Interpolation and Vector Bundles on Curves

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Interpolation and vector bundles on curves

A dissertation presented

by

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to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

April 2015

Dissertation Advisor: Professor Harris Atanas Valeryev Atanasov

Interpolation and vector bundles on curves

Abstract

Interpolation is a property of vector bundles on curves closely related to slope stability. The notion is motivated by the deformation theory of curves in projective space incident to given fixed subvarieties. If the normal bundle of a projective curve satisfies interpolation, then curves in the same component of the Hilbert scheme exhibit normal behavior with respect to incident problems.

We demonstrate how to use degeneration arguments to deduce interpolation. In particular, we show that a general connected space curve of degree d and genus g satisfies interpolation for $d \geq g + 3$ unless $d = 5$ and $g = 2$. As a second application, we show that a general elliptic curve of degree d in $\mathbb{P}^n$ satisfies a slightly weaker notion when $d \geq 7$, $d \geq n + 1$, and the remainder of $2d$ modulo $n - 1$ lies between 3 and $n - 2$ inclusive. We also show that interpolation is equivalent to the—a priori stricter—notion of strong interpolation.

The use of degeneration techniques to prove interpolation requires working with modifications of vector bundles. In the second part of this thesis, we develop a general theory of modifications for bundles over varieties of arbitrary dimensions. We explain how to apply this machinery when dealing with families of curves, and prove a number of results which allow us to deduce interpolation via short exact sequences.

Contents

I would not have been able to reach where I am today without the unconditional support and love that my family has given me throughout my life, and particularly in the last nine years that I have been been away from them.

Last but not least, I would like to express my deepest gratitude to Ivy Chen who is a constant source of joy and wisdom. My journey would not have been possible without having her by my side!

1. Introduction

The study of moduli spaces of curves is a very rich branch of algebraic geometry. There is a long history of interaction between curves in the abstract (e.g., the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$) and concrete realizations of curves (e.g., Hilbert schemes and Kontsevich spaces). Understanding of the geometry on one side could lead to a result on the other and vice versa. For example, many results about the birational geometry of $\overline{\mathcal{M}}_g$ use trigonal curves or curves on K3s and other surfaces. Interpolation properties help us understand subvarieties of the Hilbert scheme given by incidence conditions. We will use this section to give a more precise meaning of this statement and explain how these ideas can eventually be used to study the birational geometry of $\overline{\mathcal{M}}_g$.

Our study of interpolation was inspired by an interest in constructing moving curves in $\overline{\mathcal{M}}_g$. Harris-Morrison [9] produced such curves by studying ramified covers over $\mathbb{P}^1$ and used them to bound the slopes of effective divisors on $\overline{\mathcal{M}}_g$. Fedorchuk [6] continued this line of work by using curves in $\mathbb{P}^2$. A thorough overview of this topic can be found in [4]. Instead of attacking the construction question directly, our goal is to create a broader framework for studying the deformation theory of curves in projective space. Constructing moving curves is only one of several possible applications.

We proceed to outline how Hilbert schemes of curves could be used to produce moving curves in $\overline{\mathcal{M}}_g$ and motivate the definition of interpolation. The Hilbert scheme $\mathcal{H}_{d,g,n}$ parametrizes degree d and arithmetic genus g curves in $\mathbb{P}^n$. The restricted Hilbert scheme $\mathcal{I}_{d,g,n}$ is defined as the closure of the locus of smooth irreducible curves in $\mathcal{H}_{d,g,n}$. For every triple (d, g, n) there is a rational map $\varphi_{d,g,n}\colon \mathcal{I}_{d,g,n} \dashrightarrow \overline{\mathcal{M}}_g$ from the restricted Hilbert scheme to the Deligne-Mumford moduli space of curves. Given a fixed g, it is always possible to find d

and n such that $\varphi_{d,g,n}$ is dominant. If we can construct a moving curve in $\mathcal{I}_{d,g,n}$ which does not map to a point, then it maps to a moving curve in $\overline{\mathcal{M}}_g$.

Let $C \subset \mathbb{P}^n$ be a smooth irreducible curve and $[C] \in \mathcal{I}_{d,g,n}$ the corresponding point. The tangent space $T_{[C]}\mathcal{I}_{d,g,n}$ can be identified with $\mathrm{H}^0(N_C)$. If $\mathrm{H}^1(N_C) = 0$, then $[C]$ is a smooth point of $\mathcal{I}_{d,g,n}$. Pick a point $p \in C$ and consider the subvariety $\mathcal{J}(p) \subset \mathcal{I}_{d,g,n}$ consisting of all curves containing p. By construction $[C] \in \mathcal{J}(p)$ and we can identify the tangent space $T_{[C]}\mathcal{J}(p)$ with $\mathrm{H}^0(N_C(-p))$. Since the normal bundle N_C has rank $n - 1$, the expectation is that $\mathcal{J}(p)$ has codimension $n - 1$ in $\mathcal{I}_{d,g,n}$. After choosing a second point $q \in C \setminus \{p\}$, we can define analogously the variety $\mathcal{J}(p, q)$ which satisfies $T_{[C]}\mathcal{J}(p, q) \cong \mathrm{H}^0(N_C(-p - q))$ and the process continues. The idea is to use just enough points so that $\mathcal{J}(p_1, \ldots, p_m)$ is a curve. By allowing the p_i to vary, we have constructed a moving curve in $\mathcal{I}_{d,g,n}$.

While very instructive, there are several issues with this naive idea. First, we would expect $\mathcal{J}(p_i)$ to be a curve only if $\mathrm{H}^0(N_C) - 1$ is a multiple of $n - 1$. Fortunately, there is a simple generalization which circumvents this restriction. Suppose $L \subset \mathbb{P}^n$ is a linear space which intersects C transversely in a point p. Consider the space $\mathcal{J}(L)$ of curves incident to L, that is, they intersect L nontrivially. The tangent space $T_{[C]}\mathcal{J}(L)$ can be identified with $\mathrm{H}^0(N')$ where N' is a vector bundle defined by the short exact sequence

$$0 \longrightarrow N' \longrightarrow N_C \longrightarrow N_{C,p}/[T_pL] \longrightarrow 0.$$

The third term is the quotient of $N_{C,p}$ by $(T_pL + T_pC)/T_pC$, the image of T_pL in $N_{C,p}$. We expect that $\mathcal{J}(L)$ has codimension

$$\dim N_{C,p}/[T_pL] = n - 1 - \dim L.$$

The construction extends to multiple linear spaces $L_1, \ldots, L_m$ and the tangent space to $\mathcal{J}(L_1, \ldots, L_m)$ at $[C]$ is controlled by a vector bundle $N' \subset N_C$ whose cokernel is supported on $C \cap \bigcup_i L_i$. We can always choose multiple points and a single higher dimensional linear space such that the incident curves form a variety whose expected dimension is 1.

The second, and more pressing, issue lies with the difference between expectation and reality. Computing the actual dimension of a variety $\mathcal{J}(L_1, \ldots, L_m)$ often requires some significant understanding of the curves we are dealing with and their deformation theory. Intuitively speaking, the notion of interpolation formalizes the expected behavior we have been discussing.

Given a triple of integers d, g and n, we ask whether the general smooth curve in $\mathcal{I}_{d,g,n}$ satisfies interpolation. The goal of this paper is to develop the notion of interpolation rigorously and demonstrate how to use degeneration techniques in order to approach our question for certain triples (d, g, n). For example, the union of a rational curve and a secant line can be treated as an elliptic curve. This degeneration plays a key role in our proof that a general elliptic curve with $d \geq n + 1$ and $d \geq 7$ satisfies interpolation under an additional congruence condition. More generally, the addition of g secant lines to a rational curve produces a genus g curve. We combine this idea with a careful analysis of normal bundles to show that a general smooth curve in $\mathbb{P}^3$ satisfies interpolation as long as $d \geq g + 3$ and $(d, g) \neq (5, 2)$.

Summary. Section 2 introduces elementary modifications which are later used in the definition of interpolation. In Section 3, we discuss the relation between interpolation, slope stability, and section stability. We illustrate these notions using vector bundles on the projective line. Once the basic language is established, we discuss the use of degenerate curves in interpolation arguments. Section 4 establishes two results which respectively allow us to attach two curves in a single

point and attach a secant line to a curve. We show interpolation is an open property and make other remarks about Hilbert schemes in Section 5. The framework we constructed is applied to elliptic curves in Section 6 and curves in $\mathbb{P}^3$ in Section 7. The birational geometry of Hirzebruch surfaces plays a central role in our study of space curves. We prove the equivalence of strong and regular interpolation in section Section 8. The last three sections provide a deeper study of vector bundle modifications and their relation to interpolation. We begin by studying multi-modifications for vector bundles over varieties of arbitrary dimension in Section 9. Next, we specialize these results to curves in Section 10. Finally, we explain how modifications interact with interpolation in Section 11.

Conventions. Unless otherwise noted, we will consistently make the following conventions.

- We will work over an algebraically closed field K of characteristic 0.
- All varieties are reduced, separated, finite type schemes over K.
- All curves are connected and locally complete intersection (lci); all families of curves have connected lci fibers.
- All vector bundles are locally free sheaves of finite constant rank.
- A subbundle refers to a vector subbundle with locally free quotient.
- All divisors are Cartier.
- We will call a vector bundle nonspecial if it has no higher cohomology.

2. Elementary modifications of vector bundles

Let E be a vector bundle on an algebraic variety X. Consider a quotient of the form

$$E \xrightarrow{\ q\ } Q \longrightarrow 0$$

where Q is a locally free $\mathcal{O}_D$-sheaf for a Cartier divisor $D \subset X$. We will refer to the kernel of q as the *(elementary) modification of E corresponding to q* and denote it by $M(E, q)$. It can be shown that all such modifications are vector bundles.

While the definition of an elementary modification is quite general, most of the applications we care about assume $X = C$ is a curve and $D = \{p_1, \ldots, p_m\}$ is a reduced Cartier divisor. These hypotheses will be dropped in Section 9 where we develop the notion of multiple modifications for higher dimensional varieties.

If we take $Q = E|_D = \bigoplus_i E_{p_i}$, then

$$M(E, q) = E(-D).$$

Following the long exact sequence in cohomology, the global sections of this bundle admit a convenient presentation in terms of the global sections of E:

$$\mathrm{H}^0(E(-D)) = \{\sigma \in \mathrm{H}^0(E) \mid \sigma|_D = 0\}$$

$$= \{\sigma \in \mathrm{H}^0(E) \mid \sigma(p_i) = 0 \text{ for all } i\}.$$

In a generalization of the above example, we keep $D = \{p_1, \ldots, p_m\}$ and consider a subspace $V_i \subset E_{p_i}$ for each $1 \leq i \leq m$. We can assemble a sheaf by taking the quotient by V_i at each of the corresponding fibers:

$$Q = \bigoplus_i E_{p_i} / V_i.$$

By composing the evaluation at D with the natural quotient morphism

$$E \longrightarrow \oplus_i E_{p_i} \longrightarrow Q = \oplus_i E_{p_i} / V_i,$$

we construct a modification which we will denote by $M(E, V_i) = M(E, V_1, \ldots, V_m)$. The points p_i are implicitly understood. Its global sections admit an analogous presentation:

$$H^0(M(E, V_i)) = \{ \sigma \in H^0(E) \mid \sigma(p_i) \in V_i \text{ for all } i \}.$$

Setting all $V_i = 0$ recovers $E(-D)$ as above.

Remark 2.1. Note that we require the divisor $D \subset C$ to be reduced and Cartier. This means that D is contained in the smooth locus C_{sm} of C. For more information about the general case, see Section 9.

3. Interpolation and related properties

Let C be a curve and E a nonspecial vector bundle on it. Consider a point $p \in C_{sm}$ and a subspace of the fiber $V \subset E_p$. Comparing $H^0(M(E,V))$ with $H^0(E)$, we expect the dimension of the former is $\operatorname{codim}(V, E_p)$ less than the dimension of the latter. However, this is not always the case. For example, if $H^0(E) = 0$, it is impossible to have $h^0(M(E,V)) < 0$. More generally, one has to look at the long exact sequence

$$0 \longrightarrow H^0(M(E,V)) \longrightarrow H^0(E) \longrightarrow E_p/V \longrightarrow H^1(M(E,V)) \longrightarrow 0.$$

We are interested in studying the conditions under which the expected dimension is achieved. It may be the case that a specific pair (p, V) fails to satisfy the proposed condition but that behavior is isolated. It is more meaningful to ask the same question for a general point $p \in C_{sm}$ and a general subspace $V \subset E_p$ of a fixed dimension. The notion of interpolation formalizes this idea.

