Essays on the Economics of Contracts and Organizations

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Essays on the Economics of Contracts and Organizations

A dissertation presented

by

Daisuke Hirata

to

The Department of Economics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Economics

Harvard University
Cambridge, Massachusetts
April 2015
Abstract

This thesis consists of three essays on the economics of contracts and organizations. The first essay studies organizational design as the allocation of decision rights, primarily focusing on its interplay with agents’ career motives. It identifies a new tradeoff between delegation and centralization, which arises solely from career concerns: When delegated, an agent takes inefficient actions at the cost of a principal but also works harder ex post to implement his project, in order to manipulate the market expectations of his ability. Compared to the existing literature, the contribution of this study is two-fold. First, it endogenizes the agent’s bias as a result of career concerns. Second, and perhaps more importantly, it uncovers a new link between organizational design and the implementation of a decision. Both of these features are in sharp contrast to the vast majority of the existing studies, which takes the agent’s bias as given and abstracts away from the implementation stage of a decision process. Specifically, delegation can be strictly optimal in the present framework even if the agent has no information advantage over the principal.

Motivated by some entry-level labor markets, the second essay studies an incentive-contracting problem where (i) a principal learns an agent’s ability before the agent himself, and (ii) both the agent’s productivity with the principal as well as his outside option depends on his ability. I characterize the optimal contracts
for the principal, defined to be the most profitable equilibrium outcomes among those satisfying the D1 criterion; pooling at an earlier date is strictly optimal if the agent’s outside option is sufficiently sensitive to the principal’s private information, whereas separation at a later date is (weakly) optimal otherwise. Further, the principal’s profit is shown to be neither continuous nor monotone with respect to the agent’s outside option.

The third essay studies stable and (one-sided) strategy-proof matching rules in many-to-one matching markets with contracts. First, the number of such rules is shown to be at most one. Second, the doctor-optimal stable rule, whenever it exists, is shown to be the unique candidate for a stable and strategy-proof rule. Notably, these results are established without any substitutes conditions on hospitals’ choice functions, and hence, the proofs do not rely on the “rural hospital” theorem. Finally, a stable and strategy-proof rule, when exists, is shown to be second-best optimal for doctor welfare, in the sense that no individually-rational and strategy-proof rule can dominate it.
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Acknowledgments

I first thank my adviser Oliver Hart for his continuous guidance and encouragement. I am also grateful to the other members of my dissertation committee, Philippe Aghion and Tomasz Strzalecki, who have been also always accessible and supportive. In addition, I have benefited from interaction with and feedback from various people throughout my graduate study. A partial list includes Charles Angelucci, Nava Ashraf, Kirill Borusyak, Qiang Fu, Tadashi Hashimoto, Akifumi Ishihara, Sangram Kadam, Yusuke Kasuya, Divya Kirti, Fuhito Kojima, Toshihiro Matsumura, Takeshi Murooka, Fausto Panunzi, Armin Schmutzler, Yuichi Yamamoto, and Noriyuki Yanagawa. Finally, the Nakajima Foundation is also gratefully acknowledged for its generous financial support.
To Mika
Chapter 1

Organizational Design and Career Concerns

1.1 Introduction

The allocation of decision rights within an organization is an important issue both in management practice (e.g., McConkey, 1974; Steinmetz, 1976) and in organizational economics (e.g., Aghion et al., 2014; Gibbons et al., 2012). While various economic theories have been offered, most of the existing studies share a common assumption that a principal and agent(s) have misaligned preferences over the choice of certain actions. In particular, it is often assumed that agents are empire-builders; i.e., they are assumed to put too much weight (from a principal’s point of view) on the outcomes of their own divisions and undervalue the other divisions. The presence of such preference misalignment is critical in those theories, because if preferences are perfectly aligned, anyone will make the same decision and hence, the allocation of decision rights becomes irrelevant. Despite its importance, however, most existing studies take preference misalignment as an exogenously given
assumption, and its source has been largely unexplored.

The purpose of this chapter is to propose a new micro-founded theory of organizational design, which endogenizes the agent’s bias as a result of career concerns à la Holmstrom (1999). If an agent has career motives, he would naturally put a larger weight on the outcome of his own division or project, because it is the primary source of information on his ability, which will determine his future career. Yet, his ability may not necessarily be the unique determinant of the outcome. For example, one’s project may fail not because of his ability, but simply because a headquarter manager did not invest a sufficient amount of corporate resources into his project, or because he was forced to take a coordinated action that helps another division at the cost of his own performance.\(^1\) If one’s performance depends on certain decisions as well as his own ability, the quality of his performance as a signal of his ability will be not independent of who makes those decisions. That is, it is not only that career motives make the allocation of decision rights relevant by creating preference misalignment, but also that the allocation of decision rights has feedback on the effectiveness of the agents’ career motives.

By investigating such interplay between career concerns and organizational design, this chapter identifies a new tradeoff between a centralized and decentralized organization. In the presence of career concerns, an agent will take inefficient actions when he has the decision right. For example, he will overinvest corporate resources into his own project, in order to inflate its performance and the market assessment of his ability. Of course, this is a cost of delegation from the principal’s point of view. At the same time, however, it will also increase the quality

\(^1\)Indeed, Steinmetz (1976) writes: “A common lament heard by many management consultants in working with supervisory training groups and executive development programs is: ‘I could really get things done if I had the authority.’ Or, ‘If I was the boss, I could do it. But my boss won’t let me have the authority to get the job done.’”
of his performance as a signal of his ability, because the resources invested in his project should be utilized more efficiently when his ability is higher. As a result, the agent’s market value in the future becomes more sensitive to his current performance, and hence, he will also exert higher effort to manipulate the signal when he is delegated the decision right. The benefit of delegation in the present study is this incentive effect.

Compared to the existing literature, a notable feature of the present theory is that it identifies a new benefit of delegation, which arises solely from career concerns. A majority of the existing studies assumes that an agent has some “local” information that is not directly observed by a principal, and this information advantage is the source of the relative benefit of delegation in those studies. Contrastingly, delegation can strictly dominate centralization in the present study even though the agent has no private information. Instead, it is critical to assume that the decisions made within a firm is not directly observable to the outside labor maker. That is, the key informational asymmetry in this study lies between the inside and outside of a firm.

Relatedly, it is also noteworthy that this chapter focuses on a different stage of decision making process from most existing studies. Figure 1.1 illustrates this distinction in the spirit of Mintzberg (1979), who divides a decision process into four steps: from information to advice to choice to execution. As mentioned above, the main focus of the literature has been on the effects of asymmetric information (and communication) between a principal and agent, and the execution of a decision is largely ignored; i.e., payoffs are finalized once a decision is made and no further ac-

---

2Benefits of delegation are the key in the theories of organizational design, because it is non-trivial to explain why a principal in a centralized organization cannot mimic the decision rule that would be taken by a delegated agent— i.e., why centralization cannot always do at least as good as delegation.
Figure 1.1: Comparison of this chapter and existing studies in the spirit of Mintzberg (1979).

Information is taken after that. Consequently, when the agent’s effort is taken into account, it is *ex-ante* effort to gather or create information. In contrast, the key element in the present model is the agent’s *ex-post* incentive to execute or implement a project after a decision is made, whereas I abstract away from asymmetric information (and thus communication) between the principal and agent. Indeed, it is shown that in my framework delegation has opposite effects on those two types of incentives—it decreases *ex-ante* efforts while it increases *ex-post* efforts. This result would also highlight the distinction between the present study and Aghion and Tirole (1997), who illustrate the benefit of delegation through *ex-ante* incentives.

As an application, I extend the model to a multi-tasking environment à la Holmstrom and Milgrom (1991), and identify an interaction effect between organizational design (i.e., decision right allocation) and job design (i.e., task allocation).\(^4\)

\(^3\)Few exceptional studies on the execution stage and their relation to the present study will be explained in the next section.

\(^4\)To the best of my knowledge, this chapter is the first to study the interaction between those
When an agent is assigned multiple tasks, it is possible that neither the principal nor the agent has an interim incentive to follow a decision rule that is ex-ante optimal to the principal. To see this, suppose for example that the principal has two tasks, one of which is riskier than the other, and a single agent handles both tasks. Then, the agent tends to like allocating corporate resources into the safe task, because the less volatile the outcome of a task is, the larger fraction of it will be attributed to his ability rather than pure noise. Hence, even when the principal wants to commit to (over)investment to the riskier task because of its incentive effect explained above, delegation to a multi-tasking agent does not necessarily work. Even if so, however, the principal can use specialization in conjunction with delegation to make sure that the agent in charge of the risky project uses the resources himself and gets motivated. As a result, specialization can be strictly optimal even when both (i) the outcomes of the tasks are statistically independent, and (ii) there exist positive synergies between the tasks. Furthermore, delegation and specialization are strictly complementary in this scenario: starting from centralization and non-specialization, specialization without delegation is strictly harmful and delegation without specialization is neutral, whereas delegation with specialization is strictly profitable.

The rest of the chapter is organized as follows. Section 1.1.1 provides a brief overview of the related literature. Section 1.2 setups and analyzes the baseline model. Section 1.3 studies the the interaction between organizational design and job design. Section 1.4 concludes. Appendices A.1 and A.2 provide the proofs and additional results, respectively, omitted in the main text.
1.1.1 Related Literature

This study is related to a few strands of the literature in organizational economics and related fields. In the incomplete contracting literature, Aghion and Tirole (1997) first study the effects of decision-right allocation in the absence of renegotiation. One of their key assumptions is that the decision right allocation is contractible, i.e., the principal can credibly commit to delegation, whereas decisions or actions are not contractible per se. When the principal has no commitment power, Baker et al. (1999) investigate to what extent delegation is sustainable through repeated interactions. Aghion et al. (2002, 2004) introduce the concept of “transferable” control, and illustrate how delegation in the short run can be useful as a screening device, even when the principal cannot commit to delegation in the long run.

With contractible decision rights, Dessein (2002) applies the cheap talk model of Crawford and Sobel (1982) and asks whether centralization with strategic communication is better or worse than delegation when the agent is privately informed about a decision-relevant state of the world. Since then, his approach has been extended to various settings (see, e.g., Agastya et al., 2014; Alonso et al., 2008a,b, 2010; Harris and Raviv, 2005; and Rantakari, 2008). In these studies, the primary question is the tradeoff between loss of control (due to misaligned preferences) and loss of information (due to strategic communication), and the agent’s bias is

---

5 In the model of Grossman and Hart (1986), renegotiation always leads to ex-post efficient decisions and hence, decision rights are relevant only through ex-ante investments. Based on their logic, Stein (1997) argues that returns to investments should be lower in an integrated firm because of the bargaining over private benefits between a CEO and division manager. See also Aghion and Bolton (1992) for the effects of decision rights in the presence of renegotiation.

6 When actions are contractible, Holmström (1984) studies a situation where a principal can commit to a decision rule, and name it the “delegation problem” since committing to a rule that induces truth-telling is identical to letting an agent choose from a prespecified subset of actions. See, e.g., Alonso and Matouschek (2007, 2008) and Melumad and Shibano (1991) for subsequent studies in this line. See also Krishna and Morgan (2008) who study the opposite situation, where the principal can commit to a message-contingent transfer rule but not to a decision rule.
exogenously assumed.\textsuperscript{7} An exception is Rantakari (2013), who considers the setting where the principal can set the \textit{relative} weights on divisional profits in each divisional manager’s objective function. Setting biased weights may be preferable to the principal, because it will increase the incentive to gather local information. In his model, however, it remains unexplored why the principal cannot incentivize the agent by high-powered but balanced contracts.\textsuperscript{8}

The effects of decision-right allocation are also studied in the field of corporate finance, primarily in their connection to internal capital markets in multi-divisional firms. See, e.g., Brusco and Panunzi (2005), Harris and Raviv (1998, 1996), Innderst and Laux (2005), Marino and Matsusaka (2005), Scharfstein and Stein (2000), and Stein (2002). This strand of the literature commonly assumes the agent(s) to be an empire-builder, who perceives intrinsic values in the size of his own division/project.

As mentioned above, in the context of organizational design, few studies consider the ex-post efforts and incentives at the execution stage of a decision making process. Zábojník (2002) illustrates the benefit of delegation when both a principal and agent have private information about a state of the world, on which the productivity of the agent’s ex-post effort depends. In his model, the agent can be disincentivised by knowing the principal’s private information, and delegation allows the principal to avoid revealing such information through her orders. Bester and Krähmer (2008) might be the closest to the present study in that they explore the link between organizational design and ex-post incentives in the absence of asymmetric information. In their model, centralization can be beneficial as a com-

\textsuperscript{7}Jensen and Meckling (1992) also point out a similar tradeoff without a formal model.

\textsuperscript{8}In a somewhat different setting, Friebel and Raith (2010) show that when managers are protected by limited liability, inducing both high effort and truth-telling requires higher expected wage bills than inducing only high effort (without informative communication).
mitment device. However, delegation has no non-trivial benefits, and it can be optimal only when the second-best effort level is zero and the choice of organizational form boils down to a simple comparison between the bliss points of the principal and agent. Therefore, the present study and Bester and Krähmer (2008) shed light on quite different aspects of the organizational design problem.

Lastly, there also exists a vast literature on career concerns. The concept of career concerns is formalized by Holmström (1999), and Dewatripont et al. (1999a,b) further develop the general theory. Among various other applications, Ottaviani and Sørensen (2006a,b) study cheap-talk communication with career concerns, and are worth particular mention given the cheap-talk based literature described above. In their model, an agent has private information and his ability is measured by the accuracy of his signal.\footnote{See also Brandenburger and Polak (1996), Fox and Van Weelden (2012), Prat (2005) and Scharfstein and Stein (1990) for studies on this type of career concerns.} In the model of this chapter, the agent has no private information and his ability becomes relevant when implementing the decision.

\section{1.2 Baseline Model}

\subsection{1.2.1 Setup}

Suppose that a principal hires an agent to manage a project in her firm.\footnote{Following the convention, the principal and agent are referred to as “she” and “he” respectively.} Both the principal and agent are risk-neutral. The agent’s project will either succeed ($Y = 1$) or fail ($Y = 0$), and the probability of success depends on three elements: First, the project is more likely to succeed if the agent’s ability is high ($a = a_H$) than if it is low ($a = a_L$). We assume that the agent’s ability $a \in \{a_H, a_L\}$ is symmetrically
unknown to both the principal and agent, but it is common knowledge that the (prior) probability of being high type (resp. low type) is \( \mu \) (resp. \( 1 - \mu \)), which is strictly between 0 and 1. Second, the probability of success is also increasing in the amount of corporate resources, \( K \in \{0, 1\} \), allocated to this project.\(^{11}\) Third, the agent can increase the probability of success by exerting efforts, denoted by \( e \in \mathbb{R}_+ \), at the cost of \( \Psi(e) \). The level of efforts chosen by the agent is not observable to the principal or any third parties. Specifically, we will assume that the probability of success is given by

\[
\text{Prob}[Y = 1|a, K, e] = \min \{f(a, K) + e, 1\},
\]

where \( f(\cdot, \cdot) \) satisfies the following assumption.

**Assumption 1.1.** The function \( f(\cdot, \cdot) \) is positive, increasing in both arguments, and log-supermodular.

The assumption of log-supermodularity represents complementarity between the resources and ability: the effect of allocating the resources is higher if the agent’s ability is high, because such an agent can use the resources more efficiently.

The principal’s net profit is given by

\[
\Pi = Y - \gamma_\theta \cdot K - W_0,
\]

where \( \gamma_\theta \) is the opportunity cost of the corporate resources, and \( W_0 \) is the wage payment to the agent, which is determined by exogenous labor market conditions. The opportunity cost of the resources, \( \gamma_\theta \), depends on the state of the nature, \( \theta \in \{\theta_H, \theta_L\} \), and is either high (\( \gamma_H \)) or low (\( \gamma_L \)). The probability of \( \theta_L \) is denoted by

\[^{11}\text{Although we refer to } K \text{ as “resources” to fix the idea, it allows for other interpretation as well. See Section 1.2.2 below.}\]
p, and is common knowledge. To make the decision problem relevant, we further make the following assumption.

Assumption 1.2. It is socially optimal to choose $K = 1$ if and only if $\theta = L$. That is, $\gamma_L < \tilde{f}(1) - \tilde{f}(0) < \gamma_H$, where $\tilde{f}(K) := E_{a}[f(a, K)] = \mu f(a_H, K) + (1 - \mu) f(a_L, K)$.

The agent’s objective is to maximize his life-time income net of effort costs,

$$U = W_0 + W_1 - \Psi(e),$$

where $W_1$ is his future wage, which is determined by the outside labor market’s ex-post assessment of his ability. The outside market can observe the organizational form and the realization of $Y$, but not $\theta$, $K$, or $e$. (The information structure is summarized in Table 1.1.) Specifically,

$$W_1(Y, \hat{K}, \hat{e}) = \delta \cdot \text{Prob} \left[a = a_H | Y, \hat{K}, \hat{e}\right],$$

where $\hat{K} : \{\theta_H, \theta_L\} \rightarrow \{0, 1\}$ and $\hat{e} : \{\theta_H, \theta_L\} \rightarrow \mathbb{R}_+$ are the market expectations of the resource allocation and effort choice, as functions of the state $\theta$. Finally, we impose the following assumption on $\Psi(\cdot)$ so as to guarantee the existence of a unique, internal equilibrium.\(^{12}\)

Assumption 1.3. The cost of effort is $\Psi(e) = \frac{\psi}{2} e^2$ with $\psi > \delta(1 - f(a_H, 1))$.

Following the incomplete contracting literature, we assume that the state of nature, $\theta$, is hard to verify to a third party although it is observable to both the principal and agent. That is, the principal cannot write a binding contract to commit to a specific allocation rule $K(\cdot)$ as a function of $\theta$. Instead, she can commit to an organizational form, i.e., who will retain the decision right over the choice

\(^{12}\)In general, there may exist multiple equilibria. However, the key insight that delegation leads to a higher level of efforts is robust, as long as we focus on internal stable equilibria.
Table 1.1: Summary of Information Structure

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<th>Org. Form</th>
<th>State</th>
<th>Decision</th>
<th>Ability</th>
<th>Effort</th>
<th>Outcome</th>
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<td>Principal</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Agent</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Outside Market</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

of $K$ after $\theta$ realizes. We will refer to the organizational form as delegation (resp. centralization) if the agent (resp. the principal) has the decision right.

The timeline of the game is as follows.

1. The principal fixes the organizational form.
2. The state $\theta$ realizes.
3. The person with the decision right chooses $K$ after observing $\theta$.
4. The agent chooses his effort level $e$.
5. The outside labor market updates its belief on $a$, and $W_1$ is determined.

1.2.2 Discussion of the Modeling Assumptions

Interpretation of $K$. Although we refer to $K$ as the resource input above, it allows for other interpretations as well, as long as $K = 1$ is costly to the principal and complementary with the agent’s ability. For example, it can be viewed as a reduced form of a coordination problem within a multi-divisional organization. To be concrete, suppose that a firm produces two products and an agent is responsible for one of them. The decision variable is the design of a common component that is used for both of the two final products. Specifically, $K = 1$ is the design tailored for the agent’s final product that reduces the production cost of his division while...
raising the cost of the other division (compared to the other design \( K = 0 \)). Then, adopting \( K = 1 \) will raise the profit of the agent’s division at the cost of the other division, but the amount of increment will depend on the sales volume, which further depends on the agent’s marketing ability (i.e., \( a \)).

**Informational assumptions.** As we will see, two informational assumptions are critical in the subsequent analysis. First, we assume that the organization form is observable to the outside labor market. This assumption is critical for otherwise centralization will always be the unique equilibrium outcome, and it is closely related to the contractibility assumption on the decision right allocation. Although we simply assume that the principal can credibly delegate the decision right to the agent, it is often not easy to do so (see, e.g., Bolton and Dewatripont, 2012). Credible delegation might be available only by building some corporate culture or reputation, which would require substantial long-term commitment, and if so, such culture or reputation would be likely to be observable to the outside of the firm, at least in the long run. This scenario would also rationalize the timing of the first two stage, where the principal cannot adjust the organizational form in response to the short-run fluctuation of \( \theta \).

