}'_i$, $\mathcal{A} \in \mathfrak{A}_1^{\kappa_0}$, $\mathcal{B} \in \mathfrak{B}_1^{\kappa_1}$ and $\mathcal{C} \in \mathfrak{C}_1^{\kappa_1}$. We give $-s\mathfrak{c}_0^g - s\mathfrak{h}_0^g - s\mathfrak{e}_0^g + s\mathfrak{k}_{0,s}^b$ a name \mathfrak{k}^0 .

Now we can fix κ_1 forever.

8.4. Part I of the proof: Iteration of $(\eta_i, (\mathfrak{c}_i^g, \mathfrak{h}_i^g, \mathfrak{e}_i^g, \mathfrak{k}_{i,s}^b, \mathfrak{k}_{i,s}^\perp), \mathfrak{f}_i)$. In this subsection we will construct an iterative process by determining the following two constants $1 > \mathfrak{r}$ and $P \in (T^{\frac{1}{8}} + 1, T^{\frac{1}{5}})$ where $T > 512$ is any fix number. We will also use another constant $\varepsilon > 0$ which depends only on \mathfrak{r} . In addition, we will also give the upper bound for c_0 . We divide our argument into the following 6 steps.

Step 1. Suppose we have $\psi_i = \psi + s\mathfrak{k}^i \in L^2$ satisfies

$$(8.29) \quad D_{s\eta^i, pert}(\psi_i + s^2\mathfrak{k}_{i,s}^\perp) = s\mathfrak{f}_i$$

where $\eta^i = \sum_{j=0}^i \chi_j \eta_j$. Moreover, we assume the following conditions:

(8.30) **Inductive Assumptions**

1. $s\mathfrak{f}_i$ can be decomposed as

$$s\mathfrak{f}_i = s^2\mathfrak{f}_{i,A} + s^2\mathfrak{f}'_{i,B} + s\mathfrak{f}_{i,C}$$

where $\mathfrak{f}_{i,A} \in \mathfrak{A}_i^{P^i\kappa_0}$, $\mathfrak{f}'_{i,B} \in \mathfrak{B}_i^{P^i\kappa_1}$ and $\mathfrak{f}_{i,C} \in \mathfrak{C}_i^{P^i\kappa_1}$.

2. The sequence $\{(\chi_j, \eta_j)\}_{1 < j \leq i}$ satisfies (5.26), (5.27), (5.28) with $\kappa_2 = \varepsilon P^j \kappa_0$.

3. We have $\mathfrak{k}^i = \sum_{j=0}^i (-s\mathfrak{c}_j^g - s\mathfrak{h}_j^g - s\mathfrak{e}_j^g + \mathfrak{k}_{j,s}^b)$ and $\{\mathfrak{k}^i\}$ converges in L^2 sense.

In fact, $\sum_{j=0}^i (-s\mathfrak{c}_j^g - s\mathfrak{h}_j^g - s\mathfrak{e}_j^g)$ converges in L_1^2 -sense.

In order to do the iteration, we need to construct the following data

$$\begin{aligned} & (\eta_{i+1}, (\mathfrak{c}_{i+1}^g, \mathfrak{h}_{i+1}^g, \mathfrak{e}_{i+1}^g, \mathfrak{k}_{i+1,s}^b, \mathfrak{k}_{i+1,s}^\perp), \mathfrak{f}_{i+1}) \\ & \in L^2(S^1; \mathbb{C}) \times (L_1^2)^3 \times (L^2)^2 \times (s^2\mathfrak{A}_{i+1}^{P^i\kappa_0} + s^2\mathfrak{B}_{i+1}^{P^i\kappa_0} + s\mathfrak{C}_{i+1}^{P^i\kappa_0}) \end{aligned}$$

form all previous data $\{(\eta_j, (\mathfrak{c}_j^g, \mathfrak{h}_j^g, \mathfrak{e}_j^g, \mathfrak{k}_{j,s}^b, \mathfrak{k}_{j,s}^\perp), \mathfrak{f}_j)\}_{j \leq i}$. We will show that all conditions in (8.30) will be satisfied inductively.

Step 2. In this step, we will construct \mathfrak{h}_{i+1} and determine the constant c_0 in terms of ε , \mathfrak{r} and T . First of all, since $\mathfrak{f}_{i,C} \in \mathfrak{C}_i^{P^i \kappa_1}$ so we have

$$\chi_{i+1} \mathfrak{f}_{i,C} \in \mathfrak{C}_{i+1}^{T^{\frac{1}{8}} P^i \kappa_1}$$

and

$$(1 - \chi_{i+1}) \mathfrak{f}_{i,C} \in \mathfrak{B}_i^{P^i \kappa_1}.$$

Now we can rewrite

$$(8.31) \quad s \mathfrak{f}_i = s^2 \mathfrak{f}_{i,A} + s \mathfrak{f}_{i,B} + s \epsilon_i$$

where $\mathfrak{f}_{i,A} \in \mathfrak{A}_i^{P^i \kappa_0}$, $\mathfrak{f}_{i,B} = s \mathfrak{f}'_{i,B} + (1 - \chi_{i+1}) \mathfrak{f}_{i,C}$ and $\epsilon_i := \chi_{i+1} \mathfrak{f}_{i,C} \in \mathfrak{C}_{i+1}^{T^{\frac{1}{8}} P^i \kappa_1}$.

Before we start to solve \mathfrak{h}_{i+1} , we need to show that $\mathfrak{f}_{i,B} \in \frac{\varepsilon \mathfrak{r}^{\frac{5}{2}}}{4T^{\frac{5}{2}}} \mathfrak{B}_i^{P^i \kappa_0}$.

Firstly, by taking s small enough we will have $s \mathfrak{f}'_{i,B} \in \frac{\varepsilon \mathfrak{r}^{\frac{5}{2}}}{8T^{\frac{5}{2}}} \mathfrak{B}_i^{P^i \kappa_0}$. This fact can be achieved if we assume $c_0 \leq \frac{\varepsilon \mathfrak{r}^{\frac{5}{2}}}{8T^{\frac{5}{2}}} (\frac{\kappa_0}{\kappa_1})$.

Secondly, by lemma 2.6, for any $\zeta \in L_1^2$ and $\|\zeta\|_{L_1^2} = 1$, we have

$$\begin{aligned} \left| \int \langle \zeta, (1 - \chi_{i+1}) \mathfrak{f}_{i,C} \rangle \right| &= \left| \int \langle (1 - \chi_{i+1}) \zeta, \mathfrak{f}_{i,C} \rangle \right| \\ &\leq C \frac{\mathfrak{r}}{T^i} \|\mathfrak{f}_{i,C}\|_{L^2} \\ &\leq C \left(\frac{\mathfrak{r}}{T^i} \right)^{\frac{5}{2}} P^i \kappa_1 \left(\frac{\mathfrak{r}}{T^i} \right)^{\frac{1}{8}} \\ &\leq \frac{\varepsilon}{8} P^i \kappa_0 \left(\frac{\mathfrak{r}}{T^{i+1}} \right)^{\frac{5}{2}} \end{aligned}$$

by taking \mathfrak{r} small enough. Therefore we have $\|(1 - \chi_{i+1}) \mathfrak{f}_{i,C}\|_{L^2_{-1}} \leq \frac{\varepsilon P^i \kappa_0}{8T^{\frac{5(i+1)}{2}}}$, which implies that $\mathfrak{f}_{i,B} \in \frac{\varepsilon}{4T^{\frac{5}{2}}} \mathfrak{B}_i^{P^i \kappa_0}$.

