Several Theorems About Probabilistic Limiting Expressions: The Gaussian free field, symmetric Pearcey process, and strong Szegő asymptotics

Several Theorems About Probabilistic Limiting Expressions:
The Gaussian free field, symmetric Pearcey process, and strong Szegő asymptotics

A dissertation presented

by

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to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

April 2015

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Abstract

Certain probabilistic processes appear in the asymptotic scaling limit of many models. This thesis covers several theorems about such processes. Chapter 2 covers the Gaussian free field in interlacing particle systems, chapters 4 and 5 construct a non–commutative particle system and prove the Gaussian free field convergence. Chapter 3 shows the symmetric Pearcey process in a discrete–time interlacing particle system with a wall, and chapter 6 shows Strong Szegő asymptotics for the Riemann ζ funcion.

Contents

Acknowledgements

I would like to thank:

My advisor Alexei Borodin, from whom I have learned how to be a mathematician, for giving me the opportunity to pursue research even as an undergraduate, and for continuing to mentor me throughout the years.

Everyone who has contributed discussions to the research, including Maurice Duits, Manon Defosseux, Ivan Corwin, Paul Bourgade, Alexey Bufetov and Philippe Biane.

My high school physics teacher David Derbes for encouraging me to learn physics and apply mathematics to other disciplines.

The late Paul J. Sally, Jr, who gave me the opportunity to take advanced mathematics courses at the University of Chicago while I was in high school.

My parents Sing–Ling and Shih–Fan for constantly nurturing my desire to learn mathematics, and for finding opportunities for me to pursue further education. My brother Kevin as well, for being a great younger brother and teaching me the value of patience.

My in–laws Kuei and Min, as well as my brother–in–law Sebastian, for their love, support and continued interest in my research.

My son Jayson who shows me there is more to life than math and my beautiful wife Yichen for her constant support though thick and thin, through good times and bad.

Last but not least, Christ, who strengthens me.

1. Introduction

One of the most central ideas in probability theory is that of a central limit theorem. This says that if $(X_i)_{1\leq i<\infty}$ are i.i.d. random variables with mean 0 and variance $\sigma^2<\infty$, then

$$\frac{X_1+\ldots+X_n}{\sqrt{n}}\xrightarrow{d}\mathcal{N}(0,\sigma^2)$$

Notice that there are several distinct features of this theorem. The first is that the statement does not depend on the distribution of X_i. The second is that the scaling factor grows in n as $\sqrt{n}$. The third is that the limiting distribution is normal.

There are other CLT–type theorems which occur in other contexts. There will again be a scaling factor and a limiting distribution which do not depend on the specific details of the model.

This thesis will prove several theorems about probabilistic limiting expressions using tools and objects from other branches of mathematics. The first section will describe a randomly growing stepped surface that arises from the represntation theory of Lie groups. Here, we will see the Gaussian free field and the symmetric Pearcey kernel in the limit. The second section will describe a non–commutative version of this growth process, which admits a generalization of the Gaussian free field. The third section will describe the strong Szegő asymptotics of the Riemann zeta function.

1.1. **Interacting Particles.**

Let us describe an interacting particle system introduced in [2]. Consider the two–dimensional lattice $\mathbb{Z}\times\mathbb{Z}_+$. On each horizontal level $\mathbb{Z}\times\{n\}$ there are exactly n particles, with at most one particle at each lattice site. Let $X_1^{(n)}>\ldots>X_n^{(n)}$ denote the x–coordinates of the locations of the n particles. Additionally, the particles need to satisfy the *interlacing property* $X_{i+1}^{(n+1)}<X_i^{(n)}\leq X_i^{(n+1)}$. The particles can be viewed as a random stepped surface, see Figure 15. This can be made rigorous by defining the height function $h(x,n,t)$ to be the number of particles to the right of (x,n) at time t.

The dynamics on the particles are as follows. The initial condition is the *densely packed* initial condition, $\lambda_i^{(n)}=-i+1, 1\leq i\leq n$. Each particle has a clock with exponential waiting

FIGURE 1. The particles as a stepped surface. The lattice is shifted to make the visualization easier.

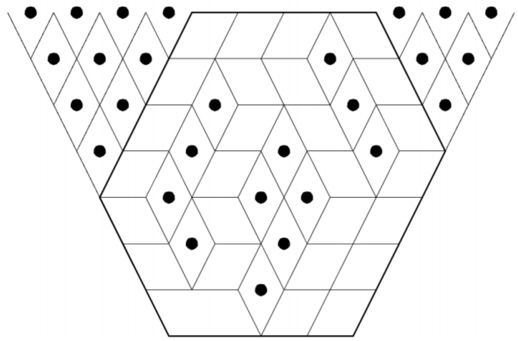
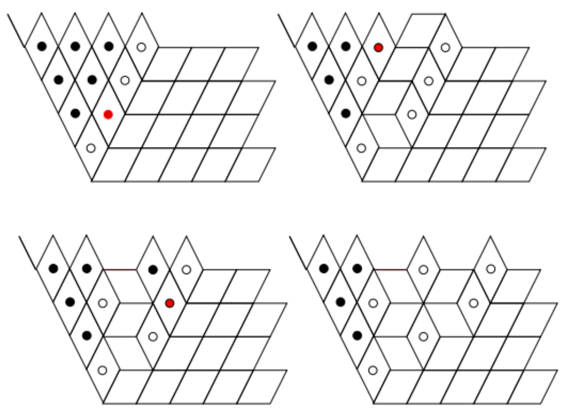


FIGURE 2. The red particle makes a jump. If any of the black particles attempt to jump, their jump is blocked by the particle below and to the right, and nothing happens. White particles are not blocked.



time of rate 1, with all clocks independent of each other. When the clock rings, the particle attempts to jump one step to the right. However, it must maintain the interlacing property. This is done by having particles push particles above it, and jumps are blocked by particles below it. One can think of lower particles as being more massive. See Figure 16 for an example.

This particle system is actually constructed from representation theory [3]. Consider $U(N)$, the group of $N \times N$ unitary matrices. It is a classical result that its characters are Schur polynomials s_λ indexed by the set of $\lambda \in \mathbb{GT}_N := \{(\lambda_1 \geq \ldots \geq \lambda_N) : \lambda_i \in \mathbb{Z}\}$. If $\chi_t(U)$ is the function on $U(N)$ defined by $\exp(t \cdot \mathrm{Tr}(U - \mathrm{Id}))$, then it decomposes as $\chi_t(U) =$

$\sum_{\lambda\in\mathbb{GT}_N} P_N(\lambda)\frac{s_\lambda(U)}{\dim\lambda}$, where $\dim\lambda$ is the dimension of the representation corresponding to λ. It turns out that $P_N(\lambda)$ is a probability measure on $\mathbb{GT}_N$. Under the map $\lambda_i \mapsto \lambda_i - i$, this probability measure pushes forward to the measure on $X^{(N)}$ at time t. In other words,

$$P_N(\lambda) = \mathbb{P}\left(X_i^{(N)} = \lambda_i - N + 1, \quad 1 \le i \le N\right)$$

More generally,

$$\mathbb{P}\left(X_i^{(n)} = \lambda_i^{(n)} - n + 1, 1 \le n \le N, 1 \le i \le n\right) = \frac{P_N(\lambda^{(N)})}{\dim\lambda^{(N)}}$$

This interacting particle system is actually a very special type of process, called a *determinantal point process*, which allows for a computation of asymptotics. Here, this means that there exists an expression $K(x_1, n_1, t_1; x_2, n_2, t_2)$ called the *correlation kernel* such that

$$\mathbb{P}\,(\text{there is a particle at } (x_i, n_i) \text{ at time } t_i, 1 \le i \le k) = \det\left[K(x_i, n_i, t_i, x_j, n_j, t_j)\right]_{1\le i,j\le k}$$

as long as $n_i \le n_j$ whenever $t_i \ge t_j$. This condition is called a *space–like* path. On a discrete space, the entire determinantal point process can be extracted from the correlation kernel.

There are several ways to find universal asymptotic behavior in this particle system using the correlation kernel. Here, we will focus on two in particular. Let x_i, n_i, t depend on a parameter L in such a way that

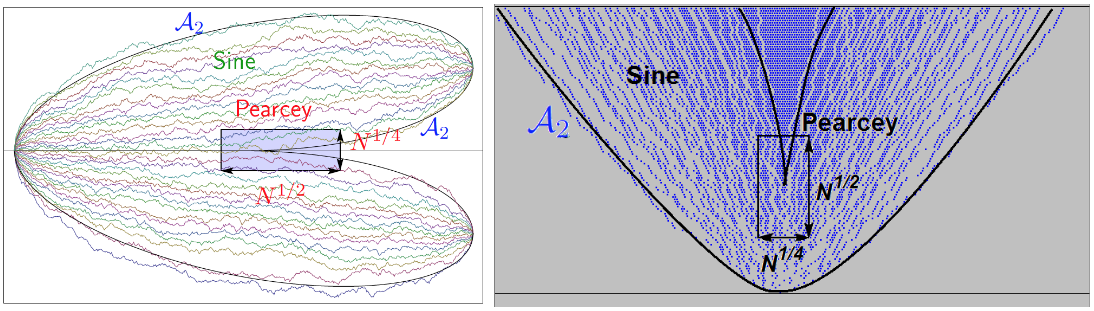
$$n_i - L \sim \eta_i L^{1/2}, \quad x_i - L - c_0 L^{1/2} \sim \nu_i L^{1/4}, \quad t \sim \tau L$$

Then it is shown in [3] that as $L \to \infty$,

$$\det[L^{1/4} K(x_i, n_i, t, x_j, n_j, t)]_{1\le i,j\le k} \to \det[P(\eta_i, \nu_i, \tau, \eta_j, \nu_j, \tau)]$$

where P is the *Pearcey kernel.* This has appeared in a number of previous works, such as [1, 4, 5, 10, 11] This is illustrated in Figure 3. We also show nonintersecting Brownian motion to demonstrate another model with the same limiting behavior.

FIGURE 3. The image on the left is nonintersecting Brownian paths. The image on the right is the particle system.



Another regime is in the height function, where we see the Gaussian free field. Let $\varkappa_j = (\nu_j, \eta_j, \tau_j)$ such that

$$\tau_1 \leq \tau_2 \leq \cdots \leq \tau_k, \quad \eta_1 \geq \eta_2 \geq \cdots \geq \eta_k.$$

Denote

$$H_L(\nu,\eta,\tau) := \sqrt{\pi}\left(h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)-\mathbb{E}h([(\nu-\eta)L]+\frac{1}{2},[\eta L],\tau L)\right).$$

Then

$$\lim_{L\to\infty}\mathbb{E}\left(H_L(\varkappa_1)\cdots H_L(\varkappa_k)\right)=\mathbb{E}\left(\mathrm{GFF}(\varkappa_1)\cdots\mathrm{GFF}(\varkappa_k)\right),$$

where GFF is the Gaussian free field. (see [?IS] for a description of the Gaussian free field). When viewed as a randomly growing surface, this process lies in the Anisotropic KPZ universality class.

What are some ways of extending these results? My thesis goes in two different directions. One is to look at slightly modified particle systems arising from different Lie groups, the second is to try to find the full three–dimensional limiting field (that is, without assuming space–like paths).

1.2. **With a wall.** Consider the construction with the orthogonal groups instead of the unitary groups. The resulting interacting particle system lives on the lattice $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_+$. There are now $\lfloor\frac{n+1}{2}\rfloor$ particles on the nth level. The particles still interlace and obey the push–block dynamics, but now the particle jumps to the right with probability 1/2 and to the left with probability 1/2. If the particle is located at a lattice point with horizontal co–ordinate 0 and attempts to jump to the left, then the particle reflects off the wall and jumps to the right. In other words, $\{0\} \times \mathbb{Z}_+$ acts as a reflecting wall.

Due to the wall, we now expect a different limiting process instead of the Pearcey process. Indeed, it was shown in CITE that with the same scaling limits, one gets the so–called *symmetric Pearcey process*. This also appears in non–intersecting squared Bessel paths near the hard edge at 0. A discrete–time interacting particle system was constructed in CITE using Pieri's rule for the orthogonal groups. In section 3, I show that the asymptotic limits of this particle system near the critical point at the wall is again the symmetric Pearcey process.

We could also ask if the Gaussian free field appears as the limit of the fluctuations of the height function. In section 2, I show that this is indeed the case.

1.3. **Noncommutative probability.** In order to determine the three–dimensional Gaussian field which generalizes the Gaussian free field, we construct the particle system using non–commutative probability theory. Instead of considering a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider an algebra $\mathcal{A}$ of $\mathcal{F}$–measurable functions on Ω. The probablity measure is now a linear functional ω on $\mathcal{A}$ defined by $\omega(X) = \mathbb{E}_{\mathbb{P}}[X]$. In this construction, the algebra $\mathcal{A}$ is necessarily commutative. This motivates the definition of a non–commutative probability space as a unital (not necessarily commutative) algebra $\mathcal{A}$ and a linear functional ω on $\mathcal{A}$ such that $\omega(1) = 1$ and $\omega(XY) = \omega(YX)$. A Markov chain on a non–commutative probability space is then a semigroup P_t of operators on $\mathcal{A}$. Additionally, there is a family of linear functionals ω_t, representing the expectation of a random variable at time t, such that $\omega_s(P_t(X)) = \omega_{t+s}(X)$.

Here, consider the algebra $\mathcal{U}(\mathfrak{gl}_N)$, the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_N$. This is isomorphic to the algebra of left–invariant differential operators on the Lie group $U(N)$. For each $X \in \mathcal{U}(\mathfrak{gl}_N)$, let D_X be the corresponding differential operator. The states ω_t are defined by

$$\omega_t(X) = (D_X \chi_t)(\mathrm{Id}), \text{ where } \chi_t(U) = e^{t \cdot \mathrm{Tr}(U - \mathrm{Id})}.$$

The non–commutative Markov chain P_t is defined by

$$P_t(X) = (\mathrm{id} \otimes \omega_t)(\Delta(X)),$$

where Δ denotes the co–product.

This is related to the interacting particles in the following way. The Harish–Chandra isomorphism maps the center $Z = Z(\mathcal{U}(\mathfrak{gl}_N))$ to the algebra of shifted symmetric polynomials in N variables, i.e. polynomials which are symmetric in the variables $\lambda_i - i, 1 \leq i \leq N$. The restriction of P_t to Z is still Markov in the sense that $P_t Z \subset Z$. This means that Z can be identified with an algebra of observables on the set $\mathbb{GT}_N$. It turns out that the restriction of ω_t to $\mathbb{Z}$ is exactly the probability measure $P_N(\cdot)$. Furthermore, the restriction of P_t to Z is the generator of the corresponding dynamics on the Nth level.

We also have the natural inclusion $\mathcal{U}(\mathfrak{gl}_{N-1}) \subset \mathcal{U}(\mathfrak{gl}_N)$. This allows to use ω_t to compute expectations observables on multiple levels. Doing so allows to to compute the full three–dimensional Gaussian field. This is the content of section 5

It is also shown in section 4 that a similar construction can be done for the group von Neumann algebra of $U(N)$.

1.4. **Riemann Zeta and random matrices.** Another source of universal limiting processes comes from number theory. The connections between the zeros of the Riemann zeta function and eigenvalues of random matrices go back to Montgomery [9] and Dyson, who conjectured that the aforementioned sine process should also appear in the asymptotic distribution of the zeros of ζ along the critical axis.

Here, we considering another observable, which are linear statistics in the mesoscopic regime. If $x_1 < \ldots < x_N$ are the eigenvalues of a random $N \times N$ Gaussian Hermitian matrix with average spacing of order N^{-1}, and f is a sufficiently nice test function, consider the random variable $\sum_{j=1}^{N} f(\lambda_N x_N)$, where λ_N is a scaling parameter satisfying $1 \ll \lambda_N \ll N$. The asymptotics as $N \to \infty$ yield the *strong Szego limit theorem*, and the resulting distribution depends only on the $H^{1/2}$ norm of f. The same distribution appears in several other random matrix models (e.g. [6–8]).

We can also consider linear statistics of ζ in the mescoscopic regime. The classical Riemann–von Mangoldt formula implies that the average spacing of the zeroes of ζ at height T are of order $2\pi/\log T$. Thus, let λ_t be in the mescoscopic scaling, or in other words, a parameter such that $1 \ll \lambda_t \ll \log t$. For a suitable test function f, consider the random variable $S_t(f) := \sum_\gamma f(\lambda_t(\gamma - \omega t))$ where the sum is over γ such that $\frac{1}{2} + i\gamma$ is a non–trivial zero of ζ and ω is a uniform random variable on $(1, 2)$.

We would then expect the limit of $S_t(f) - \mathbb{E}S_t(f)$ to match the limiting field from random matrices. Indeed, it is proved in section 6 that assuming the Riemann hypothesis, we obtain the same strong Szegő theorem. The main tool in the proof is to use Helffer-Sjőstrand formula applied to ζ'/ζ, which is analogous to the Stieltjes transform of the empirical measure of eigenvalues of random matrices.

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2. Gaussian free field in interlacing particle systems

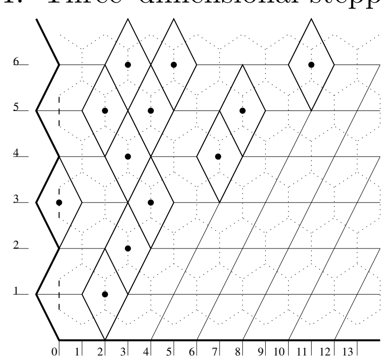
Abstract. We show that if an interlacing particle system in a two-dimensional lattice is a determinantal point process, and the correlation kernel can be expressed as a double integral with certain technical assumptions, then the moments of the fluctuations of the height function converge to that of the Gaussian free field. In particular, this shows that a previously studied random surface growth model with a reflecting wall has Gaussian free field fluctuations.

2.1. **Introduction.**

We begin by describing a particle system which was introduced in [4].

Particle System. Introduce coordinates on the plane as shown in Figure 4. Denote the horizontal coordinates of all particles with vertical coordinate m by $y_1^m > y_2^m > \cdots > y_k^m$, where $k = \lfloor (m+1)/2 \rfloor$. There is a wall on the left side, which forces $y_k^m \geq 0$ for m odd and $y_k^m \geq 1$ for m even. The particles must also satisfy the interlacing conditions $y_{k+1}^{m+1} < y_k^m < y_k^{m+1}$ for all meaningful values of k and m.

Figure 4. Three–dimensional stepped surface



By visually observing Figure 4, one can see that the particle system can be interpreted as a stepped surface. We thus define the height function at a point to be the number of particles to the right of that point.

Define a continuous time Markov chain as follows. The initial condition is a single particle configuration where all the particles are as much to the left as possible, i.e. $y_k^m = m - 2k + 1$ for all k, m. This is illustrated in the left-most iamge in Figure 5. Now let us describe the evolution. We say that a particle y_k^m is blocked on the right if $y_k^m + 1 = y_{k-1}^{m-1}$, and it is blocked on the left if $y_k^m - 1 = y_k^{m-1}$ (if the corresponding particle y_{k-1}^{m-1} or y_k^{m-1} does not exist, then y_k^m is not blocked).

Each particle has two exponential clocks of rate $\frac{1}{2}$; all clocks are independent. One clock is responsible for the right jumps, while the other is responsible for the left jumps. When the clock rings, the particle tries to jump by 1 in the corresponding direction. If the particle is blocked, then it stays still. If the particle is against the wall (i.e. $y^m_{[\frac{m+1}{2}]} = 0$) and the left jump clock rings, the particle is reflected, and it tries to jump to the right instead.

When y_k^m tries to jump to the right (and is not blocked on the right), we find the largest $r \in \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$ such that $y_k^{m+i} = y_k^m + i$ for $0 \leq i \leq r$, and the jump consists of all particles $\{y_k^{m+i}\}_{i=0}^r$ moving to the right by 1. Similarly, when y_k^m tries to jump to the left (and is not blocked on the left), we find the largest $l \in \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$ such that $y_{k+j}^{m+j} = y_k^m - j$ for $0 \leq j \leq l$, and the jump consists of all particles $\{y_{k+j}^{m+j}\}_{j=0}^l$ moving to the left by 1.

In other words, the particles with smaller upper indices can be thought of as heavier than those with larger upper indices, and the heavier particles block and push the lighter ones so that the interlacing conditions are preserved.

FIGURE 5. First three jumps

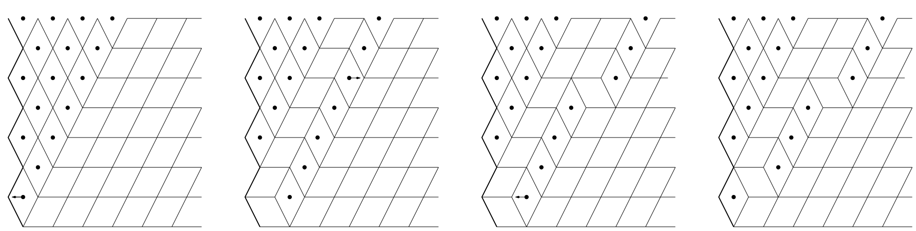


Figure 5 depicts three possible first jumps: Left clock of y_1^1 rings first (it gets reflected by the wall), then right clock of y_1^5 rings, and then left clock of y_1^1 again.

In terms of the underlying stepped surface, the evolution can be described by saying that we add possible "sticks" with base 1×1 and arbitrary length of a fixed orientation with rate

1/2, remove possible "sticks" with base 1×1 and a different orientation with rate 1/2, and the rate of removing sticks that touch the left border is doubled.[1]

A computer simulation of this dynamics can be found at
`http://www.math.caltech.edu/papers/Orth_Planch.html`.

This particle system has important connections to the representation theory of the orthogonal groups, to the Kardar–Parisi–Zhang equation from mathematical physics, and to random lozenge tilings. The interested eader is referred to the introduction of [4].

Limit shape A very natural question about this random surface is to ask if it satisfies a law of large numbers and central limit theorem. In other words, in the large N limit, the random surface should converge to a deterministic limit shape, and the fluctuations around this limit shape should be a reasonably nice object. This paper will prove that the flucutations are described by the Gaussian free field, but first let us describe the limit shape, which was proved in Proposition 5.6 of [4].

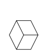
Let $H(x, n, t)$ denote the height function, i.e. the number of particles to the right of (x, n) at time t. Define h to be

$$h(\nu, \eta, \tau) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E} H(\nu N, \lfloor \eta N \rfloor, \tau N).$$

Thus, h describes the deterministic limit shape. It can be described explicitly as follows. Let $G(z) = G(\nu, \eta, \tau; z)$ be the function

$$G(\nu, \eta, \tau; z) = \tau \frac{z + z^{-1}}{2} + \eta \log \left(\frac{z + z^{-1}}{2} - 1 \right) - \nu \log z. \tag{2.1}$$

There is an explicit (in the sense that it can be written in terms of algebraic functions) connected domain $\mathcal{D}$ consisting of all triples (ν, η, τ) such that $G(\nu, \eta, \tau; z)$ has a unique critical point in the region $\mathbb{H} - \mathbb{D} = \{z : \Im z > 0 \text{ and } |z| > 1\}$. This induces a map

[1]This phrase is based on the convention that  is a figure of a $1 \times 1 \times 1$ cube. If one uses the dual convention that this is a cube-shaped hole then the orientations of the sticks to be added and removed have to be interchanged, and the tiling representations of the sticks change as well.

$\Omega : \mathcal{D} \to \mathbb{H} - \mathbb{D}$ by sending (ν, η, τ) to the critical point of $G(z)$. Then

$$h(\nu, \eta, \tau) = \Im\left(\frac{S(\Omega(\nu, \eta, \tau))}{2\pi}\right).$$

Outside of $\mathcal{D}$, the limit shape is trivial – that is, if ν is too large, then there are no particles to the right of $(\nu N, \eta N)$ at time τN, so the height function is zero. If η is too small, then all the particles are to the right of $(\nu N, \eta N)$ at time tN, so the height function is $\eta/2$. In the literaure, $\mathcal{D}$ is called the *liquid region* and the triples (ν, η, τ) outside of $\mathcal{D}$ is called the *frozen region*.

Gaussian free field fluctuations In order to describe the fluctuations, let us review the Gaussian free field. A comprehensive survey can be found in [13]. The Gaussian free field is a Gaussian probability measure on a suitable class of distributions on a domain $D \subset \mathbb{R}^d$. More precisely, given compactly supported smooth test functions $\{\phi_m\}_{m=1}^{\infty}$, the random variables $\{\mathrm{GFF}(\phi_m)\}_{m=1}^{\infty}$ are mean zero Gaussians with covariance

$$\mathbb{E}[\mathrm{GFF}(\phi_{m_1})\mathrm{GFF}(\phi_{m_2})] = \int_{D\times D} \phi_{m_1}(z_1)\phi_{m_2}(z_2)\mathcal{G}(z_1, z_2)dz_1dz_2, \tag{2.2}$$

where $\mathcal{G}(z_1, z_2)$ is the Green function for the Laplacian on D with Dirichlet boundary conditions.

Formally, one could attempt to set $\phi_m = \delta_{z_m}$ for $z_m \in D$ in order to define the Gaussian free field at a point. However (2.2) would imply that $\mathrm{GFF}(z)$ has variance $\mathcal{G}(z, z)$, which is undefined for $d \geq 2$. However, for pairwise distinct points $z_1, \ldots, z_k$ one expects from Wick's theorem

$$\mathbb{E}[\mathrm{GFF}(z_1)\ldots\mathrm{GFF}(z_k)] = \begin{cases} \sum_\sigma \prod_{i=1}^{k/2} \mathcal{G}(z_{\sigma(2i-1)}, z_{\sigma(2i)}), & \text{k even} \\ 0, & \text{k odd} \end{cases}$$

where the sum is over all fixed point free involutions σ on $\{1, \ldots, k\}$. This can be made into a rigorous statement:

$$\mathbb{E}[\mathrm{GFF}(\phi_1) \ldots \mathrm{GFF}(\phi_k)] = \int_{D^n} \mathbb{E}[\mathrm{GFF}(z_1) \ldots \mathrm{GFF}(z_k)] \prod_{i=1}^{k} \phi_i(z_i) dz_i.$$

Furthermore, these moments uniquely determine the Gaussian free field.

Theorem 2.1. *Let $\varkappa_j = (\nu_j, \eta_j, \tau) \in \mathcal{D}$ for $1 \leq j \leq k$. Define*

$$H_N(\nu, \eta, \tau) := \frac{1}{N} \left(H(\nu N, \lfloor \eta N \rfloor, \tau N) - \mathbb{E} H(\nu N, \lfloor \eta N \rfloor, \tau N) \right)$$

and let $\Omega_j = \Omega(\varkappa_j)$. Then

$$\lim_{N \to \infty} \mathbb{E}(H_N(\varkappa_1) \cdots H_N(\varkappa_k)) = \begin{cases} \sum_\sigma \prod_{i=1}^{k/2} \mathcal{G}(\Omega_{\sigma(2i-1)}, \Omega_{\sigma(2i)}), & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

where the sum is over all fixed point free involutions σ on $\{1, \ldots, k\}$ and $\mathcal{G}$ is the Green's function for the Laplacian on $\mathbb{H} - \mathbb{D}$ with Dirichlet boundary conditions:

$$\mathcal{G}(z, w) = \frac{1}{2\pi} \log \left(\frac{z + z^{-1} - \bar{w} - \bar{w}^{-1}}{z + z^{-1} - w - w^{-1}} \right). \tag{2.3}$$

Idea of proof and generalization The proof uses a very specific property of the interacting particle system, namely that it is a *determinantal point process*. There are several previous examples of determinantal point processes having Gaussian free field fluctuations [2, 6, 7, 12]. (See also [9]). The essential idea in these proofs is similar. One takes an explicit formula for the correlation kernel $K(x, y)$, and then asymptotic analysis on $K(x, x)$ provides information about the limit shape while asymptotics of $K(x, y), x \neq y$ provides information about the fluctuations. In [4], an explicit formula for the correlation kernel was proved, enabling steepest descent analysis.

It is thus natural to ask: given a determinantal point process with an explicit correlation kernel, is there a general statement that the fluctuations of the height function are governed by the Gaussian free field? The answer is yes.

Theorem 2.2. *Suppose we are given a particle system on $\mathbb{Z}\times\mathbb{Z}_{\geq 1}$ which is a determinantal point process with correlation kernel*

$$K(x_1,n_1,x_2,n_2,t)\approx\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_1}\int_{\Gamma_2}\frac{\exp(NG(\nu_1,\eta_1,\tau,u))}{\exp(NG(\nu_2,\eta_2,\tau,w))}f(u,w)dwdu,$$

where Γ_1,Γ_2 are steepest descent paths. We make certain technical assumptions about K (see Definition below).

Let $\mathcal{D}\subset\mathbb{R}^3$ be the liquid region and let $\Omega:\mathcal{D}\to\mathbb{C}$ send (ν,η,τ) to the critical point of $G(\nu,\eta,\tau,z)$. If H_N denotes the scaled and centered random height function of the particle system, then for $\varkappa_1,\ldots,\varkappa_k\in\mathcal{D}$ with $\Omega_j=\Omega(\varkappa_j)$

$$\mathbb{E}[H_N(\varkappa_1)\ldots H_N(\varkappa_k)]\to\begin{cases}\sum_\sigma\prod_{j=1}^{k/2}\mathcal{G}(\Omega_{\sigma(j)},\Omega_{\sigma(j+1)}), & k \text{ even}\\ 0, & k \text{ odd},\end{cases}$$

where

$$\mathcal{G}(z,w)=\frac{1}{2\pi}\int_{\bar z}^{z}\int_{\bar w}^{w}\frac{f(u,v)f(v,u)}{G'_\nu(u)G'_\nu(v)}dudv, \tag{2.4}$$

with G'_ν denoting $(\partial^2/\partial\nu\partial z)G$.

The rigorous details are in Section 2.2. In particular, the formula for $\mathcal{G}$ in (2.3) follows from (2.4) with S as in (2.1) and

$$f(u,v)=\frac{1}{v}\frac{1-u^{-2}}{v+v^{-1}-u-u^{-1}}.$$

Outline of paper In section 2.2.1, we state precisely the assumptions on the determinantal point process, as well as explain why these assumptions are natural. In sections 2.2.2 and 2.2.3, we prove Theorem 2.4. In section 2.3, we show that Theorem 2.1 follows once we prove that the interacting particle system with a reflecting wall satisfies the necessary technical assumptions. In section 2.3.2 and 2.3.3, we show that the necessary technical assumptions indeed hold. Section 4 collects the asymptotic analysis needed throughout the proofs.

2.2. **General Results.**

2.2.1. *Statement of the Main Theorem.* Suppose we have a family of point processes on $\mathfrak{X} = \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ which runs over time $t \in [0, \infty)$. (Note that these are different co-ordinates from the introduction). In other words, at any time t, the system selects a random subset $X \subset \mathfrak{X}$. If $(x, n) \in X$, then we say that there is a particle at (x, n). For any $k \geq 1$ and $t \geq 0$, let $\rho_k^t : \mathfrak{X}^k \to [0, 1]$ be defined by

$$\rho_k^t(x_1, n_1, \dots, x_k, n_k)$$
$$= \mathbb{P}(\text{There is a particle at } (x_j, n_j) \text{ at time } t \text{ for each } j = 1, \dots, k).$$

Assume that there is a map K on $\mathfrak{X} \times \mathfrak{X} \times [0, \infty)$ such that

$$\rho_k^t(x_1, n_1, \dots, x_k, n_k) = \det[K(x_i, n_i, x_j, n_j, t)]_{1 \leq i,j \leq k}. \tag{2.5}$$

The maps ρ_k and K are called the *kth correlation function* and the *correlation kernel*, respectively.

A function c on $\mathfrak{X} \times \mathfrak{X}$ is called a *conjugating factor* if there exists another function $\mathcal{C}$ on $\mathfrak{X}$ such that

$$c(x, n, x', n') = \frac{\mathcal{C}(x, n)}{\mathcal{C}(x', n')}.$$

Note that if c is a conjugating factor, then

$$\det[K(x_i, n_i, x_j, n_j, t)]_{1 \leq i,j \leq k} = \det[c(x_i, n_i, x_j, n_j) K(x_i, n_i, x_j, n_j, t)]_{1 \leq i,j \leq k}. \tag{2.6}$$

Two kernels K and $\tilde{K}$ are called *conjugate* if $\tilde{K} = cK$ for some conjugating factor c.

If a correlation kernel exists, the point process is called *determinantal*. On a discrete space, a point process is determined uniquely by its correlation functions (see e.g. [10]). Therefore, if we are given two determinantal point process on a discrete space with conjugate kernels, they must have the same law.

The set $\mathbb{Z} \times \{n\}$ is called the *nth level*. Given a subset $X \subset \mathfrak{X}$, let m_n be the cardinality of the set $X \cap (\mathbb{Z} \times \{n\})$. Assume that the numbers m_n take constant finite values which are independent of the time parameter t. In words, this means that the number of particles on the nth level is always m_n. Further assume that $m_n \leq m_{n+1} \leq m_n + 1$ for all n. Let $x_1^{(n)} > x_2^{(n)} > \ldots > x_{m_n}^{(n)}$ denote the elements of $X \cap (\mathbb{Z} \times \{n\})$. A subset X is called *interlacing* if

$$x_{k+1}^{(n+1)} < x_k^{(n)} \leq x_k^{(n+1)}, \quad \text{when} \quad m_{n+1} = m_n,$$

$$x_{k+1}^{(n+1)} \leq x_k^{(n)} < x_k^{(n+1)}, \quad \text{when} \quad m_{n+1} = m_n + 1.$$

Assume that at any time t, the system almost surely selects an interlacing subset. Let δ_n equal $m_{n+1} - m_n$.

Define the random *height function* by

$$h : \mathfrak{X} \times \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0},$$

$$h(x, n, t) = |\{(s, n) \in X : s > x\}|.$$

In words, h counts the number of particles to the right of (x, n) at time t.

We wish to study the large-time asymptotics of this particle system. Let $x = [N\nu], n = [N\eta], t = N\tau$, where N is a large parameter. Define $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ to be

$$\mathcal{D} := \{(\nu, \eta, \tau) : \lim_{N \to \infty} \rho_1^t(x, n) > 0\}.$$

Let H_N be defined by

$$H_N : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},$$

$$H_N(\nu, \eta, \tau) := h(x, n, t) - \mathbb{E}h(x, n, t).$$

In words, H_N is the fluctuation of the height function around its expectation.

Before stating the theorem, we need to state some more assumptions on the kernel.

Suppose the kernel K is conjugate to a kernel $\tilde{K}$ such that $\tilde{K}$ satisfies the following property: There is a number L such that whenever $x, x' \geq L$,

$$\tilde{K}(x,n,x',n',t) + \tilde{K}(x,n,x'-1+\delta_n,n'+1,t) + \tilde{K}(x,n,x'+\delta_n,n'+1,t) = \begin{cases} 1, & (x,n)=(x',n') \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

$$\tilde{K}(x,n,x',n',t) + \tilde{K}(x+1-\delta_n,n-1,x',n',t) + \tilde{K}(x-\delta_n,n-1,x',n',t) = \begin{cases} 1, & (x,n)=(x',n') \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Further suppose that for $x', x'' > L$,

$$\tilde{K}(x,n,x',n',t)\tilde{K}(x'',n'',x-1+\delta_n,n+1,t) \to 0 \text{ as } x \to \infty \tag{2.9}$$

$$\tilde{K}(x,n,x',n',t)\tilde{K}(x'',n'',x+\delta_n,n+1,t) \to 0 \text{ as } x \to \infty \tag{2.10}$$

$$\tilde{K}(x,n,x-1+\delta_n,n+1,t) \to 0 \text{ as } x \to \infty \tag{2.11}$$

$$\tilde{K}(x,n,x+\delta_n,n+1,t) \to 1 \text{ as } x \to \infty. \tag{2.12}$$

Suppose $G(\nu,\eta,\tau,z)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{C}$. To save space, we will sometimes write $G(z)$. Expressions such as G', G_ν, G'_ν will be shorthand for $\partial G/\partial z$, $\partial G/\partial \nu$ and $\partial^2 G/\partial z \partial \nu$, respectively. Assume $G(\overline{z}) = \overline{G(z)}$.Also suppose there exists a differentiable map Ω from $\mathcal{D}$ to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ such that Ω is a critical point of G. In other words,

$$G'(\nu,\eta,\tau,\Omega(\nu,\eta,\tau)) = 0 \quad \text{for all} \quad (\nu,\eta,\tau) \in \mathcal{D}. \tag{2.13}$$

Note that Ω need not be onto. For any (η, τ), if the set $\{\nu \in \mathbb{R} : (\nu, \eta, \tau) \in \mathcal{D}\}$ is nonempty, let $q_2(\eta, \tau)$ denote its supremum.

Definition 2.3. With the notation above, a determinantal point process on $\mathbb{Z} \times \mathbb{Z}_{\geq 1}$ is *normal* if all of the following hold:

- For all $\eta, \tau > 0$, the limit $\Omega(q_2(\eta, \tau) - 0, \eta, \tau)$ exists and is a positive real number.
- For all $\eta, \tau > 0$, as ν approaches $q_2(\eta, \tau)$ from the left, $G''(\nu, \eta, \tau, \Omega(\nu, \eta, \tau)) = \mathcal{O}((q_2(\eta, \tau) - \nu)^{1/2})$.
- K is conjugate to some $\tilde{K}$ such that (2.7)-(2.12) hold for some integer L.
- Set $t = N\tau$, $x_j = [N\nu_j]$ and $n_j = [N\eta_j]$ for $j = 1, 2$, where $(\nu_j, \eta_j, \tau) \in \mathcal{D}$. Let Ω_j denote $\Omega(\nu_j, \eta_j, \tau)$ and let $G_j(z)$ denote $G(\nu_j, \eta_j, \tau, z)$. If $\Omega_1 \neq \Omega_2$ and k_1,k_2 are finite integers, then as $N \to \infty$,

$$\tilde{K}(x_1, n_1 + k_1, x_2, n_2 + k_2, t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{\exp(NG(\nu_1, \eta_1, \tau, z))}{\exp(NG(\nu_2, \eta_2, \tau, w))} f_{k_1 k_2}(u, w) dw du + \mathcal{O}(e^{N\kappa}), \tag{2.14}$$

  where Γ_1 and Γ_2 are steepest descent paths, $\kappa < \Re(G_1(\Omega_1) - G_2(\Omega_2))$, and $f_{k_m k_n}$ are complex-valued meromorphic functions satisfying the identity

$$f_{k_1 k_2}(z_1, z_2) f_{k_2 k_3}(z_2, z_3) \ldots f_{k_{r-1} k_r}(z_{r-1}, z_r) f_{k_r k_1}(z_r, z_1) = f(z_1, z_2) f(z_2, z_3) \ldots f(z_{r-1}, z_r) f(z_r, z_1).$$

  Here, we have written f for f_{00}.
- For any $l \geq 3$, the following indefinite integral satisfies

$$\int \cdots \int \sum_{\sigma} \prod_{i=1}^{l} \frac{f(z_{\sigma(i)}, z_{\sigma(i+1)})}{G'_\nu(z_{\sigma(i)})} dz_i \equiv 0, \tag{2.15}$$

  where the sum is taken over all l-cycles in S_l and the indices are taken cyclically.
- For any finite interval $[a, b]$, $G \in C^2[a, b]$ and the Lesbesgue measure of the set $\{x \in [a, b] : I'(x) \in 2\pi\mathbb{Z} + [-\delta, \delta]\}$ is $\mathcal{O}(\delta^a)$ for some positive a.

The following remarks will help explain the definition.

Remark 2.4. (1) The assumption that $\Omega(q_2(\eta,\tau)-0,\eta,\tau)>0$ occurs naturally. One often finds that for $k_1 \neq k_2 \in \mathbb{Z}$,

$$\lim_{N\to\infty} K([N\nu]+k_1,[N\eta],[N\nu]+k_2,[N\eta],N\tau)=\frac{1}{2\pi i}\int_{\bar{\Omega}}^{\Omega}\frac{dz}{z^{k_1-k_2+1}}=\frac{\Im(\Omega^{k_2-k_1})}{\pi(k_2-k_1)},$$

where the contour crosses the positive real line. By setting $\Omega=e^{i\varphi}$, we see that the right hand side reduces to the ubiquitous sine kernel. When $k_1=k_2=0$, we see that

$$\lim_{N\to\infty}\rho_1^{N\tau}([N\nu],[N\eta])=\frac{1}{2\pi}(\log\Omega-\log\bar{\Omega})=\frac{\arg\Omega(\nu,\eta,\tau)}{\pi}.$$

Since the left hand side equals zero, we expect $\arg\Omega(q_2(\eta,\tau)-0,\eta,\tau)=0$.

(2) Since $G(\overline{z})=\overline{G(z)}$, this means that G has a critical point at both Ω and $\bar{\Omega}$. As Ω approaches the real line, the two critical points coalesce into a triple zero, so $G''(t,\eta,\tau)$ converges to 0 as t approaches $q_2(\eta,\tau)$. We need a control for how quickly this convergence to 0 occurs, in order to order to control the behavior near the boundary of $\mathcal{D}$. More specifically, it controls the bound in Proposition 2.23.

There is a heuristic understanding for why (2) should hold. The function G has two critical points which coalesce into a triple zero. The simplest example of such a function is $G(t,x)=x^3/3-tx$ as t approaches 0. In this case, the solution to $G'(t,x)=0$ is $\Omega(t,x)=t^{1/2}$. Then $G''(t,\Omega(t,x))=2t^{1/2}$.

(3) Assumptions (2.7)–(2.12) will be elucidated when we interpret the particles as lozenges. In particular, see remark 2.8.

(4) It is common for the kernel to be expressed in this form; previous examples are [4] and [2]. If the kernel has a different expression with the same asymptotics as in Propositions 2.19 and 2.23, the results still hold.

(5) In particular, (2.15) holds if there always exist u-substitutions and an expression Y such that

$$\int\cdots\int\prod_{i=1}^{l}\frac{f(z_i,z_{i+1})}{G'_{\nu_1}(z_i)}dz_i=\int\cdots\int\prod_{i=1}^{l}\frac{1}{Y(u_i)-Y(u_{i+1})}du_i,$$

where $z_{l+1} = z_1$ and $u_{l+1} = u_1$. This is because of Lemma 7.3 in [9], which refers back to [5], which says that

$$\sum_\sigma \prod_{i=1}^{l} \frac{1}{Y(u_{\sigma(i)}) - Y(u_{\sigma(i+1)})} = 0.$$

(6) This is a technical lemma which allows Lemma 2.16 to be applied.

We can now state the main theorem.

Theorem 2.5. *Suppose we are given a normal determinantal point process. For $1 \le j \le k$, let $\varkappa_j = (\nu_j, \eta_j, \tau)$ be distinct points in $\mathcal{D}$, and let $\Omega_j = \Omega(\nu_j, \eta_j, \tau)$. Define the function $\mathcal{G}$ on the upper half-plane to be*

$$\mathcal{G}(z,w) = \left(\frac{1}{2\pi}\right)^2 \int_{\bar{z}}^{z} \int_{\bar{w}}^{w} \frac{f(z_1,z_2)f(z_2,z_1)}{G'_\nu(z_1)G'_\nu(z_2)} dz_2 dz_1$$

Then

$$\lim_{N\to\infty} \mathbb{E}(H_N(\varkappa_1)\cdots H_N(\varkappa_k)) = \begin{cases} \displaystyle\sum_{\sigma\in\mathcal{F}_k} \prod_{j=1}^{k/2} \mathcal{G}(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & k \text{ is even} \\ 0, & k \text{ is odd,} \end{cases}$$

where $\mathcal{F}_k$ is the set of all involutions in S_k without fixed points.

Remark 2.6. We note that these are the moments of a linear family of Gaussian random variables: see Appendix A. Using the results of [14], it should be possible to show that $H_N(\varkappa)/\sqrt{\mathrm{Var} H_N(\varkappa)}$ converges to a Gaussian, but this was not pursued.

2.2.2. *Algebraic steps in proof of Theorem 3.1.* The most natural way to view $\mathfrak{X}$ is as a square lattice. However, it turns out that a hexagonal lattice is more useful. To obtain the hexagonal lattice, take the nth level and shift it to the right by $(n+1)/2 - m_n$. See Figure 6.

Figure 6 also shows that the particle system can be interpreted as lozenges. Each lozenge is a pair of adjacent equilateral triangles. See Figure 7.

By setting the location of each triangle to be the midpoint of its horizontal side, each lozenge can be viewed as a pair (x,n,x',n'), where the black triangle is located at (x,n)

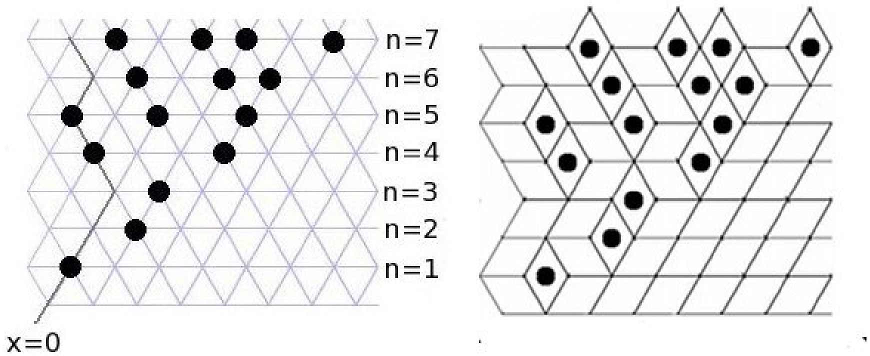


Figure 6. In this example, the integers m_n equal $1, 1, 1, 2, 3, 3, 4 \ldots$. The black line on the left represents the points where $x = 0$. Examples of $x_k^{(n)}$ are $x_1^{(3)} = 1, x_1^{(4)} = 3, x_2^{(7)} = 4$.