Fix a curve C and a nonspecial vector bundle E on it. Let

$$\lambda = (\lambda_1, \ldots, \lambda_m)$$

be a weakly decreasing sequence of integers such that $0 \le \lambda_i \le \operatorname{rank} E$ for all i, and $\sum_i \lambda_i \le h^0(E)$. Let $\Delta \subset C_{sm}^m$ denote the big diagonal. A point of the Grassmannian bundle

$$\prod_i \operatorname{Gr}(\operatorname{rank} E - \lambda_i, E)$$
$$\downarrow$$
$$C_{sm}^m \setminus \Delta$$

corresponds to the datum

$$(p_i \in C_{sm}, V_i \in \operatorname{Gr}(\operatorname{rank} E - \lambda_i, E_{p_i}))$$

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such that all p_i are distinct. Given this information, we can construct an elementary modification $M(E, V_i)$.

Definition 3.1 (λ-interpolation). We say E satisfies λ-*interpolation* if it is nonspecial and there exists a point on the above Grassmannian bundle such that the corresponding elementary modification E' satisfies

$$h^0(E) - h^0(E') = \sum_i \lambda_i.$$

Remark 3.2. Note that the definition of interpolation requires the inequality to hold for only a single point of the Grassmannian bundle. By upper semi-continuity of h^0, the notion of λ-interpolation is not strengthened if we require the equality $h^0(E) - h^0(E') = \sum_i \lambda_i$ to hold for a general point of one of the components of the Grassmannian bundle.

The (connected) components of the Grassmannian bundle correspond to (connected) components of the base $C^m_{\mathrm{sm}} \setminus \Delta$. If the curve C is reducible, then C_{sm} is not connected.

Definition 3.3 (Interpolation properties). Let us write $h^0(E) = q \cdot \operatorname{rank} E + r$ where $0 \leq r < \operatorname{rank} E$. The following table describes, in increasing strength, all interpolation-type properties we will use.

λ	Term
$((\operatorname{rank} E)^q)$	weak interpolation
$((\operatorname{rank} E)^q, \lambda_{\mathrm{tail}})$	weak interpolation with tail λ_{tail}
$((\operatorname{rank} E)^q, r)$	(regular) interpolation
all admissible tuples λ	strong interpolation

The notion of section stability is related to interpolation.

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Definition 3.4 (Section stability)**.** We call a vector bundle E *section-semistable* if all subbundles $F \subset E$ satisfy the inequality

$$\frac{\mathrm{h}^0(F)}{\mathrm{rank}\, F} \leq \frac{\mathrm{h}^0(E)}{\mathrm{rank}\, E}.$$

We will further call E *section-stable* if the strict version of the inequality above holds for all proper $F \subset E$.

Contrast the notion of section stability with the more classical slope stability.

Definition 3.5 (Slope stability)**.** A vector bundle E is called *slope-semistable* if all subbundles $F \subset E$ satisfy

$$\frac{c_1(F)}{\mathrm{rank}\, F} \leq \frac{c_1(E)}{\mathrm{rank}\, E}.$$

If the strict version of the inequality holds for all proper $F \subset E$, then we call E *slope-stable*.

The slope of a vector bundle refers to the ratio

$$\mu(E) = \frac{c_1(E)}{\mathrm{rank}\, E}.$$

Remark 3.6. As a simple application of Riemann-Roch, we can replace c_1 in the definition of slope stability with the Euler characteristic χ without changing its meaning.

Remark 3.7. Let C be a curve embedded in a smooth ambient space X. We will use its normal bundle $N_{C/X}$ to extend all properties we defined so far to C. For example, we will say that C satisfies interpolation if its normal bundle does so.

Lemma 3.8. *If E satisfies section (semi)stability and $\mathrm{h}^1(E) = 0$, then E also satisfies slope (semi)stability.*

Proof. The result is immediate from the inequality

$$\frac{\chi(F)}{\operatorname{rank} F} \leq \frac{h^0(F)}{\operatorname{rank} F} \leq \frac{h^0(E)}{\operatorname{rank} E} = \frac{\chi(E)}{\operatorname{rank} E}. \qquad \square$$

Our next goal is to illustrate the definitions of interpolation, slope stability, and section stability using $\mathbb{P}^1$.

Example 3.9 (Vector bundles on $\mathbb{P}^1$). Vector bundles on the projective line are simple enough that we can characterize their stability and interpolation properties very concretely. Let us fix a rank n vector bundle $E = \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{P}^1}(a_j)$.

Without loss of generality, we assume the a_j are in weakly decreasing order

$$a_1 \geq \cdots \geq a_n.$$

Considering the subbundle $F = \bigoplus_{j<n} \mathcal{O}_{\mathbb{P}^1}(a_j) \subset E$, we see that slope semistability implies all a_j are equal. It is easy to show this is a necessary and sufficient condition. Furthermore, the only slope-stable vector bundles have rank 1.

Section semistability is equally easy to analyze. Since

$$h^0(\mathcal{O}_{\mathbb{P}^1}(a)) = \max\{1+a, 0\},$$

the condition we are interested in can be stated as follows: for all subsets $J \subset \{1, \ldots, n\}$, we have

$$\frac{1}{|J|} \sum_{j \in J} \max\{a_j + 1, 0\} \leq \frac{1}{n} \sum_j \max\{a_j + 1, 0\}.$$

If E has sections and there exists j' such that $a_{j'} < -1$, taking $J = \{j \neq j'\}$ would violate the inequality. It follows that $a_j \geq -1$ for all j, and then we can replace $h^0(E)$ with the Euler characteristic $\chi(E)$. By Riemann-Roch, this is equivalent to slope semistability. In conclusion, $E = \bigoplus_j \mathcal{O}_{\mathbb{P}^1}(a_j)$ is section-semistable if and only if all a_j are no greater than -1 or all a_j are equal and no less than -1.

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Equivalently, we can also say that either E has no sections or E is slope-semistable and $H^1(E) = 0$.

We are ready to analyze interpolation for bundles on $\mathbb{P}^1$. Assume that $h^1(E) = 0$ which is equivalent to $a_j \geq -1$ for all j. We note that a general $(\lambda_1, \ldots, \lambda_m)$-modification is the same as a general $(\lambda_2, \ldots, \lambda_m)$-modification performed on a general (λ_1)-modification of E.

Let us consider the a_j in $E = \bigoplus_j \mathcal{O}_{\mathbb{P}^1}(a_j)$ as a tableau of height $\operatorname{rank} E$. A general (λ_1)-modification of E has the effect of removing a box from each of the top λ_1 rows. Consider the following example with $\lambda_1 = 3$.

Note that we may need to rearrange the rows of the result to arrive at a tableau. The number of sections drops by λ_1 as long as no row in the new tableau has value less than -1. The vector bundle E satisfies (λ_1)-interpolation as long as all $a_j \geq 0$ for $1 \leq j \leq \lambda_1$.

There is another way to state the condition for (λ_1)-interpolation. Consider the tableau $(b_i) = (a_j + 1)^T$, that is, we add a column of width 1 and height $n = \operatorname{rank} E$, and then take the transpose of the result. We illustrate this operation through an example; the new cells are marked with bullet points.

In these terms (λ_1)-interpolation is equivalent to the inequality $b_1 \geq \lambda_1$. Extending our logic, it is not too hard to state a condition for (λ_1, λ_2)-interpolation. It turns out we need two inequalities: $\lambda_1 \leq b_1$ and $\lambda_1 + \lambda_2 \leq b_1 + b_2$. Using

induction, one can show that E is λ-stable if and only if

$$\sum_{j' \leq j} \lambda_{j'} \leq \sum_{j' \leq j} b_{j'}$$

for all $1 \leq j \leq \operatorname{rank} E$.

Our analysis suggests we should introduce the following partial order.

Definition 3.10 (Dominance order). Let $\lambda = (\lambda_i)$ and $\lambda' = (\lambda'_i)$ be two tableaux. If $\sum_{j \leq i} \lambda'_j \leq \sum_{j \leq i} \lambda_j$ for all i, we will say that λ' is *dominated* by λ and write $\lambda' \trianglelefteq \lambda$.

We will later see that λ-interpolation implies λ'-interpolation if $\lambda' \trianglelefteq \lambda$ (see Theorem 8.2). For now, we restate our result for bundles over $\mathbb{P}^1$.

Proposition 3.11. *Let $E = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_j)$ and $a = (a_j)$. The vector bundle E satisfies λ-interpolation if and only if $\lambda \trianglelefteq (1 + a)^T$.*

We are now ready to analyze weak, regular and strong interpolation. As before, we are assuming $\operatorname{h}^1(E) = 0$. Start by expressing $\operatorname{h}^0(E)$ as

$$\operatorname{h}^0(E) = n + \sum_j a_j = n \cdot q + r$$

where $0 \leq r < n$. Weak interpolation means that E satisfies (n^q)-interpolation, so $a_j + 1 \geq q$ for all j. More strongly, regular interpolation is defined as (n^q, r)-interpolation which implies

$$a_j = \begin{cases} q & \text{if } 1 \leq j \leq r, \\ q - 1 & \text{if } r + 1 \leq j \leq n. \end{cases}$$

In other words, E is as balanced as possible. It is not hard to check that strong interpolation does not impose any additional constraints on E.

Remark 3.12. Vector bundles on $\mathbb{P}^1$ lead to another interesting observation, namely, that λ- and λ'-interpolation are equivalent notions if and only if $\lambda = \lambda'$. It is not

difficult to use this strategy and construct similar examples over a curve of any genus.

This completes our account of interpolation properties over $\mathbb{P}^1$. We return to the general case in order to prove several other statements.

Proposition 3.13 (Reducing λ). *If λ and λ' are such that $\lambda_i \geq \lambda'_i$ for all i, then λ-interpolation implies λ'-interpolation.*

Proof. Consider a vector bundle E on C which satisfies λ-interpolation. Suppose the interpolation condition holds for a specific set of points $p_i \in C$ and subspaces $V_i \subset E_{p_i}$. For each i, we pick a superspace $V_i \subset V'_i \subset E_{p_i}$ such that $\dim V'_i = \operatorname{rank} E - \lambda'_i$. The Snake Lemma implies the modifications $F = M(E, V_i)$ and $F' = M(E, V'_i)$ fit in the following diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & & \bigoplus_i V'_i / V_i & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & \bigoplus_i E_{p_i} / V_i & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & F' & \longrightarrow & E & \longrightarrow & \bigoplus_i E_{p_i} / V'_i & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \bigoplus_i V'_i / V_i & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

The λ-interpolation of E implies $h^1(F) = h^1(E) = 0$. The long exact sequence of the first column implies $h^1(F') = 0$. Putting these observations together, we get

$$h^0(F') = h^0(E) - \dim \bigoplus_i E_{p_i}/V_i' = h^0(E) - \sum_i \lambda_i'. \qquad \square$$

Proposition 3.14. *Let E be a vector bundle for which we write $n = \mathrm{rank}\, E$ and $h^0(E) = n \cdot q + r$ for $0 \le r < n$. Suppose E satisfies weak interpolation. If $F \subset E$ is a subbundle, then*

$$0 \le \frac{h^0(F)}{\mathrm{rank}\, F} \le \frac{h^0(E)}{\mathrm{rank}\, E} + r\left(\frac{1}{\mathrm{rank}\, F} - \frac{1}{\mathrm{rank}\, E}\right).$$

If E further satisfies interpolation, then

$$0 \le \frac{h^0(F)}{\mathrm{rank}\, F} \le \frac{h^0(E)}{\mathrm{rank}\, E} + \min\left\{1, \frac{r}{\mathrm{rank}\, F}\right\} - \frac{r}{\mathrm{rank}\, E}.$$

Proof. The proof of the first statement is analogous to that of the second but uses an easier construction, so we will allow ourselves to only present an argument for the second.