Suppose (8.29) and (8.31) are true for i . We can solve

$$D_{s\eta^i} \mathfrak{h}_{i+1,A} = s\mathfrak{f}_{i,A}$$

$$D_{s\eta^i} \mathfrak{h}_{i+1,B} = \mathfrak{f}_{i,B}$$

by using proposition 5.9. Since $\mathfrak{f}_{i,A}|_{N_{\frac{R}{T^{i+1}}}} = 0$ and $\mathfrak{f}_{i,B}|_{N_{\frac{R}{T^{i+1}}}} = 0$, we have

$$\begin{aligned} & \left(\frac{\mathfrak{r}}{T^{i+1}}\right) \|h_{i+1,A}^\pm\|_{L^2}^2, \left(\frac{\mathfrak{r}}{T^{i+1}}\right)^3 \|(h_{i+1,A}^\pm)_t\|_{L^2}^2, \\ & \left(\frac{\mathfrak{r}}{T^{i+1}}\right)^5 \|(h_{i+1,A}^\pm)_{tt}\|_{L^2}^2 \leq \|\mathfrak{h}_{i+1,A}\|_{L^2}^2 \leq s^2 \|\mathfrak{f}_{i,A}\|_{L_{-1}^2}^2 \leq s^2 \frac{P^{2i} \kappa_0^2}{T^{5i}}. \end{aligned}$$

This implies that

$$(8.32) \quad \begin{aligned} \|h_{i+1,A}^\pm\|_{L^2} &\leq \frac{\varepsilon P^i \kappa_0}{4T^{2(i+1)}}, \quad \|(h_{i+1,A}^\pm)_t\|_{L^2} \leq \frac{\varepsilon P^i \kappa_0}{4T^{i+1}}, \\ \|(h_{i+1,A}^\pm)_{tt}\|_{L^2} &\leq \frac{\varepsilon P^i \kappa_0}{4}, \quad \|\mathfrak{h}_{i+1,A}\|_{L^2} \leq \frac{\varepsilon P^i \kappa_0}{4T^{\frac{5(i+1)}{2}}} \end{aligned}$$

by taking $c_0 \leq \frac{\varepsilon}{4} \left(\frac{\mathfrak{r}}{T}\right)^{\frac{5}{2}}$.

Meanwhile, we have

$$(8.33) \quad \begin{aligned} \|h_{i+1,B}^\pm\|_{L^2} &\leq \frac{\varepsilon P^i \kappa_0}{4T^{2(i+1)}}, \quad \|(h_{i+1,B}^\pm)_t\|_{L^2} \leq \frac{\varepsilon P^i \kappa_0}{4T^{(i+1)}}, \\ \|(h_{i+1,B}^\pm)_{tt}\|_{L^2} &\leq \frac{\varepsilon P^i \kappa_0}{4}, \quad \|\mathfrak{h}_{i+1,B}\|_{L^2} \leq \frac{\varepsilon P^i \kappa_0}{4T^{\frac{5(i+1)}{2}}}. \end{aligned}$$

So we put these data together, denote $\overline{\mathfrak{h}}_{i+1} = \mathfrak{h}_{i+1,A}^\pm + \mathfrak{h}_{i+1,B}^\pm - s\mathfrak{k}_{i,s}^\perp$. Then we have

$$D_{s\eta^i, \text{pert}}(\psi_i - s\mathfrak{h}_{i+1}) = sT^s(\mathfrak{h}_{i+1}).$$

$sT^s(\mathfrak{h}_{i+1})$ is of order $O(s^2)$, which can be written as a term $sT^s(\mathfrak{h}_{i+1}) \in \mathfrak{A}_{i+1}^{P^{i+1} \frac{\kappa_0}{2}}$.

Step 3. Now we find $\eta_{i+1} \in \mathcal{O}(\ker(\mathcal{T}))^\perp$, c_{i+1} and its corresponding L^2 -harmonic section \mathbf{c}_{i+1} such that

$$\begin{cases} 2h_{i+1}^+ + d^+ \eta_{i+1} + c_{i+1} = k_{i+1}^+ \\ 2h_{i+1}^- + d^- \bar{\eta}_{i+1} + c_{i+1}^{aps} = k_{i+1}^- \end{cases}$$

for some $(k_{i+1}^+, k_{i+1}^-) \in \mathbb{H}_0$ and

$$(8.34) \quad \|k_{i+1}^\pm\|_{L^2}^2 \leq \frac{\varepsilon P^i \kappa_0}{2T^{2(i+1)}}, \quad \|(k_{i+1}^\pm)_t\|_{L^2}^2 \leq \frac{\varepsilon P^i \kappa_0}{2T^{i+1}}, \quad \|(k_{i+1}^\pm)_{tt}\|_{L^2}^2 \leq \frac{\varepsilon P^i \kappa_0}{2}.$$

By using proposition 5.9, there exists c_{i+1} where $D_{s\eta^i} \mathbf{c}_{i+1} = 0$ and

$$\mathbf{c}_{i+1} = \begin{pmatrix} \frac{c_{i+1}}{2\sqrt{z}} \\ \frac{c_{i+1}^{aps}}{2\sqrt{\bar{z}}} \end{pmatrix} + \mathbf{c}_{\mathfrak{R},i+1} + \mathbf{c}_{i+1}^s.$$

Moreover, since c_{i+1} satisfies $\mathcal{T}_{d^+,d^-}(c_{i+1}) = \bar{d}^-(k_{i+1}^+ - 2h_{i+1}^+) - d^+(\overline{k_{i+1}^- - 2h_{i+1}^-})$, we have

$$(8.35) \quad \begin{aligned} \|c_{i+1}\|_{L^2}^2 &\leq \frac{\varepsilon P^i \kappa_0}{2T^{2(i+1)}}, \quad \|(c_{i+1})_t\|_{L^2}^2 \leq \frac{\varepsilon P^i \kappa_0}{2T^{i+1}}, \\ \|(c_{i+1})_{tt}\|_{L^2}^2 &\leq \frac{\varepsilon P^i \kappa_0}{2}, \quad \|\mathbf{c}_{i+1}\|_{L^2} \leq \varepsilon P^i \kappa_0. \end{aligned}$$

According to these estimates, we can show that $sT^s(\mathbf{c}_{i+1}) \in \mathfrak{A}_{i+1}^{P^{i+1} \frac{\kappa_0}{2}}$.