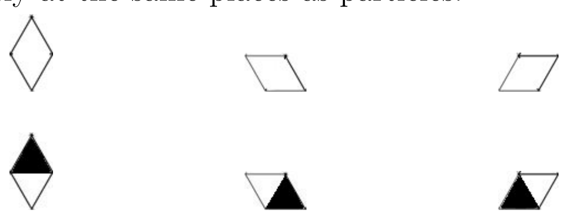


Figure 7. Lozenges of types I,II, and III, respectively. Note that lozenges of type I occur exactly at the same places as particles.

and the white triangle is located at (x', n'). For example, in Figure 6 there are lozenges $(1, 3, 1, 3), (2, 3, 2, 4)$, and $(0, 3, 1, 4)$. The three types of lozenges can be described as follows. For lozenges of type I, $(x', n') = (x, n)$. For lozenges of type II, $(x', n') = (x - 1 + \delta_n, n + 1)$. For lozenges of type III, $(x', n') = (x + \delta_n, n + 1)$. Note that a lozenge of type I is just a particle.

We say that $(x, n, x', n') \in \mathfrak{X} \times \mathfrak{X}$ is *viable* if $(x', n') = (x, n), (x - 1 + \delta_n, n + 1)$, or $(x + \delta_n, n + 1)$. A sequence $(x_1, n_1, x_1', n_1'), \ldots, (x_k, n_k, x_k', n_k')$ of viable elements is *non-overlapping* if $(x_1, n_1), \ldots, (x_k, n_k)$ are all distinct from each other and $(x_1', n_1'), \ldots, (x_k', n_k')$ are also all distinct from each other. We do, however, allow the possibility of $(x_i, n_i) = (x_j', n_j')$.

The statement and proof of the next proposition are similar to Theorem 5.1 of kn:BF.

Proposition 2.7. *Suppose the kernel K is conjugate to some $\tilde{K}$ such that* (2.7)-(2.12) *hold for some L. If $t \geq 0, x_1, x_1', \ldots, x_k, x_k' > L$, and $(x_1, n_1, x_1', n_1'), \ldots,$ (x_k, n_k, x_k', n_k') is a* *sequence of non-overlapping viable elements of* $\mathfrak{X} \times \mathfrak{X}$*, then*

$$\text{(2.16)} \quad \mathbb{P}(\textit{There is a lozenge } (x_j, n_j, x'_j, n'_j) \textit{ at time } t \textit{ for each } j = 1, \ldots, k) \\ = \det[\tilde{K}(x_i, n_i, x'_j, n'_j, t)]_{1 \le i,j \le k}.$$

Remark 2.8. The equations (2.7)–(2.12) can now be intuitively understood. Equation (2.7) says that each black triangle is located in exactly one of the three lozenges around it, and equation (2.8) makes an identical statement for white triangles. Equations (2.9) and (2.11) say that lozenges of type II almost surely do not occur far to the right of the particles, with (2.9) controlling the off-diagonal entries in the determinant and (2.11) controllling the diagonal entries. Similarly, equations (2.10) and (2.12) says that lozenges of type III almost surely do occur far to the right of the particles. This intuition will be exploited in the proof of Thereom 2.7.

Proof. We proceed by induction on the number of lozenges that are not of type I. When this number is zero, the statement reduces to (2.5) and (2.6).

For any set $S = \{(x_1, n_1, x'_1, n'_1), \ldots, (x_k, n_k, x'_k, n'_k)\}$ of non-overlapping, viable elements, let $P(S)$ and $D(S)$ denote the left and right hand sides of (2.16), respectively. First, as a preliminary statement, it is not hard to prove that if $(x_{k+1}, n_{k+1}) \neq (x_r, n_r)$ for $1 \le r \le k$, then

$$\text{(2.17)} \quad D(S \cup \{(x_{k+1}, n_{k+1}, x_{k+1}, n_{k+1})\}) + D(S \cup \{(x_{k+1}, n_{k+1}, x_{k+1} - 1 + \delta_n, n_{k+1} + 1)\}) \\ + D(S \cup \{(x_{k+1}, n_{k+1}, x_{k+1} + \delta_n, n_{k+1} + 1)\}) = D(S).$$

One simply expands the determinant in the left–hand–side as a sum over permutations $\sigma \in S_{k+1}$. One then uses (2.7) to show that the sum over the σ fixing $k+1$ equals $D(S)$, while the sum over the sigma not fixing $k+1$ equals 0. Note that if D is replaced by P in (2.17), the statement is immediate, since the black triangle at (x_{k+1}, n_{k+1}) must be contained in exactly one lozenge.

In a similar manner, if $(x'_{k+1}, n'_{k+1}) \neq (x'_r, n'_r)$ for $1 \leq r \leq k$, then (2.8) implies that

$$D(S \cup \{(x_{k+1}, n_{k+1}, x_{k+1}, n_{k+1})\}) + D(S \cup \{(x_{k+1} + 1 - \delta_n, n_{k+1} - 1, x_{k+1}, n_{k+1})\}) + D(S \cup \{(x_{k+1} - \delta_n, n_{k+1} - 1, x_{k+1}, n_{k+1})\}) = D(S). \tag{2.18}$$

Again, the statement holds if D is replaced by P.

In order to prove the induction step, it suffices to prove that D and P still agree if we add a lozenge of type II or type III to S. Let us do type II, as type III is similar. Suppose that $(x, n, x-1+\delta_n, n+1)$ is viable and that $S \cup \{(x, n, x-1+\delta_n, n+1)\}$ is non-overlapping. Then equation (2.17) is equivalent to

$$D(S \cup \{(x, n, x - 1 + \delta_n, n + 1)\}) = D(S) - D(S \cup \{(x, n, x, n)\}) - D(S \cup \{(x, n, x + \delta_n, n + 1)\}), \tag{2.19}$$

and the same holds for P instead of D. By the induction hypothesis,

$$D(S) = P(S),$$
$$D(S \cup \{(x, n, x, n)\}) = P(S \cup \{(x, n, x, n)\})$$
$$D(S \cup \{(x + \delta_n, n + 1, x + \delta_n, n + 1)\}) = P(S \cup \{(x + \delta_n, n + 1, x + \delta_n, n + 1)\}).$$

Thus, (2.19) implies

$$D(S \cup \{(x, n, x - 1 + \delta_n, n + 1)\}) - P(S \cup \{(x, n, x - 1 + \delta_n, n + 1)\})) = -D(S \cup \{(x, n, x + \delta_n, n + 1)\}) + P(S \cup \{(x, n, x + \delta_n, n + 1)\}).$$

Assume for now that $(x'_r, n'_r) \neq (x+\delta_n, n+1)$ for $1 \leq r \leq$k. Then we cam apply equation (2.18), which implies that

$$\text{(2.20)} \quad D(S \cup \{(x,n,x-1+\delta_n,n+1)\}) = D(S)$$
$$- D(S \cup \{(x+\delta_n, n+1, x+\delta_n, n+1)\}) - D(S \cup \{(x+1, n, x+\delta_n, n+1)\}),$$

and the same statement holds for P. Thus,

$$- D(S \cup \{(x,n,x+\delta_n,n+1)\}) + P(S \cup \{(x,n,x+\delta_n,n+1)\})$$
$$= D(S \cup \{(x+1,n,x+\delta_n,n+1)\}) - P(S \cup \{(x+1,n,x+\delta_n,n+1)\}).$$

If $S \cup \{(x+1,n,x+\delta_n,n+1)\}$ is non-overlappinng, then (2.19) is again applicable. We repeatedly apply (2.19) and (2.20) as often as possible. First, suppose that this can be done indefinitely. Then

$$|D(S \cup \{(x,n,x-1+\delta_n,n+1)\}) - P(S \cup \{(x,n,x-1+\delta_n,n+1)\})|$$
$$= \lim_{M\to\infty} |D(S\cup\{(x+M,n,x-1+\delta_n+M,n+1)\}) - P(S\cup\{(x+M,n,x-1+\delta_n+M,n+1)\})|.$$

Since lozenges of type II almost surely do not appear when we look far to the right of the particles,

$$\lim_{M\to\infty} P(S \cup \{(x+M,n,x-1+\delta_n+M,n+1)\}) = 0.$$

By expanding the determinant into a sum over S_{k+1}, (2.9) and (2.11) imply that

$$\lim_{M\to\infty} D(S \cup \{(x+M,n,t,x-1+\delta_n+M,n+1)\}) = 0.$$

Now suppose that (2.19) and (2.20) can only be applied finitely many times. This means that $D(S \cup \{(x,n,x-1+\delta_n,n+1)\}) - P(S \cup \{(x,n,x-1+\delta_n,n+1)\})$ equals either

$$D(S \cup \{(x+M,n,x+M+\delta_n,n+1)\}) - P(S \cup \{(x+M,n,x+M+\delta_n,n+1)\})$$

or

$$D(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}) - P(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\})$$

In the first case, $S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}$ is non non-overlapping. This implies $D(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}) = 0$ (because two of the rows are idential) and $P(S \cup \{(x+M+1, n, x+M+\delta_n, n+1)\}) = 0$ (because a triangle cannot be in two different lozenges at the same time). Thus, D and P agree. A similar argument holds in the second case. Thus, D and P agree whenever a lozenge of type II is added to S.

An identical argument holds for type III lozenges, except that we use (2.10) and (2.12) instead of (2.9) and (2.11). □

We have been describing a lozenge as a pair (x, n, x', n'). It can also be described as (x', n', λ), where (x', n') is the location of the white triangle and $\lambda \in \{I, II, III\}$ is the type of the loznege. Thus the proposition can be restated as the following statement.

Corollary 2.9. *For any non–overlapping* $(x'_1, n'_1, \lambda_1), \dots, (x'_k, n'_k, \lambda_k)$*,*

$$\mathbb{P}(\textit{There is a lozenge } (x'_j, n'_j, \lambda_j) \textit{ at time } t \textit{ for each } j = 1, \dots, k)$$
$$= \det[K_\lambda(x'_i, n'_i, \lambda_i, x'_j, n'_j, t)]_{1 \le i,j \le k},$$

where

$$K_\lambda(x, n, \lambda, x', n', t) = \begin{cases} \tilde{K}(x, n, x', n', t), & \textit{when } \lambda = I \\ \tilde{K}(x - \delta_{n-1}, n-1, x', n', t), & \textit{when } \lambda = II \\ \tilde{K}(x - \delta_{n-1} - 1, n-1, x', n', t), & \textit{when } \lambda = III \end{cases}$$

Proof. This is a result of the correspondences

$$(x', n', I) \text{ iff } (x', n', x', n'),$$

$$(x', n', II) \text{ iff } (x' - \delta_{n'-1}, n' - 1, x', n'),$$

$$(x', n', III) \text{ iff } (x' - \delta_{n'-1} - 1, n' - 1, x', n').$$

□

There are two different formulas for the height function. One formula is

$$h(x, n) = \sum_{s>x} \mathbf{1}(\text{lozenge of type I at } (s, n)). \tag{2.21}$$

It is possible to only use (2.21) to complete the proof. However, when there are multiple points on one level, i.e. not all $\eta_1, \ldots, \eta_k$ are distinct, the computation becomes much more complicated. This is because lozenges of type I will appear in multiple sums of the form (2.21). We can avoid this difficulty by introducing another formula for the height function:

$$h(x, n) = h(x + \delta_n + \delta_{n+1} + \ldots + \delta_{n'-1}, n') + H_{n,n'}(x), \tag{2.22}$$

where, for $n < n'$,

$$H_{n,n'}(x) = -\sum_{p=n+1}^{n'} \mathbf{1}(\text{lozenge of type II at } (x + \delta_n + \delta_{n+1} \ldots + \delta_{p-1}, p)). \tag{2.23}$$

Therefore, the expression

$$\mathbb{E}\left(\prod_{j=1}^{k} [h(x_j, n_j) - \mathbb{E}(h(x_j, n_j))]\right) \tag{2.24}$$

can be expressed as a sum of terms of the form

$$\mathbb{E}\left(\prod_{j=1}^{k'} [h(x_j, n_j) - \mathbb{E}(h(x_j, n_j))] \prod_{l=k'+1}^{k} [H_{n_l,n_l'}(x_l) - \mathbb{E}(H_{n_l,n_l'}(x_l))]\right). \tag{2.25}$$

Lemma 2.10. *Assume that the following sets are disjoint:*

$$\{(s,n_j) : s > x_j\},\ 1 \le j \le k'$$

$$\{(x_l + \delta_{n_l} + \delta_{n_l+1} \ldots + \delta_{p-1}, p) : n_l + 1 \le p \le n_l'\},\ k'+1 \le l \le k.$$

Then

$$(2.25) = \sum_{s_1 > x_1} \cdots \sum_{s_{k'} > x_{k'}} \sum_{p_{k'+1}=n_{k'+1}+1}^{n'_{k'+1}} \cdots \sum_{p_k = n_k+1}^{n'_k} \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.26}$$

where the matrix blocks are:

$$A_{11} = [(1-\delta_{ij})\tilde{K}(s_i, n_i, s_j, n_j, t)]_{1 \le i,j \le k'}$$

$$A_{12} = [\tilde{K}(s_i, n_i, x_j, p_j, t)]_{1 \le i \le k',\ k'+1 \le j \le k}$$

$$A_{21} = [-\tilde{K}(x_i - \delta_{p_i-1}, p_i - 1, s_j, n_j, t)]_{k'+1 \le i \le k,\ 1 \le j \le k'}$$

$$A_{22} = [-(1-\delta_{ij})\tilde{K}(x_i - \delta_{p_i-1}, p_i - 1, x_j, p_j, t)]_{k'+1 \le i,j \le k}$$

Proof. By applying Corollary 2.9 to (2.21) and (2.23), we see that

$$\mathbb{E}\left(\prod_{j=1}^{k'} h(x_j, n_j) \prod_{l=k'+1}^{k} H_{n_l, n_l'}(x_l) \right)$$

equals the right hand side of (2.26) with the $(1-\delta_{ij})$ terms removed. It is well-known that subtracting the expectation corresponds to putting zeroes on the diagonal. For example, this is noticed in the proof of Theorem 7.2 of [9]. □

Write the determinant in (2.26) as a sum over permutations σ in S_k. If the cycle decomposition of σ contains the cycle $(c_1\ c_2\ \ldots\ c_r)$ of length r and M denotes the matrix in the right hand side of (2.26), then the contribution from σ is

$$\sum_{s_1} \cdots \sum_{s_{k'}} \sum_{p_{k'+1}} \cdots \sum_{p_k} \operatorname{sgn}(\sigma) M_{c_1c_2} M_{c_2c_3} \ldots M_{c_rc_1} (\cdots)(\cdots),$$

where $(\cdots)(\cdots)$ correspond to other cycles of σ. Let ψ_{c_ι} denote s_{c_ι} if $1 \leq c_\iota \leq k'$, and p_{c_ι} if $k' < c_\iota \leq k$. Since the sum over ψ_{c_ι} only affects the matrix terms $M_{c_{\iota-1}c_\iota}$ and $M_{c_\iota c_{\iota+1}}$, the contribution from σ is

$$\left((-1)^{r-1} \sum_{\psi_{c_1}} \cdots \sum_{\psi_{c_r}} M_{c_1c_2} M_{c_2c_3} \ldots M_{c_rc_1} \right) (\ldots), \tag{2.27}$$

where $(\ldots)$ denote other cycles. In other words, the contribution from σ can be expressed as a product over the cycles in the cycle decomposition of σ.

Note that if σ fixes any points, then the correponding contribution is zero because all the diagonal entries are zero.

2.2.3. *Analysis steps in proof of Theorem 3.1.* In (2.26), set $x_j = [N\nu_j], n_l = [N\eta_l]$, and $t = N\tau$. Our goal is to find the limit of (2.26) as $N \to \infty$. Expanding the determinant into a sum over $\sigma \in S_k$, we just saw that the contribution from a fixed σ is of the form (2.27). First note that if any of the ψ_{c_i} denotes p_{c_i}, then

$$\sum_{\psi_{c_1}} \cdots \sum_{\psi_{c_r}} M_{c_1c_2} M_{c_2c_3} \ldots M_{c_rc_1} \to 0.$$

This is because each $M_{c_jc_{j+1}}$ is proportional to $1/N$ (by Proposition 2.19, so therefore $M_{c_1c_2} M_{c_2c_3} \ldots M_{c_rc_1}$ is proportional to N^{-r}, but the sum is only taken over $\mathcal{O}(N^{r-1})$ terms. Therefore, (2.24) can be expressed as a single term of the form in (2.25), and in this term $k' = k$.

Now we will prove (stated as Theorem 2.12 below) that

$$\sum_{s_{c_1}} \cdots \sum_{s_{c_r}} M_{c_1c_2} M_{c_2c_3} \ldots M_{c_rc_1} \to \left(\frac{1}{2\pi}\right)^r \int_{\bar{\Omega_1}}^{\Omega_1} dz_1 \cdots \int_{\bar{\Omega_r}}^{\bar{\Omega_r}} dz_r \frac{f(z_1, z_2)}{G'_\nu(z_1)} \cdots \frac{f(z_r, z_r)}{G'_\nu(z_r)}$$

Once this is proven, (2.15) implies that the total contribution from $S_k - \mathcal{F}_k$ equals zero. When $l = 2$, then the right hand side is just $\mathcal{G}(\Omega_1, \Omega_2)$, completing the proof of Theorem 3.1.

Recall the definitions of G and Ω from section 2.2.1. Set $\theta : \mathcal{D} \to [0, \pi)$ to be

$$\theta(\nu, \eta, \tau) = \frac{1}{2} \arg G''(\nu, \eta, \tau, \Omega(\nu, \eta, \tau)).$$

Proposition 2.11. *For $i = 1, 2, 3$, let $(\nu_i, \eta_i, \tau) \in \mathcal{D}$, $x_i = [N\nu_i]$, $n_i = [N\eta_i]$ and $t = N\tau$. For $i = 1, 3$, let $G_i(z)$ denote $G(\nu_i, \eta_i, \tau, z)$, let θ_i denote $\theta(\nu_i, \eta_i, \tau)$ and let Ω_i denote $\Omega(\nu_i, \eta_i, \tau)$. Let $\Gamma_+ := \{\Omega(\nu, \eta_2, \tau) : \nu_2 \le \nu < q_2(\eta_2, \tau)\}$ and $\Gamma_- = \bar{\Gamma}_+$. Let $G'_\nu(z) = (\partial^2/\partial z \partial v) G(\nu_2, \tau_2, \tau, z)$. Then*

$$\begin{aligned}
(2.28) \quad & \sum_{y > [N\nu_2]} K(x_1, n_1, y, n_2, t) K(y, n_2, x_3, n_3, t) \\
&= o\left(\frac{1}{N}\right) + \frac{e^{N\Re((G_1(\Omega_1) - G_3(\Omega_3)))}}{2\pi N \sqrt{|G_1''(\Omega_1)|}\sqrt{|G_3''(\Omega_3)|}} \int_{\Gamma_+ \cup \Gamma_-} \frac{dz}{2\pi G'_{\nu_1}(z)} \\
&\times \Bigg[f(\Omega_1, z) f(z, \Omega_3) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{iN\Im(G_3(\Omega_3)) + i\theta_3}} + f(\bar{\Omega}_1, z) f(z, \Omega_3) \frac{e^{-iN\Im(G_1(\Omega_1)) + i\theta_1}}{e^{iN\Im(G_3(\Omega_3)) + i\theta_3}} \\
&+ f(\Omega_1, z) f(z, \bar{\Omega}_3) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{-iN\Im(G_3(\Omega_3)) - i\theta_3}} + f(\bar{\Omega}_1, z) f(z, \bar{\Omega}_3) \frac{e^{-iN\Im(G_1(\Omega_1)) + i\theta_1}}{e^{-iN\Im(G_3(\Omega_3)) - i\theta_3}} \Bigg].
\end{aligned}$$

Proof. Let $G_2(z)$ denote $G([y/N], \eta_2, \tau, z)$, let θ_2 denote $\theta([y/N], \eta_2, \tau)$ and Ω_2 denote for short $\Omega(y/N, \eta_2, \tau)$. Fix some $\beta \in (-1/2, 0)$ and split up the sum into two parts: the first part is from $\lfloor N\nu_2 \rfloor$ to $\lfloor N(q_2 - N^\beta) \rfloor$, while the second sum is from $\lfloor N(q_2 - N^\beta) \rfloor$ to $\lfloor Nq_2 \rfloor$. Since there are no particles to the right of Nq_2 in the limit $N \to \infty$, the sum from Nq_2 to ∞ can be ignored. It is common to refer to the first sum as the bulk and the second sum as the edge. First examine the bulk. By Proposition 2.19,

$$\begin{aligned}
(2.29) \quad & K(x_1, n_1, y, n_2, t) K(y, n_2, x_3, n_3, t) \\
&= \frac{e^{N\Re((G_1(\Omega_1) - G_2(\Omega_2)))}}{2\pi N \sqrt{|G_1''(\Omega_1)|}\sqrt{|G_2''(\Omega_2)|}} \frac{e^{N\Re((G_2(\Omega_2) - G_3(\Omega_3)))}}{2\pi N \sqrt{|G_2''(\Omega_2)|}\sqrt{|G_3''(\Omega_3)|}} \\
&\times \left[f(\Omega_1, \Omega_2) f(\Omega_2, \Omega_3) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{iN\Im(G_2(\Omega_2)) + i\theta_2}} \frac{e^{iN\Im(G_2(\Omega_2)) - i\theta_2}}{e^{iN\Im(G_3(\Omega_3)) + i\theta_3}} + \circlearrowleft \right] \\
&\qquad + \mathcal{O}(G_2''(\Omega_2)^{-4} N^{-3}) + \mathcal{O}(G_2''(\Omega_2)^{-7} N^{-4}),
\end{aligned}$$

where $\circlearrowleft$ denotes the other fifteen terms that occur in the sum. First let us examine the error term in the bulk.

By (2) of Definition 2.3, each term in the error is bounded by $(N^{\beta/2})^{-4}N^{-3}$ and also by $(N^{\beta/2})^{-7}N^{-4}$, respectively. There are $\sim N$ terms, and since $\beta > -1/2$, we must have $-2\beta - 3 + 1 < -1$ and $-7\beta/2 - 4 + 1 < -1$. Therefore the sum is $o(1/N)$.

Now let us return to the main term in the bulk. For eight of the sixteen terms in $\circlearrowleft$, the expression $e^{iN\Im(G_2(\Omega_2))}$ cancels in the numerator and the denominator. By Proposition 2.17, these eight terms are $o(1/N)$. By Proposition 2.18, the other eight terms equal

$$\frac{e^{N\Re((G_1(\Omega_1)-G_3(\Omega_3)))}}{2\pi N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_3''(\Omega_3)|}}\int_{\nu_2}^{\infty}\frac{e^{-2i\theta_2}}{2\pi|G_2''(\Omega_2)|}$$
$$\times\left[f(\Omega_1,\Omega_2)f(\Omega_2,\Omega_3)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}}+\ldots\right]d\nu+o\left(\frac{1}{N}\right),$$

where $\ldots$ represent the other seven terms. Of the eight total terms, four have $f(\cdot,\Omega_2)f(\Omega_2,\cdot)$ and four have $f(\cdot,\bar{\Omega}_2)f(\bar{\Omega}_2,\cdot)$. For the four terms with the expression Ω_2, make the substitution $z=\Omega(\nu,\eta_2,\tau)$. The new integration path is Γ_+. By taking the partial of (2.13) with respect to ν and using the chain rule,

$$\frac{\partial\Omega}{\partial\nu}=-\frac{G_\nu'(\Omega)}{G''(\Omega)},$$

which implies

$$\frac{e^{-2i\theta_2}}{2\pi|G_2''(\Omega_2)|}d\nu=\frac{d\nu}{2\pi G_2''(\Omega_2)}=-\frac{dz}{2\pi G_\nu'(z)}.$$

For the four terms with $\bar{\Omega}_2$, make the substitution $z = \bar{\Omega}(\nu, \eta_2, \tau)$. The path of integration is Γ_-. Finally, the integral becomes

$$o\left(\frac{1}{N}\right) + \frac{e^{N\Re((G_1(\Omega_1)-G_3(\Omega_3)))}}{2\pi N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_3''(\Omega_3)|}} \int_{\Gamma_+\cup\Gamma_-} \frac{dz}{2\pi G'_{\nu_1}(z)}$$
$$\times \Big[f(\Omega_1, z)f(z, \Omega_3)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}} + f(\bar{\Omega}_1, z)f(z, \Omega_3)\frac{e^{-iN\Im(G_1(\Omega_1))+i\theta_1}}{e^{iN\Im(G_3(\Omega_3))+i\theta_3}}$$
$$+ f(\Omega_1, z)f(z, \bar{\Omega}_3)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{-iN\Im(G_3(\Omega_3))-i\theta_3}} + f(\bar{\Omega}_1, z)f(z, \bar{\Omega}_3)\frac{e^{-iN\Im(G_1(\Omega_1))+i\theta_1}}{e^{-iN\Im(G_3(\Omega_3))-i\theta_3}}\Big].$$

Now we sum over the edge. By Proposition 2.23 and (2) of Definition 2.3, the sum is bounded above by

$$\sum_{y=(q_2-N^\beta)N}^{q_2 N} |G_2(\Omega_2)^{-1}|N^{-2} \leq \sum_{y=0}^{N^{\beta+1}} \left(\frac{y}{N}\right)^{1/2} N^{-2} = \mathcal{O}(N^{3\beta/2-1}).$$

As long as $\beta < 0$, the sum over the edge is also $o(1/N)$. □

Theorem 2.12. *For $i = 1, \ldots, l$, let $(\nu_i, \eta_i, \tau) \in \mathcal{D}$ and set $x_i = [N\nu_i], n_i = N\eta_i$. For $i = 1, \ldots, l$, let $G_i(z)$ denote $G(\nu_i, \eta_i, z)$, let θ_i denote $\theta(\nu_i, \eta_i, \tau)$ and let Ω_i denote $\Omega(\nu_i, \eta_i, \tau)$. Let $\Gamma_i^+ := \{\Omega(\nu, \eta_i, \tau) : \nu_1 \leq \nu < q_2(\eta_i, \tau)\}$ and $\Gamma_i^- = \bar{\Gamma}_i^+$. Then*

$$\sum_{y_1>[N\nu_1]} \cdots \sum_{y_l>[N\nu_l]} \prod_{i=1}^{l} K(y_i, x_i, y_{i+1}, x_{i+1}, t)$$
$$\to \left(\frac{1}{2\pi}\right)^l \int_{\Gamma_1^+\cup\Gamma_1^-} dz_1 \cdots \int_{\Gamma_l^+\cup\Gamma_l^-} dz_l \frac{f(z_1, z_2)}{G'_\nu(z_1)} \cdots \frac{f(z_l, z_1)}{G'_\nu(z_l)}.$$

The indices are taken cyclically.

Proof. By Proposition 2.19, the product has 4^l terms. Each application of Proposition 2.11 decreases the number of terms by a factor of 4, so repeated applications of Proposition 2.11 yields the result. □

2.3. **Specific Results.**

2.3.1. *Particle system with a wall.* We now return to the particle system with a reflecting wall described in the Introduction. For notational reasons, it is more convenient to use different co-ordinates. Instead of labeling the levels as $1, 2, 3, \ldots$, it is more convenient to label them as $(1, -1/2), (1, 1/2), (2, -1/2), (2, 1/2), \ldots$. If the (n_1, a_1) is at least as high as the (n_2, a_2) level, then this will be denoted as $(n_1, a_1) \trianglerighteq (n_2, a_2)$. This happens if and only if $2n_1 + a_1 \geq 2n_2 + a_2$. Using the notation of Section 2.2.2, $m_{(n,a)} = n$ and $\delta_{(n,a)} = a + 1/2$. Along the horizontal direction, we will use a square lattice, so that the particles live on $\mathbb{N}$ instead of $2\mathbb{N}$ or $2\mathbb{N} + 1$.

Let $m_{a_1}(dz)$ be defined by

$$m_{a_1}(dz) \begin{cases} \dfrac{dz}{2iz}, & a_1 = -1/2, \\ \dfrac{-(z^{1/2} - z^{-1/2})^2 dz}{4iz}, & a_1 = 1/2. \end{cases}$$

Let $\mathsf{J}_s^{(\pm 1/2, -1/2)}$ denote the (normalized) Jacobi polynomial with parameters $(\pm 1/2, -1/2)$. The normalization is set so that for any nonzero complex number z, $\mathsf{J}_s^{(\pm 1/2, -1/2)}$ satisfies

$$\mathsf{J}_s^{(-1/2,-1/2)}\left(\frac{z + z^{-1}}{2}\right) = \frac{z^s + z^{-s}}{2}, \tag{2.30}$$

$$\mathsf{J}_s^{(1/2,-1/2)}\left(\frac{z + z^{-1}}{2}\right) = \frac{z^{s+1/2} - z^{-s-1/2}}{z^{1/2} - z^{-1/2}}. \tag{2.31}$$

Let $W^{(a,-1/2)}(s)$ be defined for nonnegative integers s by

$$W^{(a,-1/2)}(s) = \begin{cases} 2, & \text{if } \ s > 0, a = -\frac{1}{2}, \\ 1, & \text{if } \ s = 0, a = -\frac{1}{2}, \\ 1, & \text{if } \ s \geq 0, a = \frac{1}{2}. \end{cases}$$

Note that for $a = \pm 1/2$,

$$\frac{W^{(a,-1/2)}(s_1)}{\pi} \oint_{|z|=1} \mathsf{J}_{s_1}^{(a,-1/2)}\left(\frac{z + z^{-1}}{2}\right) \mathsf{J}_{s_2}^{(a,-1/2)}\left(\frac{z + z^{-1}}{2}\right) m_a(dz) = \delta_{s_1 s_2} \tag{2.32}$$

By Theorem 4.1 of [3], the correlation functions are determinantal with kernel

$$K(n_1, a_1, s_1, n_2, a_2, s_2, t) \tag{2.33}$$

$$= \frac{W^{(a_1,-1/2)}(s_1)}{2\pi^2 i} \oint \oint \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}} \mathsf{J}_{s_1}^{(a_1,-1/2)}\left(\frac{z+z^{-1}}{2}\right) \mathsf{J}_{s_2}^{(a_2,-1/2)}\left(\frac{v+v^{-1}}{2}\right)$$

$$\times \frac{(\frac{z+z^{-1}}{2}-1)^{n_1}}{(\frac{v+v^{-1}}{2}-1)^{n_2}} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} m_{a_1}(dz)dv$$

$$+ \mathbf{1}_{(n_1,a_1)\trianglerighteq(n_2,a_2)} \left(\frac{W^{(a_1,-1/2)}(s_1)}{\pi} \oint \mathsf{J}_{s_1}^{(a_1,-1/2)}\left(\frac{z+z^{-1}}{2}\right) \mathsf{J}_{s_2}^{(a_2,-1/2)}\left(\frac{z+z^{-1}}{2}\right) \times \left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2} m_{a_1}(dz)\right), \tag{2.34}$$

where the z-contour is the unit circle and the v-contour is a circle centered at the origin with radius bigger than 1.

Theorem 2.13. *The determinantal point process is normal. The Green's function is given by*

$$\mathcal{G}(z,w) = \frac{1}{2\pi} \log\left(\frac{z+z^{-1}-\bar{w}-\bar{w}^{-1}}{z+z^{-1}-w-w^{-1}}\right).$$

Once we prove the point process is normal, the expression for the Green's function follows from Theorem 3.1 with

$$G(\nu,\eta,\tau;u) = \tau\frac{u+u^{-1}}{2} + \eta\log\left(\frac{u+u^{-1}}{2}-1\right) - \nu\log u,$$

$$f(u,v) = \frac{1}{v}\frac{1-u^{-2}}{v+v^{-1}-u-u^{-1}}.$$

In section 2.3.2, we show that the third condition in Definition 2.3 is satisfied. In section 2.3.3, we show that the fourth and second conditions are satisfied. Since these are conditions are the hardest to prove, we will focus mainly on their proofs. The fifth conditions follows from the substitution $u_j = z_j + z_j^{-1}$ and (5) of Remark 2.4.

2.3.2. *Algebraic steps in proof of theorem 2.13.*

Proposition 2.14. *Let $\mathcal{C}_0(n,a,s)$ equal*

$$\mathcal{C}_0(n,a,s) = \begin{cases} (-1)^s(-2)^{n-1}, & a=-1/2 \\ (-1)^s(-2)^n, & a=1/2 \end{cases}$$

and $c_0(n_1,a_1,s_1,n_2,a_2,s_2) = \mathcal{C}_0(n_1,a_1,s_1)/\mathcal{C}_0(n_2,a_2,s_2)$. Then $\tilde{K} = c_0 K$ satisfies (2.7)–(2.12) *for $L=1$.*

Proof. Using (2.30)–(2.31) and the orthogonality relation (2.32), it is straightforward to check that (2.7) and (2.8) hold. What happens is that in the left hand side of (2.7) or (2.8), one obtains six terms, three of which come from (2.33) and three of which come from (2.34). The three terms from (2.33) always sum to 0, while the three terms from (2.34) sum to 0 or 1.

Now we will prove (2.11)-(2.12) when $a_1 = -1/2$. The term (2.34) equals zero, so we only need to look at (2.33). Explicitly, the expression is

$$K(n,-1/2,s,n,1/2,s',t) = \frac{2}{2\pi^2 i}\oint\oint_{|z|=1} \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}} \left(\frac{z^s+z^{-s}}{2}\right) \times \left(\frac{v^{s'+1/2}-v^{-s'-1/2}}{v^{1/2}-v^{-1/2}}\right) \frac{(\frac{z+z^{-1}}{2}-1)^n}{(\frac{v+v^{-1}}{2}-1)^n} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} \frac{dzdv}{2iz},$$

and we want the asymptotic result when $s,s'\to\infty$ in such a way that $s-s'$ is 0 or 1. Expand the paranthetical expression $v^{s'+1/2}-v^{-s'-1/2}$ to get two terms, each of which is a double integral. Since $1=|z|<|v|$, the term with $v^{-s'-1/2}$ goes to zero. For the remaining term, expand z^s+z^{-s} to get two terms. For the term with z^s, make the substitution $z\mapsto z^{-1}$. What remains is

$$\frac{2}{2\pi^2 i}\oint\oint_{|z|=1} \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}} \frac{v^{s'}}{z^s} \frac{v}{v-1} \frac{(\frac{z+z^{-1}}{2}-1)^n}{(\frac{v+v^{-1}}{2}-1)^n} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} \frac{dzdv}{2iz}.$$

Now deform the z-contour to the circle $|z|=1+2\epsilon$ and the v-contour to the circle $|v|=1+\epsilon$, where $\epsilon>0$. With these deformations, $|v|<|z|$, so the double integral goes to zero. However, residues are picked up when the contours pass through each other. These residues equal

$$-\frac{2}{\pi}\oint_{|z|=1+2\epsilon} z^{s'-s}\frac{z}{z-1}\frac{dz}{2iz}.$$

There is a residue at $z=1$ which equals -2, and a residue at $z=0$ which equals 0 for $s'\geq s$ and 2 for $s>s'$. Since $c_0(n,-1/2,s,n,1/2,s)=-1/2$, this proves (2.11) and (2.12) when $a_1=-1/2$. The case when $a_1=1/2$ is similar.

It remains to show (2.9) and (2.10). When considering the product of two kernels, we obtain a quadruple integral. After the substitutions $z_1\mapsto z_1^{-1}$ and $v_2\mapsto v_2^{-1}$, the part of the integrand that depends on s is just $(z_1/v_2)^s$. Therefore, deforming contours so that $|v_2|>|z_1|$ gives (2.9) and (2.10). $\square$

2.3.3. *Analysis steps in proof of theorem 2.13.* For this section, we need a slightly different expression for the kernel. By (40)–(42) of [3], the kernel equals

$$\begin{aligned}(2.35)\quad & K(n_1,a_1,s_1;n_2,a_2,s_2,t)\\ &= \frac{W^{(a_1,-1/2)}(s_1)}{2\pi^2 i}\int_{e^{-i\theta}}^{e^{i\theta}}\oint_{|z|=1}\frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}}\mathsf{J}_{s_1}^{(a_1,-1/2)}\left(\frac{z+z^{-1}}{2}\right)\mathsf{J}_{s_2}^{(a_2,-1/2)}\left(\frac{v+v^{-1}}{2}\right)\\ &\qquad\times\frac{(\frac{z+z^{-1}}{2}-1)^{n_1}}{(\frac{v+v^{-1}}{2}-1)^{n_2}}\frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}}m_{a_1}(dz)dv\end{aligned}$$

$$\begin{aligned}(2.36)\quad & +\mathbf{1}_{(n_1,a_1)\trianglerighteq(n_2,a_2)}\Bigg(\frac{W^{(a_1,-1/2)}(s_1)}{\pi}\oint_{|z|=1}\mathsf{J}_{s_1}^{(a_1,-1/2)}\left(\frac{z+z^{-1}}{2}\right)\mathsf{J}_{s_2}^{(a_2,-1/2)}\left(\frac{z+z^{-1}}{2}\right)\\ &\qquad\times\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}m_{a_1}(dz)\Bigg)\end{aligned}$$

$$
(2.37) \quad + \left(\frac{W^{(a_1,-1/2)}(s_1)}{\pi} \int_{e^{-i\theta}}^{e^{i\theta}} \mathsf{J}_{s_1}^{(a_1,-1/2)}\left(\frac{z+z^{-1}}{2}\right) \mathsf{J}_{s_2}^{(a_2,-1/2)}\left(\frac{z+z^{-1}}{2}\right) \times \left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2} m_{a_1}(dz) \right),
$$

where θ is any real number, and the arc from $e^{-i\theta}$ to $e^{i\theta}$ is outside the unit circle and does not cross $(-\infty, 0]$.

Set

$$
G(\nu,\eta,\tau,z) = \tau\frac{z+z^{-1}}{2} + \eta\log\left(\frac{z+z^{-1}}{2}-1\right) - \nu\log z
$$

By Proposition 5.1.1 of [3], we can take $\mathcal{D}$ to be

$$
\mathcal{D} = \{(\nu,\eta,\tau) : \eta,\tau > 0, q_1(\eta,\tau) < \nu < q_2(\eta,\tau)\},
$$

for some explicit algebraic functions q_1 and q_2.

Lemma 2.15. *Let* $\Omega_\pm$ *denote* $\Omega(\pm\nu,\eta,\tau)$*. Then* $\bar{\Omega}_+\Omega_- \equiv 1$.

Proof. In general,

$$
G'(z) = \frac{p(z)}{r(z)},
$$

where p and r are

$$
\begin{aligned}
p(z) &= \tau + (2\eta+2\nu-\tau)z + (2\eta-2\nu-\tau)z^2 + \tau z^3, \\
r(z) &= 2z^2(z-1).
\end{aligned}
$$

Let $p_\pm(z)$ denote the polynomial $p(z)$ corresponding to $(\pm\nu,\eta,\tau)$. Note that $z^3p_+(z^{-1}) = p_-(z)$. By definition, $\Omega_\pm$ is the zero of $p_\pm$ that is in the upper half-plane. Therefore, $\Omega_-^{-1} = \bar{\Omega}_+$. ◻

Now let us return to the proof of the fourth condition in Definition 2.3. Start by examining (2.35). Expanding the parantheses, we obtain four terms corresponding to the terms $z^{s_1}v^{s_2}, z^{s_1}v^{-s_2}, z^{-s_1}v^{-s_2}$, and $z^{-s_1}v^{s_2}$. For the two terms with z^{s_1}, make the substitution $z \to z^{-1}$. What remains are two terms, corresponding to $z^{-s_1}v^{s_2}$ and $z^{-s_1}v^{-s_2}$. Therefore, (2.35) equals

$$\text{(2.38)} \quad \frac{W^{(-1/2,-1/2)}(s_1)}{4\pi^2 i}\int_{e^{-i\theta}}^{e^{i\theta}}\oint \frac{e^{t(\frac{z+z^{-1}}{2})}}{e^{t(\frac{v+v^{-1}}{2})}} z^{-s_1}(v^{s_2}+v^{-s_2}) \times \frac{(\frac{z+z^{-1}}{2}-1)^{n_1}}{(\frac{v+v^{-1}}{2}-1)^{n_2}} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} \frac{dzdv}{2iz},$$

We now need to deform the contours in (2.38) to steepest descent paths. In other words, we need

$$\text{(2.39)} \qquad \Re(G(\nu_1,\eta_1,\tau,z)) < \Re(G(\nu_1,\eta_1,\tau,\Omega(\nu_1,\eta_1,\tau)))$$

for all z on the z-contour and

$$\text{(2.40)} \qquad \Re(G(\nu_2,\eta_2,\tau,v)) \quad > \quad \Re(G(\nu_2,\eta_2,\tau,\Omega(\nu_2,\eta_2,\tau))),$$

$$\text{(2.41)} \qquad \Re(G(-\nu_2,\eta_2,\tau,v)) \quad > \quad \Re(G(-\nu_2,\eta_2,\tau,\Omega(-\nu_2,\eta_2,\tau)))$$

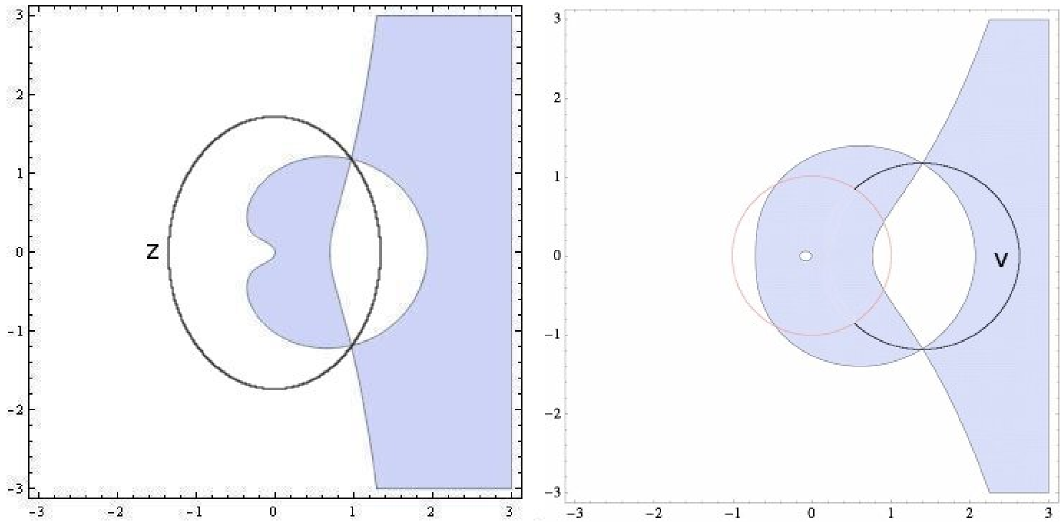
for all v on the v-contour. By Lemma 2.15 and the definition of G, we see the equality $\Re(G(\nu_2,\eta_2,\tau,\Omega(\nu_2,\eta_2,\tau))) = \Re(G(-\nu_2,\eta_2,\tau,\Omega(-\nu_2,\eta_2,\tau)))$. If $|v| \geq 1$, then we have $\Re(G(-\nu_2,\eta_2,\tau,v)) \geq \Re(G(\nu_2,\eta_2,\tau,v))$. Since the steepest descent paths can go completely outside the unit circle (see Proposition 5.1.2 of [3]), (2.41) follows from (2.40).

If we deform the contours to the steepest descent paths Γ_1 and Γ_2 in Figure 12, we get that (2.35) asymptotically becomes

$$\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{\exp(NG(\nu_1,\eta_1,\tau,z))}{\exp(NG(\nu_2,\eta_2,\tau,v))} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} \frac{dvdz}{z}$$
$$+\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{\exp(NG(\nu_1,\eta_1,\tau,z))}{\exp(NG(-\nu_2,\eta_2,\tau,v))} \frac{1-v^{-2}}{z+z^{-1}-v-v^{-1}} \frac{dvdz}{z},$$

plus possibly the residues at $z = v$. Since Γ_2 goes outside the unit circle and the critical point of $G(-\nu_2,\eta_2,\tau,v)$ lies inside the unit circle, the second double integral is negligible.

FIGURE 8. On the left is $\Re(G(\nu_1,\eta_1,\tau,z)-G(\nu_1,\eta_1,\tau,\Omega(\nu_1,\eta_1,\tau)))$, and on the right is $\Re(G(\nu_2,\eta_2,\tau,v)-G(\nu_2,\eta_2,\tau,\Omega(\nu_2,\eta_2,\tau)))$. White regions indicate $\Re<0$ and shaded regions indicate $\Re>0$. The double zero occurs at $\Omega(\nu_j,\eta_j,\tau)$. The arc v goes from $e^{-i\theta}$ to $e^{i\theta}$. The unit circle has been drawn on the right.