Second, we also assume \( K \) is unobservable to the labor market. This assumption is also critical in our analysis: If \( K \) is observable, the delegated agent will underinvest and exert even less effort than in the case of centralization. Roughly speaking, the agent can have an excuse for a bad performance by publicly choosing \( K = 0 \), which allows him to shirk. However, there are a few possible reasonings why \( K \) would not be publicly observable:

- A large part of the corporate resources would consist of worker’s firm-specific human capital, and the allocation of human capital (i.e., who works for what)
would be hard to monitor from the outside of the firm.

- Although we model the choice of $K$ as a one-shot decision, it could be interpreted as a sequence of daily decisions, each of which is hard to keep track of.

- Even if the principal can choose to publicly disclose $K$, it will be often suboptimal to do so. (See Section 1.2.4 for the details on this point.)

1.2.3 Analysis

Posterior belief and effort choice: Since the labor market cannot observe the values of $K$ and $e$, it will updates its belief based on the realized value of $Y$ and some expectations $\tilde{K}(\cdot)$ and $\tilde{e}(\cdot)$, which should be rational in equilibrium. Let $\tilde{\mu}_1(\tilde{K}, \tilde{e})$ and $\tilde{\mu}_0(\tilde{K}, \tilde{e})$ be the posterior probability that the agent is high type in the event of $Y = 1$ and $Y = 0$, respectively, i.e.,

$$\tilde{\mu}_1(\tilde{K}, \tilde{e}) = \frac{\text{Prob}[Y = 1|a_H, \tilde{K}, \tilde{e}]}{\text{Prob}[Y = 1|\tilde{K}, \tilde{e}]}$$  
and  
$$\tilde{\mu}_0(\tilde{K}, \tilde{e}) = \frac{\text{Prob}[Y = 0|a_H, \tilde{K}, \tilde{e}]}{\text{Prob}[Y = 0|\tilde{K}, \tilde{e}]}.$$

Then, after $\theta$ realizes and $K$ is chosen, the agent’s effort choice problem can be written as

$$\max_e E \left[ \delta \cdot \{ Y \cdot \tilde{\mu}_1 + (1 - Y)\tilde{\mu}_0 \} - \Psi(e)|\tilde{K}, \tilde{e} \right],$$
given $K$, $\tilde{K}$, and $\tilde{e}$. The first order condition is given by

$$\delta \cdot \Delta \tilde{\mu}(\tilde{K}, \tilde{e}) - \psi \cdot e^* = 0 \iff e^* = \frac{\delta}{\psi} \Delta \tilde{\mu}(\tilde{K}, \tilde{e}), \tag{1.2}$$
assuming this value is not too large, where $\Delta \tilde{\mu}(\tilde{K}, \tilde{e}) := \tilde{\mu}_1(\tilde{K}, \tilde{e}) - \tilde{\mu}_0(\tilde{K}, \tilde{e})$. Indeed, Assumption 1.3 guarantees this first-order condition is always necessary and sufficient.
Lemma 1.1. Under Assumption 1.3, the agent’s optimal effort level \( e^* \) is no greater than \( \bar{e} := 1 - f(a_H, 1) \) and satisfies equation (1.2), for any \( K, \tilde{K}(\cdot) \), and \( \hat{e} \).

Proof. See Appendix A.1.

Note that the market’s posterior belief is fully determined by \( Y, \tilde{K}, \text{and } \hat{e} \), and is independent of the true value of \( K \). Hence, the above Lemma implies the agent’s optimal level of effort is also independent of \( K \), and consequently, \( \hat{e}(\cdot) \) must be constant in equilibrium. In what follows, I slightly abuse notation and let \( \hat{e} \) be a scalar that represents the market expectation of the constant level of effort.

Decision under each organizational form: If the principal retains the decision right, on the one hand, the arguments in the previous paragraph imply that she cannot manipulate the agent’s effort level through the choice of \( K \). As a result, she has no incentive to deviate from the optimal decision rule for any given market expectations. If the agent is delegated the decision right, on the other hand, he will always have an incentive to choose \( K = 1 \), because \( D\hat{\mu} \) is always positive. That is, the equilibrium decision rule must be

\[
K^*_C(\cdot) = (K^*_C(\theta_H), K^*_C(\theta_L)) = (0, 1),
\]

under centralization, and,

\[
K^*_D(\cdot) = (K^*_D(\theta_H), K^*_D(\theta_L)) = (1, 1),
\]

under delegation.

Then, it remains to pin down the equilibrium effort level in each organizational form. With Assumption 1.1, we can show that the sensitivity of the posterior belief, \( \Delta\hat{\mu} \), is increasing in the amount of corporate resources (expected to be) allocated to the agent. Intuitively, by the log-supermodularity of \( f(\cdot, \cdot) \), the effect of the agent’s
ability is magnified and $Y$ becomes to convey more information regarding $a$, when $K = 1$ rather than $K = 0$.

**Lemma 1.2.** Under Assumptions 1.1 and 1.3, $\Delta \hat{u}(\tilde{K}, \tilde{e})$ is increasing in $\tilde{K}$ for any $\tilde{e} \in (0, \tilde{\varepsilon})$.

**Proof.** See Appendix A.1.

Combining this property with the equilibrium decision rules, we obtain the following proposition.

**Proposition 1.1.** Suppose that Assumptions 1.1–1.3 hold. Then, there exists a unique rational-expectation equilibrium for each organizational form. Delegation leads the agent to exert a higher level of effort than centralization, while it induces inefficient resource allocation. Delegation is more profitable to the principal if $\gamma_H$ is sufficiently close to $f(1)$.

**Proof.** See Appendix A.1.

### 1.2.4 Extensions and Discussions

**Ex-Ante Incentives**

As mentioned in the Introduction, the model presented above sheds light on the effect of organizational form on *ex-post* incentives to implement the project, while the majority of the existing studies focus on *ex-ante* incentives to gather information or to search for a good project. To highlight this distinction, this subsection extends the model incorporating both ex-ante and ex-post effort.

Suppose now that the agent chooses $p = \text{Prob}[\theta = \theta_L] \in [p_0, 1]$ at the cost of $g(p)$ after the principal fixes the organizational form. Assume that $g(\cdot)$ is strictly increasing and convex, satisfying $g'([p_0, 1)) = [0, \infty)$. If the agent has the decision rights, he has no incentive at all to increase $p$, because he can always allocate the
corporate resources to his own project even if the cost is high. In contrast, if the principal retains the decision right, the agent needs to “persuade” the principal to choose $K = 1$ by reducing its cost, i.e., centralization leads to a higher level of ex-ante effort than delegation.

**Proposition 1.2.** Suppose that the agent chooses the probability $p$ of $\theta = \theta_L$. In equilibrium, then, $e^*$ is higher but $p^*$ is lower under delegation than under centralization.

Proof. See Appendix A.1.

This proposition differentiates the present study from Aghion and Tirole (1997), although the key tradeoff in both studies can be summarized as “loss of control” versus “loss of incentives.” To see the difference, it should be noted that the benefit of delegation in Aghion and Tirole (1997) depends on a special structure of their model: They assume that the level of preference congruence is exogenously fixed and independent from the agent’s endogenous effort. Put differently, the agent cannot “persuade” the principal that his favorite decision is also beneficial to her unless she has no idea at all what to do. Indeed, it has been established that if the agent can persuade the principal, centralization induces a higher level of ex-ante effort than delegation, through its “winner picking” effect (see, e.g., Friebel and Raith (2010), Inderst and Laux (2005), and Stein (2002)). The main message of Proposition 1.2 is that delegation can have positive overall incentive effects even in the presence of “winner-picking.”

Apparently, the fact that the delegated agent has no ex-ante incentives at all heavily depends on the assumption that the state $\theta$ only affects the cost of $K = 1$. If instead the agent can increase the expected return from $K = 1$, then the agent

---

13The same is true for the soft information case in Stein (2002). The agent cannot persuade the principal if his information cannot be credibly revealed and thus no “winner-picking” exists with soft information.
will have an incentive to do so even if delegated. As a result, it is possible that delegation induces more effort than centralization both ex-ante and ex-post, and vice versa. However, in such a setting, we can still show that (i) delegation leads to a higher level of ex-post effort than centralization, taking a level of ex-ante effort as fixed, and (ii) it is never the case that delegation induces more ex-ante effort and less ex-post effort. (See Section A.2.1 for details.) Based on these results, one could still argue that delegation is relatively better at inducing ex-post efforts.

**Endogenous Transparency**

As mentioned in Section 1.2.2, a key in the above analysis is the assumption that the value of $K$ chosen by the principal or agent is not observable to the labor market. To scrutinize the role of this assumption, suppose now that the principal can also choose the “transparency” of her organization, i.e., whether to make $K$ publicly observable or not, at the first stage of the game.

On the one hand, transparency mitigates the commitment problem faced by the principal: If $K$ is publicly observable, its choice directly affects (the sensitivity of) the market belief, and thus, the principal will internalize the incentive effect of $K = 1$ at the decision-making stage. This allows her to replicate the equilibrium outcome under delegation, whenever it is profitable. On the other hand, transparency is harmful when the agent is delegated the decision. If it is observable to the labor market, the choice of $K$ does not change the expectation of the posterior belief. Thus, the agent is always better off by choosing $K = 0$ and lowering the equilibrium effort cost. Since the equilibrium effort level is lower and the decision rule is inefficient, delegation with observable $K$ is always less profitable than centralization. The following proposition summarizes these observations.

**Proposition 1.3.** Suppose that the principal can costlessly choose whether or not to make
Centralization with observable $K$ always weakly dominates delegation with unobservable $K$. Delegation with observable $K$ is always strictly dominated by centralization (with observable or unobservable $K$).

**Proof.** See above.

The first part of this proposition depends on the assumption that transparency is costless. If it is costly to make $K$ observable, no matter how small it is, delegation with unobservable $K$ becomes strictly optimal for a non-degenerated set of parameter values. Similarly, the optimality of delegation is also regained if there exists a direct cost of centralization. This would be the case, for example, if the principal incurs a cost to observe $\theta$ directly or indirectly (through the communication with the agent).

**Proposition 1.4.** Suppose that the principal can costlessly choose to make $K$ publicly observable, and that her payoff is given by $Y - \xi \cdot \mathbb{1}_C$, where $\xi > 0$ and $\mathbb{1}_C$ is an indicator function that takes 1 if and only if she chooses centralization. Then, there exists a cutoff $\gamma_H > \bar{f}(1)$ such that if $\gamma_H < \gamma_H$, then delegation with unobservable $K$ is strictly optimal for any $\xi > 0$.

**Proof.** This is an immediate corollary of Proposition 1.1.

Although $\xi$ could be seen as a consequence of asymmetric information between the principal and agent (or, the agent’s “local” information), it plays a much smaller role in the present model than in the communication-based models such as Dessein (2002). In Proposition 1.4, the threshold $\gamma_H$ is independent of $\xi$ and hence, delegation can be strictly optimal for a non-degenerate set of (the other) parameter values even when $\xi \downarrow 0$. This is not true in the communication-based models. If the principal can directly observe the state of nature in those models, centralization achieves
the first-best outcome for her. Therefore, for any values of the other parameters, such as the agent’s bias, centralization becomes optimal as the observation cost shrinks to zero.

The arguments in this subsection also has an implication for the case where the agent is motivated by internal promotions. If the agent’s career is determined by the belief of principal rather than the outside labor market, whether to disclose $K$ (in the case of delegation) could be reinterpreted as whether to monitor the decision made by the agent. Then, the above results indicate that the principal should not both delegate and monitor the decision, even if monitoring does not undermine the credibility of delegation. Roughly speaking, it could be restated that a delegated agent should be responsible for the final outcome rather than the interim choice.\[14\]

1.3 Multi-Tasking and Job Design

This section studies a variant of the baseline model in the previous section, and investigates how organizational design (i.e., decision right allocation) interacts with job design (i.e., task assignment) in a multi-tasking setting.

1.3.1 Setup

Suppose that the principal has two related tasks or projects, 1 and 2. For each task $t \in \{1, 2\}$, let $i_t$ denote the agent who is in charge of task $t$. The principal can hire either a single agent ($i_1 = i_2$) or two distinct agents ($i_1 \neq i_2$) to manage those tasks. The tasks are related in two respects. First, there is a single unit of resources that can be invested in either task. That is, the decision variable is now

\[14\] The idea that authority should be balanced with responsibility is also common in the practitioner literature of management (e.g., Steinmetz, 1976).
\[ K = (K_1, K_2) \in \{(0, 1), (1, 0)\}. \] Second, the two tasks are effort-cost complementary: an agent \( i \)'s effort cost is given by
\[
\Psi_i(e_{1,i}, e_{2,i}) = \frac{1}{2(1 - \eta^2)} \left( e_{1,i}^2 - 2\eta e_{1,i} e_{2,i} + e_{2,i}^2 \right), \tag{1.3}
\]
where \( e_{t,i} \geq 0 \) is \( i \)'s effort devoted for task \( t \), and \( \eta \in (0, 1) \) is the degree of technological complementarity.\(^{15}\)

Agent \( i \)'s ability is represented by \( a_i = (a_{1,i}, a_{2,i}) \sim \mathcal{N}(\bar{a}, \Sigma) \), where each \( a_{t,i} \) is his talent for task \( t \). The mean and variance are given by
\[
\bar{a} = (\bar{a}_1, \bar{a}_2), \quad \text{and} \quad \Sigma_a = \begin{bmatrix} \sigma_{a,1}^2 & 0 \\ 0 & \sigma_{a,2}^2 \end{bmatrix}.
\]
The assumption of independence between \( a_{1,i} \) and \( a_{2,i} \) is to isolate the interaction effect between delegation and specialization, and the case of correlated abilities will be discussed later in Section 1.3.3 and Appendix A.2.2. Further, the ability vector \( a_i \) is assumed to be drawn independently across agents.

The revenue from each task is given by
\[
Y_1 = \alpha K_1 + \beta K_1 (a_{1,i_1} - \bar{a}_1) + a_{1,i_1} + e_{1,i_1} + \varepsilon_1, \text{ and } Y_2 = \gamma K_2 + a_{2,i_2} + e_{2,i_2} + \varepsilon_2, \tag{1.4}
\]
where (i) \( \alpha, \beta, \gamma > 0 \) are positive parameters, (ii) \( \varepsilon_t \sim \mathcal{N}(0, \sigma_t^2) \) is a noise term for each task \( t \in \{1,2\} \). Each \( \varepsilon_t \) is independent from all other random variables. To focus on the most interesting cases, we also restrict the parameter values as follows.

**Assumption 1.4.** \( \alpha < \gamma \) and \( \sigma_{\varepsilon,1}^2 > (1 + \beta)\sigma_{a,1}^2 \).

\(^{15}\)Note that \( \Psi_i \) is positive and convex on \( \mathbb{R}^2_+ \), for it can be rewritten as
\[
\Psi_i = \frac{1}{2(1 - \eta^2)} \left[ (e_{1,i} - e_{2,i})^2 + 2(1 - \eta)e_{1,i} e_{2,i} \right],
\]
which is a sum of two positive convex functions.
Two points should be noted regarding the above production technology. First, only task 1 exhibits the complementarity between the agent’s ability and resource input. A possible interpretation would be that task 1 is a new, challenging line of business while task 2 is more established. That is, the efficient use of corporate resources for task 2 is well known and the agent can achieve the same return even if his ability is low, whereas he needs to find out by himself how to utilize the resource for task 1.

Second, agent $i_t$’s effort in task $t'$ does not increase $Y_{i'}$ if he is not in charge of $t'$ (i.e., if $i_t \neq i_{t'}$). This specification, along with technological complementarity in (1.3), is intended to model the coordination cost of specialization. When agent $i_t$ works for task $t$, a part of his work will be also useful for the other task $t'$. If he is assigned both tasks, on the one hand, he can utilize this by-product for task $t'$ without any communication. If $t'$ is assigned to a distinct agent $i_{t'}$, then the two agents need to communicate and coordinate with each other in order to efficiently exploit each other’s work (or, to avoid wasteful duplication). The above specification can be seen as an extreme case where there is no coordination at all in the case of specialization. The main qualitative insights will be unchanged for a broader class of specifications.\footnote{For example, $e_{t,ii}$ in equation (1.4) can be replaced with $e_{t,ii} + (1 - \zeta \cdot 1_S) e_{t,i'}$, where $\zeta \in (0, 1)$ is a parameter that captures the coordination cost, and $1_S$ is an indicator function that takes 1 if and only if $i_t \neq i_{t'}$.}

Agent $i$’s future wage is now given by

$$W_{1,i}(Y_1, Y_2, \hat{K}, \hat{\epsilon}) = \delta_0 + \delta_1 \cdot \mathbb{E}[a_{1,i}|Y_1, Y_2, \hat{K}] + \delta_2 \cdot \mathbb{E}[a_{2,i}|Y_1, Y_2, \hat{K}].$$

(1.5)

As in the baseline case, we assume $W_{0,i}$ is determined by the labor market conditions (and thus taken as given by the principal), and for notational simplicity, it is assumed to be zero. Hence, the principal and agent $i$ will maximize $Y_1 + Y_2$ and
\( W_{1,i} = \Psi_i(e_{1,i}, e_{2,i}), \) respectively.

**Organizational Forms**: We will consider the following three organizational forms:\(^{17}\)

- **Centralization (with no specialization)**: The principal hires a single agent (i.e., \( i_1 = i_2 \)), and retains the decision right over \( K \).

- **Decentralization with no specialization**: The principal hires a single agent (i.e., \( i_1 = i_2 \)), and delegates the decision right to the agent.

- **Decentralization with specialization**: The principal hires two agents (i.e., \( i_1 \neq i_2 \)) and delegates the decision right to the agent in charge of task 1 (i.e., \( i_1 \)).

**Timing**: The timing of the game is as follows.

1. First, the principal (publicly) fixes the organizational form.

2. Second, the person assigned the decision right chooses \( K \), which is unobservable to the outside market.

3. Third, effort levels are chosen by the agent(s).

4. Finally, \( Y_1 \) and \( Y_2 \) realize, the market updates its belief, and the game ends.

### 1.3.2 Analysis

To begin the analysis, note that for any given decision and effort levels, \( a_{t,i} \) and \( Y_t \) are jointly normal. By the well known formula, the conditional mean \( \mathbb{E}[a_{t,i} | Y_t, K, e] \)

\(^{17}\)With the assumptions we impose, the other organizational forms become never optimal. See also Section 1.3.3.
is linear in $Y_t$ and its slope is given by

$$\frac{\partial \mathbb{E}[a_{t,i}|Y_t,K,e]}{\partial Y_t} = \frac{\text{Cov}[Y_t,a_{t,i}|K]}{\text{Var}[Y_t|K]}.$$  

Hence, given the market expectations $\hat{K}$ and $\hat{a}_{t,i}$,

$$\frac{\partial \mathbb{E}[W_{1,i}|K,e,\hat{K},\hat{e}]}{\partial e_{t,i}} = \mathbb{E}_Y \left[ \frac{\partial W_{1,i}(Y_t,\hat{K},\hat{e})}{\partial Y_t} \cdot \frac{\partial Y_t}{\partial e_{t,i}} \right]
= \mathbb{E}_Y \left[ \delta_t \cdot \frac{\partial \mathbb{E}[a_{t,i}|Y_t,\hat{K},\hat{e}]}{\partial Y_t} \right] = \delta_t \cdot \frac{\text{Cov}[Y_t,a_{t,i}|\hat{K}]}{\text{Var}[Y_t|\hat{K}]}.$$  

(1.6)

Notice that as in the previous section, this sensitivity of the future wage depends on the market expectations, but not on the true value of $K$.

