Rational Curves on Hypersurfaces

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Rational Curves on Hypersurfaces

A dissertation presented

by

Eric Riedl

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
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We investigate the dimensions of the spaces of rational curves on hypersurfaces in characteristic 0, and various related questions.
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1. Introduction

A basic way of attempting to understand a variety is to understand its subvarieties. For example, what subvarieties appear? What are the irreducible components to their moduli spaces? What are the dimensions of these components? Much work has been done on this area, but the question appears to be extremely difficult in general, and there are many simple-sounding open questions. For instance, the dimensions of the spaces of degree $d$, genus $g$ curves in $\mathbb{P}^n$ are unknown in general.

When considering these questions, of particular interest is understanding the geometry of the rational subvarieties of a given variety, partly because these questions are sometimes more tractable and partly because the nice properties enjoyed by rational varieties give us particular insight into the geometry of the variety. For instance, Harris, Mazur and Pandharipande [21] prove that any smooth hypersurface in sufficiently high degree is unirational by looking at the space of $k$-planes contained in the hypersurface. Or, in [32] Starr, again by considering the spaces of $k$-planes contained in a hypersurface, proves that the $k$-plane sections of a smooth hypersurface in sufficiently high dimension dominate the moduli space of hypersurfaces in $\mathbb{P}^k$. Rational varieties, particularly rational curves, have an important role in birational geometry and the Minimal Model Program, as is evidenced by the proofs by Campana [7] and Kollár-Miyaoka-Mori [28] that Fano varieties are rationally connected. In a different direction, genus 0 Gromov-Witten invariants, which are important in mathematical physics, are an attempt to count the number of rational curves satisfying certain incidence conditions. Knowing the dimensions of Kontsevich spaces allows us to connect the Gromov-Witten theory calculations to actual curves.

In this document, we focus on finding the dimensions of the spaces of rational subvarieties of hypersurfaces in an algebraically closed field of characteristic 0. For most of the document, the focus is on the spaces of rational curves. The main result is the result of Section 6 proving that the spaces of rational curves on general
hypersurfaces of degree $d \leq n - 2$ in $\mathbb{P}^n$ have the expected dimension. Section 2 provides background information and context on this problem, and the following sections up to and including Section 6 are concerned with the proof of this result. In Section 7, we prove that a general hypersurface in $\mathbb{P}^n$ of degree $2n - 3 \geq d \geq \frac{3n+1}{2}$ contains lines but no other rational curves. In Section 8, we give an idea of how difficult the traditional incidence correspondence approach to these questions is by exploring some of the complexities involved in classifying the Hilbert functions of rational curves of small degree in $\mathbb{P}^3$. This is an interesting question in its own right, and the few cases we consider are by no means exhaustive. In Section 9, we continue the theme of investigating incidence correspondences by proving that the incidence correspondence of irreducible rational curves lying in Fano hypersurfaces is irreducible for $e \leq n$. The incidence correspondences will not be irreducible in general, so this result is an exploration of when things work out as well as possible. It also has implications about the Hilbert functions of rational curves of low degree in $\mathbb{P}^n$.

In Section 10, we continue to investigate the geometry of these incidence correspondences by exploring which types of normal bundles occur. We prove that for rational normal curves in $\mathbb{P}^n$, any possible kernel of a surjective map of vector bundles $\mathcal{O}_{\mathbb{P}^1}(e+2)^{e-1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)^{n-e} \to \mathcal{O}(ed)$ can occur as the normal bundle of a rational normal curve on a smooth hypersurface, and in most cases we work out the exact codimension of the locus in the incidence-correspondence with that splitting type.

In Section 11, we begin to investigate higher-dimensional rational subvarieties of hypersurfaces. Such varieties are interesting for many reasons, and represent one of the next natural steps in understanding rational subvarieties of general hypersurfaces now that many questions regarding curves on Fano hypersurfaces are answered. These questions are still at a very rudimentary stage, and much remains to be done on them. Here, we prove that for large $n$ with respect to $d$, the space of 2-planes through an
arbitrary point of a general $X$ has the expected dimension, and the space of quadric surfaces through a point in $X$ also has the expected dimension.

2. Previous work on Rational Curves

Let $X$ be a hypersurface in $\mathbb{P}^n$ of degree $d$. We are interested in the space of degree $e$ rational curves on $X$. Let $N = \binom{n+d}{d} - 1$. Let $\overline{\mathcal{M}}_{0,0}(X, e)$ be the Kontsevich space of degree $e$ stable rational maps to $X$. Let $R_e(X) \subset \overline{\mathcal{M}}_{0,0}(X, e)$ be the space of rational maps whose general element is a birational map from an irreducible curve. It is not hard to see that $\overline{\mathcal{M}}_{0,0}(X, e)$ is cut out by a section of a rank $ed + 1$ vector bundle on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$. Thus, every component of $\overline{\mathcal{M}}_{0,0}(X, e)$ will have dimension at least $(e+1)(n+1) - 4 - (ed+1) = e(n-d+1) + n - 4$. There are examples of smooth hypersurfaces $X$ with $\dim \overline{\mathcal{M}}_{0,0}(X, e) \geq e(n-d+1) + n - 4$. This leads naturally to the following open question, which was conjectured by Coskun-Harris-Starr in the special case $n \geq d + 1$.

**Conjecture 2.1.** For $X$ a general hypersurface of degree $d$ in $\mathbb{P}^n$, the dimension of $R_e(X)$ is equal to $\max\{0, e(n-d+1) + n - 4\}$, the minimum possible.

In fact, Coskun-Harris-Starr conjecture that $\overline{\mathcal{M}}_{0,0}(X, e)$ has dimension $e(n-d+1) + n - 4$ for $n \geq d+1$. Conjecture 2.1 is open for large ranges of $d$, $n$ and $e$. However, some cases are known, and we summarize them here. For $e \leq d+2$, the conjecture follows from a result of Gruson, Lazarsfeld and Peskine [19]. In the special case $d = 5$, $n = 4$, Conjecture 2.1 is a version of the well-known Clemens’ Conjecture, which has been worked on by Katz, Kleiman-Johnsen, Cotterill and others [27, 25, 26, 11, 12]. Despite all of this progress, Clemens’ Conjecture remains open for $e \geq 12$.

Some work has been done on Conjecture 2.1 for $d$ larger than $n$. Voisin [34, 35], improving on work of Clemens [8] and Ein [15, 16], proved as a special case of a more general result that if $d \geq 2n - 2$, then a general $X$ contains no rational curves. Pacienza [31] proves that for $d = 2n - 3$, $X$ contains lines but no other rational
curves. To the best of our knowledge, for \( d > n \), the other cases were open, as is stated on page 6 of Voisin [38]. We improve Pacienza’s result for \( d \geq \frac{3n+1}{2} \).

If \( d < n \), which is the original setting for Conjecture 2.1 Beheshti and Kumar [1], following work of Harris, Roth and Starr [22], proved that if \( d \leq \frac{2n+2}{3} \), a general \( X \) contains the expected dimension of curves.

In this paper, we improve on Beheshti-Kumar’s [4] and Harris-Roth-Starr’s [22] results. \(^1\)

**Theorem 2.2.** Let \( X \) be a general degree \( d \) hypersurface in \( \mathbb{P}^n \), with \( n \geq d+2 \). Then \( \mathcal{M}_{0,0}(X, e) \) is an irreducible local complete intersection stack of dimension \( e(n - d + 1) + n - 4 \).

For \( n \geq d + 1 \), Coskun, Harris and Starr [10] conjecture still more. Namely they conjecture that the evaluation morphism \( \mathcal{M}_{0,1}(X, e) \to X \) is flat.

**Conjecture 2.3.** For \( X \) a general hypersurface of degree \( d \leq n - 1 \) in \( \mathbb{P}^n \), the evaluation morphism \( \text{ev} : \mathcal{M}_{0,1}(X, e) \to X \) is flat of relative dimension \( e(n - d + 1) - 2 \).

Clearly, Conjecture 2.3 for a given \( n \), \( d \), and \( e \) implies Conjecture 2.1 for that \( n \), \( d \) and \( e \). Harris-Roth-Starr [22] showed that knowing Conjecture 2.3 for \( e \) up to a certain threshold degree \( \frac{n+1}{n-d+1} \) proves Conjecture 2.3 for all \( e \), and moreover proves that \( \mathcal{M}_{0,0}(X, e) \) is a local complete intersection stack. We prove flatness for \( n \geq d+2 \).

**Theorem 2.4.** Let \( X \) be a general degree \( d \) hypersurface in \( \mathbb{P}^n \), with \( n \geq d+2 \). Then \( \text{ev} : \mathcal{M}_{0,1}(X, e) \to X \) is flat with fibers of dimension \( e(n - d + 1) - 2 \).

The idea of the proof in Harris-Roth-Starr [22] is to prove Conjecture 2.3 for \( e = 1 \), then to show that every rational curve through a given point specializes to a reducible curve. By flatness of the evaluation morphism, the result follows

\(^1\)As we were working on this write-up, we received word that Roya Beheshti had independently proven Conjecture 2.1 for \( d < n - 2\sqrt{n} \). Her techniques seem likely to apply to hypersurfaces in the Grassmannian.
by induction. Beheshti-Kumar \cite{1} get a stronger result by proving flatness of the evaluation morphism for $e = 2$ and then using the Harris-Roth-Starr result. The key to Harris-Roth-Starr’s approach is a version of Bend-and-Break which allows them to show that there when there are enough curves in the fibers of the evaluation morphism $ev : \overline{\mathcal{M}}_{0,1}(X, e) \to X$, every curve must specialize to a reducible curve. The reason that their bound applies only for small $d$ is because when $d$ gets larger, there are not enough curves to ensure that every component of a fiber of $ev$ contains reducible curves.

Our approach builds on Harris-Roth-Starr’s by borrowing curves from nearby hypersurfaces to ensure that there are enough curves to apply Bend-and-Break. We do this by bounding the codimension of the space of hypersurfaces for which the statement of Conjecture \cite{2.1} does not hold. Crucial to our analysis are the notions of $e$-level and $e$-layered hypersurfaces, that is, hypersurfaces that have close to the right dimensions of degree-at-most-$e$ rational curves through any given point and hypersurfaces whose rational curves all specialize to reducible curves. Our proof proceeds by inductively bounding the codimension of the space of hypersurfaces that are not $e$-layered.

The next few sections of this document are organized as follows. First we state and prove the version of Bend-and-Break that we will use. Then we sketch how the version of Bend-and-Break can be used to prove Harris-Roth-Starr. Next, we introduce the concepts of $e$-levelness and $e$-layeredness and prove some important properties of them. Then we prove Theorem \cite{2.4} using these notions.

3. BACKGROUND FOR RATIONAL CURVES ON HYPERSURFACES

We will use standard facts about Kontsevich spaces, such as those found in \cite{17}. We treat $\overline{\mathcal{M}}_{0,0}(X, e)$ as a coarse moduli space. Occasionally we will need to use the
result found in Vistoli [33] which says that there is a scheme $Z$ that is a finite cover of the stack $\overline{\mathcal{M}}_{0,0}(X, e)$. We also need the following well-known result.

**Lemma 3.1.** It is at least $\binom{d+k}{k}$ conditions for a degree $d$ hypersurface to contain a given $k$-dimensional variety.

The following variant of a result whose proof we read in [4] (although it was known before this) is the version of Bend-and-Break that we will use.

**Proposition 3.2.** Let $T \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ with $\dim T = 1$. Suppose each of the maps parameterized by $T$ contains two distinct fixed points $p, q \in \mathbb{P}^n$ in its image, and that the images of the curves parameterized by $T$ sweep out a family of dimension $2$ in $\mathbb{P}^n$. Then $T$ parameterizes maps with reducible domains.

**Proof.** Suppose the result is false. Then, possibly after a finite base-change, we can find a family of Kontsevich stable maps parameterized by $T$ providing a counterexample. After normalizing, we can assume $T$ is smooth. Thus, we have a $\mathbb{P}^1$-bundle $\pi : B \to T$ and a map $\phi : B \to \mathbb{P}^n$ such that the restriction of $\phi$ to each fiber of $\pi$ is the Kontsevich stable map in question.

The Neron-Severi group of $B$ is two dimensional, generated by the fiber class and a divisor whose class is $\mathcal{O}(1)$ on each fiber. Since the image of $B$ is two-dimensional, any contracted curves must have negative self-intersection. Thus, the sections of $\pi$ corresponding to the two points $p$ and $q$ must be two disjoint curves with negative self-intersection. Thus, their classes in the Neron-Severi group must be independent, since their intersection matrix is negative definite. However, this contradicts the Neron-Severi rank being two, since it is impossible for the entire Neron-Severi group to be contracted. □

In fact, the above result is true even if the image is a curve See [14] Proposition 3.2.
Corollary 3.3. If $T$ is a complete family in $\overline{M}_{0,0}(\mathbb{P}^n, e)$ of dimension at least $2n - 1$ such that the image of a general map in the family is unique, then $T$ contains elements with reducible domains.

Proof. Consider the incidence correspondence $Y = \{(C, f, p, q) \mid (C, f) \in T \text{ and } p, q \in f(C)\}$. Then $Y$ has dimension at least $2n + 1$. Looking at the natural map $Y \to \mathbb{P}^n \times \mathbb{P}^n$, we see that the general non-empty fiber has to be at least one-dimensional. Thus, we can find a 1-parameter subfamily passing through two distinct points whose image is non-constant. □

We also need a similar result for families of curves lying on a hypersurface all passing through one point:

Proposition 3.4. Let $X$ be a hypersurface in $\mathbb{P}^n$. If $T$ is a complete family in $\overline{M}_{0,0}(X, e)$ of dimension at least $n - 1$ such that the image of a general map in the family is unique and contains a fixed point $p$, then $T$ parameterizes a map with reducible domain.

Proof. Consider the incidence correspondence $Y = \{(C, f, q) \mid (C, f) \in T \text{ and } q \in f(C)\}$. Then $Y$ has dimension at least $n$. Looking at the natural map $Y \to X$, we see that the general fiber has to be at least one-dimensional. Thus, we can find a 1-parameter subfamily passing through $p$ and another point $q$ whose image is non-constant. □

4. THE CASE OF SMALL DEGREE

Because the ideas in Harris-Roth-Starr \cite{22} are so central to our approach, we provide a sketch of the proof of a main result from Harris-Roth-Starr.

Theorem 4.1 (Harris-Roth-Starr). Let $d \leq \frac{n+1}{2}$. Then if $X$ is a general degree $d$ hypersurface in $\mathbb{P}^n$, the space of degree $e$ rational curves in $X$ through an arbitrary point $p$ has dimension $e(n - d + 1) - 2$. 7
Proof. (sketch) It follows from Proposition 5.10 that a general hypersurface has a $1 \cdot (n - d + 1) - 2 = n - d - 1$ dimensional family of lines through every point. Now we use induction. Suppose we know the result for all curves of degree smaller than $e$. Then the space of reducible curves through a point $p$ with components of degrees $e_1$ and $e_2$ has dimension $e_1(n-d+1)-2+1+e_2(n-d+1)-2 = e(n-d+1)-3$. Thus, it remains to show that every component of rational degree $e$ curves through $p$ contains reducible curves, since we know that the space of reducible curves is codimension at most 1 in the space of all rational curves.

It follows from Proposition 3.4 that any $(n-1)$-dimensional family of curves contained in $X$ passing through $p$ must contain reducible curves. Thus, we will have the result if

$$e(n - d + 1) - 2 \geq n - 1$$

for all $e \geq 2$. This simplifies to

$$ed \leq n(e - 1) + e - 1$$

or

$$d \leq \frac{(n + 1)(e - 1)}{e}.$$

The right-hand side is increasing in $e$, so if $d \leq \frac{n+1}{2}$ we have our result. \qed

From the proof, we see that if $ev : \overline{M}_{0,1}(X, k) \to X$ is flat with expected-dimensional fibers for $1 \leq k \leq e - 1$ but not for $k = e$, then $e(n - d + 1) - 2 \leq n - 1$, or $e \leq \frac{n+1}{n-d+1}$. That is, we need only check flatness for degrees up to $\frac{n+1}{n-d+1}$. Harris-Roth-Starr call $\lfloor \frac{n+1}{n-d+1} \rfloor$ the threshold degree. Note that as a Corollary of the proof, it follows that every component of degree $e$ curves contains curves with reducible domains.
Harris-Roth-Starr also prove irreducibility of the space of rational curves, and we will need this result as well, but we will describe it further in the next section, after we have talked about $e$-layeredness.

5. $e$-LEVELNESS AND $e$-LAYEREDNESS

This section is about two related concepts that underlie the ideas behind our proofs: $e$-levelness and $e$-layeredness. Roughly speaking, an $e$-level point of a hypersurface has the expected dimension of degree-up-to-$e$ rational curves through it, and an $e$-layered point is such that every degree-up-to-$e$ rational curve through it specializes to a reducible curve. The definitions are new, but they are related to ideas in [22]. Our main innovation is extending these ideas to singular hypersurfaces, so that we can try to bound the codimensions of the loci of hypersurfaces which are not $e$-layered.

**Definition 5.1.** A point $p \in X$ is $e$-level if:

- $p \in X_{\text{smooth}}$ and the space of rational curves in $X$ through $p$ has dimension at most $e(n - d + 1) - 2$ or
- $p \in X_{\text{sing}}$ and the space of rational curves in $X$ through $p$ has dimension at most $e(n - d + 1) - 1$.

A point $p \in X$ is $e$-sharp if it is not $e$-level.