Now we need to compute the possible residues at $z=v$. If the contours pass through each other, then the residues at $z=v$ equal

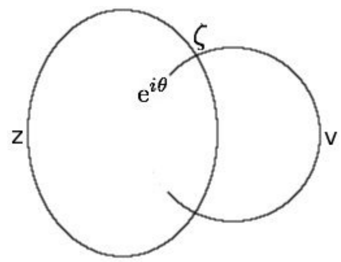
$$\frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i}\int_{e^{i\theta}}^{\zeta} z^{s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z} + \frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i}\int_{\bar{\zeta}}^{e^{-i\theta}} z^{s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z} \tag{2.42}$$

$$+\frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i}\int_{e^{i\theta}}^{\zeta} z^{-s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z} + \frac{W^{(-1/2,-1/2)}(s_1)}{4\pi i}\int_{\bar{\zeta}}^{e^{-i\theta}} z^{-s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z}, \tag{2.43}$$

where ζ is any complex number satisfying (2.39) and (2.40). See Figure 9. If the contours do not pass through each other, then there is no contribution from the residues. For notational convenience, set

$$\xi = \begin{cases} \zeta, & \text{if } \zeta \text{ exists,} \\ e^{i\theta}, & \text{otherwise.} \end{cases}$$

FIGURE 9. The z and v contours from Figure 12. They intersect at ζ.



It is important to note that ξ is arbitrarily selected. The only requirement on ζ is that it satisfies the inequalities (2.39) and (2.40), and the only requirement on $e^{i\theta}$ is that $\Re(G_2(e^{i\theta})) > \Re(G_2(\Omega_2))$. So there exists $\epsilon > 0$ such that if $|\xi_1 - \xi| < \epsilon$, then ξ_1 also satisfies those inequalities.

Now we need to compute (2.36) and (2.37). Expanding the parantheses, we get four terms corresponding to $z^{s_1+s_2}, z^{s_1-s_2}, z^{s_2-s_1}, z^{-s_1-s_2}$. For the terms corresponding $z^{-s_2-s_1}$ and $z^{s_1-s_2}$, make the substitution $z \to z^{-1}$. Therefore, the sum of (2.36),(2.37),(2.42),(2.43) equals

$$\frac{1}{4\pi i}\int_{\bar{\xi}}^{\xi} z^{s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z} + \frac{1}{4\pi i}\int_{\bar{\xi}}^{\xi} z^{-s_2-s_1}\left(\frac{z+z^{-1}}{2}-1\right)^{n_1-n_2}\frac{dz}{z}, \tag{2.44}$$

where the contour crosses $(0,\infty)$ if $n_1 \geq n_2$, and it crosses $(-\infty, 0)$ if $n_1 < n_2$. For each integral, deform the contour to a circular arc of constant radius. It is not a difficult calculus exercise to show that the absolute value of the integrand is maximized at the endpoints.

Using a standard asymptotic analysis (see e.g. chapter 3 of [11]), we get that the asymptotic expansion of (2.44) is

$$\frac{c_1}{N}\xi^{N(\nu_2-\nu_1)}\left(\frac{\xi+\xi^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)}+\frac{\overline{c_1}}{N}\bar{\xi}^{N(\nu_2-\nu_1)}\left(\frac{\bar{\xi}+\bar{\xi}^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)}$$
$$+\frac{c_2}{N}\xi^{N(-\nu_2-\nu_1)}\left(\frac{\xi+\xi^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)}+\frac{\overline{c_2}}{N}\bar{\xi}^{N(-\nu_2-\nu_1)}\left(\frac{\bar{\xi}+\bar{\xi}^{-1}}{2}-1\right)^{N(\eta_1-\eta_2)}$$

for some constants c_1, c_2. To complete the proof, notice that if

$$\left|\xi^{\pm\nu_2-\nu_1}\left(\frac{\xi+\xi^{-1}}{2}-1\right)^{\eta_1-\eta_2}\right|>e^{\Re(G_1(\Omega_1)-G_2(\Omega_2))}$$

for some selection of $\pm$, then the asymptotic expansion of the kernel would depend on ξ. But ξ was arbitrarily selected, so this is impossible.

Now that the fourth condition has been proved, it remains to show that the second condition in Definition 2.3 holds. Recall that $\Omega(\nu,\eta,\tau)$ is the root of $p(\nu,\eta,\tau,z)$ that lies in the upper half-plane, where p is the polynomial from Lemma 2.15. We thus need to solve

$$p(q_2(\eta,\tau)-\epsilon_1,\eta,\tau,\Omega(q_2(\eta,\tau),\eta,\tau)+\epsilon_2)=0.$$

Since $\Omega(q_2(\eta,\tau),\eta,\tau)$ is a double zero of $p(q_2(\eta,\tau),\eta,\tau,z)$, we thus have to solve

$$\frac{1}{2}\epsilon_2^2p''(\Omega)-2\epsilon_1(\Omega+\epsilon_2-\epsilon_2^2-2\epsilon_2\Omega-\Omega^2)+\mathcal{O}(\epsilon_2^3)=0,$$

which implies that $\epsilon_2=\mathcal{O}(\epsilon_1^{1/2})$. In other words, as ν approahces $q_2(\eta,\tau)$, $\Omega(\nu,\eta,\tau)-\Omega(q_2(\eta,\tau),\eta,\tau)=\mathcal{O}((q_2(\eta,\tau)-\nu)^{1/2})$. Plugging this into the expression for G'' gives the result.

2.4. **Asymptotic Lemmas.**

2.4.1. *Riemannian Approximations.*

Lemma 2.16. *Suppose that $g\in C^1[a,b]$ and $I\in C^2[a,b]$. Suppose that as $\delta\to 0$, the Lesbesgue measure of the set $\{x\in[a,b]:I'(x)\in 2\pi\mathbb{Z}+[-\delta,\delta]\}$ is $\mathcal{O}(\delta^a)$ for some positive* *a. Let $\epsilon_N \in [-1,1]$ depend on N. Then*

$$\lim_{N\to\infty} \sum_{k=1}^{\lfloor (b-a)N \rfloor} e^{iNI(a+(k+\epsilon_N)/N)} g\left(a+\frac{k+\epsilon_N}{N}\right)\frac{1}{N} = o(1).$$

Proof. Let t_k denote $a+(k+\epsilon_N)/N$. Note that $|t_k - t_s| = |k-s|/N$. Fix some $1+N^{1/3} \leq s \leq \lfloor (b-a)N \rfloor - N^{1/3}$ and consider

$$\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g\left(t_k\right)\frac{1}{N}.$$

We bound this sum in two cases.

Case 1. Assume $I'(t_s) \notin 2\pi\mathbb{Z} + [-\delta,\delta]$. For $s - N^{1/3} \leq k \leq s+N^{1/3}$, Taylor's theorem says that

$$I(t_k) = [I(t_s) + I'(t_s)(t_k - t_s)] + [\frac{1}{2}I''(c_k)(t_k-t_s)^2] =: [I_1(t_k)] + [I_2(t_k)]$$

for some c_k between t_s and t_k. We will prove that

$$\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g\left(t_k\right)\frac{1}{N} \leq \frac{99\delta^{-1}\|g\|_\infty}{N} + \frac{18\|g\|_\infty \|I''\|_\infty}{N} + \frac{3\|g'\|_\infty}{N^{4/3}}$$

Using the inequality

$$|g(t_k) - g(t_s)| \leq \|g'\|_\infty \cdot |t_k - t_s| = \|g'\|_\infty \frac{|k-s|}{N},$$

we have that

$$\left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)}(g\left(t_k\right) - g(t_s))\frac{1}{N}\right| \leq \sum_{k=s-N^{1/3}}^{s+N^{1/3}} \|g'\|_\infty \frac{|k-s|}{N^2} \tag{2.45}$$

$$= 2\|g'\|_\infty \frac{N^{1/3}(N^{1/3}+1)}{N^2}.$$

Furthermore, for $|k-s| \leq N^{1/3}$,

$$|1-e^{iNI_2(t_k)}| = |1-e^{iI''(c_k)(k-s)^2/(2N)}| \leq |1-e^{i\|I''\|_\infty N^{-1/3}}| \leq 9\|I''\|_\infty N^{-1/3}. \tag{2.46}$$

Also,

$$(2.47)\qquad \left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI_1(t_k)}\right| = \left|e^{iNI(t_s)}\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iI'(t_s)(k-s)}\right|$$

$$\leq \left|\frac{4}{e^{iI'(t_s)}-1}\right| \leq 99\delta^{-1}$$

Using (2.45), the definition of I_1 and I_2, (2.46) and (2.47) respectively,

$$\begin{aligned}
\left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g\left(t_k\right)\frac{1}{N}\right| &\leq \left|g\left(t_s\right)\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)}\frac{1}{N}\right| + 3\|g'\|_\infty N^{-4/3} \\
&\leq \|g\|_\infty \left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} \frac{e^{iNI_1(t_k)} + e^{iNI_1(t_k)}(e^{iNI_2(t_k)}-1)}{N}\right| + \frac{3\|g'\|_\infty}{N^{4/3}} \\
&\leq \|g\|_\infty \left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} \frac{e^{iNI_1(t_k)}}{N}\right| + \frac{18\|g\|_\infty\|I''\|_\infty}{N} + \frac{3\|g'\|_\infty}{N^{4/3}} \\
&\leq \frac{99\delta^{-1}\|g\|_\infty}{N} + \frac{18\|g\|_\infty\|I''\|_\infty}{N} + \frac{3\|g'\|_\infty}{N^{4/3}}
\end{aligned}$$

Case 2. Assume that $I'(t_s) \in 2\pi\mathbb{Z}+(-\delta,\delta)$. In this case, only a simple estimate is needed:

$$\left|\sum_{k=s-N^{1/3}}^{s+N^{1/3}} e^{iNI(t_k)} g(t_k)\frac{1}{N}\right| \leq \frac{2\|g\|_\infty}{N^{2/3}}.$$

Since the estimate in case 1 is much better than the estimate in case 2, we need an upper bound on how frequently case 2 can occur. In other words, we need an upper bound on the measure of the set $\{x \in [a,b] : I'(x) \in 2\pi\mathbb{Z} + (\delta,\delta)\}$. We assumed that

$$|\{x \in [a,b] : I'(x) \in 2\pi\mathbb{Z} + [\delta,\delta]\}| = \mathcal{O}(\delta^a).$$

Now we need to sum over all s in the set $\{N^{1/3}+1, 3N^{1/3}+1, 5N^{1/3}+1\ldots,(b-a)N-N^{1/3}\}$. There are $\mathcal{O}(\delta^a N^{2/3})$ terms for which case 2 applies. Therefore,

$$\sum_{k=1}^{\lfloor(b-a)N\rfloor} e^{iNI(a+(k+\epsilon_N)/N)} g\left(a+\frac{k+\epsilon_N}{N}\right)\frac{1}{N} = \mathcal{O}\left(\delta^{-1}N^{-1/3}\right) + \mathcal{O}(\delta^a),$$

and setting $\delta = N^{-1/6}$ yields the result. $\square$

Proposition 2.17. *Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ is a function such that for each $t > 0$,*

$$f(\lfloor tN \rfloor) = e^{iNI(t)} g(t) N^d + o(N^d) \text{ as } N \to \infty,$$

where g and I satisfy the same assumptions as in Lemma 2.16. Further suppose that the error term $o(N^d)$ is uniform, i.e.

$$\frac{f(\lfloor tN \rfloor) - e^{iNI(t)} g(t) N^d}{N^d} \to 0 \text{ uniformly on } [a, b].$$

Then as $N \to \infty$,

$$\sum_{x=\lfloor aN \rfloor + 1}^{\lfloor bN \rfloor} f(x) = o(N^{d+1}).$$

Proof. This follows quickly from Lemma 2.16. $\square$

Proposition 2.18. *Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is a function such that for each $t > 0$,*

$$f(\lfloor tN \rfloor) = g(t) N^d + o(N^d) \text{ as } N \to \infty,$$

where g is a function on $[a, b]$ of bounded variation. Further suppose that the error term $o(N^d)$ is uniform, i.e.

$$\frac{f(\lfloor tN \rfloor) - g(t) N^d}{N^d} \to 0 \text{ uniformly on } [a, b].$$

Then

$$\sum_{x=\lfloor aN \rfloor + 1}^{\lfloor bN \rfloor} f(x) = N^{d+1} \int_a^b g(t)dt + o(N^{d+1}).$$

Proof. This is an elementary, albeit somewhat tedious, exercise in approximating integrals with Riemann sums. $\square$

2.4.2. *Asymptotics.*

Proposition 2.19. *For $j = 1, 2$, let $(\nu_j, \eta_j, \tau) \in \mathcal{D}$, Ω_j denote $\Omega(\nu_j, \eta_j, \tau)$, $G_j(z)$ denote $G(\nu_j, \eta_j, \tau, z)$, and θ_j denote $\theta(\nu_j, \eta_j, \tau)$. With the assumptions from section 2.2.1,*

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{\exp(NG(\eta_1, \nu_1, \tau, u))}{\exp(NG(\eta_2, \nu_2, \tau, w))} f(u, w) dw du \\
&\qquad = \mathcal{O}\left(\frac{G_1''(\Omega_1)^{-3} + G_2''(\Omega_2)^{-3}}{G_1''(\Omega_1)^{1/2} G_2''(\Omega_2)^{1/2}} N^{-2}\right) + \mathcal{O}(G_1''(\Omega_1)^{-7/2} G_2''(\Omega_2)^{-7/2} N^{-3}) \\
&\quad + \frac{e^{N\Re((G_1(\Omega_1) - G_2(\Omega_2)))}}{2\pi N \sqrt{|G_1''(\Omega_1)|}\sqrt{|G_2''(\Omega_2)|}} \times \Big[f(\Omega_1, \Omega_2) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{iN\Im(G_2(\Omega_2)) + i\theta_2}} + f(\Omega_1, \bar{\Omega}_2) \frac{e^{iN\Im(G_1(\Omega_1)) - i\theta_1}}{e^{-iN\Im(G_2(\Omega_2)) - i\theta_2}} \\
&\qquad\qquad + f(\bar{\Omega}_1, \Omega_2) \frac{e^{-iN\Im(G_1(\Omega_1)) + i\theta_1}}{e^{iN\Im(G_2(\Omega_2)) + i\theta_2}} + f(\bar{\Omega}_1, \bar{\Omega}_2) \frac{e^{-iN\Im(G_1(\Omega_1)) + i\theta_1}}{e^{-iN\Im(G_2(\Omega_2) - i\theta_2)}} \Big].
\end{aligned}$$

Proof. First, we show that the main term is correct.

By assumption, we can deform Γ_1 and Γ_2 so that Γ_j passes through $\Omega_j, \bar{\Omega}_j$ for $j = 1, 2$. The contributions to the integral away from $\Omega_j, \bar{\Omega}_j$ are exponentially small, so we can replace Γ_j with $\gamma_j \cup \bar{\gamma}_j$, where γ_j and $\bar{\gamma}_j$ are steepest descent paths near Ω_j and $\bar{\Omega}_j$, respectively. The integration over $u \in \gamma_1 \cup \bar{\gamma}_1, w \in \gamma_2 \cup \bar{\gamma}_2$ expands into four integrations corresponding to $(u, w) \in \gamma_1 \times \gamma_2, \bar{\gamma}_1 \times \gamma_2, \gamma_1 \times \bar{\gamma}_2, \bar{\gamma}_1 \times \bar{\gamma}_2$. We explicitly do the calculation for $\gamma_1 \times \gamma_2$. The other three calculations are essentially identical.

Make the substitutions $s = G_1(\Omega_1) - G_1(u)$ and $t = G_2(\Omega_2) - G_2(w)$. In the neighborhood of $u = \Omega_1$ and $w = \Omega_2$, we have

$$f(u, w) \approx f(\Omega_1, \Omega_2), \quad s = -\frac{(u - \Omega_1)^2}{2} G_1''(\Omega_1), \quad t = -\frac{(w - \Omega_2)^2}{2} G_2''(\Omega_2),$$

which imply

$$\begin{aligned}
G_1'(u) &= -\frac{ds}{du} = (u - \Omega_1) G_1''(\Omega_1) = \sqrt{-2sG_1''(\Omega_1)}, \\
G_2'(w) &= -\frac{dt}{dw} = (w - \Omega_2) G_2''(\Omega_2) = \sqrt{-2tG_2''(\Omega_2)}.
\end{aligned}$$

Then we get

$$\begin{aligned}
&\left(\frac{1}{2\pi i}\right)^2 e^{N(G_1(\Omega_1)-G_2(\Omega_2))}\int_0^\infty\int_0^\infty e^{-N(s+t)}\frac{f(u,w)}{G_1'(u)G_2'(w)}dtds\\
&= 4\cdot\frac{e^{N(G_1(\Omega_1)-G_2(\Omega_2))}}{8\pi^2\sqrt{G_1''(\Omega_1)}\sqrt{G_2''(\Omega_2)}}f(\Omega_1,\Omega_2)\Big(\int_0^\infty s^{-1/2}e^{-Ns}ds\Big)\Big(\int_0^\infty t^{-1/2}e^{-Nt}dt\Big)\\
&= \frac{e^{N(G_1(\Omega_1)-G_2(\Omega_2))}}{2\pi N\sqrt{G_1''(\Omega_1)}\sqrt{G_2''(\Omega_2)}}f(\Omega_1,\Omega_2)\\
&= \frac{e^{N\Re((G_1(\Omega_1)-G_2(\Omega_2)))}}{2\pi N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_2''(\Omega_2)|}}\Big[f(\Omega_1,\Omega_2)\frac{e^{iN\Im(G_1(\Omega_1))-i\theta_1}}{e^{iN\Im(G_2(\Omega_2))+i\theta_2}}\Big],
\end{aligned}$$

where the last equality follows from $G(\bar{z})=\overline{G(z)}$. The 4 appears because the maps $u\mapsto s$ and $w\mapsto t$ are both two-to-one.

It still remains to show that the error term is correct. The remainder of this section is devoted to proving this. The idea is to reduce the double integral to progressively simpler forms. First, by a reparametrization, the integral over two arcs in $\mathbb{C}$ can be written as a integral in $\mathbb{R}^2$. Second, by using a Taylor approximation, the integral in $\mathbb{R}^2$ can be written as a product of two integrals in $\mathbb{R}$, each of which is of the form $\int e^{-NR(t)}\phi(t)dt$, where $R(t)$ has a maximum $t_{\max}$ in the interval of integration. Third, by using the implicit function theorem, this integral reduces to the form $\int e^{-Nt^2}g(t)$, where the interval of integration is a small neighbourhood $t_{\max}$. Fourth, this last integral is a slight generalization of $\int e^{-Nt}g(t)dt$, which is dealt with by the well-known Watson's lemma (Lemma 2.20 below). Since the first two steps have been done before (see Chapters 3 and 4 of [11]), we will focus mostly on the third and fourth steps.

Lemma 2.20. *Suppose that R and ϕ are infinitely continuously differentiable in some neighbourhood of t_{max}. Also suppose that t_{max} is a local maximum of R and $R''(t_{max})<0$. Then* *for any $N > 1$ and $s \in [0, m^2]$,*

$$\left| \int_{t_{max}-\delta_1}^{t_{max}+\delta_2} e^{NR(t)} \phi(t) dt - \phi(t_{max}) e^{NR(t_{max})} \sqrt{\frac{-2\pi}{NR''(t_{max})}} \right| \leq \frac{\sqrt{\pi}}{2} \frac{\sup_{0 \leq \tau \leq s} |g'(\tau)|}{N^{3/2}}$$

$$+ e^{-Ns} \int_s^{m^2} |h(t)| dt + e^{-Nm^2} \int_m^{\max(\alpha,\beta)} |h(t)| dt + h(0) \int_s^\infty e^{-Nt} t^{-1/2} dt$$

$$+ e^{-Ns/2} \sup_{0 \leq \tau \leq s} |g'(\tau)|,$$

where

$$\alpha = \sqrt{-R(t_{max} - \delta_1)}, \ \beta = \sqrt{-R(t_{max} + \delta_2)}, \ m = \min(\alpha, \beta),$$

$$h(s) = \phi(t_{max} + sv(s))(sv'(s) + v(s)), \ g(s) = \frac{1}{2}(h(s^{1/2}) + h(-s^{1/2})),$$

where $v(s)$ is an infinitely differentiable function solving

$$- R(t_{max}) + R(t_{max} + sv(s)) = -s^2. \tag{2.48}$$

Proof. This is a slight generalization of Watson's lemma (e.g. Proposition 2.1 of [11]), which deals with asymptotics of integrals of the form $\int_0^T e^{-Nt} \phi(t) dt$. By following pages pages 58–60 of [11], one generalizes to integrals of the form $\int_{-\alpha}^{\beta} e^{-Nt^2} \phi(t) dt$, and then it is not hard to generalize to functions $R(t)$ which behave like $-t^2$ near its maximum. $\square$

Before continuing, a few estimates on $v(s)$ are needed.

Lemma 2.21. *Let $v(s)$ be as in (2.48).*

(a)

$$R''(t_{max}) = -2v(0)^{-2}. \tag{2.49}$$

$$v'(0) = \frac{R'''(t_{max})}{12} v(0)^4 \tag{2.50}$$

(b) Set $B = \sup |R^{(4)}/24|$. Then

$$|v(s) - v(0) - v'(0)s| < \left(\frac{5}{144} R'''(t_{max})^2 v(0)^7 + Bv(0)^5 \right) s^2.$$

for

$$s < \min(|53R'''(t_{max})|/(625Bv(0)), (|R'''(t_{max})|v(0)^3)^{-1}, (50\sqrt{B}v(0)^2)^{-1}).$$

In particular, $|v(s) - v(0)| < |R'''(t_{max})|v(0)^4|s|/4$ *and* $|v(s) - v(0)| < v(0)/4$.

(c) Let

$$a_3 := \left(\frac{157}{16}Bv(0)^3 + \frac{101}{288}R'''(t_{max})^2v(0)^5\right).$$

Then

$$|v(s) + sv'(s) - v(0) - 2v'(0)s| < \left(\frac{39}{32}R'''(t_{max})^2v(0)^7 + \frac{471}{16}Bv(0)^5\right)s^2$$

$$\text{(2.51)} \qquad \textit{for} \ \ |s| < \min\left(\frac{|53R'''(t_{max})|}{625Bv(0)}, \frac{1}{50\sqrt{B}v(0)^2}, \frac{1}{6R'''(t_{max})v(0)^3}, \left|\sqrt{\frac{2v(0)^{-1}}{3a_3}}\right|\right).$$

(d) With the same bounds on $|s|$,

$$|2v'(s) + sv''(s) - 2v'(0)| < 450000(R'''(t_{max})^2v(0)^7 + Bv(0)^5)s$$

Proof. (a) The proof comes from page 69 of [11]. It follows immediately from using implicit differentiation of (2.48) and setting $s = 0$.

(b) First notice that if $R_-(t) \le R(t) \le R_+(t)$ with $R_-(t_{\max}) = R(t_{\max}) = R_+(t_{\max})$ and $v_\pm$ are the solutions to $-R(t_{\max}) + R(t_{\max} \pm sv_\pm(s)) = -s^2$, then $v_- \le v \le v_+$. We will use

$$R_\pm(t) = R(t_{\max}) + \frac{1}{2}R''(t_{\max})(t - t_{\max})^2 + \frac{1}{6}R'''(t_{\max})(t - t_{\max})^3 \pm B(t - t_{\max})^4$$

Therefore, we obtain bounds on $v(s)$ by solving $-R(t_{\max}) + R(t_{\max} \pm sv_\pm(s)) = -s^2$, which is equivalent to solving

$$Q_{\pm,s}(y) := 1 - y_0^{-2}y^2 + Asy^3 \pm Bs^2y^4 = 0, \quad A = R'''(t_{\max})/6, \quad y_0 = v(0).$$

In other words $Q_{\pm,s}(v(s)) = 0$. Taking the derivative of $Q_{\pm,s}(v(s)) = 0$ with respect to s and setting $s = 0$, observe that $v'(0) = Ay_0^4s/2$. We will use the intermediate value theorem to estimate roots of $Q_{\pm,s}$.

For $\epsilon = Ay_0^4 s/2 + (5A^2y_0^7/4 + By_0^5)s^2$ and $|s| = (HAy_0^3)^{-1}$ where H is any real number, we have

$$Q_{+,s}(y_0+\epsilon) \le \frac{1}{256A^{10}H^{10}y_0^{10}}(256B^5 + 256A^2B^4p_2(H)y_0^2 + 32A^4B^3p_4(H)y_0^4 + 16A^6B^2p_6(H)y_0^6 + A^8Bp_8(H)y_0^8 + 4A^{10}H^3p_5(H)y_0^{10})$$

where the p_i are polynomials which satisfy the following inequalities when $|H| > 6$:

$$p_2(H) < 11H^2, p_4(H) < 371H^4, p_6(H) < 1447H^6, p_8(H) < -10H^8, p_5(H) < 0.$$

Now setting $H := hy_0^{-1}A^{-1}\sqrt{B}$ where $h > 50$,

$$Q_{+,s}(y_0+\epsilon) < \frac{128 + 1408h^2 + 5936h^4 + 11576h^6 - 5h^8}{128h^{10}} < 0.$$

Since

$$Q_{+,s}(y_0 + Ay_0^4 s/2) > \frac{1}{16}s^2y_0^4(B(2+Asy_0^3)^4 + 2A^2y_0^2(10 + 6Asy_0^3 + A^2s^2y_0^6)) > 0,$$

this implies that $v_+(s) < y_0 + \epsilon = v(0) + v'(0)s + (5A^2v(0)^7/4 + Bv(0)^5)s^2$ for $|s| < \min((6|A|v(0)^3)^{-1}, (50\sqrt{B}v(0)^2)^{-1})$.

By applying a similar argument to $Q_{-,s}$, one can show that $v(0) + v'(0)s - (5A^2v(0)^7/4 + Bv(0)^5)s^2 < v_-(s) < y_0$. Thus the lower bound holds in both cases.

The last statement follows because

$$\begin{aligned} |v(s) - v(0)| &< |v'(0)| \cdot |s| + \left(\frac{5}{144}R'''(t_{\max})^2v(0)^7 + Bv(0)^5\right)s^2 \\ &< \frac{|R'''(t_{\max})|}{12}v(0)^4|s| + \frac{5}{144}\frac{R'''(t_{\max})^2v(0)^7}{|R'''(t_{\max})|v(0)^3}|s| + Bv(0)^5\frac{53|R'''(t_{\max})|}{625Bv(0)}|s| \\ &< \frac{|R'''(t_{\max})|v(0)^4}{4}|s| < \frac{v(0)}{4}. \end{aligned}$$

(c) Differentiating (2.48) yields

$$(v(s) + sv'(s)) = \frac{-2s}{R'(t_{\max} + sv(s))}.$$

To estimate this, let us first estimate R'.

By a Taylor expansion,

$$|R'(t_{\max}+sv(s))-R''(t_{\max})sv(s)-\frac{1}{2}R'''(t_{\max})s^2v(s)^2|\le 4Bs^3v(s)^3. \tag{2.52}$$

By the triangle inequality and part (b),

$$\begin{aligned}
&\left|R'(t_{\max}+sv(s))-R''(t_{\max})s(v(0)+sv'(0))-\frac{1}{2}R'''(t_{\max})s^2v(0)^2\right| \\
&\quad\le 4Bv(s)^3s^3-R''(t_{\max})((v(s)-v(0)-v'(0)s)s+\frac{1}{2}(v(s)-v(0))(v(s)+v(0))s^2 \\
&\qquad\le \frac{125}{16}Bv(0)^3s^3-R''(t_{\max})\left(\frac{5}{144}R'''(t_{\max})^2v(0)^7+Bv(0)^5\right)s^3 \\
&\qquad\qquad+\frac{1}{2}\cdot\frac{1}{4}R'''(t_{\max})^2v(0)^4\cdot\frac{9}{4}v(0)s^3,
\end{aligned}$$

which, by (2.49) and (2.50), implies

$$\begin{aligned}
&\left|R'(t_{\max}+sv(s))+2v(0)^{-1}s-4v(0)^{-2}v'(0)s^2\right| \\
&\qquad\le\left(\frac{157}{16}Bv(0)^3+\frac{101}{288}R'''(t_{\max})^2v(0)^5\right)s^3=:a_3s^3.
\end{aligned} \tag{2.53}$$

To estimate the inverse of R', use

$$\begin{aligned}
\left|\frac{1}{a_1s+a_2s^2+a_3s^3}-\frac{1}{a_1s}+\frac{a_2}{a_1^2}\right|&=\left|\frac{(a_2^2-a_1a_3)s^2+a_2a_3s^3}{a_1^2(a_1s+a_2s^2+a_3s^3)}\right|\le\left|\frac{6(a_2^2+|a_1a_3|)}{a_1^3}s\right| \\
&\qquad\text{for}\quad |s|<\min(\left|\frac{a_1}{3a_2}\right|,\left|\sqrt{\frac{a_1}{3a_3}}\right|),
\end{aligned}$$

which, by setting $a_1=2v(0)^{-1}$ and $a_2=4v(0)^{-2}v'(0)$, implies that

$$\left|\frac{1}{R'(t_{\max}+sv(s))}+\frac{v(0)}{2s}+v'(0)\right|\le\frac{R'''(t_{\max})^2v(0)^7+18v(0)^2|a_3|}{12}|s|. \tag{2.54}$$

Multiplying by $2|s|$ finishes the proof of (c).

(d) Differentiating (2.48) twice yields

$$2v'(s) + sv''(s) = \frac{-2 - R''(t_{\max} + sv(s))(v(s) + sv'(s))^2}{R'(t_{\max} + sv(s))}. \tag{2.55}$$

From part (c) and a Taylor approximation for R'',

$$\begin{aligned}\left|-2 - R''(t_{\max} + sv(s))(v(s) + sv'(s))^2\right| &< 999((R'''(t_{\max})^2 v(0)^6 + Bv(0)^4)s^2 \\ &+ (R'''(t_{\max})^3 v(0)^9 + R'''(t_{\max})Bv(0)^7)s^3 \\ &+ (R'''(t_{\max})^4 v(0)^{12} + R'''(t_{\max})^2 Bv(0)^{10} + 10B^2 v(0)^8)s^4 \\ &+ (R'''(t_{\max})^5 v(0)^{15} + R'''(t_{\max})^3 Bv(0)^{13} + 10R'''(t_{\max})B^2 v(0)^{11})s^5 \\ &+ (R'''(t_{\max})^6 v(0)^{18} + R'''(t_{\max})^4 Bv(0)^{16} + 10R'''(t_{\max})^2 B^2 v(0)^{14} + 10B^3 v(0)^{12})s^6).\end{aligned}$$

Since $|s| < (R'''(t_{\max})v(0)^3)^{-1}, R'''(t_{\max})(Bv(0))^{-1}$,

$$\left|-2 - R''(t_{\max} + sv(s))(v(s) + sv'(s))^2\right| < 99999(R'''(t_{\max})^2 v(0)^6 + Bv(0)^4)s^2. \tag{2.56}$$

By (2.54) and the estimates on $|s|$,

$$\left|\frac{1}{R'(t_{\max} + sv(s))} + \frac{v(0)}{2s} + v'(0)\right| \le 30R'''(t_{\max})v(0)^4. \tag{2.57}$$

Combining (2.55),(2.56) and (2.57),

$$\begin{aligned}|2v'(s) + sv''(s) - 2v'(0)| &< 50000(R'''(t_{\max})^2 v(0)^7 + Bv(0)^5)s \\ &+ 400000(R'''(t_{\max})^3 v(0)^{10} + R'''(t_{\max})Bv(0)^8)s^2,\end{aligned}$$

and using $|s| < (R'''(t_{\max})v(0)^3)^{-1}$ on the second term gives the result.

□

Corollary 2.22. *Suppose that R and ϕ are infinitely continuously differentiable in some neighbourhood of t_{max}. Also suppose that t_{max} is a local maximum of R and $R''(t_{max}) < 0$.*

Let δ_1 and δ_2 be positive numbers such that

$$m^2 := -R(t_{max} - \delta_1) = -R(t_{max} + \delta_2),$$

and assume m^2 equals the right-hand side of (2.51)*. Let*

$$\tilde{s} = \min\left(\frac{R'''(t_{max})}{50Bv(0)}, \frac{1}{50R'''(t_{max})v(0)^3}\right),$$

$$\Lambda := 500R'''(t_{max})\|\phi'\|_\infty v(0)^5 + 450000\|\phi\|_\infty (R'''(t_{max})^2 v(0)^7 + Bv(0)^5).$$

Then for any $N > 1$,

$$\left|\int_{t_{max}-\delta_1}^{t_{max}+\delta_2} e^{NR(t)}\phi(t)dt - \phi(t_{max})\sqrt{\frac{-2\pi}{NR''(t_{max})}}\right| \leq$$

$$\frac{\sqrt{\pi}}{2}\frac{\Lambda}{N^{3/2}} + \phi(t_{max})\sqrt{\frac{-2}{R''(t_{max})}}\frac{e^{-N\tilde{s}}}{\sqrt{\tilde{s}}N} + e^{-N\tilde{s}^2/2}\Lambda.$$

Proof. Use Lemma 2.20. By part (a) of Lemma 2.21,

$$h(0) = \phi(t_{\max})\sqrt{\frac{-2}{R''(t_{\max})}}.$$

By parts (c) and (d) of Lemma 2.21,

$$\sup_{0 \leq \tau \leq m^2} |g'(\tau)| \leq \Lambda.$$

When $v(0) > \sqrt{B}/R'''(t_{\max})$,

$$\frac{53}{625}\frac{R'''(t_{\max})}{Bv(0)} > \frac{53}{625R'''(t_{\max})v(0)^3}$$

$$\frac{1}{50\sqrt{B}v(0)^2} > \frac{1}{50R'''(t_{\max})v(0)^3}$$

$$a_3 < 16R'''(t_{\max})^2 v(0)^5$$

$$\sqrt{\frac{2v(0)^{-1}}{3a_3}} > \frac{1}{5R'''(t_{\max})v(0)^3},$$

implying $m^2 > (50R'''(t_{max})v(0)^3)^{-1} \geq \tilde{s}$. Similarly, when $v(0) < \sqrt{B}/R'''(t_{\max})$, then $m^2 > R'''(t_{max}))/(50Bv(0)) \geq \tilde{s}$. Thus $m^2 > \tilde{s}$, and

$$\int_{m^2}^{\infty} e^{-Nt} t^{-1/2} dt \leq \frac{e^{-Nm^2}}{mN} \leq \frac{e^{-N\tilde{s}}}{\sqrt{\tilde{s}}N}.$$

□

We can finally wrap up the proof of Proposition 2.19. Since s is not too small (polynomial in $v(0)^{-1}$), the exponential terms are small enough to be ignored. Therefore the error term is $\Lambda_1 = \mathcal{O}(v(0)^7 N^{-3/2}) = \mathcal{O}(R_1''(t_{\max})^{-7/2} N^{-3/2})$. The main term is of order $N^{-1/2} R_1''(t_{\max})^{-1/2}$. Thus, when multiplying two integrals of the form in Corollary 2.22, we get

$$\mathcal{O}\left(\frac{R_1''(t_{\max_1})^{-3} + R_2''(t_{\max_2})^{-3}}{R_1''(t_{\max_1})^{1/2} R_2''(t_{\max_2})^{1/2}} N^{-2}\right) + \mathcal{O}(R_1''(t_{\max_1})^{-7/2} R_2''(t_{\max_2})^{-7/2} N^{-3}),$$

as needed. □

In Proposition 2.19, the error term blows up at the edge. Therefore a better bound is needed. To get this bound, we simply use the first term in Watson's lemma, as opposed to using two terms. Since the method of the proof is identical as before and the details are simpler, the proof will be omitted. The exact statement is the following.

Proposition 2.23. *For $j = 1, 2$, let $(\nu_j, \eta_j, \tau) \in \mathcal{D}$, Ω_j denote $\Omega(\nu_j, \eta_j, \tau)$, $G_j(z)$ denote $G(\nu_j, \eta_j, \tau, z)$, and θ_j denote $\theta(\nu_j, \eta_j, \tau)$. With the assumptions in section 2.2.1,*

$$\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{\exp(NG(\eta_1, \nu_1, \tau, u))}{\exp(NG(\eta_2, \nu_2, \tau, w))} f(u, w) dw du$$
$$\leq \frac{1000}{N\sqrt{|G_1''(\Omega_1)|}\sqrt{|G_2''(\Omega_2)|}} \times \left[|f(\Omega_1, \Omega_2)| + \left|f(\Omega_1, \bar{\Omega}_2)\right| + \left|f(\bar{\Omega}_1, \Omega_2)\right| + \left|f(\bar{\Omega}_1, \bar{\Omega}_2)\right| \right]$$

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Acknowledgments. The author would like to thank Alexei Borodin and Maurice Duits for insightful comments. The author was supported by the National Science Foundation's Graduate Research Fellowship Program.

3. Symmetric Pearcey Process

Abstract. We examine a discrete-time Markovian particle system on $\mathbb{N} \times \mathbb{Z}_+$ introduced in [8]. The boundary $\{0\} \times \mathbb{Z}_+$ acts as a reflecting wall. The particle system lies in the Anisotropic Kardar-Parisi-Zhang with a wall universality class. After projecting to a single horizontal level, we take the long–time asymptotics and obtain the discrete Jacobi and symmetric Pearcey kernels. This is achieved by showing that the particle system is identical to a Markov chain arising from representations of $O(\infty)$ (introduced in [6]). The fixed–time marginals of this Markov chain are known to be determinantal point processes, allowing us to take the limit of the correlation kernel.

We also give a simple example which shows that in the multi-level case, the particle system and the Markov chain evolve differently.

3.1. **Introduction.**

In the study of random interface growth, universality is a ubiquitous topic. Informally, universality says that random growth models with similar physical properties will have identical behavior at long–time asymptotics. In particular, different models in the same universality class are expected to have the same growth exponents and limiting distributions. In this sense, the classical central limit theorem is a universality statement, where the growth exponent is 1/2 and the limiting distribution is Gaussian, regardless of the distribution of each summand.

A different universality class, called the Kardar-Parisi-Zhang (KPZ) universality class (introduced in [13]), models a variety of real–world growth processes, such as turbulent liquid crystals [15] and bacteria colony growth [19]. If $h(\vec{x}, t)$ is the height of the interface at location $\vec{x}$ and time t, then it satisfies the stochastic differential equation

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\vec{x}, t),$$

where $\eta(\vec{x}, t)$ is space–time white noise. Due to the non–linearity, however, this stochastic differential equation is not well–defined. A common mathematical approach has been to study exactly solvable models (i.e. where the finite–time probability distributions can be computed exactly) in the universality class, and then to take the long–time limits. Examples of such models include random matrix theory [17], the PNG droplet [14], ASEP [16], non–intersecting Brownian motions [1], and random partitions [5]. In all of these models, the growth exponent is $1/3$ and the limiting distribution is called the Airy process, demonstrating the universality of the KPZ equation. More recently, there have also been mathematically rigorous interpretations of a solution to the KPZ equation [2, ?H].

The universality class considered in this paper is called anisotropic Kardar-Parisi-Zhang (AKPZ) with a wall. It is a variant of KPZ in two ways: there is anisotropy and the substrate acts as a reflecting barrier. As before, the stochastic differential equation is not well–defined, so we take the approach of analyzing exactly solvable models. So far, there have only been two models which have been proven to be in this universality class: a randomly growing stepped surface in $2+1$ dimensions [6] and non–intersecting squared Bessel paths [12]. In both cases, the limiting behavior near the critical point of the barrier has growth exponent $1/4$ and limiting process the Symmetric Pearcey process (defined in section 3.8).

The exactly solvable model considered here was introduced in [8]. It is a discrete-time interacting particle system with a wall which evolves according to geometric jumps with a parameter $q \in [0,1)$. In the $q \to 1$ limit, this model also has connections to a random matrix model. The main result of this paper is that in the long–time asymptotics near the wall, the symmetric Pearcey process appears after rescaling by $N^{1/4}$. This therefore helps to establish the universality of the growth exponent $1/4$ and the Symmetric Pearcey process in the AKPZ with a wall universality class. The approach is to show that when projected to a single level and to (finite) integer times, the particle system is identical to a previously studied family of determinantal point process. Taking asymptotics of the correlation kernel then yields the desired results.

We will also show that away from the critical point and at finite distances from the wall, the discrete Jacobi kernel appears in the long–time asymptotics. This kernel also appeared in the long–time limit in [6], but has not appeared anywhere else. In particular, it did not appear in non–intersecting squared Bessel paths [11].

In section 3.2, we review the particle system from [8] and the determinantal point processes from [6]. In section 3.5, we compute the correlation kernel for the particle system on one level by showing that the two processes are identical. In section 3.6, we take the large-time asymptotics.

The models in [8] and [6] have connections to the representation theory of the orthogonal groups, but this paper is intended to be understandable without knowledge of representation theory.

It should also be true that given the initial conditions, the fixed-time distributions for the two models are identical without needing to restrict to a single level, but this is not pursued here.

Acknowledgements. The author would like to thank Alexei Borodin, Manon Defosseux, Ivan Corwin and the referees for helpful comments.

3.2. **Two Models.**

3.2.1. *Interacting Particle System.* The interacting particle system in [8] arises from a Pieri-type formula for the (finite-dimensional) orthogonal groups. Here, we briefly describe the model.

The particles live on the lattice[2] $\mathbb{N}\times\mathbb{Z}_+$. The horizontal line $\mathbb{N}\times\{k\}$ is often called the kth level. There are always $\lfloor \frac{k+1}{2}\rfloor$ particles on the kth level, whose positions at time n will be denoted $X_1^k(n)\geq X_2^k(n)\geq X_3^k(n)\geq\ldots\geq X_{\lfloor (k+1)/2\rfloor}^k(n)\geq 0$. The time can take integer or half–integer values. For convenience of notation, $X^k(n)$ will denote $(X_1^k(n),X_2^k(n),X_3^k(n),\ldots,X_{\lfloor (k+1)/2\rfloor}^k(n))\in\mathbb{N}^{\lfloor (k+1)/2\rfloor}$. More than one particle may occupy a lattice point. The particles must satisfy the *interlacing property*

$$X_{i+1}^{k+1}(n)\leq X_i^k(n)\leq X_i^{k+1}(n) \tag{3.1}$$

for all meaningful values of k and i. This will be denoted $X^k\prec X^{k+1}$. With this notation, the state space can be described as the set of all sequences $(X^1\prec X^2\prec\ldots)$ where each $X^k\in$

[2] $\mathbb{N}$ denotes the non–negative integers and $\mathbb{Z}_+$ denotes the positive integers.

$\mathbb{N}^{\lfloor (k+1)/2 \rfloor}$. The initial condition is $X_i^k(0) = 0$, called the *densely packed* initial conditions. Now let us describe the dynamics.

For $n \geq 0, k \geq 1$ and $1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$, define random variables

$$\xi_i^k(n+1/2), \quad \xi_i^k(n)$$

which are independent identically distributed geometric random variables with parameter q. In other words, $\mathbb{P}(\xi_1^1(1/2) = x) = q^x(1-q)$ for $x \in \mathbb{N}$. Let $R(x, y)$ be a Markov kernel on $\mathbb{N}$ defined by

$$R(x, y) = \frac{1-q}{1+q} \cdot \frac{q^{|x-y|} + q^{x+y}}{1 + 1_{y=0}},$$

so that $R(x, \cdot)$ is the law of the random variable $|x + \xi_1^1(1) - \xi_1^1(\frac{1}{2})|$.

At time n, all the particles except $X_{(k+1)/2}^k(n)$ try to jump to the left one after another in such a way that the interlacing property is preserved. The particles $X_{(k+1)/2}^k(n)$ do not jump on their own. The precise definition is

$$\begin{aligned} X_{(k+1)/2}^k(n + \tfrac{1}{2}) &= \min(X_{(k+1)/2}^k(n), X_{(k-1)/2}^{k-1}(n + \tfrac{1}{2})) \quad k \text{ odd} \\ X_i^k(n + \tfrac{1}{2}) &= \max(X_i^{k-1}(n), \min(X_i^k(n), X_{i-1}^{k-1}(n + \tfrac{1}{2})) - \xi_i^k(n + \tfrac{1}{2})), \end{aligned}$$

where $X_0^{k-1}(n + \frac{1}{2})$ is formally set to $+\infty$.

At time $n + \frac{1}{2}$, all the particles except $X_{(k+1)/2}^k(n + \frac{1}{2})$ try to jump to the right one after another in such a way that the interlacing property is preserved. The particles $X_{(k+1)/2}^k(n+\frac{1}{2})$ jump according to the law R. The precise definition is

$$X_{(k+1)/2}^k(n+1) = \min(|X_{(k+1)/2}^k(n) + \xi_{(k+1)/2}^k(n+1) - \xi_{(k+1)/2}^k(n + \tfrac{1}{2})|, X_{(k-1)/2}^{k-1}(n))$$

when k is odd and

$$X_i^k(n+1) = \min(X_{i-1}^{k-1}(n + \tfrac{1}{2}), \max(X_i^k(n + \tfrac{1}{2}), X_i^{k-1}(n+1)) + \xi_i^k(n+1)),$$

where $X_0^{k-1}(n+1)$ is formally set to $+\infty$.