Meanwhile we can easily check that η_{i+1} satisfies $i+1$ -th version of (5.26), (5.27) and (5.28) with $(\kappa_2, \kappa_3) = (\varepsilon P^i \kappa_0, \varepsilon P^i \kappa_1)$ and so does it satisfies the condition (5.29), (5.30) and (5.31). So the inductive assumption 2 in (8.30) is true. Also, we have the $\kappa_3 = \varepsilon P^i \kappa_1$ version of proposition 5.7 and proposition 5.8. Therefore we have

$$(8.36) \quad \int_{\{r=r_0\}} |\hat{\mathcal{A}}_s^{i+1}|^2 i_{\bar{n}} dVol(M) \leq \gamma_T^4 \varepsilon^4 P^{4i} \kappa_1^4 \left(\frac{\mathbf{r}}{T^{i+1}}\right)^{\frac{15}{16}} s^4 \leq \varepsilon^2 P^{2i} \kappa_1^2 \left(\frac{\mathbf{r}}{T^{i+1}}\right)^{\frac{1}{2}} s^2.$$

by taking $P \leq T^{\frac{1}{5}}$ and s small enough.

Remark 8.7. Here we show the estimate of the Holder norm of η_i . By the argument similar to remark 8.4, we have the following Holder estimate

$$(8.37) \quad \|\eta_i^\perp\|_{C^{1, \frac{1}{4}}} \leq C\kappa_0 P^i \left(\frac{\mathbf{r}}{T^i}\right)^{\frac{1}{4}} \leq C\kappa_0 T^{\frac{i}{5}} \left(\frac{\mathbf{r}}{T^i}\right)^{\frac{1}{4}} \leq C\kappa_0 \frac{\mathbf{r}^{\frac{1}{4}}}{T^{\frac{i}{20}}}.$$

for all i .

Step 4. In this step and the next step, we construct \mathbf{f}_{i+1} and prove the inductive assumption 1 in (8.30). Firstly, since $D_{s\eta^i} \mathbf{c}_{i+1} = 0$ we have

$$D_{s\eta^i, pert}(\psi - s\mathbf{c}_{i+1} - s\mathbf{h}_{i+1}) = sT_0^s(-\mathbf{c}_{i+1} - \mathbf{h}_{i+1}).$$

Secondly, recall that we can write

$$\begin{aligned} D_{s\eta^i + s\chi_{i+1}\eta_{i+1}} &= (1 + \varrho_{s\chi_{i+1}\eta_{i+1}})D_{s\eta^i} + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1})e_1 \partial_t \\ &\quad + \Theta_s^{i+1} + \mathcal{R}_s^{i+1} + \hat{\mathcal{A}}_s^{i+1} + \mathcal{F}_s^{i+1}. \end{aligned}$$

Now by proposition 5.9, we can decompose $\mathbf{h}_{i+1} = \mathbf{h}_{i+1}^g + \mathbf{h}_{i+1}^b + \mathbf{h}_{i+1}^s$ and $\mathbf{c}_{i+1} = \mathbf{c}_{i+1}^g + \mathbf{c}_{i+1}^b + \mathbf{c}_{i+1}^s$ as follows: recall that $\mathbf{h}_{i+1} = \mathbf{h}_{i+1}^0 + \mathbf{h}_{i+1}^s$ and $\mathbf{c}_{i+1} = \mathbf{c}_{i+1}^0 + \mathbf{c}_{i+1}^s$ such that

$$D\mathbf{h}_{i+1}^0 = s\mathbf{f}_{i,A} + \mathbf{f}_{i,B};$$

$$D\mathbf{c}_{i+1}^0 = 0.$$

Since $s\mathbf{f}_{i,A} + \mathbf{f}_{i,B} = 0$ on $N_{\frac{r}{T^{i+1}}}$, we have

$$\mathbf{h}_{i+1}^0 = \begin{pmatrix} \frac{h_{i+1}^+}{\sqrt{z}} \\ \frac{h_{i+1}^-}{\sqrt{\bar{z}}} \end{pmatrix} + \mathbf{h}_{\mathfrak{R}, i+1}; \quad \mathbf{c}_{i+1}^0 = \begin{pmatrix} \frac{c_{i+1}}{2\sqrt{z}} \\ \frac{c_{i+1}^{aps}}{2\sqrt{\bar{z}}} \end{pmatrix} + \mathbf{c}_{\mathfrak{R}, i+1}.$$

So we define

$$\mathfrak{h}_{i+1}^b = \chi_{i+1} \begin{pmatrix} \frac{h_{i+1}^+}{\sqrt{z}} \\ \frac{h_{i+1}^-}{\sqrt{\bar{z}}} \end{pmatrix}; \mathfrak{c}_{i+1}^b = \chi_{i+1} \begin{pmatrix} \frac{c_{i+1}}{2\sqrt{z}} \\ \frac{c_{i+1}^{aps}}{2\sqrt{\bar{z}}} \end{pmatrix}; \mathfrak{k}_{i+1}^b = \chi_{i+1} \begin{pmatrix} \frac{k_{i+1}^+}{\sqrt{z}} \\ \frac{k_{i+1}^-}{\sqrt{\bar{z}}} \end{pmatrix}.$$

Now we compute

$$\begin{aligned} (8.38) \quad & D_{s\eta^i} \Big|_{N_{\frac{r}{T^{i+1}}}} (s(\mathfrak{c}_{i+1}^b + \mathfrak{c}_{i+1}^s) + s(\mathfrak{h}_{i+1}^b + \mathfrak{h}_{i+1}^s)) \\ &= Ds(\mathfrak{c}_{i+1}^b + \mathfrak{h}_{i+1}^b) + (D_{s\eta^i} - D)s(\mathfrak{c}_{i+1}^b + \mathfrak{h}_{i+1}^b) + D_{s\eta^i} s(\mathfrak{c}_{i+1}^s + \mathfrak{h}_{i+1}^s) \\ &= Ds(\mathfrak{c}_{i+1}^b + \mathfrak{h}_{i+1}^b) + (D_{s\eta^i} - D)s(\mathfrak{c}_{i+1}^g + \mathfrak{h}_{i+1}^g) \end{aligned}$$

For the first term on the right hand side of (8.38), we can follow the argument in step 3 in subsection 8.3 to get

$$\mathfrak{e}_{i+1} = \chi_{i+1} \begin{pmatrix} -id^- \bar{\eta}_{i+1} \sqrt{\bar{z}} \\ -id^+ \eta_{i+1} \sqrt{z} \end{pmatrix}$$

such that

$$Ds(-\mathfrak{c}_{i+1}^b - \mathfrak{h}_{i+1}^b) = \Theta_s^{i+1}(\psi) + D(s\mathfrak{e}_{i+1}) - se_1 \partial_t(\mathfrak{e}_{i+1}) - sD(\chi_{i+1}) \left(\frac{\mathfrak{e}_{i+1}}{\chi_{i+1}} \right) - D(s\mathfrak{k}_{i+1}^b).$$

For the second term on the right hand side of (8.38), since

$$\begin{aligned} (8.39) \quad & (D_{s\eta^i} - D) \Big|_{N_{\frac{r}{T^{i+1}}}} = \sum_{j=0}^i \Theta_s^j + \mathcal{A}_s^i \\ &= s \left(\sum_{j=0}^i \dot{\eta}_j \partial_z + \sum_{j=0}^i \dot{\bar{\eta}}_j \partial_{\bar{z}} \right) + \mathcal{A}_s^i, \end{aligned}$$

we have

$$(D_{s\eta^i} - D)s(\mathfrak{h}_{i+1}^g + \mathfrak{c}_{i+1}^g) = s \sum_{j=0}^i \Theta_s^j (\mathfrak{h}_{i+1}^g + \mathfrak{c}_{i+1}^g) + s\mathcal{A}_s^i (\mathfrak{h}_{i+1}^g + \mathfrak{c}_{i+1}^g).$$