The hypothesis on E implies that there exists an effective divisor D of degree q such that $h^0(E(-D)) = r$. Furthermore, there exists a point $p \in C \setminus D$ and a subspace $V \subset E_p$ of codimension r such that the modification $E' = M(E(-D), V)$ has no sections. We construct the modification $F' = M(F(-D), W)$ for $W = V \cap F_p$. A basic fact about dimension of intersections implies $\max\{0, \mathrm{rank}\, F - r\} \le \dim W$, so

$$\dim(F_p/W) \le \min\{\mathrm{rank}\, F, r\}.$$

The constructions above ensure that there is a injection $F' \to E'$, hence $h^0(F') \le h^0(E') = 0$. Furthermore, the short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F|_D \oplus F_p/W \longrightarrow 0$$

implies

$$h^0(F) \leq h^0(F') + h^0(F|_D \oplus F_p/W) = q \cdot \operatorname{rank} F + \min\{\operatorname{rank} F, r\}.$$

Dividing by rank F, we arrive at the desired inequality. □

Setting $r = 0$ in the proof of Proposition 3.14, we obtain the following result.

Corollary 3.15. *If E satisfies weak interpolation and its rank divides $h^0(E)$, then E is section-semistable.*

Lemma 3.16. *If E is a vector bundle whose global sections span a subbundle of rank r, then E satisfies (r)-interpolation.*

Proof. Let $F \subset E$ be the subbundle spanned by global sections (the saturation of the subsheaf generated by global sections). If p is a general point on the underlying curve, then the evaluation morphism $F \to F_p$ is surjective on global sections. By picking a subspace $V \subset E_p$ complimentary to F_p we arrive at the following diagram with two exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F(-p) & \longrightarrow & F & \longrightarrow & F_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & M(E,V) & \longrightarrow & E & \longrightarrow & E_p/V & \longrightarrow & 0
\end{array}
$$

By construction $F \to E$ is an isomorphism on global sections, so $E \to E_p/V$ must be surjective on global sections which proves the required claim. □

Lemma 3.17. *If E satisfies λ-interpolation, then E also satisfies $(\lambda, 1^k)$-interpolation for $k = h^0(E) - \sum_i \lambda_i$.*

Proof. If F denotes a general λ-modification of E, then it suffices to show F satisfies $(1^{h^0(F)})$-interpolation. Furthermore, it suffices to show that if $h^0(F) \geq 1$, then F satisfies (1)-interpolation. By considering a non-zero section of F, it follows that

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the subbundle spanned by global sections has rank at least 1 and we can apply Lemma 3.16. $\qquad\square$

Corollary 3.18. *For rank* 2 *bundles, weak, regular and strong interpolation are equivalent notions.*

Proof. Weak interpolation means that E satisfies (2^q)-interpolation where $q = \lfloor h^0(E)/2 \rfloor$. To demonstrate strong interpolation, we need to consider all tableaux $(2^a, 1^b)$ such that $2a + b \leq h^0(E)$. Since $a \leq q$, Proposition 3.13 implies that E satisfies (2^a)-interpolation. To arrive at $(2^a, 1^b)$ we apply Lemma 3.17. $\qquad\square$

In fact, the equivalence of strong and regular interpolation extends to arbitrary rank vector bundles; we postpone the discussion of this result to Section 8.

The following two diagrams summarize the implications we described so far.

In genus 0

In general

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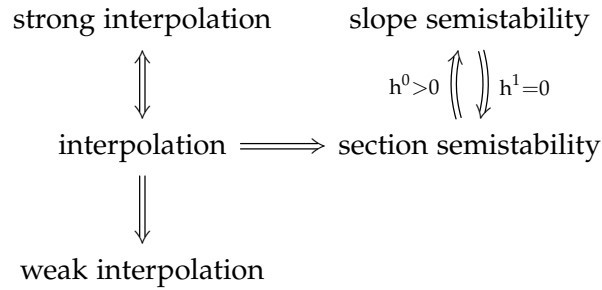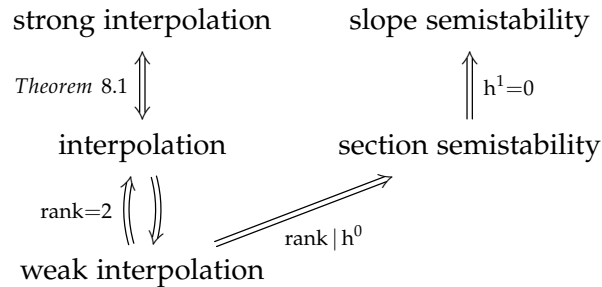
4. Interpolation via nodal curves

We aim to prove the following two results.

Theorem 4.1. *Let C and D be curves in some ambient projective space $\mathbb{P}^n$ which meet transversely in a single point p, smooth in each curve. Assume*

(a) *C satisfies λ-interpolation,*

(b) *D satisfies $(n-1, \mu)$-interpolation.*

Then, up to changing the point of attachment on D through a rigid motion (via the action of $\mathrm{PGL}(n+1)$ on $\mathbb{P}^n$), the union curve $X = C \cup D$ satisfies (λ, μ)-interpolation.

Theorem 4.2. *Let $C \subset \mathbb{P}^n$ be a curve, and L a secant line which meets C transversely in two points p, q, both smooth in C. If C satisfies λ-interpolation, then these properties also hold for the union $X = C \cup L$.*

If the two tangent lines to C at the points of $C \cap L$ are skew, then X satisfies $(\lambda, 2)$-interpolation. This condition holds for a general secant line as long as C is not planar.

The proofs use techniques similar to [11] and [15], so we start by recalling several preliminary results these sources present.

**Proposition 4.3.** *Let $X \subset \mathbb{P}^n$ be a nodal curve, and let $\nu\colon X' \to X$ denote its normalization. Then the normal bundles of the two curves fit in a short exact sequence*

$$0 \longrightarrow N_{X'} \longrightarrow \nu^* N_X \longrightarrow \nu^* T_X^1 \longrightarrow 0,$$

where T_X^1 stands for the Lichtenbaum-Schlessinger T^1-functor of X. It is supported on $S = \mathrm{Sing}\, X$, so we will also denote it by T_S^1.

Remark 4.4. Let $i\colon X \to \mathbb{P}^n$ denote the inclusion morphism as in the previous result. While the composition $i \circ \nu\colon X' \to \mathbb{P}^n$ is not an embedding, it is an embedding locally on X'. It follows that the differential $d(i \circ \nu)\colon T_{X'} \to (i \circ \nu)^* T_{\mathbb{P}^n}$

an injective morphism of vector bundles, so we can define the normal bundle $N_{X'}$
as its cokernel.

Corollary 4.5. *Let* $X = C \cup D$ *be the union of two smooth curves* C, D *intersecting transversely in* S. *Then there is a short exact sequence*

$$0 \longrightarrow N_C \longrightarrow N_X|_C \longrightarrow T_S^1 \longrightarrow 0.$$

We are now ready to substantiate our claims.

Proof of Theorem 4.1. First pick a λ-modification datum $V_i \subset N_{C,p_i}$ away from $C \cap D = \{p\}$, so $h^0(M(N_C, V_i)) = h^0(N_C) - \sum_i \lambda_i$. We do the same for D but only using a μ-modification: $W_j \subset N_{D,q_j}$ is a datum away from p such that $h^0(M(N_D, W_j)) = h^0(N_D) - \sum_j \mu_j$ and $M(N_D, W_j)$ satisfies $(n-1)$-interpolation.

After applying a modification to the sequence from Corollary 4.5, we obtain the following diagram with two exact rows and two exact columns.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M(N_C, V_i) & \longrightarrow & M(N_X|_C, V_i) & \longrightarrow & T_S^1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N_C & \longrightarrow & N_X|_C & \longrightarrow & T_S^1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & \oplus_i N_{C,p_i}/V_i & = & \oplus_i N_{C,p_i}/V_i & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

A diagram chase argument shows that all sheaves present have $H^1 = 0$, so the diagram remains exact after applying H^0 to it. The exactness of the second column

implies

$$h^0(M(N_X|_C, V_i)) = h^0(N_X|_C) - \sum \lambda_i.$$

The analogous argument for D produces

$$h^0(M(N_X|_D, W_j)) = h^0(N_X|_D) - \sum \mu_j.$$

Next, we need to relate $N_X|_C$ and $N_X|_D$ to N_X. Tensor the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_D \longrightarrow \mathcal{O}_p \longrightarrow 0,$$

with the morphism $M(N_X, V_i, W_j) \to N_X$.

$$
\begin{array}{ccccccccc}
0 & \to & M(N_X, V_i, W_j) & \to & M(N_X|_C, V_i) \oplus M(N_X|_D, W_j) & \to & M(N_X, V_i, W_j)|_p & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow {\scriptstyle \cong} & & \\
0 & \longrightarrow & N_X & \longrightarrow & N_X|_C \oplus N_X|_D & \longrightarrow & N_X|_p & \longrightarrow & 0
\end{array}
$$

If we assume $H^1(M(N_X, V_i, W_j)) = 0$, then $H^1(N_X) = 0$ and we deduce

$$h^0(M(N_X, V_i, W_j)) = h^0(M(N_X|_C, V_i)) + h^0(M(N_X|_D, W_j)) - h^0(M(N_X, V_i, W_j)|_p)$$

$$= h^0(N_X|_C) - \sum \lambda_i + h^0(N_X|_D) - \sum \mu_j - h^0(N_X|_p)$$

$$= h^0(N_X) - \sum \lambda_i - \sum \mu_j,$$

which proves that X satisfies (λ, μ)-interpolation.

We are left to show that $H^1(M(N_X, V_i, W_j)) = 0$. From the previous diagram, it suffices to show that

$$M(N_X|_D, W_j) \longrightarrow M(N_X, V_i, W_j)|_p = M(N_X, W_j)|_p$$

is surjective on H^0. This follows if we show that

$$M(N_D, W_j) \longrightarrow M(N_D, W_j)|_p$$

is surjective on H^0, but that is a consequence of the fact $M(N_D, W_j)$ satisfies $(n - 1)$-interpolation. A rigid motion of one of the curves relative to the other may be necessary since the point of modification p for D has to be general. $\square$

Taking D to be a line leads to a very useful result.

Corollary 4.6. *Let $C \subset \mathbb{P}^n$ be a curve, and L a line which meets C transversely in a single point p smooth on C. If C satisfies λ-interpolation, then $X = C \cup L$ satisfies $(n - 1, \lambda)$-interpolation.*

Proof. The normal bundle to L is $N_L \cong \mathcal{O}_L(1)^{\oplus(n-1)}$, so it satisfies $((n - 1)^2)$-interpolation. No reattachment is necessary since two points on a line have no moduli. $\square$

We need several "selection" results to complete the proof of Theorem 4.2. These statements allow us to choose objects (e.g., secant lines, tangent lines) with sufficiently nice properties.

Proposition 4.7. *Let $C \subset \mathbb{P}^n$ be a reduced irreducible curve and $L \subset \mathbb{P}^n$ be a line. If C is non-planar, then a general tangent line of C does not meet L.*

Proof. For contradiction, assume all tangent lines of C meet L. There exists $p \in L \setminus C$, otherwise $L \subset C$ and C is planar. We consider the projection $\pi \colon \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$ away from p. Let $\ell \in \mathbb{P}^{n-1}$ be the image of L and C' the image of C. The incidence condition for C and L implies that all tangent lines of C' pass through the point ℓ. Hartshorne calls such curves *strange* and proves they are either a line (in any characteristic) or a conic (in characteristic 2) [10, Theorem IV.3.8] (the original result is attributed to [17]). We are excluding the latter case, so C' must be a line. Then C lies in the preimage of this line which is a plane, contradicting our hypotheses. $\square$

Corollary 4.8. *Let $C \subset \mathbb{P}^n$ be a reduced irreducible curve. If C is non-planar, then any finite set of general tangent lines is pairwise disjoint.*

Proof. We induct on the number of tangent lines we need to construct. In the inductive step we invoke Proposition 4.7 for each of the previously chosen lines. $\square$

Proposition 4.9. *Let $C \subset \mathbb{P}^n$ be a reduced irreducible curve and $L \subset \mathbb{P}^n$ a line. If C is non-planar, then a general secant line does not meet L.*

Proof. Since C is non-planar, there exists a point $p \in C \setminus L$. If all secants through p meet L, then C is contained in the plane spanned by L and p which is a contradiction. $\square$

Corollary 4.10. *Let $C \subset \mathbb{P}^n$ be a reduced irreducible curve. If C is non-planar, then any finite set of general secant lines is pairwise disjoint.*

Proof. The argument is analogous to Corollary 4.8. $\square$

Proposition 4.11 (Trisecant lemma). *Let C be a smooth curve. If C is non-planar, then*

(a) *a general secant line is not trisecant, and*

(b) *a general tangent line intersects C in a single point only.*

Proof. This is a combination of [10, Proposition IV.3.8] and [10, Theorem IV.3.9]. $\square$

The next result can be stated more easily using the following notion.