We say that the space $T$ of degree $e$ rational curves in $X$ through $p$ has the expected dimension if $p$ is singular and $\dim T = e(n - d + 1) - 1$ or $p$ is smooth and $\dim T = e(n - d + 1) - 2$. The reason that the condition is different for singular points is that through a singular point, there will always be at least a $e(n - d + 1) - 1$ dimensional family of rational curves, as we can see from writing out how many conditions it is for an explicit map from $\mathbb{P}^1$ to $\mathbb{P}^n$ to lie in $X$. Points are $e$-level if they have the expected dimension of degree $e$ rational curves through them.
Definition 5.2. A hypersurface \( X \) is \( e \)-level if for all \( k \leq e \) the following two conditions hold:

- There are no rational curves of degree \( k \) contained in \( X_{\text{sing}} \).
- There are no \( k \)-sharp points of \( X \).

We say a hypersurface is \( e \)-sharp if it is not \( e \)-level.

Let 

\[
\Phi = \{(p, X)|p \in X\} \subset \mathbb{P}^n \times \mathbb{P}^N.
\]

Let \( \Phi_{\text{smooth}} \) and \( \Phi_{\text{sing}} \) be the respectively open and closed subsets given by

\[
\Phi_{\text{smooth}} = \{(p, X)|p \in X \text{ such that } p \in X_{\text{smooth}}\} \subset \Phi
\]

and

\[
\Phi_{\text{sing}} = \{(p, X)|p \in X \text{ such that } p \in X_{\text{sing}}\} \subset \Phi.
\]

Let \( \Phi_{e,\text{sharp}} \subset \Phi \) be the locus of pairs \((p, X)\) where \( p \) is an \( e \)-sharp point of \( X \). Notice that \( \Phi_{e,\text{sharp}} \) is not closed in \( \Phi \). To see this, consider the family of cubics in \( \mathbb{P}^5 \) cut out by \( f_t = tx_0^2x_1 + x_0x_1x_2 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 \). For all \( t \neq 0 \), \( X_t = V(f_t) \) is smooth at \( p \), which means that \( p \) is a 1-sharp point of \( X_t \). However, for \( t = 0 \), \( X_0 \) is singular at \( p \), which means \( p \) is a 1-level point of \( X_0 \). Although \( \Phi_{e,\text{sharp}} \) is not closed in \( \Phi \), it is the case that \( \Phi_{e,\text{sharp}} \cap \Phi_{\text{smooth}} \) is closed in \( \Phi_{\text{smooth}} \) and \( \Phi_{e,\text{sharp}} \cap \Phi_{\text{sing}} \) is closed in \( \Phi_{\text{sing}} \) (which means that it is also closed in \( \Phi \), since \( \Phi_{\text{sing}} \) is closed in \( \Phi \)).

Definition 5.3. A point \( p \in X \) is \( e \)-layered if:

- It is 1-level.
- For \( k \leq e \), every component of degree \( k \) rational curves through \( p \) contains reducibles.
A point \( p \in X \) is \( e \)-uneven if it is not \( e \)-layered.

As before, we define \( \Phi_{e, \text{layered}} \) to be \( \{(p, X)|p\text{ is an }e\text{-layered point of }X\} \) and \( \Phi_{e, \text{uneven}} \) to be \( \{(p, X)|p\text{ is an }e\text{-uneven point of }X\} \). The definition of \( e \)-layered hypersurfaces is analogous to that of \( e \)-level hypersurfaces.

**Definition 5.4.** We say that a hypersurface \( X \) is \( e \)-layered if:

- All of its points are \( e \)-layered.
- It contains no rational curves of degree less than or equal to \( e \) in its singular locus.

Proposition 5.5 will allow us to relate layeredness and levelness. We wish to specialize arbitrary rational curves to reducible curves in order to get a bound on the dimension of the space of curves through a point. The specialization will not be useful unless we know that the space of reducible curves has sufficiently small dimension. The notion of \( e \)-levelness is exactly tailored so that this is the case.

**Proposition 5.5.** Let \( X \) be an \((e - 1)\)-level degree \( d \) hypersurface. Let \( ev : \overline{M}_{0,1}(X, e) \to X \). Then if \( p \) is a point of \( X \), the subspace of reducible curves in \( ev^{-1}(p) \) has dimension at most \( e(n - d + 1) - 3 \) if \( p \in X_{\text{smooth}} \) and at most \( e(n - d + 1) - 2 \) if \( p \in X_{\text{sing}} \).

**Proof.** We use strong induction on \( e \). It is obvious for \( e = 1 \), as there are no reducible curves in \( \overline{M}_{0,1}(X, 1) \).

Denote by \( \overline{M}_{0,\{a\}}(X, e) \) the Kontsevich space of stable degree \( e \) maps from rational curves with a marked point \( a \). The subspace of reducible curves in \( \overline{M}_{0,\{a\}}(X, e) \) is covered by maps from \( B_{e_1} = \overline{M}_{0,\{a,b\}}(X, e_1) \times_X \overline{M}_{0,\{c\}}(X, e_2) \), where \( e_1 + e_2 = e \), \( e_1, e_2 \geq 1 \), and the maps \( ev_b : \overline{M}_{0,\{a,b\}}(X, e_1) \to X \) and \( ev_c : \overline{M}_{0,\{c\}}(X, e_2) \to X \) are used to define the fiber products. The map from \( \overline{M}_{0,\{a,b\}}(X, e_1) \times_X \overline{M}_{0,\{c\}}(X, e_2) \) to \( \overline{M}_{0,\{a\}}(X, e) \) is defined by gluing the domain curves together along \( b \) and \( c \) (see [5] for details on how the gluing map works). The marked point \( a \) of the first curve
becomes the point $a$ of the resulting curve. Let $pr_1$ and $pr_2$ be the projection maps of $B_{e_1}$ onto the first and second components.

Fix a point $p \in X$, and let $Y = (ev_a \circ pr_1)^{-1}(p) \subset B_{e_1}$. We are thus reduced to bounding the dimension of $Y$. Let $Z = ev_a^{-1}(p) \subset \overline{\mathcal{M}}_{0,\{a\}}(X, e_1)$, and $Z' = ev_a^{-1}(p) \subset \overline{\mathcal{M}}_{0,\{a,b\}}(X, e_1)$, so that we have a natural sequence of maps $Y \to Z' \to Z$ given by $pr_1: Y \to Z'$ and the forgetful map $\pi : Z' \to Z$ which forgets the point $b$.

$$
\begin{array}{ccc}
Y & \longrightarrow & B_{e_1} \\
\downarrow & & \downarrow pr_1 \\
Z' & \longrightarrow & \overline{\mathcal{M}}_{0,\{a,b\}}(X, e_1) \\
\downarrow & & \downarrow \pi \\
Z & \longrightarrow & \overline{\mathcal{M}}_{0,\{a\}}(X, e_1) \\
\downarrow & & \downarrow ev_a \\
\text{Spec } k & \longrightarrow & X
\end{array}
$$

Given a tuple $(f, C, p_a) \in Z \subset \overline{\mathcal{M}}_{0,\{a\}}(X, e_1)$, we wish to analyze the fibers of $pr_1 \circ \pi$. The fibers of $\pi$ are all 1-dimensional, and the fibers of $pr_1$ are all at most $e_2(n - d + 1) - 1$ dimensional by $(e - 1)$-levelness. If $C$ is irreducible, then since there are no degree at most $e - 1$ rational curves in $X_{\text{sing}}$, the general fiber of $pr_1$ over a point $(f, C, p_a, p_b) \in Z'$ has dimension $e_2(n - d + 1) - 2$. By induction, we know that for a general $(f, C, p_a) \in Z$ with $f(p_a) = p$, $C$ is irreducible. Therefore, $\dim Y = \dim Z + 1 + e_2(n - d + 1) - 2 = \dim Z + e_2(n - d + 1) - 1$.

Putting it all together, by $e$-levelness, the dimension of $Z$ is at most $e_1(n - d + 1) - 2$ if $p \in X_{\text{smooth}}$, and $e_1(n - d + 1) - 1$ if $p \in X_{\text{sing}}$. Thus, the dimension of $Y$ is $e_1(n - d + 1) - 2 + e_2(n - d - 1) - 1 = e(n - d + 1) - 3$ if $p \in X_{\text{smooth}}$ or at most $e_1(n - d + 1) - 1 + e_2(n - d + 1) - 1 = e(n - d + 1) - 2$ if $p \in X_{\text{sing}}$, as desired.

Since reducible curves are codimension at most one in $\overline{\mathcal{M}}_{0,1}(X, e)$, we obtain an immediate corollary.

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**Corollary 5.6.** If $X$ is $e$-layered, then $X$ is $e$-level.

It follows that if $p \in X$ is $(e-1)$-level, but $e$-sharp, than $p$ must be $e$-uneven.

**Corollary 5.7.** The space $\Phi_{e,\text{uneven}}$ is closed in $\Phi_{e-1,\text{level}}$.

*Proof.* To get a contradiction, suppose we have a 1-parameter family of $(e - 1)$-level pairs $(p_t, X_t)$ such that a general point is $e$-uneven, while $(p_0, X_0)$ is $e$-layered. Since $(p_0, X_0)$ is $(e - 1)$-level, this means that the family of reducible curves in $X_t$ passing through $p_t$ is codimension at least 2 in the family of all degree curves in $X_t$ passing through $p_t$. This is a contradiction. \[\square\]

Using Proposition 5.5, we immediately obtain a generalization of the result of Harris-Roth-Starr. The idea behind this corollary is related to ideas found in Harris, Roth, and Starr’s treatment of families of curves on smooth cubic hypersurfaces [23].

**Corollary 5.8.** If $n \geq d + 2$, $X$ is an $e$-level, degree $d$ hypersurface in $\mathbb{P}^n$ which has no rational curves of any degree in its singular locus and $(e + 1)(n - d + 1) - 2 \geq n - 1$, then $X$ is $k$-level for all $k$. It follows that if $X$ contains no rational curves in its singular locus and is $\lfloor \frac{n+1}{n-d+1} \rfloor$-level, then $X$ is $k$-level for all $k$.

As in Section 4, we refer to $\lfloor \frac{n+1}{n-d+1} \rfloor$ as the threshold degree.

*Proof.* Because for $k \geq e + 1$ the space of any component of degree $k$ rational curves through a point must have dimension at least $(e + 1)(n - d + 1) - 2 \geq n - 1$, we see by Proposition 3.4 that every component of the space of degree $k$ rational curves through an arbitrary point of $X$ must contain reducible curves. Since the space of reducible curves is a divisor in $\overline{M}_{0,0}(\mathbb{P}^n,e)$, by Proposition 5.5 we see that every component of the space of degree $k$ rational curves through an arbitrary point must have the expected dimension. \[\square\]

The following result is essentially proven in [22], although they do not have the term $e$-layered. We sketch their proof here for convenience.
**Theorem 5.9.** If \( n \geq d + 2 \) and \( X \subset \mathbb{P}^n \) is a smooth, degree \( d \), \( e \)-layered hypersurface such that the space of lines through a general point is irreducible, then \( \mathcal{M}_{0,1}(X,e) \) is irreducible of the expected dimension.

**Proof.** By induction, it follows that the space of degree \( e \) rational curves through a general point \( p \) contains curves that are trees of lines with no nodes at \( p \). By 1-levelness, it follows that any tree of lines can be specialized to a “broom” of lines, that is, a set of \( e \) lines all passing through the same point which is distinct from the fixed point \( p \). By irreducibility of the space of lines through a point, it follows that the space of brooms is irreducible, and considering codimensions shows that every component of the space of rational curves through \( p \) that contains a broom contains the entire space of brooms.

For a general broom, the lines that it contains will all have balanced normal bundle in \( X \). It follows that the pullback of the tangent sheaf \( T_X \) of \( X \) twisted by \(-p\) will have no \( H^1 \) for a general element of the family of brooms. Thus, the family of brooms is contained in a unique irreducible component of \( \mathcal{M}_{0,1}(X,e) \). However, since we showed that it is contained in every component of \( \mathcal{M}_{0,1}(X,e) \), our result is proven. \( \square \)

Let \( S_1 \) be the closure of the set of 1-sharp hypersurfaces, and let \( S_e \subset \mathbb{P}^N \) be the closure of the union of \( S_1 \) with the set of \( e \)-uneven hypersurfaces. Note that \( S_k \subset S_e \) for \( k \leq e \).

The way we prove Theorem 6.1 is by bounding the codimension of the locus of \( e \)-uneven hypersurfaces. We prove the base case \( e = 1 \) here. The ideas are similar to those found in Section 2 of [22], but we need more precise dimension estimates so we restate and re-prove the result.

**Proposition 5.10.** If \( n \geq d + 2 \), then the codimension of \( S_1 \) in \( \mathbb{P}^N \) is at least \( \min\{n(d - 2) + 3, \binom{n}{2} - n + 1\} \).
Proof. Note that $S_1$ will simply be the space of hypersurfaces singular along a line union the closure of the space of hypersurfaces having a sharp point.

First consider the space of hypersurfaces everywhere singular along a given line. Let $f$ be the polynomial cutting out our hypersurface. We examine what conditions are imposed on the coefficients of $f$ when we insist that $V(f)$ be everywhere singular along a given line. If we choose coordinates so that the line is $x_1 = \cdots = x_{n-1} = 0$, then the coefficients of $x_0^j x_n^{d-j}$ will have to vanish for $0 \leq j \leq d$, as will the coefficients of $x_i x_0^j x_n^{d-j-1}$ for $0 \leq j \leq d - 1$. This is $nd + 1$ conditions. Since there is a $(2n - 2)$-dimensional family of lines in $\mathbb{P}^n$, this means that for a hypersurface to be singular along a line is $nd + 1 - (2n - 2) = n(d - 2) + 3$ conditions.

Thus, it remains to bound the codimension of the space of hypersurfaces with a sharp point. By considering the natural projection map $\Phi \to \mathbb{P}^n$, it will suffice to show that the codimension of $\Phi_{1,\text{sharp}} \subset \Phi$ is at least $\binom{n}{2}$.

We do this by considering the fibers of the map $\phi : \Phi \to \mathbb{P}^n$. Because all points of $\mathbb{P}^n$ are projectively equivalent, it will suffice to work with a single fiber of $\phi$, say the fiber over $p$. Choose an affine coordinate chart containing $p$ such that $p = (0, \cdots, 0)$ in this chart, and let $f$ be an equation cutting out a hypersurface $X$ which contains $p$. We want to understand how many conditions are imposed on the coefficients of $f$ when we insist that $p$ be a 1-sharp point of $X$. Take the Taylor expansion of $f$ at $p$ in this affine coordinate chart, writing $f = f_1 + f_2 + \cdots + f_d$, where $f_i$ is a homogeneous polynomial of degree $i$.

Note that if we identify the space of lines in $\mathbb{P}^n$ passing through $p$ with $\mathbb{P}^{n-1}$, then the space of such lines that lie in $X$ will be the intersection of the $V(f_i)$. We therefore only need to analyze when this intersection has larger dimension than would be expected (given that $X$ is smooth or singular at $p$.)
By Lemma 3.1 it is at least \( \binom{d+k}{d} \) conditions for a hypersurface to contain a \( k \)-dimensional subvariety. We consider separately the case where \( X \) is singular at \( p \) and \( X \) is smooth at \( p \).

First, suppose \( X \) is singular at \( p \), i.e., \( f_1 = 0 \). Then \( f_2 \) will be non-zero outside of a codimension \( \binom{n+1}{2} \) variety, and given that \( f_2 \neq 0 \), \( V(f_3) \) will not contain any component of \( V(f_2) \) outside of a codimension \( \binom{n-2+3}{3} = \binom{n+1}{3} \) variety. Similar, if \( \bigcap_{1<i<j} V(f_i) \) has dimension \( n-j+1 \), \( V(f_j) \) will not contain any component of \( \bigcap_{1<i<j} V(f_i) \) outside of a \( \binom{n-j+1+j}{j} = \binom{n+1}{j} \)-codimensional variety. For \( n \geq d+2 \), \( \binom{n+1}{j} \geq \binom{n}{2} \) for \( 2 \leq j \leq d \).

Now suppose \( X \) is nonsingular at \( p \), i.e., \( f_1 \neq 0 \). Then \( f_2 \) will not contain any component of \( V(f_1) \) outside of a codimension \( \binom{n-2+2}{2} = \binom{n}{2} \) variety. Similarly, if \( \bigcap_{i<j} V(f_i) \) has dimension \( n-j \), \( V(f_j) \) will not contain any component of \( \bigcap_{i<j} V(f_i) \) outside of a codimension \( \binom{n-j+j}{j} = \binom{n}{j} \) variety. For \( n \geq d+2 \) and \( 2 \leq j \leq d \), we see that \( \binom{n}{j} \geq \binom{n}{2} \).

\[ \square \]

For the proof of our main result, we need to understand which hypersurfaces contain small-degree rational curves in their singular loci. The following Proposition bounds the codimension of such hypersurfaces.

**Proposition 5.11.** The space of degree \( d \) hypersurfaces singular along a degree \( e \) rational curve has codimension at least \( n(d-e-1)-e+4 \).