Let us explain the particle system. The particles preserve the interlacing property in two ways: by pushing particles above it, and being blocked by particles below it. So, for example, in the left jumps, the expression $\min(X_i^k(n), X_{i-1}^{k-1}(n+\frac{1}{2}))$ represents the location of the particle after it has been pushed by a particle below and to the right. Then the particle attempts to jump to the left, so the term $\xi_i^k(n+\frac{1}{2})$ is subtracted. However, the particle may be blocked a particle below and to the left, so we must take the maximum with $X_i^{k-1}(n)$.

While $X_i^k(n)$ is not simple, applying the shift $\tilde{X}_i^k(n) = X_i^k(n) + \lfloor \frac{k+1}{2} \rfloor - i$ yields a simple process. In other words, $\tilde{X}$ can only have one particle at each location.

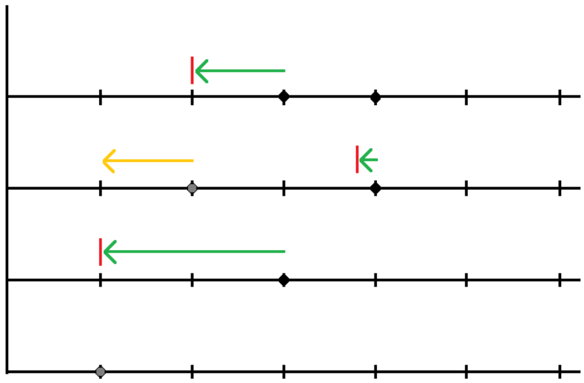
Figure 10 shows an example of $\tilde{X}$. Additionally, an interactive animation can be found at `http://www.math.harvard.edu/~jkuan/DiscreteTimeWithAWall.html`

By drawing lozenges around the particles as in Figure 11, one can see that the particle system can be interpreted as a two–dimensional stepped surface. This can be made rigorous by defining the height function at a point to be the number of particles to the right of that point. With this interpretation, the jumping of the particles corresponds to adding and removing sticks, and therefore the interacting particle system is equivalent to a randomly growing surface. The anisotropy is shown with the observation that only sticks of one type may be added or removed. The necessity of the interlacing condition is also visually apparent: it guarantees that the lozenges can be drawn in a way to make the figure three–dimensional.

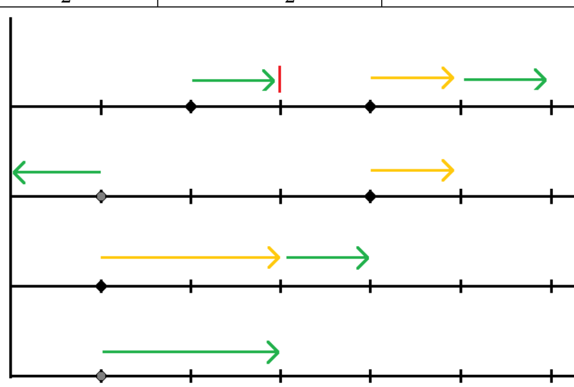
3.3. **Determinantal Point Processes.** In [6], the authors introduce a family of determinantal point processes, indexed by a time parameter $n \in \mathbb{N}$, which arise from representations of the infinite-dimensional orthogonal group. (See [3] for background on determinantal point processes.) This family depends on a function $\phi \in C^1[-1,1]$. Each determinantal point process also lives on the lattice $\mathbb{N} \times \mathbb{Z}_+$, with exactly $\lfloor \frac{k+1}{2} \rfloor$ particles on the kth level, and the particles must also satisfy the interlacing property.

Remark on notation. It is convenient to re-label the levels. For $a = \pm 1/2$, one should think of (r,a) as corresponding to the $2r + a + \frac{1}{2}$ level. Throughout this paper, the letter k

FIGURE 10. The top figure shows left jumps and the bottom figure shows right jumps. A yellow arrow means that the particle has been pushed by a particle below it. A green arrow means that the particle has jumped by itself. A red line means that the particle has been blocked by a particle below. Color online. In the table, keep in mind that $\xi^k_{(k+1)/2}(n+1/2)$ actually correspond to left jumps, but occur at the same time as the right jumps.

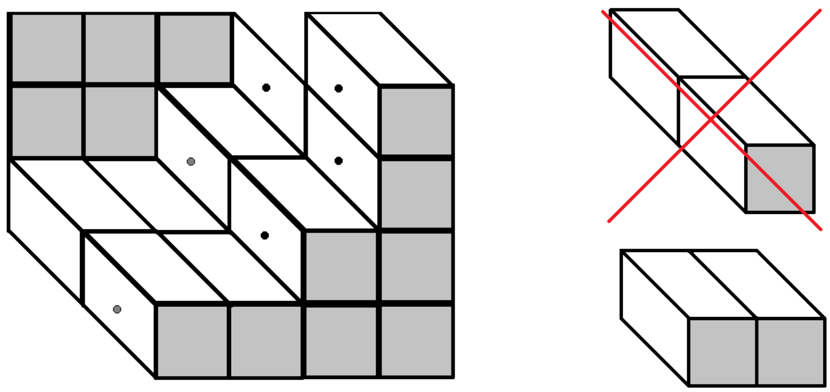


$\tilde{\boldsymbol{X}}(n)$	Left Jumps	$\tilde{\boldsymbol{X}}(n+\frac{1}{2})$	Right jumps	$\tilde{\boldsymbol{X}}(n+1)$
$\tilde{X}^1_1(n)=1$		$\tilde{X}^1_1(n+\frac{1}{2})=1$	$\xi^1_1(n+\frac{1}{2})=1$ $\xi^1_1(n+1)=3$	$\tilde{X}^1_1(n+1)=3$
$\tilde{X}^2_1(n)=3$	$\xi^2_1(n+\frac{1}{2})=3$	$\tilde{X}^2_1(n+\frac{1}{2})=1$	$\xi^2_1(n+1)=1$	$\tilde{X}^2_1(n+1)=4$
$\tilde{X}^3_2(n)=2$		$\tilde{X}^3_2(n+\frac{1}{2})=1$	$\xi^3_2(n+\frac{1}{2})=1$ $\xi^3_2(n+1)=0$	$\tilde{X}^3_2(n+1)=0$
$\tilde{X}^3_1(n)=4$	$\xi^3_1(n+\frac{1}{2})=1$	$\tilde{X}^3_1(n+\frac{1}{2})=4$	$\xi^3_1(n+1)=0$	$\tilde{X}^3_1(n+1)=5$
$\tilde{X}^4_2(n)=3$	$\xi^4_2(n+\frac{1}{2})=2$	$\tilde{X}^4_2(n+\frac{1}{2})=2$	$\xi^4_2(n+1)=2$	$\tilde{X}^4_2(n+1)=3$
$\tilde{X}^4_1(n)=4$	$\xi^4_1(n+\frac{1}{2})=0$	$\tilde{X}^1_1(n+\frac{1}{2})=4$	$\xi^4_1(n+1)=1$	$\tilde{X}^4_1(n+1)=6$



will denote the level and the letter r will denote the number of particles. Set $\mathbb{J}_r$ to be the set of nonincreasing sequences of integers $(\lambda_1 \geq \ldots \geq \lambda_r \geq 0)$. The superscript $\lambda^{(k)}$ will mean that $\lambda^{(k)}$ lives on the kth level. To save space, bold greek letters such as $\boldsymbol{\lambda}$ will denote

FIGURE 11. The figure on the left shows lozenges corresponding to the top figure in Figure 10. The top right figure shows sticks that are never added or removed with each jump, while the bottom right figure shows sticks that are added or removed.



$\boldsymbol{\lambda} = (\lambda^{(1)} \prec \lambda^{(2)} \prec \ldots \prec \lambda^{(k)})$, and similarly for $\boldsymbol{X}$, and $\mathbf{0}$ will denote the densely packed initial conditions.

Let $\boldsymbol{Y}(n)$ denote the positions of the particles in this determinantal point process at time n. Proposition 3.11 of [6] establishes that there is a Markov chain[3] T^{ϕ} connecting $\boldsymbol{Y}(n)$, in the sense that

$$\mathbb{P}(\boldsymbol{Y}(n+1) = \boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda}} \mathbb{P}(\boldsymbol{Y}(n) = \boldsymbol{\lambda}) T^{\phi}(\boldsymbol{\lambda}, \boldsymbol{\mu}). \tag{3.2}$$

Now let us give the formula for T^{ϕ}.

Let $\mathsf{J}_s^{(a,b)}(x)$ denote the (normalized) s-th Jacobi polynomial with parameters a, b. These are polynomials of degree s which are orthogonal with respect to the measure $(1-x)^a(1+x)^b dx$ on $[-1,1]$. In this paper, we just need the equations

$$\begin{aligned} \mathsf{J}_s^{(1/2,-1/2)}\left(\frac{z+z^{-1}}{2}\right) &= \frac{z^{s+1/2} - z^{-s-1/2}}{z^{1/2} - z^{-1/2}}, \\ \mathsf{J}_s^{(-1/2,-1/2)}\left(\frac{z+z^{-1}}{2}\right) &= \frac{z^s + z^{-s}}{2}. \end{aligned}$$

[3]Strictly speaking, Proposition 3.11 proves (3.2) without showing that T^{ϕ} has non–negative entries. A better term would be "signed Markov chain," but this is not standard terminology. In any case, Proposition 3.3 below will show that for the ϕ studied in this paper, T^{ϕ} is a bona–fide Markov chain.

Also define

$$W^{(a,b)}(s) = \begin{cases} 2, & \text{if } \; s > 0, a = b = -\frac{1}{2} \\ 1, & \text{if } \; s = 0, a = b = -\frac{1}{2} \\ 1, & \text{if } \; s \geq 0, a = \frac{1}{2}, b = -\frac{1}{2} \end{cases}$$

For a function $\phi \in C^1[-1,1]$, define

$$I_a^\phi(l,s) = \frac{W^{(a,-1/2)}(s)}{\pi} \int_{-1}^{1} \mathsf{J}_s^{(a,-1/2)}(x) \mathsf{J}_l^{(a,-1/2)}(x) \phi(x)(1-x)^a (1+x)^{-1/2} dx.$$

For $a = \pm\frac{1}{2}$, define the matrix $T_{r,a}^\phi$ with nonnegative entries, and rows and columns parameterized by $\mathbb{J}$:

$$T_{r,a}^\phi(\mu,\lambda) = \det[I_a^\phi(\mu_i - i + r, \lambda_j - j + r)]_{1 \leq i,j \leq r} \frac{\dim_{2r+1/2+a} \lambda}{\dim_{2r+1/2+a} \mu}.$$

Here dim is the dimension of the corresponding representation of $SO(2r+1/2+a)$ – but for the purposes of this paper, it suffices just to know that dim is a positive integer. In the proof of Proposition 3.3, the dim terms will cancel immediately anyway. Set

$$T_k^\phi = \begin{cases} T^\phi_{\lfloor (k+1)/2 \rfloor, 1/2}, & k \;\; even \\ T^\phi_{\lfloor (k+1)/2 \rfloor, -1/2}, & k \;\; odd \end{cases}$$

For λ on the kth level and μ on the $k-1$ level, let $\varkappa_{k-1}^k$ be

$$\varkappa_{k-1}^k(\lambda,\mu) = \begin{cases} 0, & \mu \not\prec \lambda \\ 1, & \mu \prec \lambda \;\; \textit{and } k \textit{ odd} \\ 1, & \mu \prec \lambda, \; \mu_{r/2} = 0, \;\; \textit{and } k \textit{ even} \\ 2, & \mu \prec \lambda, \; \mu_{r/2} > 0, \;\; \textit{and } k \textit{ even} \end{cases}$$

and T_{k-1}^k be

$$T_{k-1}^k(\lambda,\mu) = \frac{\dim_k \mu}{\dim_{k+1} \lambda} \varkappa_{k-1}^k(\lambda,\mu)$$

and

$$\Delta^k_{k-1}(\lambda,\mu)=\sum_\nu T_k(\lambda,\nu)T^k_{k-1}(\nu,\mu)$$

The matrix of transition probabilities is

$$T^\phi(\boldsymbol{\mu},\boldsymbol{\lambda})=T_1^\phi(\mu^{(1)},\lambda^{(1)})\prod_{j=2}^k\frac{T_j^\phi(\mu^{(j)},\lambda^{(j)})T^j_{j-1}(\lambda^{(j)},\lambda^{(j-1)})}{\Delta^j_{j-1}(\lambda^{(j)},\lambda^{(j-1)})}.$$

For the densely packed initial conditions, T^ϕ satisfies a semingroup property. More specifically, if $T^{\phi_1}T^{\phi_2}$ denotes matrix multiplication, then ([6])

$$T^{\phi_1\phi_2}(\mathbf{0},\boldsymbol{\mu})=[T^{\phi_1}T^{\phi_2}](\mathbf{0},\boldsymbol{\mu}).$$

When projected to the kth level, the matrix of transition probabilities is just T_k^ϕ.

Certain functions ϕ arise naturally from the representations of $O(\infty)$ – see section 2.1 of [6]. For our purposes, it suffices to consider the function:

$$\phi_\alpha(x)=(1+\alpha(1-x)+\alpha^2(1-x)/2)^{-1},\quad \alpha\geq 0.$$

By Proposition 4.1 from [6], $\boldsymbol{Y}(n)$ is determinantal with the correlation kernel given by $K(r_1,a_1,s_1;r_2,a_2,s_2)$, equal to

(3.3)

$$\begin{aligned}1_{2r_1+a_1\geq 2r_2+a_2}\frac{W^{(a_1,-1/2)}(s_1)}{\pi}\int_{-1}^1\mathsf{J}_{s_1}^{(a_1,-1/2)}(x)\mathsf{J}_{s_2}^{(a_2,-1/2)}(x)(x-1)^{r_1-r_2}(1-x)^{a_1}(1+x)^{-1/2}dx\\+\frac{W^{(a,-1/2)}(s_1)}{\pi}\frac{1}{2\pi i}\int_{-1}^1\oint_C\frac{\phi_\alpha(x)^n}{\phi_\alpha(u)^n}\mathsf{J}_{s_1}^{(a_1,-1/2)}(x)\mathsf{J}_{s_2}^{(a_2,-1/2)}(u)\\\times\frac{(x-1)^{r_1}}{(u-1)^{r_2}}\frac{(1-x)^{a_1}(1+x)^{-1/2}dudx}{x-u}.\end{aligned}$$

3.4. **Finite–time distributions.** The next theorem, which will be proved in the next section, establishes that $\tilde{X}^k$ is a determinantal point process.

Theorem 3.1. *Let* $\alpha=2q/(1-q)$. *Then* $\tilde{X}^k(n)=Y^k(n)$. *In particular,* $\tilde{X}^k(n)$ *is a determinantal point process on* $\mathbb{N}$ *with kernel* $K(r,a,s_1;r,a,s_2)$, *where* $2r+1/2+a=k$.

Numerical calculations made by the author indicate that the fixed time marginals on multiple levels should also be identical, assuming the densely packed initial conditions. The exact statement is below:

Conjecture 3.2. *For any time $n \geq 0$,*

$$\mathbb{P}(\boldsymbol{X}(n) = \boldsymbol{\lambda}) = T^{\phi_\alpha^n}(\mathbf{0}, \boldsymbol{\lambda}).$$

Note that without the fixed-time assumption, the conjecture is false. For example,

$$\mathbb{P}(\boldsymbol{X}(n+1) = (0, 0, (0, 0)) | \boldsymbol{X}(n) = (0, 1, (1, 0))) = 0,$$

by the fact that X_1^2 prevents X_1^3 from jumping to 0. However,

$$T^{\phi_\alpha}((0, 1, (1, 0)), (0, 0, (0, 0))) \neq 0,$$

since none of the terms in the definition of T^{ϕ_α} is zero.

3.5. **Proof of theorem 3.1.** Let $P_k(\lambda, \beta)$ denote the transition kernel of X on the kth level. In other words

$$P_k(\lambda, \beta) = \mathbb{P}(X^k(n+1) = (\beta_1, \beta_2, \ldots, \beta_{\lfloor (k+1)/2 \rfloor}) | X^k(n) = (\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor (k+1)/2 \rfloor}))$$

By Theorem 7.1 of [8],

$$P_{2r}(\lambda, \beta) = \sum_{c \in \mathbb{N}^r, c \prec \lambda, \beta} (1-q)^{2r} \frac{\dim_{2r+1} \beta}{\dim_{2r+1} \lambda} q^{\sum_{i=1}^r \lambda_i + \beta_i - 2c_i} \left(1_{c_r > 0} + \frac{1_{c_r = 0}}{1+q} \right)$$

$$P_{2r+1}(\lambda, \beta) = \sum_{c \in \mathbb{N}^r, c \prec \lambda, \beta} (1-q)^{2r+1} \frac{\dim_{2r+2} \beta}{\dim_{2r+2} \lambda} q^{\sum_{i=1}^r \lambda_i + \beta_i - 2c_i} R(\lambda_{r+1}, \beta_{r+1}).$$

In this section, we prove Theorem 3.1. Set $\phi = \phi_\alpha$, where $\alpha = \frac{2q}{1-q}$. Since $\boldsymbol{X}(0) = \boldsymbol{Y}(0) = 0$, the following proposition suffices.

Proposition 3.3. *For $a = \pm\frac{1}{2}$, $T_k^{\phi_\alpha} = P_k$.*

We start with a few lemmas.

Lemma 3.4. *Let*

$$\phi(x) = \frac{1}{1+\alpha(1-x)+\frac{\alpha^2}{2}(1-x)}, \quad \alpha = \frac{2q}{1-q}.$$

Then $I^{\phi}_{-1/2}(l,k) = R(l,k)$ *and*

$$I^{\phi}_{1/2}(l,k) = \frac{q-1}{q+1}\left(q^{k+l+1} - q^{|k-l|}\right)$$

Proof. Substitute $x = (z+z^{-1})/2$. Then

$$I^{\phi}_{-1/2}(l,k) = \frac{W^{(-1/2,-1/2)}(k)}{2\pi i}\oint_{|z|=1} \frac{z^k+z^{-k}}{2}\frac{z^l+z^{-l}}{2}\frac{(1-q)^2}{(1-qz)(z-q)}dz,$$

which has residues at q and 0. The residue at $z=q$ is

$$W^{(-1/2,-1/2)}(k)\frac{q^k+q^{-k}}{2}\frac{q^l+q^{-l}}{2}\frac{1-q}{1+q}.$$

Using the expansion

$$\frac{1}{(1-qz)(z-q)} = \sum_{m=0}^{\infty}\frac{q^{m+1}-q^{-m-1}}{1-q^2}z^m,$$

the residue at $z=0$ is

$$W^{(-1/2,-1/2)}(k)\frac{1-q}{1+q}\left(\frac{q^{k+l}-q^{-k-l}}{4} + \frac{q^{|k-l|}-q^{-|k-l|}}{4}\right),$$

so the total contribution is

$$I^{\phi}_{-1/2}(l,k) = \frac{W^{(-1/2,-1/2)}(k)}{2}\frac{1-q}{1+q}(q^{k+l}+q^{|k-l|}) = R(l,k).$$

For $a = 1/2$,

$$-\frac{1}{4\pi i}\oint_{|z|=1}(z^{k+1/2}-z^{-k-1/2})(z^{l+1/2}-z^{-l-1/2})\frac{(1-q)^2}{(1-qz)(z-q)}dz.$$

The residue at $z=q$ is

$$-\frac{1}{2}(q^{k+1/2}-q^{-k-1/2})(q^{l+1/2}-q^{-l-1/2})\frac{1-q}{1+q}$$

and the residue at $z = 0$ is

$$-\frac{1}{2}\frac{1-q}{1+q}\left(q^{k+l+1} - q^{-k-l-1} - q^{|k-l|} + q^{-|k-l|}\right).$$

□

Lemma 3.5. *Let* $c = (c_1 \geq c_2 \geq \ldots \geq c_r)$ *and* $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r)$. *Set*

$$\psi_m(s,l) = \begin{cases} m, & \text{if } l \geq s = 0 \\ 1, & \text{if } l \geq s > 0 \\ 0, & \text{if } l < s. \end{cases}$$

Then

$$\det[\psi_m(c_i - i + r, \lambda_j - j + r)] = \begin{cases} m, & \text{if } c \prec \lambda,\ c_r = 0 \\ 1, & \text{if } c \prec \lambda,\ c_r > 0 \\ 0, & \text{if } c \not\prec \lambda. \end{cases}$$

Proof. The proof is standard, see e.g. Lemma 3.8 of [6]. □

Now return to the proof of Theorem 3.1. Start with the odd case. By the lemma, we can write

$$P_{2r+1}(\lambda,\beta) = (1-q)^{2r}\frac{\dim_{2r+1}\beta}{\dim_{2r+1}\lambda}\sum_{s_1 > \ldots > s_r \geq 0}\det[f_{s_i,1}(\lambda_j - j + r)]\det[f_{s_i,\frac{1}{1+q}}(\beta_j - j + r)],$$

where

$$f_{s,m}(l) = q^{l-s}\psi_m(s,l).$$

Thus, by Lemma 2.1 of [6],

$$P_{2r+1}(\lambda,\beta) = (1-q)^{2r}\frac{\dim_{2r+1}\beta}{\dim_{2r+1}\lambda}\det\left[\sum_{s=0}^{\infty} f_{s,1}(\lambda_i - i + r) f_{s,\frac{1}{1+q}}(\beta_j - j + r)\right].$$

A simple calculation shows that

$$\sum_{s=0}^{\infty} f_{s,1}(x) f_{s,\frac{1}{1+q}}(y) = \begin{cases} \dfrac{1}{1+q} q^{x+y}, \text{ if } & \min(x,y)=0, \\ \dfrac{q^{x+y+1} - q^{|x-y|}}{q^2-1}, \text{ otherwise,} \end{cases}$$

which, by Lemma 3.4, equals $(1-q)^{-2} I^{\phi}_{1/2}(x,y)$.

Now proceed to the even case. Lemma 2.1 from [6] is not immediately applicable, because we are summing over elements of $\mathbb{N}^{r-1}$ while the determinants are of size r. Notice, however, that $c \prec \lambda, \beta$ if and only if $c \prec \lambda_{\text{red}}, \beta_{\text{red}}$ (where $\lambda_{\text{red}}, \beta_{\text{red}}$ denote $(\lambda_1, \dots, \lambda_{r-1}), (\beta_1, \dots, \beta_{r-1})$) and $c_{r-1} \geq \max(\lambda_r, \beta_r)$. Thus

$$(1-q)^{2r-1} \frac{\dim_{2r} \beta}{\dim_{2r} \lambda} \frac{q^{|\lambda_r - \beta_r|} + q^{\lambda_r + \beta_r}}{1 + 1_{\beta_r = 0}}$$
$$\times \sum_{s_1 > s_2 > \dots > s_{r-1} \geq \max(\lambda_r, \beta_r)} \det[f_{s_i,1}(\lambda_j - j + r - 1)]_1^{r-1} \det[f_{s_i,1}(\beta_j - j + r - 1)]_1^{r-1}$$
$$= (1-q)^{2r-2} R(\lambda_r, \beta_r) \frac{\dim_{2r} \beta}{\dim_{2r} \lambda} \det \left[\sum_{s=\max(\lambda_r,\beta_r)}^{\infty} f_{s,1}(\lambda_i - i + r - 1) f_{s,1}(\beta_j - j + r - 1) \right]_1^{r-1}.$$

A straightforward calculation shows that if $\max(\lambda_r, \beta_r) \leq \min(x,y)$, then

$$\sum_{s=\max(\lambda_r,\beta_r)}^{\infty} f_{k,1}(x) f_{k,1}(y) = \frac{q^{x+y-2\max(\lambda_r,\beta_r)+2} - q^{|x-y|}}{q^2-1}.$$

To deal with the case $\max(\lambda_r, \beta_r) > \min(x,y)$, we use the following lemma.

Lemma 3.6. *If* $\max(\lambda_r, \beta_r) > \min(\lambda_{r-1}, \beta_{r-1})$, *then* $P_{2r-1}(\lambda, \beta) = T^{\phi}_{2r-1}(\lambda, \beta) = 0$.

Proof. The fact that $P_{2r-1}(\lambda, \beta) = 0$ follows immediately from the description of the interacting particle system, or from the fact that $\{c \in \mathbb{N}^{r-1} : c \prec \lambda, \beta\}$ is empty.

Now it remains to show that $T^{\phi}_{r,-1/2} = 0$. If $\lambda_r > \beta_{r-1}$, then $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > \beta_{r-1} \geq \beta_r$, so

$$(r-1)\text{th column} = \frac{R(\lambda_{r-1}+1, \beta_{r-1}+1)}{R(\lambda_r, \beta_r)} (r\text{th column}),$$

implying the determinant is zero. An identical argument holds if $\beta_r > \lambda_{r-1}$. ☐

For the rest of the proof, assume that $\max(\lambda_r, \beta_r) \leq \min(\lambda_{r-1}, \beta_{r-1})$.

Notice now that the determinant in T^{ϕ}_{2r-1} is of size r, which needs to be compared to a determinant of size $r-1$. To show that the larger determinant is $(1-q)^{2r-2}R(\lambda_r, \beta_r)$ times the smaller determinant, we perform a sequence of operations to the smaller matrix. These operations are slightly different for $\lambda_r > \beta_r$ and $\lambda_r \leq \beta_r$. Consider $\lambda_r > \beta_r$ for now.

First, add a row and a column to the matrix of size $r-1$. The rth column is given by $[0, 0, 0, \ldots, 0, R(\lambda_r, \beta_r)]$ and the rth row is $[R(\lambda_r, \beta_1 - 1 + r), R(\lambda_r, \beta_2 - 2 + r), \ldots, R(\lambda_r, \beta_{r-1} + 1), R(\lambda_r, \beta_r)]$. This multiplies the determinant by $R(\lambda_r, \beta_r)$.

Second, for $1 \leq i \leq r-1$, perform row operations by replacing the ith row with

$$i\text{th row} + \frac{1}{(q-1)^2}\frac{R(\lambda_i - i + r, \beta_r)}{R(\lambda_r, \beta_r)}(r\text{th row}).$$

For $1 \leq i, j \leq r-1$ and letting $(x, y) = (\lambda_i - i + r, \beta_j - j + r)$, the (i, j) entry is

$$\begin{aligned} & \frac{q^{x+y-2\max(\lambda_r,\beta_r)} - q^{|x-y|}}{q^2-1} + \frac{1}{(q-1)^2}\frac{R(x, \beta_r)}{R(\lambda_r, \beta_r)}R(\lambda_r, y) \\ = \quad & \frac{q^{x+y-2\max(\lambda_r,\beta_r)} - q^{|x-y|}}{q^2-1} - \frac{q^{x-\lambda_r}}{(1+1_{y=0})(q^2-1)}(q^{\lambda_r+y} + q^{|\lambda_r-y|}) \\ = \quad & \frac{-q^{x+y} - q^{|x-y|}}{q^2-1} = (1-q)^{-2}R(x, y). \end{aligned}$$

Here, we used the fact that $y \geq \beta_{r-1} \geq \min(\lambda_{r-1}, \beta_{r-1}) \geq \lambda_r$ and $y \geq \lambda_r > \beta_r \geq 0$. For $j = r$, $\lambda_r > y = \beta_r$, so the (i, j) entry is

$$\frac{-q^{x+y} - q^{|x-y|}}{(1+1_{y=0})(q^2-1)} = (1-q)^{-2}R(x, y).$$

Thus, the larger determinant is $(1-q)^{2(r-1)}R(\lambda_r, \beta_r)$ times the larger determinant.

Now consider $\lambda_r \leq \beta_r$. First, add the rth row, which is $[0, 0, \ldots, 0, R(\lambda_r, \beta_r)]$, and add the rth column which is $[R(\lambda_1 - 1 + r, \beta_r), R(\lambda_2 - 2 + r, \beta_r), \ldots, R(\lambda_r, \beta_r)]$. This multiplies the determinant by $R(\lambda_r, \beta_r)$.

Second, for $1 \le j \le r-1$, perform column operations by replacing tSecond, for $1 \le j \le r-1$, perform column operations by replacing the jth column with

$$j\text{th column} + \frac{1}{(q-1)^2}\frac{R(\lambda_r, \beta_j - j + r)}{R(\lambda_r, \beta_r)}(r\text{th column}).$$

Once again, this yields a matrix whose entries are $(1-q)^{-2}R(\lambda_i - i + r, \beta_j - j + r)$, except for the last column, which is $R(\lambda_i - i + r, \beta_r)$.

3.6. **Asymptotics.** Thus far, we have shown that $\tilde{X}^k(n)$ is determinantal with correlation kernel $K(r, a, s_1; r, a, s_2)$. In this section, we will take asymptotics of $K(r_1, a_1, s_1; r_2, a_2, s_2)$, with (r_1, a_1) not necessarily equal to (r_2, a_2). This is because the asymptotic analysis is not much more difficult, and this would be the appropriate limit if Conjecture 3.2 were true. Recall that (r, a) corresponds to the $2r + 1/2 + a$ level.

3.7. **Discrete Jacobi Kernel.** For $-1 < u < 1$ and $a_1, a_2 = \pm\frac{1}{2}$, define the *discrete Jacobi kernel* $L(r_1, a_1, s_1, r_2, a_2, s_2; u)$ as follows. If $2r_1 + a_1 \ge 2r_2 + a_2$, then

$$L(r_1, a_1, s_1, r_2, a_2, s_2; u) = \frac{W^{(a_1,-1/2)}(s_1)}{\pi}\int_u^1 \mathsf{J}_{s_1}^{(a_1,-1/2)}(x)\mathsf{J}_{s_2}^{(a_2,-1/2)}(x)(x-1)^{r_1-r_2}(1-x)^{a_1}(1+x)^{-1/2}dx.$$

If $2r_1 + a_1 < 2r_2 + a_2$, then

$$L(r_1, a_1, s_1, r_2, a_2, s_2; u) = -\frac{W^{(a_1,-1/2)}(s_1)}{\pi}\int_{-1}^u \mathsf{J}_{s_1}^{(a_1,-1/2)}(x)\mathsf{J}_{s_2}^{(a_2,-1/2)}(x)(x-1)^{r_1-r_2}(1-x)^{a_1}(1+x)^{-1/2}dx.$$

Theorem 3.7. *Let n depend on N in such a way that $n/N \to t$. Let $r_1, \ldots, r_l$ depend on N in such a way that $r_i/N \to l$ and their differences $r_i - r_j$ are fixed finite constants. Here, $t, l > 0$. Fix $s_1, s_2, \ldots, s_l$ to be finite constants. Let*

$$\theta = 1 + \frac{2l}{(l-t)(2\alpha + \alpha^2)}$$

Then

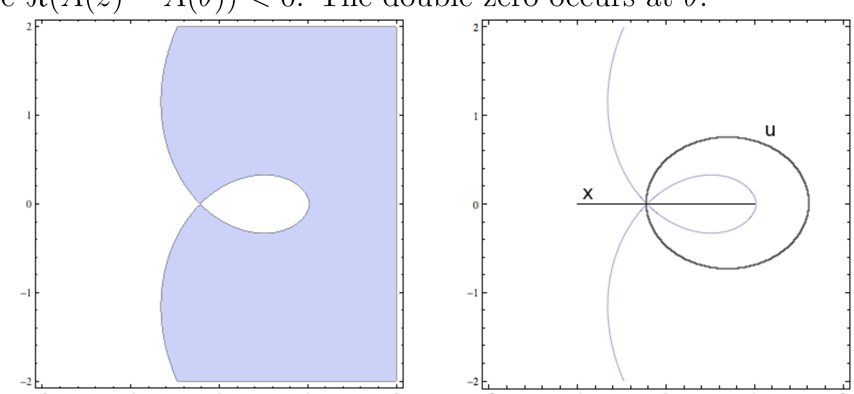
$$\lim_{N\to\infty} \det[K(r_i,a_i,s_i,r_j,a_j,s_j)]_{i,j=1}^{l} = \begin{cases} 1, & l \geq (1-(1+\alpha)^{-2})t \\ \det[L(r_i,a_i,s_i,r_j,a_j,s_j;\theta)]_{i,j=1}^{l}, & l < (1-(1+\alpha)^{-2})t \end{cases}$$

Proof. First consider the case when $l < t$. Let $A(z) = -t\log(1+\alpha(1-z)+\alpha^2/2\cdot(1-z)) + l * \log(z-1)$. Then the kernel asymptotically equals

$$1_{2r_1+a_1\geq 2r_2+a_2}\frac{W^{(a_1,-1/2)}(s_1)}{\pi}\int_{-1}^{1}\mathsf{J}_{s_1}^{(a_1,-1/2)}(x)\mathsf{J}_{s_2}^{(a_2,-1/2)}(x)(x-1)^{r_1-r_2}(1-x)^{a_1}(1+x)^{-1/2}dx$$
$$+\frac{W^{(a_1,-1/2)}(s_1)}{\pi}\frac{1}{2\pi i}\int_{-1}^{1}\oint_C \frac{e^{N(A(x)-A(\theta))}}{e^{N(A(u)-A(\theta))}}\mathsf{J}_{s_1}^{(a_1,-1/2)}(x)\mathsf{J}_{s_2}^{(a_2,-1/2)}(u)$$
$$\times (x-1)^{r_1-r_2}\frac{(1-x)^{a_1}(1+x)^{-1/2}dudx}{x-u}.$$

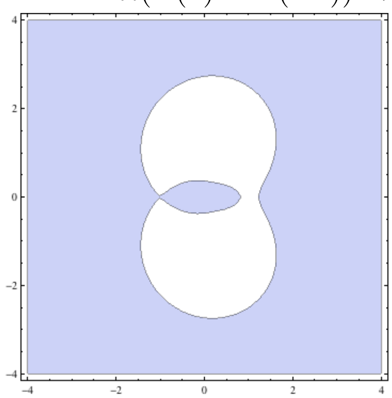
Deform the contours as in Figure 12. With these deformations, the double integral converges to zero, but residues are picked up at $u = x$. For $l > (1-(1+\alpha)^{-2})t$, the parameter θ is less than -1, so no residues are picked up. We arrive at a triangular matrix with diagonal entries equal to 1, so the determinant is 1. For $l < (1-(1+\alpha)^{-2})t$, the parameter θ is in $(-1,1)$, and the residues give the discrete Jacobi kernel.

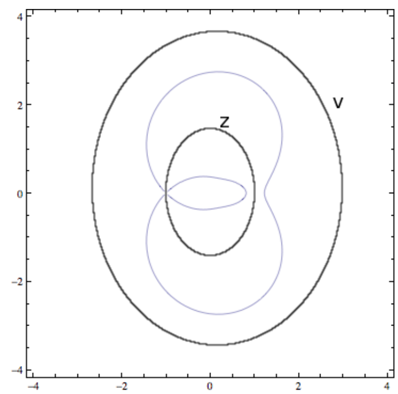
FIGURE 12. Shaded regions indicate $\Re(A(z)-A(\theta)) > 0$ and white regions indicate $\Re(A(z)-A(\theta)) < 0$. The double zero occurs at θ.



For $l > t$, the situation is quite different, due to the discontinuity in θ at $l = t$. Make the substitutions $x = (z + z^{-1})/2$ and $u = (v + v^{-1})/2$. Now the x-contour is the unit circle and the v contour is a simple loop that goes outside the unit circle. After deforming as shown in Figure 13, the double integral converges to 0, with no residues picked up. Again, we obtain a triangular matrix with diagonal entries equal to 1.

FIGURE 13. Shaded regions indicate $\Re(A\left(\frac{z+z^{-1}}{2}\right) - A(-1)) > 0$ and white regions indicate $\Re(A(z) - A(-1)) < 0$. The double zero occurs at -1.





□

3.8. **Symmetric Pearcey Kernel.** Define the *symmetric Pearcey kernel* $\mathcal{K}$ on $\mathbb{R}_+ \times \mathbb{R}$ as follows. In the expressions below, the u-contour is integrated on rays from $\infty e^{i\pi/4}$ to 0 to $\infty e^{-i\pi/4}$. Let

$$\mathcal{K}(\sigma_1, \eta_1, \sigma_2, \eta_2) =$$
$$\frac{2}{\pi^2 i} \int \int_0^\infty \exp(-\eta_1 x^2 + \eta_2 u^2 + u^4 - x^4) \cos(\sigma_1 x) \cos(\sigma_2 u) \frac{u}{u^2 - x^2} dx du$$
$$- \frac{1_{\eta_2 < \eta_1}}{2\sqrt{\pi(\eta_1 - \eta_2)}} \left(\exp \frac{(\sigma_1 + \sigma_2)^2}{4(\eta_2 - \eta_1)} + \exp \frac{(\sigma_1 - \sigma_2)^2}{4(\eta_2 - \eta_1)} \right).$$

Theorem 3.8. *Let c_α be the constant $(1 + \alpha)(\alpha(2 + \alpha))^{-1/4}$. Let s_1 and s_2 depend on N in such a way that $s_i/N^{1/4} \to 2^{-5/4}\sigma_i c_\alpha^{-1} > 0$ as $N \to \infty$. Let n and r_1, r_2 also depend on N in* *such a way that $n/N \to 1$ and $(r_j - (1-(1+\alpha)^{-2})N)/\sqrt{N} \to 2^{-1/2}\eta_j$. Then*

$$(-2)^{r_2-r_1}(-1)^{s_1-s_2}\frac{N^{1/4}}{c_\alpha 2^{5/4}}K(r_1,a_1,s_1,r_2,a_2,s_2) \to \mathcal{K}(\sigma_1,\eta_1,\sigma_2,\eta_2).$$

Proof. Since the proof is almost identical to the proof of Theorem 5.8 from [6], the details will be omitted. The only difference is that now

$$A(z) = \log\phi_\alpha(z) + (1-(1+\alpha)^{-2})\log(z-1),$$

with asymptotic expansion

$$A(z) - A(-1) = -\frac{\alpha(2+\alpha)}{8(1+\alpha)^4}(z+1)^2 + O((z+1)^3).$$

□

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4. Quantum random walks on $vN(U(N))$

Abstract. We provide two new constructions of Markov chains which had previously arisen from the representation theory of $U(\infty)$. The first construction uses the combinatorial rule for the Littlewood–Richardson coefficients, which arise from tensor products of irreducible representations of the unitary group. The second arises from a quantum random walk on the von Neumann algebra of $U(n)$, which is then restricted to the center. Additionally, the restriction to a maximal torus can be expressed in terms of weight multiplicities, explaining the presence of tensor products.

4.1. **Introduction.** In [3], the authors introduce a family of Markov chains on the Gelfand–Tsetlin set $\mathbb{GT}$. This is the set of infinite sequences $\lambda^{(1)} \prec \lambda^{(2)} \prec \ldots$, where $\lambda^{(k)} = (\lambda_1^{(k)} \geq \ldots \geq \lambda_k^{(k)})$ is a k–tuple of nonincreasing integers and $\lambda \prec \mu$ denotes the condition $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \mu_n \geq \lambda_n$. By considering the map

$$\lambda^{(1)} \prec \lambda^{(2)} \prec \ldots \mapsto \{(\lambda_i^{(k)} - i, k)\}_{1 \leq i \leq k < \infty} \subset \mathbb{Z} \times \mathbb{Z}_+,$$

these Markov chains define an interacting particle system on $\mathbb{Z} \times \mathbb{Z}_+$. Drawing lozenges around each particle yields a random tiling of the half plane. Furthermore, the condition $\lambda^{(k)} \prec \lambda^{(k+1)}$ ensures that there is a natural interpretation as a randomly growing stepped surface. This random growth belongs to the $2+1$ anisotropic KPZ class of stochastic growth models. This universality class is a variant of the KPZ universality class, which has seen many results in recent years (see [7] for a survey). By considering suitable projections, the Markov chains also reduce to TASEP, PushASEP (introduced in [4]) , and the Charlier process from [12].

The Gelfand–Tsetlin set parametrises the branching of irreducible representations of the unitary group. Additionally, the family of Markov chains can be constructed from the representation theory of the infinite–dimensional unitary group $U(\infty)$ [6]. Tools from representation theory have yielded a rich variety of two–dimensional dynamics (e.g. [9, ?WW]). One of the most general processes arising from representation theory are Macdonald processes [5].

In this paper, we hope to deepen the connections between probability theory and representation theory. To this end, we give two new representation–theoretic constructions for these Markov chains. The first involves the Littlewood–Richardson rule for decomposing tensor products of irreducible representations of $U(n)$. The second involves a quantum random walk on the von Neumann algebra of $U(n)$, which is then restricted to the center. These constructions have the advantage of not requiring infinite–dimensional representation theory and are therefore more generalisable to other simple Lie groups.

The structure of the paper is as follows. In section 4.2, we review the representation theory of $U(n)$ and introduce the Markov chains from [3]. In section 4.3, we provide a construction the combinatorial description of the Littlewood–Richardson coefficients. In section 4.4, we provide another construction, this time using a quantum random walk on the von Neumann algebra of $U(n)$. This will also give a representation theoretic explanation (i.e. using tensor products of representations instead of combinatorics) for the occurence of the Littlewood–Richardson coefficients.

4.2. **Markov chains.**

4.2.1. *Background.* Before defining the Markov chains, let us review some background on the unitary groups. Let $U(n)$ denote the compact group of $n\times n$ unitary matrices. Occasionally, to clean up notation, G will also refer to $U(n)$. Let $\mathbb{T}^n\subset U(n)$ be the subgroup of diagonal unitary matrices, which is a maximal torus of $U(n)$. With respect to this maximal torus, the weight lattice of $U(n)$ is easy to describe. The Lie algebra of $\mathbb{T}^n$, denoted $L\mathbb{T}^n$, consists of imaginary diagonal matrices. The weight lattice $P\subset(L\mathbb{T}^n)^*$ is the n–dimensional lattice generated by the elements $\epsilon_1,\ldots,\epsilon_n$, where $\epsilon_j(\mathrm{diag}(u_1,\ldots,u_n))=u_j/(2\pi i)$. Each $\lambda_1\epsilon_1+\ldots+\lambda_n\epsilon_n, \lambda_j\in\mathbb{Z}$ defines a character of $\mathbb{T}^n$ by sending $(z_1,\ldots,z_n)$ to $z_1^{\lambda_1}\cdots z_n^{\lambda_n}$. In this way, there is an isomorphism $e:P\to\widehat{\mathbb{T}^n}$. For $x\in P$ and $\boldsymbol{\theta}\in\mathbb{T}^n$, write $x(\theta)=e(x)(\theta)$. Note that with this notation, $x(\boldsymbol{\theta})y(\boldsymbol{\theta})=(x+y)(\boldsymbol{\theta})$.

The roots of $U(n)$ with respect to $\mathbb{T}^n$ are $e_i - e_j, 1 \leq i \neq j \leq n$. The Weyl group is generated by the reflections with respect to the roots. It is isomorphic to the group S_n acting on $\{\epsilon_1, \ldots, \epsilon_n\}$, where the reflection with respect to $e_i - e_j$ is the transposition $(\epsilon_i \ \epsilon_j)$. The Weyl chamber is thus $\mathcal{W} := \{\lambda_1\epsilon_1 + \ldots + \lambda_n\epsilon_n : \lambda_1 \geq \ldots \geq \lambda_n, \lambda_j \in \mathbb{Z}\}$.

Recall that any irreducible representation of any compact, connected, simple Lie group is generated by a highest weight vector, which must lie in the Weyl chamber. Conversely, any weight in the Weyl chamber generates an irreducible representation by sucessively applying negative roots. Therefore the irreducible unitary representations of $U(n)$ is parameterised by $\mathcal{W}_n$.

Let $m_1^{n_1} m_2^{n_2} \ldots$ denote the sequence $(\underbrace{m_1, \ldots, m_1}_{n_1}, \underbrace{m_2, \ldots, m_2}_{n_2}, \ldots)$. For $\lambda, \mu \in \mathcal{W}_n$, let $\lambda \prec \mu$ denote the condition $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \mu_n \geq \lambda_n$.

For each $\lambda \in \mathcal{W}_n$, let $\pi_\lambda : U(n) \to GL(V_\lambda), \chi_\lambda$ and $\dim \lambda$ denote the corresponding representation, character and dimension. Let $\widetilde{\chi}_\lambda$ denote the normalized character $\chi_\lambda / \dim \lambda$. Recall that the conjugacy class of a matrix in $U(n)$ is given by its eigenvalues. Therefore, χ^λ is a function of $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$. Explicitly, χ^λ is just the Schur polynomial s_λ. Useful formulae are

$$\chi_\lambda(\theta_1, \ldots, \theta_n) = s_\lambda(\theta_1, \ldots, \theta_n) = \frac{\det[\theta_i^{\lambda_j + n - j}]_{1 \leq i,j \leq n}}{\det[\theta_i^{n-j}]_{1 \leq i,j \leq n}}. \tag{4.1}$$

and

$$\chi_\lambda(\theta_1, \ldots, \theta_n) = s_\lambda(\theta_1, \ldots, \theta_n) = \det[h_{\lambda_i - i + j}(\boldsymbol{\theta})], \tag{4.2}$$

where h_k is the k–th complete homogeneous symmetric polynomial:

$$h_k(\boldsymbol{\theta}) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \theta_{i_1} \cdots \theta_{i_n}.$$

Equation (4.2) is called *the first Giambelli formula.* The elementary homogeneous symmetric polynomial e_k will also appear:

$$e_k(\boldsymbol{\theta}) = \sum_{1<i_1<\cdots<i_k<n} \theta_{i_1} \cdots \theta_{i_n}.$$

Observe that

$$\det[\theta_i^{\lambda_j+n-j}]_{1\leq i,j\leq n} = \begin{cases} h_k, \quad \text{when} \quad \lambda = k0^{n-1} \\ e_k, \quad \text{when} \quad \lambda = 1^k 0^{n-k} \\ h_k(\theta_1^{-1}, \ldots, \theta_n^{-1}), \quad \text{when} \quad \lambda = 0^{n-1}(-k) \\ \theta_1^{-1} \ldots \theta_n^{-1} e_{n-k} = e_k(\theta_1^{-1}, \ldots, \theta_n^{-1}), \quad \text{when} \quad \lambda = 0^{n-k}(-1)^k \end{cases}$$

The third formula follows from the first Giambelli formula. A formula for the dimension is

$$\dim \lambda = \prod_{i<j} \frac{\lambda_i - i - (\lambda_j - j)}{j - i},$$

which extends formally to $\dim : P \to \mathbb{R}$.