At the effort choice stage, agent $i$ solves

$$\max_{e_{1,i},e_{2,i}} \left\{ \mathbb{E}_Y [W_{1,i}] - \Psi_i(e_{1,i},e_{2,i}) \right\}.$$  

The first order conditions are given by $\nabla \mathbb{E}[W_{1,i}] - \nabla \Psi_i = 0$, which can be rewritten as

$$\begin{bmatrix} e_{1,i}^* \\ e_{2,i}^* \end{bmatrix} = \begin{bmatrix} 1 & \eta \\ \eta & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial W_{1,i}}{\partial Y_1} \\ \frac{\partial W_{1,i}}{\partial Y_2} \end{bmatrix},$$  

(1.7)

where each $\frac{\partial W_{1,i}}{\partial Y_i}$ depends on the organizational form and market expectations.

**Centralization versus Delegation with no specialization:** Suppose first that a principal assigns both tasks to a single agent $i$ while retaining the decision right. Then, since the agent’s effort choice is independent of $K$, the assumption of $\gamma > \alpha$ implies that the principal prefers investing the corporate resources into task 2, for any market expectation. That is, $K^* = \hat{K}^* = (0,1)$ must hold in equilibrium.
Plugging this into (1.6), we obtain

$$\frac{\partial W_{1,i}}{\partial Y_1} = \delta_1 \cdot \frac{\sigma_{a_1}^2}{\sigma_{a_1}^2 + \sigma_{\xi_1}^2} \equiv S_1,$$

and

$$\frac{\partial W_{1,i}}{\partial Y_2} = \delta_2 \cdot \frac{\sigma_{a_2}^2}{\sigma_{a_2}^2 + \sigma_{\xi_2}^2} \equiv S_2.$$

This implies that the principal’s expected profit under centralization is

$$\Pi^C = \gamma + (1 + \eta)(S_1 + S_2). \quad (1.8)$$

Next, suppose for a moment that the principal could commit to investing the resource into task 1 (i.e., $K = (1,0)$) while assigning both tasks to a single agent. Then the principal would face the same tradeoff as in the baseline case: Expecting $K = (1,0)$, the market would update the belief about $a_1$, so that

$$\frac{\partial W_{1,i}}{\partial Y_1} = \delta_1 \cdot \frac{(1 + \beta)\sigma_{a_1}^2}{(1 + \beta)^2 \sigma_{a_1}^2 + \sigma_{\xi_1}^2} \equiv S_1 + \Delta S_1,$$

where $\Delta S_1 > 0$ by the second part of Assumption 1.4. This increase in the sensitivity of $W_{1,i}$ would further incentivize the agent, while the principal would incur the cost of $\gamma - \alpha > 0$ by committing to $K = (1,0)$. Specifically, by the first-order condition (1.7), the total effort of the agent would increase by $(1 + \eta)\Delta S_1$ when $K = (1,0)$. That is, the principal would want to commit to allocating the resource to task 1 if and only if

$$(1 + \eta)\Delta S_1 > \gamma - \alpha \Longleftrightarrow \alpha > \gamma - (1 + \eta)\Delta S_1 \equiv \bar{\alpha}. \quad (1.9)$$

The question is, therefore, when delegation leads a non-specialized agent to invest in task 1. Given the market expectation $\hat{K}$, $W_{1,i}$ will increase by $\alpha \frac{\partial W_{1,i}}{\partial Y_1}$ if the agent chooses $K = (1,0)$, and by $\gamma \frac{\partial W_{1,i}}{\partial Y_2}$ if $K = (0,1)$. Therefore, under delegation without specialization, an equilibrium can support $K = (1,0)$ if

$$\alpha(S_1 + \Delta S_1) \geq \gamma S_2 \Longleftrightarrow \alpha \geq \frac{\gamma S_2}{S_1 + \Delta S_1} \equiv \alpha^*, \quad (1.10)$$

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and \( K = (0, 1) \) if
\[
\alpha S_1 \leq \gamma S_2 \iff \alpha \leq \frac{\gamma S_2}{S_1} \equiv \alpha^{**},
\]
where \( \alpha^* < \alpha^{**} \) by their definitions. Note that multiple equilibria exist if \( \alpha \in [\alpha^*, \alpha^{**}] \). In such a case, we will pick the more profitable equilibrium to the principal. Then, the principal’s (highest) equilibrium profit under delegation with no specialization will be
\[
\Pi^{DN} = \begin{cases} 
\Pi^C & \text{if } \alpha < \alpha^*, \\
\Pi^C + \max \{0, (1 + \eta) \Delta S_1 - (\gamma - \alpha)\} & \text{if } \alpha^* \leq \alpha \leq \alpha^{**}, \text{ and} \\
\Pi^C + (1 + \eta) \Delta S_1 - (\gamma - \alpha) & \text{if } \alpha^{**} < \alpha.
\end{cases}
\]
Comparing this expression with \( \Pi^C \) given by (1.8), we obtain the following intermediate result.

**Lemma 1.3.** If \( \alpha \in (\alpha^{**}, \bar{\alpha}) \), centralization strictly dominates delegation with no specialization. If \( \alpha > \max\{\bar{\alpha}, \alpha^*\} \), delegation with no specialization strictly dominates centralization. Otherwise, the two organizational forms are equally profitable.

**Proof.** Immediate from equations (1.8) and (1.12).

The role of specialization:

Comparing equations (1.9) and (1.10), notice that \( \bar{\alpha} < \alpha^* \) may hold. If \( \alpha \in (\bar{\alpha}, \alpha^*) \), neither centralization nor delegation with no specialization can induce \( K = (1, 0) \), even though it is the (ex-ante) optimal decision to the principal. In such a case, the role of specialization emerges despite the positive synergies from non-specialization.

Under delegation with specialization, the agent in charge of task 1 (i.e., \( i_1 \)) never has an incentive to allocate the resource to task 2, because in the case of specializa-
tion $Y_2$ has no statistical link to $i_1$’s ability and thus his wage is independent of $Y_2$. Therefore, the market rationally expects $K = (1, 0)$, which leads to

$$
\left( \frac{\partial W_{1,i_1}}{\partial Y_1}, \frac{\partial W_{1,i_1}}{\partial Y_2} \right) = (S_1 + \Delta S_1, 0) \quad \text{and} \quad \left( \frac{\partial W_{1,i_2}}{\partial Y_1}, \frac{\partial W_{1,i_2}}{\partial Y_1} \right) = (0, S_2).
$$

Plugging these into the first-order condition (1.7), it follows that

$$(e^*_{1,i_1}, e^*_{2,i_2}) = (S_1 + \Delta S_1, S_2),$$

and thus, the equilibrium profit is given by

$$
\Pi^{DS} = \alpha + (S_1 + \Delta S_1) + S_2
= \Pi^C + \Delta S_1 - (\gamma - \alpha) - \eta(S_1 + S_2).
$$

(1.13)

Therefore, the principal gets strictly better off by specialization if and only if

$$
\Delta S_1 - (\gamma - \alpha) - \eta(S_1 + S_2) > 0 \iff \alpha > \gamma + \eta(S_1 + S_2) - \Delta S_1 \equiv \bar{\alpha}.
$$

Recalling that specialization can be optimal only if $\alpha \in (\bar{\alpha}, \alpha^*)$, we can make the following observation.

**Lemma 1.4.** Delegation with specialization is optimal if $\alpha \in (\bar{\alpha}, \alpha^*)$. Otherwise, it is dominated by delegation with no specialization.

**Proof.** Immediate from equations (1.8), (1.12), and (1.13).

**Optimal organizational form:**

Combining Lemmas 1.3–1.4, the following proposition characterizes the optimal organizational form(s). It should be noted that when delegation with specialization is optimal, delegation and specialization are strictly complementary. That is, starting from centralization (with no specialization), (i) delegation without specialization
has no impact on the principal’s profit, and (ii) hiring two agents while retaining
the decision right strictly lowers the profit.\footnote{By the assumption of \( \eta > 0 \), centralization with specialization is always strictly dominated by
centralization with no specialization.}

**Proposition 1.5.** Fix parameters \((S_1, S_2, \Delta S_1, \beta, \gamma, \eta)\). Then,

- Centralization is uniquely optimal if \( \alpha \in (\alpha^{**}, \bar{\alpha}) \).
- Delegation with no specialization is uniquely optimal if \( \alpha > \max\{\bar{\alpha}, \alpha^*\} \).
- Delegation with specialization is uniquely optimal if \( \alpha \in (\bar{\alpha}, \alpha^*) \).
- Otherwise, centralization and delegation with no specialization are both optimal.

**Proof.** Immediate from Lemmas 1.3–1.4.

Figure 1.2 (a) illustrates the optimal organizational forms on the \((S_2, \alpha)\)-space,
holding the other parameters as fixed. It is noteworthy that a change in \( \eta \) has
opposite effects on the two forms of delegation, as shown in Figure 1.2 (b). On the
one hand, \( \eta \) represents the degree of technological complementarity, which cannot
be utilized in the case of specialization. Hence, it can been seen as the cost of
specialization, and delegation with specialization is less likely to be optimal when
\( \eta \) is larger. On the other hand, \( \eta \) also captures the benefit of delegation with no
specialization (over centralization), for it measures how much the non-specialized
agent will be motivated for task 2 by the increase in \( \frac{\partial [W_{i2}]}{\partial T_i} \). Therefore, delegation
with no specialization is more likely to optimal when \( \eta \) is larger. The following
proposition summarizes these comparative statics results.

**Proposition 1.6.** Fix parameters \((S_1, S_2, \Delta S_1, \alpha, \beta, \gamma)\). If delegation with no specialization
is strictly optimal for \( \eta \in (0, 1) \), then it is so for any \( \eta' \in (\eta, 1) \). If delegation with
specialization is strictly optimal for \( \eta \in (0, 1) \), then it is so for any \( \eta' \in (0, \eta) \).
(a) Typical case with intermediate value of $\eta$.

(b) Effect of lowering $\eta$.

**Figure 1.2**: Optimal organizational forms.
Proof. Since $\bar{a}$ is decreasing and $\bar{a}$ is increasing in $\eta$, the statement immediately follows from Proposition 1.5.

Figure 1.3 further illustrates the effects of a change in $\eta$. In the limit of $\eta = 0$, delegation with no specialization becomes never (uniquely) optimal, for its outcome is always identical either to centralization or to delegation with specialization. In contrast, delegation with specialization can be never optimal for any values of $\alpha$ and $S_2$, when $\eta$ is sufficiently close to 1 (holding the other parameters constant).

Finally, it is also noteworthy that the second-best effort level for task 1 is not necessarily decreasing in $\sigma_{\varepsilon_1}^2$. When $\sigma_{\varepsilon_1}^2$ increases, $\Delta S_1$ can also increase although both $S_1$ and $S_1 + \Delta S_1$ always decrease. If so, the principal may want to switch the organizational form, thereby discontinuously raising the second-best level of $e_{1,i_1}$.

**Proposition 1.7.** Fix all the parameters but $\sigma_{\varepsilon_1}^2$. The level of efforts for task 1, $e_{1,i_1}$, under the optimal organizational form may not be monotonically decreasing in $\sigma_{\varepsilon_1}^2$.

Proof. See Appendix A.1.

### 1.3.3 Discussion: Correlated Abilities

In the previous subsection, we assume that an agent’s abilities for the two tasks are statistically independent. Of course, however, it would be more realistic that $a_{1,i}$ and $a_{2,i}$ are (positively) correlated. If this is the case, specialization can have its own benefit even when the principal retains the decision right (i.e., even though $K = (0,1)$ no matter if specialized or not). Specifically, it can be shown that specialization is more profitable than no specialization if $\eta$ is sufficiently small.\(^{20}\)

---

\(^{19}\)Note that this is in contrast with the standard CARA-normal model of moral hazard (see, e.g., Bolton and Dewatripont (2004, Chapter 6.2)).

\(^{20}\)See Proposition A.3 in Section A.2.2 for details.
(a) Limit of $\eta = 0$.

(b) Sufficiently high $\eta$.

Figure 1.3: Optimal organizational forms (continued).
This is because the market’s posterior belief become less sensitive with respect to each $Y_t$ when the agent is assigned two tasks. This can be easily seen if we divide the update process into two steps. Suppose that starting from the unconditional prior, the market updates its belief on $a_{1,i_1}$ first to $a_{1,i_1} \mid Y_2$ and then to $a_{1,i_1} \mid Y_1, Y_2$. If $i_1$ specializes in task 1, $Y_2$ apparently contains no information about his ability, and thus the interim belief, $a_{1,i_1} \mid Y_2$, remains exactly the same as the prior. If he is assigned both tasks, in contrast, $Y_2$ conveys some information on $a_{1,i_1}$ (through $a_{2,i_1}$) and hence, the interim belief becomes less variable and less sensitive to further information (i.e., $Y_1$). This logic is parallel to Dewatripont et al. (1999b, Proposition 3.1–2), although their results are not directly comparable to ours.\(^{21}\) The purpose of the independence assumption above is to isolate the new kind of benefit of specialization, which arises only in the presence of delegation.

### 1.4 Conclusion

This chapter proposes a new model of organizational design based on agents’ career concerns, and illustrates a new possible explanation for the benefit of delegation. My arguments rely on a different set of assumptions from existing theories, and thus could apply to separate situations. For example, my theory could better explain delegation in small firms, where a principal can closely monitor agents’ activities and thereby directly alleviate the problems caused by the “local” information. Yet the analysis in this chapter has been kept very simple, and a number of important aspects are abstracted away. One of such missing aspects is the dynamic effects of organizational form. While we have restricted our attention to its impact on the agent’s current effort level, a change in signal quality should also affect

\(^{21}\)Namely, Dewatripont et al. (1999b) do not consider the possibility to hire multiple agents.
his future effort choice, optimal job assignment, and so on. An interesting avenue for future research would be to construct a tractable model accommodating such dynamics and to explore the design of career paths in organizations.
Chapter 2

Incentive Contracts with Signaling

2.1 Introduction

This chapter studies the incentive-contracting problem when a principal is better informed of an agent’s ability than the agent himself.\textsuperscript{1} While economists often assume the contrary, such a situation would naturally arise particularly in entry-level job markets. On the one hand, employers would know from their past experiences what attributes are important in determining one’s prospects in their jobs, and their recruitment process (e.g., interview questions) must be designed to best infer an applicant’s productivity by inspecting those important characteristics. On the other hand, although new workers might know well what attributes they do or do not have, they would be much more uncertain about how those attributes are converted into their productivity and/or how they are evaluated by potential employers. Then, it would be possible that at the contracting stage (after some screening process) an employer has a better estimate of a worker’s ability or pro-

\textsuperscript{1}I follow the convention of referring to a principal/employer as she and to an agent/employee as he.
ductivity than the worker himself.

Furthermore, if job-hunting is a sequential process, such informational asymmetry is not necessarily resolved by the time when a worker decides to accept or reject an offer. That is, even though he would be better informed of his (expected) ability after he observes the offers made by multiple employers, he may not be allowed to postpone his decision until he collects sufficient information. If so, he can optimally accept an offer and quit the search and learning process while he is still uncertain about his prospects. Actually, in many entry-level job markets such as the ones for new MBAs, law graduates, and clinical psychologists, employers often make an exploding offer, which expires unless a worker accepts within a very short time window (Roth and Xing, 1994, 1997), so as to prevent the worker from comparing multiple offers.

If an agent is uncertain about his own ability, his subjective belief could have two opposite effects. On the one hand, the higher his belief, the more effort he will exert conditional on the acceptance of an offer, because he believes his effort is likely to yield a better outcome and thus to lead a higher bonus payment. On the other hand, however, the higher his belief, the more rent he will require to accept a contract offered by the principal, since he believes he can get a better outside option by continuing his job search. Accordingly, the principal could also have two opposite incentives to manipulate the agent’s belief through her offer. That is, she might want to raise his belief in order to induce more effort, by offering a contract which is “appropriate” for a high-ability agent (e.g., high-powered incentives). Or she might want to lower it so as to let him accept an unfavorable offer, by offering a contract for a low-ability agent.

The purpose of this chapter is to investigate how those forces are balanced at an equilibrium and shape the optimal contracts and timing for the principal. Specif-
ically, I study a simple model of moral hazard with two dates, where a principal learns the agent’s ability at date 1 but the agent can do so only at date 2. I analyze equilibria satisfying the D1 criterion (Cho and Kreps, 1986; Cho and Sobel, 1990), which is a standard equilibrium refinement in the signaling-game literature, and characterize the most profitable outcomes to the principal among those equilibria. As a result, it turns out that the principal optimally makes a same exploding offer to any type of agent at date 1, if the agent’s outside option is sufficiently sensitive to the agent’s ability; otherwise, it is weakly optimal to delay an offer until date 2. Interestingly, the principal’s highest profit is not monotonically decreasing in the agent’s outside option. If the outside option for a high-ability agent increases, it decreases the incentive for the principal to wait until date 2 because she will need to give a larger rent to him after the information reveals. Hence, pooling at date 1 becomes easier to sustain as an equilibrium, which is profitable for the principal because she can induce a higher level of effort from a low-ability agent. Conversely, if the incentive compatibility binds with the high-ability agent, a marginal decrease in his outside option forces the principal to always wait until date 2, even though pooling is strictly profitable when she face the low-type agent. Consequently, her profit may discontinuously decrease by such a marginal change in the agent’s reservation wage.

The rest of this chapter is organized as follows. Section 2.1.1 briefly discusses the related literature. Section 2.2 sets up the model and Section 2.3 presents the analysis. Section 2.4 makes a few concluding remarks. Appendix B.1 contains omitted proofs.
2.1.1 Related Literature

The most related to this chapter are the studies in the contract theory literature that investigate the effect of a principal’s private information about the profitability or production technology of her business (see, e.g., Bénabou and Tirole, 2003; Inderst, 2001; Silvers, 2012; Spier, 1992). This is because both the profitability in those papers and the agent’s ability in this chapter are modeled as a parameter of a production function. However, a key distinction is that the agent’s ability in this chapter is also correlated with his outside option while the principal’s information is purely firm-specific in those papers. The correlation between the principal’s knowledge and the agent’s outside options creates additional effects of the agent’s belief, and thereby makes it optimal for the principal to make early, exploding offers. Relatedly, De la Rosa (2011) and Santos-Pinto (2008) study how the agent’s over-confidence or positive self-image affects the optimal incentive scheme, but they also assume that the agent’s reservation utility is independent of the agent’s belief.

More broadly, this chapter is related to the literature on the mechanism design problem with an informed principal (e.g., Myerson, 1983; Maskin and Tirole, 1992). Compared to this literature, it should be noted that the analysis in this chapter (as well as the studies mentioned above) implicitly assumes that the principal cannot force the agent to commit to work for her, without fully specifying the contract. In contrast, the mechanism design literature typically assume that the principal can delay to reveal her private information, as long as a mechanism satisfies participation constraints in expectation. In the current context, this means that the principal

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2Put differently, the model in this chapter could be alternatively viewed as the situation where a principal privately knows about non-firm-specific profitability (e.g., the forecasts of business conditions) that affects both the production function inside the firm and the outside option for the agent.
can offer a contract saying, e.g., “if you sign this contract you must work for me, and I will choose the incentive scheme from this pre-specified set after you sign.” In the environment of this chapter, allowing such contracts may or may not be beneficial to the principal depending on parameter values, but it generally increases the incentive for early contracting.

As this chapter implicitly assumes that the principal learns the agent’s type before other employers make competitive offers, it also loosely relates to the personnel economics literature that examine the asymmetric information about a worker’s ability between the current and other potential employers (e.g., Waldman, 1984; Gibbons and Katz, 1991). However, it should be noted that these papers study the competition for mid-career workers, while this chapter is intended to model entry-level labor markets. Note also that workers’ beliefs are irrelevant in those papers, for they are assumed to simply take the offer with the highest (fixed) wage.