Definition 4.12. Let $C \subset \mathbb{P}^n$ be a smooth curve and E a subbundle of the restriction of the ambient tangent bundle $T_{\mathbb{P}^n}|_C$. Consider a second curve D which shares no components with C and the intersection $C \cap D$ lies in the smooth locus of D.

We will say that D is *away* from E at a point $p \in C \cap D$ if the tangent line $T_p D$ is not contained in the fiber E_p. We will say that D is *away* from E if the previous condition holds at all points of intersection $p \in C \cap D$.

If instead E is a subbundle of the normal bundle N_C, we make the same definitions by replacing it with its preimage in $T_{\mathbb{P}^n}|_C$.

Proposition 4.13. *Let $C \subset \mathbb{P}^n$ be a smooth irreducible curve and $E \subset T_{\mathbb{P}^n}|_C$ a proper subbundle, that is, its rank is strictly less than n. If C spans the ambient space $\mathbb{P}^n$, then a general secant line is away from E.*

Proof. Let $\mathrm{pr}_1 \colon C \times C \to C$ denote the projection to the first factor. We construct a line bundle $M \subset \mathrm{pr}_1^* T_{\mathbb{P}^n}|_C$ as follows. Consider a point $(p, q) \in C \times C$. If $p \neq q$, let $L_{p,q}$ denote the secant line joining p and q, and if $p = q$, then $L_{p,p}$ will stand for the tangent line of C at p. The bundle M is defined by saying its fiber over (p, q) is $T_{L_{p,q}, p} \subset T_{\mathbb{P}^n, p}$. This construction can be made precise by assembling the lines $L_{p,q}$ into a $\mathbb{P}^1$-bundle over $C \times C$.

Note that $L_{p,q}$ is away from E at p if and only if $M_{p,q}$ is not contained in E_p. Let $U \subset C \times C$ denote the open locus where M is transverse to $\mathrm{pr}_1^* E$. Since the problem is symmetric in the two points on the secant, it is easy to reverse it. Let $\sigma \colon C \times C \to C \times C$ denotes the swap morphism. A secant line $L_{p,q}$ is away from E if and only if (p, q) lies in $U \cap \sigma(U)$.

It suffices to show that U is not empty to infer that neither is $U \cap \sigma(U)$ which completes our argument. For contradiction, assume that $M \subset \mathrm{pr}_1^* E$. Fix a point $p \in C$ and let H be the unique linear space through p of dimension rank E such that $T_{H,p} = E_p$. Consider another point $q \in C$ and the secant (tangent) line $L_{p,q}$. Since $M \subset \mathrm{pr}_1^* E$ and $M_{p,q} = T_{L_{p,q}, p}$, it follows that $L_{p,q}$ is contained in H. A fortiori, the point q lies in H, hence so does the entire curve C. We are given that C spans the ambient $\mathbb{P}^n$, so $H = \mathbb{P}^n$ contradicting the properness of E. □

We are now ready to prove the second main result of this section.

Proof of Theorem 4.2. We proceed analogously to the Proof of Theorem 4.1. Pick a modification datum $V_i \subset N_{C,p_i}$ away from $S = C \cap L$, so $h^0(M(N_C, V_i)) = h^0(N_C) - \sum_i \lambda_i$ and $h^1(M(N_C, V_i)) = 0$. Since N_X and N_C are identical on $C \setminus S$, this gives us a modification datum for N_X and we aim to show the analogous statement $h^0(M(N_X, V_i)) = h^0(N_X) - \sum_i \lambda_i$.

First, we deduce

$$h^0(M(N_X|_C, V_i)) = h^0(N_X|_C) - \sum \lambda_i.$$

If we assume $H^1(M(N_X, V_i)) = 0$, then

$$h^0(M(N_X, V_i)) = h^0(N_X) - \sum \lambda_i,$$

which is the desired interpolation property for N_X. It suffices to show the morphism $H^0(N_X|_L) \to H^0(N_X|_S)$ is surjective. But $N_L \cong \mathcal{O}_L(1)^{\oplus(n-1)}$ and $N_X|_L \cong \mathcal{O}_L(1)^{\oplus(n-3)} \oplus \mathcal{O}_L(2)^{\oplus 2}$ or $N_X|_L \cong \mathcal{O}_L(1)^{\oplus(n-2)} \oplus \mathcal{O}_L(3)$. In each of the cases, the degrees of all components are bounded from below by 1 and the statement holds.

For the second claim we need to prove, consider a secant line L to C whose two tangent lines are skew. Corollary 4.8 shows this condition is satisfied for a general secant line. If we assume C satisfies λ-interpolation, our goal is to show that $X = C \cup L$ satisfies $(\lambda, 2)$-interpolation via the addition of a point and codimension 2 space on L. It suffices to show that $N_X|_L \cong \mathcal{O}_L(1)^{\oplus(n-3)} \oplus \mathcal{O}_L(2)^{\oplus 2}$ and not $N_X|_L \cong \mathcal{O}_L(1)^{\oplus(n-2)} \oplus \mathcal{O}_L(3)$. The normal bundle $N_X|_L$ remains unchanged if we replace C with the union $L_1 \cup L_2$ of the two tangent lines to C at $C \cap L$. Let P_1 denote the 2-plane spanned of L and L_1, P_2 the 2-plane spanned by L and L_2, and P the 3-plane spanned by L, L_1 and L_2. We can use these to show that

$$N_X|_L \cong N_{(L \cup L_1)/P_1}|_L \oplus N_{(L \cup L_2)/P_2}|_L \oplus N_{P/\mathbb{P}^n}|_L.$$

By studying the union of two distinct lines in the plane, we arrive at the conclusion that the three summands are isomorphic to $\mathcal{O}_L(2)$, $\mathcal{O}_L(2)$ and $\mathcal{O}_L(1)^{\oplus(n-3)}$ respectively. This completes our claim. $\qquad\square$

5. Remarks on Hilbert schemes

The results from Section 4 allow us to prove interpolation for some special reducible nodal curves. Building on knowledge about simpler curves, we can use these ideas to deduce interpolation properties for a wider class of reducible nodal curves. On the other hand, we would like to prove statements about smooth curves. In this section we demonstrate how to transfer information using a family of curves, hence opening the opportunity for the degeneration arguments customary in algebraic geometry. Hilbert schemes play a central role in this discussion.

For any triple d, g and n, we can construct the Hilbert scheme $\mathcal{H}_{d,g,n}$ whose points parametrize curves of degree d and genus g in $\mathbb{P}^n$. We consider the locus of smooth irreducible curves $\mathcal{I}'_{d,g,n} \subset \mathcal{H}_{d,g,n}$, and its closure $\mathcal{I}_{d,g,n}$ also referred to as the *restricted Hilbert scheme*.

Suppose we have a nodal curve C such that $[C] \in \mathcal{H}_{d,g,n}$. In order to use C to reach any statements about smooth curves, we need to verify it lies in $\mathcal{I}_{d,g,n}$. Hartshorne and Hirschowitz consider this problem in [11]. We recall several useful statements from their work.

Definition 5.1. We call a curve $C \subset \mathbb{P}^n$ *strongly smoothable* if there exists a flat family $\mathcal{C} \to \mathbb{P}^n \times B$ with special fiber C such that

(a) the general member $\mathcal{C}_t$ is smooth,

(b) the total space $\mathcal{C}$ is smooth and irreducible, and

(c) the base B is smooth.

Remark 5.2. Being strongly smoothable is stronger than containment in $\mathcal{I}_{d,g,n}$, also called (regular) smoothability.

Proposition 5.3. *Let C be a smooth curve in $\mathbb{P}^n$ with $\mathrm{H}^1(N_C) = 0$. If L is a line meeting C transversely in one or two points, then $\mathrm{H}^1(N_X) = 0$ and $X = C \cup L$ is strongly smoothable.*

Proposition 5.4. *Let $X = C \cup L_1 \cup \cdots \cup L_a \subset \mathbb{P}^n$ be the union of a smooth irreducible curve C with lines, each meeting C transversely in a single point. If $a \geq h^1(N_C) + 1$, then X is strongly smoothable. If in addition $H^1(N_C) = 0$ or the lines L_i are general, then $H^1(N_X) = 0$.*

By studying the proofs of these statements, there is a simple way to combine some aspects from each.

Proposition 5.5. *Let $C \subset \mathbb{P}^n$ be a smooth irreducible curve, and $L_1, \ldots, L_a, L'_1, \ldots, L'_b$ be lines such that*

(a) *each L_i meets C in a single point,*

(b) *each L'_j is a secant to C (it meets C in exactly two points),*

(c) *all lines L_i, L'_j are disjoint, and*

(d) *any line intersects C transversely.*

Consider the union curve $X = C \cup L_1 \cup \cdots \cup L_a \cup L'_1 \cup \cdots \cup L'_b$. If $H^1(N_C) = 0$, then $H^1(N_X) = 0$ and X is strongly smoothable.

It is often convenient to refer to a general (smooth) curve. Since we are discussing the Hilbert scheme of curves, this is an appropriate place to elaborate on the meaning of "general". If $\mathcal{I}_{d,g,n}$ is irreducible, then referring to a general curve with the given characteristics in unambiguous. If however that is not the case, we might want to make separate claims for each component. In such cases, we avoid using the adjective "general" and attempt to be more specific. The good news is most cases of interest fall in the former category.

Combining the irreducibility of $\mathcal{M}_g$ (see [5]) with Riemann-Roch yields an irreducibility statement for $\mathcal{I}_{d,g,n}$. For further discussion of this and other related results, we refer to [8].

Theorem 5.6. *If $d \geq 2g - 1$ and $d \geq n + g$, the space $\mathcal{I}_{d,g,n}$ is irreducible.*

Note that when $g = 1$, the hypotheses of Theorem 5.6 are always satisfied for an elliptic curve which spans the ambient space.

Space curves are another interesting source of examples. There has been a sequence of results extending the range from Theorem 5.6. The strongest version we are aware of is due to S.J. Keem and C. Kim.

Theorem 5.7 (Kim-Keem [14, 13])**.** *The space $\mathcal{I}_{d,g,3}$ is irreducible in the following cases:*

(a) $d \geq g + 3$ and $g \geq 0$,

(b) $d = g + 2$ and $g \geq 5$,

(c) $d = g + 1$ and $g \geq 9$.
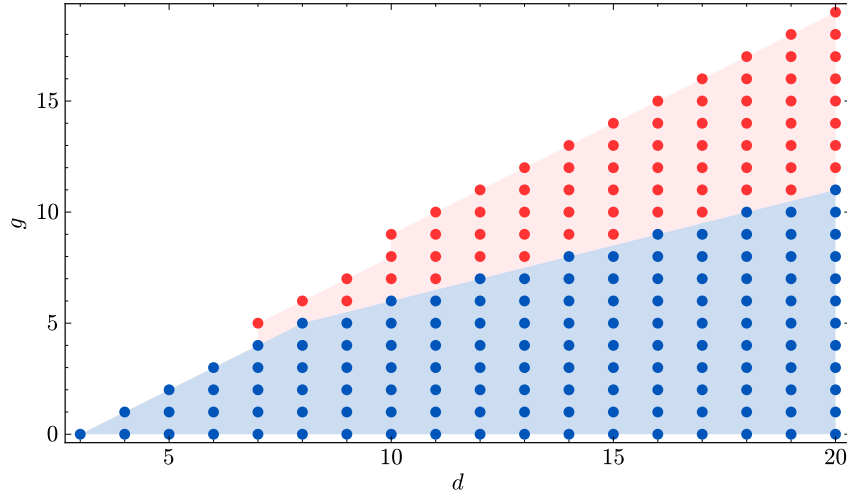
FIGURE 1. The shaded region of the (d, g)-plane illustrates the cases for which we know $\mathcal{I}_{d,g,3}$ is irreducible. Theorem 5.6 corresponds to the blue region (darker in grayscale). The additional part covered by Theorem 5.7 is in red (lighter in grayscale).