**Proof.** The space of degree \( e \) rational curves in \( \mathbb{P}^n \) has dimension \( (n+1)(e+1) - 4 \), so we just need to check that the space of hypersurfaces singular along a given degree \( e \) rational curve \( C \) has codimension at least \( nd+1 \), the codimension of the space of hypersurfaces singular along a line. We will reduce to this case by deforming \( C \) to a line.
Without loss of generality, we may assume that $C$ does not intersect the $(n - 2)$-plane $a_0 = a_1 = 0$. Now consider the closed subvariety $\mathcal{F}^o$ of $\mathbb{P}^n \times \mathbb{A}^1 - \{0\}$ whose fiber above a point $r$ of $\mathbb{A}^1 - \{0\}$ is the image of $C$ under the automorphism $[a_0 : a_1 : \cdots : a_n] \rightarrow [a_0 : a_1 : ra_2 : \cdots : ra_n]$ of $\mathbb{P}^n$. Let $\mathcal{F}$ be the closure of $\mathcal{F}^o$ in $\mathbb{P}^n \times \mathbb{A}^1$.

The set theoretic fiber of $\mathcal{F}$ over $0$ is the line $a_2 = a_3 = \cdots = a_n = 0$. We thus see that the dimension of the space of hypersurfaces singular everywhere along $C$ is at most the dimension of the space of hypersurfaces singular everywhere along a line, and we already worked out the dimension of the space of hypersurfaces singular along a line in the proof of Proposition 5.10. Thus, the codimension of the space of degree $d$ hypersurfaces singular along any degree $e$ rational curve is at most $nd + 1 - ((n + 1)(e + 1) - 4) = n(d - e - 1) - e + 4$.

\[ \square \]

For technical reasons in the proof of the main theorem we will need to show that an $e$-level hypersurface will contain lots of curves that aren’t multiple covers of other curves, which is a result of independent interest.

**Proposition 5.12.** If $n \geq d + 2$, $e \geq 2$ and $X$ is $(e - 1)$-level, then in any component of the family of degree $e$ rational curves through $p$, there is a pair $(f, C) \in \mathcal{M}_{0,0}(X, e)$ such that $f$ is generically injective.

**Proof.** Let $k > 1$ be a factor of $e$. We claim that the dimension of the space of degree $k$ covers of a degree $\frac{e}{k}$ curve is smaller than the dimension of curves through $p$. We assume that $p$ is a smooth point of $X$, since the computation is similar if $p$ is a singular point (for $p$ singular, everything works the same except in the exceptional case $k = e = 2$, we need to use the fact that there will be a $(n - d)$-family of lines through a singular point). The space of degree $k$ covers of a degree $\frac{e}{k}$ curve through $p$ has dimension

\[ \frac{e}{k}(n - d + 1) - 2 + 2k - 2 = \frac{e}{k}(n - d + 1) + 2k - 4. \]
Any component of the family of degree $e$ rational curves through $p$ will have dimension at least

$$e(n - d + 1) - 2.$$ 

Thus, we need only show that

$$e(n - d + 1) - 2 > \frac{e}{k}(n - d + 1) + 2k - 4.$$ 

Rearranging, we obtain

$$e \left(1 - \frac{1}{k}\right)(n - d + 1) > 2k - 2.$$ 

or

$$e(n - d + 1) > 2k,$$

which is clear, as $e \geq k$ and $n - d + 1 \geq 3$.

\[\square\]

6. Proof of Main Result

The proof of our main result proceeds by inductively bounding the codimensions of the spaces of $e$-uneven hypersurfaces. To do this, we show that if codim $S_{e-1} - \text{codim } S_e$ is too large, then we can find a large family of hypersurfaces and points with no reducible curves through the point. We then apply Bend-and-Break to the family of curves in those hypersurfaces through those points. We can imagine “borrowing” the curves from nearby hypersurfaces to have enough to apply Bend-and-Break.

**Theorem 6.1.** Suppose $M = \text{codim } S_{e-1}$. Then the codimension of $S_e$ in $\mathbb{P}^N$ is at least $\min\{M, M - 2n + e(n - d + 1) - 1, n(d - e - 1) - e + 4\}$.

**Proof.** Note that $S_e$ is the union of three (possibly overlapping) sets: $S_{e-1}$, the space of hypersurfaces singular along a degree $e$ rational curve, and the closure of the space of hypersurfaces with an $e$-uneven point. The codimension of $S_{e-1}$ is at least $M$
by assumption, and the codimension of the space of hypersurfaces singular along a
degree $e$ rational curve is at least $n(d - e - 1) - e + 4$ by Proposition 5.11. Thus,
it remains to bound the codimension of the space of hypersurfaces with an $e$-uneven
point. If $e(n-d+1)-1 \geq 2n$ (or indeed, if $e(n-d+1) \geq n+1$), then by Corollary 5.8
we see that any $(e-1)$-level hypersurface not singular along a degree $e$ rational curve
will be $e$-level, so we need only consider the case $e(n-d+1)-1 < 2n$. The statement
is vacuous if $M \leq 2n - e(n-d+1) - 1$, so we can assume $M > 2n - e(n-d+1) - 1$.
We show that the closure of the space of hypersurfaces with an $e$-uneven point is
codimension at least $M - 2n + e(n-d+1) - 1$, which will suffice to prove the theorem.

Suppose the result is false. That is, suppose that the closure of the space of
hypersurfaces with an $e$-uneven point has codimension at most $M - 2n + e(n-d+1) - 2$.
Then if

$\mathcal{A} = \{(p, f, C, X) \mid p \in f(C) \subset X, \text{the fiber of } ev \text{ over } p \text{ contains a component}
containing } (f, C) \text{ that is disjoint from the boundary } \Delta\}$

$\subset \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, e) \times \mathbb{P}^N$

we can find an irreducible component $\mathcal{N}_e$ of the closure of $\mathcal{A}$ such that the dimension
of the projection of $\mathcal{N}_e$ onto the space of hypersurfaces has codimension at most
$M - 2n + e(n-d+1) - 2$. Let $\mathcal{C}$ be the Chow variety of rational degree $e$ curves
in $\mathbb{P}^n$, and let $\pi : \mathcal{N}_e \to \mathcal{C}$ be the natural map. Let $\psi : \mathcal{N}_e \to \mathbb{P}^N$, $\phi : \mathcal{N}_e \to \Phi$ and
$\psi_1 : \Phi \to \mathbb{P}^N$ be given by the natural maps. Note $\psi = \psi_1 \circ \phi$.

$$
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \mathcal{N}_e \\
\downarrow & & \downarrow \pi \\
\Phi & \longrightarrow & \mathbb{P}^n \\
\downarrow \psi_1 & & \\
\mathbb{P}^N & & \\
\end{array}
$$
We claim that we can find an irreducible family $F \subset \mathcal{N}_e$ of dimension $2n - 1$ with the following properties:

1. $\psi(F) \cap S_{e-1} = \emptyset$
2. If $(p, f, C, X) \in F$, then $C$ is irreducible.
3. $\dim \pi(F) = 2n - 1$

First we prove the theorem assuming the claim. Since $\pi(F)$ has dimension at least $2n - 1$, by Corollary 3.3 we see that $F$ must parameterize points $(p, f, C, X)$ with $C$ reducible, which contradicts property 2. (Condition 1 is needed to prove Condition 2).

Thus, it remains to prove the claim. We start by proving that $\pi(\mathcal{N}_e)$ has dimension at least $3n - 3$. By the definition of $\mathcal{A}$, $\mathcal{A}$ is invariant under automorphisms of $\mathbb{P}^n$. We thus have a map $\text{PGL}_{n+1} \times \mathcal{N}_e \to \mathcal{A}$. As $\text{PGL}_{n+1}$ is irreducible, so is $\text{PGL}_{n+1} \times \mathcal{N}_e$, and thus the image of this map must be irreducible. But the image of this map contains $\mathcal{N}_e$, so the image of this map must be $\mathcal{N}_e$. Thus, $\mathcal{N}_e$ is preserved by automorphisms of $\mathbb{P}^n$.

By Proposition 5.12 there is a point $(p, f, C, X) \in \mathcal{N}_e$ such that $f(C)$ is not a line. Then, if we choose three points $p_1, p_2, p_3$ on $f(C)$ which are not collinear, then those three points can be sent to any other three non-collinear points in $\mathbb{P}^n$ by an automorphism of $\mathbb{P}^n$. This shows that the dimension of the space $C_3 \subset \mathcal{C} \times \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ of triples $(C', p_1', p_2', p_3')$ which can be obtained by applying an automorphism of $\mathbb{P}^n$ to $(C, p_1, p_2, p_3)$ is at least $3n$. But the fibers of the projection $\mathcal{B} \to \mathcal{C}$ are 3-dimensional, as $p_1, p_2, p_3$ must lie on $C$, so the image of $\pi$ will have dimension at least $3n - 3$.

We now construct $F$. Let $c$ be the dimension of a generic fiber of $\phi$, and let $a = 2n - 1 - c$. If $c \geq 2n - 1$, then choose $F$ to be a general $(2n - 1)$-dimensional subvariety of a general fiber of $\phi$. Otherwise, let $H$ be a general plane in $\mathbb{P}^{N}$ of dimension $N - \dim \psi(\mathcal{N}_e) + a$, so that $H \cap \psi(\mathcal{N}_e)$ has dimension $a$. Let $F'$ be a component of a general linear section (under some projective embedding of $\Phi$) of
\(\psi^{-1}(H) \cap \phi(N_e)\), with \(\dim F' = a\). Let \(F\) be an irreducible component of \(\phi^{-1}(F')\) that dominates \(F'\). By construction, \(F\) has dimension \(a + c = 2n - 1\).

We now check the three conditions. We start with condition 1. Because \(c \geq e(n-d+1)-2\), \(a \leq 2n - 1 - (e(n-d+1)-2) = 2n - e(n-d+1) + 1\). By hypothesis \(\dim \psi(N_e) - \dim S_{e-1} \geq 2n - e(n-d+1) + 2 > a\), so it follows that \(\psi(F) = \psi(N_e) \cap H\) is disjoint from \(S_{e-1}\).

For condition 2, we see from generality of \(F\) and definition of \(N_e\) that for a general \((p,X) \in \phi(F)\), every \(C\) with \((p,f,C,X) \in F\), will be irreducible. Additionally, every hypersurface in \(\psi(F)\) is \((e-1)\)-level by condition 1. From an argument similar to the proof of Corollary 5.7, it follows that every \(C\) with \((p,f,C,X) \in F\) will be irreducible.

To see this, suppose some of the \(C\) were reducible. By condition 1, they would have to be reducible over a codimension 1 subset of \(F\), which contradicts the general fiber of \(\phi\) having only \((p,f,C,X)\) with irreducible \(C\).

To prove condition 3, we show that \(\pi|_F\) is generically finite. Let \((p,f,C,X) \in F\) be general. We claim \(\pi\) is finite at \((p,f,C,X)\). By Proposition 5.12 and generality of \((p,f,C,X)\), \(f\) will be generically injective. Let \(B = \{(p',X') \in \Phi \mid p' \in f(C) \subset X'\} \cap \phi(N_e)\). Then since the image of \(\pi\) had dimension at least \(3n-3\), \(B\) has codimension at least \(3n-3\) in \(\phi(N_e)\). Since the fibers of \(\psi_1|_B\) are 1-dimensional, this means \(\psi_1(B)\) has codimension at least \(2n-1\) in \(\psi(N_e)\), which means that \(H\) intersects \(\psi_1(B)\) in at most finitely many points by generality of \(H\). Since \(\psi_1|_{F'}\) is finite, this shows that \(\pi\) is finite at \((p,f,C,X)\). This suffices to show condition 3.

\(\square\)

The rest is just working out the numbers. We know the result for \(d \leq \frac{n+1}{2}\), so it remains to consider \(d \geq \frac{n+1}{2}\). If \(n \leq 5\) then \(d \leq n - 2\) means \(d \leq 3 = \frac{n+1}{2}\), so without loss of generality, we may assume \(n \geq 6\).
Corollary 6.2. If \( d \geq \frac{n+1}{2} \) and \( e \leq \frac{n+1}{n-d+1} \) then

\[
\text{codim } S_e \geq \left( \frac{n}{2} \right) + d - 2en + \frac{e(e+1)}{2} (n - d + 1) - e + 1.
\]

Proof. First, we show that \( n(d - e - 1) - e + 4 \geq \left( \frac{n}{2} \right) + d - 2en + \frac{e(e+1)}{2} (n - d + 1) \).

We will first show that the inequality is strict for \( e \leq \frac{n}{n-d+1} \). For \( e = 1 \) we have

\[
n(d - 2) + 3 > \left( \frac{n}{2} \right) - n + 1
\]

is equivalent to

\[
(d - 2) > \frac{1}{n} \left( \frac{n}{2} \right) - 1 - \frac{2}{n}
\]

which is equivalent to

\[
d > \frac{1}{n} \left( \frac{n}{2} \right) + 1 - \frac{2}{n} = \frac{n - 1}{2} + 1 - \frac{2}{n} = \frac{n + 1}{2} - \frac{2}{n}.
\]

For \( e \leq \frac{n}{n-d+1} \), note that each time we replace \( e - 1 \) with \( e \), the left-hand side decreases by \( n + 1 \), while the right hand side decreases by \( 2n - e(n - d + 1) + 1 \). We see that \( 2n - e(n - d + 1) + 1 \geq n + 1 \), which together with the base case \( e = 1 \) shows that \( n(d - e - 1) - e + 4 > \left( \frac{n}{2} \right) + d - 2en + \frac{e(e+1)}{2} (n - d + 1) \) for this range of \( e \). For \( e = \frac{n+1}{n-d+1} \), we see that replacing \( e - 1 \) with \( e \) decreases the right-hand side by at least \( n \). Together with the fact that the inequality was strict for \( e - 1 \), this proves \( n(d - e - 1) - e + 4 \geq \left( \frac{n}{2} \right) + d - 2en + \frac{e(e+1)}{2} (n - d + 1) \) for \( e \leq \frac{n+1}{n-d+1} \).

Now we prove the entire statement of the corollary by induction. For the base case \( e = 1 \), we need to show \( \text{codim } S_1 \geq \left( \frac{n}{2} \right) + d - 2n + (n - d + 1) = \left( \frac{n}{2} \right) - n + 1 \), which follows from Proposition 5.10 and the above discussion.

Finally, we proceed with the induction step. Suppose \( \text{codim } S_{e-1} \geq \left( \frac{n}{2} \right) + d - 2(e - 1)n + \frac{e(e-1)}{2} (n - d + 1) - e + 2 \). By Theorem 6.1, we see that \( \text{codim } S_e \geq \min\{\text{codim } S_{e-1}, \text{codim } S_{e-1} - 2n + e(n - d + 1) - 1, n(d - e - 1) - e + 4\} \). Using the induction hypothesis and the fact that \( n(d - e - 1) - e + 4 \geq \left( \frac{n}{2} \right) + d - 2en + \frac{e(e+1)}{2} (n - d + 1) - e + 2 \).
\[ \frac{e(e+1)}{2} (n - d + 1), \]

we see that

\[
\operatorname{codim} S_e \geq \binom{n}{2} + d - 2en + \frac{e(e+1)}{2} (n - d + 1) - e + 1.
\]

\[ \square \]

**Corollary 6.3.** If \( X \) is a general hypersurface of degree \( d \) in \( \mathbb{P}^n \) and \( n \geq d + 2 \), then the space of rational, degree \( e \) curves through an arbitrary point \( p \in X \) has the expected dimension for all \( e \).

**Proof.** By Corollary 5.8 the threshold degree is \( \lceil \frac{n+1}{n-d+1} \rceil \), so by Corollary 6.2, it remains to show that \( \binom{n}{2} + d - 2en + \frac{e(e+1)}{2} (n - d + 1) - e + 1 \) is positive for \( 1 \leq e \leq \frac{n+1}{n-d+1} \). Multiplying by two, it suffices to show

\[
n(n-1) - 4en + e(e+1)(n-d+1) + 2d - 2e + 2 > 0.
\]

The expression on the left is decreasing in \( e \) for \( e \leq \frac{n+1}{n-d+1} \), so it suffices to prove the result for \( e = \frac{n+1}{n-d+1} \). Dividing by \( e \) gives

\[
\frac{n(n-1)}{n+1} (n - d + 1) - 4n + (e+1)(n - d + 1) + \frac{2d}{e} - 2 + \frac{2}{e} > 0
\]

which can be rearranged to obtain

\[
\frac{n^2 + 1}{n+1} (n - d + 1) - 4n + n + 1 + \frac{2d}{e} - 2 + \frac{2}{e} > 0
\]

or

\[
\frac{n^2 + 1}{n+1} (n - d + 1) + \frac{2d}{e} + \frac{2}{e} > 3n+1.
\]

Multiplying both sides by \( n+1 \), we get

\[
(n^2 + 1)(n - d + 1) + 2d(n - d + 1) + 2(n - d + 1) > 3n^2 + 4n + 1,
\]

23
or

\[(n^2 + 2d + 3)(n - d + 1) > 3n^2 + 4n + 1.\]

The left-hand side is quadratic in \(d\) with negative coefficient of \(d^2\), so we need only check the endpoints to minimize it. If \(d = n - 2\), the left-hand side becomes

\[3(n^2 + 2n - 1) = 3n^2 + 6n - 3.\]

This will be greater than \(3n^2 + 4n + 1\) precisely when \(2n > 4\), or \(n > 2\).

If \(d = \frac{n+1}{2}\), the left-hand side is

\[(n^2 + n + 4)\frac{n+1}{2}.

For \(n \geq 5\), we have

\[(n^2 + n + 4)\frac{n+1}{2} = (n^2 + n)\frac{n+1}{2} + 2(n+1) \geq 3n^2 + 3n + 2n + 2 > 3n^2 + 4n + 1.

This concludes the proof. \(\square\)

7. Rational Curves on General Type Hypersurfaces

The goal of this section is to prove the following theorem.