Therefore we can derive from (8.35) the following equality

(8.40)

$$\begin{aligned}
& D_{s\eta^i} \Big|_{N_{\frac{\mathbf{r}}{T^{i+1}}}} (-s(\mathbf{c}_{i+1}^b + \mathbf{c}_{i+1}^s) - s(\mathbf{h}_{i+1}^b + \mathbf{h}_{i+1}^s)) \\
&= \Theta_s^{i+1}(\psi) + D(s\mathbf{e}_{i+1}) - se_1 \partial_t(\mathbf{e}_{i+1}) - sD(\chi_{i+1}) \left(\frac{\mathbf{e}_{i+1}}{\chi_{i+1}} \right) \\
&\quad + s \sum_{j=0}^i \Theta_s^j(\mathbf{h}_{i+1}^g - \mathbf{c}_{i+1}^g) + s\mathcal{A}_s^i(\mathbf{h}_{i+1}^g - \mathbf{c}_{i+1}^g) - D_{pert}(s\mathbf{k}_{i+1}^b).
\end{aligned}$$

Recall that the Dirac operator $D_{s\eta^{i+1}}$ can be written as

$$\begin{aligned}
D_{s\eta^{i+1}} &= (1 + \varrho^{i+1})D_{s\eta^i} + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_0) e_1 \partial_t \\
&\quad + \Theta_s^{i+1} + \hat{\mathcal{A}}_s^{i+1} + \mathcal{R}_s^{i+1} + \mathcal{F}_s^{i+1} \\
&= (1 + \varrho^{i+1})D_{s\eta^i} + s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_0) e_1 \partial_t \\
&\quad + \hat{\Theta}_s^{i+1} + \mathcal{W}_s^{i+1} + \hat{\mathcal{A}}_s^{i+1} + \mathcal{R}_s^{i+1} + \mathcal{F}_s^{i+1}
\end{aligned}$$

where

$$\begin{aligned}
\hat{\Theta}_s^{i+1} &= e_1(s\chi_{i+1} \dot{\eta}_{i+1} \partial_z + s\chi_{i+1} \dot{\bar{\eta}}_{i+1} \partial_{\bar{z}}), \\
\mathcal{W}_s^{i+1} &= e_2(s(\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1} \partial_z - s(\chi_{i+1})_z \eta_{i+1} \partial_{\bar{z}}) \\
&\quad + e_3(-s(\chi_{i+1})_{\bar{z}} \eta_{i+1} \partial_z + s(\chi_{i+1})_z \eta_{i+1} \partial_{\bar{z}}).
\end{aligned}$$

We use the following notations to simplify the upcoming equation:

$$(8.41) \quad \mathcal{W}_s^{i+1}(\psi_i - \psi) = s^2 \mathcal{B}_0;$$

$$(8.42) \quad \mathcal{W}_s^{i+1}(s\mathbf{e}_{i+1}) + \varrho^{i+1}(e_2\partial_z + e_3\partial_{\bar{z}})(s\mathbf{e}_{i+1}) = s^2 \mathcal{B}_1;$$

$$(8.43) \quad -\mathcal{W}_s^{i+1}(s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) - \varrho^{i+1}(e_2\partial_z + e_3\partial_{\bar{z}})(s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) = s^2 \mathcal{B}_2;$$

$$(8.44) \quad s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) e_1 \partial_t \psi_i + s\mathcal{A}_s^i(s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) = s\mathcal{C}_0;$$

$$-sD(\chi_{i+1})\left(\frac{\mathbf{e}_{i+1}}{\chi_{i+1}}\right) + (\varrho^{i+1} - 1)e_1 \partial_t (s\mathbf{e}_{i+1})$$

$$(8.45) \quad +s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) e_1 \partial_t (s\mathbf{e}_{i+1}) + s\mathcal{A}_s^i(s\mathbf{e}_{i+1}) = s\mathcal{C}_1;$$

$$(8.46) \quad -\varrho^0 e_1 \partial_t (s\mathbf{c}_0^g - s\mathbf{h}_0^g) - s((\chi_{i+1})_z \eta_{i+1} + (\chi_{i+1})_{\bar{z}} \bar{\eta}_{i+1}) e_1 \partial_t (s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) = s\mathcal{C}_2;$$

$$(8.47) \quad \hat{\Theta}_s^{i+1}(\psi_i - \psi) + \sum_{j=0}^i \Theta_s^j(s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) = s^2 \mathcal{Q}_0;$$

$$(8.48) \quad \hat{\Theta}_s^{i+1}(s\mathbf{e}_{i+1}) + \sum_{j=0}^i \Theta_s^j(s\mathbf{e}_{i+1}) = s^2 \mathcal{Q}_1;$$

$$(8.49) \quad -\hat{\Theta}_s^{i+1}(s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) = s^2 \mathcal{Q}_2.$$

Put all these data together, (8.40) yields

$$\begin{aligned}
D_{s\eta^i, \text{pert}}(\psi_i - s\mathbf{c}_{i+1} - \mathbf{h}_{i+1}) &= sT^s(-\mathbf{c}_{i+1} - \mathbf{h}_{i+1}) + s\epsilon_i \\
&= D_{s\eta^i}(\psi_i - s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) + (D_{s\eta^{i+1}} - D_{s\eta^i})\psi_i \\
&\quad + D_{s\eta^{i+1}}(s\mathbf{c}_{i+1}) - (\hat{\mathcal{A}}_s^{i+1} + \mathcal{R}_s^{i+1})(\psi_i + s\mathbf{c}_{i+1}) \\
&\quad + s^2(\mathcal{B}_0 + \mathcal{B}_1) + s(\mathcal{C}_0 + \mathcal{C}_1) + s(\mathcal{Q}_0 + \mathcal{Q}_1) \\
&\quad - D_{\text{pert}}(s\mathbf{k}_{i+1}^b) \\
&= D_{s\eta^i}(\psi_i - s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) + (D_{s\eta^{i+1}} - D_{s\eta^i})\psi_i \\
&\quad + D_{s\eta^{i+1}}(s\mathbf{c}_{i+1}) + (D_{s\eta^{i+1}} - D_{s\eta^i})(s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g) \\
&\quad - (\hat{\mathcal{A}}_s^{i+1} + \mathcal{R}_s^{i+1})(\psi_i + s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{c}_{i+1}) \\
&\quad + s^2\left(\sum_{j=0}^2 \mathcal{B}_j\right) + s\left(\sum_{j=0}^2 \mathcal{C}_j\right) + s\left(\sum_{j=0}^2 \mathcal{Q}_j\right) \\
&\quad - D_{\text{pert}}(s\mathbf{k}_{i+1}^b).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
D_{s\eta^{i+1}, \text{pert}}(\psi_i - s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{c}_{i+1}) &= sT^s(\mathbf{c}_{i+1} - \mathbf{h}_{i+1}) + s\epsilon_i \\
&\quad - (\hat{\mathcal{A}}_s^{i+1} + \mathcal{R}_s^{i+1})(\psi_i - s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{c}_{i+1}) \\
(8.50) \quad &\quad - s^2\left(\sum_{j=0}^2 \mathcal{B}_j\right) - s\left(\sum_{j=0}^2 \mathcal{C}_j\right) - s\left(\sum_{j=0}^2 \mathcal{Q}_j\right) \\
&\quad - D_{\text{pert}}(s\mathbf{k}_{i+1}^b).
\end{aligned}$$