### 2.2 Model

Suppose that a principal hires an agent to run a project. If the agent decides to work for the principal, the project will either succeed ($Y = 1$) or fail ($Y = 0$). The probability of success is given by

$$\text{Prob}[Y = 1] = \min\{e \cdot \theta + 2F, 1\},$$

where $e \geq 0$ is the agent’s effort and $\theta \in \{H, L\} \subseteq \mathbb{R}_+$ is his ability. Without any loss, we assume $H > L$.\(^3\) The prior probability of $\theta = H$ (resp. $\theta = L$) is denoted by $p$ (resp. $1 - p$) and is common knowledge between the principal and

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\(^3\) However, $L$ should not be interpreted as the lowest productivity among a pool of qualified agents rather than among all potential workers. See Section 2.4 for details.
agent. The prior mean of $\theta$ is denoted by $M = pH + (1 - p)L$. To induce effort, the principal can offer a contingent bonus $b \in [0, 1]$, which is paid if and only if $Y = 1$.\(^4\) Taking the incentive bonus $b$ as fixed, the principal’s profit and the agent’s utility are $\Pi(b, Y) = (1 - b)Y$ and $U(b, Y, e) = bY - \frac{c}{2}e^2$, where $c$ is a cost parameter.

To guarantee an internal solution to the effort choice problem, we assume $c$ is sufficiently high:

**Assumption 2.1.** The cost parameter is sufficiently high: $c > H^2/(1 - 2F)$.

If the agent decides not to work for the principal, he will return to an outside labor market and find another employer. The principal’s profit in this case is assumed to be zero.\(^5\) The value of the outside option for the agent, contingent on $\theta$, is denoted by $u_\theta$, with $u_H \geq u_L$. The prior expectation of $u_\theta$ is $u_M = pu_H + (1 - p)u_L$.

To simplify the analysis and focus on the main insights, we also assume the following:

**Assumption 2.2.** The expected value of the agent’s outside option is sufficiently low: $u_M \leq 0$.\(^6\)

As motivated in the introduction, the principal could better estimate $\theta$ during the hiring process, whereas the agent would be uncertain what outside options he could expect until he indeed goes through the process with other employers. To model such a situation, suppose that there are two dates, 1 and 2, and that

\(^4\)An implicit assumption here is that the agent is wealth-constrained.

\(^5\)This does not necessarily mean that the principal cannot find another employee. As mentioned in footnote 3, the agent in this model should be seen as sufficiently qualified even when $\theta = L$, and it could be very costly to find an equally qualified worker even if it is possible.

\(^6\)All of the results will remain qualitatively the same under a much weaker assumption that $\beta_{H \theta}^{PC} < \beta_H^b$ (see equations (2.1)–(2.2) for the definitions of these variables). However, minor changes will be required in both the statements and proofs, because we need to consider more subcases when the participation constraint is binding for the $L$-type.
the principal and agent can directly learn $\theta$ at the beginning of date 1 and date 2, respectively. Further, the principal can make a contract offer either at date 1 or date 2, and in the case of an early offer, she can also force the agent to take it or leave it by the end of date 1 (i.e., the offer is exploding). Although the agent cannot directly observe $\theta$ at date 1, he rationally updates his belief if an early offer is made. Without any loss, I identify his subjective belief with the expectation of $\theta$ according to that belief. In what follows, let $\theta(b) \in [L, H]$ denote the agent’s belief after observing an exploding offer of $b \in [0,1]$, and $u_\theta := \frac{\theta-L}{H-L} u_H + \frac{H-\theta}{H-L} u_L$ be the subjective expectation of the outside option when the belief is $\theta$.\footnote{It is unnecessary to specify the belief after no offer at date 1, because he will know $\theta$ for sure at the beginning of date 2.} To summarize, the timeline of the game is as follows:

0. The nature draws $\theta$ from the prior distribution.

1. After the principal observes $\theta$ through the screening process, she decides whether to make an exploding offer $b \in [0,1]$ to the agent, or wait until date 2 (denoted by $b = \emptyset$). If she makes an offer, the agent decides to accept or reject it after updating his belief.

   (a) If the agent accepts the offer, he choose his effort level, the output $Y$ realizes, and the payoffs are finalized.

   (b) If the agent rejects, the agent gets the outside option $u_\theta$ and the principal’s profit is zero.

2. If the principal did not make an exploding offer in date 1, she offers $b \in [0,1]$ after $\theta$ becomes publicly observable.
(a) If the agent accepts the offer, he choose his effort level, the output $Y$ realizes, and the payoffs are finalized.

(b) If the agent rejects, the agent gets the outside option $u_\theta$ and the principal’s profit is zero.

2.3 Analysis

Preliminaries

To analyze the model backwardly, first consider the agent’s effort choice problem. If the agent accepts a contract $b$ with belief $\theta \in [L, H]$, he will solve

$$\max_e \left[ b\theta e - \frac{c}{2}e^2 \right],$$

which yields $e^*(b, \theta) = \frac{b\theta}{c}$ under Assumption 2.1. Hence, the principal’s expected profit will be

$$\Pi(b, \theta, \theta) := (2\Delta - b) \left[ \frac{b\theta}{c} + 2F \right].$$

Given belief $\theta$, the agent optimally accepts an offer $b$ if and only if

$$b \in \text{Ac}(\theta) := \left\{ b \in [0, 1] : \frac{1}{2} \left( \frac{b\theta}{c} \right)^2 \geq u_\theta \right\}.$$  

When $\theta$ becomes publicly observable at date 2, the principal’s problem is to maximize $\Pi(b, \theta, \theta)$ subject to $b \in \text{Ac}(\theta)$. The optimal offer is $b = \beta^F_\theta := \max \{ \beta^e_\theta, \beta^{PC}_\theta \}$, where

$$\beta^e_\theta := \arg \max_{b \in [0,1]} \Pi(b, \theta, \theta) = \frac{1}{2} - \frac{cF}{\theta^2}, \quad \text{and} \quad \beta^{PC}_\theta := \min \left( \text{Ac}(\theta) \cup \{1\} \right),$$

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and the principal’s unique equilibrium profit with full information is \( \tilde{\Pi}(\beta^F, \theta, \theta) \).\(^8\)

In what follows, we characterize the most profitable (pure strategy) perfect Bayesian equilibria to the principal among those satisfying the D1 criterion (Cho and Kreps, 1986; Cho and Sobel, 1990). Since the principal can always wait until date 2, each \( \theta \)-type principal must also earn non-negative expected profits in any equilibrium. Therefore, the principal can never profitably deviate from an equilibrium by offering \( b \) that will be rejected. Taking this observation into account, we can define the D1 criterion in this model as follows.

**Definition 2.1.** Fix an equilibrium and let \( \Pi^*_\theta \) denote the \( \theta \)-type principal’s equilibrium profit, for each \( \theta \in \{ H, L \} \). A pair \((b, \theta)\) is said to be deleted by the D1 criterion if (i) for all \( \theta \in [L, H] \), \( \tilde{\Pi}(b, \theta, \theta) \geq \Pi^*_\theta \) implies \( \tilde{\Pi}(b, \theta, \theta') > \Pi^*_\theta \), and (ii) there exists \( \theta \in [L, H] \) such that \( b \in \text{Ac}(\theta) \) and \( \tilde{\Pi}(b, \theta, \theta') > \Pi^*_\theta \), where \( \theta' \in \{ H, L \} - \{ \theta \} \).\(^9\)

A belief system \( \theta^*(\cdot) : [0, 1] \rightarrow [L, H] \) is said to satisfy the D1 criterion if \( \theta^*(b) = \theta' \) whenever \((b, \theta)\) is deleted. The equilibrium is said to satisfy the D1 criterion if its associated belief system satisfies the D1 criterion.

The following Lemma is useful in the subsequent analysis to restrict the belief systems that satisfy the D1 criterion. Although this corresponds to the standard sorting condition in the signaling game literature, it is effective only when the contracts are accepted by the agents. As a consequence, it does not always eliminate the possibility of pooling equilibria.\(^{10}\)

\(^8\)Notice that \( \beta^F_\theta \) is defined to be 1 when \( \text{Ac}(\theta) \) is empty, and \( \tilde{\Pi}(1, \theta, \theta) = 0 \). Hence, the principal’s full-information profit is still \( \tilde{\Pi}(\beta^F_\theta, \theta, \theta) \) even in the case where no contract is signed at an equilibrium.

\(^9\)Notice that this definition is based only on the agent’s pure best replies while the standard definition uses mixed best replies. However, this difference is irrelevant here because the pure best reply is generically unique.

\(^{10}\)Technically, a key property of the present model is that the principal’s profit is not continuous
Lemma 2.1. Suppose that \( b' > b \). If \( \tilde{\Pi}(b', \theta', L) \geq \tilde{\Pi}(b', \theta, L) \), then \( \tilde{\Pi}(b', \theta', H) > \tilde{\Pi}(b', \theta, H) \).

Proof. See Appendix B.1.

Optimal Equilibria

Now we are ready to characterize the most profitable equilibrium outcomes among those satisfying the D1 criterion. The first result establishes that the full-information outcome is always supportable by an equilibrium.

Proposition 2.1. There always exists an equilibrium satisfying the D1 criterion such that the principal never offers an exploding offer at date 1.

Proof. See Appendix B.1.

Notice that the equilibrium outcome in Proposition 2.1 is always (weakly) more profitable than any separating equilibria, because the agent must correctly and certainly know \( \theta \) on the equilibrium-path of such equilibria. The question is, therefore, whether and when pooling equilibria exist. The next proposition fully characterizes the condition for pooling at date 1 to be supportable. This condition immediately implies that whenever a pooling equilibrium exists, it yields a weakly higher profit than any separating equilibria.\(^{11}\)

Proposition 2.2. There exists a pooling equilibrium satisfying the D1 criterion, where the principal offers a same \( b \in [0, 1] \) to both types of the agent at date 1, if and only if \( \tilde{\Pi}(b, M, \theta) \geq \tilde{\Pi}(b^F, \theta, \theta) \) for each \( \theta \in \{L, H\} \).

with respect to the agent’s belief, because a marginal increase in \( \theta \) can make the agent to switch from accepting to rejecting an offer and thereby discontinuously lower the profit. This is why the results in Cho and Sobel (1990) do not apply here.

\(^{11}\)See Section 2.4 for a brief discussion regarding semi-pooling equilibria.
Proof. See Appendix B.1.

**Corollary 2.1.** Suppose that some pooling equilibrium at date 1 satisfies the D1 criterion. Then, there exists a pooling equilibrium at date 1 that yields a (weakly) higher profit for the principal than any other pure strategy equilibrium surviving the D1 criterion.

**Proof.** The is an immediate corollary of Proposition 2.2.

**Comparative Statics**

Since Propositions 2.1 and 2.2 pin down the principal’s highest equilibrium profit for a fixed set of parameters, the next question should be how it varies with changes in the parameters. Actually, the principal’s profit has a few interesting comparative statics properties with respect to changes in the agent’s outside option(s). So as to keep Assumption 2.2 intact against such variations, now take $u_M$ as fixed and let $D := u_H - u_L \geq 0$ be a free parameter. That is, higher $D$ means both higher $u_H$ and lower $u_L$. Note that $D$ would increase when the principal’s assessment of $\theta$ becomes more accurate or more correlated with the assessments by other potential employers.

As $D$ increases from 0, the participation constraint for the $H$-type becomes more and more binding where as that for the $L$-type remains slack. Therefore, the principal’s profit from separating at date 2 is (weakly) decreasing in $D$. In contrast, the constraint $\bar{\Pi}(b, M, H) \geq \bar{\Pi}(\beta^F_L, H, H)$ becomes less demanding when $D$ rises, for the right-hand side decreases in $u_H$. Consequently, the set of supportable pooling equilibria at date 1 is (weakly) increasing in $D$, and so is the profit at the optimal equilibrium as long as it is non-empty.

To further look into the behavior of optimal pooling equilibria, let $\beta^1_M := \arg\max_b \bar{\Pi}(b, M, H)$ and $\beta^2_M$ be such that $\beta^2_M > \beta^F_L$ and $\bar{\Pi}(\beta^2_M, M, L) = \bar{\Pi}(\beta^F_L, L, L)$. 

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That is, $\beta^1_M$ is the point that minimizes the principal’s incentive to reveal $\theta = H$, and $\beta^2_M$ is the maximal point at which the principal has no incentive to reveal $\theta = L$. Also define $\beta^*_M := \arg \max_b \tilde{\Pi}(b, M, M)$ to denote the best possible pooling outcome. Observe that $\beta^*_M < \beta^1_H$ must hold by definition. Based on these variables, then, we need to consider three cases.

- First, suppose that $\beta^*_M < \beta^1_M < \beta^2_M$: The set of supportable pooling contract becomes non-empty when $D$ hits $D^*$, which is defined to be the point such that $\tilde{\Pi}(\beta^1_M, M, H) \geq \tilde{\Pi}(\beta^*_H, H, H)$ holds with equality. At this point, the unique supportable pooling equilibrium yields the expected profits of $\tilde{\Pi}(\beta^1_M, M, M)$. Furthermore, the assumption of $\beta^1_M < \beta^2_M$ implies $\tilde{\Pi}(\beta^1_M, M, L) > \tilde{\Pi}(\beta^*_L, L, L)$ and hence,

$$\tilde{\Pi}(\beta^1_M, M, M) = p \cdot \tilde{\Pi}(\beta^1_M, M, H) + (1 - p) \tilde{\Pi}(\beta^1_M, M, L) > p \cdot \tilde{\Pi}(\beta^*_H, H, H) + (1 - p) \tilde{\Pi}(\beta^*_L, L, L),$$

i.e., the principal’s profit at this unique pooling equilibrium is strictly higher than her full-information profits. Since no pooling equilibrium is supportable at any $D < D^*$, the profits at the optimal equilibrium jumps up at $D = D^*$. When $D$ further increases to $D^* + \delta$, pooling becomes supportable on a larger interval of $b$, containing $\beta^1_M$ in its interior. As $\beta^1_M > \beta^*_M$ by definitions, we can conclude that the principal’s profit is strictly right-increasing at $D = D^*$.

- Second, suppose that $\beta^*_M < \beta^2_M \leq \beta^1_M$: Then, the first supportable pooling is $b_H = b_L = \beta^2_M$, which becomes supportable at $D = D^*$, which is now defined by $\tilde{\Pi}(\beta^2_M, M, H) = \tilde{\Pi}(\beta^*_H, H, H)$. When $D$ further increases, pooling at $\beta^2_M - \epsilon$ becomes supportable, and it is strictly more profitable because $\tilde{\Pi}(\cdot, M, M)$ is decreasing at $\beta^2_M$ by the assumption of $\beta^*_M < \beta^2_M$. That is, the principal’s
profit is strictly right-increasing at $D = D^*$. In this case, however, it is con-
tinuous because $\bar{\Pi}(\beta_M^2, M, L) = \bar{\Pi}(\beta_L^F, L, L)$ holds by definition, as well as
$\bar{\Pi}(\beta_M^2, M, H) = \bar{\Pi}(\beta_H^F, H, H)$ at $D = D^*$.

- Lastly, suppose that $\beta_M^2 \leq \beta_M^*$: As in the second case, the first supportable
pooling is at $\beta_M^2$ and the principal’s profit is continuous at $D = D^*$. Yet, the
profit is not right-increasing in this case, because $\bar{\Pi}(\cdot, M, M)$ is increasing on
$[0, \beta_M^2]$, and pooling at $b > \beta_M^2$ becomes never supportable.

For each of these three cases, the trajectory of the principal’s profit at the optimal
equilibrium is illustrated in Figure 2.1. To summarize, we have established the
following proposition.

**Proposition 2.3.** Fix $u_M$ as given and take $D$ to be the free parameter. There exists a cutoff
$D^*$ such that in the optimal equilibrium, the principal offers a same exploding offer to both
types of agents if and only if $D \geq D^*$. With respect to $D$, the principal’s profit at the optimal
equilibrium is weakly (resp. strictly) decreasing on $[0, D^*)$ (resp. in a left neighborhood of
$D^*), and weakly increasing on $[D^*, \infty]$. Further, it is strictly right-increasing at $D^*$ if
$\beta_M^2 > \beta_M^*$, and discontinuously jumps up at $D^*$ if $\beta_M^2 > \beta_M^1$.

**Proof.** See the above arguments. $\blacksquare$

Since the decrease in $u_L$ is relevant only by keeping $u_M$ constant, the principal’s
profit has similar comparative statics properties with respect $u_H$ as long as $u_M$ is
sufficiently low. In particular, we have the following corollary.

**Corollary 2.2.** Given $u_L$ is sufficiently low, the principal’s profit at the optimal equilibrium
may be discontinuous and increasing in $u_H$.

**Proof.** The is an immediate corollary of Proposition 2.3. $\blacksquare$
Figure 2.1: Principal’s highest profit as a function of $D$. 

(a) $\beta_M^* \beta^1_M < \beta^2_M$. 

(b) $\beta_M^* \beta^2_M \beta^1_M$. 

(c) $\beta^2_M \beta^*_M$. 

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2.4 Discussions

This chapter studies an incentive-contracting problem with the assumption that the principal can learn about the agent’s productivity before the agent himself. I characterize the most profitable equilibria to the principal and illustrate how such (adversely) asymmetric information can cause the principal to make early, exploding offers. It should be noted that in the present model, the time length between date 1 and 2 could be very short, as is often observed in reality, if the hiring schedule in the outside market is sufficiently tight. This could be seen as a possible advantage compared with the existing models of unraveling (e.g., Halaburda, 2010; Li and Rosen, 1998; Li and Suen, 2000), where the time window is defined by the evolution of public information on match qualities. One might wonder, however, if the principal’s information is really essential in the above analysis, because when the optimal equilibrium is pooling, she would be (weakly) better off by offering $\beta^*_M$ before learning $\theta$ herself. This argument is completely valid if the above model is taken literally, but not necessarily in general. Suppose, as briefly mentioned in footnotes 3 and 5, that there exists a third type of agents and the principal prefers vacancy to hiring such agents. If the cost of hiring those unqualified agents is sufficiently high, the principal would strictly prefer to make offers after she learns $\theta$ but before the agent does.

In terms of comparative statics, this study suggests, unlike the standard moral hazard models, an increase in the ($H$-type) agent’s reservation wage may benefit the principal. An interesting implication of this result is that wage competitions at later dates may actually enhance unraveling. If an employer wants to hire more qualified agents but such agents are scarce at date 2, she might consider to raise the wage she offers, hoping it would let more agents to remain until date 2. However,
our model would suggest that such a wage increase would actually lead more competitors to make an exploding offer, and consequently, the qualified agents could become even more scarce at date 2. This argument is of course informal, because the present model abstracts away from such strategic interactions among employers. It would be an interesting avenue for future research to extend the model and study market equilibria, endogenizing the agent’s outside options.