**Theorem 7.1.** Let \(X \subset \mathbb{P}^n\) be a general hypersurface of degree \(d\), with \(\frac{3n+1}{2} \leq d \leq 2n - 3\). Then \(X\) contains lines but no other rational curves.

The proof has three main ingredients. The first is a result of Clemens and Ran.

**Theorem 7.2.** (\cite{9}) Let \(X\) be a very general hypersurface of degree \(d\) in \(\mathbb{P}^n\) and let \(k\) be an integer. Let \(d \geq n\), \(\frac{d(d+1)}{2} \geq 3n - 1 - k\) and let \(f : Y \to Y_0\) be a desingularization of an irreducible subvariety of dimension \(k\). Let \(t = \max(0, -d + n + 1 + \lfloor \frac{n-k}{2} \rfloor )\). Then either \(h^0(\omega_Y(t)) \neq 0\) or \(Y_0\) is contained in the subvariety of \(X\) swept out by lines.
If we take the special case where $Y$ is a rational curve, then we can see that for $t = 0$, $h^0(\omega_Y(t)) = 0$. Thus, $Y_0$ must be contained in the subvariety swept out by lines.

**Corollary 7.3.** Let $X \subset \mathbb{P}^n$ be a general degree $d$ hypersurface. Then if $\frac{d(d+1)}{2} \geq 3n - 2$ and $d \geq \frac{3n+1}{2}$, then any rational curve in $X$ is contained in the union of the lines on $X$.

Note that if $n \geq 3$, then $d \geq 5$ and $\frac{d(d+1)}{2} \geq 3n - 2$ automatically. The second ingredient is a new result about the Fano scheme of lines on $X$.

**Theorem 7.4.** If $X \subset \mathbb{P}^n$ is general and $n \leq \frac{d(d+2)}{6} - 1$, then $F_1(X)$ contains no rational curves.

Note that if $d \geq \frac{3n+1}{2}$, then $\frac{d(d+2)}{6} - 1 \geq \frac{9n^2+18n+5}{24} - 1 = \frac{9n^2+18n-19}{24} \geq n$ for $n \geq 3$. The third ingredient is a result about reducible conics.

**Theorem 7.5.** If $X \subset \mathbb{P}^n$ is general and $d \geq \frac{3n}{2}$, then $X$ contains no reducible conics.

Using these three results, the Main Theorem follows easily.

**Proof.** (Theorem 7.1) Let $Y \subset X$ be the subvariety of $X$ swept out by lines. Let $U \rightarrow F_1(X)$ be the universal line on $X$. Then since there are no reducible conics in $X$, the natural map $U \rightarrow X$ is injective, and hence, an isomorphism. By Theorem 7.3, any rational curve in $X$ has to lie in $Y$. However, this contradicts Theorem 7.4.

Thus it remains to prove Theorem 7.4 and Theorem 7.5. Theorem 7.5 is substantially easier, and is probably well-known. However, we offer a different proof from a typical proof involving Hilbert functions, one that will set up some of the ideas for the proof of Theorem 7.4. The idea of this proof is to realize $X$ as a hyperplane section of
a higher-dimensional hypersurface, and show that as we vary the hyperplane section, a general one will contain no reducible conics.

**Proof.** (Theorem 7.5) First note that a general hypersurface $X$ in $\mathbb{P}^n$ will be a hyperplane section of a general hypersurface $Y$ in $\mathbb{P}^{d+1}$ (for $n < d$). By Proposition 5.10 it follows that there are at most finitely many lines through any point $p \in Y$. Consider

$$C_{n,d} = \{(p, X) | X \subset \mathbb{P}^n, p \in X \text{ with a reducible conic in } X \text{ with node at } p\}.$$ 

We aim to show that $C_{n,d}$ is codimension at least $n$ in $\{(p, X) | p \in X \subset \mathbb{P}^n\}$ at $(X, p)$, which will suffice to prove the result. Choose a general hypersurface $Y \subset \mathbb{P}^{d+1}$ such that $X$ is a hyperplane section of $Y$. Let $Z$ be the set of parameterized $n$-planes in $\mathbb{P}^{d+1}$ passing through $p$. Let $Z' \subset Z$ be the closure of the subset of $n$-planes that contain a reducible conic with node at $p$. It suffices to show that $Z'$ has codimension at least $n$ in $Z$.

The dimension of $Z$ will simply be $6 + n(d - n)$. Since $p$ has a 0-dimensional family of lines through it, it follows that there will be a 0-dimensional family of 2-planes through $p$ that contain a reducible conic with node at $p$. Thus, the dimension of $Z'$ will simply be the dimension of the space of parameterized $n$-planes through $p$ containing a 2-plane, or $6 + (n - 2)(d + 1 - n - 1) = 6 + (n - 2)(d - n)$. This shows that the codimension of $Z'$ in $Z$ is

$$6 + n(d - n) - (6 + (n - 2)(d - n)) = 2(d - n).$$

This will be at least $n$ precisely when

$$2(d - n) \geq n$$

or

$$d \geq \frac{3n}{2}.$$
The proof of Theorem 7.4 relies on the following result about Grassmannians, which is of independent interest.

**Proposition 7.6.** Let $m \leq n$. Let $B \subset \mathbb{G}(m,n)$ be codimension at least $\epsilon \geq 1$ (i.e., every component of $B$ has codimension at least $\epsilon$). Let $C \subset \mathbb{G}(m-1,n)$ be a nonempty subvariety satisfying the following condition: $\forall c \in C$, if $b \in \mathbb{G}(m,n)$ has $c \subset b$, then $b \in B$. Then it follows that the codimension of $C$ in $\mathbb{G}(m-1,n)$ is at least $\epsilon + 1$.

The proof makes use of the following elementary Lemma.

**Lemma 7.7.** Suppose $B \subset \mathbb{G}(m,n)$ and $C \subset \mathbb{G}(m-1,n)$ are nonempty subvarieties satisfying the following two conditions:

1. $\forall c \in C$, if $b \in \mathbb{G}(m,n)$ has $c \subset b$, then $b \in B$.
2. $\forall b \in B$, if $c \in \mathbb{G}(m-1,n)$ has $c \subset b$, then $c \in C$.

Then $B = \mathbb{G}(m,n)$ and $C = \mathbb{G}(m-1,n)$.

**Proof.** (Lemma 7.7) Let $\Lambda \in B$, and let $\Phi \in \mathbb{G}(n,m)$. We will show $\Phi \in B$. Let $k = m - \dim(\Lambda \cap \Phi)$. Let $\Lambda = \Lambda_0, \Lambda_1, \ldots, \Lambda_k = \Phi$ be a sequence of $m$-planes such that $\dim(\Lambda_i \cap \Lambda_{i+1}) = m - 1$. Then we see by Condition 2, if $\Lambda_i \in B$, then $\Lambda_i \cap \Lambda_{i+1} \in C$ and, hence, by Condition 1, $\Lambda_{i+1} \in B$. Since $\Lambda_0 = \Lambda$ was in $B$ by assumption, we see that $\Phi \in B$. This shows $B = \mathbb{G}(m,n)$, and it follows that $C = \mathbb{G}(m-1,n)$. □

**Proof.** (Proposition) We show that $\dim C \leq m(n - m + 1) - (\epsilon + 1)$. Consider the incidence correspondence $\mathcal{I} = \{(b,c) | c \in C, b \in \mathbb{G}(n,m), c \subset b\}$. Note that if $(b,c) \in \mathcal{I}$, then necessarily $b \in B$. Let $\pi_B : \mathcal{I} \rightarrow B$, $\pi_C : \mathcal{I} \rightarrow C$ be the projection maps. Then by the Lemma, the fibers of $\pi_B$ have dimension at most $m - 1$, since if they had dimension $m$, then both Conditions 1 and 2 would be satisfied, contradicting $\epsilon \geq 1$. Thus, $\dim \mathcal{I} \leq \dim B + m - 1 \leq (m+1)(n-m) + m - 1 - \epsilon$. By the Condition on
\( C \), we see that the fibers of \( \pi_C \) have dimension at least \( n - m \), so \( \dim I \geq \dim C + n - m \).

Putting the two inequalities together gives

\[
\dim C + n - m \leq \dim I \leq (m + 1)(n - m) + m - 1 - \epsilon
\]

or

\[
\dim C \leq (m + 1)(n - m) - (n - m) - \epsilon = m(n - m) + m - 1 - \epsilon = m(n - m + 1) - (\epsilon + 1).
\]

The result follows. \( \square \)

Using Proposition \ref{7.6}, we can prove Theorem \ref{7.4}

**Proof.** (Theorem \ref{7.4}) First we find the \( n \) and \( d \) for which \( F_1(X) \) is general type. Note that since \( X \) is general, \( F_1(X) \) is the expected-dimensional vanishing of a section of the vector bundle \( \text{Sym}^d(S^*) \). Recall that \( c(S^*) = 1 + \sigma_1 + \sigma_{1,1} \). We use the splitting principle to work out \( c_1(\text{Sym}^d(S^*)) \). Suppose \( c(S^*) = (1 + \alpha)(1 + \beta) \). Then

\[
c(\text{Sym}^d(S^*)) = \sum_{k=0}^d (1 + \alpha)^k (1 + \beta)^{d-k}.
\]

Counting the coefficients of \( \alpha \) in the product, we see that

\[
c_1(\text{Sym}^d(S^*)) = \sum_{k=0}^d k\sigma_1 = \frac{d(d+1)}{2}\sigma_1.
\]

The canonical bundle of \( \mathbb{G}(1, n) \) is \( -(n+1)\sigma_1 \). Thus, the canonical bundle of \( F_1(X) \) is \( -n - 1 + \frac{d(d+1)}{2} \). This will be ample if

\[
-n - 1 + \frac{d(d+1)}{2} \geq 1
\]

or

\[
n \leq \frac{d(d+1)}{2} - 2.
\]

Let \( m = \frac{d(d+1)}{2} - 2 \). Consider

\[
R_{n,d} = \{(\ell, X) | \ell \subset X \subset \mathbb{P}^n, \exists \text{ rational curve in } F_1(X) \text{ containing } [\ell]\}.
\]
We claim that $R_{k,d} \subset \{(\ell, X) | \ell \subset X \}$ is codimension at least $2k - d - 2$ for some $k$ that we will determine, which will prove that a general hypersurface $X \subset \mathbb{P}^k$ has no rational curves in its Fano scheme.

Let $(\ell, X) \in R_{k,d}$. We find a family $F$ with $(\ell, X) \in F \subset \{(\ell, X) | \ell \subset X \}$ such that $F \cap R_{k,d} \subset F$ is codimension at least $2k - d - 2$. Let $Y' \subset \mathbb{P}^m$ be a general hypersurface, and $\ell' \subset Y'$ a general line in $Y$. Note that there are no rational curves in $F_1(Y')$ through $\ell'$ since $F_1(Y')$ is smooth and general type. Let $Y$ be a hypersurface containing a line $\ell$ such that $(\ell, X)$ is a $k$-plane section of $Y$ and $(\ell', Y')$ is an $m$-plane section of $Y$. Let $Z_r$ be the set of parameterized $r$-planes containing $\ell$, and let $Z'_r \subset Z_r$ be the set of parameterized $r$-planes containing $\ell$ such that the corresponding linear section of $Y$ has no rational curve through $\ell$ in its Fano scheme. We see that $Z'_m$ is codimension at least one in $Z_m$. By Proposition 7.6, we see that $Z'_k$ is codimension at least $m - k + 1$ in $Z_k$. Thus, our result will hold for

$$m - k + 1 \geq 2k - d - 2$$

or

$$3k \leq m + d + 3$$

or equivalently

$$k \leq \frac{m + d}{3} + 1 = \frac{d(d + 2)}{6} - 1.$$

The techniques here can be used to prove that a general hypersurface in sufficiently high degree contains no copies of a class of subvarieties, provided that there exists a degree such that a general hypersurface is not swept out by these subvarieties. When combined with a result of Voisin [37], this can be a useful way to prove that given any class of varieties and fixed dimension of hypersurface, there exists a degree large enough so that a very general hypersurface contains no varieties in that class.
**Theorem 7.8.** Suppose that a very general hypersurface of degree $d$ in $\mathbb{P}^{2m}$ is not rationally swept out by varieties in the family $Y \to B$. Then a very general hypersurface of degree $d$ in $\mathbb{P}^m$ contains no varieties from the family $Y \to B$.

**Proof.** Consider the components of the incidence correspondence

$$\Gamma_{n,\beta} = \{(p,X) | p \in X \subset \mathbb{P}^n, p \text{ lies on a subvariety swept out by varieties of } Y\}.$$

We see that this incidence correspondence will have countably many irreducible components. Let $\Gamma$ be one such component. We show that $\Gamma_m$ will have codimension at least $m$ in the incidence correspondence

$$I_m = \{(p,X) | p \in X \subset \mathbb{P}^m\}.$$

We use the same technique as in the proof of Theorem 7.4. Let $(p, X_0) \in \Gamma_m$ be general, and let $(p, X_1) \in \Gamma_{2m}$ be a pair such that $X_0$ is an $m$-plane section of $X_1$ by a plane through $p$. Let $S \subset I_m$ be the space of pairs $(p, X)$ that are $m$-plane sections of $X_1$ by a plane through $p$. Then by Proposition 7.6 we see that $S \cap \Gamma_m \subset S$ is codimension at least $2m - m = m$. This concludes the proof. \(\square\)

Using a theorem of Voisin [37], we obtain an immediate corollary.

**Theorem 7.9.** (Voisin [37]) Let $n$ and $S \to B$ be given. Then there exists $d$ sufficiently large so that a very general degree $d$ hypersurface in $\mathbb{P}^n$ is not rationally swept by varieties in the family $S \to B$.

**Corollary 7.10.** Let $n$ and $S \to B$ be given. Then there exists a $d$ sufficiently large so that a general degree $d$ hypersurface in $\mathbb{P}^n$ admits no finite maps from varieties in the fibers of $S \to B$.  

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8. Examples in \( \mathbb{P}^3 \)

In order to give a sense of the complexity of questions about rational curves on hypersurfaces in \( \mathbb{P}^n \), we consider some small examples, and show that there is a wide range of possible behaviors even in these relatively simple cases. We consider the possible range of Hilbert functions of rational curves in \( \mathbb{P}^3 \). We will completely classify the possible Hilbert functions of rational curves of degree up to 7.

A first basic result that will inform our investigation is one of Gruson-Lazarsfeld-Peskine \([19]\).

**Theorem 8.1.** Let \( C \subset \mathbb{P}^n \) be a non-degenerate integral curve of degree \( e \), and let \( d \geq e + 1 - n \). Then the map \( H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_C(d)) \) is surjective. Moreover, if \( d = e - n \) and \( H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_C(d)) \) is not surjective, then \( C \) is smooth and rational and either \( e = n + 1 \) or \( e > n + 1 \) and \( C \) has a \( e + 2 - n \) secant line.

In our special case of smooth rational curves in \( \mathbb{P}^3 \), we can restate their result, as well as use their classification of the boundary cases.

**Corollary 8.2.** If \( C \subset \mathbb{P}^3 \) is a smooth rational curve of degree \( e \), then the Hilbert function is equal to the Hilbert polynomial \( p_C(d) \) for \( d \geq e - 2 \). That is, \( h_C(d) = ed + 1 \) for \( d \geq e - 2 \). If \( h_C(e - 3) \neq p_C(e - 3) \), then either \( e = 4 \) or \( e \geq 5 \) and \( C \) has a \( e - 1 \) secant line. If \( C \) does not lie on a smooth quadric, then \( p_C(e - 3) - h_C(e - 3) \leq 1 \).

Another basic tool that we will use is a partial result from \([24]\) on the maximum rank conjecture. We state it only in the context of rational curves in \( \mathbb{P}^3 \).

**Theorem 8.3** (\([24]\)). For \( C \subset \mathbb{P}^3 \) a general rational curve of degree \( e \) and for \( ed + 1 \geq \binom{d+3}{3} \), the map \( H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\mathcal{O}_C(d)) \) is injective.

We now commence the classification of rational curves in \( \mathbb{P}^3 \) by Hilbert function. Recall that for \( e = 1 \) we are simply talking about lines in \( \mathbb{P}^3 \). All lines are projectively...
equivalent, and the Hilbert function $h_C(d)$ is equal to the Hilbert polynomial $p_C(d) = d + 1$ for all $d$.

For $e = 2$, we are talking about smooth conics in $\mathbb{P}^3$. All conics are planar, and all plane conics are projectively equivalent in $\mathbb{P}^3$. Just as was the case for lines, the Hilbert function $h_C(d)$ is equal to the Hilbert polynomial $p_C(d) = 2d + 1$ for all $d \geq 1$.

For $e = 3$, we are talking about smooth cubics in $\mathbb{P}^3$, that is rational normal cubics. All of them are projectively equivalent to each other, and we have $h_C(d) = p_C(d) = 3d + 1$ for all $d \geq 1$.

For $e = 4$, things are a bit different, although there is still only one possible Hilbert function by Corollary 8.2. The Hilbert function is

<table>
<thead>
<tr>
<th>$d$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_C(d)$</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>$4d + 1$</td>
</tr>
<tr>
<td>$h^0(\mathcal{O}_{\mathbb{P}^3}(d))$</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>20</td>
<td>35</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

For $e = 5$, we have a wider range of possible behaviors. Corollary 8.2 completely determines the Hilbert function for all $d \geq 3$. Since we are only considering smooth curves, we know that our curve will be non-degenerate, which means that $h_C(1) = 4$. However, there are two possibilities for $h_C(2)$: either $C$ lies on a quadric or $C$ does not. It follows from Theorem 8.3 that a general degree 5 rational curve in $\mathbb{P}^3$ will lie on no quadrics. However, we see that there are degree 5 rational curves that lie on smooth quadrics.