Now we prove that

$$(8.51) \quad \hat{\mathcal{A}}_s^{i+1}(\psi_i - s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{e}_{i+1}) + s\left(\sum_{j=0}^2 \mathcal{C}_j\right) + s\epsilon_{i+1} \in s\mathfrak{C}_{i+1}^{((1+C\varepsilon)P^i)\kappa_0};$$

$$(8.52) \quad \mathcal{R}_s^{i+1}(\psi_i - s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{e}_{i+1}) + s^2\left(\sum_{j=0}^2 \mathcal{B}_j\right) \in s^2\mathfrak{B}_{i+1}^{C\varepsilon P^i \kappa_0};$$

$$(8.53) \quad sT_0^s(-\mathbf{c}_{i+1} - \mathbf{h}_{i+1}) \in s^2\mathfrak{A}_{i+1}^{P^{i+1}\kappa_0}.$$

We already prove (8.53) in step 2 and step 3. By using $\kappa_3 = \varepsilon P^i \kappa_1$ version of (5.34), we can prove that

$$\|\hat{\mathcal{A}}_s^{i+1}(\psi_i + s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{e}_{i+1})\|_{L^2(N_r - N_s)} \leq \varepsilon P^i \kappa_1 (r^3 - s^3).$$

Meanwhile, by $(\kappa_2, \kappa_3) = (\varepsilon P^i \kappa_0, \varepsilon P^i \kappa_1)$ version of (5.26) - (5.31), we have $s(\sum_{j=0}^2 \mathcal{C}_j) \in s\mathfrak{C}_{i+1}^{C\varepsilon P^i \kappa_1}$. So we get (8.51).

Finally, by using $(\kappa_2, \kappa_3) = (\varepsilon P^i \kappa_0, \varepsilon P^i \kappa_1)$ version of proposition 5.8, we have

$$\begin{aligned} \|\mathcal{R}_s^{i+1}(\psi_i + s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{e}_{i+1})\|_{L^2} &\leq C\gamma_T^{\frac{3}{2}}\varepsilon^2 P^{2i} \kappa_1^2 \left(\frac{\mathbf{r}}{T^{i+1}}\right)^2 \\ &\leq \frac{C\gamma_T^{\frac{3}{2}}\varepsilon P^i \kappa_1 \mathbf{r}^{\frac{1}{2}}}{T^{\frac{i+1}{2}}}\varepsilon P^i \kappa_1 \left(\frac{\mathbf{r}}{T^{i+1}}\right)^{\frac{3}{2}} \\ &\leq \varepsilon P^i \kappa_1 \left(\frac{\mathbf{r}}{T^{i+1}}\right)^{\frac{3}{2}} \end{aligned}$$

by taking $P \leq \sqrt{T}$ and $\varepsilon \leq \frac{1}{C\gamma_T^{\frac{3}{2}}\kappa_1 \mathbf{r}^{\frac{1}{2}}}$. So we have $\mathcal{R}_s^{i+1}(\psi_i + s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{e}_{i+1}) \in s^2\mathfrak{B}_{i+1}^{C\varepsilon P^i \kappa_1}$. Meanwhile, by $(\kappa_2, \kappa_3) = (\varepsilon P^i \kappa_0, \varepsilon P^i \kappa_1)$ version of (5.26) - (5.31) again, we have $s(\sum_{j=0}^2 \mathcal{B}_j) \in s^2\mathfrak{B}_{i+1}^{C\varepsilon P^i \kappa_1}$. So we prove (8.52).

Step 5. In this step, we state the following lemma which is the $i + 1$ -th version of lemma 8.6 in previous subsection. The proof of this lemma can follow from the argument of lemma 8.6 directly. So we omit the proof.

Lemma 8.8. *Suppose \mathcal{Q} be either the following 4 types:*

$$s^2 \chi_{i+1} \begin{pmatrix} \frac{q^+(t)}{\sqrt{z}} \\ \frac{q^-(t)}{\sqrt{\bar{z}}} \end{pmatrix}, s^2 \chi_{i+1} \begin{pmatrix} \frac{q^+(t)}{\sqrt{\bar{z}}} \\ \frac{q^-(t)}{\sqrt{z}} \end{pmatrix}, s^2 \begin{pmatrix} \frac{q^+(t)}{\sqrt{z}} \\ \frac{q^-(t)}{\sqrt{\bar{z}}} \end{pmatrix} \text{ or } s^2 \begin{pmatrix} \frac{q^+(t)}{\sqrt{\bar{z}}} \\ \frac{q^-(t)}{\sqrt{z}} \end{pmatrix}$$

where $\|q^\pm\|_{L^2} \leq \kappa_3 \frac{\tau}{T^{i+1}}$, $\|(q^\pm)_t\|_{L^2} \leq \kappa_3$. Then there exists a L_1^2 section \mathbf{e}' which can be written as

$$\begin{aligned} \mathbf{e}' &= \sum_{j \geq 0} s^j \chi_{i+1}^{j+1} \begin{pmatrix} e_j^+(t) \sqrt{\bar{z}} \\ e_j^-(t) \sqrt{z} \end{pmatrix}, \sum_{j \geq 0} s^j \chi_{i+1}^{j+1} \begin{pmatrix} e_j^+(t) \sqrt{z} \\ e_j^-(t) \sqrt{\bar{z}} \end{pmatrix} \\ &, \sum_{j \geq 0} s^j \chi_{i+1}^j \begin{pmatrix} e_j^+(t) \sqrt{\bar{z}} \\ e_j^-(t) \sqrt{z} \end{pmatrix} \text{ or } \sum_{j \geq 0} s^j \chi_{i+1}^j \begin{pmatrix} e_j^+(t) \sqrt{z} \\ e_j^-(t) \sqrt{\bar{z}} \end{pmatrix} \end{aligned}$$

for the each type respectively such that $D_{s\eta^{i+1}}(s^2 \mathbf{e}') = s^2 \mathcal{Q} + s^2 \mathcal{B} + s\mathcal{C}$ for some $\mathcal{B} \in \mathfrak{B}_{i+1}^{\kappa_3}$ and $\mathcal{C} \in \mathfrak{C}_{i+1}^{\kappa_3}$ for all $s \leq \frac{T^{\frac{i+1}{2}}}{2\gamma_T^2 \kappa_3 \tau^{\frac{1}{2}}}$. Furthermore, we have $\|\mathbf{e}'\|_{L_1^2} \leq 2\kappa_3$.