To conclude, let us briefly discuss the possibility of semi-pooling equilibria at date 1. Indeed, semi-pooling can be an equilibrium and moreover, strictly optimal for the principal. To see this, suppose that $\beta^*_M < \beta^2_M < \beta^1_M$ and $D = D^*$. Then, pooling is profitable to the principal but $\tilde{\Pi}(\beta^2_M, M, H) = \tilde{\Pi}(\beta^F_H, H, H)$ is a binding constraint. If the $L$-type principal randomize between $b_L = \beta^*_L$ and $b^L = \beta^2_M + \varepsilon$, it is both strictly profitable and incentive compatible for the $H$-type principal to offer $b_H = \beta^2_M + \varepsilon$ with probability one, because $\tilde{\Pi}(\beta^2_M + \varepsilon, \theta, H) > \tilde{\Pi}(\beta^F_H, H, H)$, where $\theta > M$ is the conditional belief at $b = \beta^2_M + \varepsilon$. Further, randomization is also incentive compatible for the $L$-type principal if $\varepsilon$ is sufficiently small and $\theta$ is sufficiently close to $M$. Thus, such a mixed strategy forms an equilibrium and is more profitable than full pooling at $\beta^2_M$. However, semi-pooling equilibria may not necessarily fill the “gap” in the principal’s profit, because randomization by the $L$-type principal cannot increase her own profit, whereas it is the jump in the $L$-type’s profit that gives rise to the discontinuity in Proposition 2.3.12

12It is easy to check randomization by the $H$-type principal is never profitable, because it will decrease the conditional expectation at the point where both types pool.
Chapter 3

Stable and Strategy-Proof Rules in Matching Markets with Contracts

3.1 Introduction

Since its birth by Gale and Shapley (1962), the theory of two-sided matching markets has been centered around the deferred acceptance algorithm, which is known to satisfy a number of desiderata. Specifically, in the classic setup, it is not only stable but also (one-sided) strategy-proof (Dubins and Freedman, 1981; Roth, 1982), and moreover, it is the unique such rule (Alcalde and Barberà, 1994). More recently, Hatfield and Milgrom (2005) propose a more general model of matching with contracts and, among many other things, verify that the deferred acceptance is stable and strategy-proof under certain conditions, called substitutable contracts and the law of aggregated demand.¹ With those conditions, Sakai (2011) also generalizes the uniqueness result and hence, the study of stable and strategy-proof rules is

¹See Definitions C.1 and C.4 in Appendix C.1 for the definitions of those conditions.
necessarily the study of the deferred acceptance. At the same time, however, the rapid developments of the matching with contracts literature has started to cover, in both theory and practice, the cases that violate those pre-known conditions for the deferred acceptance to be stable and/or strategy-proof (e.g., Dimakopoulos and Heller, 2014; Hatfield and Kojima, 2010; Kominers and Sönmez, 2013, 2014).

With those developments in mind, the purpose of this chapter is to study stable and (one-sided) strategy-proof rules as generally as possible, and to disentangle some nature of such rules that arise purely from the two properties. Specifically, our results only require the choice functions on the hospital side to satisfy the irrelevance of rejected contracts (henceforth, IRC) condition, which is a very mild rationality requirement, but not any kind of “substitutes” condition.

The first and second results (Theorems 3.1–3.2) are on the uniqueness of stable and strategy-proof rules and extend the existing results mentioned above: Theorem 3.1 states that the number of such rules is at most one in general (as long as the IRC is satisfied), although there may or may not exist one without additional assumptions. Further, Theorem 3.2 establishes that the doctor-optimal stable rule is the unique candidate for a stable and strategy-proof rule, whenever it is well-defined, although it may or may not be strategy-proof without additional assumptions. From the technical point of view, it is noteworthy that our proofs of Theorems 3.1–3.2 do not rely on the rural hospital theorem, which has been playing a central role in the existing studies on stable and strategy proof rules. Instead, our proofs exploit a weaker property of stable allocations, which follows solely from the IRC condition, and shed new light on the fundamental tension between stability and strategy-proofness.

Our last main result, Theorem 3.3, is on the constrained optimality of a stable

\footnote{See the discussion after Corollary 3.2 for details.}
and strategy-proof rule. Namely, we show (again, under the IRC condition) that a stable and strategy-proof rule, if exists, is never dominated in terms of doctor welfare by any other individually rational and strategy-proof rule. Further, we also show in an appendix that the same holds true even if stability is weakened to non-wastefulness in the above statement (Theorem C.1). These results (partially) generalize similar existing results in the school choice literature (e.g., Abdulkadiroglu et al., 2009; Kesten, 2010; Kesten and Kurino, 2013).

The rest of this chapter is organized as follows. Section 3.2 describes the model and introduces key concepts. Section 3.3 provides the results. Section 3.4 concludes. Appendix C.1 introduces a number of definitions that are omitted in this chapter but useful to relate our results to the literature. Appendix C.2 presents two variants of Theorem 3.3. Appendix C.3 contains a few examples.

### 3.2 Preliminaries

We study the standard setting of a many-to-one matching market with contracts. Let $D$ and $H$ be finite sets of doctors and hospitals, respectively. The finite set of possible contracts is given by $X \subseteq D \times H \times \Theta$ for some $\Theta$.\(^3\) For each contract $x \in X$, let $d(x)$ and $h(x)$ be its projections onto $D$ and $H$, i.e., $x = (d(x), h(x), \theta)$ for some $\theta \in \Theta$. In other words, $x$ is a bilateral contract between $d(x) \in D$ and $h(x) \in H$.

A subset $X' \subseteq X$ of contracts is said to be an allocation if it includes at most one contract for each doctor, i.e., if $x, x' \in X'$ and $x \neq x'$ imply $d(x) \neq d(x')$. The set of all possible allocations is denoted by $\mathcal{X} \subseteq 2^X$. For each allocation

\(^3\)For example, $\Theta$ can be interpreted as the set of possible wage levels (Kelso and Crawford, 1982) and/or job descriptions (Roth, 1984).
and doctor $d \in D$, let $x(d, X')$ denote the contract that $X'$ assigns to $d$; i.e., $x(d, X') = x$ if $x \in X'$ and $d(x) = d$. If there is no such contract in $X'$, doctor $d$ is said to be assigned a null-contract and we write $x(d, X') = \emptyset$. Similarly, let $X(h, X') = \{x \in X' : h(x) = h\}$ be the set of (non-null) contracts that $X'$ assigns to hospital $h \in H$.

Each doctor $d \in D$ has a strict preference relation $\succ_d$ over $\{x \in X : d(x) = d\} \cup \{\emptyset\}$. The domain of all possible preferences for doctor $d$ is denoted by $\mathcal{P}_d$. Given his preference relation $\succ_d$, a non-null contract $x$ is said to be acceptable to doctor $d$ if $x \succ_d \emptyset$. The set of acceptable contracts to doctor $d$, as a function of $\succ_d$, is given by

$$\text{Ac}(\succ_d) := \{x \in X : d(x) = d \text{ and } x \succ_d \emptyset\}.$$ 

The profile of the doctors’ preference relations is denoted by $\succ_D = (\succ_d)_{d \in D}$. Let $\mathcal{P}_D := \prod_{d \in D} \mathcal{P}_d$ be the domain of all possible preference profiles. Each hospital $h \in H$ has a choice function $C_h : 2^X \to \mathcal{X}$ such that for all $X' \subset X$, (i) $C_h(X') \in 2^{X' \cap \mathcal{X}}$ and (ii) $h(x) = h$ for all $x \in C_h(X')$. Throughout this chapter, we assume that the choice functions satisfy the following mild requirement: Hospital $h$’s choice function $C_h(\cdot)$ is said to satisfy the irrelevance of rejected contracts (henceforth, IRC) condition if $x \notin C_h(X' \cup \{x\})$ implies $C_h(X' \cup \{x\}) = C_h(X')$ for all $X' \subset X$ and $x \in X$. The profile of the hospitals’ choice functions is denoted by $C_H(\cdot) = (C_h(\cdot))_{h \in H}$.

Given $\succ_D$ and $C_H(\cdot)$, we define the following concepts on $\mathcal{X}$: An allocation

---

4Note that $x \succ_d \emptyset$ implies $d(x) = d$ since $\succ_d$ is defined over $\{x \in X : d(x) = d\} \cup \{\emptyset\}$.

5Aygün and Sönmez (2012, 2013) point out the importance of this condition, which is implicitly assumed in Hatfield and Milgrom (2005) and Hatfield and Kojima (2010). Note that this condition is satisfied if a choice function is induced by a strict preference over subsets of contracts (as in some of the examples below).
$X' \in \mathcal{X}$ is said to be individually rational if (i) $x(d; X') \succeq_d \emptyset$ for all $d \in D$, and (ii) $C_h(X') = X(h, X')$ for all $h \in H$. A pair of a hospital $h \in H$ and a subset $X'' \subset X$ of contracts is said to block an allocation $X'$ if (i) $C_h(X' \cup X'') = X'' \neq C_h(X')$ and (ii) $x((d, C_h(X' \cup X'')) \succeq_d x((d, X'))$ for all $d \in \{d(x)\}_{x \in C_h(X' \cup X'')}$.

An allocation $X'$ is said to be stable if it is individually rational and not blocked by any $(h, X'') \in H \times 2^X$. An allocation $X'$ is said to strictly dominate another allocation $X'' \neq X'$ if $x((d, X')) \succeq_d x((d, X''))$ for all $d \in D$. A stable allocation $X^*$ is said to be doctor-optimal if it strictly dominates any other stable allocation.

Given $C_H(\cdot)$ as well as $(D, H, X)$, a matching rule is a mapping $f : \mathcal{P}_D \rightarrow \mathcal{X}$, which associates each possible preference profile of doctors with an allocation. A rule $f(\cdot)$ is said to be stable (resp. individually rational) if for all $\succ_D \in \mathcal{P}_D$, its output $f(\succ_D)$ is stable (resp. individually rational) with respect to $\succ_D$. Similarly, the doctor-optimal stable rule, denoted by $X^*(\cdot)$ if exists, is a rule such that for all $\succ_D \in \mathcal{P}_D$, its output $X^*(\succ_D)$ is the doctor-optimal stable allocation with respect to $\succ_D$. A rule $f(\cdot)$ is said to strictly dominate another rule $g(\cdot) \neq f(\cdot)$ if $x((d, f(\succ_D))) \succeq_d x((d, g(\succ_D)))$ for all $d \in D$ and $\succ_D \in \mathcal{P}_D$. Finally, a rule $f(\cdot)$ is said to be strategy-proof if $x((d, f(\succ_D))) \succeq_d (d, f(\succ_D'))$ for all $d \in D$, $\succ_D \in \mathcal{P}_D$, and $\succ_D' \in \mathcal{P}_d$, where $\succ_D-\{d\} = (\succ_D')_{d' \in D-\{d\}}$.

### 3.3 Results

To start our analysis, we introduce the following weaker notion of blocking coalitions: We say that a pair $(h, X'') \in H \times 2^X$ weakly blocks an allocation $X'$ if

---

6 Note that $X' \neq X''$ and $x((d, X')) \succeq_d x((d, X''))$ for all $d$ imply $x((d', X')) \succ_d x((d', X''))$ for some $d'$.

7 Again, $g(\cdot) \neq f(\cdot)$ implies $x((d', f(\succ_D))) \succ_d x((d', g(\succ_D)))$ for some $d'$ and $\succ_D$ if $f(\cdot)$ strictly dominates $g(\cdot)$.

---
(i) \( C_h(X' \cup X'') \neq C_h(X') \) and (ii) \( x(d, C_h(X' \cup X'')) \succeq_d x(d, X') \) for all \( d \in \{d(x)\}_{x \in C_h(X' \cup X'')} \). This definition is weak in that the first part does not require \( C_h(X' \cup X'') = X'' \). Under the IRC condition, however, it is straightforward to verify that the two blocking concepts are equally effective in the following sense.

Lemma 3.1. Suppose that hospital \( h \)'s choice function \( C_h(\cdot) \) satisfies the IRC condition. For any allocation \( X' \in \mathcal{X} \), then, there exists \( X'' \subset X \) such that \( (h, X'') \) blocks \( X' \) if and only if there exists \( X''' \subset X \) such that \( (h, X''') \) weakly blocks \( X' \).

Proof. The “only if” part is immediate from the definitions. To see the “if” part, suppose that \( (h, X''') \) weakly blocks \( X' \), and let \( X'' := C_h(X' \cup X''') \). Then, the IRC condition implies \( C_h(X' \cup X'') = C_h(X' \cup X''') = X'' \) and hence, the first requirement for \( (h, X'') \) to block \( X' \) is satisfied. The second requirement is also trivially satisfied by the assumption that \( (h, X''') \) weakly blocks \( X' \).

Lemma 3.1 leads to the following observation, which will be the key in the proofs of Theorems 3.1–3.2.

Lemma 3.2. Suppose that every hospital \( h \in H \) has a choice function \( C_h(\cdot) \) that satisfies the IRC condition, and that \( X' \) and \( X'' \) are two distinct stable allocations at \( (C_H, \succ_D) \). Then, there exists a doctor \( d \in D \) who is assigned distinct non-null contracts by \( X' \) and \( X'' \), i.e., \( \emptyset \neq x(d, X') \neq x(d, X'') \neq \emptyset \).

Proof. The proof is by contraposition. Assume the negation of the consequent, i.e.,

\[
\left[ x(d, X') \neq x(d, X'') \right] \implies \left[ \emptyset \in \{x(d, X'), x(d, X'')\} \right], \text{ for all } d \in D,
\]

where \( X' \) and \( X'' \) are two (possibly identical) stable allocations at \( (C_H(\cdot), \succ_D) \). Since \( X'' \) is stable (and thus individually rational), this implies for all \( d \in D \),

\[
\left[ x(d, X'') \neq \emptyset \right] \implies \left[ x(d, X'') \succ_d \emptyset = x(d, X') \text{ or } x(d, X') = x(d, X'') \right],
\]

54
and hence,

\[ x(d, X'') \neq \emptyset \implies [x(d, X'') \succeq_d x(d, X')]. \]

For an arbitrary hospital \( h \in H \), then, \((h, X'')\) satisfies the second requirement to weakly block \( X' \). Since \((h, X'')\) cannot weakly block \( X' \) by stability and Lemma 3.1, it must violate the first requirement; i.e., \( C_h(X' \cup X'') = X(h, X') \) must hold. As the symmetric arguments also imply \( C_h(X' \cup X'') = X(h, X'') \) for all \( h \in H \), it follows that \( X(h, X') = X(h, X'') \) for all \( h \in H \) and thus \( X' = X'' \). \[\Box\]

Our first main result generalizes the existing results on the uniqueness of a stable and strategy-proof rule by Alcalde and Barberà (1994, Theorem 3) and Sakai (2011, Theorem 1). While this theorem does not require any substitutes condition, its proof depends on Lemma 3.2, which in turn necessitates the IRC condition. See Example C.1 in Appendix C.3 for a counterexample in the absence of the IRC condition.

**Theorem 3.1.** Suppose that every hospital \( h \in H \) has a choice function \( C_h(\cdot) \) that satisfies the IRC condition. Then, there exists at most one stable and strategy-proof rule; i.e., if \( f(\cdot) \) and \( g(\cdot) \) are both stable and strategy-proof, \( f(\succeq_D) = g(\succeq_D) \) for all \( \succeq_D \in \mathcal{P}_D \).

**Proof.** Towards a contradiction, suppose that there exist two distinct stable and strategy-proof rules, \( f(\cdot) \) and \( g(\cdot) \). Let \( \succeq_D^* \in \mathcal{P}_D \) be a preference profile such that \( f(\succeq_D^*) \neq g(\succeq_D^*) \) and

\[
\left[ f(\succeq_D) \neq g(\succeq_D) \implies \sum_{d \in D} |\text{Ac}(\succeq_d)| \geq \sum_{d \in D} |\text{Ac}(\succeq_d^*)| \right] \text{ for all } \succeq_D \in \mathcal{P}_D,
\]

which exists by assumption. Then, by Lemma 3.2, there must exist a doctor \( d^* \) such that \( \emptyset \neq x(d^*, f(\succeq_D^*)) \neq x(d^*, g(\succeq_D^*)) \neq \emptyset \). Note that this also implies \( |\text{Ac}(\succeq_{d^*})| \geq 2 \).
Now, suppose without loss of generality that $x(d^*, f(\succ_D^{*})) \succ_D^{*} x(d^*, g(\succ_D^{*}))$, and let $\succ_D^{**} := (\succ_D^{**}, \succ_D-\{d^*\})$, where $\succ_D^{**}$ is a preference relation of doctor $d^*$ such that only $x(d^*, f(\succ_D^{*}))$ is acceptable, i.e., $\text{Ac}(\succ_D^{**}) = \{x(d^*, f(\succ_D^{*}))\}$. Notice that

$$x(d^*, f(\succ_D^{**})), x(d^*, g(\succ_D^{**})) \in \{x(d^*, f(\succ_D^{*})), \emptyset\},$$

since $f(\cdot)$ is assumed to be stable (and hence individually rational). Then, the strategy-proofness of $f(\cdot)$ and $g(\cdot)$ implies

$$x(d^*, f(\succ_D^{**})) = x(d^*, f(\succ_D^{*})), \text{ and}$$

$$\emptyset = x(d^*, g(\succ_D^{**})) \neq x(d^*, g(\succ_D^{*})),$$

respectively, and hence, $f(\succ_D^{**}) \neq g(\succ_D^{**})$. This, however, contradicts the definition of $\succ_D^{**}$, since $|\text{Ac}(\succ_D^{**})| = 1 < 2 \leq |\text{Ac}(\succ_D^{*})|$ and $\succ_D^{**} = \succ_D-\{d^*\}$, and the proof is complete. 

Theorem 1, together with the results by Kominers and Sönmez (2013, 2014), immediately implies the following corollary.\footnote{See Definitions C.5 and C.6 in Appendix C.1 for the definitions of slot-specific priorities and the cumulative offer process.}

**Corollary 3.1.** Suppose that every hospital $h \in H$ has a choice function $C_h(\cdot)$ that is induced by some slot-specific priorities. Then, the cumulative offer process induces the unique stable and strategy-proof rule.

**Proof.** In the case of slot-specific priorities, Kominers and Sönmez (2013, 2014) show that the rule induced by the cumulative offer process is both stable and strategy-proof. The uniqueness follows from Theorem 3.1. 

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8See Definitions C.5 and C.6 in Appendix C.1 for the definitions of slot-specific priorities and the cumulative offer process.
Following the same line of proof, we can also show that whenever it exists, the doctor-optimal stable rule is the unique candidate for a stable and strategy rule. Note, however, that this unique candidate may or may not be strategy-proof without additional assumptions.

**Theorem 3.2.** Suppose that every hospital \( h \in H \) has a choice function satisfying the IRC condition, and that the doctor-optimal stable allocation \( X^* (\succ_D) \) exists for all \( \succ_D \in \mathcal{P}_D \). If \( f (\cdot) \) is a stable and strategy-proof rule, then \( f (\succ_D) = X^* (\succ_D) \) for all \( \succ_D \in \mathcal{P}_D \).

**Proof.** Towards a contradiction, suppose that the doctor-optimal stable rule \( X^* (\cdot) \) is well-defined, and that \( f (\cdot) \neq X^* (\cdot) \) is a stable and strategy-proof rule. Let \( \succ^*_D \in \mathcal{P}_D \) be a preference profile such that \( f (\succ_D) \neq X^* (\succ_D) \) and

\[
\left[ f (\succ_D) \neq X^* (\succ_D) \implies \sum_{d \in D} |\text{Ac}(\succ_d)| \geq \sum_{d \in D} |\text{Ac}(\succ^*_d)| \right] \text{ for all } \succ_D \in \mathcal{P}_D,
\]

which exists by assumption. Then, by Lemma 3.2, there must exist a doctor \( d^* \) such that \( \emptyset \neq x (d^*, f (\succ^*_D)) \neq x (d^*, X^* (\succ^*_D)) \neq \emptyset \). Note that this also implies \( x (d^*, X^* (\succ^*_D)) \succ^*_d x (d^*, f (\succ^*_D)) \succ^*_d \emptyset \).