**Proposition 8.4.** A smooth degree $e$ curve lies on at most one quadric for $e \geq 5$.

*If $Q$ is a smooth quadric, then the curves of class $(e - 1, 1)$ or $(1, e - 1)$ on $Q$ are precisely the degree $e$ rational curves. The space of rational curves lying on a smooth quadric will have dimension $2e + 8$.  

*Proof. If $C$ were on more than one quadric, it would lie in a complete intersection of degree 4, which contradicts $C$ having degree at least 5.*
We see that $Q$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and it is a well-known fact that a curve of class $(a, b)$ will have genus $(a-1)(b-1)$ and degree $a+b$. There are many ways to see that the space of rational curves on a quadric surface has the expected dimension, but the simplest is probably to compute its normal bundle and show that it has degree $2e - 2$. To see this, recall the short exact sequence

$$0 \to N_{C/Q} \to N_{C/P^3} \to N_{Q/P^3}|_C \to 0.$$ 

Since $Q$ is a quadric, $N_{Q/P^3}|_C = \mathcal{O}_{\mathbb{P}^1}(2e)$, and the degree of $N_{C/P^n}$ is $4e - 2$, it follows that $N_{C/Q}$ has degree $2e - 2$. Since there is a 9-dimensional space of quadrics in $\mathbb{P}^3$, the result follows.

Since the curve has class $(e-1, 1)$ or $(1, e-1)$ on the quadric, we see that it will lie on no other hypersurfaces of degree $d < e - 1$ except those given by multiples of the quadric, so we completely understand the Hilbert function of rational curves lying on a smooth quadric. However, there are singular quadrics in $\mathbb{P}^3$. Any singular quadric in $\mathbb{P}^3$ will either be a double plane, a union of two planes, or a quadric cone. We know that the rational curve will not lie in any planes, but we still need to consider the case of a quadric cone.

**Proposition 8.5.** There are no smooth rational curves of degree $e > 3$ in a quadric cone.

**Proof.** By blowing up the cone point, we see that a singular quadric $Q$ is birational to the Hirzebruch surface $\mathbb{F}_2$. We write $\pi : \mathbb{F}_2 \to Q$ for the blowup map. Recall that the Picard group of $\mathbb{F}^2$ is $\mathbb{Z}^2$, generated by $E$, the class of the exceptional divisor of the blowup, and $f$, the class of a line in the quadric. The intersection products are $E^2 = -2$, $E \cdot f = 1$ and $f^2 = 0$. The canonical divisor $K_{\mathbb{F}_2}$ is $-4f - 2E$. The pullback of the hyperplane class from $\mathbb{P}^3$ is simply $H = 2f + E$. We will work out the possible
Picard classes of the strict transforms of a smooth, degree $e$ rational curve and show that they must correspond to singular curves in the quadric.

Let $C = af + bE$ be the class of the curve. Then we wish for the curve to intersect $H$ with multiplicity $e$, so $e = C \cdot H = (af + bE) \cdot (2f + E) = 2b + a - 2b = a$. Thus, $a = e$.

We know that $C$ is rational, so by adjunction, we see that

$$-2 = (K_F + C) \cdot C = (-2E - 4f + ef + bE) \cdot (ef + bE) = ((e - 4)f + (b - 2)E) \cdot (ef + bE)$$

$$= b(e - 4) + e(b - 2) - 2b(b - 2) = -2b^2 + 2eb - 2e.$$

Rearranging, we obtain

$$b^2 - eb + e - 1 = 0$$

or

$$(b - (e - 1))(b - 1) = 0.$$

Thus, there are two possible classes for $C$: $ef + E$, and $ef + (e - 1)E$. We intersect both of these possibilities with $E$. Intersection $ef + E$ with $E$, we see that $(ef + E) \cdot E = e - 2$, which means that $\pi(C)$ will have high multiplicity at the cone point of the quadric for $e > 3$, contradicting $\pi(C)$ smooth. Intersecting $ef + (e - 1)E$ with $E$, we get $e - 2(e - 1) = 2 - e < 0$ for $e > 3$. This is impossible for $C$ the strict transform of a smooth curve on $Q$. 

Proposition 8.4 and Proposition 8.5 completely characterize the possible Hilbert functions of smooth, degree 5 rational curves in $\mathbb{P}^3$: either $C$ will have the Hilbert function of a general rational curve, or it will lie on a smooth quadric. The possible Hilbert functions are given below.
For $e = 6$, we have more possibilities. We first consider the Hilbert function of a general curve $C$. Corollary 8.2 tells us what the Hilbert function will be for $d \geq 4$. A simple dimension count shows that there will be curves that do not lie on a quadric hypersurface. However, it is not immediately clear what $h_C(3)$ will be, since we do not know whether a general curve will have a 5-secant line or not. This is cleared up by the following result.

**Proposition 8.6.** For $e \geq 5$, the space of degree $e$ rational curves in $\mathbb{P}^3$ with an $e - 1$ secant line has dimension $3e + 5$.

*Proof.* Any degree $e$ rational curve in $\mathbb{P}^3$ is a projection of a rational normal curve from $\mathbb{P}^e$. We count the codimension of such curves that have an $e - 1$-secant line. Fix an embedding of a rational normal curve in $C_0 \subset \mathbb{P}^e$. To project it into $\mathbb{P}^3$, we need to project it from an $(e - 4)$-plane. The $e - 1$ secant line will correspond to an $e - 2$ plane in $\mathbb{P}^e$ that meets $C_0$ $e - 1$ times. Thus, in order to get a curve with an $e - 1$ secant line, we need to project from an $(e - 4)$-plane contained in an $(e - 2)$-plane meeting $C_0$ at least $e - 1$ times. There are $e - 1$ dimensions of $(e - 2)$-planes meeting $C_0$ $e - 1$ times (simply pick the $e - 1$ points of intersection with $C_0$) and $(e - 3)(e - 2 - (e - 4)) = 2e - 6$ dimensions of $(e - 4)$-plane contained in a fixed $(e - 2)$-plane, for a total of $3e - 7$. Since $\dim \mathcal{G}(e - 4, e) = (e - 3)(4) = 4e - 12$, this means that for a given rational normal curve $C_0$, the space of projections giving rise to a curve with an $e - 1$ secant line has codimension $4e - 12 - (3e - 7) = e - 5$. Since every degree $e$ rational curve in $\mathbb{P}^e$ is such a projection, it follows that the codimension of such curves with an $e - 1$ secant line will be $e - 5$. Thus, the dimension of the space of such curves will be $4e - (e - 5) = 3e + 5$. \qed
Note that this result implies that every degree 5 rational curve will have a 4-secant line. In the case $e = 6$, we see that the space of rational curves with a 5-secant line has dimension 23, which is smaller than the 24-dimensional family of rational sextics. Thus, a general rational sextic will only lie on one cubic surface. This completely determines the Hilbert function in this case.

Next we consider the Hilbert functions of curves lying on a smooth quadric. As we see from Proposition 8.4, there will exist such rational sextics. They cannot also lie on an independent cubic, since that would mean that they were a complete intersection of a quadric and a cubic, which would have higher genus. Thus, they lie only on the four dimensional vector space of cubics consisting of multiples of the equation of the quadric. This completely determines the Hilbert function of rational sextics lying on a quadric.

Since the dimension of rational sextics with a 5-secant line is larger than the 20 dimensional family of curves lying on a smooth quadric, we see that there will be smooth rational sextics with a 5-secant line that do not lie on a quadric, which means that they will lie on exactly two cubics.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$d$ & 0 & 1 & 2 & 3 & 4 & $d$ \\
\hline
$h_C(d)$ for $C$ general & 1 & 4 & 10 & 19 & 25 & $6d + 1$ \\
\hline
$h_C(d)$ for $C$ on a quadric & 1 & 4 & 9 & 16 & 25 & $6d + 1$ \\
\hline
$h_C(d)$ for $C$ on two cubics & 1 & 4 & 10 & 18 & 25 & $6d + 1$ \\
\hline
$h^0(O_{\mathbb{P}^3}(d))$ & 1 & 4 & 10 & 20 & 35 & $\cdots$ \\
\hline
\end{tabular}
\end{table}

For the case $e = 7$, we have still more possibilities. Septic rational curves in $\mathbb{P}^3$ will lie on the expected number of quintics. However, they could lie on more than expected quadrics, cubics or quartics. If a septic rational curve lies on a smooth quadric, then we have completely determined its Hilbert function, since a $(1, 6)$ curve on a quadric will not lie on any quadrics, cubics or quartics that are not multiples of the quadric.
Now we wish to investigate whether a septic rational curve can lie on a cubic hypersurface. There are four different types of irreducible cubic surfaces in \( \mathbb{P}^3 \) (see [6] for more details): cubics with at worst rational double points, cubics that are a cone over a smooth cubic plane curve, cubics that are the cone of a singular cubic plane curve, and cubics that are projections of a cubic scroll in \( \mathbb{P}^4 \).

**Lemma 8.7.** A smooth rational curve of degree \( e \geq 7 \) in \( \mathbb{P}^3 \) lies on at most one cubic surface.

**Proof.** We can see that curves of that class will lie on a unique irreducible cubic by liaison theory. If it were on two independent cubics, we can see that it would be linked to a curve of degree \( 3 \cdot 3 - e = 9 - e \), and genus \( g = \frac{3+3-4}{2}(9 - e - e) = 9 - 2e \leq -5 \). However, there are no such curves on a cubic surface.

The only possible ambiguity is in the case \( e = 7 \), and in this case, the only possible linked curve would be a type of degree 2 scheme supported on a line called a ribbon. However, any such curve in a cubic surface will have genus at most \(-2\). See [30] for more details.

\( \square \)

However, we need more specific information about the geometry of these surfaces. In particular, we need to understand whether smooth septics on cubic hypersurfaces have 6-secant lines. In order to do this we need to understand the class in the Picard group of the surface. Any cubic surface with isolated singularities is either a cone or contains at worst rational double points. We consider the latter case first.

**Theorem 8.8.** Consider the set \( U \subset \mathbb{P}^{19} \) of cubic hypersurfaces with at worst rational double points. The incidence correspondence

\[ \{(C, X) | X \in U, C \subset X \text{ smooth septic rational curve}\} \]
is an irreducible variety of dimension 25. Any septic in the image of this incidence correspondence has a 6-secant line. The space of smooth septic rational curves on a general cubic surface in \( \mathbb{P}^3 \) has 432 components.

**Proof.** Recall that any cubic with at worst rational double points will admit a map from the successive blowup of \( \mathbb{P}^2 \) at six points. The Picard group of such a surface will be the free \( \mathbb{Z} \) module on \( \ell, E_1, \cdots, E_6 \), with \( \ell^2 = 1, E_i^2 = -1, E_i \cdot \ell = 0 \) and \( E_i \cdot E_j = 0 \) for \( i \neq j \). Note that if \( S \) is a smooth cubic surface, then \( \ell \) will be the class of a line on \( \mathbb{P}^2 \), and \( E_1, \cdots, E_6 \) will be the six exceptional divisors of the blowup of \( \mathbb{P}^2 \) at distinct points. We have \( H = 3\ell - \sum_{i=1}^{6} E_i \) is the restriction of the hyperplane class from \( \mathbb{P}^3 \).

We start by computing the dimension of the space of any component of degree 7 rational curves on \( X \). Let \( C \) be a class of a degree 7 rational curve on \( S \). By Riemann-Roch, we see that

\[
\chi(\mathcal{O}_S(C)) = \frac{C^2 - K_S \cdot C}{2} + \chi(\mathcal{O}_S),
\]

and since \( S \) is rational, we see that \( \chi(\mathcal{O}_S) = 1 \). This means that \( \chi(\mathcal{O}_S(C)) \) is completely determined by the intersection theory data of its class in the Picard group.

We claim furthermore that \( h^1(\mathcal{O}_S(C)) = h^2(\mathcal{O}_S(C)) = 0 \). For \( h^2(\mathcal{O}_S(C)) = 0 \), note that by Serre duality, \( h^2(\mathcal{O}_S(C)) = h^0(\mathcal{O}_S(K_S - C)) = 0 \), since \( K_S \) is anti-ample and \( C \) is effective. For \( h^1(\mathcal{O}_S(C)) \), consider the short exact sequence

\[
0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.
\]

Since \( S \) is rational, we know that \( h^1(\mathcal{O}_S) = 0 \), and so it remains to show \( h^1(\mathcal{O}_C(C)) = 0 \). Since \( C \) is rational, we see by adjunction that \( C^2 + K_S \cdot C = -2 \), which gives that \( C^2 = -2 - K_S \cdot C \). Since \( K_S \) is ample, we see that \( C^2 \geq -1 \), which shows that \( \mathcal{O}_C(C) \) is a line bundle of degree at least \(-1\) on \( \mathbb{P}^1 \), which shows that \( h^1(\mathcal{O}_C(C)) = 0 \). Thus,
the dimension of the linear series of $C$ is completely determined by intersection-theoretic data, which means that it is independent of which cubic surface with at worst rational double points we select. Thus, we may assume that $S$ is smooth.

Let $C$ have class $a\ell + \sum b_i E_i$. For the curve $C$ to be degree $e$, we need

$$e = 3a - \sum_{i=1}^{6} b_i.$$ 

For the curve to be rational, we need

$$\left( a - 1 \right) = \sum_{i=1}^{6} \left( \frac{b_i}{2} \right).$$

Now, there will be many different classes $a$, and $b_1, \ldots b_6$ that satisfy these two conditions. However, as we move around in the space of smooth cubic hypersurfaces, we obtain a monodromy action on the Picard group of the cubic surfaces. We claim that up to this monodromy action, there is only one class of septic rational curve on a cubic hypersurface. Recall that this monodromy action allows for permuting the exceptional divisors, and also for applying Cremona transforms (see [20] for more details on the monodromy action on the space of lines of a smooth cubic surface). First we permute the $b_i$ so that $b_1 \geq b_2 \geq \cdots \geq b_6$. Now, for the curve to be irreducible, we need it to have nonnegative intersection with any line in the cubic. In particular,

$$0 \leq C \cdot (\ell - E_1 - E_2) = a - b_1 - b_2$$

or

$$a \geq b_1 + b_2.$$ 

Additionally,

$$0 \leq C \cdot (2\ell - E_1 - E_2 - E_3 - E_4 - E_5) = 2a - b_1 - b_2 - b_3 - b_4 - b_5.$$
or
\[ 2a \geq b_1 + b_2 + b_3 + b_4 + b_5. \]

Working with these equations, we can obtain bounds for how large \( a \) can be. Using
\[ e = 3a - \sum_{i=1}^{6} b_i \text{ and } 2a \geq b_1 + b_2 + b_3 + b_4 + b_5 \]
we have
\[ e = a + (2a - \sum_{i=1}^{6} b_i) \geq a - b_6 \]
which gives
\[ b_6 \geq a - e. \]
Since \( b_i \geq b_6 \) for all \( i \), we see from
\[ 2a \geq b_1 + b_2 + b_3 + b_4 + b_5 \]
that
\[ 2a \geq 5b_6 \geq 5(a - e). \]
Rearranging, we get
\[ 3a \leq 5e, \]
or
\[ a \leq \frac{5e}{3}. \]
For each \( e \), this gives a finite list of \( a, b_1, \cdots, b_6 \) to check. We do a computer search for \( e = 7 \) and obtain the following possible values:
Using the Cremona transforms of the plane, we can reduce the list further. We can reduce \( a \) using a Cremona transform if \( a < b_1 + b_2 + b_3 \). Looking at the chart, we see that the first entry is the only one with \( a \geq b_1 + b_2 + b_3 \). Thus, the space

\[
\{(C,X)| C \subset X \text{ is a smooth septic rational curve in cubic } X\}
\]

is irreducible, and we can learn about the dimension and Hilbert function by studying rational septics of class \( 3\ell - 2E_1 \). From this it follows that the space of septic curves in \( \mathbb{P}^3 \) that lie on a smooth cubic surface is dimension 6, so the incidence-correspondence will be irreducible of dimension 25. We see that these curves will all have 6-secant lines, because they intersect the line \( 2\ell - E_2 - E_3 - E_4 - E_5 - E_6 \) six times.

From the list of possible classes of \( C \), we can compute the number of components of the space of smooth septic rational curves on \( S \). Note that every such curve will be one of the above types, up to permuting \( E_1, \ldots, E_6 \), and we can work out exactly how many ways there are to do this. There are

\[
\binom{6}{1,5} + \binom{6}{1,2,3} + \binom{6}{1,4,1} + \binom{6}{3,1,2} + \binom{6}{1,3,1,1} + \binom{6}{3,1,2} + \binom{6}{1,4,1} + \binom{6}{1,2,3}
\]
\[ + \binom{6}{1,5} = 432. \]

This classifies the Hilbert functions of rational curves that lie on cubic surfaces with at worst rational double points. Next, we show that irreducible cubic cones contain no smooth degree 7 rational curves.

**Proposition 8.9.** An irreducible cubic cone in \( \mathbb{P}^3 \) contains no smooth degree 7 rational curves.

**Proof.** First we consider cones over smooth cubic plane curves. Any curve lying on that cone will either be contained in a finite union of lines through the cone point or will admit a map to the genus 1 plane curve. Thus, there are no smooth septic rational curves on such surfaces.