By using this lemma, we can show that there exist $\mathbf{e}'_{i+1,j}$, $j = 0, 1, 2$, such that

$$D_{s\eta^{i+1}, \text{pert}}(s^2 \mathbf{e}'_{i+1,j}) = s^2 \mathcal{Q}_j + s^2 \mathcal{B}_j + s\mathcal{C}_j.$$

Meanwhile, by the proposition 5.9 and proposition 5.4, we can show that there exist $\mathbf{k}_{i+1,s}^b$ and $\mathbf{k}_{i+1,s}^\perp$ satisfying $D_{s\eta^{i+1}, \text{pert}}(\mathbf{k}_{i+1,s}^b + s\mathbf{k}_{i+1,s}^\perp) = D_{\text{pert}} \mathbf{k}_{i+1}^b$, $B(\mathbf{k}_{i+1,s}^b) \in \mathbb{H}_0$ and

(8.54)

$$\|B(\mathbf{k}_{i+1,s}^\perp)\|_{L^2}^2 \leq \frac{\varepsilon P^i \kappa_0}{2T^{2(i+1)}}, \|(B(\mathbf{k}_{i+1,s}^\perp))_t\|_{L^2}^2 \leq \frac{\varepsilon P^i \kappa_0}{2T^{i+1}}, \|(B(\mathbf{k}_{i+1,s}^\perp))_{tt}\|_{L^2}^2 \leq \frac{\varepsilon P^i \kappa_0}{2}.$$

Therefore we can rewrite

(8.55)

$$D_{s\eta^{i+1}, \text{pert}}(\psi_i - s\mathbf{c}_{i+1}^g - s\mathbf{h}_{i+1}^g + s\mathbf{e}_{i+1}^g + s\mathbf{k}_{i+1,s}^b + s^2 \mathbf{k}_{i+1,s}^\perp) = s^2 \mathcal{A} + s^2 \mathcal{B} + s\mathcal{C} \equiv \mathbf{f}_{i+1}.$$

with $\mathbf{e}_{i+1}^g = \mathbf{e}_{i+1} + \sum_j \mathbf{e}'_{i+1,j}$ and $\mathcal{A} \in \mathfrak{A}_{i+1}^{(1+C\varepsilon)P^i\kappa_1}$, $\mathcal{B} \in \mathfrak{B}_{i+1}^{C\varepsilon P^i\kappa_1}$ and $\mathcal{C} \in \mathfrak{C}_{i+1}^{C\varepsilon P^i\kappa_1}$. So by taking $\varepsilon \leq \frac{P-1}{C}$, we prove the inductive assumption 1 in (8.30).

Step 6. Finally, we should prove the inductive assumption 3 in (8.30). To prove this part, we notice that both \mathfrak{h}_{i+1}^g and \mathfrak{c}_{i+1}^g vanish on Σ , therefore we can do the integration by parts to get

$$\|\mathfrak{h}_{i+1}^g\|_{L_1^2}^2 \leq \|D_{s\eta^i} \mathfrak{h}_{i+1}^g\|_{L^2}^2 + C\|\mathfrak{h}_{i+1}^g\|_{L^2}^2$$

for some constant C depending on the curvature of M . Now by the fact $D_{s\eta^i} \mathfrak{h}_{i+1} = 0$ on $N_{\frac{\varepsilon}{T^{i+1}}}$ and corollary 4.6, we have

$$\|D_{s\eta^i} \mathfrak{h}_{i+1}^g\|_{L^2} \leq |\sigma(\chi_{i+1})| \|\mathfrak{h}_{i+1}\|_{L^2} + \|D_{s\eta^i} \begin{pmatrix} h_{i+1}^+ \sqrt{z} \\ h_{i+1}^- \sqrt{\bar{z}} \end{pmatrix}\|_{L^2(N_{\frac{\varepsilon}{T^i}})} \leq C \frac{P^{i+1} \kappa_1}{T^{4(i+1)}}$$

and by (8.32) and (8.33) and corollary 4.6, we have

$$\|\mathfrak{h}_{i+1}^g\|_{L^2} \leq C\|\mathfrak{h}_{i+1}\|_{L^2} \leq C \frac{P^{i+1}}{T^{i+1}} \kappa_1.$$

So we have

$$\|\mathfrak{h}_{i+1}^g\|_{L_1^2} \leq C \frac{P^{i+1} \kappa_1}{T^{(i+1)}}.$$

Similarly, we have

$$\|\mathfrak{c}_{i+1}^g\|_{L_1^2} \leq C \frac{P^{i+1} \kappa_1}{T^{i+1}}.$$

For the L^2 -part, we have

$$\begin{aligned} \|\mathfrak{e}_{i+1,s}^b\|_{L^2} &\leq C\|\mathfrak{e}_{i+1}^b\|_{L^2} \leq C \frac{P^i \kappa_0}{T^{2(i+1)}}; \\ \|\mathfrak{e}_{i+1,s}^\perp\|_{L^2} &\leq C\|\mathfrak{e}_{i+1}^b\|_{L^2} \leq C \frac{P^i \kappa_0}{T^{2(i+1)}}. \end{aligned}$$

So $\mathfrak{k}_{i+1,s}^\perp \rightarrow 0$ in L^2 -sense. Therefore we finish the proof of the inductive assumption 3 in (8.30).

By induction, we get a sequence $\psi_i \in L^2$ and a family of perturbations $\eta^i = \sum_{j=0}^i \chi_j \eta_j$ such that

$$D_{s\eta^i, \text{pert}}(\psi_i + s^2 \mathfrak{k}_{i+1,s}^\perp) \rightarrow 0$$

as $i \rightarrow \infty$ in L^2_{-1} sense. Moreover, since $\|\psi_{i+1} - \psi_i\|_{L^2} \leq C\kappa_3(\frac{P}{T})^i$ for some $C > 0$, so we have $\psi_i \rightarrow \psi_s$ in L^2 sense. Meanwhile, since $\|\eta_i\|_{L^2_1} \leq C\kappa_3(\frac{P}{T})^i$ for some $C > 0$, we have $\sum \eta_i \rightarrow \eta_s$ in L^2_1 sense.

To prove that η^i converges to a C^1 circle, we only need to use the Holder estimates in remarks 8.4 and 8.7. We have

$$\|\eta_i\|_{C^{1, \frac{1}{4}}} \leq C\kappa_0 \left(\frac{\mathfrak{r}^{\frac{1}{4}}}{T^{\frac{i}{20}}} \right).$$

for all i . Therefore, by Arzela-Ascoli theorem, there is a subsequence of the partial sum $\{\eta^i\}$ converging in C^1 sense. So the limit, η , will be a C^1 circle.

In the case that $B(\psi_s) = 0$, ψ will vanish on Σ and $D_{s\eta, \text{pert}}(\psi_s) = 0$. So $\psi \in L^2_1$.

Remark 8.9. Suppose we consider a smaller neighborhood of $((g, \Sigma, e), \psi)$ to parametrize. This means we can take \mathfrak{r} , c_0 smaller. In this case, the constant ε can be chosen smaller, too. We can see that

$$\frac{1}{\mathfrak{r}^{\frac{1}{4}}} \left\| \sum_{j=1}^{\infty} \eta_j \right\|_{C^1} \rightarrow 0$$

as \mathfrak{r} goes to 0. Similarly, we have $\mathfrak{k}_s - \mathfrak{k}^0$ is $O(\varepsilon)$. So all these terms we derived in this iteration process is $o(s)$ -order.