Now let \( \succ^{**}_D \) := \( \left( \succ^{*+}_D, \succ^*_D - \{d^*\} \right) \), where \( \succ^{*+}_D \) is a truncation of \( \succ^*_D \) at \( x (d^*, X^* (\succ^*_D)) \), i.e., a preference relation such that

\[
\text{Ac}(\succ^{**}_d) = \left\{ x \in X : x \succeq^{**}_d x (d^*, X^* (\succ^*_D)) \right\},
\]

and

\[
[x \succ^{**}_d y \iff x \succ^*_d y] \text{ for all } x, y \in \text{Ac}(\succ^{**}_d).
\]

Notice that \( X^* (\succ^{**}_D) = X^* (\succ^*_D) \) by construction.\(^9\) Together with the strategy-

---

\(^9\)To see this, note first that any stable allocation at \( \succ^*_D \), including \( X^* (\succ^*_D) \), is also stable at \( \succ^{**}_D \). For \( X^* (\succ^{**}_D) \neq X^* (\succ^*_D) \) to hold, therefore, \( X^* (\succ^{**}_D) \) cannot be stable at \( \succ^*_D \). Yet, \( x (d^*, X^* (\succ^{**}_D)) = \emptyset \) must hold if \((h, X')\) blocks \( X^* (\succ^{**}_D) \) at \( \succ^*_D \) but not at \( \succ^{**}_D \). Since \( x (d^*, X^* (\succ^{**}_D)) \succ^*_d \emptyset \), this means that \( X^* (\succ^{**}_D) \) cannot dominate \( X^* (\succ^*_D) \) at \( \succ^{**}_D \), which is a contradiction.
proofness of \( f(\cdot) \), this further implies
\[
  x(d^*, X^*(\succ_D^{**}))) = x(d^*, X^*(\succ_D^{**})) \succ_d^* x(d^*, f(\succ_D^{**})) \succeq^*_d x(d_1, f(\succ_D^{**})) ,
\]
and hence, \( f(\succ_D^{**}) \neq X^*(\succ_D^{**}) \). This, however, contradicts the definition of \( \succ_D^* \), since
\[
  \text{Ac}(\succ_d^*) \subset \text{Ac}(\succ_d^*) - \{x(d^*, f(\succ_D^{**}))\} \quad \text{and} \quad \succ_{D-\{d\}}^* = \succ_{D-\{d\}}^*,
\]
and the proof is complete. 

Along with Theorem 5 of Hatfield and Kojima (2010), Theorem 3.2 leads to the following corollary.\(^{10}\)

**Corollary 3.2.** Suppose that every hospital \( h \in H \) has a choice function \( C_h(\cdot) \) that satisfies the unilateral substitutes condition and the IRC condition. Then, if a stable and strategy-proof rule exists, it is induced by the doctor-proposing deferred acceptance algorithm.

**Proof.** Under the supposition, a doctor-optimal stable allocation always exists and is induced by the doctor-proposing deferred acceptance algorithm (Hatfield and Kojima, 2010, Theorem 5).\(^{11}\) Hence, the claim immediately follows from Theorem 3.2.

Compared to the existing uniqueness results, Theorems 3.1–3.2 above are technically novel for two related reasons. First, our proof of Theorem 3.1 requires no dominance relation between \( f(\cdot) \) and \( g(\cdot) \). Consequently, it is applicable even when the doctor-optimal stable allocation does not always exist. In contrast, the uniqueness results by Alcalde and Barberà (1994, Theorem 3) and Sakai (2011, Theorem 1) are established by showing any stable rule that is strictly dominated by the doctor-optimal stable rule cannot be strategy-proof and hence, the existence of the

---

\(^{10}\)See Definition C.2 in Appendix C.1 for the definition of unilateral substitutes.

\(^{11}\)See also Aygün and Sönmez (2012).
doctor-optimal stable rule is critical in their proofs. Second, our proofs do not call for the \textit{rural hospital theorem} either, which states that every agent (i.e., every doctor and every hospital) signs the same number of non-null contracts across all stable allocations. Instead we utilize Lemma 3.2, which could be seen as a weaker version of the rural hospital theorem but holds true without any restrictions on $C_H(\cdot)$ other than the IRC condition.\textsuperscript{12} It is this distinction that makes the proof of Theorem 3.2 non-trivial, although its statement might look very close to the previous results.

Given the second point in the previous paragraph, one might wonder if the doctor-optimal stable rule can be strategy-proof even when the rural hospital theorem fails to hold. Theorem 3.2 would be vacuous if there is no such case, since the doctor-optimal stable rule has been shown to be strategy-proof whenever the rural hospital theorem holds.\textsuperscript{13} Indeed, there exist such cases and hence, Theorem 3.2 applies to a strictly larger domain of choice functions than the previous results.

\textbf{Fact 3.1.} The doctor-optimal stable rule can be strategy-proof even if there exists a preference profile such that not all doctors sign the same number of non-null contracts across all stable allocations (and hence, even if the rural hospital theorem fails to hold).

\textit{Proof.} The proof is by example. Suppose that $D = \{d_1, d_2, d_3\}$, $H = \{h, h'\}$, and $X = \{x_i, x'_i\}_{i \in \{1,2,3\}}$, where $x_i$ (resp. $x'_i$) represents a contract between doctor $d_i$ and hospital $h$ (resp. $h'$). The choice functions of the hospitals, $C_h(\cdot)$ and $C_{h'}(\cdot)$, are induced by preference profiles

\[
\succsim_h : \{x_1\} \succsim_h \{x_2, x_3\} \succsim_h \{x_2\} \succsim_h \{x_3\} \succsim_h \emptyset, \text{ and}
\]

\[
\succsim_{h'} : \{x'_2\} \succsim_{h'} \{x'_1\} \succsim_{h'} \emptyset,
\]

\textsuperscript{12}Note that the conclusion of Lemma 3.2 immediately follows if the rural hospital theorem holds.

\textsuperscript{13}See the proofs of Hatfield and Kojima (2010, Theorem 7) and Hatfield and Milgrom (2005, Theorem 11).
Table 3.1: Doctor optimal stable allocations in the proof of Fact 3.1. The rows and columns represent the preferences of doctor $d_1$ and $d_2$, respectively.

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(a) Case of $x_3 \in \text{Ac}(\succ_{d_3})$.

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(b) Case of $x_3 \notin \text{Ac}(\succ_{d_3})$.

Respectively. Notice that the resulting choice functions satisfy the substitutes condition (Definition C.1 in Appendix C.1) and thus, the doctor-optimal stable allocation exists for any $\succ_D$, as summarized in Table 3.1.

To see the rural hospital theorem fails to hold in this market, fix a preference $\succ_h$ and $\succ_{h'}$, respectively. That is, for each $X' \subset X$, $C_h(X')$ and $C_{h'}(X')$ are the most preferred subsets of $X' \cap X(h, X)$ and $X' \cap X(h', X)$ according to $\succ_h$ and $\succ_{h'}$, respectively.
profile $\succeq_D$ such that

\begin{align*}
\succeq_{d_1} : & \quad x_1' \succeq_{d_1} x_1 \succeq_{d_1} \emptyset, \\
\succeq_{d_2} : & \quad x_2 \succeq_{d_1} x_2' \succeq_{d_1} \emptyset, \text{ and}
\end{align*}

$x_3 \in \text{Ac}(\succeq_{d_3})$. As shown in the colored cell in Table 3.1, the doctor-optimal stable allocation at such $\succeq_D$ is $X^* = \{x_1', x_2, x_3\}$, whereas there exists another stable allocation $X_* = \{x_1, x_2'\}$. Note that doctor $d_3$ is assigned a non-null contract at $X^*$ but not at $X_*$, and hence, the rural hospital theorem fails.

To complete the proof, it remains to verify that the doctor-optimal stable rule, $X^*(\cdot)$, is strategy-proof in this market. For doctors $d_1$ and $d_2$, note that $x(d_1, X^*(\succeq_D))$ and $x(d_2, X^*(\succeq_D))$ are independent of $\succeq_{d_3}$. Hence, the incentives for doctors $d_1$ and $d_2$ to manipulate will remain unchanged if $d_3$ is excluded from the market. Actually, once $d_3$ is omitted, the remaining market reduces to a standard one-to-one matching market without contracts and thus, $d_1$ and $d_2$ have no incentive to manipulate the doctor-optimal stable rule (Dubins and Freedman, 1981; Roth, 1982). For doctor $d_3$, observe that $x(d_3, X^*(\succeq_D))$ is either $x_3$ or $\emptyset$, and that it depends on $\succeq_{d_3}$ only through whether or not $x_3 \in \text{Ac}(\succeq_{d_3})$. Therefore, $d_3$ has no incentive to report that $x_3$ is acceptable when it is not, and vice versa. In sum, the doctor-optimal stable rule is strategy-proof in this market, and the proof is complete.

Another natural question that stems from Theorem 3.2 would be whether or not the necessary condition for a stable and strategy-proof rule can be replaced with a weaker notion of optimality. Specifically, one might wonder if a stable and strategy-proof rule always chooses an allocation that is not dominated by another
stable allocation. Actually, the answer to this question is known to be negative.\textsuperscript{15}

**Fact 3.2** (Kominers and Sönmez, 2014). *A stable and strategy-proof rule may choose an allocation that is strictly dominated by another stable allocation.*

**Proof.** In the case of slot-specific priorities, Kominers and Sönmez (2014) show that the cumulative offer process may not choose the doctor-optimal stable allocation even when it exists (Example 4), whereas it always induces a stable and strategy-proof rule (Theorem 4).

Given that the outcomes of a stable and strategy-proof rule may not necessarily be doctor-optimal even among the stable allocations, it could be of policy interest whether the doctor welfare can be Pareto-improved. Since Theorem 3.1 implies such improvement is impossible maintaining both stability and strategy-proofness, it would be natural to ask if it becomes possible once we weaken the stability requirement. Our last main result, Theorem 3.3, shows that such improvement is generally impossible. This extends the existing results in the school choice literature that the student-optimal stable rule is second-best optimal among strategy-proof rules (see, Abdulkadiroğlu et al., 2009; Kesten, 2010; Kesten and Kurino, 2013).\textsuperscript{16}

**Theorem 3.3.** *Suppose that every hospital $h$ has a choice function $C_h(\cdot)$ that satisfies the IRC condition. Then, no individually-rational and strategy-proof rule strictly dominates a stable and strategy-proof rule.*

**Proof.** Towards a contradiction, suppose that $f(\cdot)$ is individually rational and strategy-proof, $g(\cdot)$ is stable and strategy-proof, and that $f(\cdot)$ strictly dominates $g(\cdot)$. Let

\textsuperscript{15}In matching markets without contracts, contrastingly, Pathak and Sönmez (2013, Lemma 1) establish that the dominance in terms of outcomes between two stable rules implies the dominance in terms of manipulability. For an extension of this result, see also Chen et al. (2014).

\textsuperscript{16}See also Erdil (2014) and Anno and Kurino (2014) for related results in assignment problems.
\( \succ_D^* \in \mathcal{P}_D \) be a preference profile such that \( f(\succ_D^*) \neq g(\succ_D^*) \) and

\[
\left[f(\succ_D) \neq g(\succ_D) \implies \sum_{d \in D} |\text{Ac}(\succ_d)| \geq \sum_{d \in D} |\text{Ac}(\succ_d^*)| \right] \quad \text{for all } \succ_D \in \mathcal{P}_D,
\]

which exists by assumption. Then, there must exist \( d^* \in D \) such that

\[
x(d^*, f(\succ_D^*)) \succ_d^* x(d^*, g(\succ_D^*)) \succ_d^* \emptyset.
\]

(3.1)

To see this, suppose contrarily that for all \( d \in D \), \( x(d, f(\succ_D^*)) \succ_d^* x(d, g(\succ_D^*)) \) implies \( x(d, g(\succ_D^*)) = \emptyset \). This requires \( f(\succ_D^*) \supseteq g(\succ_D^*) \) and hence, for some \( h \in H \),

\[
C_h(f(\succ_D^*) \cup g(\succ_D^*)) = C_h(f(\succ_D^*)) = X(h, f(\succ_D^*))
\]

\[
\neq X(h, g(\succ_D^*)) = C_h(g(\succ_D^*)),
\]

where the second and last equalities hold by the individual rationality of \( f(\cdot) \) and \( g(\cdot) \), respectively. Therefore, \( (h, f(\succ_D^*)) \) weakly blocks \( g(\succ_D^*) \), but by Lemma 3.1, this contradicts the stability of \( g(\cdot) \).

Now, take a new preference relation \( \succ_{d^*}^{**} \) of \( d^* \) such that \( \text{Ac}(\succ_{d^*}^{**}) = \{x(d^*, f(\succ_D^*))\} \), and let \( \succ^{**} = (\succ_{d^*}^{**}, \succ_D^* \setminus \{d^*\}) \). Then, the strategy-proofness of \( f(\cdot) \) and \( g(\cdot) \) implies

\[
x(d^*, f(\succ_{d^*}^{**})) = x(d^*, f(\succ_D^*)) \succ_d^* x(d^*, g(\succ_D^*)) \succ_d^* x(d^*, g(\succ_{d^*}^{**})),
\]

and thus, \( f(\succ_{D}^{**}) \neq g(\succ_{D}^{**}) \). However, this contradicts the definition of \( \succ_D^* \) since \( |\text{Ac}(\succ_{d^*}^{**})| < |\text{Ac}(\succ_{d^*}^*)| \) and \( \succ_{D^*}^{**} \setminus \{d^*\} = \succ_{D^*} \setminus \{d^*\} \), and the proof is complete.

In Appendix C.2, we provide two variants of this theorem. First, we show that the same claim holds even if stability is weaken to non-wastefulness. Second, we also show that the stable and strategy-proof rule in the above statement can be
replaced with the doctor-optimal stable rule. That is, if the doctor-optimal rule is not strategy-proof, two possible policy goals, to achieve strategy-proofness and to unambiguously improve doctor welfare (by relaxing stability requirement), are incompatible with each other.

3.4 Concluding Remarks

This chapter studies the model of many-to-one matching with contracts, and derives a number of properties that a stable and strategy-proof rule must generally satisfy. A notable feature of our approach is that we only impose a minimal structure on hospitals’ choice function, i.e., the IRC condition, and do not rely on any algorithmic properties of matching rules. On the one hand, this allows us to directly illuminate the joint implication of stability and strategy-proofness. On the other hand, our abstract approach does not tell much about how to identify a stable and strategy-proof rule in practical applications.

It remains for future research, for example, to characterize conditions with which the cumulative offer process is the unique candidate for a stable strategy-proof rule. As shown by Kominers and Sönmez (2013, 2014), it is both stable and strategy-proof in the case of slot-specific priorities, even though such markets fail to satisfy some pre-known conditions that guarantee the deferred acceptance algorithm to be stable, doctor-optimal, or strategy-proof.\footnote{Namely, both the unilateral substitutes condition and the law of aggregate demand may fail in those markets.} Furthermore, our Example C.2 in Appendix C.3 show that similar cases arise even if the choice functions are not generated by slot-specific priorities. This might suggest that the condition for the cumulative offer process to be the unique candidate could be substantially gen-

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eralized.

Finally, it is also worth mentioning that all of our proofs share a common technique to derive a contradiction starting from a “minimal” preference profile in terms of the number of acceptable contracts. While this technique could be also useful elsewhere, it should be noted that it necessitates the existence of a null-contract as well as the full preference domain. Consequently, our results do not directly extend to an environment where the null-contract does not exist or the preference domain is restricted, e.g., as in Kesten and Kurino (2013).
Appendix A

Appendix to Chapter 1

A.1 Proofs

Lemma A.1. Let $a, b, c, d \in \mathbb{R}^{++}$. Then, $\frac{a}{b} < \frac{b + c}{b + d}$ if and only if $\frac{a}{b} < \frac{c}{a}$.

Proof. It is immediate to see

$$\frac{a}{b} < \frac{c}{d} \iff ad < bc \iff ab + ad < ab + bc \iff a(b + d) < b(a + c) \iff \frac{a}{b} < \frac{a + c}{b + d}.$$  

Proof of Lemma 1.1. Note that the first order condition (1.2) is not valid if and only if

$$\frac{\partial}{\partial e} \text{Prob}[Y = 1|K, e] \bigg|_{e = e^*} < 1,$$

which in turn is the case if $e^* > 1 - f(a_H, K)$. Since $\Delta\hat{\mu} < 1$ by definition, Assumption 1.3 implies $e^* = \frac{\delta}{\phi} \Delta\hat{\mu} < 1 - f(a_H, 1)$, and the proof is complete.

Proof of Lemma 1.2. First, we show that $\hat{\mu}_1$ is (strictly) increasing in $\hat{K}(\cdot)$ for any
Differentiating \(\ell_1(\hat{K}, \hat{e})\) for any fixed \(\hat{e} \in (0, \bar{e})\). The likelihood ratio when \(Y = 1\) is

\[
\ell_1(\hat{K}, \hat{e}) = \frac{p \cdot f(a_H, \hat{K}_L) + (1 - p)f(a_H, \hat{K}_H) + \hat{e}}{p \cdot f(a_L, \hat{K}_L) + (1 - p)f(a_L, \hat{K}_H) + \hat{e}}.
\]

Now suppose that \(\hat{K}_H\) increases from 0 to 1. (The case of increase in \(\hat{K}_L\) is perfectly symmetric.) Then, the numerator and denominator increase by \(f(a_H, 1) - f(a_H, 0)\) and \(f(a_L, 1) - f(a_L, 0)\), respectively, and

\[
\frac{f(a_H, 1) - f(a_H, 0)}{f(a_L, 1) - f(a_L, 0)} \cdot \frac{f(a_L, 1) - f(a_L, 1)}{f(a_H, 1) - f(a_H, 0)} \geq \frac{f(a_H, 1)}{f(a_L, 1)},
\]

where the inequality holds by log-supermodularity.\(^1\) Since \(\ell_1 < \frac{f(a_H, 1)}{f(a_L, 1)}\) if \(\hat{K}_H = 0\), by Lemma A.1, this implies \(\ell_1\) strictly increases in \(\hat{K}_H\). Then, since \(\hat{\mu}_1\) is increasing in \(\ell_1\), so is it in \(\hat{K}_H\).

Then, it suffices to show that \(\hat{\mu}_0\) is decreasing in \(\hat{K}\). The law of iterated expectations implies

\[
\text{Prob}[a = a_H|\hat{K}, \hat{e}] \equiv \text{Prob}[Y = 1|\hat{K}, \hat{e}] \cdot \hat{\mu}_1 + (1 - \text{Prob}[Y = 1|\hat{K}, \hat{e}]) \cdot \hat{\mu}_0.
\]

Notice that both \(\text{Prob}[Y = 1|\hat{K}, \hat{e}]\) and \(\hat{\mu}_1\) are increasing in \(\hat{K}\). Hence, \(\hat{\mu}_0\) is decreasing in \(\hat{K}\) and the proof is complete. \(\blacksquare\)

**Lemma A.2.** For any fixed \(\hat{K}\), \(\Delta \hat{\mu} \left( \hat{K}, e \right)\) is a convex function of \(e\) on \([0, \bar{e}]\).

**Proof.** Differentiating \(\Delta \hat{\mu} \left( \hat{K}, e \right)\) with respect to \(e\), we obtain

\[
\frac{\partial}{\partial e} \Delta \hat{\mu} \left( \hat{K}, e \right) = \mu \frac{(\bar{F} + e) - (F_H + e)}{[\bar{F} + e]^2} - (1 - \mu) \frac{[1 - (F_H + e)] - [1 - (\bar{F} + e)]}{[1 - (\bar{F} + e)]^2} = (\bar{F} - F_H) \left\{ \frac{\mu}{[\bar{F} + e]^2} - \frac{1 - \mu}{[1 - (\bar{F} + e)]^2} \right\},
\]

\(^1\)Log-supermodularity implies \(f(a_L, 0) \geq \frac{f(a_H, 1)}{f(a_L, 1)} f(a_H, 0)\).
where
\[ F_H := \mathbb{E}_\theta \left[ f(a_H, \bar{K}(\theta)) \right] \quad \text{and} \quad \bar{F} := \mathbb{E}_\theta \left[ f(\bar{K}(\theta)) \right] \]
are independent of \( e \). Then, it is immediate to see that the first-order derivative is increasing in \( e \), as \( \bar{F} - F_H \) is a positive constant, \( \bar{F} + e \) is increasing in \( e \), and \( 1 - (\bar{F} + e) \) is decreasing in \( e \).

\[ \text{Proof of Proposition 1.1.} \]

First we show the uniqueness of equilibrium for each organizational form. Since the uniqueness of equilibrium decision rule is demonstrated in the main text, it suffices to show the existence and uniqueness of equilibrium \( e^* \). Once \( \bar{K}^* \) is fixed, the equilibrium \( e^* = \hat{e}^* \) must satisfy
\[
e^* = \frac{\delta}{\psi} \cdot \Delta \hat{\mu} \left( \bar{K}, e^* \right). \tag{A.1}
\]
The existence of a solution to this equation follows from the standard fixed-point arguments. The uniqueness is guaranteed by Lemmas 1.1 and A.2.