Next we consider cones over singular cubic plane curves, i.e., either cones over nodal or cuspidal plane cubics. In either case, the surface admits a birational map from the Hirzebruch surface \( \mathbb{F}_3 \), which will have Picard group \( \mathbb{Z}f \oplus \mathbb{Z}E \), with \( E^2 = -3 \), \( E \cdot f = 1 \), and \( f^2 = 0 \). The class \( E \) maps to the cone point, and the curves of class \( f \) correspond to the lines through the cone point. The canonical bundle of \( \mathbb{F}_3 \) is \( K = -2E - 5f \), and the hyperplane class \( h \) will be determined by \( h \cdot f = 1 \), and \( h^2 = 3 \), which gives \( h = 3f + E \).

We work out the class of a curve \( C \) on \( \mathbb{F}_3 \) which maps to a septic rational curve on the cone. Let \( C = af + bE \). Then \( C \cdot h = 7 \) gives

\[ C \cdot h = 3b + a - 3b = a = 7. \]

By adjunction, we see that any rational curve \( C \) will have \( -2 = K \cdot C + C^2 \) which gives

\[ K \cdot C + C^2 = -14 - 5b + 6b + 14b - 3b^2 = -3b^2 + 15b - 14 = -2, \]
or
\[(b - 1)(b - 4) = 0.\]

Thus, the class of \(C\) is either \(7f + E\) or \(7f + 4E\).

If the class of \(C\) is \(7f + 4E\), then \(E \cdot C = 7 - 12 = -5 < 0\), which means that \(C\) is reducible, which contradicts \(C\) being smooth and integral. If the class of \(C\) is \(7f + E\), then \(C \cdot E = 4\), which means that the image of \(C\) in \(\mathbb{P}^3\) is multiple at the cone point, which contradicts \(C\) having smooth image. \(\square\)

We now turn to the Hilbert functions of rational curves lying on projections of cubic scrolls.

**Proposition 8.10.** There are irreducible degree 7 rational curves lying on a cubic singular along a line (which is birational to a cubic scroll). Such curves will have a 6-secant line. They form a family of dimension 21.

**Proof.** Let \(S\) be the scroll, and \(g : S \to \mathbb{P}^3\) be the map to \(\mathbb{P}^3\). Recall that cubic scrolls have Picard group \(\mathbb{Z}f \oplus \mathbb{Z}E\), with \(E^2 = -1\), \(f^2 = 0\), and \(f \cdot E = 1\). They naturally live in \(\mathbb{P}^4\) and the map to \(\mathbb{P}^3\) is given by projecting from a point. They will be singular along a double line which is the image of a curve of class \(f + E\), which will be a conic in \(\mathbb{P}^4\). The canonical bundle is \(K_S = -3f - 2E\), and the pull-back of the hyperplane class on \(\mathbb{P}^3\) is \(2f + E\). We wish to investigate the classes of degree 7 rational curves on \(S\). Suppose \(C = aE + bf\) is the class of a rational degree 7 curve on \(S\). Then since \(C\) has degree 7, we have
\[7 = C \cdot H = (aE + bf)(E + 2f) = -a + 2a + b = a + b.\]

Since \(C\) is rational, we have by adjunction
\[-2 = C \cdot (C + K_S) = (aE + bf) \cdot ((a - 2)E + (b - 3)f)\]
\[= -a(a - 2) + a(b - 3) + b(a - 2) = -a^2 + 2a + ab - 3a + ab - 2b = -a^2 + 2ab - a - 2b.\]
Using $a = 7 - b$, we have

$$-2 = -(7 - b)^2 + 2b(7 - b) - (7 - b) - 2b = -49 + 14b - b^2 + 14b - 2b^2 - 7 + b - 2b$$

$$= -3b^2 + 27b - 56$$

or

$$b^2 - 9b + 18 = (b - 3)(b - 6) = 0.$$ 

This gives two possible classes for $C$: $3f + 4E$ or $6f + E$. If $C$ has class $3f + 4E$, then $C \cdot E$ is $3 - 4 = -1$, which means that $C$ is not irreducible. However, if $C$ is $6f + E$, we have no such problem. In fact, $6f + E$ should have a general element smooth, since by Riemann-Roch (and the fact that $h^2(6f + E) = 0$) we see that it moves in a family of dimension at least $C^2 - K_S \cdot C + \chi(O_S) - 1 = \frac{(6f + E)^2 + (3f + 2E) \cdot (6f + E)}{2} = \frac{11 + 15 - 2}{2} = 12,$

while $f$ moves in a basepoint free one-dimensional family and $E$ is fixed. Thus, the linear series will be basepoint free, and hence, a general element will be smooth. We can see that the linear series is exactly 12-dimensional, by considering the sequence

$$0 \to O_S \to O_S(6f + E) \to O_C(6f + E) \cong O_{\mathbb{P}^1}(11) \to 0$$

and showing that $h^1(6f + E) = 0$.

The space of cubic scrolls in $\mathbb{P}^3$ has dimension $19 - 10 = 9$ (since it is $3 \cdot 3 + 1 = 10$ conditions for a cubic surface to be singular along a line), so the incidence-correspondence will have dimension 21.

We see that $C$ has the 6 secant line given by the image of the curve $E + f$ that is collapsed, since $(E + f) \cdot (E + 6f) = 6$. It remains to argue that the image of $C$ is smooth, i.e., that $C$ does not meet the curve $D$ (the preimage of the double line) in any pair of points that are identified under the projection from $\mathbb{P}^4$. To see this, we claim that the natural map $H^0(O_S(C)) \to H^0(O_D(C)) \cong H^0(O_{\mathbb{P}^1}(6))$ is surjective. To do this, we need only show that $H^1(O_S(5f)) = 0$. But this follows from taking
cohomology of the sequences

\[ 0 \to \mathcal{O}_S((m-1)f) \to \mathcal{O}_S-mf) \to \mathcal{O}_f(f) = \mathcal{O}_f \to 0 \]

for \( m = 1, \cdots, 5 \).

This completes the analysis for curves that lie on a cubic surface. Note that the space of septic rational curves with a 6-secant line will have dimension \( 3 \cdot 7 + 5 = 26 \), which is larger than the 25-dimensional family of septic rational curves lying on a smooth cubic surface. Thus, there must be curves that do not lie on a cubic but that do have a 6-secant line. This completely determines their Hilbert function.

| \( d \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( d \) |
|---------|-------|-------|-------|-------|-------|-------|
| \( h_C(d) \) for \( C \) general | 1 | 4 | 10 | 20 | 29 | 36 | \( 7d + 1 \) |
| \( h_C(d) \) for \( C \) on a quadric | 1 | 4 | 9 | 16 | 25 | 36 | \( 7d + 1 \) |
| \( h_C(d) \) for \( C \subset \) cubic, \( \not\subset \) quadric | 1 | 4 | 10 | 19 | 28 | 36 | \( 7d + 1 \) |
| \( h_C(d) \) for \( C \not\subset \) quadric, with 6-secant | 1 | 4 | 10 | 20 | 28 | 36 | \( 7d + 1 \) |
| \( h^0(\mathcal{O}_{\mathbb{P}^3}(d)) \) | 1 | 4 | 10 | 20 | 35 | 56 | \( \cdots \) |

Thus, we work out the possible Hilbert functions for rational curves in \( \mathbb{P}^3 \) for degrees up to 7. For all \( e \geq 7 \), there will be smooth curves lying on cubic surfaces and smooth curves lying on quadrics. One might expect that the curves lying on quadrics would be limits of curves lying on smooth cubic surfaces, since they have more special Hilbert functions. However, this is not the case. The space of curves on a quadric surface has dimension \( 2e + 8 \), while the space of curves on smooth cubic surface has dimension \( e + 18 \), which is smaller than \( 2e + 8 \) for large \( e \). As is apparent, there is much more work to be done in this area.
In this section, we consider the incidence correspondence $\Phi = \{(C, X)|C \subset X\} \subset B \times \mathbb{P}^N$, where $B$ is the space of smooth degree $e$ rational curves in $\mathbb{P}^n$, $N$ is the dimension of the space of degree $d$ hypersurfaces in $\mathbb{P}^n$, and $\mathbb{P}^N$ is the space of degree $d$ hypersurfaces in $\mathbb{P}^n$. If $\Phi$ were irreducible, it would follow that the space of rational curves on $X$ had the expected dimension for a general $X$. Indeed, a careful analysis of $\Phi$ describes the only known approach to the Clemens Conjecture used in [27, 25, 26, 11, 12]. It is known that for $e$ large with respect to $n$ and $d$ that $\Phi$ will be reducible. However, one can hope that for $e$ small with respect to $n$ and $d$ that $\Phi$ will be irreducible. Indeed, it follows from Theorem 8.1 that $\Phi$ will be irreducible for $d \geq e - 2$. In positive characteristic, Furukawa proves that $\Phi$ will be irreducible for $e \leq 3$ and for $d \geq 2e - 3$ [18].

In this section, we prove that in characteristic 0 $\Phi$ will be irreducible for $e \leq n$ and $n \geq d$. Note that this is an improvement on the bound that follows from Theorem 8.1. Our argument works by considering the stratification of the space of rational curves by the dimension of $k$-plane that they span.

We state a simple Corollary of Theorem 8.1 which outlines the way that we will use it in this section.

**Corollary 9.1.** If $C$ is a smooth rational curve spanning a $\mathbb{P}^k$ and $d \geq e - k + 1$ then the set of degree $d$ hypersurfaces containing $C$ has the expected dimension.

Let $\pi_B$ be the projection from $\Phi$ to $B$. Let $B_k \subset B$ be the subvariety of curves spanning a $k$-plane. Using Corollary 9.1 we describe a component of $\Phi$ which we think of as the “main” component. It will be the component containing all rational normal curves.

**Corollary 9.2.** The subvariety $\pi_B^{-1}\left(\bigcup_{k \geq e - d + 1} B_k\right) \subset \Phi$ is irreducible.
Thus, it remains to consider rational curves that lie in small linear subspaces of $\mathbb{P}^n$.

**Lemma 9.3.** Let $C \in B$ be a curve which is contained in a $k$-plane in $\mathbb{P}^n$. If $n \geq d+k$, then there exists a hypersurface $X \supseteq C$ with $N_{C/X}$ globally generated.

**Proof.** By enlarging $k$ if necessary, we may assume $n = d+k$. Now we choose coordinates so that $C$ is contained in the $k$-plane $\Gamma$ cut out by $x_{k+1} = \cdots = x_{k+d} = 0$, and so that $x_0$ and $x_1$ do not both simultaneously vanish on $C$. Then let

$$F = x_{k+1}x_0^{d-1} + x_{k+2}x_0^{d-2}x_1 + \cdots + x_{k+d-1}x_0^{d-2} + x_{k+d}x_1^{d-1};$$

and set $X = V(F)$. Note that the singular locus of $X$ in $\Gamma$ is simply the $(k-2)$-plane $x_0 = x_1 = 0$ and that it is disjoint from $C$. We claim that $N_{C/X}$ is globally generated. Since $X$ is smooth along $C$, we have the following short exact sequence:

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^n} = N_{C/\Gamma} \oplus \mathcal{O}_C(1)^{n-k} \xrightarrow{\alpha \oplus \beta} N_{X/\mathbb{P}^n}|_C \longrightarrow 0.$$

Thus, it remains to determine the map $\alpha$. Since $\Gamma \subset X$, we can see that $\alpha = 0$. Moreover, we can see from $N_{Y/X} = \mathcal{I}_{Y/X}|_{\mathbb{P}}$ that $\beta = (x_0^{d-1}|_C, x_0^{d-2}x_1|_C, \cdots, x_0x_1^{d-2}|_C, x_1^{d-1}|_C)$. There are clearly $n-k-1$ independent degree-$e$ relations among the entries of the map $\beta$, so we see that $N_{C/X} = N_{C/\Gamma} \oplus \mathcal{O}_C^{n-k-1}$. However, $N_{C/\Gamma}$ is globally generated, so $N_{C/X}$ is as well.

Notice that in the construction, we did not assume anything about $C$ other than that it lies in a $k$-plane. Thus, using the same $X$ we can deform any $C_1$ lying in a $k$-plane to any other $C_2$ that does not intersect the $(k-2)$-plane $x_0 = x_1 = 0$, where $(C_1, X)$ and $(C_2, X)$ are both smooth points of $\Phi$. Using projective automorphisms of $\mathbb{P}^k$, it is clear that the pre-image of $B_k$ in $\Phi$ will be irreducible.

**Theorem 9.4.** If $n \geq d$, $e \leq n$, then the space $\Phi$ is irreducible.
Proof. For the given range of \( n, d, \) and \( e \), we have described two potentially distinct irreducible components of \( \Phi \), the component lying over \( \bigcup_{k \geq e-d+1} B_k \) and the component lying over \( \bigcup_{k \leq n-d} \). If \( e \leq n - 1 \), the bases of the two components will overlap, and they will hence be the same component since the incidence correspondence is smooth over \( \bigcup_{k \geq e-d+1} B_k \). However, we still need to analyze the case \( e = n \) and show that the two components are the same in this instance. In order to do this, we construct an explicit deformation between one element of each family. If \( n - d \leq 2 \) then any curve not in \( B_k \) will be singular. Thus, we may assume \( n - d \geq 3 \). Furthermore, since \( d \geq 3 \), it is clear that \( n - d < n - 1 \). We construct a deformation between a curve-hypersurface pair with globally generated normal bundle having the curve in a \( \mathbb{P}^{n-d} \) and a curve-hypersurface pair where the curve spans a \( \mathbb{P}^{n-1} \).

Let \( C_\epsilon \) be the curve given by the map \( f : \mathbb{P}^{1} \to \mathbb{P}^{n} \) with coordinates

\[
\begin{align*}
  f_0 &= s^e \\
  f_1 &= t^e \\
  f_2 &= s^{e-1}t \\
  f_3 &= st^{e-1} \\
  f_4 &= s^2 t^{e-2} \\
  &\quad \vdots \\
  f_k &= s^{k-2} t^{e-k+2} \\
  f_{k+1} &= \epsilon s^{e-d} t^d \\
  f_{k+2} &= \epsilon s^{e-d+1} t^{d-1} \\
  &\quad \vdots \\
  f_{d+k-1} &= \epsilon s^{e-2} t^{2} \\
  f_{d+k} &= \epsilon s^{e-1} t.
\end{align*}
\]
Note that $C_0$ spans a $\mathbb{P}^k$, and that $C_\epsilon$ for $\epsilon \neq 0$ are all projectively equivalent curves spanning a $\mathbb{P}^{n-1}$.

Let $X_\epsilon$ be the hypersurface given by

$$F = x_{k+1}x_0^{d-1} + x_{k+2}x_0^{d-2}x_1 + \cdots + x_{d+k-1}x_0x_1^{d-2} + x_{d+k}x_1^{d-1} - \epsilon(x_2^d + x_2^{d-1}x_0 + \cdots + x_2x_0^{d-1}).$$

Note that $N_{C_0/X_0}$ is globally generated. Thus, for a general $\epsilon$, $N_{C_\epsilon/X_\epsilon}$ will also be globally generated. Since pairs $(C, X)$ with globally generated normal bundle are smooth points of $\Phi$, we can see that $\Phi$ is irreducible, reduced and of the expected dimension.

\[\square\]

10. Normal Bundles

One natural question related to this area involves normal bundles. Fix a degree $e$ rational curve $C$. Then, as $X$ varies over all hypersurfaces containing $C$ and smooth along $C$, what are the possible normal bundles $N_{C/X}$? As we saw in Section 9, the normal bundle $N_{C/X}$ carries important information about the deformation theory of how $C$ is embedded in $X$, and it would be very interesting to understand the possible normal bundles for a given $C$. This question has not been well-explored, even for lines. In this section, we get a start by investigating the possible normal bundles that can occur for rational normal curves. We completely characterize the possible normal bundles, and prove bounds on the codimension of each possible normal bundle in the space of hypersurfaces containing the rational normal curve. Our bounds exactly characterize the dimensions of almost all of the loci in this incidence correspondence, and completely characterize the dimensions of all the different normal bundles for lines. At the very end we note some interesting applications, although we expect there to be more.
We start by getting an explicit description of $N_{C/P^n}$, the normal bundles of rational normal curves in projective space. Let $C$ be the rational normal curve defined by $[s : t] \mapsto [s^n : s^{n-1}t : \cdots : t^n]$. The ideal of $C$ is cut out by the $2 \times 2$ minors of

$$
\begin{pmatrix}
x_1 & x_2 & \cdots & x_n \\
x_0 & x_1 & \cdots & x_{n-1}
\end{pmatrix}.
$$

Let $q_{ij} = x_ix_{j-1} - x_jx_{i-1}$. Then the relations between the $q_{ij}$ are completely described by the $3 \times 3$ minors of

$$
\begin{pmatrix}
x_1 & x_2 & \cdots & x_n \\
x_1 & x_2 & \cdots & x_n \\
x_0 & x_1 & \cdots & x_{n-1}
\end{pmatrix}
$$
and

$$
\begin{pmatrix}
x_0 & x_1 & \cdots & x_{n-1} \\
x_1 & x_2 & \cdots & x_n \\
x_0 & x_1 & \cdots & x_{n-1}
\end{pmatrix},
$$
i.e., they are all of the form $x_iq_{jk} - x_jq_{ik} + x_kq_{ij} = 0$ or $x_{i-1}q_{jk} - x_{j-1}q_{ik} + x_{k-1}q_{ij} = 0$.