8.5. Part I of the proof: The set $\pi_1(\mathcal{N})$. Here we should say more about the neighborhood \mathcal{N} . We define the topology on \mathcal{Y} as follows. Let $((g, \Sigma, e), \psi) \in \mathfrak{M}$, we

use refine the notation used in section 5.2 in the following way:

$$\mathcal{V}_{\Sigma,r,C} = \{\eta : S^1 \rightarrow \mathbb{C} \mid \|\eta\|_{C^1} \leq C; (\eta(t), t) \in N_r\}$$

and define

$$\mathcal{V}_{g,r,C'} = \{\hat{g} \in \mathcal{X} \mid \|\hat{g} - g\|_{C^2} \leq C'; \text{dist}(\Sigma, \text{supp}(\hat{g} - g)) \leq r\}.$$

So we can generate the topology on \mathcal{Y} by the family of open sets $\{\mathcal{V}_{g,r,C'} \times \mathcal{V}_{\Sigma,r,C}\}$ for $r < R$, $C, C' \in \mathbb{R}^+$.

Now we define our $\mathcal{N} = \bigcup_{r>\mathfrak{r}} \mathcal{V}_{g,r,Cr^{5/2}} \times \mathcal{V}_{\Sigma,r,C}$ for some C small enough. Reader can double check the argument in step 2 of section 8.3: By taking \mathcal{N} in this way, we have all elements in $\pi_1(\mathcal{N})$ will follows the argument in this section.

Remark 8.10. It seems to be impossible to take \mathcal{N} to be $\bigcup_{r>0} \mathcal{V}_{g,r,Cr^{5/2}} \times \mathcal{V}_{\Sigma,r,C}$ because the map f will not differentiable on this set. However, the choice of \mathfrak{r} can be arbitrary small.

9. PROOF OF THE MAIN THEOREM: PART II

In this section, we prove two statements. Firstly, we have to show that the choice of (η_s, ψ_s) have dimension equaling $\dim(\ker(D|_{L_1^2}))$. Secondly, we have to show that the function f we defined in previous section is C^1 .

9.1. Part II of the proof: parametrization of (η_s, ψ_s) . First of all, by the argument in the previous section. After we fix a $\xi \in \mathbb{H}_1$, we have the choice of η_s is unique. Also, we have $B(\psi_s)$ is unique.

According to this observation, we can prove the following proposition instead.

Proposition 9.1. For any two solutions (η_s, ψ_s) and (η_s, ψ_s^*) satisfying $D(\psi_0 - \psi_0^*) = 0$, then $\psi_s - \psi_s^* = 0$.

Proof. We can write $D_{s\eta_s, pert} = D + P(s)$ where $P(s)$ is an analytic operator with respect to s . Meanwhile, since we have $\psi_s - \psi_s^* \in C^\omega([0, c_0]; L_1^2)$, so we have

$$D_{s\eta_s, pert}(\psi_s - \psi_s^*) = D(\psi_s - \psi_s^*) - P(s)(\psi_s - \psi_s^*) = 0$$

So inductively, we have $(\psi_s - \psi_s^*) = O(s^k)$ for all k . This implies $(\psi_s - \psi_s^*) = 0$. \square

By this proposition, we know that we can parametrize the data ψ_s by elements in $\ker(D|_{L_1^2})$. Therefore, we can define a map $\mathcal{K} : s\hat{\psi} \mapsto \psi_s$ where $\hat{\psi} \in \ker(D|_{L_1^2})$ and $\|\hat{\psi}\|_{L_1^2} = 1$.

9.2. Part II of the proof: C^1 regularity of f . Since the function f is defined on a infinity dimensional space, so the definition of C^1 will be in the sense of Frechet C^1 . Here we recall the definition of Frechet C^1 .

Definition 9.2. Let B_1, B_2 are two Banach space. $\mathcal{F} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a bounded operator. Then \mathcal{F} is differentiable at p if and only if there exists a bounded linear operator $d_p\mathcal{F} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ such that

$$\|\mathcal{F}(x) - d_p\mathcal{F}(x) - \mathcal{F}(p)\|_{\mathcal{B}_2} = o(\|x\|_{\mathcal{B}_1}).$$

In addition, if \mathcal{F} is differentiable everywhere and $d_p\mathcal{F}$ vary continuous. Then we call \mathcal{F} a C^1 map.

Now let \mathcal{F} maps from $\mathbb{R}^n \times \mathcal{B}$ to \mathbb{R}^m . Suppose we have

$$(9.1) \quad \frac{\partial}{\partial x_i} \mathcal{F}(p) := h_i(p) \text{ is continuous near } 0.$$

$$(9.2) \quad \text{The family of directional derivatives } \{D_v\mathcal{F} := j_v(p) | v \in \mathcal{B}, \|v\| = 1\}$$

is equicontinuous near 0,

$$(9.3) \quad \{D_v\mathcal{F} = k_p(v) | p \in \mathbb{R}^n \times \mathcal{B}\} \text{ is equicontinuous on}$$

$\{v \in \mathcal{B} | \|v\| = 1\}$.

Then we can define the linear operator as follows:

$$(9.4) \quad \mathcal{L}_p(x, v) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \mathcal{F}(p) x_i + D_{\frac{v}{\|v\|}} \mathcal{F}(p) v$$

To prove this is the linear approximation, we need to do more. However, this is the only possible linear operator tangential to \mathcal{F} at 0.

Now, suppose we already show that these linear operators are the differential of \mathcal{F} . To show \mathcal{F} is actually C^1 , it is sufficient to show that \mathcal{L}_p varies continuously. So the condition (9.1) and (9.2) are exactly what we need to show.

Here I divide my proof into two parts. In first part, I will assume that f is differentiable at every point and then showing that f is C^1 . In the second part, I will prove that f is differentiable.

Step 1. Since \mathfrak{k}_s^b is analytic, the family of directional derivatives of f is actually equicontinuous at any point except $p = 0$ with any direction fixed. Therefore we only need to show the conditions (9.1) and (9.2) hold near 0.

Since we have

$$(9.5) \quad s\mathfrak{k}_s^b = \sum_{i=0}^{\infty} s\mathfrak{k}_{i,s}^b = \sum_{i=0}^{\infty} s\mathfrak{k}_i^b + O(s^2).$$

We can further simplify this equation by using the conclusion in remark 8.9.

$$(9.6) \quad s\mathfrak{k}_s^b = s\mathfrak{k}_0^b + o(s)$$

Now, recall the way we construct k_0^\pm in Step 1 and Step 2 in section 8.3. In the case that we have no perturbation for g , $k_0^\pm = 0$. That is to say, $s\mathfrak{k}_s^b = o(s)$. Therefore we have the directional derivatives of f along \mathbb{H}_1 will be 0. Meanwhile, it is obvious that they are continuous by using (9.6).

To prove (9.2), we use (9.6) again. Here we can check that if we perturb the metric along the opposite direction, then the corresponding \mathfrak{k}_0^b will only change the sign. So the directional derivatives along $\pi_1(\mathcal{N})$ also exist and are continuous at 0.

Furthermore, since the estimates showed in section 8 are independent of the choice of g^s , so it also doesn't depend on v . Therefore we have $\{j_v(p)\}$ is equicontinuous at 0.

So we finish our argument in this step.