Let $L^2(G, dg)^G$ denote the square–integrable class functions on G. By the Peter–Weyl theorem, $\{\chi_\lambda\}_{\lambda \in \mathcal{W}_n}$ is an orthonormal basis for $L^2(G, dg)^G$. For any $\kappa \in L^2(G, dg)^G$ and any $\lambda \in \mathcal{W}_n$, let $\widehat{\kappa}(\lambda)$ be the Fourier coefficient

$$\widehat{\kappa}(\lambda) = \int_G \kappa(g) \overline{\chi_\lambda}(g) dg,$$

so that

$$\kappa(g) = \sum_{\lambda \in \mathcal{W}_n} \widehat{\kappa}(\lambda) \chi_\lambda(g). \tag{4.3}$$

Formally, this means that $\sum_{\lambda \in \mathcal{W}_n} \overline{\chi_\lambda(g)} \chi_\lambda(g')$ is the Dirac delta function $\delta_{g^{-1}g'}$.

Restricting the highest weight representation V_λ to $\mathbb{T}^n$ yields a decomposition into one–dimensional subspaces

$$V_\lambda = \bigoplus_{x \in P} U_x^{\oplus n_\lambda(x)}, \tag{4.4}$$

where

$$U_x = \{v \in V_\lambda : \boldsymbol{\theta} \cdot v = x(\boldsymbol{\theta})v \quad \text{for all} \quad \boldsymbol{\theta} \in \mathbb{T}^n\}$$

and $n_\lambda(x)$ are non–negative integers. In terms of characters, this means that

$$\chi_\lambda(\boldsymbol{\theta}) = \sum_{x \in P} n_\lambda(x) x(\boldsymbol{\theta}).$$

For $\kappa \in L^2(G, \mathbb{C})^G$, define $n_\kappa(x)$ by linear extension, i.e.

$$n_\kappa(x) = \sum_{\lambda \in \mathcal{W}_n} \widehat{\kappa}(\lambda) n_\lambda(x).$$

4.2.2. *Markov chains.* Now review the Markov chains from [3]. Let $\theta_1, \ldots, \theta_n$ be fixed nonzero complex numbers. and let F be an analytic function in an annulus A which contains all the α_j^{-1} such that each $F(\alpha_j^{-1})$ is nonzero. Given such an F, define

$$f(m) := \frac{1}{2\pi i} \oint \frac{F(z)}{z^{m+1}} dz, \tag{4.5}$$

where the integral is taken over any positively oriented simple loop in A. Section 2.3 of [3] defines matrices T_n with rows and columns parameterised by $\mathcal{W}_n$:

$$T_n(\boldsymbol{\theta}; F)(\lambda, \mu) := \frac{s_\mu(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta})} \frac{\det[f(\lambda_j + j - \mu_i - i)]_1^n}{\prod F(\theta_j^{-1})}$$

Proposition 4.1. *There is the commuting relation* $T_n(\boldsymbol{\theta}; F_1) T_n(\boldsymbol{\theta}; F_2) = T_n(\boldsymbol{\theta}; F_1 F_2)$. *For* $\boldsymbol{\theta} = (1, 1, \ldots, 1)$, $T_n(\boldsymbol{\theta}; F)$ *is a stochastic matrix.*

Proof. Proposition 2.10 of [3] gives the commuting relation. Proposition 2.8 of [3] gives the stochastic matrix result. □

Let us now describe the functions F to be considered. Define the functions

$$F_{\alpha^+,q}(z) = (1 - qz)^{-1}, \qquad F_{\alpha^-,q}(z) = (1 - qz^{-1})^{-1}, \quad 1 > q \geq 0$$

$$F_{\beta^+,p}(z) = 1 + pz, \qquad F_{\beta^-,p}(z) = 1 + pz^{-1}, \quad 1 \geq p \geq 0.$$

$$F_{\gamma^+,t}(z) = e^{tz}, \qquad F_{\gamma^-,t}(z) = e^{tz^{-1}}, \quad t \geq 0.$$

Lemma 4.2. *For these functions,*

$$T_n(\boldsymbol{\theta}; F_{\beta^-,p})(\lambda,\mu) = \frac{p^k}{\prod F(\theta_j^{-1})}\frac{s_\mu(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta})}, \text{ if each } \mu_j - \lambda_j \in \{0,1\} \text{ and } \sum(\mu_j - \lambda_j) = k,$$

$$0, \text{ otherwise.}$$

$$T_n(\boldsymbol{\theta}; F_{\beta^+,p})(\lambda,\mu) = \frac{p^k}{\prod F(\theta_j^{-1})}\frac{s_\mu(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta})}, \text{ if each } \mu_j - \lambda_j \in \{-1,0\} \text{ and } \sum(\mu_j - \lambda_j) = -k,$$

$$0, \text{ otherwise.}$$

$$T_n(\boldsymbol{\theta}; F_{\alpha^-,q})(\lambda,\mu) = \frac{q^k}{\prod F(\theta_j^{-1})}\frac{s_\mu(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta})}, \text{ if } \lambda \prec \mu \text{ and } \sum(\mu_j - \lambda_j) = k,$$

$$0, \text{ otherwise.}$$

$$T_n(\boldsymbol{\theta}; F_{\alpha^+,q})(\lambda,\mu) = \frac{q^k}{\prod F(\theta_j^{-1})}\frac{s_\mu(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta})}, \text{ if } \mu \prec \lambda \text{ and } \sum(\mu_j - \lambda_j) = -k,$$

$$0, \text{ otherwise.}$$

Proof. These are Lemmas 2.12 and 2.13 from [3]. □

Use the variable ξ to denote one of the symbols $\alpha^\pm, \beta^\pm, \gamma^\pm$. For $\xi = \beta^\pm$ and $\xi = \gamma^\pm$, the process with transition probabilities $T_n(\mathbf{1}, F_\xi)$ are respecitvely the *Krawtchouk* and *Charlier* processes from [12]. These can be described respecitvely as the Doob h–transform (where $h(\lambda) = \dim\lambda$) of n independent Bernoulli walks and n independent exponential random walks of rate 1.

There is a general construction for building multivariate Markov chains out of $\{T_n : n = 1, 2, 3, \ldots\}$. This construction requires a intertwining relation between the transition probabilities (see section 2 of [3]). It is still an open problem to find a representation–theoretic interpretation of the commutation relation.

Let $\mathbb{P}_n(\boldsymbol{\theta}; F)(\mu) = T_n(\boldsymbol{\theta}; F)(\mathbf{0}, \mu)$. From Lemma 4.2, we see that for $F = F_{\alpha^\pm}, F_{\beta^\pm}$, these are geometric random variables weighted by the dimension of the representation. The construction then proceeds as follows. First, for $F = F_{\alpha^\pm}$ or $F_{\beta^\pm}$, construct $T_n(F)$ out of $\mathbb{P}_n(F)$ (Lemmas 4.4 and 4.10 below). Second, we show that there is a commuting relation that is analagous to the one in Proposition 4.1 (Proposition 4.5 and Lemma 4.11 below). Finally, use a continuity argument (Lemmas 4.7 and 4.12 below) to give $F_{\gamma^\pm}$.

4.3. **Littlewood–Richardson Coefficients.** Let us briefly now recall the definition of the Littlewood–Richardson coefficients. For any two paritions λ, τ such that $\lambda_j \leq \tau_j$ for each j, the *skew diagram* of $\tau \backslash \lambda$ is the set–theoretic difference of the Young diagrams of λ and τ. A *skew Tableau* of shape $\tau \backslash \lambda$ and weight μ is obtained by filling in the skew diagram of $\tau \backslash \lambda$ with positive integers such that the integer k appears μ_k times. A skew Tableau is *semistandard* if it the entries weakly increase along each row and strictly increase down each column. A *Littlewood–Richardson tableau* is a semistandard skew Tableau with the additional property that in the sequence obtained by concatenating the reversed rows, every initial part of the sequence contains any number k at least as often as it contains $k+1$. See figure 15 for an example.

In the special case $\mu = k0^{n-1}$, the Littlewood–Richardson rule is known as *Pieri's formula.* In this case, the semistandard skew Tableau can only be filled with 1's, so the condition on the concatenated sequence is automatically satisfied. The only requirement is that the skew diagram of $\tau \backslash \lambda$ does not contain two boxes in the same column. In other words,

$$\text{when } \mu = k0^{n-1}, \quad c^{\tau}_{\lambda\mu} = \begin{cases} 1, & \text{if } \lambda \prec \tau \text{ and } \sum(\tau_j - \lambda_j) = k, \\ 0, & \text{else.} \end{cases} \tag{4.6}$$

Figure 14. For $\tau = (4,3,2), \lambda = (2,1,0)$ and $\mu = (3,2,1)$, here is a Littlewood–Richardson tableau of shape $\tau \backslash \lambda$ and weight μ. The sequence obtained by concatenating the reversed rows is 112132.

		1	1
	1	2	
2	3		

We also need the special case $\mu = 1^k 0^{n-k}$. The integers appearing in the semistandard skew Tableau are $\{1, 2, \ldots, k\}$, so the condition on the concatenated sequence can only hold if the skew diagram of $\tau \backslash \lambda$ does not contain two boxes in the same *row*. In other words,

$$\text{when } \mu = 1^k 0^{n-k}, \quad c^\tau_{\lambda\mu} = \begin{cases} 1, & \text{if } \tau_j - \lambda_j \in \{0,1\} \text{ and } \sum(\tau_j - \lambda_j) = k, \\ 0, & \text{else.} \end{cases} \tag{4.7}$$

The Littlewood-Richardson coefficients are related to representation theory by the decomposition

$$V_\lambda \otimes V_\mu = \bigoplus_{\tau \in GT_n} c^\tau_{\lambda\mu} V_\tau.$$

Since the character of V_λ is the Schur polynomial s_λ it is equivalent to say

$$s_\lambda s_\mu = \sum_{\tau \in \mathcal{W}_n} c^\tau_{\lambda\mu} s_\tau.$$

Also define the coefficients $c^\tau_{\lambda\sigma\mu}$ by

$$s_\lambda s_\sigma s_\nu = \sum_{\tau \in \mathcal{W}_n} c^\tau_{\lambda\sigma\nu} s_\tau.$$

It follows immediately that

$$\sum_{\mu \in \mathcal{W}_n} c^\tau_{\lambda\mu} c^\mu_{\sigma\nu} = c^\tau_{\lambda\sigma\nu}.$$

In [8], the author considers a discrete–time particle system which arises from Pieri's formula for the orthogonal groups. In [13], it is proven that this discrete–time particle system also arises from representations of $O(\infty)$. Thus, the next theorem is a generalization in the unitary group case.

Theorem 4.3. *For* $F = F_\xi$,

$$\sum_{\mu \in \mathcal{W}_n} \mathbb{P}_n(\boldsymbol{\theta}, F)(\mu) c^{\tau}_{\lambda\mu} \frac{s_\tau(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta}) s_\mu(\boldsymbol{\theta})} = T_n(\boldsymbol{\theta}, F)(\lambda, \tau). \tag{4.8}$$

Proof. Let $\mathcal{C}$ be the set of all functions $F : A \to \mathbb{C}$ such that (4.8) holds.

Lemma 4.4. *The functions* $F = F_{\alpha^\pm}, F_{\beta^\pm}$ *are in* $\mathcal{C}$.

Proof. Start with $1 + pz^{-1}$. By lemma 4.2,

$$\mathbb{P}_n(\boldsymbol{\theta}; F)(\mu) = \begin{cases} s_\mu(\boldsymbol{\theta}) \frac{p^k}{\prod F(\theta_j^{-1})}, & \mu = 1^k 0^{n-k}, \\ 0, & \text{else.} \end{cases} \tag{4.9}$$

Thus it suffices to consider at the the value of $c^\tau_{\lambda\mu}$ when $\mu = 1^k 0^{n-k}$. By using Pieri's formula (4.6) and another application of lemma 4.2, $F_{\beta^+} \in \mathcal{C}$.

Now let consider $1 + pz$. Since $\tilde{f}(m) = f(-m)$,

$$\mathbb{P}_n(\boldsymbol{\theta}; F_{\beta^-})(\mu) = \begin{cases} s_\mu(\boldsymbol{\theta}) \frac{p^k}{\prod F(\theta_j^{-1})}, & \mu = 0^{n-k}(-1)^k, \\ 0, & \text{else.} \end{cases}$$

Since $c^\tau_{\lambda\mu} = c^{\tau+\mathbf{1}}_{\mu+\mathbf{1},\lambda}$, then for $\mu = 0^{n-k}(-1)^k$,

$$c^\tau_{\lambda\mu} = \begin{cases} 1, & \text{if each } \tau_i - \lambda_i \in \{-1, 0\} \text{ and } \sum(\tau_j - \lambda_j) = -k, \\ 0, & \text{else.} \end{cases}$$

Therefore, by lemma 4.2, $F_{\beta^-} \in \mathcal{C}$.

Now consider the function $F(z) = (1 - qz^{-1})^{-1}$ By lemma 4.2,

$$\mathbb{P}_n(\boldsymbol{\theta}; F)(\mu) = \begin{cases} s_\mu(\boldsymbol{\theta}) \frac{q^k}{\prod F(\alpha_j^{-1})}, & \mu = k0^{n-1}, \\ 0, & \text{else.} \end{cases}$$

By (4.7) and lemma 4.2, $F_{\alpha^+} \in \mathcal{C}$.

Finally let $\tilde{F}(z) = (1-qz)^{-1}$. Then

$$\mathbb{P}_n(\boldsymbol{\theta}; F)(\mu) = \begin{cases} s_\mu(\boldsymbol{\theta}) \frac{q^k}{\prod F(\theta_j^{-1})}, & \mu = 0^{n-1}(-k), \\ 0, & \text{else.} \end{cases}$$

Using the identity

$$s_\lambda(\boldsymbol{\theta}^{-1}) = (\theta_1 \dots \theta_n)^{-\lambda_1} s_{(\lambda_1-\lambda_n,\dots,\lambda_1-\lambda_2,0)}(\boldsymbol{\theta}),$$

we get for $\mu = 0^{n-1}(-k)$

$$\begin{aligned} s_{(\lambda_1-\lambda_n,\dots,\lambda_1-\lambda_2,0)}(\boldsymbol{\theta}) s_{k0^{n-1}}(\boldsymbol{\theta}) &= (\theta_1 \dots \theta_n)^{\lambda_1} s_\lambda(\boldsymbol{\theta}^{-1}) \cdot s_\mu(\boldsymbol{\theta}^{-1}) \\ &= (\theta_1 \dots \theta_n)^{\lambda_1} \sum_\tau c_{\lambda\mu}^\tau s_\tau(\boldsymbol{\theta}^{-1}) \\ &= \sum_\tau c_{\lambda\mu}^\tau (\theta_1 \dots \theta_n)^{\lambda_1-\tau_1} s_{(\tau_1-\tau_n,\dots,\tau_1-\tau_2,0)}(\boldsymbol{\theta}) \\ &= \sum_\tau c_{\lambda\mu}^\tau s_{(\lambda_1-\tau_n,\dots,\lambda_1-\tau_1)}(\boldsymbol{\theta}), \end{aligned}$$

so

$$c_{\lambda\mu}^\tau = \begin{cases} 1, & \text{if } (\lambda_1-\lambda_n,\dots,\lambda_1-\lambda_2,0) \prec (\lambda_1-\tau_n,\dots,\lambda_1-\tau_1) \text{ and } \sum(-\tau_j+\lambda_j) = k, \\ 0, & \text{else.} \end{cases}$$

Equivalently,

$$c_{\lambda\mu}^\tau = \begin{cases} 1, & \text{if } \tau \prec \lambda \text{ and } \sum(\tau_j - \lambda_j) = -k \\ 0, & \text{else.} \end{cases}$$

Therefore, by lemma 4.2, $F_{\alpha^-} \in \mathcal{C}$.

□

Proposition 4.5. *If $F_1, F_2 \in \mathcal{C}$, then $F_1 F_2 \in \mathcal{C}$.*

Proof. We start with a lemma.

Lemma 4.6. *Suppose $\mathrm{Pr}_X, \mathrm{Pr}_Y, \mathrm{Pr}_Z$ are complex valued measures on the countable sets X, Y, Z. Suppose Pr_f is a complex valued measure on Y^X with total weight 1, (one should* *think of $f : X \to Y$ as a random map). Suppose $h : X \to Z$ and $g : Y \to Z$ are deterministic maps such that $h = g \circ f$ almost surely. If $h_* \Pr_X = \Pr_Z$ and $f_* \Pr_X = \Pr_Y$ (in the sense that $\Pr_Y(B) = \sum_{x \in X} \Pr_X(x) \Pr_f(f(x) \in B)$), then $g_* \Pr_Y = \Pr_Z$.*

Proof. Since $h = g \circ f$, then for a fixed $x \in X$ and $E \subset Z$,

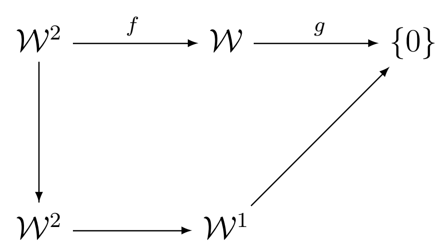
$$Pr_f(f(x) \in g^{-1}E) = \begin{cases} 1, & x \in h^{-1}E, \\ 0, & x \notin h^{-1}E. \end{cases}$$

Thus

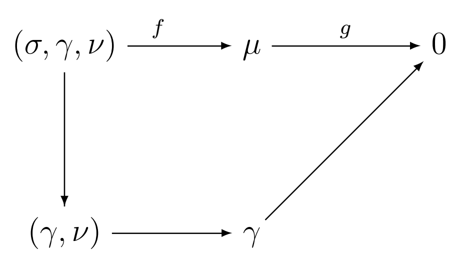
$$Pr_Z(E) = Pr_X(h^{-1}E) = \sum_{x \in X} Pr_X(x) Pr_f(f(x) \in g^{-1}E) = Pr_Y(g^{-1}E),$$

i.e. $g_* \Pr_Y = \Pr_Z$. $\square$

Fix λ and τ. To apply the lemma, use the following commutative diagram:



defined by



Every map is projection except for f, which is defined by

$$Pr_f(f(\sigma, \gamma, \nu) = \mu) = \begin{cases} \dfrac{c^{\mu}_{\sigma\nu} c^{\tau}_{\lambda\mu}}{c^{\tau}_{\lambda\sigma\nu}}, & \text{if } c^{\tau}_{\lambda\sigma\nu} \neq 0, \\ \\ 0, & \text{if } c^{\tau}_{\lambda\sigma\nu} = 0 \end{cases}$$

Let $h : \mathcal{W}^3 \to \{0\}$ be the composition along the bottom row of the diagram. Define the measures $\Pr_X$ on $\mathcal{W}_N^3$, $\Pr_Y$ on $\mathcal{W}_N$ and $\Pr_Z$ on $\{0\}$ by

$$Pr_X(\sigma,\gamma,\nu) = \mathbb{P}_n(\boldsymbol{\theta}; F_1)(\sigma) c^{\gamma}_{\lambda\sigma} \frac{s_\gamma(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta}) s_\sigma(\boldsymbol{\theta})} \mathbb{P}_n(\boldsymbol{\theta}; F_2)(\nu) c^{\tau}_{\gamma\nu} \frac{s_\tau(\boldsymbol{\theta})}{s_\gamma(\boldsymbol{\theta}) s_\nu(\boldsymbol{\theta})}.$$

$$Pr_Y(\mu) = \mathbb{P}_n(\boldsymbol{\theta}; F_1F_2)(\mu) c^{\tau}_{\lambda\mu} \frac{s_\tau(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta}) s_\mu(\boldsymbol{\theta})}$$

$$Pr_Z(\{0\}) = T_n(\boldsymbol{\theta}; F_1F_2)(\lambda,\tau).$$

In this formulation, the proposition states that $g_* \Pr_Y = \Pr_Z$ for all $\lambda \in \mathcal{W}_N$. Thus the proposition follows if the conditions of the lemma hold.

First, it is immediate from the definitions that the weights of f sum to 1 and that $h = g \circ f$ almost surely.

Second, we need to check that $h_* \Pr_X = \Pr_Z$. Since $F_1 \in \mathcal{C}$, the first projection sends $\Pr_X$ to

$$T_n(\boldsymbol{\theta}; F_1)(\lambda,\gamma) \mathbb{P}_n(\boldsymbol{\theta}; F_2)(\nu) c^{\tau}_{\gamma\nu} \frac{s_\tau(\boldsymbol{\theta})}{s_\gamma(\boldsymbol{\theta}) s_\nu(\boldsymbol{\theta})}.$$

Since $F_2 \in \mathcal{C}$, the second projection sends this to

$$T_n(\boldsymbol{\theta}; F_1)(\lambda,\gamma) T_n(\boldsymbol{\theta}; F_2)(\gamma,\tau).$$

Since $T_n(\boldsymbol{\theta}; F_1) T_n(\boldsymbol{\theta}; F_2) = T_n(\boldsymbol{\theta}; F_1F_2)$, the third projection sends this to $T_n(\boldsymbol{\theta}; F_1F_2)(\lambda,\tau) = \Pr_Z$, as needed.

Finally, we need to check that $f_* \Pr_X = \Pr_Y$. By definition,

$$f_* Pr_X(\mu) = \sum_{\sigma,\nu,\gamma \in \mathcal{W}_n} \mathbb{P}_n(\boldsymbol{\theta}; F_1)(\sigma) c^{\gamma}_{\lambda\sigma} \frac{s_\gamma(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta}) s_\sigma(\boldsymbol{\theta})} \mathbb{P}_n(\boldsymbol{\theta}; F_2)(\nu) c^{\tau}_{\gamma\nu} \frac{s_\tau(\boldsymbol{\theta})}{s_\gamma(\boldsymbol{\theta}) s_\nu(\boldsymbol{\theta})} \frac{c^{\mu}_{\sigma\nu} c^{\tau}_{\lambda\mu}}{c^{\tau}_{\lambda\sigma\nu}}$$

By summing over γ,

$$f_* Pr_X(\mu) = \sum_{\sigma,\nu,\in \mathcal{W}_n} \mathbb{P}_n(\boldsymbol{\theta}; F_1)(\sigma) c^{\tau}_{\lambda\sigma\nu} \frac{s_\tau(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta}) s_\sigma(\boldsymbol{\theta}) s_\nu(\boldsymbol{\theta})} \mathbb{P}_n(\boldsymbol{\theta}; F_2)(\nu) \frac{c^{\mu}_{\sigma\nu} c^{\tau}_{\lambda\mu}}{c^{\tau}_{\lambda\sigma\nu}}$$

Now look at $Pr_Y(\mu)$. Since $F_2 \in \mathcal{C}$,

$$\begin{aligned}\mathbb{P}_n(\boldsymbol{\theta}; F_1F_2)(\mu) &= \sum_{\sigma\in\mathcal{W}_n} \mathbb{P}_n(\boldsymbol{\theta}; F_1)(\sigma) T_n(\boldsymbol{\theta}; F_2)(\sigma,\mu) \\ &= \sum_{\sigma\in\mathcal{W}_n} \mathbb{P}_n(\boldsymbol{\theta}; F_1)(\sigma) \sum_{\nu\in\mathcal{W}_n} \mathbb{P}_n(\boldsymbol{\theta}; F_2)(\nu) c^{\mu}_{\sigma\nu} \frac{s_\mu(\boldsymbol{\theta})}{s_\sigma(\boldsymbol{\theta}) s_\nu(\boldsymbol{\theta})}.\end{aligned}$$

Thus,

$$Pr_Y(\mu) = \sum_{\sigma,\nu\in\mathcal{W}_n} \mathbb{P}_n(\boldsymbol{\theta}; F_1)(\sigma) P_n(\boldsymbol{\theta}; F_2)(\nu) c^{\mu}_{\sigma\nu} c^{\tau}_{\lambda\mu} \frac{s_\tau(\boldsymbol{\theta})}{s_\lambda(\boldsymbol{\theta}) s_\sigma(\boldsymbol{\theta}) s_\nu(\boldsymbol{\theta})}$$

so $\Pr_Y = f_* \Pr_X$, as needed. This proves the proposition. □

Lemma 4.7. *If $\{F_k\}$ is a sequence of functions in $\mathcal{C}$ which converges to F uniformly in A, then $F \in \mathcal{C}$.*

Proof. It is immediate that $\{f_k\}$ converges to f uniformly. Since the determinant is a continuous function of its entries, $T_n(\boldsymbol{\theta}; F_k)(\lambda,\tau)$ converges to $T_n(\boldsymbol{\theta}; F)(\lambda,\tau)$. Since sum in the left–hand side of (4.8) only has finitely many terms, convergence must hold as well. □

Finally, since $e^x = \lim_{k\to\infty}(1+x/k)^k = \lim_{k\to\infty}(1-x/k)^{-k}$, Lemma 4.4, Proposition 4.5 and Lemma 4.7 prove Theorem 4.3. □

4.4. **Quantum Random Walk.** Let us introduce some notation, which will folllow [1] closely.

Let G be a compact topological group, let dg denote its Haar measure (normalized to have total weight 1), and let $H = L^2(G, dg)$ be the Hilbert space of square–integrable functions. Let α denote the representation of G on H by left translation. In other words, for $f \in B(H)$ a unitary operator on H, the map $\alpha : G \to B(H)$ is defined by $[\alpha(g)(f)](x) = f(xg)$. The von Neumann algebra of G, denoted $vN(G)$, is the closure (under the strong operator topology) of the $*$–subalgebra of $B(H)$ generated by $\alpha(G)$.

Let κ be a continuous, positive type function on G which sends the identity to 1. This defines a state φ on $vN(G)$ by $\varphi(\alpha(g)) = \kappa(g)$, and also defines a completely positive map on $vN(G)$ by $Q(\alpha(g)) = \kappa(g)\alpha(g)$.

Since $vN(G)$ is a unital C^*–algebra, we can define the infinite tensor product $vN(G)^{\otimes\infty}$, which is also a C^*–algebra. Let $\varphi^{\otimes\infty}$ be the state on $vN(G)^{\otimes\infty}$ defined by $\varphi^{\otimes\infty}(x_1 \otimes x_2 \otimes \cdots) = \varphi(x_1)\varphi(x_2)\ldots$. The Gelfand–Naimark–Segal construction produces a von Neumann algebra $\mathcal{W}$. For nonnegative integers n, define $j_n : vN(G) \to \mathcal{W}$ by $j_0(\alpha(g)) = Id_W$. and $j_n(\alpha(g)) = \alpha(g)^{\otimes n} \otimes Id \otimes Id \otimes \cdots$. The j_n form what is called a "non-commutative Markov process". There is a projection map E_n from $\mathcal{W}$ to $\mathcal{W}_n$, the von Neumann subalgebra generated by the images of $j_0, \ldots, j_n$. For $n \leq m$, there is the Markov property $E_n \circ j_m = j_n \circ Q^{m-n}$. One could think of these objects with the following analogy:

Classical	State Space S	$(\Omega, \mathcal{F})$	$(\Omega, \mathcal{F}_n)$	$X_n : \Omega \to S$	$\mathbb{E}(\cdot \vert \mathcal{F}_n)$	$\mathbb{E}$
Quantum	$vN(G)$	$\mathcal{W}$	$\mathcal{W}_n$	j_n	$E_n(\cdot)$	$\varphi^{\otimes\infty}$

4.4.1. *Restriction to Center.* Let $Z(vN(G))$ be the center of $vN(G)$. The Peter–Weyl theorem gives an isomorphism $\chi : Z(vN(G)) \to L^\infty(\widehat{G})$, where $\widehat{G}$ is the set of equivalence classes of irreducible representations of G. If κ is constant on conjugacy classes, then Q sends $Z(vN(G))$ to itself. Because it is completely positive, the map $\chi \circ Q \circ \chi^{-1}$ defines a transition matrix for a (classical) Markov chain with state space $\widehat{G}$. By a slight abuse of notation, let $Q_n(\kappa)(x, y)$ denote the transition probabilities.

Now let $G = U(n)$. Define $\kappa_F : U(n) \to \mathbb{C}$ to be the class function defined by

$$\kappa_F(\boldsymbol{\theta}) = \prod_{j=1}^{n} \frac{F(\theta_j)}{F(1)}.$$

Here, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ are the eigenvalues of the unitary matrix on which κ_F is applied. If $F = F_\xi$, write $\kappa_\xi = \kappa_{F_\xi}$.

Here is some useful information about $Q_n(\kappa)$:

Proposition 4.8. *1. For any $\kappa \in L^2(G, dg)^G$,*

$$Q_n(\kappa)(\lambda, \mu) = \frac{\dim \mu}{\dim \lambda} \int_{U(n)} \chi_\lambda(g)\overline{\chi_\mu(g)}\kappa(g)dg. \tag{4.10}$$

2. The map $Q_n : BC(G,\mathbb{C})^G \to Mat(\mathcal{W}_n \times \mathcal{W}_n)$ from complex–valued bounded continuous class functions on G to matrices with rows and columns indexed by $\mathcal{W}_n$ is a morphism of $^* -$ algebras.

Proof. 1. This is Theorem 3.2 from [2]. Although the result there is only stated for certain κ, by following the proof one sees that it holds more generally.

2. The fact that Q_n preserves linearity and * follows immediately from (4.10). By applying (4.3) to (4.10), it is immediate that multiplication is also preserved. Another way to see this is to use the quantum random walk: let Q_1, Q_2, and Q_{12} be the maps $vN(G) \to vN(G)$ defined by sending $\alpha(g)$ to $\kappa_1(g)\alpha(g)$, $\kappa_2(g)\alpha(g)$ and $\kappa_2(g)\kappa_1(g)\alpha(g)$ respectively. By construction, $Q_n(\kappa_1)$, $Q_n(\kappa_2)$, and $Q_n(\kappa_1\kappa_2)$ are the respective restrictions to $Z(vN(G))$. Since $Q_1 \circ Q_2 = Q_{12}$, the result follows.

□

Now specialize to κ_ξ.

Theorem 4.9. *For any symbol ξ, $Q_n(\kappa_\xi) = T_n(\mathbf{1}, F_\xi)$.*

Proof. Start with:

Lemma 4.10. *Theorem 4.9 holds for $\xi = \alpha^\pm, \beta^\pm$.*

Proof. By Weyl's integration formula and (4.1), equation (4.10) implies

$$Q_n(\kappa)(\lambda,\mu) = \frac{\dim\mu}{\dim\lambda}\frac{1}{n!}\int_{\mathbb{T}^n} \det[\theta_i^{\lambda_j+n-j}]\overline{\det[\theta_i^{\mu_j+n-j}]}\kappa(\theta_1,\ldots,\theta_n)d\theta_1\cdots d\theta_n.$$

Changing to complex analytic notation and using that the Haar measure $d\theta$ on $\mathbb{T}$ is $dz/2\pi iz$ implies

$$(4.11)\quad Q_n(\kappa)(\lambda,\mu) = \frac{\dim\mu}{\dim\lambda}\frac{1}{n!}\left(\frac{1}{2\pi i}\right)^n\int_{\mathbb{T}^n} \det[z_i^{\lambda_j+n-j}]\overline{\det[z_i^{\mu_j+n-j}]}\kappa(z_1,\ldots,z_n)\frac{dz_1\cdots dz_n}{z_1\cdots z_n}.$$

Note that

$$\text{(4.12)} \qquad \kappa_{\alpha^+}(\boldsymbol{z}) = \prod_{j=1}^{n} \frac{(1-qz_j)^{-1}}{(1-q)^{-1}} = \prod_{j=1}^{n} \frac{1+qz_j+(qz_j)^2+\dots}{(1-q)^{-1}} = \sum_{k=0}^{\infty} \frac{q^k h_k(\boldsymbol{z})}{(1-q)^{-n}}.$$

$$\kappa_{\alpha^-}(\boldsymbol{z}) = \prod_{j=1}^{n} \frac{(1-qz_j^{-1})^{-1}}{(1-q)^{-1}} = \prod_{j=1}^{n} \frac{1+qz_j^{-1}+(qz_j^{-1})^2+\dots}{(1-q)^{-1}} = \sum_{k=0}^{\infty} \frac{q^k h_k(\boldsymbol{z^{-1}})}{(1-q)^{-n}}.$$

$$\text{(4.13)} \qquad \kappa_{\beta^+}(\boldsymbol{z}) = \prod_{j=1}^{n} \frac{1+pz_j}{1+p} = \sum_{k=0}^{\infty} \frac{p^k e_k(\boldsymbol{z})}{(1+p)^n}.$$

$$\kappa_{\beta^-}(\boldsymbol{z}) = \prod_{j=1}^{n} \frac{1+pz_j^{-1}}{1+p} = \sum_{k=0}^{\infty} \frac{p^k e_k(\boldsymbol{z}^{-1})}{(1+p)^n}.$$

By expanding the determinant in (4.11),

$$\text{(4.14)} \quad Q_n(\kappa_\xi)(\lambda,\mu) = \frac{\dim\mu}{\dim\lambda}\frac{1}{n!}\left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}^n} \left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) z_{\sigma(1)}^{\lambda_1+n-1}\cdots z_{\sigma(n)}^{\lambda_n}\right)$$
$$\times \left(\sum_{\tau\in S_n} \operatorname{sgn}(\tau) z_{\tau(1)}^{-\mu_1-n+1}\cdots z_{\tau(n)}^{-\mu_n}\right) \kappa_\xi(z_1,\dots,z_n)\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

Observe that for any ξ, the only contributions to the integral come from the constant terms after expanding the product.

First consider when $\xi = \alpha^+$. Define the *contribution* from σ and τ to be

$$\operatorname{Con}(\sigma,\tau) = \begin{cases} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau), & \text{if each } \ \lambda_j - j \le \mu_{\tau^{-1}(\sigma(j))} - \tau^{-1}(\sigma(j)), \\ 0, & \text{else.} \end{cases}$$

so that by (4.12)

$$Q_n(\kappa_{\alpha^+})(\lambda,\mu) = \frac{\dim\mu}{\dim\lambda}\frac{1}{n!}\frac{q^{\sum_{i=1}^n \mu_i-\lambda_i}}{(1-q)^{-n}} \sum_{\sigma,\tau} \operatorname{Con}(\sigma,\tau).$$

Proceed with three steps:

I If $\mu_i < \lambda_i$ for some $1 \le i \le n$, then $\operatorname{Con}(\sigma,\tau) = 0$ for all $\sigma,\tau \in S_n$.

II If $\mu_i > \lambda_{i-1}$ for some $1 < i \le n$, then $Q_n(\kappa)(\lambda,\mu) = 0$.

III If $\lambda \prec \mu$, then $\frac{1}{n!}\sum_{\sigma,\tau} \mathrm{Con}(\sigma,\tau) = 1$.

For I, fix an i such that $\mu_i < \lambda_i$ and suppose $\mathrm{Con}(\sigma,\tau)$ is nonzero for some $\sigma, \tau \in S_n$. If there is a $j < i$ which satisfies $\tau(\sigma^{-1}(j)) \geq i$, then $\lambda_j - j \leq \mu_{\tau\sigma^{-1}(j)} - \tau\sigma^{-1}(j) \leq \mu_i - i$, implying that $\mu_i > \lambda_j \geq \lambda_i$, which contradicts $\mu_i < \lambda_i$. Therefore $\tau\sigma^{-1}$ sends the set $\{1, 2, \ldots, i-1\}$ to itself, so $\tau(\sigma^{-1}(i)) \geq i$. Thus $\lambda_i - i \leq \mu_{\tau\sigma^{-1}(i)} - \tau\sigma^{-1}(i) \leq \mu_i - i$, so $\lambda_i \leq \mu_i$. Again, this is a contradiction. Therefore, the only possibility is that all $\mathrm{Con}(\sigma,\tau)$ are zero.

For II, supose that $Q_n(\kappa)(\lambda,\mu) \neq 0$. Fix an i such that $\mu_i > \lambda_{i-1}$. I claim that for some $j \leq i$, there is no k such that $\mu_j - j < \lambda_k - k \leq \mu_{j-1} - (j-1)$. This is simply because there are $i-1$ intervals $(\mu_j - j, \mu_{j-1} - (j-1)]$, but only $i-2$ numbers $\lambda_k - k$ that can fit into these intervals, so at least one interval must be empty. The claim implies that the inequality $\lambda_k - k \ \leq \mu_j - j$ holds if and only if the inequality $\lambda_k - k \leq \mu_{j-1} - (j-1)$ holds. Therefore $\mathrm{Con}(\sigma,\tau) + \mathrm{Con}(\sigma, (j \quad j-1)\cdot\tau) = 0$, so the sum $\sum_{\sigma,\tau} \mathrm{Con}(\sigma,\tau)$ is zero.

For III, suppose that $\mathrm{Con}(\sigma,\tau) \neq 0$. Then, using that $\lambda \prec \mu$, a strong induction argument on j implies that $\tau^{-1}\sigma(j) = j$ for all j. In other words, $\mathrm{Con}(\sigma,\tau) \neq 0$ implies that $\sigma = \tau$. Since the converse is immediate, the sum $\sum_{\sigma,\tau} \mathrm{Con}(\sigma,\tau)$ simplifies to $\sum_{\sigma\in S_n} \mathrm{Con}(\sigma,\sigma)$, which equals $|S_n| = n!$.

Together, I, II and III imply that

$$Q_n(\kappa_{\alpha^+})(\lambda,\mu) = \frac{q^{\sum_{i=1}^n \mu_i - \lambda_i}}{(1-q)^{-n}} \frac{\dim\mu}{\dim\lambda} 1_{\lambda\prec\mu},$$

which is just $T_n(\mathbf{1}, F_{\alpha^+})$.

Now move on to $\xi = \beta^+$. Define the *contribution* from σ and τ to be

$$\mathrm{Con}'(\sigma,\tau) = \begin{cases} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau), & \text{if each } \lambda_j - j - (\mu_{\tau\sigma^{-1}(j)} - \tau\sigma^{-1}(j)) \in \{0,-1\} \\ 0, & \text{else,} \end{cases}$$

so that

$$Q_n(\kappa_{\beta^+})(\lambda,\mu) = \frac{\dim\mu}{\dim\lambda}\frac{1}{n!}\frac{p^{\sum_{i=1}^n \mu_i - \lambda_i}}{(1+p)^n}\sum_{\sigma,\tau}\mathrm{Con}'(\sigma,\tau).$$

Again we prove three steps:

(1) If $\mu_i < \lambda_i$ for some $1 \le i \le n$, then $\mathrm{Con}'(\sigma, \tau) = 0$ for all $\sigma, \tau \in S_n$.
(2) If $\mu_i - \lambda_i \notin \{0, 1\}$ for some $1 < i \le n$, then $Q_n(\kappa)(\lambda, \mu) = 0$.
(3) If all $\mu_i - \lambda_i$ are 0 or 1, then $\frac{1}{n!} \sum_{\sigma,\tau} \mathrm{Con}'(\sigma, \tau) = 1$.

For 1, notice that $\mathrm{Con}(\sigma, \tau) = 0$ implies that $\mathrm{Con}'(\sigma, \tau) = 0$, and I above shows that $\mathrm{Con}(\sigma, \tau)$ is always 0.

For 2, we already know that $Q_n(\kappa)(\lambda, \mu) = 0$ if some $\mu_i < \lambda_i$, so we can assume that all $\mu_i \ge \lambda_i$. Now fix some j such that $\mu_j - \lambda_j \ge 2$, and suppose $\mathrm{Con}'(\sigma, \tau) \ne 0$. Then $\tau\sigma^{-1}(j) \le j$ would imply $\mu_{\tau\sigma^{-1}(j)} - \tau\sigma^{-1}(j) \ge \mu_j - j$, which implies that $\lambda_j - j - (\mu_{\tau\sigma^{-1}(j)} - \tau\sigma^{-1}(j)) \le \lambda_j - \mu_j \le -2$, which contradicts $\mathrm{Con}'(\sigma, \tau) \ne 0$. So $\tau\sigma^{-1}(j) > j$. Thus there must be some $i > j$ such that $\tau\sigma^{-1}(i) \le j$ (or else $\tau\sigma^{-1}$ would map $\{j, \dots, n\}$ to $\{j+1, \dots, n\}$). This implies that $\lambda_i - i < \lambda_j - j \le \mu_j - j - 2 \le \mu_{\tau\sigma^{-1}(i)} - \tau\sigma^{-1}(i) - 2$, which again contradicts $\mathrm{Con}'(\sigma, \tau) \ne 0$. Thus, $\mathrm{Con}'(\sigma, \tau)$ must always be zero.

For 3, suppose that $\mathrm{Con}'(\sigma, \tau) \ne 0$ with $\sigma \ne \tau$, and let j be the smallest integer such that $\sigma(j) \ne \tau(j)$. Then $\tau\sigma^{-1}(j) > j$ and there is some $i > j$ such that $\tau\sigma^{-1}(i) = j$. This implies that $\lambda_i - i < \lambda_j - j \le \mu_{\tau\sigma^{-1}(j)} - \tau\sigma^{-1}(j) < \mu_j - j = \mu_{\tau\sigma^{-1}(i)} - \tau\sigma^{-1}(i)$, which implies that $\lambda_i - i - (\mu_{\tau\sigma^{-1}(i)} - \tau\sigma^{-1}(i)) \le -2$, which is a contradiction. Therefore $\mathrm{Con}'(\sigma, \tau) = 1$ exactly when $\tau = \sigma$, and the result follows.

For the α^- case, it is almost identical to the α^+ case.

Now move on to the β^- case. Since

$$e_k(\boldsymbol{z}^{-1}) = z_1^{-1} \dots z_n^{-1} e_{n-k}(\boldsymbol{z}),$$

the contribution is now

$$\mathrm{Con}''(\sigma, \tau) = \begin{cases} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau), & \text{if each } \ \lambda_j - j - (\mu_{\tau\sigma^{-1}(j)} - \tau\sigma^{-1}(j)) \in \{0, 1\} \\ 0, & \text{else,} \end{cases}$$

and

$$Q_n(\kappa_{\beta^-})(\lambda,\mu) = \frac{\dim\mu}{\dim\lambda}\frac{1}{n!}\frac{p^{\sum_{i=1}^n \lambda_i-\mu_i}}{(1+p)^n}\sum_{\sigma,\tau}\mathrm{Con}''(\sigma,\tau).$$

From here, the proof is the essentially indentical as the β^+ case, except with negative signs inserted and inequalities reversed. $\square$

Lemma 4.11. *If Theorem 4.9 holds for two functions F_1 and F_2, then it holds for F_1F_2.*

Proof. This follows from Proposition 4.8. $\square$

Lemma 4.12. *If Theorem 4.9 holds for a sequence of functions F_k which converge uniformly to a function F on A, then the theorem also holds for F.*

Proof. With f_k defined as in (4.5), it is immediate that f_k converges to f uniformly. Since the determinant is a continuous function of its entries, $T_n(\mathbf{1};F_k)(\lambda,\mu)$ converges to $T_n(\mathbf{1};F)(\lambda,\mu)$. By (4.11), $Q_n(\kappa_{F_k})(\lambda,\mu)$ converges to $Q_n(\kappa_F)(\lambda,\mu)$ as well. $\square$

Finally, since $e^x = \lim_{k\to\infty}(1+x/k)^k = \lim_{k\to\infty}(1-x/k)^{-k}$, Lemmas 4.10, 4.11 and 4.12 finish the proof of Theorem 4.9. $\square$

Let us also prove a statement similar to Theorem 4.3.