Thus $\mathcal{I}_{C/P^n} = < q_{ij} >$. Now we wish to connect this description to the normal bundle. We have $N_{C/P^n} = \text{Hom}_{\mathcal{O}_{P^n}}(\mathcal{I}_{C/P^n}, \mathcal{O}_C)$. Since $\mathcal{I}_{C/P^n}$ is generated by the $q_{ij}$, an element $\phi \in \text{Hom}_{\mathcal{O}_{P^n}}(\mathcal{I}_{C/P^n}, \mathcal{O}_C)$ will be completely determined by $\phi(q_{ij})$, where the only constraints on $b'_{ij} = \phi(q_{ij})$ are those coming from the relations between the $q_{ij}$.

**Lemma 10.1.** We have $s^{n-i-1}t^{i-1}$ divides $b'_{i,i+1}$.

**Proof.** Applying $\phi$ to the relation

$$
x_1q_{i,i+1} - x_iq_{1,i+1} + x_{i+1}q_{1,i} = 0,
$$
we get
\[ s^{n-1}tb'_{i,i+1} - s^{n-i}tb'_{1,i+1} + s^{n-i-1}t^{i+1}b'_{1,i} = 0. \]

From this, we see that \( t^{i-1} \) divides \( b'_{i,i+1} \).

Similarly, we can apply \( \phi \) to the relation
\[ x_{i}q_{i+1,n} - x_{i+1}q_{i,n} + x_{n}q_{i,i+1} = 0 \]

to get
\[ s^{n-i}tb'_{i+1,n} - s^{n-i-1}t^{i+1}b'_{i,n} + t^{n}b'_{i,i+1} \]

which shows that \( s^{n-i-1} \) divides \( b'_{i,i+1} \). □

In light of this lemma, we write \( b'_{i,i+1} = s^{n-i}t^{i-1}b_{i,i+1} \).

**Proposition 10.2.** The \( b_{i,i+1} \) completely determine \( \phi \). In fact,

\[ b'_{i,k} = \sum_{\ell=i}^{k-1} s^{n-k-i+\ell}t^{k+i-\ell-2}b_{\ell,\ell+1} \]

*Proof.* We show that we can find all of the \( b'_{i,j} \) given the \( b_{i,i+1} \). We use induction on \( i-j \). The base case \( i-j = 1 \) follows from the definition of the \( b_{i,i+1} \). Now suppose that we have proved the formula for \( b'_{r,m} \) for all \( r, m \) with \( r-m < i-k \). We find \( b_{i,k} \).

To do this, we apply \( \phi \) to the relation
\[ x_{i}q_{i+1,k} - x_{i+1}q_{i,k} + x_{k}q_{i,i+1} = 0, \]

obtaining
\[ s^{n-i}tb'_{i+1,k} - s^{n-i-1}t^{i+1}b'_{i,k} + s^{n-k}t^{k}b'_{i,i+1} = 0. \]

Solving for \( b'_{i,k} \), we get
\[ b'_{i,k} = \frac{s_{k}b'_{i,k}}{t} + \frac{s_{i-k+1}}{t^{i-k+1}}b'_{i,i+1}. \]
Using induction, we have

\[ b'_{i,k} = \frac{s}{\ell} \left( \sum_{\ell=i+1}^{k-1} s^{n-k-i-\ell}\ell^{k+i-\ell-1}b_{\ell,\ell+1} \right) + \frac{s^{i-1}}{\ell^{i-1+k-1}} s^{n-i-1}\ell^{i-1}b_{i,i+1} \]

\[ = \sum_{\ell=i+1}^{k-1} s^{n-k-i+\ell}\ell^{k+i-\ell-2}b_{\ell,\ell+1} + s^{n-k}\ell^{k-2}b_{i,i+1} \]

\[ = \sum_{\ell=i}^{k-1} s^{n-k-i+\ell}\ell^{k+i-\ell-2}b_{\ell,\ell+1} \]

\[ \square \]

**Proposition 10.3.** For any choice of \(b_{i,i+1}\), we get an element of \(H^0(N_{C/P^n})\).

**Proof.** It remains to show that if we use the formulas from Proposition 10.2 for the \(b'_{i,j}\) that they satisfy the relations that the \(q_{i,j}\) satisfy. Let \(x_iq_{jk} - x_jq_{ik} + x_kq_{ij}\) (the other case is almost identical). We apply \(\phi\) to the relation to obtain

\[ s^{n-i}t^i b'_{jk} - s^{n-j}t^j b'_{ik} + s^{n-k}t^k b'_{jh} \]

\[ = s^{n-i}t^i \sum_{\ell=j}^{k-1} s^{n-k-j+\ell}\ell^{k+j-\ell-2}b_{\ell,\ell+1} - s^{n-j}t^j \sum_{\ell=i}^{k-1} s^{n-k-i+\ell}\ell^{k+i-\ell-2}b_{\ell,\ell+1} + \]

\[ s^{n-k}t^k \sum_{\ell=i}^{j-1} s^{n-j-i+\ell}\ell^{j+i-\ell-2}b_{\ell,\ell+1} \]

\[ = \sum_{\ell=i}^{k-1} (s^{2n-i-j-k+\ell}\ell^{j+k-\ell-2} - s^{n-i-j-k+\ell}\ell^{j+k-\ell-2})b_{\ell,\ell+1} = 0. \]

Thus, we may select arbitrary \(b_{i,i+1}\).

\[ \square \]

Note that this proves directly that \(N_{C/P^n} = \mathcal{O}_{\mathbb{P}^1(n)}^{n-1}\). Now we wish to investigate maps from \(N_{C/P^n}\) to \(\mathcal{O}_C(d) = \mathcal{O}_{\mathbb{P}^1}(nd)\). In particular, given a hypersurface \(V(f)\) containing \(C\), we wish to understand the map \(N_{C/P^n} \to \mathcal{O}_{\mathbb{P}^1}(ed)\).
Proposition 10.4. If $f = \sum_{i,j} a_{i,j} q_{i,j}$, then the corresponding map

$$\phi = (\phi_{1,2}, \ldots, \phi_{n-1,n}) : N_{C/P^n} = \mathcal{O}_{\mathbb{P}^1}(n+2)^{n-1} \rightarrow N_{X/P^n} |_C = \mathcal{O}_{\mathbb{P}^1}(nd)$$

is given by

$$\phi_{i,i+1} = \sum_{k=1}^{i} \sum_{j=i+1}^{n} s^{n-j-k+i} t^{j+k-i-2} a_{kj} |_C.$$

Proof. This follows from Proposition 10.2 after some reindexing. □

Proposition 10.5. If $d \geq 3$, then given an arbitrary map $\phi = (\phi_{1,2}, \ldots, \phi_{n-1,n}) : \mathcal{O}_{\mathbb{P}^1}(n+2)^{n-1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(nd)$, there exists a hypersurface $V(f)$ such that $\phi$ is the map $N_{C/P^n} \rightarrow N_{X/P^n} |_C$.

Proof. Since $C$ is a normal curve, we see that the natural map $H^0(\mathcal{O}_{\mathbb{P}^n}(1) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ given by restricting to $C$ is surjective. In an abuse of notation, we write $a_{i,j}$ for $a_{i,j} | C$.

We wish to show that given $\phi$, we can find polynomials $a_{i,j}$ so that

$$\phi_{i,i+1} = \sum_{k=1}^{i} \sum_{j=i+1}^{n} s^{n-j-k+i} t^{j+k-i-2} a_{kj} |_C.$$

We work in stages. Let $V$ be the image in $\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(n+2)^{n-1}, \mathcal{O}_{\mathbb{P}^1}(nd))$ of the natural map $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(n+2)^{n-1}, \mathcal{O}_{\mathbb{P}^1}(nd))$ given by taking $f$ to the natural map of normal bundles. We wish to show that $V$ is all of $\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(n+2)^{n-1}, \mathcal{O}_{\mathbb{P}^1}(nd))$.

Let

$$V_0 = \{ (\phi_{1,2}, \ldots, \phi_{n-1,n}) | s^{n-2} \text{ divides } \phi_{1,2}, s^{n-3} t \text{ divides } \phi_{2,3}, \ldots, t^{n-2} \text{ divides } \phi_{n-1,n} \}.$$

We observe that $V_0 \subset V$, because we can simply take $a_{i,i+1} = \frac{\phi_{i,i+1}}{s^{n-i-1} t^{n-i-2}}$ to obtain a polynomial $f$ with the desired property. Thus, we can work mod $V_0$.

Now let $(\phi_{1,2}, \ldots, \phi_{n-1,n})$ be arbitrary. We first show that we can select $a_{i,j}$ that give a map of the form $(\phi_{1,2}, *, \ldots, *)$. Take $\phi_{1,2}$, cancel out any terms involving
multiples of \( s^{n-2} \), divide the result by \( t^{n-2} \), and set the result to be \( a_{1,n} \). Let all the other \( a_{i,j} \)'s be 0. Mod \( V_0 \), this gives a map of the desired form.

Now suppose we have selected \( a_{i,j} \) that give a map of the form \((\phi_{1,2}, \cdots, \phi_{k-1,k}, \phi_{k,k+1} - h, \cdots)\), where \( h \neq 0 \). We show how to modify the \( a_{i,j} \) to get a map of the form we want. Let \( g_0 \) be given by starting with \( h \), canceling out all terms divisible by \( s^{n-k-1} \) and dividing by \( t^{n-2} \). Let \( g_1 \) be given by starting with \( h \), canceling out all terms divisible by \( t \), and dividing the result by \( s^{n-2} \). Note that \( g_1 \) is divisible by \( s^n \). Let \( g_2 \) be given by starting with \( h - s^{n-2}g_1 \), canceling out all terms divisible by \( t^2 \) and dividing the result by \( ts^{n-3} \). Note that \( g_2 \) is divisible by \( s^n \). Similarly, let \( g_r \) be given by starting with \( h - s^{n-2}g_1 - s^{n-3}tg_2 - \cdots - s^{n-r-2}g_{r-1} \), canceling out all terms divisible by \( t^r \) and dividing the result by \( s^{n-r-1}t^{r-1} \). Note that \( g_r \) will be divisible by \( s^n \).

Add \( g_0 \) to \( a_{k,n} \), \( g_1 \) to \( a_{1,k+1} \), \( g_2 \) to \( a_{2,k+1} \), \( g_3 \) to \( a_{3,k+1} \), and so forth, all the way up to adding \( g_{k-1} \) to \( a_{k-1,k+1} \). Then we claim that mod \( V_0 \), this gives us a map of the form \((\phi_{1,2}, \cdots, \phi_{k,k+1}, *, \cdots, *)\). To see this, note that we have canceled out all of the terms of \( h \) that are not divisible by \( s^{n-k-1}t^{k-1} \). Adding \( g_0 \) to \( a_{k,n} \) has no effect on \( \phi_{1,2}, \cdots, \phi_{k-1,k} \). Finally, adding \( g_{r} \) to \( a_{r,k+1} \) will only change \( \phi_{j,j+1} \) by terms divisible by \( s^n t^{n-2} \), so mod \( V_0 \), this new \( f \) gives rise to the map \((\phi_{1,2}, \cdots, \phi_{k,k+1}, \cdots)\).

Induction completes the proof.

\[ \square \]

Having described the normal bundles of rational normal curves of degree \( n \) in \( \mathbb{P}^n \), we can easily extend this to rational normal curves of degree \( e \leq n \) in \( \mathbb{P}^n \) by noting that if \( C \subset \Lambda \cong \mathbb{P}^k \), then \( N_{C/\mathbb{P}^n} = N_{C/\Lambda} \oplus N_{\Lambda/\mathbb{P}^n}|_C \). We can see that for curves of this form, any surjective map \( N_{C/\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^1}(ed) \) for \( d \geq 3 \) comes from a the map of normal bundles for a hypersurface \( X \), i.e., comes from the sequence

\[ 0 \to N_{C/X} \to N_{C/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_C \to 0. \]
Thus, in order to understand the possible normal bundles of rational normal curves in degree \(d\) hypersurfaces, we simply need to understand things about maps from
\[
\mathcal{O}_{\mathbb{P}^1}(e + 2)^{e-1} \oplus \mathcal{O}(e)^{n-e} \rightarrow \mathcal{O}_{\mathbb{P}^1}(ed).
\]

Given a rational curve \(C\) and a hypersurface \(X\) containing it, we can imagine deforming \(X\) while still insisting that it contain \(C\). As we do this, we get a deformation of the normal bundle \(N_{C/X}\), which leads to the following natural question: What is the Kodaira-Spencer map for these deformations of \(N_{C/X}\)? Using Proposition 10.5, we see that we need only consider how the kernel of the map \(\alpha : \mathcal{O}_{\mathbb{P}^1}(e + 2)^{e-1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)^{n-e} \rightarrow \mathcal{O}_{\mathbb{P}^1}(ed)\) deforms as we deform the map \(\alpha\). Deformations of \(\alpha\) are simply given by the tangent space to \(\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(e + 2)^{e-1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)^{n-e}, \mathcal{O}_{\mathbb{P}^1}(ed))\), which is itself isomorphic to \(\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(e + 2)^{e-1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)^{n-e}, \mathcal{O}_{\mathbb{P}^1}(ed))\). Thus, the Kodaira-Spencer map will be given by a map
\[
\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(e + 2)^{e-1} \oplus \mathcal{O}_{\mathbb{P}^1}(e)^{n-e}, \mathcal{O}_{\mathbb{P}^1}(ed)) \rightarrow \text{Ext}^1(N_{C/X}, N_{C/X}).
\]

**Proposition 10.6.** If \(C\) is a rational normal curve and \(X\) a hypersurface containing \(C\) in its smooth locus, then the Kodaira-Spencer map described above will have cokernel \(\text{Ext}^1(N_{C/X}, N_{C/X})\).

**Proof.** For simplicity of notation, let \(N = N_{C/X}\), \(A = N_{C/\mathbb{P}^n}\), and \(B = \mathcal{O}_{\mathbb{P}^1}(ed)\). Then we have the two short exact sequences
\[
0 \rightarrow N \rightarrow A \rightarrow B \rightarrow 0
\]
and
\[
0 \rightarrow B^\vee \rightarrow A^\vee \rightarrow N^\vee \rightarrow 0.
\]

Taking long exact sequences in homology, we obtain the following diagram.
\[
\begin{array}{c}
\text{Hom}(A, B) \\
\downarrow^\alpha \\
\text{Hom}(N, B) \rightarrow^\beta \text{Ext}^1(N, N) \rightarrow^\gamma \text{Ext}^1(N^\vee \otimes A) \rightarrow \text{Ext}^1(B, B) \\
\downarrow \quad \\
\text{Ext}^1(N, A) \rightarrow H^1(N^\vee \otimes B)
\end{array}
\]

We can relate the Kodaira-Spencer map to the map on tangent spaces to the Quot scheme \( Q \) of quotients of \( \text{Hom}(O_{P^1}(e + 2)^e - 1 \oplus O_{P^1}(e)^{n - e}) \) with chern character \( \text{ch}(O_{P^1}(ed)) \). For more details see Le Poitier [29]. We can identify \( \alpha \) with the map on tangent spaces induced by the natural map \( \text{Hom}(A, B) \) to \( Q \), and we can identify \( \beta \) with the Kodaira-Spencer map of deformations of \( N \) coming from the Quot scheme.

Since \( B \) is a line bundle, we see that \( \text{Ext}^1(B, B) \) is trivial and hence, \( \alpha \) is surjective. Since \( ed \) will be larger than the degree of every component of \( A \), we see that \( \text{Ext}^1(N, B) = 0 \) by the local-global spectral sequence and hence, \( \gamma \) is surjective. Since \( \beta \circ \alpha \) is the Kodaira-Spencer map, we see that the cokernel is \( \text{Ext}^1(N, A) \cong H^1(N^\vee \otimes A) \) since \( N \) and \( A \) are vector bundles.

Thus, we have completely characterized the deformations of \( N_{C/X} \) given by fixing \( C \) and varying \( X \). As an immediate corollary, we can describe the relative dimensions of the various splitting types of normal bundle in the incidence correspondence of rational normal curves on hypersurfaces.

**Theorem 10.7.** Let \( \Phi_{n,d,e} \) be the incidence correspondence of pairs \((C, X)\) where \( C \) is a degree \( e \) rational normal curve in \( \mathbb{P}^n \) lying in the smooth locus of the hypersurface \( X \) of degree \( d \). Then the Kodaira-Spencer map will be surjective for all pairs \((C, X)\) with \( H^1(N_{C/X} \otimes N_{C/P^n}) = 0 \), and if the Kodaira-Spencer map is not surjective, then the cokernel will have dimension \( h^1(N_{C/X} \otimes N_{C/P^n}) \).
Theorem 10.7 has immediate implications about what types of normal bundles can occur in a general hypersurface, and in what codimensions they occur. It is particularly interesting in the case $e = 1$, where the rational curves in question are all lines. For instance, it is known that a general hypersurface will have a smooth Fano scheme of lines. However, our techniques let us bound the codimension of lines which have normal bundles with at least one $O_{P^1}(-1)$ factor.

**Corollary 10.8.** If $X$ is a general hypersurface, then the space of lines in $X$ with normal bundle $N_{C/X} = O_{P^1}(-1)^k \oplus O_{P^1}^{d-1-2k} \oplus O_{P^1}(1)^{n-d-1+k}$ has codimension at least $k(n - d - 1 + k)$ in the Fano scheme of lines on $X$. If the codimension exceeds the dimension, then there will not be any lines of that splitting type on $X$.

**Proof.** For lines, we see that $H^1(N_{C/X}^\vee \otimes N_{C/P^n}) = 0$. Thus, the Kodaira-Spencer map will always be surjective, and so the codimension of the space of lines with a given splitting type will just be $h^1(\End(N_{C/X}))$. For $N_{C/X} = O_{P^1}(-1)^k \oplus O_{P^1}^{d-1-2k} \oplus O_{P^1}(1)^{n-d-1+k}$, we see that $h^1(\End(N_{C/X})) = k(n - d - 1 + k)$. □

**Corollary 10.9.** The space of lines in a general hypersurface $X$ with unbalanced normal bundle has dimension at most $n - 3$.