Next we show that \( e^*_C < e^*_D \), where \( e^*_C \) (resp. \( e^*_D \)) denotes the unique equilibrium effort level under centralization (resp. delegation). Note that Lemma A.2 implies that
\[
e < e^*_D \Leftrightarrow e < \frac{\delta}{\psi} \cdot \Delta \hat{\mu} \left( K^*_D, e \right).
\]
Since Lemma 1.2 and the equilibrium condition (A.1), implies
\[
e^*_C = \frac{\delta}{\psi} \cdot \Delta \hat{\mu} \left( K^*_C, e^*_C \right) < \frac{\delta}{\psi} \cdot \Delta \hat{\mu} \left( K^*_D, e^*_C \right),
\]
it follows that \( e^*_C < e^*_D \).

For both the equilibrium decision rule and effort level are independent of \( \gamma_H \), the last claim of the statement is immediate and the proof is complete. \[ \blacksquare \]
Proof of Proposition 1.2. As argued in the main text, it is immediate to see the agent chooses $p_C^* > p_0$ in an equilibrium under centralization and $p_D^* = p_0$ under delegation.\(^2\) Hence it remains to show that $e^*$ is higher under delegation than under centralization. Note also that the equilibrium decision rule must be $K_C^*$ under centralization and $K_D^*$ under delegation as in the baseline case.

Define $\hat{\Delta}$ and $\hat{\ell}_1$ as in the baseline case. Since the assumption of $g'(1) = \infty$ implies $p_C^* < 1$ in any equilibrium under centralization,

$$\hat{\ell}_1(K_D, \hat{e}, p_0) = \frac{f(a_H, 1) + \hat{e}}{f(a_L, 1) + \hat{e}} > \frac{p_C^* \cdot f(a_H, 1) + (1 - p_C^*) f(a_H, 0) + \hat{e}}{p_C^* \cdot f(a_L, 1) + (1 - p_C^*) f(a_L, 0) + \hat{e}} = \hat{\ell}_1(K_C, \hat{e}, p_C^*).$$

It then follows that $\Delta \hat{\mu}(K_C, \hat{e}, p_C^*) < \Delta \hat{\mu}(K_D, \hat{e}, p_0)$ for any $\hat{e}$. The rest of the proof is exactly parallel to that of Proposition 1.1 and thus omitted. \(\blacksquare\)

Proof of Proposition 1.7. The proof is by example. Suppose $\eta = 0$, $\beta = 3$, $\sigma_{a,1}^2 = 1$, $\delta_1 = 1$, and $\gamma - \alpha = \frac{1}{17}$. First, suppose also that $\sigma_{\epsilon,1}^2 = 6$. Then, $S_1 = \frac{1}{7}$ and $\Delta S_1 = \frac{3}{9 + 6} - \frac{1}{7} = \frac{2}{35}$. Comparing the equilibrium profit under centralization and delegation with specialization, $\Pi^C - \Pi^{DS} = (\gamma - \alpha) - \Delta S_1 = \frac{1}{17} - \frac{2}{35} > 0$.

Therefore, centralization is the optimal organizational form and induces $e_{i_1}^* = S_1 = \frac{1}{7}.\(^3\)$ Second, suppose that $\sigma_{\epsilon,1}^2 = 7$. Then, $S_1 = \frac{1}{8}$, $\Delta S_1 = \frac{3}{9 + 7} - \frac{1}{8} = \frac{1}{16}$, and hence, $\Pi^C - \Pi^{DS} = \frac{1}{17} - \frac{1}{16} < 0$. Therefore, delegation with specialization is optimal, which leads to $e_{i_1}^* = S_1 + \Delta S_1 = \frac{1}{8} > \frac{1}{7}.\(\blacksquare\)

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\(^2\)The existence of equilibrium under centralization follows from the standard fixed-point argument.

\(^3\)Recall that delegation with no specialization is never strictly optimal when $\eta = 0$. 

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A.2 Additional Results

A.2.1 State-Dependent Returns

In the baseline model in Section 1.2, we assume that the state of nature affects the cost of \( K = 1 \) but not its return. The purpose of this section is to check the robustness of the results when the expected return \( K = 1 \) varies across states. To do so, suppose that instead of (1.1), the probability of success is given by

\[
\text{Prob}[Y = 1|a, K, e, \theta] = \begin{cases} 
\min \{a \cdot K + f(a, K) + e, 1\} & \text{if } \theta = \theta_G, \text{ and} \\
\min \{f(a, K) + e, 1\} & \text{if } \theta = \theta_B.
\end{cases}
\]

Let \( p = \text{Prob}[\theta = \theta_G] \). The cost of \( K = 1 \) is \( \gamma \), independently of \( \theta \). The following assumptions are analogues to Assumptions 1.1–1.3 in Section 1.2.

Assumption A.1. The function \( f(\cdot, \cdot) \) is positive, increasing in both arguments, and log-supermodular.

Assumption A.2. It is first-best efficient to choose \( K = 1 \) if and only if \( \theta = \theta_G \): \( \bar{f}(1) < \gamma < \alpha + \bar{f}(1) \).

Assumption A.3. The cost of effort is \( \Psi(e) = \frac{\psi}{a}e^2 \) with \( \psi > \delta(1 - f(a_H, 1) - \alpha) \).

Then, we can show that the quality of the signal is increasing in \( \hat{K}(\theta_B) \), which is a partial extension of Lemma 1.2.

Lemma A.3. Taking \( \hat{e} \) as fixed, \( \Delta \hat{\mu} \) is increasing in \( \hat{K}_B \).

Proof. The proof is exactly the same as of Lemma 1.2 and thus omitted. 

When \( p = \text{Prob}[\theta = \theta_G] \) is fixed, we obtain an analogue of Proposition 1.1 as follows.
**Proposition A.1.** For any fixed $p$, there exists a unique equilibrium for each organizational form. The equilibrium level of ex-post effort $e^*$ is higher under delegation than under centralization.

*Proof.* The proof is perfectly analogue to that of Proposition 1.1 and thus omitted.

When $p$ is chosen by the agent in this case, no unambiguous ranking exists between the effort levels under centralization and delegation, as discussed in Section 1.2.4. We can show, however, a weaker version of Proposition 1.2.

**Proposition A.2.** Suppose that the agent chooses $p$ as well as $e$. Let $p_C^*$ and $e_C^*$ (resp. $p_D^*$ and $e_D^*$) be the levels of ex-ante and ex-post efforts in an equilibrium under centralization (resp. under delegation). Then, it is never the case that both $p_C^* \leq p_D^*$ and $e_C^* \geq e_D^*$.

*Proof.* Suppose that $p_C^*$ and $e_C^*$ be equilibrium levels of ex-ante and ex-post efforts in an equilibrium under centralization. Then, the first order conditions imply

$$e_C^* = \frac{\delta_1}{\alpha} \cdot \Delta \tilde{\mu}(K_C^*, e_C^*, p_C^*),$$

and

$$g'(p_C^*) = \left[ \alpha + \bar{f}(1) \right] \Delta \tilde{\mu}(K_C^*, e_C^*, p_C^*),$$

which further lead to

$$g'(p_C^*) = \frac{c}{\delta} \left[ \alpha + \bar{f}(1) \right] e_C^*.$$

Similarly, for an equilibrium under delegation, it is immediate to check

$$g'(p_D^*) = \frac{c}{\delta} \cdot \bar{f}(1) \cdot e_D^*,$$

must hold. Now suppose that $e_C^* \geq e_D^*$. Then, the above two equations imply $g'(p_C^*) > g'(p_D^*)$, which is equivalent to $p_C^* > p_D^*$ by the convexity of $g$. Therefore, $e_C^* \geq e_D^*$ and $p_C^* \leq p_D^*$ cannot simultaneously hold.

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A.2.2 Direct Benefit of Specialization under Correlated Abilities

This section considers an extension of the model in Section 1.3, and identifies a benefit of specialization that is independent from the equilibrium choice of $K$. Throughout this section, I assume that the principal retains the decision right and thus, $K = \hat{K} = (0, 1)$ must hold in equilibrium.\footnote{More generally, it can be shown specialization dominates no specialization if $\eta$ is sufficiently small, assuming that the two organizational forms induce the same equilibrium $K$. The proof for the case of $K = (1,0)$ is completely analogous and thus omitted.} The setup is the same as in Section 1.3 except that the covariance matrix $\Sigma_a$ is now given by

$$
\Sigma_a = \begin{bmatrix}
\sigma^2_{a,1} & r\sigma_{a,1}\sigma_{a,2} \\
 r\sigma_{a,1}\sigma_{a,2} & \sigma^2_{a,2}
\end{bmatrix},
$$

where $r \in [0, 1]$ is the correlation coefficient.

Suppose first that a single agent $i$ handles both tasks. Then, for any given $e_i = (e_{1,i}, e_{2,i})$,

$$
\begin{bmatrix}
a_i \\
y
\end{bmatrix} = \begin{bmatrix}
a_{1,i} \\
a_{2,i} \\
y_1 \\
y_2
\end{bmatrix} \sim \mathcal{N}(\mu, \Sigma),
$$

where

$$
\mu = \begin{bmatrix}
\mu_a \\
\mu_y
\end{bmatrix} = \begin{bmatrix}
\bar{a}_1 \\
\bar{a}_2 \\
\bar{a}_1 + e_{1,i} \\
\gamma + \bar{a}_2 + e_{2,i}
\end{bmatrix},
$$
and,

\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix} = \begin{bmatrix}
\sigma_{a,1}^2 & r\sigma_{a,1}\sigma_{a,2} & \sigma_{a,1}^2 & r\sigma_{a,1}\sigma_{a,2} \\
\sigma_{a,2}^2 & r\sigma_{a,1}\sigma_{a,2} & \sigma_{a,2}^2 & r\sigma_{a,1}\sigma_{a,2} \\
\sigma_{a,1}^2 + \sigma_{\epsilon,1}^2 & r\sigma_{a,1}\sigma_{a,2} & \sigma_{a,1}^2 + \sigma_{\epsilon,2}^2 & r\sigma_{a,1}\sigma_{a,2}
\end{bmatrix}.
\]

It is well known that

\[
\mathbb{E}[a_i|Y_1, Y_2, \hat{\epsilon}_i, \hat{K}] - \mathbb{E}[a_i] = \Sigma_{12}\Sigma_{22}^{-1}(Y - \mu_Y).
\]

Therefore,

\[
J^N := \frac{\partial \mathbb{E}[a_i|Y_1, Y_2, \hat{\epsilon}_i, \hat{K}]}{\partial Y} = \frac{1}{\det(\Sigma_{22})} \begin{bmatrix}
\sigma_{a,1}^2 \left(\sigma_{a,2}^2 + \sigma_{\epsilon,2}^2\right) - r^2\sigma_{a,1}^2\sigma_{a,2}\sigma_{\epsilon,1}^2 & r\sigma_{a,1}\sigma_{a,2}\sigma_{\epsilon,1}^2 \\
\sigma_{a,2}^2 \sigma_{\epsilon,1}^2 - r\sigma_{a,1}\sigma_{a,2}\sigma_{\epsilon,1}^2 & \sigma_{a,1}^2 \left(\sigma_{a,2}^2 + \sigma_{\epsilon,2}^2\right) - r^2\sigma_{a,1}^2\sigma_{a,2}\sigma_{\epsilon,2}^2
\end{bmatrix},
\]

where

\[
\det(\Sigma_{22}) = (\sigma_{a,1}^2 + \sigma_{\epsilon,1}^2)(\sigma_{a,2}^2 + \sigma_{\epsilon,2}^2) - r^2\sigma_{a,1}^2\sigma_{a,2}^2.
\]

Next, suppose that the principal hires two specialized agents, i.e., \(i_1 \neq i_2\). Then, it is clear that \(Y_2\) (resp. \(Y_1\)) contains no information regarding \(a_{i_1}\) (resp. \(a_{i_2}\)). The Jacobian matrix for \(a_{i_1}\) is thus given by

\[
J_{\bullet,1}^S := \frac{\partial \mathbb{E}[a_{i_1}|Y_1, Y_2, \hat{\epsilon}_i, \hat{K}]}{\partial Y_1} = \frac{1}{\sigma_{a,1}^2 + \sigma_{\epsilon,1}^2} \begin{bmatrix}
\sigma_{a,1}^2 \\
r\sigma_{a,1}\sigma_{a,2}\sigma_{\epsilon,1}^2
\end{bmatrix},
\]

and

\[
J_{\bullet,2}^S := \frac{\partial \mathbb{E}[a_{i_2}|Y_1, Y_2, \hat{\epsilon}_i, \hat{K}]}{\partial Y_2} = \frac{1}{\sigma_{a,2}^2 + \sigma_{\epsilon,2}^2} \begin{bmatrix}
r\sigma_{a,1}\sigma_{a,2} \\
\sigma_{a,1}^2
\end{bmatrix}.
\]

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Comparing these with the columns of $J^N$, it is can be verified that the posterior becomes more sensitive when an agent is specialized, unless $r = 0$. As a consequence, specialization is more profitable when $\eta$ is sufficiently small. (Recall that $\eta$ represents the cost of specialization, as it measures the degree of synergies that cannot be utilized under specialization.)

**Proposition A.3.** Suppose that the principal has the decision right and thus $K = \hat{K} = (0, 1)$. If $r > 0$, there exists a cutoff $\eta^* > 0$ such that specialization dominates no specialization if and only if $\eta \leq \eta^*$.

**Proof.** By definitions, the sum of effective efforts is given by

$$e^*_{1,i} + e^*_{2,i} = (1 + \eta) \left[ \delta_1 (J^N_{1,1} + J^N_{1,2}) + \delta_2 (J^N_{2,1} + J^N_{2,2}) \right], \quad (A.4)$$

in the case of no specialization (i.e., $i_1 = i_2 = i$), and

$$e^*_{1,i_1} + e^*_{2,i_2} = \left[ \delta_1 J^S_{11} + \delta_2 J^S_{21} \right] + \left[ \delta_1 J^S_{12} + \delta_2 J^S_{22} \right], \quad (A.5)$$

in the case of specialization (i.e., $i_1 \neq i_2$). For the value of (A.4) is increasing in $\eta$ while that of (A.5) is constant, the statement holds true if the former is smaller than the latter at $\eta = 1$.

To complete the proof, thus, it suffices to show that $J^N_{*,t} \ll J^S_{*,t}$ for each $t \in \{1, 2\}$, as long as $r > 0$. For $t = 1$, it follows from equations (A.2)–(A.3) that

$$J^N_{11} = \frac{\sigma_{a,1}^2 (\sigma_{\varepsilon,2}^2 + \sigma_{\varepsilon,1}^2) - r^2 \sigma_{a,1}^2 \sigma_{\varepsilon,2}^2}{(\sigma_{a,1}^2 + \sigma_{\varepsilon,1}^2)(\sigma_{a,2}^2 + \sigma_{\varepsilon,2}^2) - r^2 \sigma_{a,1}^2 \sigma_{\varepsilon,2}^2} < \frac{\sigma_{a,1}^2}{\sigma_{a,1}^2 + \sigma_{\varepsilon,1}^2} = J^S_{11},$$

and

$$J^N_{21} = \frac{r \sigma_{a,1} \sigma_{a,2} \sigma_{\varepsilon,2}^2}{(\sigma_{a,1}^2 + \sigma_{\varepsilon,1}^2)(\sigma_{a,2}^2 + \sigma_{\varepsilon,2}^2) - r^2 \sigma_{a,1}^2 \sigma_{\varepsilon,2}^2} \quad \text{and} \quad \frac{r \sigma_{a,1} \sigma_{a,2} \sigma_{\varepsilon,2}^2}{(\sigma_{a,1}^2 + \sigma_{\varepsilon,1}^2) + \frac{1}{\sigma_{\varepsilon,2}^2}[(1 - r^2)\sigma_{a,1}^2 \sigma_{a,2}^2 + \sigma_{a,2}^2 \sigma_{\varepsilon,1}^2]} < J^S_{12},$$

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since $1 - r^2 \geq 0$. The proof for $t = 2$ is perfectly symmetric and thus omitted. □
Appendix B

Appendix to Chapter 2

B.1 Proofs

Proof of Lemma 2.1. By definition,
\[
\Pi(b', \theta', L) - \Pi(b', \theta, L) = 2F(b - b') + \left[ (2\Delta - b') \frac{b'\theta'}{c} - (2\Delta - b) \frac{b\theta}{c} \right] L.
\]
Since \( b' > b \), the first term on the RHS is negative and hence, the second term must be strictly positive if the LHS is non-negative. Then, the statement immediately follows.

Proof of Proposition 2.1. To begin, define \( \beta^L_{IC} > \beta^F_L \) by \( \Pi(\beta^L_{IC}, H, L) = \Pi(\beta^F_L, L, L) \), i.e., \( \beta^L_{IC} \) is the highest possible offer that a \( L \)-type principal can have an incentive to offer at date 1. Note that by definition, the D1 criterion never deletes \((b, H)\) for \( b \geq \beta^L_{IC} \).

First, suppose that \( \beta^L_{IC} \geq \beta^F_H \). Define a belief system \( \theta^*(\cdot) \) by \( \theta^*(b) = L \) if \( b < \beta^L_{IC} \) and \( \theta^*(b) = H \) if \( b \geq \beta^L_{IC} \). Associated with this \( \theta^*(\cdot) \), it is apparent that \((b_H, b_L) = (\emptyset, \emptyset)\) forms a PBE. For any \( b < \beta^L_{IC} \), \( \Pi(b, \theta, H) \geq \Pi(\beta^F_H, H, H) \) implies \( \Pi(b, \theta, H) \geq \Pi(\beta^L_{IC}, H, H) \) and hence by Lemma 2.1, \( \Pi(b, \theta, L) > \Pi(\beta^L_{IC}, H, L) \).
Therefore, \((b, L)\) is not deleted for \(b < \beta_{L}^{IC}\) and \(\theta^*(\cdot)\) survives the D1 criterion.

Second, suppose that \(\beta_{L}^{IC} < \beta_{H}^{F} = \beta_{H}^{*}\), and let \(\theta^*(\cdot)\) be the same belief system as in the previous case. Again, it is immediate to check \((b_{H}, b_{L}) = (\emptyset, \emptyset)\) with this \(\theta^*(\cdot)\) is an equilibrium. In this case, \(b < \beta_{L}^{IC}\) directly implies \(\Pi(b, \emptyset, H) \leq \Pi(b, H, H) < \Pi(\beta_{L}^{*}, H, H)\) and hence, \((b, L)\) is not deleted for any \(b < \beta_{L}^{IC}\).