Proposition 4.13. *For $\kappa \in L^2(G,\mathbb{C})^G$,*

$$\sum_{\mu\in\mathcal{W}_n} Q_n(\kappa)(0,\mu)c^{\tau}_{\lambda\mu}\frac{\dim\tau}{\dim\lambda\dim\mu} = Q_n(\kappa)(\lambda,\tau).$$

Proof. By linearity, it suffices to prove the result when $\kappa = \chi_\beta$. By (4.10),

$$\begin{aligned}\sum_{\mu\in\mathcal{W}_n} Q_n(\chi_\beta)(0,\mu)c^{\tau}_{\lambda\mu}\frac{\dim\tau}{\dim\lambda\dim\mu} &= \sum_{\mu\in\mathcal{W}_n}\frac{\dim\tau}{\dim\lambda}c^{\tau}_{\lambda\mu}\int_{U(n)}\overline{\chi_\mu(g)}\chi_\beta(g)dg \\ &= \frac{\dim\tau}{\dim\lambda}c^{\tau}_{\lambda\beta}.\end{aligned}$$

On the other side,

$$
\begin{aligned}
Q_n(\chi_\beta)(\lambda,\tau) &= \frac{\dim\tau}{\dim\lambda}\int_{U(n)} \chi_\lambda(g)\overline{\chi_\tau(g)}\chi_\beta(g)dg \\
&= \frac{\dim\tau}{\dim\lambda}\int_{U(n)} \overline{\chi_\tau(g)} \sum_{\mu\in\mathcal{W}_n} c^\mu_{\lambda\beta}\cdot\chi_\mu(g)dg \\
&= \frac{\dim\tau}{\dim\lambda} c^\tau_{\lambda\beta}.
\end{aligned}
$$

□

4.4.2. *Restriction to Maximal Torus.* The purpose of this subsection is to demonstrate that there is a natural representation theoretic reason for the occurence of tensor products in the transition probabilities. To see this, we will consider the restriction of the quantum random walk to the von Neumann algebra of the maximal torus. This is a natural restriction to consider: in [12], it is shown that the Krawtchouk and Charlier processes and Doob h–transforms of Bernoulli and exponential random walks; while in [2], it is shown that representations whose highest weight is miniscule, the restriction of the quantum random walk to the center is the Doob h–transform of the restriction to the maximal torus.

Let $\mathcal{T}$ be the subalgebra of $vN(G)$ generated by $\{\alpha(\theta) : \theta \in \mathbb{T}^n\}$. Since every element of G is conjugate to exactly one element of $\mathbb{T}^n$, we can decompose the Haar measure on G as a measure on $\mathbb{T}^n \times \mathbb{T}^n\backslash G$. Thus $L^2(G, dg) \cong L^2(\mathbb{T}^n, d\theta) \otimes L^2(\mathbb{T}^n\backslash G)$, where $d\theta$ is Haar measure on $\mathbb{T}^n$. With this isomorphism, $\alpha(\theta)$ acts as the identity element on $L^2(\mathbb{T}^n\backslash G)$. Therefore $\mathcal{T}$ is isomorphic to the group von Neumann algebra of $\mathbb{T}^n$.

Since the character group of $\mathbb{T}^n$ is isomorphic to P, there is an isomorphism of W^*–algebras $\zeta : \mathcal{T} \to L^\infty(P)$ such that $\zeta(\alpha(\theta))$ sends $x \in P$ to $e(x)(\theta)$. Since Q sends $\mathcal{T}$ to itself, the map $\zeta \circ Q \circ \zeta^{-1}$ defines a classical Markov chain with state space P. Identity P with $\mathbb{Z}^n$ naturally, and write $P_n(\kappa)(x, y), x, y \in \mathbb{Z}^n$ for the transition matrix of this Markov chain.

Proposition 4.14. *1. For any* $\kappa \in L^2(G, \mathbb{C})^G$,

$$
P_n(\kappa)(x, y) = n_\kappa(y - x) \tag{4.15}
$$

Furthermore, for any σ in the Weyl group, $P_n(\kappa)(x,y) = P_n(\kappa)(\sigma x, \sigma y)$.

2. The map $P_n : BC(G,\mathbb{C})^G \to Mat(P \times P)$ is a morphism of $^*-$ algebras.

Proof. 1. By Proposition 3.1 in [1],

$$P_n(\kappa)(x,y) = \int_{\mathbb{T}^n} e(x)(\theta)\overline{e(y)(\theta)}\kappa(\theta)d\theta, \tag{4.16}$$

which implies that

$$\begin{aligned} P_n\left(\kappa\right)\left(x,y\right) &= \int_{\mathbb{T}^n} \overline{e(y-x)(\theta)} \sum_{\lambda\in\mathcal{W}_n} \widehat{\kappa}(\lambda)\chi_\lambda(\theta)d\theta \\ &= \int_{\mathbb{T}^n} \overline{e(y-x)(\theta)} \sum_{\lambda\in\mathcal{W}_n} \widehat{\kappa}(\lambda) \sum_{z\in P} n_\lambda(z)\cdot e(z)(\theta)d\theta \\ &= \int_{\mathbb{T}^n} \overline{e(y-x)(\theta)} \cdot e(y-x)(\theta) \sum_{\lambda\in\mathcal{W}_n} \widehat{\kappa}(\lambda)n_\lambda(y-x)d\theta \\ &= \sum_{\lambda\in\mathcal{W}_n} \widehat{\kappa}(\lambda)n_\lambda(y-x) = n_\kappa(y-x). \end{aligned}$$

Furthermore, since the weight multiplicities are invariant under the action of the Weyl group, it follows that the transition probabilities are invariant under the Weyl group.

2. The fact that P_n is linear and preserves * follows from (4.16). Since $\sum_{z\in P} e(z)(\theta)\overline{e(z)(\theta')}$ is the Dirac delta $\delta_{\theta\theta'^{-1}}$, it follows that from (4.16) multiplication is also preserved. This can also be seen from the construction of the quantum random walk, as in the proof of Proposition 4.8.2.

There is also a proof which illuminates the occurence of tensor products. To show that P_n preserves multiplication, by (4.15) it suffices to show that the map $n : BC(G,\mathbb{C})^G \to B(P,\mathbb{C})$ defined by $n(\kappa) = n_\kappa$ from bounded, continuous complex–valued class functions on G to bounded complex–valued functions on P is a morphism of *–algebras, where the multiplication in $B(P,\mathbb{C})$ is usual convolution. By definition, n is linear, so it suffices to show that

$$n_{\chi_\lambda\chi_\mu} = n_{\chi_\lambda} * n_{\chi_\mu}.$$

Letting $W(\pi)$ denote the multiset of weight multiplicities (i.e. the number of times that $x \in P$ appears in $W(\pi)$ is $n_{\chi_\pi}(x)$, which is the multiplicity of the weight x in the representation V_π), this is equivalent to

$$W(\pi_1 \otimes \pi_2) = W(\pi_1) + W(\pi_2),$$

where $A + B$ denotes the usual addition of multisets, $A + B = \{a + b : a \in A, b \in B\}$. However, by (4.4), this follows immediately. □

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5. Three–dimensional Gaussian flucutations of non–commutative random surfaces

Abstract. We construct a continuous–time non–commutative random walk on $U(\mathfrak{gl}_N)$ with dilation maps $U(\mathfrak{gl}_N) \to L^2(U(N))^{\otimes\infty}$. This is an analog of a continuous–time non–commutative random walk on the group von Neumann algebra $vN(U(N))$ constructed in [15], and is a variant of discrete–time non–commutative random walks on $U(\mathfrak{gl}_N)$ [2, ?CD].

It is also shown that when restricting to the Gelfand–Tsetlin subalgebra of $U(\mathfrak{gl}_N)$, the non–commutative random walk matches a (2+1)–dimensional random surface model introduced in [7]. As an application, it is then proved that the moments converge to an explicit Gaussian field along time–like paths. Combining with [7] which showed convergence to the Gaussian free field along space–like paths, this computes the entire three–dimensional Gaussian field. In particular, it matches a Gaussian field from eigenvalues of random matrices [5].

5.1. Introduction.

Let us review some results in the mathematical and physics literature in order to motivate the problem.

The *Anisotropic Kardar–Parisi–Zhang* (AKPZ) equation, which was introduced in [20] and is a variant of the KPZ equation first considered in [12], describes a universal class of random surface growth models. Letting $h(t)$ denote the height of the surface at time t, the equation in two dimensions is

$$\partial_t h = \nu_x \partial_x^2 h + \nu_y \partial y^2 h + \frac{1}{2}\lambda_x(\partial_x h)^2 + \frac{1}{2}\lambda_y(\partial_y h)^2 + \eta,$$

where η is space–time white noise and λ_x, λ_y have different signs. (When λ_x and λ_y have the same sign, the equation is just the usual KPZ equation in two dimensions). Using non–rigorous methods, it was predicted (e.g. [13]) that the stationary distribution for the AKPZ dynamics would be the Gaussian free field (see [17] for a mathematical approach to the Gaussian free field). The question about the full three–dimensional process across different time variables remained open.

However, the equation is mathematically ill–defined, due to the non–linear term. One mathematical approach is to consider exactly solvable models in the AKPZ universality class. There have been two models considered, an interacting particle system and the eigenvalue process of a random matrix. Both will be described now.

The interacting particle system, studied in [7], lives on the lattice $\mathbb{Z}\times\mathbb{Z}_+$. It was shown that along *space–like paths*, the particle system is a determinantal point process. (See Theorem 5.7 for the definition of space–like paths). By computing the correlation kernel and taking asymptotics, it was shown that the fluctuations of the height function of the particle system indeed converge to the Gaussian free field. But due to the space–like path restriction, the problem of computing the limiting three–dimensional field remained unsolved.

The random matrix model looks at the eigenvalues of minors of a large random matrix whose entries are evolving as Ornstein–Uhlenbeck processes. By a combinatorial argument, [5] was able to compute the limiting three–dimensional Gaussian field, which has the Gaussian free field as a stationary distribution. The asymptotics at the edge were also computed in [18]. However, one drawback is that the eigenvalues are not Markovian, as shown in [1].

With these two models in mind, it is natural to want to consider an exactly solvable model that "combines" both models, and which is both Markovian and allows for the limiting three–dimensional field to be computed. This paper will construct such a model.

Let us outline the body of the paper. First, the model will be constructed as a continuous–time non–commutative random walk, which is a non–commutative version of the usual random walk in classical probability. The "state space" is $U(\mathfrak{gl}_N)$, the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_N$ of $N \times N$ matrices. The dilation maps are algebra homomorphisms $j_n : U(\mathfrak{gl}_N) \to (M^{\otimes\infty}, \omega)$, where M is a von Neumann sub–algebra of the $U(\mathfrak{gl}_N)$–module $L^2(U(N))$ and ω is a state on $M^{\otimes\infty}$. These j_n are a non–commutative analog of the usual definition of a stochastic process as a family of maps X_n from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a state space S. It is proved below (Theorem 5.1) that there is a semigroup of non–commutative Markov operator $\{P_t\}_{t\geq 0}$ on $U(\mathfrak{gl}_N)$ which is consistent with j_n.

This model is analogous to a previously constructed non–commutative random walk on the group von Neumann algebra $vN(U(N))$ with dilation maps $vN(U(N)) \to vN(U(N))^{\otimes\infty}$ [15]. Additionally, it preserves the states from [6]. All of these construction involve a continuous family of characters of the infinite–dimensional unitary group $U(\infty)$. There have also been previous non–commutative random walks using the basic representation of $U(N)$ as input [2, ?CD].

It also turns out that P_t preserves $Z := Z(U(\mathfrak{gl}_N))$, the centre of $U(\mathfrak{gl}_N)$. This means that $P_t\Big|_Z$ is a Markov operator in the usual (classical) sense. This Markov operator has a natural description: By using the Harish–Chandra isomorphism which identifies Z with the ring of shifted symmetric polynomials in N variables, P_t can be identified with the Markov operator Q_t of an interacting system of N particles on $\mathbb{Z}$. This is shown in Proposition 5.5 below. This interacting system is known as the *Charlier Process*, see [14]. In fact, the projection of the interacting particle system from [7] onto $\mathbb{Z} \times \{N\}$ is exactly Q_t. When restricting our non–commutative random walk to the Gelfand–Tsetlin subalgebra, which is the subalgebra of $U(\mathfrak{gl}_N)$ generated by the centres $Z(U(\mathfrak{gl}_k)), 1 \leq k \leq N$, it also matches the two–dimensional particle system along space–like paths; see Theorem 5.6 for the precise statement. It is worth mentioning that the matching most likely does not hold along time–like paths.

We then take asymptotics of certain elements of the Gelfand–Tsetlin subalgebra and prove convergence to jointly Gaussian random variables. These elements correspond to moments of the random surface. Here, there is no requirement that the paths be space–like, allowing for convergence to Gaussians along time–like paths as well. The explicit covariance formula is given in Theorem 5.8.

At first glance, it appears to be slightly different from the covariance formula for eigenvalues of random matrices. However, the process here corresponds to Brownian motion (see e.g. [4, ?CD]). Indeed, after applying the usual rescaling from Brownian motion to Ornstein–Uhlenbeck, the covariance from [5] is recovered.

Acknowledgments. The author would like to thank Alexei Borodin, Alexey Bufetov, Philippe Biane and Ivan Corwin for enlightening discussions.

5.2. **Preliminaries.** Let us review some background about representation theory and non–commutative random walks. See [3] for an introduction to non–commutative random walks.

5.2.1. *Representation Theory.* The universal enveloping algebra $U(\mathfrak{gl}_N)$ is the unital algebra over $\mathbb{C}$ generated by $\{E_{ij}, 1 \leq i,j \leq N\}$ with relations $E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{il}E_{kj}$. It carries a natural $*$–operation induced from complex conjugation on $\mathbb{C}$. The coproduct $\Delta : U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_N)$ is the algebra morphism sending E_{ij} to $E_{ij} \otimes 1 + 1 \otimes E_{ij}$. There is a natural one–to–one correspondence between finite–dimensional $U(\mathfrak{gl}_N)$–modules, finite–dimensional Lie algebra representations of $\mathfrak{gl}_N$, and finite–dimensional representations of the Lie group $G := U(N)$.

Let $L^2(G)$ be the Hilbert space of square–integrable complex–valued functions on G. Recall that by the Peter–Weyl theorem, this Hilbert space has an orthogonal basis given by the matrix coefficients of all irreducible representations of G, i.e.

$$\{g \mapsto \eta(\pi_\lambda(g)\xi)\},$$

where π_λ runs over all irreducible representations of G, $\{\xi\}$ runs over a basis for V_λ and $\{\eta\}$ runs overs a basis for V_λ^*. Denote this basis by $\{f_{\xi\eta}\}$. Then there is a non–degenerate pairing $\langle \cdot, \cdot \rangle$ between $U(\mathfrak{gl}_N)$ and $L^2(G)$ given by

$$\langle X, f_{\xi\eta} \rangle = \eta(X\xi).$$

This can be heuristically understood as $\langle X, f \rangle = f(X)$, since $f_{\xi\eta}(g) = \eta(g\xi)$. This pairing defines an injection $U(\mathfrak{gl}_N) \hookrightarrow L^2(G)^*$. Let us review the algebra structure of $L^2(G)^*$.

There is a co–algebra structure on $L^2(G)$ given by the co–product $\Delta : L^2(G) \to L^2(G) \otimes L^2(G) \cong L^2(G \times G)$ defined by $\Delta(f)(x,y) = f(xy)$. The multiplication μ on $L^2(G)^*$ is the composition

$$L^2(G)^* \otimes L^2(G)^* \xrightarrow{\rho} (L^2(G) \otimes L^2(G))^* \xrightarrow{\Delta^*} L^2(G)^*,$$

where $\rho(\phi \otimes \psi)(f \otimes h) = \phi(f)\psi(h)$. Use Sweedler's notation to write

$$\Delta(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}.$$

Evaluating both sides at $(x, y) \in G \times G$ shows

$$f(xy) = \sum_{(f)} f_{(1)}(x) f_{(2)}(y) \text{ for all } x, y \in G.$$

Then

$$\mu(\phi \otimes \psi)(f) = \Delta^* \rho(\phi \otimes \psi)(f) = \rho(\phi \otimes \psi)(\Delta f) = \sum_{(f)} \phi(f_{(1)})\psi(f_{(2)}).$$

In particular, if $\phi_x \in L^2(G)^*$ denotes evaluation at x, i.e. $\phi_x(f) = f(x)$, then

$$(\phi_x \phi_y)(f) = \sum_{(f)} \phi_x(f_{(1)})\phi_y(f_{(2)}) = \sum_{(f)} f_{(1)}(x) f_{(2)}(y) = f(xy).$$

So $\phi_x \phi_y = \phi_{xy}$. We also write $\phi_X(\cdot)$ for$\langle X, \cdot \rangle$.

With the pairing between $U(\mathfrak{gl}_N)$ and $L^2(G)$ above, define the action π of $U(\mathfrak{gl}_N)$ on $L^2(G)$ by

$$\pi(a) : f \mapsto \langle \mathrm{id} \otimes a, \Delta f \rangle.$$

The symbol π will sometimes be repressed, in the sense that af means $\pi(a)f$. Observe that π preserves each summand in the Peter–Weyl decomposition $L^2(G) = \bigoplus_\lambda V_\lambda^{(1)} \oplus \cdots \oplus V_\lambda^{(\dim(\lambda))}$. To see this, suppose we are given some matrix coefficient in an irreducible representation V_λ, that is, an $f \in L^2(G)$ of the form

$$f(g) = \langle gv, w \rangle \text{ for fixed } v, w \in V_\lambda.$$

Then by the definition of the co–product

$$\sum_{(f)} f_{(1)}(g_1) f_{(2)}(g_2) = \langle g_1 g_2 v, w \rangle.$$

Since $\langle X, f_{(2)}\rangle = f_{(2)}(X)$, we see that

$$(\pi(X)f)(g) = \langle g \cdot Xv, w\rangle. \tag{5.1}$$

Thus, $\pi(X)$ is of the form $\langle gv', w\rangle$ for $v', w \in V_\lambda$, so the summand is preserved. Letting be the von Neumann algebra consisting of the elements of $\mathrm{Hom}_{\mathbb{C}}(L^2(G), L^2(G))$ which preserve each summand in the Peter–Weyl decomposition, we have that π sends $U(\mathfrak{gl}_N)$ to M. From the definition of the co–product in $U(\mathfrak{gl}_N)$, the n–th tensor power $\pi^{\otimes n} : U(\mathfrak{gl}_N) \to M$ is defined by

$$\pi^{\otimes n}(X) = \sum_{i=1}^{n} \mathrm{Id}^{\otimes i-1} \otimes \pi(X) \otimes \mathrm{Id}^{\otimes n-i}.$$

In general, any Lie group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint action

$$\mathrm{Ad}(g)x = gxg^{-1}, \quad g \in G, x \in \mathfrak{g}.$$

This action extends naturally to $U(\mathfrak{gl}_N)$. For a subgroup K of G, let $U(\mathfrak{gl}_N)^K = \{x \in U(\mathfrak{gl}_N) : \mathrm{Ad}(g)x = x \text{ for all } g \in K\}$. In particular, $U(\mathfrak{gl}_N)^G = Z(U(\mathfrak{gl}_N))$, the centre of $U(\mathfrak{gl}_N)$.

Recall that the Harish–Chandra isomorphism identifies $Z(U(\mathfrak{gl}_N))$ with shifted symmetric polynomials. Explicitly, each $X \in Z(U(\mathfrak{gl}_N))$ acts as some constant $p_X(\lambda)$ on the irreducible representation V_λ. It turns out that p_X is symmetric in the shifted variables $\lambda_i - i$.

5.2.2. *Non–commutative probability.* A non–commutative probability space $(\mathcal{A}, \phi)$ is a unital *–algebra $\mathcal{A}$ with identity 1 and a state $\phi : \mathcal{A} \to \mathbb{C}$, that is, a linear map such that $\phi(a^*a) \geq 0$ and $\phi(1) = 1$. Elements of $\mathcal{A}$ are called *non–commutative random variables.* This generalises a classical probability space, by considering $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $\phi(X) = \mathbb{E}_{\mathbb{P}}X$. We also need a notion of convergence. For a large parameter L and $a_1, \ldots, a_r \in \mathcal{A}, \phi$ which depend on L, as well as a limiting space $(\mathbb{A}, \Phi)$, we say that $(a_1, \ldots, a_r)$ converges to $(\mathbf{a_1}, \ldots, \mathbf{a_r})$ with respect to the state ϕ if

$$\phi(a_{i_1}^{\epsilon_1} \cdots a_{i_k}^{\epsilon_k}) \to \Phi(\mathbf{a}_{\mathbf{i_1}}^{\epsilon_1} \cdots \mathbf{a}_{\mathbf{i_k}}^{\epsilon_\mathbf{k}})$$

for any $i_1, \ldots, i_k \in \{1, \ldots, r\}, \epsilon_j \in \{1, *\}$ and $k \geq 1$.

There is also a non–commutative version of a Markov chain. If $X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to E$ denotes the Markov process with transition operator $Q : L^\infty(E) \to L^\infty(E)$, then the Markov property is

$$\mathbb{E}[Yf(X_{n+1})] = \mathbb{E}[YQf(X_n)]$$

for $f \in L^\infty(E)$ and Y a $\sigma(X_1, \ldots, X_n)$–measurable random variable. Letting $j_n : L^\infty(E) \to L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be defined by $j_n(f) = f(X_n)$, we can write the Markov property as

$$\mathbb{E}[j_{n+1}(f)Y] = \mathbb{E}[j_n(Qf)Y]$$

for all $f \in L^\infty(E)$ and Y in the subalgebra of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ generated by the images of $j_0, \ldots, j_n$.

Translating into the non–commutative setting, we define a *non–commutative Markov operator* to be a semigroup of completely positive unital linear maps $\{P_t : t \in T\}$ from a *–algebra U to itself (not necessarily an algebra morphism). The set T indexing time can be either $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$, that is to say, the Markov process can be either discrete or continuous time. If for any times $t_0 < t_1 < \ldots \in T$ there exists algebra morphisms j_{t_n} from U to a non–commutative probability space (W, ω) such that

$$\omega(j_{t_n}(f)w) = \omega(j_{t_{n-1}}(P_{t_n - t_{n-1}}f)w)$$

for all $f \in U$ and w in the subalgebra of W generated by the images of $\{j_t : t \leq t_{n-1}\}$, then j_t is called a *dilation* of P_t.

5.3. **Non–commutative random walk on $U(\mathfrak{gl}_N)$.** The first thing that needs to be done is to define states on $U(\mathfrak{gl}_N)$. Note that is already has a natural *–algebra structure,

Given any positive type, normalized (sending the identity to 1), class function $\kappa \in L^2(G)$, we have the decomposition

$$\kappa = \sum_{\lambda \in \widehat{G}} \widehat{\kappa}(\lambda) \frac{\chi_\lambda}{\dim \lambda},$$

where $\widehat{G}$ denotes the set of equivalence classes of irreducible representations of G, and χ_λ are the characters corresponding to λ. By the orthogonality relations, $\widehat{\chi_\lambda}(\lambda) = 1$. This defines a state κ on M by

$$\kappa(X) = \sum_\lambda \widehat{\kappa}(\lambda) \sum_{i=1}^{\dim \lambda} \mathrm{Tr}(X|_{V_\lambda^{(i)}}), \quad X \in M.$$

This naturally pulls back via $\pi : U(\mathfrak{gl}_N) \to M$ to a state $\kappa(\cdot)$ on $U(\mathfrak{gl}_N)$.

Recall an equivalent definition of these states from [6]. There is a canonical isomorphism $D : U(\mathfrak{gl}(N)) \to \mathcal{D}(N)$ where $\mathcal{D}(N)$ is the algebra of left–invariant differential operators on $U(N)$ with complex coefficients. Then the state $\langle \cdot \rangle_\kappa$ on $U(\mathfrak{gl}(N))$ is defined by

$$\langle X \rangle_\kappa = D(X)\kappa(U)|_{U=I}.$$

The state can be computed using the formula (see e.g. page 101 of [19]) for $X = E_{i_1 j_1} \cdots E_{i_k j_k}$:

$$D(X)\kappa(U) = \partial_{t_1} \cdots \partial_{t_k} \kappa \left(U e^{t_1 E_{i_1 j_1}} \cdots e^{t_k E_{i_k j_k}} \right) \Big|_{t_1 = \cdots = t_k = 0}. \tag{5.2}$$

Comparing (5.2) and (5.1) shows that $D = \pi$. Here, $e^{tE_{ij}}$ is just the usual exponential of matrices, which has the simple expression

$$e^{tE_{ij}} = \begin{cases} Id + tE_{ij}, & i \neq j \\ Id + (e^t - 1)E_{ii} & i = j \end{cases} \tag{5.3}$$

Note that since (5.2) only involves linear terms in the t_j, one can replace $e^{tE_{ii}}$ with $Id + tE_{ii}$ without changing the value of the right hand side of (5.2). This is a slightly different approach from [6], which used the (equivalent) formula

$$E_{ij} \mapsto \sum_k x_{ik} \partial_{jk}.$$

It is not hard to see that the two definitions of $\langle\cdot\rangle_\kappa$ are equivalent. For each $\lambda \in \widehat{G}$ and $X = E_{i_1j_1}\cdots E_{i_kj_k}$, and letting $v_1,\ldots,v_d$ be a basis of V_λ,

$$\begin{aligned}\langle X\rangle_{\chi_\lambda} &= \partial_{t_1}\cdots\partial_{t_k}\chi_\lambda\left(e^{t_1E_{i_1j_1}}\cdots e^{t_kE_{i_kj_k}}\right)\Big|_{t_1=\cdots=t_k=0}. \\ &= \partial_{t_1}\cdots\partial_{t_k}\sum_{r=1}^{d}\left\langle e^{t_1E_{i_1j_1}}\cdots e^{t_kE_{i_kj_k}}v_r, v_r\right\rangle\Big|_{t_1=\cdots=t_k=0} \\ &= \sum_{r=1}^{d}\left\langle E_{i_1j_1}\cdots E_{i_kj_k}v_r, v_r\right\rangle \\ &= \mathrm{Tr}\left(X\big|_{V_\lambda}\right)\end{aligned}$$

By linearity, this holds for all κ and all X.

Now that the states have been defined, we define the non–commutative Markov process. In order to define a continuous–time non–commutative Markov process, there needs to be a semigroup $\{\kappa_t : t \geq 0\}$ in $L^2(G)$. Indeed, such a semigroup exists: for any $t \geq 0$, let

$$\kappa_t(U) = e^{t\mathrm{Tr}(U-\mathrm{Id})}.$$

Now fix times $t_1 < t_2 < \ldots$. Let $\mathcal{W}$ be the infinite tensor product of von Neumann algebras $M^{\otimes\infty}$ with respect to the state $\omega = \kappa_{t_1}\otimes\kappa_{t_2-t_1}\otimes\kappa_{t_3-t_2}\otimes\ldots$. For $n \geq 1$ define the morphism $j_{t_n} : U(\mathfrak{gl}_N) \to \mathcal{W}$ to be the map $j_{t_n}(X) = \pi^{\otimes n}(X)\otimes\mathrm{Id}^{\otimes\infty}$, and let $\mathcal{W}_n$ be the subalgebra generated by the images of $j_{t_1},\ldots,j_{t_n}$. Define $P_t : U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ by $(\mathrm{id}\otimes\kappa_t)\circ\Delta$. Note that P is linear as a map of complex vector spaces, but is not an algebra morphism, since the trace is not preserved under multiplication of matrices. To simplify notation, write $\langle\cdot\rangle_t$ for $\langle\cdot\rangle_{\kappa_t}$ and j_n for j_{t_n}.

Theorem 5.1. *(1) The maps (j_n) are a dilation of the non–commutative Markov operator P_t. In other words,*

$$\omega(j_n(X)w) = \omega(j_{n-1}(P_{t_n-t_{n-1}}X)w), \quad X \in U(\mathfrak{gl}(N)), \quad w \in \mathcal{W}_{n-1}.$$

(2) The pullback of ω under j_n is the state $\langle\cdot\rangle$ on $U(\mathfrak{gl}(N))$, i.e. $\langle X\rangle_{t_n} = \omega(j_n(X))$.

(3) For $n \leq m$, we have

$$\omega(j_n(X)j_m(Y)) = \langle X \cdot P_{t_m - t_n} Y \rangle_{t_n}.$$

(4) The non–commutative markov operators P_t satisfy the semi–group property $P_{t+s} = P_t \circ P_s$.

(5) For any subgroup $K \subset U(N)$, the restriction of P_t to $U(\mathfrak{gl}_N)^K$ is a non–commutative transition kernel. In other words, $P_t U(\mathfrak{gl}_N)^K \subset U(\mathfrak{gl}_N)^K$. In particular, $P_t Z(U(\mathfrak{gl}_N)) \subset Z(U(\mathfrak{gl}_N))$.

Proof. (1) This is essentially identical to Proposition 3.1 from [9]. It is reproduced here for completeness. The left–hand–side is

$$\begin{aligned}\omega((\pi^{\otimes n-1} \otimes \pi)\Delta X w) &= \sum_{(X)} \omega(\pi^{\otimes n-1}(X_{(1)}) \otimes \pi(X_{(2)})w) \\ &= \sum_{(X)} \omega(\pi^{\otimes n-1}(X_{(1)})w) \left\langle X_{(2)} \right\rangle_{t_n - t_{n-1}}.\end{aligned}$$

The right–hand–side is

$$\sum_{(X)} \omega(j_{n-1}(\left\langle X_{(2)} \right\rangle_{t_n - t_{n-1}} X_{(1)}w) = \sum_{(X)} \omega(j_{n-1}(X_{(1)})w) \left\langle X_{(2)} \right\rangle_{t_n - t_{n-1}}.$$

(2) Let m_n denote the n–fold multiplication $L^2(G)^{\otimes n} \to L^2(G)$ that sends $f_1 \otimes \cdots \otimes f_n$ to $f_1 \cdots f_n$. We will show that

$$m_n(\pi^{\otimes n}(X)(f_1 \otimes \cdots \otimes f_n)) = \pi(X)(f_1 \cdots f_n). \tag{5.4}$$

The case of general n follows inductively from $n = 2$. The left hand side is

$$\sum_{(X)} (\pi(X^{(1)})f_1) \cdot (\pi(X^{(1)})f_2) = \sum_{(X, f_1, f_2)} \langle X^{(1)}, f_1^{(2)} \rangle f_1^{(1)} \cdot \langle X^{(2)}, f_2^{(2)} \rangle f_2^{(1)}.$$

The right hand side is

$$\langle \mathrm{id} \otimes X, \Delta(f_1 \cdot f_2) \rangle = \sum_{(f_1, f_2)} \langle X, f_1^{(2)} f_2^{(2)} \rangle f_1^{(1)} \cdot f_2^{(1)}.$$

So it suffices to show that

$$\sum_{(X)} \langle X^{(1)}, f_1^{(2)} \rangle \langle X^{(2)}, f_2^{(2)} \rangle = \langle X, f_1^{(2)} f_2^{(2)} \rangle.$$

But this is just the definition of multiplication in a dual Hopf algebra. So (5.4) is true.

Now recall that if A is a Hopf algebra with co–unit $\epsilon : A \to \mathbb{C}$, then the n–fold tensor power $A^{\otimes n}$ is also a Hopf algebra, with co–unit $\epsilon^{(n)} : A^{\otimes n} \to \mathbb{C}$ defined by the composition

$$A^{\otimes n} \xrightarrow{m_n} A \xrightarrow{\epsilon} \mathbb{C}.$$

In other words,

$$\epsilon^{(n)}(a_1 \otimes \cdots \otimes a_n) = \epsilon(a_1 \cdots a_n) = \epsilon(a_1) \cdots \epsilon(a_n).$$

The second equality holds because ϵ is a morphism of $\mathbb{C}$–algebras.

When $A = L^2(G)$, then $\epsilon : L^2(G) \to \mathbb{C}$ is defined by $\epsilon(f) = f(Id_G)$. I now claim that for $X \in U(\mathfrak{gl}_N)$

$$\omega(\pi^{\otimes n}(X)) = \epsilon^{(n)} X(\kappa_{t_n}). \tag{5.5}$$

For $n = 1$, this follows immediately from the definitions. For $n \geq 2$, write as usual

$$\pi^{\otimes n}(X) = \sum_{(X)} X_{(1)} \otimes \cdots \otimes X_{(n)} \in M^{\otimes n}.$$

Then

$$\begin{aligned} \omega(\pi^{\otimes n}(X)) &= \sum_{(X)} \kappa_{t_1}(X_{(1)}) \cdots \kappa_{t_n - t_{n-1}}(X_{(n)}) \\ &= \sum_{(X)} \epsilon X_{(1)}(\kappa_{t_1}) \cdots \epsilon X_{(n)}(\kappa_{t_n - t_{n-1}}). \end{aligned}$$

At the same time,

$$\begin{aligned}\epsilon^{(n)}X(\kappa_{t_n}) &= \epsilon^{(n)}X(\kappa_{t_1}\cdots\kappa_{t_n-t_{n-1}})\\ &= \epsilon^{(n)}\sum_{(X)}X_{(1)}(\kappa_{t_1})\otimes\cdots\otimes X_{(n)}(\kappa_{t_n-t_{n-1}})\\ &= \sum_{(X)}\epsilon X_{(1)}(\kappa_{t_1})\cdots\epsilon X_{(n)}(\kappa_{t_n-t_{n-1}}).\end{aligned}$$

So (5.5) is true.

Finally, we can combine the results to obtain

$$\begin{aligned}\omega(j_n(X)) = \omega(\pi^{\otimes n}(X)) &= \epsilon^{(n)}\pi^{\otimes n}X(\kappa_{t_1}\otimes\cdots\otimes\kappa_{t_n-t_{n-1}})\\ &= \epsilon m_n(\pi^{\otimes n}X(\kappa_{t_1}\otimes\cdots\otimes\kappa_{t_n-t_{n-1}}))\\ &= \epsilon\pi(X)(\kappa_{t_1}\otimes\cdots\otimes\kappa_{t_n-t_{n-1}}) = \langle X\rangle_{\kappa_{t_n}}.\end{aligned}$$

This is exactly what (2) stated.

(3) By repeated applications of (1),

$$\omega(j_n(X)j_m(Y)) = \omega(j_n(X)j_n(P_{t_m-t_n}Y)).$$

Since j_n is a morphism of algebras, this equals $\omega(j_n(X\cdot P_{t_m-t_n}Y))$, which by (2) equals the right–hand–side of (3).

(4) By linearity, it suffices to prove this for monomials of the form $E = E_{i_1j_1}\cdots E_{i_kj_k}$. Introduce some notation: let $K = \{1,\cdots,k\}$ and for any subset $S\subseteq K$, define $E_S = \prod_{s\in S}E_{i_sj_s}$, where the product is taken in increasing order. The term $E_\emptyset$ is understood to be 1. So, for example, if $E = E_{13}E_{42}E_{55}E_{12}$ and $S = \{1,2,4\}$ then $E_S = E_{13}E_{42}E_{12}$. With this notation,

$$\Delta E = \sum_{S\subseteq K}E_S\otimes E_{K\setminus S}.$$

And therefore

$$P_{t_2-t_1}E = \sum_{S\subseteq K} \langle E_{K\setminus S}\rangle_{t_2-t_1} E_S,$$

$$P_{t_1}P_{t_2-t_1}E = \sum_{\substack{S\subseteq K\\ R\subseteq S}} \left\langle E_{S\setminus R}\right\rangle_{t_1} \left\langle E_{K\setminus S}\right\rangle_{t_2-t_1} E_R.$$

Since

$$P_{t_2}E = \sum_{R\subseteq K} \left\langle E_{K\setminus R}\right\rangle_{t_2} E_R,$$

it suffices to show

$$\left\langle E_{K\setminus R}\right\rangle_{t_2} = \sum_{R\subseteq S\subseteq K} \left\langle E_{K\setminus S}\right\rangle_{t_2-t_1} \left\langle E_{S\setminus R}\right\rangle_{t_1} \quad \text{for all } R\subseteq K,$$

or equivalently

$$\langle E_K\rangle_{t_2} = \sum_{S\subseteq K} \left\langle E_{K\setminus S}\right\rangle_{t_2-t_1} \langle E_S\rangle_{t_1}.$$

This follows from (5.2) and the general Leibniz rule applied to the derivatives of the product $\kappa_{t_1}\cdot\kappa_{t_2-t_1}$.

(5) This is Proposition 4.3 from [9]. Here is also a bare–bones proof when $K = G$ Let $X \in Z(U(\mathfrak{gl}_N))$. The goal is to show that $P(X)Y = YP(X)$ for all $Y \in U(\mathfrak{gl}_N)$. It suffices to show this when $Y \in \mathfrak{g}$. In this case,

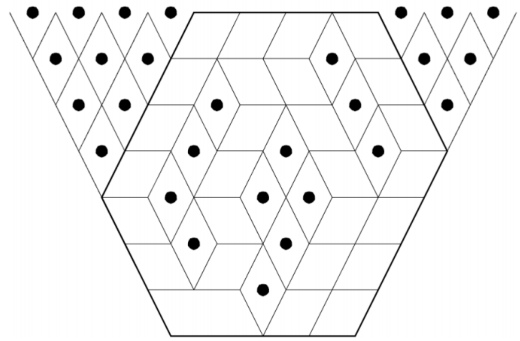
$$XY = YX \Longrightarrow \Delta(XY) = \Delta(YX) \Longrightarrow \Delta(X)\Delta(Y) = \Delta(Y)\Delta(X)$$
$$\Longrightarrow \sum_{(X)} X_{(1)}Y\otimes X_{(2)} + X_{(1)}\otimes X_{(2)}Y = \sum_{(X)} YX_{(1)}\otimes X_{(2)} + X_{(1)}\otimes YX_{(2)}.$$

Now apply the linear map $id\otimes\langle\cdot\rangle$ to both sides to get

$$\sum_{(X)} \left\langle X_{(2)}\right\rangle X_{(1)}Y + \left\langle X_{(2)}Y\right\rangle X_{(1)} = \sum_{(X)} \left\langle X_{(2)}\right\rangle YX_{(1)} + \left\langle YX_{(2)}\right\rangle X_{(1)}.$$

Since the state $\langle\cdot\rangle$ is tracial, the second summand on both sides are equal. The first summand on the left–hand–side is $P(X)Y$ while the first summand on the right–hand–side is $YP(X)$, so $P(X) \in Z(U(\mathfrak{gl}_N))$ as needed. $\square$

FIGURE 15. The particles as a stepped surface. The lattice is shifted to make the visualization easier.

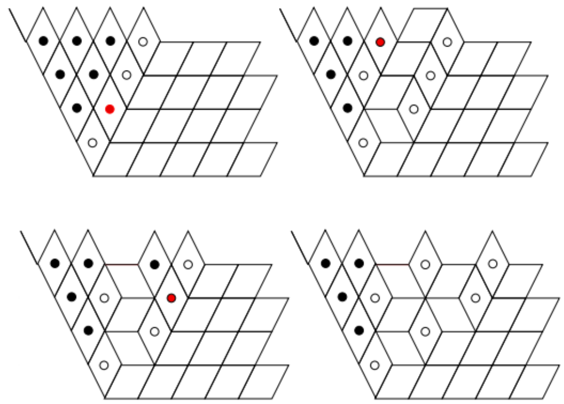


5.4. **Connections to classical probability.** In this section, we will show that restricting to the centres $Z(U(\mathfrak{gl}_1), \ldots, Z(U(\mathfrak{gl}_N))$ reduces the non–commutative random walk to a (2+1)–dimensional random surface growth model. First, here is a description of the model, which was introduced in [7].

5.4.1. *Random surface growth.* Consider the two–dimensional lattice $\mathbb{Z} \times \mathbb{Z}_+$. On each horizontal level $\mathbb{Z} \times \{n\}$ there are exactly n particles, with at most one particle at each lattice site. Let $X_1^{(n)} > \ldots > X_n^{(n)}$ denote the x–coordinates of the locations of the n particles. Additionally, the particles need to satisfy the *interlacing property* $X_{i+1}^{(n+1)} < X_i^{(n)} \leq X_i^{(n+1)}$. The particles can be viewed as a random stepped surface, see Figure 15. This can be made rigorous by defining the height function at (x, n) to be the number of particles to the right of (x, n).

The dynamics on the particles are as follows. The initial condition is the *densely packed* initial condition, $\lambda_i^{(n)} = -i+1, 1 \leq i \leq n$. Each particle has a clock with exponential waiting time of rate 1, with all clocks independent of each other. When the clock rings, the particle attempts to jump one step to the right. However, it must maintain the interlacing property. This is done by having particles push particles above it, and jumps are blocked by particles below it. One can think of lower particles as being more massive. See Figure 16 for an example.

Figure 16. The red particle makes a jump. If any of the black particles attempt to jump, their jump is blocked by the particle below and to the right, and nothing happens. White particles are not blocked.



The projection to $\mathbb{Z} \times \{n\}$ is still Markovian, and is known as the *Charlier process* [14]. It can be described by as a continuous–time Markov chain on $\mathbb{Z}^n$ with independent increments $e_i/n, 1 \leq i \leq n$, (where $\{e_i\}$ is the canonical basis for $\mathbb{Z}^n$) conditioned to stay in the Weyl chamber $(x_1 > x_2 > \ldots > x_n)$. Equivalently, the conditioned Markov chain is the Doob h–transform for some harmonic function h. There is a nice description of h in terms of representation theory, namely, $h(x_1, \ldots, x_n)$ is the dimension of the irreducible representation of $\mathfrak{gl}_n$ with highest weight $(x_1, x_2 + 1, \ldots, x_n + n - 1)$. Explicitly,

$$\dim \lambda = \prod_{i<j} \frac{\lambda_i - i - (\lambda_j - j)}{j - i}.$$

Below, let $Q_t^{(N)}$ denote the Markov operator of this Markov chain.

The construction of the full particle system is based on a general multi–variate construction from [7], which is based on [10]. Suppose there are two Markov chains with state spaces $\mathcal{S}, \mathcal{S}^*$ and transition probabilities P, P^*. Also assume there is a Markov operator $\Lambda : \mathcal{S}^* \to \mathcal{S}$ which intertwines with P, P^* in the sense that $\Lambda P^* = P\Lambda$. In other words, there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{S}^* & \xrightarrow{P^*} & \mathcal{S}^* \\
\downarrow \Lambda & & \downarrow \Lambda \\
\mathcal{S} & \xrightarrow{P} & \mathcal{S}
\end{array}
\tag{5.6}
$$

Then the state space is $\{(x^*, x) \in \mathcal{S}^* \times \mathcal{S} : \Lambda(x^*, x) \neq 0\}$ with transition probabilities

$$
\mathrm{Prob}((x^*, x) \to (y^*, y)) = \begin{cases} \frac{P(x,y)P^*(x^*,y^*)\Lambda(y^*,y)}{\Delta(x^*,y)}, & \Delta(x^*, y) \neq 0 \\ 0, & \Delta(x^*, y) = 0 \end{cases}
$$

Additionally, if the intial condition is a *Gibbs measure*, that is, a probability distribution of the form $\mathbb{P}(x^*)\Lambda(x^*, x)$, then the dynamics preserves Gibbs measures. All constructions and definitions extend naturally to any finite number of Markov chains.

Here, $Q_t^{(N)}$ and $Q_t^{(N-1)}$ will play the roles of P^*, P, and the projection Λ is

$$
\Lambda(x_1 > \ldots > x_N, y_1 > \ldots > y_{N-1}) = \frac{h(y)}{h(x)}.
$$

The construction implies that

$$
\mathbb{P}(X^{(N)}(t) = x^{(N)} | X^{(M)}(t) = x^{(M)}) = \frac{h(x^{(N)})}{h(x^{(M)})}, \forall N \leq M, t \geq 0 \tag{5.7}
$$

$$
\begin{aligned}
&\mathbb{P}(X^{(N)}(t) = x^{(N)} | X^{(M)}(s) = y^{(M)}, X^{(N)}(s) = y^{(N)}) \\
&\qquad = \mathbb{P}(X^{(N)}(t) = x^{(N)} | X^{(N)}(s) = y^{(N)}), \forall N \leq M, s \leq t
\end{aligned}
\tag{5.8}
$$

The intuition behind (5.8) is that since particles on lower levels push and block the particles on higher levels, the evolution of the N–th level is independent of the evolution M–th level. Equation (5.7) is a mathematical formulation of the statement that the dynamics preserves Gibbs measures.

5.4.2. *Restriction to centre.* Before continuing, we need to compute the states of certain observables.

Proposition 5.2. *Let* Π *denote the set of partitions of the set* $\{1,\ldots,m\}$*, let* $|\pi|$ *denote the number of blocks of the partition* $\pi \in \Pi$ *and let* $B \in \pi$ *mean that* B *is a block in* π*. Then*

$$\langle E_{i_1 j_1} \cdots E_{i_m j_m} \rangle_t = \sum_{\pi \in \Pi} t^{|\pi|} \prod_{\substack{B \in \pi \\ B=\{b_1 \ldots, b_k\}}} 1_{j_{b_1}=i_{b_2}, j_{b_2}=i_{b_3}, \ldots, j_{b_k}=i_{b_1}}$$

Example 1

$$\langle E_{21}E_{12}E_{21}E_{12} \rangle_t = 2t^2 + t$$

with two contributing partitions having two blocks: $\{1,2\} \cup \{3,4\}, \{1,4\} \cup \{2,3\}$, and one contributing partition having one block $\{1,2,3,4\}$.

Example 2

$$\left\langle E_{11}^3 E_{22} \right\rangle_t = t^4 + 3t^3 + t^2$$

with one contributing partition having four blocks:$\{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, three contributing partitions having three blocks: $\{1,2\} \cup \{3\} \cup \{4\}, \{1,3\} \cup \{2\} \cup \{4\}, \{2,3\} \cup \{1\} \cup \{4\}$, and one contributing partition having two blocks: $\{1,2,3\} \cup \{4\}$.

Example 3 For any m,

$$\left\langle E_{jj}^m \right\rangle_t = B_m(t)$$

where $B_m(t)$ is the m–th Bell polynomial. These are also the moments of a Poisson random variable with mean t, so under the state $\langle \cdot \rangle_t$, each E_{jj} can be heuristically understood to be distributed as Poisson(t).