Step 2. In this step, we need to show that f is differentiable. By definition 9.2, we need to show that for any $p = (y, w) \in \mathbb{R}^n \times \mathcal{B}$,

$$(9.7) \quad \|f(y+x, w+v) - \mathcal{L}_{(y,w)}(x, v) - f(y, w)\| \leq o(\sqrt{\|x\|^2 + \|v\|_{C^2}^2})$$

where $x, y \in \mathbb{K}_1$ and $w, v+w \in \pi_1(\mathcal{N})$. All we need to show is the "small o" in (9.7) will converge to zero uniformly. Namely, we are going to prove (9.3) here. Now, since we already prove that the directional derivatives of f are all continuous, so we can obtain (9.7) by showing that $\{k_p(v)\}$ is equicontinuous.

By using the conclusion in 8.5, we suppose that $\|\partial_s g^s\|_{C^2} = C\mathfrak{r}^{\frac{5}{2}}$, then the directional derivative of f along $v = \frac{\partial_s g^s}{\|\partial_s g^s\|}$ at g^{s_0} will be $\frac{1}{C\mathfrak{r}^{\frac{5}{2}}} \frac{\partial}{\partial s}(B(s\mathfrak{k}_s^b))|_{s=s_0}$. Now we can prove (9.3) by using the fact that \mathfrak{k}_s^b is analytic and the estimates (8.7) and (8.33).

Therefore, we complete the proof of this part.

9.3. Summary of the proof. Let me summarize what we proved in this section: For any $((g, \Sigma, e), \psi)$, there exist a neighborhood of $y = (g, \Sigma, e)$, $\mathcal{N} \subset \mathcal{Y}$, finite dimensional ball $\mathbb{B} \in \mathbb{K}_1$ and finite dimensional vector space \mathbb{K}_0 all defined as above such that \mathfrak{M} will locally homeomorphic to the kernel of f where

$$f(g^s, s\xi, s\hat{\psi}) = B(\mathcal{K}(s\hat{\psi}))$$

Moreover, f is a C^1 function.

We complete our proof.

10. APPENDIX

10.1. Remark of the proof when the metric is not Euclidean around Σ .

Here I will sketch the proof for the metric which is not Euclidean near Σ . The idea is to replace the propositions 5.5 and 5.7 by propositions 6.4 and 6.5 in the arguments containing section 8.

First of all, let me summarize what I have done in section 8. We start with a perturbation g^s and which will give us an extra term \mathbf{f}_0 such that $D_{pert}\psi = \mathbf{f}_0$. Then in the next step we construct a triple $(\mathbf{h}_0, \mathbf{c}_0, \eta_0)$ such that $D\mathbf{h}_0 = \mathbf{f}_0$, $D\mathbf{c}_0 = 0$ and "eliminate" the $\frac{1}{\sqrt{r}}$ part in \mathbf{h}_0 by (\mathbf{c}_0, η_0) . Then we repeat this process. Each time we will produce a new \mathbf{f} which can be decomposed into 3 parts, which belongs to \mathfrak{A} , \mathfrak{B} and \mathfrak{C} defined in definition 8.2 (We omit all subscripts here).

Now, we restart the process of producing $(\mathbf{h}_0, \mathbf{c}_0, \eta_0)$ for the general case, but this time we replace the Dirac operator D by $D^{(1)}$ defined in section 6.2. So $D^{(1)}\mathbf{h}_0 = \mathbf{f}_0$ and $D^{(1)}\mathbf{c}_0 = 0$. By using the same argument, we will still generate \mathbf{f}_1 . The only difference will be an extra term in \mathfrak{C} , which is something we can deal with. This part is generated by the operator $\delta^{(1)}$ defined in proposition 6.3.

Now we do this process step by step. We replace D by $D^{(i)}$ in i -th step, then we will get the same result. So the whole argument works for the general case.

10.2. Upper semi-continuity of $\dim(\text{coker}(p^-|_{\ker(p^+)}))$. In this final part, I will answer the question about the upper semi-continuity of $\dim(\text{coker}(p^-|_{\ker(p^+)}))$.

Since $p^-|_{\ker(p^+)}$ is a Fredholm operator, we can decompose $Exp^- = \text{range}(p^+) \oplus \mathbb{W}$ where \mathbb{W} is finite dimensional. Now, for any $c^\pm \in \text{range}(p^-)$, there exists $\mathbf{c} \in \ker(D|_{L^2})$ such that $B(\mathbf{c}) = c^\pm$. Suppose we have a perturbed Dirac operator D_{pert} . We can follow the argument in the proof of proposition 5.9 to get a \mathbf{c}' such that $D_{pert}(\mathbf{c}') = 0$ and $\|B(\mathbf{c} - \mathbf{c}')\| \leq \varepsilon \|B(\mathbf{c})\|$.

To prove $\text{coker}(p^-|_{\ker(p^+)})$ is upper semi-continuous, we need to show that the dimension of cokernel under a small perturbation will be less or equal than the dimension of \mathbb{W} . We can prove this fact by showing that $\text{range}(p_{\text{pert}}^-) + \mathbb{W} = \text{Exp}^-$.

Suppose this is not the case, then we can find $v \in \text{Exp}^-$, $\|v\| = 1$ such that $v \perp \mathbb{W}$ and $v \perp \text{range}(p_{\text{pert}}^-)$. So we have

$$\langle v, B(\mathbf{c}') \rangle = 0 = \langle v, B(\mathbf{c}') \rangle + O(\varepsilon)$$

This means that, if we decompose $v = v_0 + v_1$ where $v_0 \in \text{range}(p^-)$ and $v_1 \perp \mathbb{W}$, then we have $\|v_0\| \leq O(\varepsilon)$ and $v_1 = 0$. Therefore, we have $\|v\| = O(\varepsilon)$, which is a contradiction.

Therefore we prove the upper semi-continuity of $\dim(\text{coker}(p^-|_{\ker(p^+)}))$.

REFERENCES

- [1] T. W. Andriy Haydys. A compactness theorem for the seiberg-witten equation with multiple spinors in dimension three. *arXiv:1406.5683 [math.DG]*, 2014.
- [2] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.
- [3] B. Boß-Bavnbek and K. P. Wojciechowski. *Elliptic boundary problems for Dirac operators*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1993.
- [4] F. Bowman. *Introduction to Bessel functions*. Dover Publications Inc., New York, 1958.
- [5] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [6] T. Friedrich. *Dirac operators in Riemannian geometry*, volume 25 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.
- [7] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

- [8] P. B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1995.
- [9] H. B. Lawson, Jr. and M.-L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [10] P. D. Lax. *Functional analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2002.
- [11] F.-H. L. Qing Han. *Nodal Sets of Solutions of Elliptic Differential Equations*. Courant Institute Lecture.
- [12] C. H. Taubes. $\mathrm{PSL}(2; \mathbb{C})$ connections on 3-manifolds with L^2 bounds on curvature. *Camb. J. Math.*, 1(2):239–397, 2013.
- [13] C. H. Taubes. The zero loci of $\mathbb{Z}/2$ harmonic spinors in dimension 2, 3 and 4. *arXiv:1407.6206*, 2014.
- [14] K. K. Uhlenbeck. Connections with L^p bounds on curvature. *Comm. Math. Phys.*, 83(1):31–42, 1982.
- [15] F. Yang. *Dirac operators and monopoles with singularities*. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)—Massachusetts Institute of Technology.