**Proof.** Simply plug $k = 1$ into the proof of the previous theorem. □

11. Flatness of 2-planes

If one hopes to understand the geometry of Fano hypersurfaces, after understanding the rational curves on them, it is natural to next try to understand the rational surfaces that lie on them. Research in this direction is just starting, with, for example, the results of Starr and Beheshti showing that very general hypersurfaces do not contain certain types of rational surfaces [3]. Results of this form are particularly interesting, since it is conjectured that for $n \geq 5$, hypersurfaces of degree $n$ in $P^n$ are
not swept out by rational surfaces, which would provide examples of varieties that are rationally connected but not unirational.

In this section, we start to work on results of the opposite form, showing that very high dimensional hypersurfaces have the expected dimension of 2-planes through a point and the expected dimension of quadric surfaces. This begins to explore the theme that low-degree hypersurfaces in high dimension look “more regular.” However, our work just begins to explore this, and there is much left to do in this area.

**Lemma 11.1.** Let \( S_d = H^0(\mathcal{O}_{\mathbb{P}^m}(d)) \). Consider the space \( A = (S_1)^r \times S_{d_1} \times \cdots \times S_{d_t} \), where \( r + t < m \) and \( d_i \geq 2 \forall i \). Suppose \( \min_i \left\{ \binom{d_i + m - r - t}{d_i} \right\} \geq m + 1 \). Then the locus of tuples of polynomials which cut out a subvariety of \( \mathbb{P}^m \) of codimension \( r + t - k \) where \( k \leq r \) is codimension at least \( k(m - r + k + 1) \). Furthermore, if we restrict to the locus where the \( r \) linear forms intersect in a subvariety of codimension at least \( r - k + 1 \), then the codimension of the locus of tuples of polynomials which cut out a subvariety of \( \mathbb{P}^m \) of codimension \( r + t - k \) is codimension at least \( \min_i \left\{ \binom{d_i + m - r - t}{d_i} \right\} \).

**Proof.** We write \( \ell_1, \ldots, \ell_r \) for the linear forms and \( f_1, \ldots, f_t \) for the forms of degree \( d_1, \ldots, d_t \). We build up the intersection by first intersecting the \( V(\ell_i) \) one-by-one, then intersecting with the \( V(f_i) \)'s, again one-by-one. We wish to bound the codimensions loci in which these intersections fail to be proper.

First we consider the loci in which the linear forms intersect non-properly. For a linear form to vanish along a \( s \)-dimensional variety is at least \( s + 1 \) conditions, so for the linear forms to intersect in dimension \( m - r + k \) is \( k(m - r + k + 1) \) conditions.

Now we consider the higher degree polynomials. Suppose the varieties cut out by the linear forms and the varieties cut out by \( f_1, \ldots, f_{j-1} \) intersect in a variety of dimension \( s \). Then for \( f_j \) to vanish on a component of this intersection is \( \binom{s + d_i}{d_k} \) conditions. This will be smallest when \( s \) is as small as possible, i.e., when \( s = m - r - j + 1 \).
For a hypersurface $X$, let $F_k(X)$ be the Fano scheme of $k$-planes on $X$, and let $F_k(X, p)$ be the subscheme of $k$-planes in $X$ passing through $p$. We need a result from Beheshti-Kumar [4] bounding the locus of singular points of $F_1(X, p)$ for a general $X$.

**Theorem 11.2** (Implied by Theorem 1.5 from [4]). Let $X$ be a general degree $d$ hypersurface in $\mathbb{P}^n$, and let $p \in X$ be arbitrary. Let

$$a = \min_b \left\{ b \geq 0 \mid \frac{b + 3}{2} \geq n \right\}.$$ 

Then the singular locus of $F_1(X, p)$ has dimension at most $a$.

We wish to have an upper bound for $a$.

**Lemma 11.3.** With notation as above, we have

$$a \leq \lceil \sqrt{2n} \rceil.$$ 

**Proof.** We have

$$\frac{(b + 3)}{2} \leq \frac{b^2}{2}.$$ 

Thus, $\lceil \sqrt{2n} \rceil \in \{ b \geq 0 \mid \frac{b + 3}{2} \geq n \}$, so $a \leq \lceil \sqrt{2n} \rceil$. □

**Proposition 11.4.** Let $X$ be a general hypersurface of degree $d \geq 3$ in $\mathbb{P}^n$. Then if

$$\frac{n - \frac{d(d+1)}{2}}{2} \geq 2n - d - 1$$

the space of 2-planes through an arbitrary point of $X$ has the expected dimension.

**Proof.** Let $\Phi = \mathbb{P}^N$ be the space of degree $d$ hypersurfaces in $\mathbb{P}^n$. Let $\Phi_{0,1} \subset \mathbb{P}^n \times \mathbb{G}(1, n) \times \Phi$ be the incidence correspondence $\{(p, \ell, X) \mid p \in \ell \subset X\}$. There is a natural map $\pi : \Phi_{0,1} \to \Phi$. By a simple dimension count, a subvariety $S \subset \Phi_{0,1}$ will not dominate $\Phi$ if its codimension is at least $2n - d - 1$. We start by showing that
the space $\mathcal{B} \subset \Phi_{0,1}$ given by

$$\mathcal{B} = \{(p, \ell, X) | \ell \text{ is a smooth point of } F_1(X, p) \text{ with more than expected 2-planes containing it}\}$$

has codimension at least $2n - d - 1$. We do this by analyzing the fibers of the projection map $\psi : \Phi_{0,1} \to \{(p, \ell) | p \in \ell\} \subset \mathbb{P}^n \times G(1, n)$, which will be equidimensional by the symmetries of projective space.

Fix a point $p = [1 : 0 : \cdots : 0]$ and a line $L = \{x_2 = \cdots = x_n = 0\}$ containing $p$, and consider the space of hypersurfaces containing $L$. If we let $f(x_0, \cdots, x_n)$ be the equation of the hypersurface, then we see that the space of lines through $p$ will be cut out by the homogeneous pieces of $f$ with respect to the variable $x_n$. That is, if

$$f = f_1(x_1, \cdots, x_n)x_0^{d-1} + f_2(x_1, \cdots, x_n)x_0^{d-2} + \cdots + f_d(x_1, \cdots, x_n),$$

then the locus of lines in $X$ passing through $p$ is the subvariety $Y = \cap_i V(f_i) \subset \mathbb{P}^{n-1}$. The space of 2-planes through $p$ in $X$ is simply the space of lines in $Y$, and the space of 2-planes in $X$ containing $L$ is simply the space of lines in $Y$ containing the point $[1 : 0 : \cdots : 0] \in \mathbb{P}^{n-1}$, where the coordinates on $\mathbb{P}^{n-1}$ are given by $x_1, \cdots, x_n$. This, in turn, will be cut out by the homogeneous pieces of the $f_i$ with respect to $x_1$. That is, if we let $f_i = \ell_i x_1^{i-1} + g_{(i-1)+1}x_1^{i-2} + \cdots + g_{(i)}$, then the space of 2-planes in $X$ containing $L$ is simply $\cap_i V(\ell_i) \cap \cap_i V(g_i) \subset \mathbb{P}^{n-2}$. We wish to bound the codimensions of the loci where the space of 2-planes through $L$ is larger than expected.

Suppose the space of 2-planes in $X$ containing $L$ jumps when $L$ is a smooth point of $F_1(X, p)$. We see that $L$ is a smooth in $F_1(X, p)$ precisely when the planes cut out by the $\ell_i$ intersect properly. By our condition on $n$, we see that Lemma 11.1 applies with $r = d$, $t = \frac{d(d-1)}{2}$, and $m = n - 2$. Using the Lemma, we see that this will happen in codimension at least $\min_i (\{(d_i^+m-r-t)\})$ which is at least $2n - d - 1$ by our assumption on $n$ and $d$. 

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Thus, it remains to consider what happens to the fibers when $L$ is singular in $F_1(X, p)$. By Lemma 11.1, the codimension of hypersurfaces where the space of 2-planes through $L$ jumps by at least 3 is codimension at least $3(n - 2 - d + 3 + 1) = 3(n - d + 2) = 3n - 3d + 6$. This last is greater than $2n - d - 1$ for $n \geq 2d - 7$, which is certainly the case given our condition on $n$. Thus, the space of 2-planes containing a point will not jump by 3 for a general hypersurface.

It remains to show that for a general $X$ and for an arbitrary $q \in X$, $F_1(X, p)$ is singular in codimension at least 2. This follows easily from Theorem 11.2, since $\sqrt{2n} < n - d - 3$ for $n$ in the range we are considering. Thus, the result is proven.

\[ \square \]

Note that it falls out of the proof that the space of 2-planes through a line with globally generated normal bundle has the expected dimension $n - 2 - (\frac{d+1}{2})$, and the dimension jumps by at most two for singular lines.

We now turn to quadric surfaces that lie in a $\mathbb{P}^3$. We can show that the space of quadrics through a point of a general hypersurface is the expected dimension. Note that a simple dimension counting argument shows that for large $n$, the dimension will be at least the expected dimension.

**Theorem 11.5.** For a general hypersurface $X$ of degree $d$ in $\mathbb{P}^n$, with $n$ sufficiently large compared to the degree, the space of quadric surfaces in $X$ is of the expected dimension, $4(n - 3) + 9 - (d + 1)^2 = 4n - 3 - (d + 1)^2$, and the space of quadrics through a point is also the expected dimension, $3n - 1 - (d + 1)^2$.

The idea of the proof is to show that every sufficiently large family of quadric surfaces contains singular quadrics. Singular quadrics correspond to conics in the space of lines through a point, which must contain reducibles for $d$ sufficiently large. Thus, we can use the results on planes.
Note that since there is a $4(n - 3)$-dimensional family of 3-planes in $\mathbb{P}^n$, and a 9-dimensional family of quadrics in a fixed $\mathbb{P}^3$, we are talking about a $4n - 3$ dimensional family of quadric surfaces in $\mathbb{P}^n$.

**Proposition 11.6.** If $Q \to B$ parameterizes a family of quadric surfaces of dimension at least $3n - 6$, then $Q$ parameterizes some singular quadrics.

*Proof.* To get a contradiction, let $B$ be a $3n - 6$ dimensional family of smooth quadrics in $\mathbb{P}^n$. Since $B$ has dimension at least $3n - 6$, we can find a subfamily $B_0$ with $\dim B_0 \geq 2n - 4$ of quadrics all passing through a fixed point $p$. Consider the family of tangent planes $T_pQ_b$, and consider their intersections with each quadric $C_b = Q_b \cap T_pQ_b$. Each $C_b$ will be a degree 2 curve in $\mathbb{P}^n$ that is singular at $p$, which means that the $C_b$ will each either be a union of two distinct lines or a double line. If $C_b$ is a double line, then $Q_b$ must be singular, which contradicts our original assumption. Thus, $C_b$ must be a union of two distinct lines.

We can imagine the $C_b$ to be a family of points in the $\text{Sym}^2\mathbb{P}^{n-1}$ of pairs of lines through $p$. Because the two lines must be distinct, the $C_b$ must miss the diagonal locus $\Delta \subset \text{Sym}^2\mathbb{P}^{n-1}$ of double lines. However, any subvariety of $\text{Sym}^2\mathbb{P}^{n-1}$ of dimension at least $n - 1$ must meet $\Delta$. Therefore, the curves $C_b$ sweep out a subvariety of $\text{Sym}^2\mathbb{P}^{n-1}$ of dimension at most $n - 2$, which means that we can find a subfamily $B_1 \subset B_0$ of quadrics all containing two fixed lines, with $\dim B_1 \geq n - 2$. Let $\Lambda_b$ be the family of $\mathbb{P}^3$'s spanned by the quadrics $Q_b$. Clearly $\Lambda_b$ contains $C_b$. Since there is only a $n - 3$ dimensional family of 3-planes containing the plane spanned by a the fixed $C_b$, we see that there is at least a one-parameter family $B_2$ of quadrics lying in the same 3-plane. However, this is impossible, since any positive-dimensional family of quadrics in a fixed 3-plane must parameterized some singular quadrics. \[\square\]

As an aside, note that we can find a complete $n - 3$ dimensional family of smooth quadrics by taking a smooth quadric hypersurface in $\mathbb{P}^n$, and considering the family
of intersections of that quadric with all of the 3-planes containing a fixed general 2-plane. The only way such an intersection can be singular is if the 3-plane is contained in the tangent plane to the hypersurface at some point, but that is impossible given that the intersection of the 2-plane and the hypersurface will be a smooth quadric.

Note that we can obtain a similar result for arbitrary dimensional quadric hypersurfaces in a $\mathbb{P}^k$ using induction.

**Proof.** (Theorem 11.5) Note that the space of quadric surfaces in $\mathbb{P}^n$ will be a projective bundle over the Grassmannian $G(3,n)$ of 3-planes in $\mathbb{P}^n$. A quadric surface will lie on a hypersurface $X$ if the corresponding section of the rank $(d+1)^2$ vector bundle vanishes identically. Therefore, the space of quadrics on $X$ will have dimension at least $4n - 3 - (d + 1)^2$, which we refer to as the expected dimension.

For $n$ large compared to $d$, we see that every component of quadrics in $X$ will have dimension at least $3n - 5$, so every component of the space of quadrics on $X$ will parameterize singular elements. Singular quadrics correspond to conic curves in the space of lines through a point. For sufficiently large $n$, there will be a large enough family of conic curves in the space of lines through each point that it will break into two components. This corresponds to the singular quadrics degenerating to pairs of 2-planes meeting in a line. Thus, it remains to bound the dimension of the space of pairs of 2-planes meeting in a line.

The space of 2-planes through a given line will have dimension $n - 2 - \binom{d+1}{2}$ if the line has globally generated normal bundle, and can jump by at most 2 if the normal bundle is not globally generated. For a general hypersurface $X$, we see by Theorem 11.2 that the space of lines through $p$ with unbalanced normal bundle has dimension at most $a$, which for large $n$ will have high codimension in the space of lines in $X$ through $p$. Thus, we see that the space of 2-planes in $X$ meeting in a line through $p$
has dimension
\[
    n - d - 1 + 2 \left( n - 2 - \frac{d + 1}{2} \right) = 3n - 5 - d^2 - 2d = 3n - 4 - (d + 1)^2.
\]

However, the space of 2-planes meeting in a line has codimension at most 3 in any family of quadrics, so the space of quadrics in $X$ through $p$ has dimension at most $3n - 1 - (d + 1)^2$, which concludes the proof. \(\square\)

12. Conclusion

In Conclusion, we hope it might be useful to sum up some of the many questions relating to subvarieties of hypersurfaces that remain open.

This paper covers Conjecture 2.1 for much of the Fano range. However, as we point out in the introduction, there remain many ranges of $n$, $d$, and $e$ for which we do not know whether Conjecture 2.1 is true. In the Fano range, there are the two cases $n = d + 1$ and $n = d$. In the general type range, the question is open for $d < \frac{3n}{2}$. Even the (hopefully less difficult question) of whether the scheme of rational curves on a general $X$ is finite type remains open for $d < \frac{3n}{2}$, and this and related questions have been the subject of a fair bit of study. See the survey article by Voisin [36] for more details. The Calabi-Yau range remains mostly a mystery, and even low-dimensional cases like the Clemens Conjecture remain very much open. We hope that more progress will be made in the future.

Another important subject of investigation is what rational surfaces lie in Fano hypersurfaces. There has been some work on this, such as [3, 1, 13] to name a few, but there are many remaining open questions. One extremely interesting (and potentially quite difficult question) is to determine precisely which hypersurfaces are swept out by rational varieties of dimension $r$ for each $r$. We know that any Fano variety is uniruled by rational curves, but even for rational surfaces the question is open and would yield interesting insight into the birational geometry of hypersurfaces.
One possible approach to understanding rational surfaces, and one that would be interesting in its own right, is to better understand families of rational surfaces in $\mathbb{P}^n$. For instance, how large can a complete family of smooth rational surfaces in $\mathbb{P}^n$ be? How large can a complete family of irreducible rational surfaces in $\mathbb{P}^n$ be? For quadric surfaces, we prove upper and lower bounds, there is a substantial range in the middle, and the question seems to be completely open for arbitrary surfaces.

A third important set of questions, which we have not touched on here is the question of what subvarieties can lie on arbitrary smooth hypersurfaces. Here the questions are much more difficult, and even for the case of lines, it is not known what dimension and degree range guarantee the expected dimension. Coskun and Starr [10] prove that the dimensions of rational curves on smooth cubics, but for higher-degree hypersurfaces the question remains very much open. For lines, the main conjecture is the de Jong-Debarre conjecture.

**Conjecture 12.1** (de Jong-Debarre). *Any smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^n$ contains the expected dimensional family of lines.*

This conjecture is only known for degrees up to 8 [2], and the best known bounds are exponential in $n$ [21].

Another interesting question is categorizing the possible Hilbert functions of rational curves (or even more general curves) in $\mathbb{P}^n$. Even in $\mathbb{P}^3$, there are many possibilities, and the geometry of the incidence correspondence can get quite complex. Studying this is an interesting question in its own right, and it also could yield insight into the types of behaviors experienced by spaces of rational curves on hypersurfaces. Connecting the Hilbert functions of rational curves to their normal bundles would be particularly interesting, and remains almost completely open. We hope much more is known in the future.
References