Example 4

$$\langle E_{11}E_{12} \rangle_t = 0$$

with no contributing partitions.

Proof. Recall the defintion of $\langle \cdot \rangle_t$ in (5.2). By Faa di Bruno formula,

$$\langle E_{i_1 j_1} \cdots E_{i_m j_m} \rangle_t = \sum_{\pi \in \Pi} f^{(|\pi|)}(y) \prod_{B \in \pi} \frac{\partial^{|B|} y}{\prod_{b \in B} \partial x_b} \Bigg|_{x_1 = \cdots = x_m = 0}$$

where

$$f(y) = e^{ty}, \quad y = Tr(e^{x_1 E_{i_1 j_1}} \cdots e^{x_m E_{i_m j_m}} - Id).$$

Note that

$$f^{(|\pi|)}(y)\Bigg|_{x_1=\cdots=x_m=0} = t^{|\pi|} f(y)\Bigg|_{y=0} = t^{|\pi|}.$$

Since we are taking the derivative with respect to x_b and setting equal to 0, we only need the linear terms in x_b, so it is equivalent to replace y with

$$y = Tr((Id + x_1 E_{i_1 j_1}) \cdots (Id + x_m E_{i_m j_m}) - Id).$$

Here, E_{ij} are the usual $N \times N$ matrices acting on $\mathbb{C}^N$, not the generators of $U(\mathfrak{gl}_N)$. Expanding the parantheses, all terms other than $Tr\left(\prod_{b\in B} x_b E_{i_b j_b}\right)$ do not contribute, since these do not survive differentiation with respect to $x_b, b \in B$. Finally, since

$$Tr\left(\prod_{\substack{b\in B \\ B=\{b_1\ldots,b_k\}}} E_{i_b j_b}\right) = 1_{j_{b_1}=i_{b_2}, j_{b_2}=i_{b_3}, \ldots, j_{b_k}=i_{b_1}},$$

the proof is finished. □

In section 7 of [11], explicit generators of the centre $Z(U(\mathfrak{gl}_N))$ were found. See also chapter 7 of [16] for an exposition.

Let $\mathcal{G}_m$ denote the directed graph with vertices and edges

$$\{1, \ldots, m\} \quad \{(i,j) : 1 \leq i, j \leq m\}.$$

Let $\Pi_k^{(m)}$ denote the set of all paths in $\mathcal{G}_m$ of length k which start and end at the vertex m. For $\pi \in \Pi_k^{(m)}$ let $r(\pi)$ denote the length of the first return to m. Let $E(\pi) \in U(\mathfrak{gl}_m)$ denote the element with coefficient $r(\pi)$ obtained by taking the product when labeling the edge (i,j) with E_{ij} when $i \neq j$, and the edge (i,i) with $E_{ii} - m + 1$. For example, the path

$$\pi = \{5 \to 3 \to 3 \to 1 \to 5 \to 5 \to 2 \to 5\}$$

is in $\Pi_7^{(5)}$ with $r(\pi) = 4$ and

$$E(\pi) = 4E_{53}(E_{33} - 4)E_{31}E_{15}(E_{55} - 4)E_{52}E_{25}.$$

Define the elements

$$\Psi_k := \sum_{m=1}^{N} \sum_{\pi \in \Pi_k^{(m)}} E(\pi) \in U(\mathfrak{gl}_N).$$

For example,

$$\Psi_1 = \sum_{m=1}^{N} (E_{mm} - m + 1), \quad \Psi_2 = \sum_{m=1}^{N} (E_{mm} - m + 1)^2 + 2 \sum_{1 \le l < m \le N} E_{ml}E_{lm}.$$

When we wish to emphasize that $\Psi_k \in U(\mathfrak{gl}_N)$, the notation $\Psi_k^{(N)}$ will be used. [4]

Theorem 5.3. [11] *The centre $Z(U(\mathfrak{gl}_N))$ is generated by the elements $1, \{\Psi_k\}_{k \ge 1}$. Furthermore, the Harish–Chandra isomorphism maps Ψ_k to the shifted symmetric polynomial $\sum_{m=1}^{N} (\lambda_m - m + 1)^k$.*

Remark. Writing $\mathfrak{gl}_N = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_+, \mathfrak{n}_-$ are the upper and lower nilpotent subalgebras and $\mathfrak{h}$ is the diagonal subalgebra, the Harish–Chandra homomorphism is the projection

$$U(\mathfrak{gl}_N) = (\mathfrak{n}_- U(\mathfrak{gl}_N) + U(\mathfrak{gl}_N)\mathfrak{n}_+) \oplus U(\mathfrak{h}) \to U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\lambda_1, \ldots, \lambda_N].$$

This sends

$$\Psi_k = \sum_{m=1}^{N} (E_{mm} - m + 1)^k + (\text{other terms}) \mapsto \sum_{m=1}^{N} (E_{mm} - m + 1)^k = \sum_{m=1}^{N} (\lambda_m - m + 1)^k.$$

Of course, $\sum_{m=1}^{N} (E_{mm} - m + 1)^k$ is in general not central.

Now it is time to explicitly state the relationship between the non–commutative random walk and the growing stepped surface. One may be tempted to think that

[4]Caution: This notation is consistent with notation from integrable probability but different from notation in representation theory.

$$\begin{array}{ccc} U(\mathfrak{gl}_N) & \xrightarrow{P_t} & U(\mathfrak{gl}_N) \\ \uparrow & & \uparrow \\ U(\mathfrak{gl}_{N-1}) & \xrightarrow{P_t} & U(\mathfrak{gl}_{N-1}) \end{array}$$

is a non–commutative version of (5.6). However, care needs to be taken because the inclusion map does not send $Z(U(\mathfrak{gl}_{N-1}))$ to $Z(U(\mathfrak{gl}_N))$.

A slight change of variables will make statements cleaner. If $p(\lambda)$ is a shifted symmetric polynomial, then by definition it is symmetric in the variables $x_i = \lambda_i - i + 1$, and let $\bar{p}(x)$ denote the corresponding symmetric polynomial.

Proposition 5.4. *If $Y \in Z(U(\mathfrak{gl}_N))$ is sent to the symmetric polynomial $p_Y(x)$ by the Harish–Chandra isomorphism, then*

$$\langle Y \rangle_t = \mathbb{E}\left[\bar{p}_Y(X_1^{(N)}(t), \ldots, X_N^{(N)}(t))\right].$$

Proof. This is not new, see [6], but the proof is similar to Theorem 5.6 below, so will be repeated for clarity. By a result from [8],

$$e^{t\mathrm{Tr}(U-\mathrm{Id})} = \sum_{\lambda} \mathrm{Prob}(X_i^{(N)}(t) = \lambda_i - i + 1, 1 \leq i \leq N) \frac{\chi_\lambda(U)}{\dim \lambda}$$

where χ_λ and $\dim \lambda$ are the character and dimension of the highest weight representation λ. Thus, by linearity,

$$\begin{aligned} \langle Y \rangle_t &= \sum_{\lambda} \mathrm{Prob}(X_i^{(N)}(t) = \lambda_i - i + 1, 1 \leq i \leq N) \frac{\langle Y \rangle_{\chi_\lambda}}{\dim \lambda} \\ &= \sum_{\lambda} \mathrm{Prob}(X_i^{(N)}(t) = \lambda_i - i + 1, 1 \leq i \leq N) p_Y(\lambda_1, \ldots, \lambda_N) \\ &= \sum_{x} \mathrm{Prob}(X_i^{(N)}(t) = x_i, 1 \leq i \leq N) \bar{p}_Y(x_1, \ldots, x_N) \end{aligned}$$

The last line is simply the right–hand–side of the proposition. □

Proposition 5.5. *Suppose that P_t and Q_t are two semigroups which preserve $Z(U(\mathfrak{gl}_N))$ and satisfy Theorem 5.1(1), then $P_tX = Q_tX$ for all $X \in Z(U(\mathfrak{gl}_N))$. In particular, P_t is the Markov operator of the process $(X_1^{(N)}(t) > \ldots > X_N^{(N)}(t))$.*

Proof. Theorem 5.1(1) and (2) imply that $\langle P_tX \rangle_s = \langle Q_tX \rangle_s = \langle X \rangle_{t+s}$ for all $s, t \geq 0$. In order to show $P_tX = Q_tX$, it suffices to show that if $Y \in Z$ satisfies $\langle Y \rangle_t = 0$ for all $t \geq 0$, then $Y = 0$. Suppose this is not true, and let Y be a counterexample of minimal degree. But then $\langle P_tY \rangle_s = \langle Y \rangle_s = 0$ and Theorem 5.3 implies that $P_tY = Y + t$(lower degree terms). By assumption, $P_tY - Y \in Z$ also satisfies $\langle P_tY - Y \rangle_s$ for all $s \geq 0$. Thus, since Y is of minimal degree, $P_tY - Y = 0$.

If Y has degree $d > 1$, then by Theorem 5.3 the term E_{11}^d appears in Y. Thus P_tY has a tdE_{11}^{d-1} term which cannot cancel with any term in Y. If Y has degree 1 then $Y = a_1\Psi_1 + a_0$ and $P_tY - Y = a_1tN$. Thus, there is a contradiction, so no such Y can exist.

The second art of the proposition follows if we show that Q_t preserves shifted symmetric polynomials. But this follows because the process is the Doob h–transform of a random walk which is invariant under permuting the co–ordinates, and h is anti–symmetric. □

Example 5 For $N = 2$, one can explicitly compute (after a long calculation)

$$P_t\Psi_4 = \Psi_4 + 4t\Psi_3 + (6t^2 + 8t)\Psi_2 + 2t\Psi_1^2 \\ + (4t^3 + 24t^2 + 10t)\Psi_1 + (2t^4 + 24t^3 + 38t^2 + 6t).$$

For instance, the only appearance of the monomial $E_{11}E_{22}$ in the right hand side is in Ψ_1^2. The only monomial in Ψ_4 that can lead to $E_{11}E_{22}$ is $4E_{22}E_{21}E_{11}E_{12}$. The co–product $\Delta(E_{ij}) = 1 \otimes E_{ij} + E_{ij} \otimes 1$ sends E_{ij} either to the left tensor factor or the right tensor factor. In order to get $E_{11}E_{22}$, we must send $E_{22}E_{11}$ to the left and $E_{21}E_{12}$ to the right. Since $\langle E_{21}E_{12} \rangle_t = t$, the coefficient of $E_{22}E_{11}$ in $P_t\Psi_4$ must be $4t$. Since the coefficient of $E_{22}E_{11}$ in Ψ_1^2 is 2, this implies that the coefficient of Ψ_1^2 in Ψ_4 is $2t$. Similar considerations can be applied to produce the other terms.

Now, evaluating this at $(4,2)$ with $t=3$ would predict that

$$P_t\Psi_4(4,2) = 257 + 12*65 + 78*17 + 6*5^2 + 354*5 + 1170 = 5453.$$

And indeed, the explicit determinantal formula from [7] for $N=2$ yields

$$\frac{\sum_{b=x}^{\infty}\sum_{a=y}^{b}(b^k+(a-1)^k)(b-a+1)t^{b+a}\det\begin{bmatrix}(b-x)!^{-1} & (b-(y-1))!^{-1}\\ (a-1-x)!^{-1} & (a-y)^{-1}\end{bmatrix}}{\sum_{b=x}^{\infty}\sum_{a=y}^{b}(b-a+1)t^{b+a}\det\begin{bmatrix}(b-x)!^{-1} & (b-(y-1))!^{-1}\\ (a-1-x)!^{-1} & (a-y)^{-1}\end{bmatrix}}.$$

A numerical computation at $(x,y)=(4,2), t=3, k=4$ with the sum from $b=x$ to $b=50\approx\infty$ yields 5452.99999999999999999999999999999418... Note that it is not obvious from the summation that the answer would even be an integer.

Example 6 For general N,

$$P_t\Psi_1 = \Psi_1 + tN, \quad P_t\Psi_2 = \Psi_2 + 2t\Psi_1 + (t^2+Nt)N,$$
$$P_t\Psi_3 = \Psi_3 + 3t\Psi_2 + 3(t^2+Nt)\Psi_1 + N(t^3+3t^2N+\frac{1}{2}t(N^2+1)).$$

One can check explicitly that the semigroup property holds.

Example 7 We wish to take asymptotics $N\approx \eta L$ and $t\approx\tau L$. We would get

$$P_t\Psi_{1,N} = \Psi_1 + \text{const},$$

$$P_t\Psi_{2,N} = \Psi_2 + 2\tau L\Psi_1 + \text{const},$$

$$P_t\Psi_{3,N} = \Psi_3 + 3\tau L\Psi_2 + 3(\tau^2+\eta\tau)L^2\Psi_1 + \text{const}$$

$$P_t\Psi_{4,N} = \Psi_4 + 4\tau L\Psi_3 + (6\tau^2+4\tau\eta)L^2\Psi_2 + (4\tau^3+12\tau^2\eta+2\tau\eta^2)L^3\Psi_1 + 2\tau L\Psi_1^2 + \text{const}$$

$$P_t\Psi_{1,N}^2 = \Psi_{1,N}^2 + 2\eta\tau L^2\Psi_{1,N} + \text{const}$$

Again, one can check that the semigroup property holds.

Theorem 5.6. *Suppose $Y_1 \in Z(U(\mathfrak{gl}_{N_1})), \ldots, Y_r \in Z(U(\mathfrak{gl}_{N_r}))$ are mapped to the symmetric polynomials $\bar{p}_{Y_1}, \ldots \bar{p}_{Y_r}$ under the Harish–Chandra isomorphism. Assume that $N_1 \geq \ldots \geq N_r$ and $t_1 \leq \ldots \leq t_r$. Then*

$$\langle Y_1 P_{t_2-t_1} Y_2 \cdots P_{t_r-t_1} Y_r \rangle_{t_1} = \mathbb{E}\left[\bar{p}_{Y_1}(X^{(N_1)}(t_1)) \cdots \bar{p}_{Y_r}(X^{(N_r)}(t_r))\right].$$

Proof. In order to simplify notation and elucidate the idea of the proof, assume $r = 2$. The more general case follows from exactly the same argument.

First prove it for $t_1 = t_2$. Assume $N_1 = N \geq M = N_2$. Let $m(\lambda, \mu)$ denote the multiplicity of μ in the restricted representation $V_\lambda\Big|_{U(M)}$. Use $\bar{m}(\cdot, \cdot)$ to denote the same quantity in the shifted co–ordinates $x_i = \lambda_i - i + 1$. Then by the Gibbs property, that is (5.7),

$$\begin{aligned} \text{RHS} &= \sum_{x^{(N)}, x^{(M)}} \text{Prob}(X^{(N)}(t) = x_i^{(N)}, X^{(M)} = x_j^{(M)}) \bar{p}_{Y_1}(x^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \\ &= \sum_{x^{(N)}, x^{(M)}} \text{Prob}(X^{(N)}(t) = x_i^{(N)}) \frac{\bar{m}(x^{(N)}, x^{(M)}) h(x^{(M)})}{h(x^{(N)})} \bar{p}_{Y_1}(x^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \end{aligned}$$

At the same time,

$$\langle Y_1 Y_2 \rangle_t = \sum_{\lambda^{(N)}} \text{Prob}(X^{(N)}(t) = \lambda_i^{(N)} - i + 1) \frac{1}{\dim \lambda^{(N)}} \text{Tr}(Y_1 Y_2\Big|_{V_{\lambda^{(N)}}}).$$

Since Y_1 is central, it acts as $p_{Y_1}(\lambda^{(N)})\text{Id}$ on $V_{\lambda^{(N)}}$, so this equals

$$\sum_{\lambda^{(N)}} \text{Prob}(X^{(N)}(t) = \lambda_i^{(N)} - i + 1) \frac{p_{Y_1}(\lambda^{(N)})}{\dim \lambda^{(N)}} \text{Tr}(Y_2\Big|_{V_{\lambda^{(N)}}}).$$

By restricting $V_{\lambda^{(N)}}$ to $U(M)$ and using that Y_2 acts as $p_{Y_2}(\lambda^{(M)})\text{Id}$ on $V_{\lambda^{(M)}}$, we get

$$\sum_{\lambda^{(N)}, \lambda^{(M)}} \text{Prob}(X^{(N)}(t) = \lambda_i^{(N)} - i + 1) \frac{m(\lambda^{(N)}, \lambda^{(M)}) \dim \lambda^{(M)}}{\dim \lambda^{(N)}} p_{Y_1}(\lambda^{(N)}) p_{Y_2}(\lambda^{(M)}).$$

This is equal to the right–hand–side from above.

Now consider when $t = t_1 \leq t_2 = s$. Write $P_{s-t}Y_2$ as a sum over basis elements, that is $P_{s-t}Y_2 = \sum_\rho c_\rho Y_\rho$. Then

$$\begin{aligned}
\langle Y_1 P_{s-t} Y_2 \rangle_t &= \sum_\rho c_\rho \langle Y_1 Y_\rho \rangle_t \\
&= \sum_\rho c_\rho \mathbb{E}\left[\bar{p}_{Y_1}(X^{(N)}(t))\bar{p}_{Y_\rho}(X^{(M)}(t))\right] \\
&= \mathbb{E}\left[\bar{p}_{Y_1}(X^{(N)}(t))(P_{s-t}\bar{p}_{Y_2})(X^{(M)}(t))\right]
\end{aligned}$$

Thus, it suffices to prove that

$$\mathbb{E}\left[\bar{p}_{Y_1}(X^{(N)}(t))\bar{p}_{Y_2}(X^{(M)}(s))\right] = \mathbb{E}\left[\bar{p}_{Y_1}(X^{(N)}(t))(P_{s-t}\bar{p}_{Y_2})(X^{(M)}(t))\right]$$

We have

$$\begin{aligned}
&\mathbb{E}\left[\bar{p}_{Y_1}(X^{(N)}(t))\bar{p}_{Y_2}(X^{(M)}(s))\right] \\
&= \sum_{y^{(N)},x^{(M)}} \bar{p}_{Y_1}(y^{(N)})\bar{p}_{Y_2}(x^{(M)})\mathbb{P}(X^{(M)}(s) = x^{(M)}, X^{(N)}(t) = y^{(N)}) \\
&= \sum_{y^{(N)},y^{(M)},x^{(M)}} \bar{p}_{Y_1}(y^{(N)})\bar{p}_{Y_2}(x^{(M)})\mathbb{P}(X^{(M)}(s) = x^{(M)}, X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)}) \\
&= \sum_{y^{(N)},y^{(M)},x^{(M)}} \bar{p}_{Y_1}(y^{(N)})\bar{p}_{Y_2}(x^{(M)})\mathbb{P}(X^{(M)}(s) = x^{(M)} | X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)}) \\
&\qquad \times \mathbb{P}(X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)})
\end{aligned}$$

By (5.8) and the fact that $P_t = Q_t$, this then equals

$$\begin{aligned}
&= \sum_{y^{(N)},y^{(M)},x^{(M)}} \bar{p}_{Y_1}(y^{(N)})\bar{p}_{Y_2}(x^{(M)})\mathbb{P}(X^{(M)}(s) = x^{(M)} | X^{(M)}(t) = y^{(M)}) \\
&\qquad \times \mathbb{P}(X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)})) \\
&= \sum_{y^{(N)},y^{(M)}} \bar{p}_{Y_1}(y^{(N)})(P_{s-t}\bar{p}_{Y_2})(y^{(M)})\mathbb{P}(X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)})) \\
&= \mathbb{E}\left[\bar{p}_{Y_1}(X^{(N)}(t))(P_{s-t}\bar{p}_{Y_2})(X^{(M)}(t))\right]
\end{aligned}$$

□

We wrap up this section by giving an example showing that although $P_t = Q_t$ on $Z(U(\mathfrak{gl}_N))$, they are not equal on subalgebras generated by different $Z(U(\mathfrak{gl}_N))$. The determinantal formula from [7] yields

$$Q_1(\Psi_1^{(2)}\Psi_1^{(1)})(\lambda^{(2)},\lambda^{(1)}) \approx 2.37\ldots, \text{ when } \lambda^{(2)} = (1,0), \lambda^{(1)} = (0).$$

However,

$$P_t(\Psi_1^{(2)}\Psi_1^{(1)}) = \Psi_1^{(2)}\Psi_1^{(1)} + 2t\Psi_1^{(1)} + t\Psi_1^{(2)} + 2t^2 + t,$$

and when evaluated at $\lambda^{(2)} = (1,0), \lambda^{(1)} = (0), t = 1$ yields 3.

5.5. **Covariance Structure.** In this section, it will be shown that the central elements are asymptotically Gaussian with an explicit covariance that generalizes the Gaussian free field. Let us review some previously known results.

Theorem 5.7. [6, ?BF] *Suppose $N_j = \lfloor \eta_j L \rfloor, t_j = \tau_j L$ for $1 \le j \le r$. Assume they lie on a space–like path, that is $N_1 \ge \ldots \ge N_r$ and $t_1 \le \ldots \le t_r$. Then as $L \to \infty$,*

$$\left(\frac{\Psi_{k_1}^{(N_1)} - \left\langle \Psi_{k_1}^{(N_1)} \right\rangle_{t_1}}{L^{k_1}}, \ldots, \frac{P_{t_r - t_1}\Psi_{k_r}^{(N_r)} - \left\langle P_{t_r - t_1}\Psi_{k_r}^{(N_r)} \right\rangle_{t_1}}{L^{k_r}} \right) \to (\xi_1, \ldots, \xi_r),$$

where the convergence is with respect to the state $\langle \cdot \rangle_{t_1}$, and $(\xi_1, \ldots, \xi_r)$ is a Gaussian vector with covariance

$$\mathbb{E}[\xi_i\xi_j] = \left(\frac{1}{2\pi i}\right)^2 \int\int_{|z|>|w|} (\eta_i z^{-1} + \tau_i + \tau_i z)^{k_i} (\eta_j w^{-1} + \tau_j + \tau_j w)^{k_j} (z-w)^{-2} dz dw.$$

The proof uses that the particle system is a determinantal point process along space–like paths. This condition is necessary due to the construction using (5.6). In particular, there are no maps going up from S to S^*. A natural question is to ask what happens along time–like paths, that is, $N_1 \le N_2, t_1 \le t_2$. The main theorem is

Theorem 5.8. *Suppose $N_j = \lfloor \eta_j L \rfloor, t_j = \tau_j L$ for $1 \leq j \leq r$. Assume $\min(\gamma_1, \ldots, \gamma_r) = \gamma_1$. Then as $L \to \infty$*

$$\left(\frac{\Psi_{k_1}^{(N_1)} - \left\langle \Psi_{k_1}^{(N_1)} \right\rangle_{t_1}}{L^{k_1}}, \ldots, \frac{P_{t_r - t_1} \Psi_{k_r}^{(N_r)} - \left\langle P_{t_r - t_1} \Psi_{k_1}^{(N_1)} \right\rangle_{t_1}}{L^{k_r}} \right) \to (\xi_1, \ldots, \xi_r),$$

where the convergence is with respect to the state $\langle \cdot \rangle_{t_1}$, and $(\xi_1, \ldots, \xi_r)$ is a Gaussian vector with covariance

$$\mathbb{E}[\xi_i \xi_j] = \begin{cases} \left(\frac{1}{2\pi i} \right)^2 \int\int_{|z|>|w|} (\eta_i z^{-1} + \tau_i + \tau_i z)^{k_i} (\eta_j w^{-1} + \tau_j + \tau_j w)^{k_j} (z - w)^{-2} dz dw, \\ \qquad \eta_i \geq \eta_j, \gamma_i \leq \gamma_j \\ \left(\frac{1}{2\pi i} \right)^2 \int\int_{|z|>|w|} (\eta_j \frac{\tau_j}{\tau_i} z^{-1} + \tau_j + \tau_i z)^{k_j} (\eta_i w^{-1} + \tau_i + \tau_i w)^{k_i} (z - w)^{-2} dz dw, \\ \qquad \eta_i < \eta_j, \gamma_i \leq \gamma_j \end{cases}$$

Example 8 The double integral can be computed using resides and the Taylor series

$$(z - w)^{-2} = z^{-2} \left(1 + 2\frac{w}{z} + 3\frac{w^2}{z^2} + \ldots \right).$$

So for instance,

$$\left\langle \left(\frac{\Psi_1^{(\eta_1 L)} - \left\langle \Psi_1^{(\eta_1 L)} \right\rangle_{\gamma_1 L}}{L^1} \cdot \frac{P_{(\gamma_2 - \gamma_1)L} \Psi_1^{(\eta_2 L)} - \left\langle P_{(\gamma_2 - \gamma_1)L} \Psi_1^{(\eta_2 L)} \right\rangle_{\gamma_1 L}}{L^1} \right) \right\rangle_{\gamma_1 L}$$

$$\to \gamma_1 \min(\eta_1, \eta_2).$$

This can be checked using Proposition 5.2. Assume without loss of generality that $\eta := \eta_1 \leq \eta_2$ and set $\gamma = \gamma_1$. Since $\langle E_{ii} E_{jj} \rangle = \langle E_{ii} \rangle \langle E_{jj} \rangle$ for $i \neq j$, then

$$
\begin{aligned}
&\lim_{L\to\infty} L^{-2}\left\langle \left(\sum_{i=1}^{\lfloor \eta_1 L\rfloor}(E_{ii}-\gamma_1 L)\right)\left(\sum_{j=1}^{\lfloor \eta_2 L\rfloor}(E_{jj}-\gamma_1 L)\right)\right\rangle_{\gamma_1 L} \\
&= \lim_{L\to\infty} L^{-2}\left\langle \left(\sum_{i=1}^{\lfloor \eta_1 L\rfloor}E_{ii}-\eta_1\gamma_1 L^2\right)\left(\sum_{j=1}^{\lfloor \eta_1 L\rfloor}E_{jj}-\eta_1\gamma_1 L^2\right)\right\rangle_{\gamma_1 L} \\
&= \lim_{L\to\infty}\frac{(\eta L(\tau^2L^2+\tau L)+\eta L(\eta L-1)(\tau L)^2-2\eta\tau\cdot\eta\tau L^4+\eta^2\tau^2L^4)}{L^2} \\
&= \eta\tau
\end{aligned}
$$

The remainder of this section will prove Theorem 5.8. By Theorem 5.1 of [6] and the fact that P_t preserves the centre, it is immediate that convergence to a Gaussian vector holds. It only remains to compute the covariance.

From the presence of Ψ_1^2 in $P_t\Psi_4$, it is necessary to understand products of Ψ_k. Heuristically, if $\Psi_1 \approx cL^2+\xi L$, where ξ is a Gaussian random variable, then $\Psi_1^2 \approx c^2L^4+2c\xi L^3$. Here are two examples which demonstrate this:

Example 9 Since

$$
\lim_{L\to\infty} L^{-2}\left\langle \Psi_1^{(\eta L)}\right\rangle_{\tau L} = (\tau\eta-\frac{1}{2}\eta^2),
$$

the heuristics would predict that

$$
\begin{aligned}
&\lim_{L\to\infty}\left\langle \frac{\left(\Psi_1^{(\eta L)}-\left\langle\Psi_1^{(\eta L)}\right\rangle_{\tau L}\right)\left(\left[\Psi_1^{(\eta L)}\right]^2-\left\langle\left[\Psi_1^{(\eta L)}\right]^2\right\rangle_{\tau L}\right)}{L^4}\right\rangle_{\tau L} \\
&= \left(2\tau\eta-\eta^2\right)\lim_{L\to\infty}\left\langle \frac{\left(\Psi_1^{(\eta L)}-\left\langle\Psi_1^{(\eta L)}\right\rangle_{\tau L}\right)\left(\Psi_1^{(\eta L)}-\left\langle\Psi_1^{(\eta L)}\right\rangle_{\tau L}\right)}{L^2}\right\rangle_{\tau L} \\
&= \left(2\tau\eta-\eta^2\right)\eta\tau
\end{aligned}
$$

And indeed, an explicit calcuation yields

$$\lim_{L\to\infty} L^{-4}\left\langle \left(\sum_{i=1}^{\eta L} E_{ii} - \eta\tau L^2\right)\sum_{j,k=1}^{\eta L} E_{jj}E_{kk} + \left(\sum_{i=1}^{\eta L} E_{ii} - \eta\tau L^2\right)(-\eta^2 L^2)\sum_{j=1}^{\eta L} E_{jj}\right\rangle_{\tau L}$$

$$= \lim_{L\to\infty} L^{-4}\Big((\eta\tau L^2 + 3L^4\eta^2\tau^2 + \eta^3\tau^3 L^6) - \eta\tau L^2(\eta^2\tau^2 L^4 + \eta\tau L^2)$$

$$- (\eta^2 L^2)(\eta^2\tau^2 L^4 + \eta\tau L^2 - \eta\tau L^2 \cdot \eta\tau L^2\Big)$$

$$= (2\tau\eta - \eta^2)\eta\tau.$$

Example 10 Consider Theorem 5.7 with $r = 2, k_1 = 3, k_2 = 4$. Using the formula for $P_t\Psi_4$ from Example 7 and replacing Ψ_1^2 with $(2\tau_1\eta_2 - \eta_2^2)\Psi_1$ yields

$$12\eta_2\tau_1^2\tau_2(\eta_2^2\tau_1 + \tau_1(3\eta_2\tau_2 + 2\eta_2^2) + \tau_2(\tau_2 + 3\eta_2)(\tau_1 + \eta_1))$$

$$= 12\eta_2\tau_1^3(\eta_2^2\tau_1 + \tau_1(3\eta_2\tau_1 + 2\eta_2^2) + \tau_1(\tau_1 + 3\eta_2)(\tau_1 + \eta_1))$$

$$+ 4(\tau_2 - \tau_1)\cdot 3\eta_2\tau_1^2(\eta_2^2\tau_1 + 6\eta_2\tau_1^2 + 3\tau_1(\eta_2 + \tau_1)(\eta_1 + \tau_1))$$

$$+ (6(\tau_2 - \tau_1)^2 + 4(\tau_2 - \tau_1)\eta_2)\cdot 6\eta_2\tau_1^2(\tau_1(\tau_1 + \eta_1) + \eta_2\tau_1)$$

$$+ (4(\tau_2 - \tau_1)^3 + 12(\tau_2 - \tau_1)^2\eta_2 + 2(\tau_2 - \tau_1)\eta_2^2)\cdot 3\eta_2\tau_1^2(\tau_1 + \eta_1)$$

$$+ 2(\tau_2 - \tau_1)\cdot(2\tau_1\eta_2 - \eta_2^2)\cdot 3\eta_2\tau_1^2(\tau_1 + \eta_1),$$

which can be checked computationally.

Given a partition $\rho = (\rho_1, \ldots, \rho_l)$, let its *weight* $\mathrm{wt}(\rho)$ denote $|\rho| + l(\rho) = \rho_1 + \ldots + \rho_l + l$, and let $\Psi_\rho = \prod_{i=1}^{l}\Psi_{\rho_i}$. In the asymptotic limit, we should be able to replace Ψ_ρ with a linear combination of Ψ_{ρ_i}. In the examples above, Ψ_1^2 was replaced with $(2\tau\eta - \eta^2)\Psi_1$.

Proposition 5.9. *Let $\eta, \tau > 0$ be fixed. (1) Set $N = \lfloor \eta L \rfloor$ and $t = \tau L$. Then $\langle \Psi_{\rho,N}\rangle_t = \Theta(L^{\mathrm{wt}(\rho)})$.*

(2) There exist constants $c'_{k,\rho}(\tau,\eta)$ such that

$$P_{\tau L}\Psi_{k,N} = \sum_\rho (c'_{k,\rho}(\tau,\eta) + o(1))L^{k+1-\mathrm{wt}(\rho)}\Psi_\rho.$$

where the sum is over ρ with weight $\mathrm{wt}(\rho) \le k+1$.

(3) For any $\tau_1 > \tau_0$, there exist constants $c_{kj}(\tau_1, \tau_0, \eta)$ such that

$$\lim_{L\to\infty} \left\langle \frac{\Psi_m - \langle \Psi_m \rangle_{\tau_0 L}}{L^m} \cdot \frac{P_{(\tau_1-\tau_0)L}\Psi_k - \left\langle P_{(\tau_1-\tau_0)L}\Psi_k \right\rangle_{(\tau_1-\tau_0)L}}{L^k} \right\rangle_{\tau_0 L}$$
$$= \lim_{L\to\infty} \sum_{j=1}^{k} c_{kj}(\tau_1,\tau_0,\eta) \left\langle \frac{\Psi_m - \langle \Psi_m \rangle_{\tau_0 L}}{L^m} \cdot \frac{\Psi_j - \langle \Psi_j \rangle_{\tau_0 L}}{L^j} \right\rangle_{\tau_0 L}$$

Proof. (1) This can be proved from [6], but this will be an alternative proof.

By definition,

$$\Psi_\rho = \sum_{m_1=1}^{\rho_1} \cdots \sum_{m_l=1}^{\rho_l} \sum_{\pi_1 \in \Pi_{\rho_1}^{(m_1)}} \cdots \sum_{\pi_l \in \Pi_{\rho_l}^{(m_l)}} E(\pi_1)\cdots E(\pi_l).$$

Consider the sum over l–tuples $(\pi_1, \ldots, \pi_l)$ such that the paths $\pi_1, \ldots, \pi_l$ cross over a total of exactly ν distinct vertices. There are $\binom{N}{\nu} = \Theta(L^\nu)$ such l–tuples, so it remains to estimate $\langle E(\pi_1)\cdots E(\pi_l)\rangle_{\tau L}$. Let π be the union of the paths $\pi_1, \ldots, \pi_l$. Decompose π into the union of s simple cycles. By Proposition 5.2, $\langle E(\pi_1)\cdots E(\pi_l)\rangle_{\tau L} = \mathcal{O}L^s$. Decomposing π_j into s_j simple cycles, it is clear that $s = s_1 + \ldots + s_l$. If π_j covers exactly ν_j vertices, then elementary graph theory gives $s_j = \rho_j - \nu_j + 1$. Since $\nu_1 + \ldots + \nu_l \geq \nu$, thus

$$\langle \Psi_{\rho,N} \rangle_t = \mathcal{O}L^\nu L^{s_1+\ldots+s_l} = \mathcal{O}L^{\rho_1+\ldots+\rho_l+l}.$$

To get a lower bound, just observe that the constant term in $\Psi_{\rho,N}$ is $\Theta(L^{\mathrm{wt}(\rho)})$.

(2) By Theorem 5.1(5), $P_{\tau L}\Psi_k$ can expressed as a linear combination of Ψ_ρ. Taking $\langle\cdot\rangle_L$ and using that $\langle P_{\tau L}X\rangle_L = \langle X\rangle_{(1+\tau)L}$, it follows from (1) that only $\mathrm{wt}(\rho) \leq k+1$ terms have nonzero coefficieints.

(3) First apply (2) to the left–hand–side. Then, by (1),

$$\Psi_{\rho_1}\ldots\Psi_{\rho_l} - \langle \Psi_{\rho_1}\ldots\Psi_{\rho_l}\rangle_{\tau_0 L}$$
$$= \sum_{j=1}^{l} \langle \Psi_{\rho_1}\rangle_{\tau_0 L} \cdots \widehat{\left\langle \Psi_{\rho_j} \right\rangle_{\tau_0 L}} \cdots \langle \Psi_{\rho_l}\rangle_{\tau_0 L} \left(\Psi_{\rho_j} - \left\langle \Psi_{\rho_j} \right\rangle_{\tau_0 L} \right) + \text{smaller order terms}$$

□

Given a Laurent polynomial $p(w)$, let $p(w)[w^r]$ denote the cofficient of w^r in $p(w)$. Using the expansion $(z-w)^{-2} = z^{-2}(1+2(w/z)+3(w/z)^2+\ldots)$ in Theorem 5.7 and taking residues, one obtains

$$\sum_{l=1}^{k} c_{kl}(\tau_2,\tau_1,\eta_2)(\eta_2 w^{-1}+\tau_1+\tau_1 w)^l[w^r] = (\eta_2 w^{-1}+\tau_2+\tau_2 w)^k[w^r], r \leq -1.$$

For example, for $k=3$ and $r=-1$, and using the expansion of $P_t\Psi_3$, this says

$$1\cdot(3\eta_2^2\tau_1+3\eta_2\tau_1^2)+3(\tau_2-\tau_1)\cdot 2\eta_2\tau_1+3((\tau_2-\tau_1)^2+(\tau_2-\tau_1)\eta_2)\cdot\eta_2 = 3\eta_2^2\tau_2+3\eta_2\tau_2^2. \tag{5.9}$$

We need a formula for $r \geq 1$. Theorem 5.8 follows from the proposition below.

Proposition 5.10. *For $r \geq 1$,*

$$\sum_{l=1}^{k} c_{kl}(\tau_2,\tau_1,\eta_1)(\eta_1 z^{-1}+\tau_1+\tau_1 z)^l[z^r] = (\eta_1\frac{\tau_2}{\tau_1}z^{-1}+\tau_2+\tau_1 z)^k[z^r].$$

Proof. We start with an illustrative example. For $k=3$ and $r=1$ we would want to show

$$1\cdot(3\eta_1\tau_1^2+3\tau_1^3)+3(\tau_2-\tau_1)\cdot 2\tau_1^2+3((\tau_2-\tau_1)^2+(\tau_2-\tau_1)\eta_1)\cdot\tau_1 = 3\eta_1\tau_1\tau_2+3\tau_1\tau_2^2. \tag{5.10}$$

This can be checked directly, but in general the coefficients c_{kl} are difficult to work with. Instead, we would like to show that it follows directly from the covariance formula along space–like paths. Indeed, this can be done just by multiplying (5.9) by $(\tau_1/\eta_1)^r$. (And recall that $\eta_2 < \eta_1$ in (5.9) while $\eta_1 < \eta_2$ in (5.10)).

Let $S_l^{(r)} = \{(\epsilon_1,\ldots,\epsilon_l) \in \{-1,0,+1\}^l : \epsilon_1+\ldots+\epsilon_l = r\}$ and define

$$\chi(j) = \begin{cases} \eta_1, j=-1 \\ \tau_1, j=0 \\ \tau_1, j=1 \end{cases} \qquad \chi'_{\mathrm{ti}}(j) = \begin{cases} \eta_1\frac{\tau_2}{\tau_1}, j=-1 \\ \tau_2, j=0 \\ \tau_1, j=1 \end{cases} \qquad \chi'_{\mathrm{sp}}(j) = \begin{cases} \eta_1, j=-1 \\ \tau_2, j=0 \\ \tau_2, j=1 \end{cases}$$

With this notation, what we want to show is that

$$\sum_{l=1}^{k} c_{kl}(\tau_2,\tau_1,\eta_1) \sum_{\vec{\epsilon}\in S_l^{(r)}} \prod_{j=1}^{l} \chi(\epsilon_j) = \sum_{\vec{\epsilon'}\in S_k^{(r)}} \prod_{j=1}^{k} \chi'_{\mathrm{ti}}(\epsilon'_j), r \geq 1. \tag{5.11}$$

From (5.10),

$$\sum_{l=1}^{k} c_{kl}(\tau_2,\tau_1,\eta_1) \sum_{\vec{\epsilon}\in S_l^{(-r)}} \prod_{j=1}^{l} \chi(\epsilon_j) = \sum_{\vec{\epsilon'}\in S_k^{(-r)}} \prod_{j=1}^{k} \chi'_{\mathrm{sp}}(\epsilon'_j), r \geq 1.$$

By sending $\epsilon_j \mapsto -\epsilon_j$, this is equivalent to

$$\sum_{l=1}^{k} c_{kl}(\tau_2,\tau_1,\eta_1) \sum_{\vec{\epsilon}\in S_l^{(r)}} \prod_{j=1}^{l} \chi(-\epsilon_j) = \sum_{\vec{\epsilon'}\in S_k^{(r)}} \prod_{j=1}^{k} \chi'_{\mathrm{sp}}(-\epsilon'_j), r \geq 1.$$

And since for all r,

$$\sum_{l=1}^{k} c_{kl}(\tau_2,\tau_1,\eta_1) \sum_{\vec{\epsilon}\in S_l^{(r)}} \prod_{j=1}^{l} \chi(-\epsilon_j) = \left(\frac{\eta_1}{\tau_1}\right)^r \sum_{l=1}^{k} c_{kl}(\tau_2,\tau_1,\eta_1) \sum_{\vec{\epsilon}\in S_l^{(r)}} \prod_{j=1}^{l} \chi(\epsilon_j)$$

it thus follows that the left–hand–side of (5.11) equals

$$\left(\frac{\tau_1}{\eta_1}\right)^r \sum_{\vec{\epsilon'}\in S_k^{(r)}} \prod_{j=1}^{k} \chi'_{\mathrm{sp}}(-\epsilon'_j), r \geq 1.$$

So it suffices to show that

$$\left(\frac{\tau_1}{\eta_1}\right)^r \prod_{j=1}^{k} \chi'_{\mathrm{sp}}(-\epsilon'_j) = \prod_{j=1}^{k} \chi'_{\mathrm{ti}}(\epsilon'_j), \quad \text{for all } \vec{\epsilon'} \in S_k^{(r)}, r \geq 1.$$

Since $r = |\{\epsilon_j = 1\}| - |\{\epsilon_j = -1\}|$, it follows that the left–hand–side is

$$\left(\frac{\tau_1}{\eta_1}\right)^r \cdot \eta_1^{|\epsilon_j=1|} \tau_2^{|\epsilon_j=0|} \tau_2^{|\epsilon_j=-1|} = \tau_1^r \eta_1^{|\epsilon_j=-1|} \tau_2^{|\epsilon_j=0|} \tau_2^{|\epsilon_j=-1|}.$$

And similarly, the right–hand–side is

$$\tau_1^{|\epsilon_j=1|} \tau_2^{|\epsilon_j=0|} \left(\eta_1 \frac{\tau_2}{\tau_1}\right)^{|\epsilon_j=-1|} = \tau_1^r \eta_1^{|\epsilon_j=-1|} \tau_2^{|\epsilon_j=0|} \tau_2^{|\epsilon_j=-1|}.$$

□

The formula in Theorem 5.8 appears to be different from the formula in [5]. In particular, the covariance along space–like paths is different from the covariance along time–like paths. However, after rescaling from Brownian Motion to Ornstein–Uhlenbeck, i.e. replacing γ_i, γ_j with $e^{2\gamma_i}, e^{2\gamma_j}$ and multiplying by $e^{-\gamma_j k_j} e^{-\gamma_i k_i}$, the formula becomes

$$\mathbb{E}[\xi_i \xi_j] = \begin{cases} -\frac{1}{\pi}\frac{e^{\gamma_j}}{e^{\gamma_i}} \int\int_{|z|>|w|} (\eta_i z^{-1} + e^{\tau_i} + z)^{k_i} (\eta_j w^{-1} + e^{\tau_j} + w)^{k_j} (\frac{e^{\gamma_j}}{e^{\gamma_i}} z - w)^{-2} dz dw, \\ \qquad \eta_i \geq \eta_j, \gamma_i \leq \gamma_j \\ -\frac{1}{\pi}\frac{e^{\gamma_j}}{e^{\gamma_i}} \int\int_{|z|>|w|} (\eta_j z^{-1} + e^{\tau_j} + z)^{k_j} (\eta_i w^{-1} + e^{\tau_i} + w)^{k_i} (\frac{e^{\gamma_j}}{e^{\gamma_i}} z - w)^{-2} dz dw, \\ \qquad \eta_i < \eta_j, \gamma_i \leq \gamma_j \end{cases}$$

In both expressions, the z–contour is larger and corresponds to the higher level (η_i in the first case and η_j in the second). Hence, by switching the subscripts i and j in η, the formula is the same in both cases. It also matches the formula in [5] with the expression $e^{\gamma_j - \gamma_i}$ playing the role of $c(t_p, t_q)$.

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6. Strong Szegő asymptotics of the Riemann ζ function

Abstract. Assuming the Riemann hypothesis, we prove the weak convergence of linear statistics of the zeros of L-functions to a Gaussian field, with covariance structure corresponding to the $\mathrm{H}^{1/2}$-norm of the test functions. For this purpose, we obtain an approximate form of the explicit formula, relying on Selberg's smoothed expression for ζ'/ζ and the Helffer-Sjöstrand functional calculus. Our main result is an analogue of the strong Szegő theorem, known for Toeplitz operators and random matrix theory.

6.1. **Introduction.** A connection between ζ zeros and random matrix theory was discovered by Montgomery [25], who examined the pair correlation of the zeta zeros. Dyson was the first to notice that this pair correlation agrees with the pair correlation of the eigenvalues of stochastic Hermitian matrices with properly distributed Gaussian entries. Assuming the Riemann hypothesis (we denote by $1/2 \pm \mathrm{i}\gamma_j$, $\gamma_j \in \mathbb{R}$, $0 \leq \gamma_1 \leq \gamma_2 \leq \dots$, the set of non-trivial zeros), Montgomery proved that

$$\frac{1}{x} \sum_{1 \leq j,k \leq x, j \neq k} f(\tilde{\gamma}_j - \tilde{\gamma}_k) \underset{x \to \infty}{\longrightarrow} \int_{-\infty}^{\infty} f(y) \left(1 - \left(\frac{\sin \pi y}{\pi y}\right)^2\right) \mathrm{d}y,$$

where the $\tilde{\gamma}$'s are the rescaled zeta zeros ($\tilde{\gamma} = \frac{\gamma}{2\pi} \log \gamma$: at height t, the average gap between zeros is $2\pi/\log t$), and the test function f has a smooth Fourier transform supported in $(-1,1)$. A fundamental conjecture in analytic number theory concerns removing this support condition. This would imply, for example, new estimates on large gaps between primes, but it seems out of reach with available techniques. In particular, this requires a better understanding of some asymptotic correlations between primes, such as the Hardy-Littlewood conjectures, as shown in [3]. Further examples of this connection appear in [29] for the correlation functions of order greater than 2, in [23] for the function-field L-functions, and in [24] for the conjectured asymptotics of the moments of ζ along the critical axis.