The most useful result of this section says that interpolation is an open property. We can then use degeneration arguments to demonstrate interpolation.

Theorem 5.8. *Let $\pi \colon \mathcal{C} \to B$ be a flat family of curves and E a vector bundle over $\mathcal{C}$. For each point $b \in B$, let $\mathcal{C}_b = \pi^{-1}(b)$ denote the curve over b and $E_b = E|_{\mathcal{C}_b}$ be the*

associated vector bundle. Assume that both B and C are irreducible, and in addition, B is reduced. If there exists a point $b_0 \in B$ such that

(a) *the curve C_{b_0} is reduced,*

(b) *the value $H^0(E_{b_0})$ is minimal among $H^0(E_b)$ for $b \in B$, and*

(c) *vector bundle E_{b_0} satisfies λ-interpolation,*

then E_b satisfies λ-interpolation for a general point $b \in B$.

Proof. We start by disposing of the non-reduced curves in the family. Since C_{b_0} is reduced and $\pi \colon C \to B$ is flat, then the locus of reduced curves $B' \subset B$ is a non-empty open. In what follows, we replace the morphism π with its restriction

$$\pi|_{\pi^{-1}(B)} \colon \pi^{-1}(B') \longrightarrow B'.$$

The key ingredient in this argument is upper semi-continuity of cohomology. To present a more readable account, we will first assume $\lambda = (\lambda_1)$ and later explain how the general case is handled.

The strategy is to construct a space whose points parametrize possible modifications. Let $U \subset C$ denote the smooth locus of π. In other words, U consists of points p such that p is a smooth point of the fiber $C_{\pi(p)}$ it belongs to. The space U models all points at which a modification can be performed. Given $p \in U$, the datum of a (λ_1)-modification also requires a subspace $V_1 \subset E_p$ of codimension λ_1. The configurations (p, V_1) are parametrized by the total space of the Grassmannian bundle $\gamma \colon G = \mathrm{Gr}(\mathrm{rank}\, E - \lambda_1, E|_U) \to U$. Let $\gamma^* E \to Q$ denote the universal quotient over G. When there is no ambiguity, we will use E to refer to the restriction $E|_U$.

Since we are not assuming the total space of the family C is irreducible, it is possible that the smooth locus U of $\pi \colon C \to B$ is reducible and disconnected (see Remark 3.2). By hypothesis, there exists a modification datum over b_0 such

that E_{b_0} satisfies λ-interpolation. Let U' denote the (connected) component of U containing the base of this datum and set $G' = \gamma^{-1}(G')$. Since $\gamma|_{G'}: G' \to U'$ is a Grassmannian bundle and U' is connected and irreducible, then so is G'. For the sake of simplicity, we replace $\gamma: U \to G$ with $\gamma|_{G'}: G' \to U'$ for the rest of the argument. This will allow us to refer to general points without any ambiguity.

We will use the following diagram of morphisms in the rest of the argument: p_0 and p_1 denote projections on the first and second factors respectively, q is id_C in the first factor and the Grassmannian projection $\gamma: G \to U$ in the second, $q_i = p_i \circ q$ for $i = 0, 1$, and r is projection onto the second factor.

The diagonal $\Delta \subset U \times_B U$ sits naturally in $C \times_B U$ where it is a Cartier divisor. Note that the diagonal of $C \times_B C$ is not Cartier, and this is the precise reason we need to restrict C to U on the right. There is a natural isomorphism $(p_0^* E)|_\Delta \cong (p_1^* E)|_\Delta$. The composition

$$q_0^* E = q^* p_0^* E \longrightarrow q^*(p_0^* E)|_\Delta \cong q^*(p_1^* E)|_\Delta = (q_1^* E)|_{q^{-1}(\Delta)} = (r^* \gamma^* E)|_{q^{-1}(\Delta)} \longrightarrow (r^* Q)|_{q^{-1}(\Delta)}$$

is surjective and we will use E' to denote its kernel. We generalized the construction of modifications to a family, so the expectation is that for every $g \in G$ the restriction $E'|_{r^{-1}(g)}$ is isomorphic to the modification $M(E_{\pi \circ \gamma(g)}, g)$ over $C_{\pi \circ \gamma(g)}$. We ensured this the case since $\gamma(g) \in U \subset C$.

Since $\Delta \subset C \times_B U$ is Cartier, it follows that E' is locally free, and in particular, E' is flat over G. Cohomology and base change implies that the function $\varphi: G \to$

$\mathbb{Z}_{\geq 0}$ sending g to

$$h^0(r_* E'|_g) = h^0(E'|_{r^{-1}(g)}) = h^0(M(E_{\pi \circ \gamma(g)}, g))$$

is upper semi-continuous. In a similar note, sending $b \in B$ to $h^0(E_b)$ is also upper semi-continuous. Since B is irreducible, $h^0(E_b)$ attains a minimum value c over a dense open $V \subset B$. The short exact sequence defining a modification implies that φ is bounded from below by $c - \lambda_1$. We already know there is a point $g \in (\pi \circ \gamma)^{-1}(b_0)$ which satisfies $\varphi(g) = c - \lambda_1$, so this minimum is attained over a dense open G' in G. The original claim amounts to saying the image $\pi \circ \gamma(G')$ in B is open. This follows from Chevalley's Theorem on constructible sets (see [10, p. 94])

The argument above demonstrates the desired claim when $\lambda = (\lambda_1)$. In the general case $\lambda = (\lambda_1, \ldots, \lambda_m)$ we replace the second factor U with the complement V of the big diagonal in the m-fold product $U \times_B \cdots \times_B U$. Likewise, the moduli space of all modifications will be a product of m Grassmannian bundles

$$G = \prod_{i=1}^m \mathrm{Gr}(\mathrm{rank}\, E - \lambda_i)$$

and $V \subset U \times_B \cdots \times_B U$ will parametrize points $(p_1, \ldots, p_m)$ such that all p_i are distinct and they are smooth points of the corresponding fiber. Other than these changes, the argument goes through without any major modifications. $\qquad \square$

Corollary 5.9. *Let $\mathcal{I}$ be an irreducible component of some reduced Hilbert scheme $\mathcal{I}_{d,g,n}$. Consider a curve C in $\mathcal{I}$ such that $\mathrm{H}^1(N_C) = 0$. If C satisfies λ-interpolation, then the general curve in $\mathcal{I}$ also satisfies λ-interpolation.*

Proof. Let $\pi \colon \mathcal{C} \to \mathcal{I}$ be the universal curve over $\mathcal{I}$. Since $\mathcal{I}$ is irreducible and the general member fiber of π is smooth, it follows that $\mathcal{C}$ is irreducible. Since $[C] \in \mathcal{H}_{d,g,n}$ is a smooth point parametrizing a smoothable curve, we also know that B

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is reduced and $\mathcal{C}$ is irreducible. In addition, $\mathrm{H}^1(N_C) = 0$ so the value $\mathrm{H}^0(N_C)$ is minimal. The desired claim follows directly by applying Theorem 5.8. $\qquad\square$

6. Elliptic curves

We investigate elliptic curves as an application of degeneration arguments to interpolation. Instead of stating the most general result first, our philosophy is to start with a simpler one, identify the points which can be improved, and build towards stronger statements gradually.

Corollary 6.1. *A general elliptic normal curve of degree $d \geq 7$ satisfies weak interpolation.*

Proof. The idea behind this proof is to study a degeneration consisting of a rational normal curve $C \subset \mathbb{P}^{d-1}$ and a secant line L. The normal bundle of C is perfectly balanced $N_C \cong \mathcal{O}_{\mathbb{P}^1}(d+1)^{\oplus(d-2)}$ (see Remark 6.2), so $H^1(N_C) = 0$ and it satisfies $((d-2)^{d+2})$-interpolation. Theorem 4.2 then implies that $X = C \cup L$ satisfies weak interpolation. In fact, X satisfies the stronger $((d-2)^{d+2}, 2, 2)$-interpolation.

The rank of the normal bundle N_X is $d-2$ and $h^0(N_X) = d^2$. Weak interpolation is equivalent to $((d-2)^{d+2})$-interpolation as long as

$$h^0(N_X) - (d-2)(d+2) < \operatorname{rank} N_X.$$

After simplification the inequality becomes $d > 6$, hence the hypothesis on d.

By Proposition 5.3 the curve X is smoothable and $H^1(N_X) = 0$ (hence $[X]$ is a smooth point of the corresponding Hilbert scheme). We covered the hypotheses of Corollary 5.9, so the interpolation statement holds for a general elliptic normal curve. $\square$

Remark 6.2. Corollary 4.8 provides a strengthening of Theorem 4.2 which implies that a general elliptic normal curve satisfies weak interpolation with tail (2). In fact, the high symmetry of rational normal curves implies that all pairs of points are alike, so we do not need to choose a general secant line.

By studying the normal bundle of rational normal curves, we can extend the tail in question to $(2,2)$. Let $C \subset \mathbb{P}^{d-1}$ be a rational normal curve. The splitting of its normal bundle $N_C = N_{C/\mathbb{P}^{d-1}} \cong \mathcal{O}_{\mathbb{P}^1}(d+1)^{\oplus(d-2)}$ can be realized as follows. Pick distinct points $q_1, \ldots, q_{d-2}$ on C. For each q_j we consider the cone S_j of C over q_j. It is clear that S_j is smooth away from q_j, so the normal bundle $N_j = N_{C/S_j}$ is a line bundle over $C \setminus q_j$ sitting in N_C. The curve-to-projective extension theorem [10, I.6.8] allows us to extend N_j and its inclusion over q_j. At a point $q \neq q_j$, the fiber of N_j corresponds to the 2-plane through q_j which contains the tangent line to q. When $q = q_j$ the fiber of N_j comes from the osculating plane to C at q_j. By studying the bundles N_j in more detail one can show that each has degree $d+1$, and they furnish a splitting $N_C \cong \bigoplus_{j=1}^{d-2} N_j \cong \mathcal{O}_{\mathbb{P}^1}(d+1)^{\oplus(d-2)}$.

Let L be a secant line L to C through two of the points q_j, say q_1 and q_2. We consider a modification $N = N_{C \cup L}|_C(-\sum_i p_i)$ given by $d+2$ general points p_i on C. To show that $C \cup L$ satisfies weak interpolation with tail $(2,2)$ it suffices to show that $N \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ and not $N \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(d+1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. This follows from the fact that $N_1 \oplus N_2$ is a subbundle of N_C.

It is possible to generalize the argument from Corollary 6.1 to a much broader class of elliptic curves.

Theorem 6.3. *A general elliptic curve of degree d in $\mathbb{P}^n$ satisfies weak-interpolation if*

(a) $d \geq 7$,

(b) $d \geq n+1$, *and*

(c) *the remainder of $2d$ modulo $n-1$ lies between 3 and $n-2$ inclusive.*

The proof of Theorem 6.3 rests on the following result.