Finally, suppose that \(\beta_{L}^{IC} < \beta_{H}^{F} = \beta_{H}^{PC}\). Let \(\theta^*(\cdot)\) be a belief system satisfying (i) \(\theta^*(b) = H\) for all \(b \geq \beta_{L}^{IC}\) and all \(b \in B_{H}^{IC}\), and (ii) \(\theta^*(b) \in \{H, L\}\) for all \(b\), where

\[
B_{H}^{IC} := \left\{ b \in [0, 1] : \Pi(b, L, H) \geq \Pi(\beta_{H}^{*}, H, H) \right\}.
\]

Notice that \(\sup B_{H}^{IC} < \beta_{H}^{F}\) always holds (as long as \(B_{H}^{IC}\) is non-empty). Then, we can check any such belief system can support \((b_{H}, b_{L}) = (\emptyset, \emptyset)\) as an equilibrium. Since either \((b, H)\) or \((b, L)\) must survive the D1 criterion for each \(b \in [0, 1]\), we can always pick \(\theta^*(\cdot)\) that satisfies the D1 criterion.

\[\blacksquare\]

Proof of Proposition 2.2. The “only if” part is immediate because the principal can always secure the profits of \(\Pi(\beta_{\emptyset}^{F}, \emptyset, \emptyset)\) by delay her offer. To show the “if” part, suppose that an arbitrary \(b \in [0, 1]\) satisfies the condition, and let \(\theta^*(\cdot)\) be the belief system such that \(\theta(b') = L\) if \(b' < b\), \(\theta(b') = M\) if \(b' = b\), and \(\theta(b') = H\) if \(b' > b\). Then, we can easily check that it is optimal for each \(\theta\)-type principal to make an exploding offer \(b_{\theta} = b\), given the agent’s belief system \(\theta^*(\cdot)\): First, if \(\theta = L\) and the principal offers \(b_{L} = b' < b\), her profit will be \(\Pi(b', L, L) \leq \Pi(\beta_{L}^{F}, L, L) \leq \Pi(b, M, L)\), where the first and second inequalities hold by the definition of \(\beta_{L}^{F} \equiv \beta_{L}^{*}\) and by assumption, respectively. Hence the \(L\)-type principal has no incentive to deviate by \(b_{L} = b' < b\). Similarly, it is never profitable for the \(H\)-type principal to offer \(b_{H} = b' > b\), because by definitions, either \(b' \notin Ac(H)\) or \(\Pi(b', H, H) \leq \Pi(\beta_{H}^{F}, H, H) \leq \Pi(b, M, H)\). Next, suppose that the principal offers \(b_{L} = b' > b\). If this is strictly profitable, i.e., if \(b' \in Ac(H)\) and \(\Pi(b', H, L) > \Pi(b, M, L)\), then
Lemma 2.1 implies $\tilde{\Pi}(b', H, H) > \tilde{\Pi}(b, M, H)$, but this is a contradiction to the previous argument. The last case, $\theta = H$ and $b_H = b' < b$, is also guaranteed to be never profitable by Lemma 2.1. In sum, $(b_H, b_L) = (b, b)$ is an equilibrium outcome with the belief system $\theta^*(\cdot)$. Since Lemma 2.1 directly implies $\theta^*(\cdot)$ satisfy the D1 criterion, the proof is complete.
Appendix C

Appendix to Chapter 3

C.1 Additional Definitions

This section provides the definitions of several existing concepts that are referred to but not defined in the main text. To begin, the following substitutes conditions, introduced by Hatfield and Milgrom (2005) and Hatfield and Kojima (2010), play a central role in the matching with contracts literature.¹

**Definition C.1.** Hospital $h$’s choice function satisfies the *substitutes* condition if there do not exist contacts $x$ and subsets $X', X'' \subset X$ of contracts such that

1. $x \not\in C_h(X' \cup \{x\})$ and
2. $x \in C_h(X' \cup X'' \cup \{x\})$.

**Definition C.2.** Hospital $h$’s choice function $C_h(\cdot)$ satisfies the *unilateral substitutes* condition if there do not exist contracts $x, y \in X$ and a subset $X' \subset X$ of contracts such that

1. $d(x) \not\in \{d(x')\}_{x' \in X'}$,
2. $x \not\in C_h(X' \cup \{x\})$, and
3. $x \in C_h(X' \cup \{x, y\})$.

¹See also Afacan and Turhan (2015) for the relationships among these conditions.
**Definition C.3.** Hospital $h$’s choice function $C_h(\cdot)$ satisfies the *bilateral substitutes* condition if there do not exist contracts $x, y \in X$ and a subset $X' \subset X$ of contracts such that (i) $d(x), d(y) \notin \{d(x')\}_{x' \in X'}$, (ii) $x \notin C_h(X' \cup \{x\})$, and (iii) $x \in C_h(X' \cup \{x,y\})$.

Hatfield and Milgrom (2005) also introduce the following condition.

**Definition C.4.** Hospital $h$’s choice function $C_h(\cdot)$ satisfies the *law of aggregated demand* if $X' \subset X''$ implies $|C_h(X')| \leq |C_h(X'')|$ for all $X', X'' \subset X$.

Kominers and Sönmez (2013, 2014) study the markets where the hospitals’ choice functions are induced by slot-specific priorities. They show that such choice functions may violate the unilateral substitutes condition, but always satisfy the bilateral substitutes condition.

**Definition C.5.** A collection $(\succ_h, s)_{1 \leq s \leq q_h}$ of linear orders, called *slot-specific priorities*, over $\{x \in X : h(x) = h\} \cup \{\emptyset\}$ induces a choice function for hospital $h$ as follows: For each $X' \subset X$, $C_h(X') = \{x_1, \ldots, x_{q_h}\} - \{\emptyset\}$, where $x_s$ is recursively defined by letting

- $X_s' = X_{s-1}' - \{x_{s-1}\}$, where $X_0'$ and $x_{s-1}$ are defined to be $X'$ and $\emptyset$, and,
- $x_s$ be the maximal element in $X_s' \cup \{\emptyset\}$, with respect to $\succ_h, s$,

for each $s = 1, \ldots, q_h$.

Hatfield and Kojima (2010) show that the *cumulative offer process*, originally defined by Hatfield and Milgrom (2005), can find a stable allocation whenever the bilateral substitutes condition is satisfied, under which the deferred acceptance algorithm may fail to do so.\(^2\)

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\(^2\)The following definition is equivalent to the one by Hatfield and Milgrom (2005). Although Hatfield and Kojima (2010) adopt a slightly different definition, Hirata and Kasuya (2014) show that
Definition C.6. Given $(C_H(\cdot), \succ_D)$, the cumulative offer process proceeds as follows.

- Initial condition: Let $D_0 = D$ and $P_0 = \emptyset$.

- Step $t \geq 1$: Each $d \in D_{t-1}$ offers her best contract, $x^d_t$, among those remaining (i.e., among $X \setminus P_{t-1}$). Let $P_t = P_{t-1} \cup \{x^d_t\}_{d \in D_{t-1}}$ be the pool of contracts that have been offered up to this step. Among $P_t$, each hospital $h$ holds the best combination of contracts, $C_h(P_t)$. Finally, let $D_t$ be the set of doctors for whom (i) no contract is currently held by any hospital and (ii) not all acceptable contracts have been offered yet, i.e.,

$$D_t = \{d \in D : d \not\in [C_h(P_t)]_D \text{ for all } h \in H \text{ and } Ac(\succ_d) \setminus P_t \neq \emptyset\}. \quad (C.1)$$

Proceed to step $t+1$ if $D_t$ is non-empty and terminate otherwise.

- Outcome: When the process terminates at step $T$, its outcome is $\bigcup_{h \in H} C_h(P_T)$.

\[ \square \]

C.2 Additional Results

This section provides two variants of Theorem 3.3 in Chapter 3. First, we show that stability in Theorem 3.3 can be replaced with the following weaker requirement:

An individually rational allocation $X'$ is said to be non-wasteful if there is no other individually rational allocation $X''$ with $X'' \supseteq X'$. If we take $C_h(X') = X(h, X')$

the two definitions are outcome-equivalent if every hospital’s choice function satisfies the bilateral substitutes condition and the IRC condition. For the properties of the cumulative offer process, see also Afacan (2014).

\[ \text{3It is immediate to verify that under the IRC condition, stability implies non-wastefulness as defined above. This is not the case in general; see Example C.3 in Appendix C.3.} \]

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as a feasibility constraint, the above condition roughly says no hospital can accommodate any unmatched doctors without crowding out other (already matched) doctors. It reduces to the standard definition in assignment problems, if we endow each hospital (or object) $h$ with $C_h(\cdot)$ such that $C_h(X') = X(h, X')$ if and only if $|X(h, X')| \leq q_h$, where $q_h$ is the “quota” of $h$.\footnote{Of course, the feasibility constraint does not uniquely pin down $C_{H(\cdot)}$, because it imposes no restriction on $C_h(X')$ when $X(h, X')$ is infeasible. However, this additional degree of freedom is irrelevant for the current purpose (see Corollary C.1 below).} In addition, the current matching-with-contracts setup allows a richer class of constraints that could be of practical relevance. For example, suppose that a school offers two distinct programs, and they require some common resources at different factor intensity (e.g., one is mathematics-teacher intensive while the other is English-teacher intensive). Then, the total number of students that these programs can accommodate is not constant but depends on its composition.

**Theorem C.1.** Suppose that every hospital $h \in H$ has a choice function $C_h(\cdot)$ that satisfies the IRC condition. Then, no individually-rational and strategy-proof rule strictly dominates a non-wasteful and strategy-proof rule.

**Proof.** In the proof of Theorem 3.3, the stability of $g(\cdot)$ is needed only to guarantee the existence of $d^*$ satisfying (3.1). Hence it suffices to prove this part from non-wastefulness. Indeed, if $x(d, f(\succ_D^*)) \succ_d x(d, g(\succ_D^*))$ implies $x(d, g(\succ_D^*)) = \emptyset$ for all $d \in D$, it follows that $f(\succ_D^*) \not\supset g(\succ_D^*)$, which directly contradicts non-wastefulness.\hfill\blacksquare

Actually, this theorem can be slightly more generalized, since the choice functions are relevant for individual rationality and non-wastefulness only through whether $C_h(X') = X(h, X')$ or not.
Corollary C.1. Suppose that \( C_H(\cdot) \) and \( C'_H(\cdot) \) are two profiles of choice functions satisfying the IRC condition such that \( C_h(X') = X(h, X') \Rightarrow C'_h(X') = X(h, X') \) for all \( h \in H \) and \( X' \subset X \). Then, no strategy-proof rule that is individually rational with respect to \( C_H(\cdot) \) strictly dominates a strategy-proof rule that is non-wasteful with respect to \( C'_H(\cdot) \).

Proof. The proof is exactly the same as of Theorems 3.3 and C.1 and thus omitted.

Second, we can show the second-best optimality of the doctor-optimal stable rule, given its existence, no matter whether it is strategy-proof or not.

Theorem C.2. Suppose that every hospital \( h \in H \) has a choice function \( C_h(\cdot) \) that satisfies the IRC condition. Then, no individually-rational strategy-proof rule strictly dominates the doctor-optimal stable rule (whether strategy-proof or not).

Proof. Towards a contradiction, suppose that \( f(\cdot) \) is individually rational and strategy-proof, the doctor-optimal stable rule \( X^*(\cdot) \) is well-defined, and that \( f(\cdot) \) strictly dominates \( X^*(\cdot) \). Let \( D \in \mathcal{P} \) be a preference profile such that \( f(D) \neq X^*(D) \) and

\[
\left[f(\succ_D) \neq X^*(\succ_D) \implies \sum_{d \in D} |\text{Ac}(\succ_d)| \geq \sum_{d \in D} |\text{Ac}(\succ^*_d)|\right] \quad \text{for all } D \in \mathcal{P}_D,
\]

which exists by assumption. Then, for the same reasoning as in the proof of Theorem 3.3, it follows from the stability of \( X^*(\cdot) \) that there exists \( d^* \in D \) with

\[
x(d^*, f(\succ^*_D)) \succ^*_{d^*} x(d^*, X^*(\succ^*_D)) \succ^*_d \emptyset.
\]

Now, let \( \succ^*_D = \left(\succ^*_{d^*}, \succ^*_{D-\{d^*}\} \right) \), where \( \succ^*_{d^*} \) is a truncation of \( \succ^*_D \) above \( x(d^*, X^*(\succ^*_D)) \), i.e., a preference such that

\[
\text{Ac}(\succ^*_d) = \{x \in X : x \succ^*_{d^*} x(d^*, X^*(\succ^*_D))\},
\]

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and

\[ x \succ_{d^*} y \iff x \succ_{d^*} y \] for all \( x, y \in \text{Ac}(\succ_{d^*}). \]

On the one hand, the strategy-proofness of \( f(\cdot) \) implies

\[ x(d^*, f(\succ_D)) = x(d^*, f(\succ_D)) \neq \emptyset. \]

On the other hand, \( x(d^*, X(\succ_D)) = \emptyset \) must also hold by doctor-optimality.\(^5\)

These together imply \( f(\succ_D) \neq g(\succ_D) \), which contradicts the definition of \( \succ_D \), and the proof is complete.

C.3 Examples

This section provides the examples that are referred to in Chapter 3 and the previous section. The first example illustrates that Lemmas 3.1–3.2 do not generally hold true without the IRC condition. As a consequence, multiple stable and strategy-proof rules exist in this example.

**Example C.1.** Let \( D = \{d_1, d_2\} \), \( H = \{h\} \), and \( X = \{x_1, x_2\} \), where for each \( i \in \{1, 2\} \), \( x_i \) is a contract between \( d_i \) and \( h \). Suppose that each doctor \( d_i \) has a preference relation with \( x_i \succ_{d_i} \emptyset \), and that hospital \( h \)'s choice function is such that \( C_h(\{x_1\}) = x_1 \), \( C_h(\{x_2\}) = x_2 \), but \( C_h(\{x_1, x_2\}) = \emptyset \). It is immediate to check \( C_h(\cdot) \) violates the IRC condition. Note also that at this \( (C_H(\cdot), \succ_D) \), allocation \( \{x_1\} \) is not blocked by any coalition, although it is weakly blocked by \( (h, \{x_1, x_2\}) \).\(^6\) That is, the conclusion of Lemma 3.1 fails to hold. Consequently, both \( \{x_1\} \) and \( \{x_2\} \) are

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\(^5\)If \( x(d^*, X^*(\succ_D)) \neq \emptyset \), by construction, any \( (h, X) \) that blocks \( X^*(\succ_D) \) at \( \succ_D^* \) must also block \( X^*(\succ_D^*) \) at \( \succ_D^* \). Thus \( X^*(\succ_D^*) \) is stable at \( \succ_D^* \), but this contradicts the doctor-optimality of \( X^*(\succ_D^*) \) for \( x(d^*, X^*(\succ_D^*)) \succ_{d^*} x(d^*, X^*(\succ_D^*)) \).

\(^6\)The second requirement of weak blocking is vacuously satisfied since \( C_H(\{x_1, x_2\}) = \emptyset. \)
stable at this profile, and the conclusion of Lemma 3.2 also fails. Lastly, define for each \( i \in \{1, 2\} \) a rule \( f^i(\cdot) \) by

\[
f^i(\succeq^*_D) = \begin{cases} 
\{ x_i \} & \text{if } x_i \succ^*_d \emptyset, \\
\{ x_j \} & \text{if } \emptyset \succ^*_d x_i \text{ and } x_j \succ^*_d \emptyset \\
\emptyset & \text{otherwise},
\end{cases}
\]

where \( j \in \{1, 2\} - \{i\} \). Then it can be easily checked that both \( f^1(\cdot) \) and \( f^2(\cdot) \) are stable and strategy-proof. \( \square \)

The second is an example of a market where (i) the rule induced by the cumulative offer process is stable and strategy-proof but not doctor-optimal, and (ii) a hospital’s choice function cannot be induced by any slot-specific priorities.

Example C.2. Let \( D = \{d_1, d_2\} \), \( H = \{h\} \), and \( X = \{x_1, y_1, x_2, y_2\} \), where \( x_i \) and \( y_i \) denote two distinct contracts that involve doctor \( d_i \) (and hospital \( h \)). Suppose that hospital \( h \)'s choice function is induced by

\[
\succ_h: \{y_1, y_2\} \succ_h \{x_1, y_2\} \succ_h \{y_1\} \succ_h \{x_2\} \succ_h \{x_1\} \succ_h \{y_2\} \succ_h \emptyset.
\]

Notice that the resulting choice function (vacuously) satisfies the bilateral substitutes condition and hence, the cumulative offer process defines a stable rule, as summarized in Table C.1.\(^7\)

To begin, we verify that the choice function induced by the above \( \succ_h \) (over the subsets of contracts) cannot be induced by any slot-specific priorities (over contracts). To see this, suppose contrarily that \( (\succ_{h,s})_{1 \leq s \leq q_h} \) induces the same choice function. Since \( C_h(\{x_1, x_2\}) = \{x_2\} \) and \( C_h(\{y_1, x_2\}) = \{y_1\} \), there must exist \( s^* \)

\(^7\)See Definitions C.3 and C.6 in Appendix C.1 for the definitions of the bilateral substitutes condition and the cumulative offer process.
such that (i) $y_1 \succ_{h,s^*} x_2 \succ_{h,s^*} x_1$ and (ii) $x_1 \not\in \text{Ac}(\succ_{h,s})$ for all $s \neq s^*$. Such priorities, however, cannot rationalize $C_h(\{x_1, x_2, y_2\}) = \{x_1, y_2\}$.

Next, we show that this rule is strictly dominated by another stable rule, while it is strategy-proof. To check the first claim, suppose that $\succ_D$ is such that

$$\succ_{d_1} : x_1 \succ_{d_1} y_1 \succ_{d_1} \emptyset, \text{ and}$$

$$\succ_{d_2} : x_2 \succ_{d_2} y_2 \succ_{d_2} \emptyset.$$ 

The outcome of the cumulative offer process at this preference profile is $\{y_1, y_2\}$, as shown in the colored cell in Table C.1. This allocation is strictly dominated by $\{x_1, y_2\}$, which is also stable.

It remains to verify strategy-proofness.\(^8\) First, suppose $\succ_{d_2} = (x_2, y_2, \emptyset)$ or $(x_2, \emptyset)$. In this case, doctor $d_1$ is assigned $y_1$ if $y_1 \in \text{Ac}(\succ_{d_1})$ and $\emptyset$ otherwise. Second, suppose that $\succ_{d_2} = (y_2, x_2, \emptyset), (y_2, \emptyset)$, or $(\emptyset)$. In this case, doctor $d_1$ is assigned his most preferred contract. In either case, it is apparent that $d_1$ has no incentive to manipulate. Symmetrically, doctor $d_2$ is assigned $y_2$ if $y_2 \in \text{Ac}(\succ_{d_2})$.

---

\(^8\)For the remainder of this proof, we slightly abuse notation and identify a preference relation with an ordered list of acceptable contracts. For example, $\succ_{d_1} = (x_1, y_1, \emptyset)$ means $\succ_{d_1}$ is a preference such that $x_1 \succ_{d_1} y_1 \succ_{d_1} \emptyset$. 

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**Table C.1:** Outcomes of the cumulative offer process in the proof of Example C.2.
and $\emptyset$ otherwise when $\succ_{d_1} = (x_1, y_1, \emptyset)$, $(y_1, x_1, \emptyset)$, or $(y_1, \emptyset)$; he can obtain his best contract by truth-telling when $\succ_{d_1} = (x_1, \emptyset)$ or $(\emptyset)$). Hence, doctor $d_2$ does not have an incentive to manipulate either, and the proof is complete.

The last example shows that non-wastefulness as defined in Appendix C.2 does not necessarily follow from stability in the absence of IRC condition.

**Example C.3.** Let $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$, and $X = \{x_1, x_2\}$, where for each $i \in \{1, 2\}$, $x_i$ is a contract between $d_i$ and $h_i$. Suppose that each $d_i$ has a preference relation such that $x_i \succ_{d_i} \emptyset$, and each $h_i$ has a choice function $C_{h_i}(\cdot)$ such that $C_{h_i}(\{x_1, x_2\}) = x_i$ and $C_{h_i}(X') = \emptyset$ for all $X' \subsetneq X$. Then, allocation $X' = \emptyset$ is wasteful but stable.
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