By looking at linear statistics, Hughes and Rudnick [15] demonstrate another way to exhibit the repulsion between the ζ zeros at the microscopic scale. They showed that if the function f has a smooth Fourier transform supported on $(-2/m, 2/m)$, then the first m

moments of the linear statistics (here and in the following ω is a uniform random variable on $(1, 2)$)

$$\sum_{\gamma} f(\tilde{\gamma} - \omega t) \tag{6.1}$$

converge[5] to those of a Gaussian random variable as $t \to \infty$. We propose to look at linear statistics at a larger (mesoscopic) scale.

Contrary to the Dyson-Montgomery analogy, observed at the microscopic level of nearest zeros spacings, the mesoscopic regime involves a larger window and yields Gaussian fluctuations. Indeed, Selberg proved, unconditionally, the following central limit theorem [30–32]: if ω is uniform on (1,2), as $t \to \infty$,

$$\frac{\log \zeta\left(\frac{1}{2} + \mathrm{i}\omega t\right)}{\sqrt{\log \log t}} \to \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2,$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent standard normal random variables. This is related to the fluctuations between the number of zeros with imaginary part in $[0, t]$ and their expected number. The very small normalization in this convergence in law indicates the repulsion of the zeros. This central limit theorem was extended by Fujii to the fluctuations when counting zeros in smaller (but still mesoscopic) intervals [12]. Central limit theorems concerning counting the number of eigenvalues of random matrices appeared originally in [6] for Gaussian ensembles and [24, 38] for the circular unitary ensemble.

In this paper, we extend (conditioned on the Riemann hypothesis) these results on Gaussian fluctuations of zeros of L-functions to smoother statistics than indicator functions of intervals. This includes an analogue of the strong Szegő theorem, seen originally as the second-order asymptotics of Toeplitz determinants as the dimension increases. It is also related, by Heine's formula, to linear statistics of eigenangles of Haar-distributed unitary matrices, $S_n(f) = \sum_1^n f(\theta_n)$. Indeed, for f with mean 0 on $[0, 2\pi]$ satisfying $f(0) = f(2\pi)$,

[5]By analogy with what is known in random matrix theory [16], the higher moments supposedly do not converge towards those of this Gaussian random variable (see also [14] for a similar rigorous fact about non-Gaussianness in the context of low–lying zeros of L–functions).

and $\lambda \in \mathbb{R}$, then the strong Szegő theorem states that

$$\mathbb{E}\left(e^{\lambda S_n(f)}\right) \underset{n\to\infty}{\longrightarrow} \exp\left(\frac{1}{2}\lambda^2 \sum_{k=-\infty}^{\infty} |k| \cdot |\hat{f}_k|^2\right),$$

where the $\hat{f}_k$'s are the Fourier coefficients of f ($\hat{f}_j = \frac{1}{2\pi}\int f(\theta) e^{-\mathrm{i}j\theta}\mathrm{d}\theta$). In probabilistic terms, the convergence of the above Laplace transform means that the linear statistics of the eigenvalues converge with no normalization to a normal random variable with variance $\sum_{\mathbb{Z}} |k||\hat{f}_k|^2$; the only restriction is that this limiting variance is finite. This was extended by Johansson in the context of Coulomb gases on the unit circle [17] and on the real line [18]. Other proofs of the strong Szegő theorem were given, relying for example on combinatorics [22], on representation theory [9, 10], on the steepest descent method for Riemann-Hilbert problems [8], on the Borodin-Okounkov formula [4] (see [33] for many on these distinct proofs).

These linear statistics asymptotics were extended by Diaconis and Evans [9], to the general setting of more irregular test functions. Under the hypothesis $f \in L^2(\mathbb{T})$, denoting $\sigma_n^2 = \sum_{j=-n}^{n} |j||\hat{f}_j|^2$, they proved that if $(\sigma_n)_{n\geq 1}$ is slowly varying then, as $n \to \infty$, $\frac{S_n - \mathbb{E}(S_n)}{\sigma_n}$ converges in distribution to a standard normal random variable. This wide class of possible test functions includes the smooth and indicator cases. For many determinantal point processes, a similar central limit theorem was obtained by Soshnikov under weak assumptions on the regularity of f [35]. Moreover, for smoother test functions f, he proved a local version of the strong Szegő theorem [34]: the linear statistics are of type (e.g. for the unitary group) $\sum_{k=1}^{n} f(\lambda_n \theta_k)$, for a parameter λ_n satisfying[6] $1 \leq \lambda_n \ll n$. The last inequality means that we keep in the mesoscopic regime.

Our purpose consists of an analogue of the above results for linear functionals of zeros of the zeta function. This concerns linear statistics of type

$$\sum_{\gamma} f(\lambda_t(\gamma - \omega t)), \tag{6.2}$$

[6] In the following $a \ll b$ means $a = \mathrm{o}(b)$.

where ω is uniform on $(1,2)$, as in (6.1), but now the condition $\lambda_t \ll \log t$ gives the mesoscopic regime: the number of zeros visited by f goes to infinity.

In the following statements, ω is uniform on $(1,2)$, we denote by $\{1/2+\mathrm{i}\gamma\}$ ($\gamma\in\mathbb{R}$, we assume the Riemann hypothesis) the multiset of non-trivial zeros of ζ, counted with repetition. We define $\gamma_t = \lambda_t(\gamma-\omega t)$ and $\sigma_t(f)^2 = \int_{-\lambda_t}^{\lambda_t}|u||\hat{f}(u)|^2\mathrm{d}u$, where $\hat{f}(u) = \frac{1}{\pi}\int f(x)e^{-\mathrm{i}ux}\mathrm{d}x$. Moreover, the centered, normalized linear statistics are denoted

$$S_t(f) = \frac{1}{\sigma_t(f)}\left(\sum_\gamma f(\gamma_t) - \frac{\log t}{2\pi\lambda_t}\int f(u)\mathrm{d}u\right).$$

Our first result states that, for functions with sufficient regularity, the linear statistics converge to a Gaussian field with covariance function given by (f,g are real functions, for notational simplicity)

$$\langle f,g\rangle_{\mathrm{H}^{1/2}} = \Re\int_{\mathbb{R}}|u|\hat{f}(u)\overline{\hat{g}(u)}\mathrm{d}u = -\frac{2}{\pi^2}\int f'(x)g'(y)\log|x-y|\mathrm{d}x\mathrm{d}y,$$

where we refer to [11] equation (18) for the last equality. Our technical assumptions on f are the following: sufficient decay of f at $\pm\infty$, bounded variation of f, and sufficient decay of $\hat{f}$ at $\pm\infty$. More specifically,

(6.3) for some $\delta>0$ and $|x|$ large enough, $f(x), f'(x), f''(x)$ exist and are $\mathrm{O}\left(x^{-2-\delta}\right)$,

(6.4) $g(x) := f'(x)$, $\int(1+|u\log u|)|\mathrm{d}g(u)|<\infty$,

(6.5) $\xi|\hat{f}(\xi)|^2$, $(\xi|\hat{f}(\xi)|^2)' = \mathrm{O}(\xi^{-1})$.

Our assumptions on f easily include the cases of compactly supported $\mathscr{C}^2$ functions, for example. We will also assume that $\|f\|_{\mathrm{H}^{1/2}}<\infty$, and note that, as discussed in [9], there is no good characterization of the space $\mathcal{H}^{1/2} = \{\|f\|_{\mathrm{H}^{1/2}}<\infty\}$ in terms of the local regularity of f. In particular, it is likely that our assumption (6.4) may be slightly relaxed.

The assumption (6.5) appears necessary in a second moment calculation (Lemma 6.5), and there is no analogous restriction in the case of random matrices [9]; it is certainly possible to slightly weaken it but we do not pursue this goal here, as (6.5) obviously already allows smooth functions but also indicators.

Theorem 6.1. *Let $f_1, \dots, f_k$ be functions in $\mathcal{H}^{1/2}$ satisfying properties (6.3), (6.4), (6.5). Assume the Riemann hypothesis and that $1 \ll \lambda_t \ll \log t$. Then we have that the random vector $(S_t(f_1), \dots, S_t(f_k))$ converges in distribution to the k–dimensional centered Gaussian vector $(S(f_1), \dots, S(f_k))$ with correlation structure*

$$\mathbb{E}(S(f_h)S(f_\ell)) = \langle f_h, f_\ell \rangle_{\mathrm{H}^{1/2}}.$$

The absence of normalization for the above convergence in law is a tangible sign of repulsion between the ζ zeros. However, there are differences between our result and the strong Szegő theorem: in particular, the rate of convergence to the limiting Gaussian is expected to be slow in our situation, while it is extremely fast in the case of random unitary matrices [19].

In the following theorem, for diverging variance of linear statistics, the bounded variation assumption is weaker:

$$\int (1 + |u \log u|)|\mathrm{d}f(u)| < \infty. \tag{6.6}$$

Theorem 6.2. *Suppose f satisfies (6.3), (6.6), (6.5), and that $\sigma_t(f)$ diverges. Assume the Riemann hypothesis and that $1 \ll \lambda_t \ll \log t$. Then, as $t \to \infty$, $S_t(f)$ converges in distribution to a standard Gaussian random variable.*

Although there is a normalization in the above theorem, it is typically very small. For example, in the allowed case when f is an indicator function and λ_t grows very slowly, σ_t^2 will be of order $\log \log t$, agreeing with the central limit theorem proved (unconditionally) by Selberg.

As we already noted, the condition $\lambda_t \ll \log t$ implies the mesoscopic scale. The condition $1 \ll \lambda_t$ is less natural. Supposedly, asymptotic normality does not hold if $\lambda_t = \mathrm{O}(1)$, for test functions in $\mathrm{H}^{1/2}$. This is related to the phenomenon of variance saturation explained by Berry [2], which happens at the same transition of the parameter λ_t. Motivated by this result, Johansson exhibited determinantal point processes satisfying the same phenomenon [20]. Note that such a transition, where limiting normality fails, also appears for sums of random exponentials [1], in particular for the Random Energy Model. The ultrametric structure for this model also appears for the counting measure of the ζ zeros [5]. Interesting conjectures relating long-range dependence particle systems, like the REM, and extreme values of L-functions were developed in [13].

The technique employed in the proof of both theorems involves an approximate version of the Weil explicit formula relating the zeros and primes (Section 2). This uses the Helffer-Sjöstrand functional calculus, which enables us to consider non-analytic test functions, and Selberg's seminal formula for ζ'/ζ.

Finally, we want to mention that while finishing this manuscript we discovered, in the draft [28], a preliminary proof of Theorem 6.2 which seems not to require Selberg's formula (6.18).

6.2. **Approximate explicit formula.** In this section, we consider a function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^2$, satisfying (6.3) as $x \to \pm\infty$. We aim at proving the following approximate version of the Weil explicit formula, relying on Selberg's smoothed expression for ζ'/ζ and the Helffer-Sjöstrand functional calculus. Remember that ω is a uniform random variable on $(1, 2)$, $\gamma_t = \lambda_t(\gamma - \omega t)$, and we use Selberg's smoothed von Mangoldt function,

$$\Lambda_u(n) = \begin{cases} \Lambda(n) & \text{if } 1 \leq n \leq u, \\ \Lambda(n)\frac{\log(u^2/n)}{\log n} & \text{if } u \leq n \leq u^2, \\ 0 & \text{otherwise .} \end{cases}$$

Proposition 6.3. *Assume the Riemann hypothesis. For any $f \in \mathscr{C}^2$ satisfying the initial assumptions, and any $u = t^\alpha$, $\alpha > 0$ fixed,*

$$\sum_\gamma f(\gamma_t) - \frac{\log t}{2\pi\lambda_t}\int f = \frac{1}{2\lambda_t}\sum_{n\geq 1}\frac{\Lambda_u(n)}{\sqrt{n}}\left(\hat{f}\left(\frac{\log n}{\lambda_t}\right)n^{\mathrm{i}\omega t} + \hat{f}\left(-\frac{\log n}{\lambda_t}\right)n^{-\mathrm{i}\omega t}\right) + E(\omega, t),$$

where the chosen Fourier normalization is $\hat{f}(\xi) = \frac{1}{\pi}\int f(x)e^{-\mathrm{i}\xi x}\mathrm{d}x$ *and the error term* $E(\omega, t)$ *is of type*

$$(6.7)\quad X(\omega,t)\frac{\lambda_t}{\log t}\,\mathrm{O}\left(\|f\|_1+\|f'\|_1+\|f''\|_1+\frac{1}{t\log t}\|x\log x f\|_1+\frac{1}{t\log t}\|x\log x f'\|_1 + \frac{1}{t\log t}\|x\log x f''\|_1\right),$$

where $\mathbb{E}(|X(\omega,t)|)$ *is uniformly bounded and does not depend on* f.

It is clear that if f is a fixed compactly supported $\mathscr{C}^2$ function, the error term converges in probability to 0 as $t\to\infty$. However, in our application of Proposition 6.3, f can depend on t.

Moreover, we state this approximate version of the explicit formula in a probabilistic setting for convenience, as this is what is needed in the proof of Theorems 6.1 and 6.2. One could also state a deterministic version, for functions with compact support along the critical axis.

Proof. All of the integrals in $\mathrm{d}x\mathrm{d}y$ in this paper are on the domain $\mathcal{D} := \{x\in\mathbb{R}, y>0\}$. The following formula, from the Helffer-Sjöstrand functional calculus, will be useful for us: for any $q\in\mathbb{R}$,

$$(6.8)\qquad f(q) = \frac{1}{\pi}\iint_{\mathcal{D}}\frac{\mathrm{i}yf''(x)\chi(y)+\mathrm{i}(f(x)+\mathrm{i}yf'(x))\chi'(y)}{q-(x+\mathrm{i}y)}\mathrm{d}x\mathrm{d}y,$$

where χ is a smooth cutoff function equal to 1 on $[0,1/2]$, 0 on $[1,\infty)$. This is one of many possible formulas aiming originally at evaluating $\operatorname{Tr} f(H)$ from resolvent estimates of H, for general self-adjoint operators H and test function f (see e.g.[7]). We follow this idea here, the *resolvent estimate* being Selberg's expression for ζ'/ζ.

Let $\gamma_t = \lambda_t(\gamma-\omega t)$, and $N(t)$ be the number of γ's in $[0,t]$ (counted with multiplicity). It is well-known (see e.g. [37]) that, as $t\to\infty$,

$$(6.9)\qquad N(t) = \frac{t}{2\pi}\log t - \frac{1+\log(2\pi)}{2\pi}t + \mathrm{O}(\log t).$$

From (6.8), taking real parts, we obtain (here $z = x + \mathrm{i}y$)

$$\sum_{|\gamma|<M} f(\gamma_t) = -\frac{1}{\pi}\iint_{\mathcal{D}} y f''(x)\chi(y) \sum_{|\gamma|<M} \mathrm{Im}\left(\frac{1}{\gamma_t - z}\right)\mathrm{d}x\mathrm{d}y \tag{6.10}$$

$$-\frac{1}{\pi}\iint_{\mathcal{D}} f(x)\chi'(y) \sum_{|\gamma|<M} \mathrm{Im}\left(\frac{1}{\gamma_t - z}\right)\mathrm{d}x\mathrm{d}y \tag{6.11}$$

$$-\frac{1}{\pi}\iint_{\mathcal{D}} y f'(x)\chi'(y) \sum_{|\gamma|<M} \mathrm{Re}\left(\frac{1}{\gamma_t - z} - \frac{1}{\lambda_t}\frac{\gamma}{\gamma^2+\frac{1}{4}}\right)\mathrm{d}x\mathrm{d}y \tag{6.12}$$

(it will soon be clear why we add the $\gamma/(\gamma^2+1/4)$ term, which makes no contribution in the integral). We now prove that by dominated convergence, the above three terms converge as $M \to \infty$. First, note that $y \mapsto y\,\mathrm{Im}((\gamma - (x+\mathrm{i}y))^{-1})$ is increasing, so using (6.3),

$$(6.10) \leq \int |f''(x)| \sum_{\gamma} \frac{1}{1+(\gamma_t - x)^2}\mathrm{d}x \leq \int \sum_{\gamma} \frac{1}{1+x^2}\frac{1}{1+(\gamma_t-x)^2}\mathrm{d}x \leq \sum_{\gamma}\frac{1}{1+\gamma_t^2} < \infty,$$

where we used

$$\int \frac{1}{1+(a-x)^2}\frac{1}{1+x^2}\mathrm{d}x \leq \frac{1}{1+a^2}, \tag{6.13}$$

where all the above (and following) inequalities are up to universal constants. Moreover χ' is supported on [1/2,1], still using (6.3) and (6.13) it is immediate that (6.11) converges as well. Finally, concerning (6.12), grouping for example γ with $-\gamma$ in the sum, we can bound it by

$$\iint_{\mathcal{D}} y\frac{1}{1+x^2}|\chi'(y)| \sum_{0\leq\gamma\leq M}\left|\mathrm{Re}\left(\frac{1}{\gamma_t - z} + \frac{1}{(-\gamma)_t - z}\right)\right|\mathrm{d}x\mathrm{d}y$$
$$\leq \int \frac{1}{1+x^2}\sum_{0\leq\gamma\leq M}\left|\mathrm{Re}\left(\frac{1}{\gamma_t-(x+\mathrm{i})} + \frac{1}{(-\gamma)_t-(x+\mathrm{i})}\right)\right|\mathrm{d}x$$

and it is an integration exercise to prove that the contribution of each γ in this integral is $\mathrm{O}(\gamma^{-3/2})$ for example. Note that from (6.3) and (6.9), $\sum_\gamma f(\gamma_t)$ is absolutely summable for each fixed t. We therefore proved that

$$\sum_\gamma f(\gamma_t) = -\frac{1}{\pi}\iint_{\mathcal{D}} y f''(x)\chi(y)\sum_\gamma \operatorname{Im}\left(\frac{1}{\gamma_t - z}\right)\mathrm{d}x\mathrm{d}y \tag{6.14}$$

$$-\frac{1}{\pi}\iint_{\mathcal{D}} f(x)\chi'(y)\sum_\gamma \operatorname{Im}\left(\frac{1}{\gamma_t - z}\right)\mathrm{d}x\mathrm{d}y \tag{6.15}$$

$$-\frac{1}{\pi}\iint_{\mathcal{D}} y f'(x)\chi'(y)\sum_\gamma \operatorname{Re}\left(\frac{1}{\gamma_t - z} - \frac{1}{\lambda_t}\frac{\gamma}{\gamma^2+\frac{1}{4}}\right)\mathrm{d}x\mathrm{d}y, \tag{6.16}$$

where all sums are absolutely convergent. Now, the above sums can be written in terms of ζ'/ζ: it is known from Hadamard's factorization formula that, denoting ρ's for the non-trivial ζ-zeros, for any $s \notin \{\rho\}$, we have (see e.g. p 398 in [27])

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2}\log \operatorname{Im}(s) + \mathrm{O}(1).$$

By taking real and imaginary parts, and identifying $s = 1/2 + \frac{y}{\lambda_t} + \mathrm{i}\left(\omega t + \frac{x}{\lambda_t}\right)$, we get

$$\sum_\gamma \operatorname{Im}\left(\frac{1}{\gamma_t - (x+\mathrm{i}y)}\right) = \frac{1}{\lambda_t}\operatorname{Re}\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{y}{\lambda_t} + \mathrm{i}\left(\omega t + \frac{x}{\lambda_t}\right)\right) + \frac{1}{2}\frac{\log t}{\lambda_t}$$
$$+ \frac{1}{\lambda_t}\mathrm{O}\left(\left|\log\left(\omega + \frac{x}{t\lambda_t}\right)\right|\right),$$
$$\sum_\gamma \operatorname{Re}\left(\frac{1}{\gamma_t - (x+\mathrm{i}y)} - \frac{1}{\lambda_t}\frac{\gamma}{\gamma^2+\frac{1}{4}}\right) = \frac{1}{\lambda_t}\operatorname{Im}\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{y}{\lambda_t} + \mathrm{i}\left(\omega t + \frac{x}{\lambda_t}\right)\right) + \mathrm{O}\left(\frac{1}{\lambda_t}\right).$$

Still relying on (6.8), using the fact that

$$\lim_{M\to\infty}\operatorname{Im}\int_{-M}^{M}\frac{\mathrm{d}u}{u - x + \mathrm{i}y} = \int\frac{\mathrm{d}v}{x^2+1} = \pi,\ \lim_{M\to\infty}\operatorname{Re}\int_{-M}^{M}\frac{\mathrm{d}u}{u - x + \mathrm{i}y} = 0,$$

we have

$$\frac{\log t}{2\pi\lambda_t}\int f(u)\mathrm{d}u = -\frac{1}{\pi}\int y f''(x)\chi(y)\frac{1}{2}\frac{\log t}{\lambda_t}\mathrm{d}x\mathrm{d}y - \frac{1}{\pi}\int f(x)\chi'(y)\frac{1}{2}\frac{\log t}{\lambda_t}\mathrm{d}x\mathrm{d}y,$$

so we obtained,

$$\sum_\gamma f(\gamma_t) - \frac{\log t}{2\pi\lambda_t}\int f = -\frac{1}{\pi\lambda_t}\iint_{\mathcal{D}} y f''(x)\chi(y)\,\mathrm{Re}\,\frac{\zeta'}{\zeta}\left(\frac12+\frac{y}{\lambda_t}+\mathrm{i}\left(\omega t+\frac{x}{\lambda_t}\right)\right)\mathrm{d}x\mathrm{d}y$$
$$-\frac{1}{\pi\lambda_t}\iint_{\mathcal{D}} f(x)\chi'(y)\,\mathrm{Re}\,\frac{\zeta'}{\zeta}\left(\frac12+\frac{y}{\lambda_t}+\mathrm{i}\left(\omega t+\frac{x}{\lambda_t}\right)\right)\mathrm{d}x\mathrm{d}y$$
$$(6.17)\qquad -\frac{1}{\pi\lambda_t}\iint_{\mathcal{D}} y f'(x)\chi'(y)\,\mathrm{Im}\,\frac{\zeta'}{\zeta}\left(\frac12+\frac{y}{\lambda_t}+\mathrm{i}\left(\omega t+\frac{x}{\lambda_t}\right)\right)\mathrm{d}x\mathrm{d}y$$
$$+\,\mathrm{O}\left(\frac{1}{\lambda_t}\right),$$

where the above $\mathrm{O}(\lambda_t^{-1})$ is understood in the sense that its L^1 norm is bounded by λ_t^{-1}. We now substitute $\frac{\zeta'}{\zeta}$, in the above expression, with its smooth approximation by Selberg: for any $u > 0$ and $s \notin \{\rho, 1, -2\,\mathbb{N}\}$,

$$(6.18)\qquad \frac{\zeta'}{\zeta}(s) = A_u(s) + B_u(s) + C_u(s) + D_u(s)$$

where

$$A_u(s) = -\sum_{n\le u^2}\frac{\Lambda_u(n)}{n^s},$$
$$B_u(s) = \frac{1}{\log u}\sum_\rho \frac{u^{\rho-s}-u^{2(\rho-s)}}{(\rho-s)^2},$$
$$C_u(s) = \frac{1}{\log u}\sum_{n\ge1}\frac{u^{-2n-s}-u^{-2(2n+s)}}{(2n+s)^2},$$
$$D_u(s) = \frac{1}{\log u}\frac{u^{2(1-s)}-u^{1-s}}{(1-s)^2}.$$

First, it is elementary that the contribution from the terms D_u and C_u in (6.17) is negligible. For D_u, we bound by $\frac{1}{\log u}\int\frac{u}{1+(\omega t)^2}\frac{\mathrm{d}x}{1+x^2} \le \frac{1}{\log u}\frac{u}{1+t^2} \le \frac{c}{t}$, under the constraint $1 \ll u \le t$ (by the end we will choose $u = t^{1/2}$). The term involving C_u is also $\mathrm{O}(t^{-1})$ easily.

The main errors involve B_u. First, as χ' is supported on $(1/2,1)$, we have

$$\begin{aligned}(6.19)\quad &\frac{1}{\lambda_t}\iint_{\mathcal{D}}(|f(x)\chi'(y)|+|yf'(x)\chi'(y)|)\left|B_u\left(\frac{1}{2}+\frac{y}{\lambda_t}+\mathrm{i}\left(\omega t+\frac{x}{\lambda_t}\right)\right)\right|\mathrm{d}x\mathrm{d}y\\ &\leq \frac{1}{\lambda_t\log u}e^{-\frac{\log u}{2\lambda_t}}\int(|f(x)|+|f'(x)|)\sum_{|\gamma|<4t+\frac{4|x|}{\lambda_t}}\mathbb{E}\frac{1}{(1/\lambda_t)^2+(\omega t+\gamma-x/\lambda_t)^2}\mathrm{d}x\\ &\quad+\frac{1}{\lambda_t\log u}e^{-\frac{\log u}{2\lambda_t}}\int(|f(x)|+|f'(x)|)\sum_{|\gamma|>4t+\frac{4|x|}{\lambda_t}}\mathbb{E}\frac{1}{(1/\lambda_t)^2+(\omega t+\gamma-x/\lambda_t)^2}\mathrm{d}x\end{aligned}$$

Using (6.9), the first sum is at most

$$\frac{|\{|\gamma|<4t+\frac{4|x|}{\lambda_t}|\}|}{t}\int\frac{\mathrm{d}v}{(1/\lambda_t)^2+v^2}\leq(\log t+\frac{|x|\log|x|}{t})\lambda_t,$$

and the second at most $\sum 1/\gamma^2<\infty$, so this error is of type (6.7), for $u=t^{\alpha}$. Finally, the error from B_u in the expectation of the term (6.14), which is closer to the critical axis, is bounded by

$$\begin{aligned}&\frac{1}{\lambda_t\log u}\iint_{\mathcal{D}}ye^{-\frac{y}{\lambda_t}\log u}|f''(x)|\sum_{|\gamma|<4t+\frac{4|x|}{\lambda_t}}\mathbb{E}\frac{1}{(y/\lambda_t)^2+(\omega t+\gamma-x/\lambda_t)^2}\mathrm{d}x\mathrm{d}y\\ +&\frac{1}{\lambda_t\log u}\iint_{\mathcal{D}}ye^{-\frac{y}{\lambda_t}\log u}|f''(x)|\sum_{|\gamma|>4t+\frac{4|x|}{\lambda_t}}\mathbb{E}\frac{1}{(y/\lambda_t)^2+(\omega t+\gamma-x/\lambda_t)^2}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

The first sum is at most

$$\frac{|\{|\gamma|<2t+\frac{2|x|}{\lambda_t}\}|}{t}\int\frac{1}{(y/\lambda_t)^2+u^2}\mathrm{d}u\leq(\log t+\frac{|x|\log|x|}{t})\frac{\lambda_t}{y},$$

and the second at most $\sum 1/\gamma^2<\infty$, so all together this error term is of type (6.7).

Finally, the $A_u(s)$ term can be simplified observing, by successive integrations by parts [7], that for any $\delta > 0$ we have

$$\frac{1}{\pi}\int (yf''(x)\chi(y) + (f(x) - \mathrm{i}yf'(x))\chi'(y))e^{-\mathrm{i}\delta x}e^{-\delta y}\mathrm{d}x\mathrm{d}y = -\frac{1}{\pi}\int f(x)e^{-\mathrm{i}\delta x}\mathrm{d}x. \tag{6.20}$$

This completes the proof of Proposition 6.3. □

6.3. **Strong Szegő theorem.** We first prove that, in Proposition 6.3, the terms n of type p^k, for $k \geq 2$, make no contribution.

Lemma 6.4. *For $u = t^\alpha$, $\alpha \leq 1$, and a family of functions (f_t) uniformly bounded in L^1, the random variable*

$$\frac{1}{\lambda_t}\sum_{n=p^k, p\in\mathcal{P}, k\geq 2} \frac{\Lambda_u(n)}{\sqrt{n}}\hat{f}_t\left(\frac{\log n}{\lambda_t}\right)n^{\mathrm{i}\omega t}$$

converges to 0 in L^2.

Proof. For the terms corresponding to $k \geq 3$, this is obvious by absolute summability. For $k = 2$, we can use the Montgomery-Vaughan inequality [26]: for any complex numbers a_r and real numbers λ_r, and setting $\delta_r = \min_{s\neq r}|\lambda_r - \lambda_s|$,

$$\frac{\|\hat{f}\|_\infty^2}{t}\int_t^{2t}\left|\sum_r a_r e^{\mathrm{i}\lambda_r s}\right|^2 \mathrm{d}s \leq \sum_r |a_r|^2\left(1 + \frac{c}{t\delta_r}\right) \tag{6.21}$$

for some universal $c > 0$. Consequently, in our situation, taking $\lambda_p = 2\log p$, and bounding uniformly $\hat{f}$, we get

$$\mathbb{E}\left|\frac{1}{\lambda_t}\sum_{p\in\mathcal{P}}\frac{\log p}{p}\hat{f}\left(\frac{\log p^2}{\lambda_t}\right)p^{2\mathrm{i}\omega t}\right|^2 \leq \frac{1}{\lambda_t^2}\sum_{p\in\mathcal{P}, p\leq t}\frac{(\log p)^2}{p^2}\left(1 + \frac{cp}{2t}\right) \to 0,$$

---

[7] In detail,

$$\begin{aligned}\frac{1}{\pi}\int f''(x)e^{-\mathrm{i}\delta x}\mathrm{d}x\int y\chi(y)e^{-\delta y}\mathrm{d}y &= \frac{1}{\pi}\int \mathrm{i}\delta y\chi(y)e^{-\delta y}\mathrm{d}y\int f'(x)e^{-\mathrm{i}\delta x}\mathrm{d}x\\ =&\frac{1}{\pi}\int(\mathrm{i}\chi(y) + \mathrm{i}y\chi'(y))e^{-\delta y}\mathrm{d}y\int f'(x)e^{-\mathrm{i}\delta x}\mathrm{d}x.\end{aligned}$$

And notice that the $\mathrm{i}y\chi'(y)$ term cancels, and the other term equals

$$\frac{1}{\pi}\int \mathrm{i}\chi(y)e^{-\delta y}\mathrm{d}y\int f'(x)e^{-\mathrm{i}\delta x}\mathrm{d}x = -\frac{1}{\pi}\int\delta\chi(y)e^{-\delta y}\mathrm{d}y\int f(x)e^{-\mathrm{i}\delta x}\mathrm{d}x.$$

and a final integration by parts gives (6.20).

where we just used $|\log p_1 - \log p_2| > 2p_1^{-1}$ for prime numbers $p_1 < p_2$. $\square$

Concerning the terms $n = p$ appearing in Proposition 6.3, the following lemma computes the asymptotics of the diagonal terms from the second moment for a fixed function f. This will be the asymptotics of the variance.

Lemma 6.5. *Let $b_{pt} = \lambda_t^{-1}\Lambda_u(p)/\sqrt{p}\,\hat{f}(\log p/\lambda_t)$. Suppose $\xi \hat{f}(\xi)^2$ and $(\xi\hat{f}(\xi)^2)'$ have the asymptotic bound $O(\xi^{-1})$ as $\xi \to \pm\infty$. Then as $t \to \infty$, for $u = t^{1/2}$,*

$$\sum_{p\in\mathcal{P}} |b_{pt}|^2 = (1+o(1)) \int_0^{(\log t)/(2\lambda_t)} \xi|\hat{f}(\xi)|^2 \mathrm{d}\xi + \mathrm{O}\left(\int_{(\log t)/(2\lambda_t)}^{(\log t)/\lambda_t} \xi|\hat{f}(\xi)|^2 \mathrm{d}\xi\right).$$

Proof. This lemma relies on a simple asymptotic estimate based on the prime number theorem. Let p_k denote the kth prime, q_k denote $\log p_k$, with $q_0 = 0$ by convention, and $\Delta_k = q_k - q_{k-1}$. First consider the sum over $1 \leq p \leq t^{1/2}$. By the mean value theorem,

$$\left| \int_{q_{k-1}/\lambda_t}^{q_k/\lambda_t} \xi|\hat{f}(\xi)|^2 \mathrm{d}\xi - \frac{\Delta_k}{\lambda_t}\frac{q_k}{\lambda_t}|\hat{f}\left(\frac{q_k}{\lambda_t}\right)|^2 \right| \leq \mathrm{Var}(\xi|\hat{f}(\xi)|^2 1_{[q_{k-1}/\lambda_t, q_k/\lambda_t]}(\xi))\frac{\Delta_k}{\lambda_t}.$$

which implies

$$\left| \int_0^{(\log t)/(2\lambda_t)} \xi|\hat{f}(\xi)|^2 \mathrm{d}\xi - \frac{1}{\lambda_t^2}\sum_{p_k < t^{1/2}} q_k\Delta_k|\hat{f}\left(\frac{q_k}{\lambda_t}\right)|^2 \right| \leq \sum_k \mathrm{Var}(\xi|\hat{f}(\xi)|^2 1_{[q_{k-1}/\lambda_t, q_k/\lambda_t]}(\xi))\frac{\Delta_k}{\lambda_t}.$$

Since the derivative of $w|\hat{f}(w)|^2$ is bounded by a constant M, then the right hand side is bounded by $\sum_k M\Delta_k^2/\lambda_t^2$, which converges to 0.

Moreover, using summation by parts, and letting π denote the usual prime-counting function,

$$\frac{1}{\lambda_t}\sum_{k=1}^{\pi(t^{1/2})} \frac{q_k}{\lambda_t}|\hat{f}\left(\frac{q_k}{\lambda_t}\right)|^2(\Delta_k - k^{-1}) = \frac{1}{\lambda_t}\frac{q_{\pi(t^{1/2})}}{\lambda_t}|\hat{f}\left(\frac{q_{\pi(t^{1/2})}}{\lambda_t}\right)|^2(q_{\pi(t^{1/2})} - \log\pi(t^{1/2}) + \mathrm{O}(1))$$

$$- \frac{1}{\lambda_t}\sum_{k=1}^{\pi(t^{1/2})}(q_k - \log k)\left[\frac{q_{k+1}}{\lambda_t}|\hat{f}\left(\frac{q_{k+1}}{\lambda_t}\right)|^2 - \frac{q_k}{\lambda_t}\hat{f}|\left(\frac{q_k}{\lambda_t}\right)|^2\right]$$

Using the prime number theorem and $\xi\hat{f}(\xi)^2 = O(\xi^{-1})$, the first term is bounded above by

$$c\,\frac{q_{\pi(t^{1/2})} - \log\pi(t^{1/2})}{q_{\pi(t^{1/2})}} + \mathrm{o}(1) = c\,\frac{\log\log\pi(t^{1/2})}{\log\pi(t^{1/2})},$$

which converges to 0. Now look at the second term. Using $(\xi\hat{f}(\xi)^2)' = \mathrm{O}(\xi^{-1})$, the term in brackets can be bounded, so there is the upper bound

$$c\,\frac{1}{\lambda_t}\sum_{k=1}^{\pi(t^{1/2})}\log\log k\,\frac{\lambda_t}{q_k}\,\frac{\Delta_{k+1}}{\lambda_t}.$$

Using the well-known result on prime gaps, $p_{k+1} - p_k < p_k^\theta$ for sufficiently large k and for some $\theta < 1$,

$$q_{k+1} < q_k + \log(1 + p_k^{\theta-1}) < q_k + 4p_k^{\theta-1} < q_k + 8k^{\theta-1}.$$

Thus the upper bound

$$c\,\frac{1}{\lambda_t}\sum_{k=1}^{\pi(t^{1/2})}\frac{\log\log k}{k^{2-\theta}\log k},$$

holds, which also converges to 0.

The sum over $t^{1/2} \le p \le t$ follows from a similar argument and the fact that $\Lambda_{t^{1/2}}(p) = \log t - \log p \le \log p$. □

Our proof of Theorem 6.1 and Theorem 6.2 relies on a mollification f_ε of f in order to apply the approximate explicit formula, Proposition 6.3, to the following result from [5] (using an idea from [36]).

Proposition 6.6. *Let* a_{pt} $(p \in \mathcal{P}, t \in \mathbb{R}^+)$ *be given complex numbers, such that* $\sup_p |a_{pt}| \to 0$, $\sum_p |a_{pt}|^2 \to \sigma^2$ *as* $t \to \infty$. *Assume the existence of some* (m_t) *with* $\log m_t/\log t \to 0$ *and*

$$\sum_{p > m_t} |a_{pt}|^2 \left(1 + \frac{p}{t}\right) \underset{t\to\infty}{\longrightarrow} 0. \tag{6.22}$$

Then, if ω *is a uniform random variable on* $(1, 2)$,

$$\sum_{p\in\mathcal{P}} a_{pt}p^{-\mathrm{i}\omega t} \overset{\text{(weakly)}}{\longrightarrow} \sigma\mathcal{N}$$

as $t \to \infty$, $\mathcal{N}$ being a standard complex normal variable.

Proof of Theorem 6.2. Let $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)$ be a bump function, and $f_\varepsilon = f * \phi_\varepsilon$. Moreover, remember that we defined $\sigma_t^2 = \int_{-\lambda_t}^{\lambda_t} |\xi||\hat{f}(\xi)|^2$. We know that $\sigma_t \to \infty$ as $t \to \infty$. We will choose $\varepsilon = \varepsilon_t$ by the end of this proof, and use $u = t^{1/2}$.

First step. The difference $\sigma_t^{-1}\sum_\gamma (f_\varepsilon(\gamma_t) - f(\gamma_t))$ converges to 0 in probability if $\varepsilon \ll \frac{\lambda_t}{\log t}\sigma_t$. Indeed

$$\begin{aligned}\mathbb{E}\,|f_\varepsilon(\gamma_t) - f(\gamma_t)| &\leq \varepsilon^{-1}\mathbb{E}\int |f(\gamma_t - y) - f(\gamma_t)|\phi(y/\varepsilon)\mathrm{d}y \leq c\varepsilon^{-1}\int_1^2 \mathrm{d}\omega \int_0^\varepsilon \mathrm{d}y \left|\int_{\gamma_t - y}^{\gamma_t} |\mathrm{d}f(u)|\right| \\ &\leq c\int_1^2 \mathrm{d}\omega \int_{\gamma_t-\varepsilon}^{\gamma_t} |\mathrm{d}f(u)| \leq c\int |\mathrm{d}f(u)| \int_{-\frac{u+\varepsilon}{t\lambda_t}+\frac{\gamma}{t}\leq\omega\leq-\frac{u}{t\lambda_t}+\frac{\gamma}{t}, |\gamma|\leq 2t+\frac{|u|}{\lambda_t}} \mathrm{d}\omega \\ &\leq c\frac{\varepsilon}{t\lambda_t}\int 1_{|\gamma|\leq 2t+\frac{|u|}{\lambda_t}} |\mathrm{d}f(u)|.\end{aligned}$$

Hence, using (6.9), we get that $\sum_\gamma \mathbb{E}\,|f_\varepsilon(\gamma_t) - f(\gamma_t)| \leq \varepsilon\frac{\log t}{\lambda_t}\int(1 + |u\log u|)|\mathrm{d}f(u)|$, and this last integral is bounded from the hypothesis (6.6).

Second step. Let $Y_\varepsilon = \frac{1}{\sigma_t\lambda_t}\sum_{p\in\mathcal{P}} \frac{\Lambda_u(p)}{\sqrt{p}}\hat{f}_\varepsilon\left(\frac{\log p}{\lambda_t}\right)p^{\mathrm{i}\omega t}$ and $Y = \frac{1}{\sigma_t\lambda_t}\sum_{p\in\mathcal{P}} \frac{\Lambda_u(p)}{\sqrt{p}}\hat{f}\left(\frac{\log p}{\lambda_t}\right)p^{\mathrm{i}\omega t}$. Then we have $\|Y_\varepsilon - Y\|_{L^2} = \mathrm{O}\left(\varepsilon\frac{\log t}{\lambda_t}\right)$ as $t \to \infty$. Indeed, we can bound $\|Y_\varepsilon - Y\|_{L^2}^2$ by the diagonal terms in the expansion because of Montgomery-Vaughan inequality, stated in (6.21). In our case, taking $\lambda_p = \log p$, $a_p = \frac{1}{\sigma_t\lambda_t}\frac{\Lambda_u(p)}{\sqrt{p}}\left(\hat{f}_\varepsilon\left(\frac{\log p}{\lambda_t}\right) - \hat{f}\left(\frac{\log p}{\lambda_t}\right)\right)$ using that $|\hat{f}_\varepsilon(u) - \hat{f}(u)| \leq c\,u\varepsilon|\hat{f}(u)|$, we get

$$\|Y_\varepsilon - Y\|_{L^2}^2 \leq c\left(\varepsilon\frac{\log t}{\lambda_t}\right)^2 \frac{1}{\sigma_t^2}\sum_p \left|\frac{1}{\lambda_t}\frac{\Lambda_u(p)}{\sqrt{p}}\hat{f}\left(\frac{\log p}{\lambda_t}\right)\right|^2$$

and this last sum is asymptotically equivalent to σ_t^2, by Lemma 6.5.

Third step. We can easily find some m_t so that $\log m_t = \mathrm{o}(\log t)$ and the tail condition (6.22) is satisfied, for

$$a_{pt} = \frac{1}{\sigma_t\lambda_t}\frac{\Lambda_u(p)}{\sqrt{p}}\hat{f}\left(\frac{\log p}{\lambda_t}\right).$$

Indeed, as f has bounded variation, $\hat{f}(x) = \mathrm{O}(x^{-1})$, so

$$\sum_{m_t<p<t} |a_{pt}|^2 \leq \frac{1}{\sigma_t^2} \sum_{m_t<p<t} \frac{1}{p} \sim \frac{1}{\sigma_t^2}(\log\log t - \log\log m_t).$$

A possible choice is $m_t = \exp(\log t/\sigma_t)$.

Fourth step. The error term (6.7) in the approximate explicit formula for f_ε can be controlled in the following way. As f has bounded variation, and $f'' = \phi_\varepsilon'' * f$, it is a standard argument that

$$\begin{aligned}\int |f_\varepsilon''(u)|\mathrm{d}u &= \varepsilon^{-3} \int \left| \int \phi'' \left(\frac{x}{\varepsilon}\right) (f(u-x) - f(u))\mathrm{d}x \right| \mathrm{d}u \\ &\leq \varepsilon^{-3} \iint \left| \int_{u-x}^{u} |\mathrm{d}f(v)| \right| \mathrm{d}v 1_{|x|<\varepsilon} \mathrm{d}x\mathrm{d}u \leq \varepsilon^{-3} \int |\mathrm{d}f(v)| \iint 1_{|x|<\varepsilon, v\in[u-x,u]} \mathrm{d}x\mathrm{d}u \\ &\leq \varepsilon^{-1} \int |\mathrm{d}f(v)|,\end{aligned}$$

so the error term related to f_ε'' in Proposition 6.3 is of order $\frac{\lambda_t}{\log t}\varepsilon^{-1}$. All of the other error terms can be bounded in the same way, and have order at most $\frac{\lambda_t}{\log t}\varepsilon^{-1}$ as well.

Conclusion. From the previous steps, the conclusion of Theorem 6.2 holds if we can find some ε_t such that

$$\frac{\lambda_t}{\sigma_t \log t} \ll \varepsilon_t \ll \frac{\lambda_t}{\log t}\sigma_t,$$

which obviously holds for $\varepsilon_t = \lambda_t/\log t$. Indeed, using the First and Second steps, to conclude we then just need

$$Y \overset{\text{(weakly)}}{\longrightarrow} \mathcal{N}, \tag{6.23}$$

$$\frac{1}{\sigma_t} \sum_\gamma f_\varepsilon(\gamma_t) - Y_\varepsilon \overset{\text{(weakly)}}{\longrightarrow} 0, \tag{6.24}$$

where $\mathcal{N}$ is a standard complex Gaussian random variable. The convergence (6.23) is a consequence of the Third step and Lemma 6.5, to apply Proposition 6.6. The convergence (6.24) holds thanks to Proposition 6.3, the Fourth step and Lemma 6.4. $\square$

Proof of Theorem 6.1. We closely follow the proof of Theorem 6.2, except that now all of the errors due to the mollification need to vanish without normalization. This is possible because of the extra regularity assumptions (6.4) we chose to assume in Theorem 6.1.

The error from the first and second steps are controlled exactly in the same way, and they will be negligible if $\varepsilon \ll \frac{\lambda_t}{\log t}$. The error from the third step vanishes if $\frac{\lambda_t}{\log t}\|f_\varepsilon''\|_{L^1} \to 0$ (for the f_ε'' term in (6.7), for example). As in the proof of Theorem 6.2, $\|f_\varepsilon''\|_{L^1}$ can be bounded by the total variation of f', so this error goes to 0 anyway. $\square$

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