Theorem 6.4 (Ran [15]). *A general spanning rational curve satisfies interpolation.*

Remark 6.5. A general rational $C \subset \mathbb{P}^n$ of degree d is spanning as long as $d \geq n$.
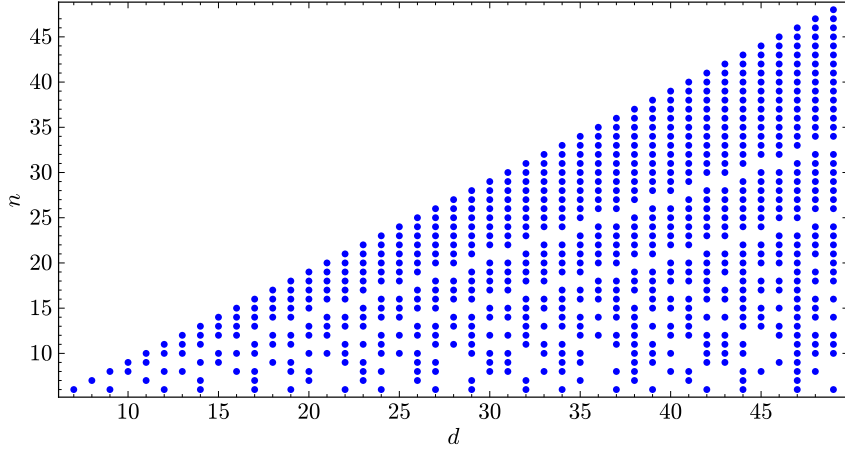
FIGURE 2. Values of d and n which satisfy the hypotheses of Theorem 6.3

Remark 6.6. Let us derive the precise splitting type of a general degree d rational curve $C \subset \mathbb{P}^n$. We assume $d \geq n$. The Euler sequence restricted to C reads

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n}|_C \longrightarrow 0,$$

so we have equality of degrees

$$\deg T_{\mathbb{P}^n}|_C = \deg \mathcal{O}_C^{\oplus(n+1)} = (n+1)d.$$

The normal bundle of C sits in the sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^n}|_C \longrightarrow N_C \longrightarrow 0,$$

and we deduce

$$\deg N_C = \deg T_{\mathbb{P}^n}|_C - \deg T_C = (n+1)d - 2.$$

If we write $2d - 2 = q \cdot (n-1) + r$ and $0 \leq r < n - 1$, then Theorem 6.4 implies

$$N_C \cong \mathcal{O}_{\mathbb{P}^1}(d + q + 1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}(d + q)^{\oplus(n-1-r)}.$$

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Proof of Theorem 6.3. We consider the union of a general degree $d - 1$ rational curve C and a secant line L. Following Remark 6.6, we deduce

$$N_C = \mathcal{O}_{\mathbb{P}^1}(d+q)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}(d+q-1)^{\oplus(n-1-r)}$$

where $2d - 4 = q \cdot (n-1) + r$ for $0 \le r < n - 1$. Theorem 4.2 implies $C \cup L$ satisfies $((n-1)^{d+q})$-interpolation. This is the definition of weak interpolation for $C \cup L$ as long as

$$\left\lfloor \frac{h^0(N_C)}{\operatorname{rank} N_C} \right\rfloor = \left\lfloor \frac{h^0(N_{C \cup L})}{\operatorname{rank} N_{C \cup L}} \right\rfloor.$$

We know that $h^0(N_{C \cup L}) = h^0(N_C) + 4$, so we can rewrite the equality as

$$d + q = d + q + \left\lfloor \frac{r+4}{n-1} \right\rfloor.$$

Since $2d - 4 = q \cdot (n-1) + r$, the condition $r + 4 < n - 1$ is equivalent to the cases

$$2d \equiv 4, \dots, n - 2 \ (\operatorname{mod} n - 1).$$

It remains to show our claim holds when $2d \equiv 3 \ (\operatorname{mod} n - 1)$. Consider the subbundle

$$E = \mathcal{O}_{\mathbb{P}^1}(d+q)^{\oplus r} \subset N_C$$

consisting of all pieces of the larger degree $d + q$. Alternatively, we can construct E as the largest proper subbundle in the Harder-Narasimhan filtration of N_C. Our choice of r ensures E is proper of corank 1. Proposition 4.13 shows that the general secant is away from E. The decomposition of $N_{C \cup L}|_C$ is obtained from the decomposition of N_C by adding 1 to the degrees of two of the components (possibly the same one). By choosing L to be away from E, we are ensuring that the single component of lower degree $d - q - 1$ is bumped up, so $C \cup L$ satisfies interpolation.

The statement about weak interpolation of a general elliptic curve follows by applying Corollary 5.9. $\qquad\square$

Remark 6.7. The congruence condition on $2d \bmod n-1$ excludes all cases when $n = 3$ and 4. We will later show that the general elliptic curve in $\mathbb{P}^3$ satisfies interpolation as long as $d \geq 4$ (see Theorem 7.1).

7. Curves in $\mathbb{P}^3$

The study of space curves in algebraic geometry has a long history and there are many strong results we can rest on. This makes for an exciting application of interpolation ideas. We aim to show the following result.

Theorem 7.1. *A general connected genus g and degree d curve in $\mathbb{P}^3$ satisfies interpolation if $d \geq g + 3$ unless $d = 5$ and $g = 2$.*

Our proof of Theorem 7.1 depends on the following analysis of secant lines to rational curves in $\mathbb{P}^3$.

Proposition 7.2. *Let $C \subset \mathbb{P}^3$ be a general rational curve of degree $d \geq 3$. Consider k general secant lines $L_1, \ldots, L_k$ and the union curve $X = C \cup \bigcup_i L_i$. Unless $d = 3$ and $k = 2$, the vector bundle $N_X|_C \cong \mathcal{O}_{\mathbb{P}^1}(2d - 1 + k)^{\oplus 2}$ is balanced. In the exceptional case $d = 3$, $k = 2$, the bundle is $N_X|_C \cong \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(8)$.*

Let us assume this result and deduce Theorem 7.1 from it.

Proof of Theorem 7.1. The normal bundle of a space curve has rank 2, so we can focus on demonstrating weak interpolation by Corollary 3.18. Ran's Theorem 6.4 guarantees that a general rational curve C of degree $d - g \geq 3$ has balanced normal bundle, so it satisfies weak interpolation. Let $L_1, \ldots, L_g$ be g secant lines to C and set $X = C \cup L_1 \cup \cdots \cup L_g$. By choosing L_i general, we can assume

(a) all L_i are disjoint (by Corollary 4.10),

(b) the tangent lines to C at $C \cap (L_1 \cup \cdots \cup L_g)$ are all skew (by Corollary 4.8), and

(c) the vector bundle $N_X|_C$ is balanced (by Proposition 7.2).

We apply Proposition 5.5 to conclude that X is smoothable and $H^1(N_X) = 0$. In other words, $[X]$ is a smooth point of the Hilbert scheme $\mathcal{H}_{d,g,3}$. Since interpolation is an open condition (Corollary 5.9), it suffices to show X satisfies weak

interpolation. We do this by twisting back N_X at $2d - g$ points q_j on C and one point p_i on each L_i. We will choose all p_i and q_j to be smooth points of X. We have a short exact sequence

$$0 \longrightarrow N_X(-\textstyle\sum_i p_i - \sum_j q_j) \longrightarrow N_X|_C(-\sum_j q_j) \oplus \bigoplus_i N_X|_{L_i}(-p_i) \longrightarrow N_X|_{\operatorname{Sing} X} \longrightarrow 0.$$

Since the tangent lines of C to all nodal points $\operatorname{Sing} X$ are skew, it follows that $N_X|_{L_i} \cong \mathcal{O}_{L_i}(2)^{\oplus 2}$ and $N_X|_{L_i}(-p_i) \cong \mathcal{O}_{L_i}(1)^{\oplus 2}$. This ensures that the morphism

$$\bigoplus_i N_X|_{L_i}(-p_i) \longrightarrow N_X|_{\operatorname{Sing} X} = \bigoplus_i N_X|_{L_i \cap C}$$

is surjective on H^0. But $N_X|_C$ is balanced, so $\mathrm{H}^1(N_X|_C(-\sum_j q_j)) = 0$ and $\mathrm{H}^1(N_X(-\sum_i p_i - \sum_j q_j)) = 0$. This completes the proof of the desired statement. $\qquad\square$

The proof of Proposition 7.2 rests on understanding elementary modifications of rank 2 bundles. This topic is very classical, so our discussion will be succinct. We refer to [2] and [16] for a detailed discussion of ruled and rational surfaces.

Example 7.3 (Elementary modifications in rank 2). Let C be a smooth curve and E a rank 2 bundle over it. We consider a modification at a point $p \in C$ given by a line $L \subset E_p$:

$$0 \longrightarrow E' = M(E, L) \longrightarrow E \longrightarrow E_p/L \longrightarrow 0.$$

The line L corresponds to a point $\ell \in \mathbb{P}E$ lying over p. If $\pi \colon \mathbb{P}E \to C$ is the natural projection morphism, let $F = \pi^{-1}(p)$ denote the fiber containing ℓ. The blow-up $S = \mathrm{Bl}_\ell \mathbb{P}E$ of $\mathbb{P}E$ at ℓ is a smooth surface with exceptional locus E. Let $\widetilde{F}$ denote the proper transform of F in S. Since $E^2 = -1$ and $(\widetilde{F} + E)^2 = F^2 = 0$, it follows that $\widetilde{F}^2 = -1$. We can contract $\widetilde{F}$, and the resulting surface is a $\mathbb{P}^1$-bundle over C, naturally isomorphic to $\mathbb{P}E'$. In general, every ruled surface can be presented as $\mathbb{P}E$ for some C and E.

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Note that the image of the contracted curve $\widetilde{F}$ is a distinguished point on $\mathbb{P}E'$. The process is symmetric, that is, if we blow up this point in $\mathbb{P}E'$ and blow down the proper transform of the fiber we arrive back at $\mathbb{P}E$.

It should not come as a surprise that the operation we described above is called an *(elementary) modification* of a ruled surface. If S is a ruled surface and p a point on it, we will use $M(S, p)$ to denote the resulting modification and $\widetilde{p} \in M(S, p)$ the distinguished point corresponding to p. More generally, given a list of points $p_1, \ldots, p_k$ lying in distinct fibers, we can consider the simultaneous modification $M(S, p_1, \ldots, p_k)$. The construction furnishes a birational equivalence $m_{S, p_1, \ldots, p_k} : S \dashrightarrow M(S, p_1, \ldots, p_k)$. When there is no ambiguity, we may shorten our notation m_{S, p_i} to m_{p_i}.

Let us specialize to $C = \mathbb{P}^1$. Since $\mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(b - a))$, it suffices to study the *Hirzebruch surfaces* $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ for $n \geq 0$. In $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ there is a unique 0-section through each point on the surface; there are no negative sections. When $n > 0$, the surface $\mathbb{F}_n$ has a distinguished section B_n with self-intersection $B_n^2 = -n$. Our goal is to describe the modification at any point $p \in \mathbb{F}_n$.

Case 1, $n = 0$.: All points on $p \in \mathbb{F}_0$ are alike, and any modification leads to an $\mathbb{F}_1$. The (-1)-section is the proper transform of the 0-section through $p \in \mathbb{F}_0$.

Case 2, $n > 0$.: If $p \in B_n$, then the resulting surface is $\mathbb{F}_{n+1}$. Its distinguished section is the proper transform of B_n. If $p \notin B_n$, then we arrive at $\mathbb{F}_{n-1}$. The section B_n maps isomorphically to the distinguished section of the new surface.

The following diagram summarizes the behavior of Hirzebruch surfaces under elementary modifications. Solid arrows represent general modifications while dotted ones stand for modifications based at a point on the distinguished section.

$$\mathbb{F}_0 \mathrel{\text{\reflectbox{$\rightleftharpoons$}}} \mathbb{F}_1 \quad \mathbb{F}_2 \quad \mathbb{F}_3 \quad \cdots$$
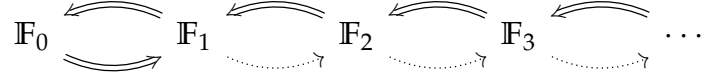
Let us investigate the result of repeated modifications applied to a Hirzebruch surface $\mathbb{F}_n$ where $n > 0$. If the first point p_1 is general (away from the $(-n)$-section), then $M(\mathbb{F}_n, p_1) \cong \mathbb{F}_{n-1}$. In choosing the second point $p_2 \in \mathbb{F}_n$, we will require that it is not in the fiber of p_1 and away from B_n (the proper transform of $B_{n-1} \subset M(\mathbb{F}_n, p_1)$ in $\mathbb{F}_n$). Under these conditions $M(\mathbb{F}_n, p_1, p_2) \cong \mathbb{F}_{n-2}$. Continuing this argument we can show that if the first n points are taken to be general, then $M(\mathbb{F}_n, p_1, \ldots, p_n) \cong \mathbb{F}_0$. Instead of adding more points, we will reset the process and present the case $n = 0$ separately.

For any $p_1 \in \mathbb{F}_0$, the resulting modification is $M(\mathbb{F}_0, p_1) \cong \mathbb{F}_1$. The (-1)-section in $M(\mathbb{F}_0, p_1)$ is the transform of the horizontal section through p_1 in $\mathbb{F}_0$. As long as p_2 avoids it as well as the fiber of p_1, we will have $M(\mathbb{F}_0, p_1, p_2) \cong \mathbb{F}_0$. In choosing the third point we avoid the fibers of the first two to obtain $M(\mathbb{F}_0, p_1, p_2, p_3) \cong \mathbb{F}_1$. The (-1)-section in this surface is the transform of the unique 2-section through the first three points, so avoiding it we are back to $\mathbb{F}_0$. It is clear how the process continues and the final result depends on the parity of the number of points we modify at. We demonstrated the following result.

Proposition 7.4. *If $p_1, \ldots, p_k \in \mathbb{F}_n$ are general points, then*

$$M(\mathbb{F}_n, p_1, \ldots, p_k) \cong \begin{cases} \mathbb{F}_{n-k} & \text{if } k \leq n, \\ \mathbb{F}_0 & \text{if } k > n \text{ and } k - n \equiv 0 \ (\mathrm{mod}\ 2), \\ \mathbb{F}_1 & \text{if } k > n \text{ and } k - n \equiv 1 \ (\mathrm{mod}\ 2). \end{cases}$$

Our discussion so far explored the effect of modifying at a single general point. What if we are interested in modifying at two points which are not general, i.e.,

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there is a relation between them? We already saw some of these ideas, but we will take the opportunity to develop and state them more clearly. Our goal is to arrive at a more refined version of Proposition 7.4.

Example 7.5 (Modifying at two points at a time). To review notation, B_n will denote the distinguished $(-n)$-section in $\mathbb{F}_n$ when $n > 0$. We will use $\pi_n \colon \mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbb{P}^1$ to denote the natural projection. When $n = 0$ and $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we will treat π_0 as the projection to the first factor.

> **Case 1, $n \geq 2$.:** If we choose $p_1 \notin B_n$, then $M(\mathbb{F}_n, p_1) \cong \mathbb{F}_{n-1}$. In order to arrive at $\mathbb{F}_{n-2}$, the second point should avoid the fiber of the first and the transform of B_{n-1}, that is B_n.
>
> **Case 2, $n = 1$.:** The first point should avoid B_1, but the second only need to avoid the fiber of the first.
>
> **Case 3, $n = 0$.:** There is no restriction on the first point; all modifications lead to $M(\mathbb{F}_0, p_1) \cong \mathbb{F}_1$. In choosing the second point, we would like to avoid the fiber of the first and the transform of the (-1)-curve which is the horizontal section through p_1.
>
> The horizontal sections on $M(\mathbb{F}_0, p_1, p_2) \cong \mathbb{F}_0$ are even more interesting. They form a pencil of 0-sections. Tracing them back through m_{p_1, p_2}, their transforms in the original surface remain sections, pass through p_1 and p_2, and have self-intersection 2 (see Figure 3). The only line bundle which fits the bill is $\mathcal{O}_{\mathbb{F}_0}(1,1)$; its global sections form a 4-dimensional vector space. Passing through each of the points p_1 and p_2 imposes a single linear condition, so indeed we are left with a pencil of $(1,1)$-curves passing through p_1 and p_2.

The pattern we described generalizes as follows.
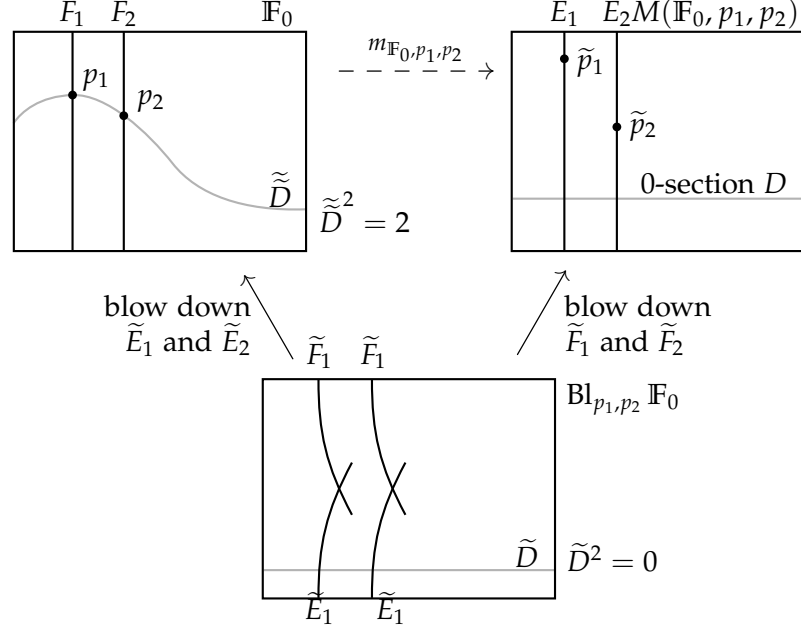
FIGURE 3. A horizontal section D in $M(\mathbb{F}_0, p_1, p_2) \cong \mathbb{F}_0$ transforms to a $(1,1)$-curve $\widetilde{\widetilde{D}}$ in $\mathbb{F}_0$ with self-intersection 2.

Proposition 7.6. *Let $p_1, \ldots, p_{2k} \in \mathbb{F}_0$ be points lying in distinct fibers of $\pi_0 \colon \mathbb{F}_0 \to \mathbb{P}^1$. If the modified surface $S = M(\mathbb{F}_0, p_1, \ldots, p_{2k})$ is isomorphic to $\mathbb{F}_0$, then under $\mathbb{F}_0 \dashrightarrow S$ the pencil of horizontal sections in S transforms to the pencil of $(k,1)$-curves passing through all $p_i \in \mathbb{F}_0$.*

Remark 7.7. Note that we only impose a condition on the final modification $M(\mathbb{F}_0, p_1, \ldots, p_{2k})$. The result applies even if one of the intermediate modifications $M(\mathbb{F}_0, p_1, \ldots, p_i)$ is not isomorphic to $\mathbb{F}_0$ or $\mathbb{F}_1$.

The result above will be useful to us in the following form.

Corollary 7.8. *Let $p_1, \ldots, p_{2k+2} \in \mathbb{F}_0$ be points lying on distinct fibers of $\pi_0 \colon \mathbb{F}_0 \to \mathbb{P}^1$. Suppose the surface $M(\mathbb{F}_0, p_1, \ldots, p_{2k})$ is isomorphic to $\mathbb{F}_0$. Then the surface $M(\mathbb{F}_0, p_1, \ldots, p_{2k+2})$ is isomorphic to $\mathbb{F}_0$ if and only if $p_1, \ldots, p_{2k+2}$ do not lie on a $(k,1)$-curve.*

Proof of Proposition 7.6. Horizontal sections on $S = M(\mathbb{F}_0, p_1, \ldots, p_{2k}) \cong \mathbb{F}_0$ form a pencil and have self-intersection 0. Reversing a modification involves blowing down $2k$ fibers, so the self-intersection of a transformed section increases by 1 for each point of modification. It follows that the curves which correspond to horizontal sections in S have self-intersection $2k$ on $\mathbb{F}_0$. Since sections remain sections, these are curves on $\mathbb{F}_0$ of the form $(k', 1)$ for some $k' \geq 0$. Their self-intersection is $k' + k' = 2k$, so $k' = k$.

Starting with a $(k, 1)$-curve in $\mathbb{F}_0$ passing through all p_i, we can reverse the argument and show that its transform in S is a 0-section. This completes the argument. $\qquad\square$

There is another useful result worth isolating from the proof of Proposition 7.2.

Proposition 7.9. *Let X, Y be normal varieties, X is irreducible, and $f, g \colon X \to Y$ are two regular morphisms. Consider a pencil $\{H_t\}_{t \in \mathbb{P}^t}$ on Y with base locus $B \subset Y$ and associated line bundle L. If*

(a) f^*L is not isomorphic to g^*L, and

(b) the codimension of $f^{-1}(B) \cup g^{-1}(B) \subset X$ is at least two,

then for a general point $x \in X$ the images $f(x), g(x) \in Y$ lie in distinct members of the pencil $\{H_t\}$.

Proof. Since Y is normal, the base locus B of $\{H_t\}$ has codimension at least two. The pencil has an associated map $\varphi \colon V \to \mathbb{P}^1$ where $V = Y \setminus B$. Let $U = X \setminus (f^{-1}(B) \cup g^{-1}(B))$. We will use $i_U \colon U \to X$ and $i_V \colon V \to Y$ to denote the natural inclusions.

$$
\begin{array}{ccccc}
U & \overset{f|_U}{\underset{g|_U}{\rightrightarrows}} & V & \overset{\varphi}{\longrightarrow} & \mathbb{P}^1 \\
{\scriptstyle i_U}\downarrow & & \downarrow {\scriptstyle i_V} & & \\
X & \underset{g}{\overset{f}{\rightrightarrows}} & Y & &
\end{array}
$$

Since the complements of both U and V are of codimension at least two, it follows that $i_U^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(U)$ and $i_V^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$ are both isomorphisms. By construction $\varphi^*\mathcal{O}_{\mathbb{P}^1}(1) \cong i_V^* L$. Combining this with the hypothesis $f^*L \not\cong g^*L$, it follows that

$$(\varphi \circ f|_U)^*\mathcal{O}_{\mathbb{P}^1}(1) \not\cong (\varphi \circ g|_U)^*\mathcal{O}_{\mathbb{P}^1}(1).$$

This demonstrates that $\varphi \circ f|_U$ and $\varphi \circ g|_U$ are not identical, hence proving our claim. $\qquad\square$

We are now ready for the central argument of this section.

Proof of Proposition 7.2. Our first goal is to translate the question into the language of rational surfaces. Consider the rational morphism $f\colon C \times C \dashrightarrow \mathbb{P}N_C$ which sends the point $(p,q) \in C \times C$ to $T_p[p,q]$, the tangent space to the secant line joining p and q, viewed as a point of $\mathbb{P}N_{C,p}$. This definition makes sense when $p \neq q$ and the secant $[p,q]$ is not tangent at p. Trisecant lines which are also tangent, called *tangential trisecants*, have been a classical topic of interest. If C is a general rational curve, then using osculating planes f extends to the entire diagonal and the number of problematic points, corresponding to tangential trisecants, is $2(d-2)(d-3)$ (see Remark 7.10).

We would like to resolve f and $f \circ \sigma$ simultaneously, where $\sigma\colon C \times C \to C \times C$ is the *swap* morphism given by $\sigma(p,q) = (q,p)$. Let $s_1,\ldots,s_{2(d-2)(d-3)} \in C \times C$ comprise the complement of the locus where f is well-defined, and let $s_i' = \sigma(s_i)$ be their swaps. The blow-up $\varepsilon\colon S = \mathrm{Bl}_{\{s_i,s_i'\}} C \times C \to C \times C$ is the minimal simultaneous resolution of f and $f \circ \sigma$. If $\widetilde{\sigma}\colon S \to S$ is the lift of σ and $\widetilde{f}\colon S \to \mathbb{P}N_C$ is the resolution of f, then the resolution of $f \circ \sigma$ is $\widetilde{f} \circ \widetilde{\sigma}$.

$$\begin{array}{ccc} & S & \\ \widetilde{f} \swarrow & \downarrow \varepsilon & \searrow \widetilde{f}\circ\widetilde{\sigma} \\ \mathbb{P}N_C \xleftarrow[f]{} & C \times C & \xdashrightarrow[f\circ\sigma]{} \mathbb{P}N_C \end{array}$$

Note that to resolve f we only need to blow up $C \times C$ at isolated points. This follows since the morphism $f \colon C \times C \dashrightarrow \mathbb{P}N_C$ is given by two pencils, one of them (the projection on the first factor) has no base locus, and the second one has reduced base locus for general C (see [7, p. 291] and [12]).

Let us compute the degree of $f \colon C \times C \dashrightarrow \mathbb{P}N_C$ (and hence of $\widetilde{f}$). A point on $\mathbb{P}N_C$ corresponds to a point $p \in C$ and a line in the normal space $N_{C,p}$. The preimage of this line under the quotient morphism $T_{\mathbb{P}^3,p} \to N_{C,p}$ yields a plane in the tangent space at p containing the line T_pC. We can treat the 2-plane in the tangent space as an actual plane $H \subset \mathbb{P}^3$. The degree problem amounts to counting the number of secant lines to C contained in H. We already know that H contains $p \in C$. We have the freedom to pick a general H as long as $T_pC \subset T_pH$. Since H is tangent to C and the point p is not a flex of C, the intersection $H \cap C$ has multiplicity 2 at p. The curve C has degree d, hence H intersects C transversely in $d - 2$ points away from $p \in C$ (see Figure 4). This demonstrates that $\deg f = \deg \widetilde{f} = d - 2$. When $d = 3$ the map $f \colon C \times C \to \mathbb{P}N_C$ is an isomorphism.

Our next goal is to determine how line bundles behave under pullbacks via $\widetilde{f}$ and $\widetilde{f} \circ \widetilde{\sigma}$. Set $E = \varepsilon^{-1}(\{s_i\})$ and $E' = \varepsilon^{-1}(\{s_i'\}) = \widetilde{\sigma}(E)$. Since C is rational, all line bundles on $C \times C$ are of the form $\mathcal{O}_{C\times C}(a,b)$. For convenience, we will denote $\mathcal{O}_S(a,b) = \varepsilon^*\mathcal{O}_{C\times C}(a,b)$. It is easy to characterize $\widetilde{\sigma}$ using this notation:

$$\widetilde{\sigma}^*\mathcal{O}_S(a,b) = \mathcal{O}_S(b,a), \qquad \widetilde{\sigma}^*\mathcal{O}_S(E) = \mathcal{O}_S(E'), \qquad \widetilde{\sigma}^*\mathcal{O}_S(E') = \mathcal{O}_S(E).$$
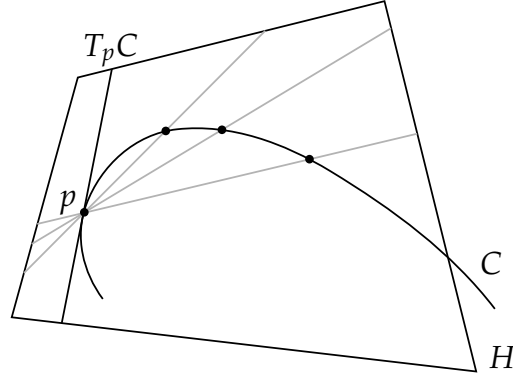
FIGURE 4. Computing the degree of $f \colon C \times C \dashrightarrow \mathbb{P}N_C$

Since we can choose C to be general, its normal bundle N_C is balanced (Theorem 6.4) and $\mathbb{P}N_C \cong C \times \mathbb{P}^1 \cong \mathbb{F}_0$. Since f, and hence $\widetilde{f}$, respect the projection to the first factor, the pullback of a fiber is a fiber and $\widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(1,0) = \mathcal{O}_S(1,0)$. By the construction of $\widetilde{f}$, it follows that $\widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(0,1) = \mathcal{O}_S(a,b)(-E)$ for some $a, b \in \mathbb{Z}$. To compute a and b, we use the intersection forms on $\operatorname{Pic} S$ and $\operatorname{Pic} \mathbb{P}N_C$. Consider a point in $\mathbb{P}N_C$ as the intersection of a $(0,1)$ and a $(1,0)$ fiber.

$$
\begin{aligned}
b &= \mathcal{O}_S(1,0) \cdot \mathcal{O}_S(a,b)(-E) \\
&= \widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(1,0) \cdot \widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(0,1) \\
&= (\deg \widetilde{f}) \left(\mathcal{O}_{\mathbb{P}N_C}(1,0) \cdot \mathcal{O}_{\mathbb{P}N_C}(0,1) \right) \\
&= d - 2
\end{aligned}
$$

The self-intersection of a horizontal section in $\mathbb{P}N_C$ allows us to compute a.

$$
\begin{aligned}
0 &= (\deg \widetilde{f})(\mathcal{O}_{\mathbb{P}N_C}(0,1)^2) \\
&= (\widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(0,1))^2 \\
&= \mathcal{O}_S(a, d-2)(-E)^2 \\
&= 2a(d-2) + E^2
\end{aligned}
$$

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Since E consists of $2(d-2)(d-3)$ (-1)-curves, it follows that $a = d-3$. In summary of our results

$$\widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(i,j) = \mathcal{O}_S(i + (d-3)j, (d-2)j)(-jE),$$

$$(\widetilde{f} \circ \widetilde{\sigma})^* \mathcal{O}_{\mathbb{P}N_C}(i,j) = \mathcal{O}_S((d-2)j, i + (d-3)j)(-jE')$$

for all $i, j \in \mathbb{Z}$.

The reason we constructed f and σ is the following. Let $L_1, \ldots, L_k$ be secants to C corresponding to the points $\ell_i = (a_i, b_i) \in C \times C \setminus (\Delta \cup \{s_i, s_i'\})$. Then the projectivization of $N_{C \cup L_1 \cup \cdots \cup L_k}|_C$ is isomorphic to the modification of $\mathbb{P}N_C$ at the images of s_i under f and $f \circ \sigma$. Our goal is to show that for general points s_i we have an isomorphism

$$M(\mathbb{P}N_C, f(\ell_1), f \circ \sigma(\ell_1), \ldots, f(\ell_k), f \circ \sigma(\ell_k)) \cong \mathbb{F}_0.$$

We will use induction on k. At each step, given that

$$\widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(k,1) \not\cong (\widetilde{f} \circ \widetilde{\sigma})^* \mathcal{O}_{\mathbb{P}N_C}(k,1)$$

we will apply Corollary 7.8 and Proposition 7.9. Our description of pullbacks above implies that

$$\widetilde{f}^* \mathcal{O}_{\mathbb{P}N_C}(k,1) = \mathcal{O}_S(k+d-3, d-2)(-E),$$

$$(\widetilde{f} \circ \widetilde{\sigma})^* \mathcal{O}_{\mathbb{P}N_C}(k,1) = \mathcal{O}_S(d-2, k+d-3)(-E').$$

Case 1, $d \geq 4$.: Recall that E consists of $2(d-2)(d-3) > 0$ curves. Since E, E', and $\varepsilon^* \operatorname{Pic}(C \times C)$ are all linearly independent in $\operatorname{Pic}(S)$, it follows that the line bundles above are not isomorphic. This holds for all $k \geq 0$, so we have covered both the base and inductive cases.

Case 2, $d = 3$.: When $C \subset \mathbb{P}^3$ is a twisted cubic, f is an isomorphism so both E and E' are empty. For convenience, we will identify $C \times C$ and $\mathbb{P}N_C$.

The two line bundles above are non-isomorphic except when $k = 1$. We can summarize the modifications of $\mathbb{P}N_C$ as follows. Choosing a first point $\ell_1 \in C \times C$ away from the diagonal leads to a modification $M(\mathbb{P}N_C, \ell_1, \sigma(\ell_1)) \cong \mathbb{F}_0$. No matter where we choose the second point ℓ_2, the four points $\ell_1, \sigma(\ell_1), \ell_2, \sigma(\ell_2)$ always lie on a $(1,1)$-curve D and $\mathbb{P}N_{C \cup L_1 \cup L_2}|_C \cong \mathbb{F}_2$. Fortunately, this makes the choice of the third point quite easy. As long as ℓ_3 avoid the fibers of the previous four points and the curve D (the transform of the (-2)-curve in $\mathbb{P}N_{C \cup L_1 \cup L_2}|_C$), the resulting modification will be balanced. At this point, the induction argument goes through analogously to the case $d \geq 4$. $\qquad \square$

Remark 7.10. We refer to [7, Chapters 2.4 and 2.5] for a more rigorous treatment of the following material. For a discussion of the general secant plane formula, see [1, VIII.4].

Let $C \subset \mathbb{P}^3$ be a smooth curve and $p \in C$ a point. We call $H \subset \mathbb{P}^3$ an *osculating plane* to C at p if H and C have order of contact 3 at p. If the tangent line $T_p C$ at p has order of contact 2 with C (which is the general case in characteristic 0), then there is a unique osculating plane through p. On the other hand, if C meets $T_p C$ with order 3 or higher, then every plane containing $T_p C$ meets the criteria, so there is no unique osculating plane.

There is an alternative way to construct the osculating plane at a point p. Consider two other points $q, r \in C$ and the plane $H_{q,r}$ spanned by p, q, and r. If a unique osculating plane exists, it is the limit of $H_{q,r}$ as both q and r approach p. Since an osculating plane contains the tangent line $T_p C$, it corresponds to a point in $\mathbb{P}N_C$. This comment allows us to extend $f \colon C \times C \dashrightarrow \mathbb{P}N_C$ to points (a, a) on the diagonal as long as $a \in C$ is not a flex.

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We claim that a general degree $d \geq 3$ rational curve $C \subset \mathbb{P}^3$ has no flex points. Since all points on the twisted cubic are alike, this is certainly the case for $d = 3$. For higher values of d, we note that every smooth rational curve is the image of a rational normal curve $C' \subset \mathbb{P}^d$ away from a $(d-4)$-plane $\Lambda \subset \mathbb{P}^d$. Let $\pi \colon \mathrm{Bl}_\Lambda \mathbb{P}^d \to \mathbb{P}^3$ denote the projection morphism. Every point of $p' \in C'$ has a well-defined osculating plane $H_{p'}$ and these sweep out a 3-fold $X \subset \mathbb{P}^d$. A point $p \in C$ is a flex if and only if the Λ intersects $H_{p'}$ nontrivially where p' is the unique point of C' lying above p. In general the intersection $\Lambda \cap H_{p'}$ is a point, and $T_p C$ is the image of $H_{p'} \subset \mathbb{P}^d$. Since Λ and X have complementary dimensions, we can choose Λ away from X. It follows that the corresponding projection C will have no flex points.

Our analysis of flex points demonstrates that given a general rational curve C, we can define $f \colon C \times C \dashrightarrow \mathbb{P}N_C$ along the diagonal. Tangential trisecants are the actual obstacle to extending f to a regular morphism. Unfortunately, as soon as $d \geq 4$ they always exist. To illustrate our point, let us consider $d = 4$. A rational quartic curve C lies on a smooth quadric surface $Q \subset \mathbb{P}^3$ as a $(1,3)$-curve. Bezout's theorem implies that all trisecants to C must be contained in Q. In fact, they are precisely the lines from one of the rulings on Q (to be precise, the pencil of $(1,0)$-curves). Contracting the lines of the trisecant ruling produces a 3-sheeted morphism $C \to \mathbb{P}^1$. As long as C has no flex points, this morphism is simply ramified. The Riemann-Hurwitz formula implies there are exactly 4 simple ramification points, each corresponding to a tangential trisecant.

While the method we presented for $d = 4$ does not generalize, there is another approach to counting tangential trisecants via what is called the trisecant correspondence (see [7, p. 290-294]). In modern language, this is a curve $D \subset C \times C$ defined as the closure of points (p, q) such that $p \neq q$ and the line $[p, q]$ is trisecant to C. By projecting C away from a general point on it and counting the number

of nodes on the image curve in $\mathbb{P}^2$ we can compute the projection degrees

$$\deg(\mathrm{pr}_1 \colon D \to C) = \deg(\mathrm{pr}_2 \colon D \to C) = (d-2)(d-3).$$

Tangential trisecants correspond to points in $D \cap \Delta$ where $\Delta \subset C \times C$ is the diagonal. In conclusion, the number of tangential trisecants is

$$\#(D \cap \Delta) = \deg(\mathrm{pr}_1) + \deg(\mathrm{pr}_2) = 2(d-2)(d-3).$$

Remark 7.11. Just because the union of a twisted cubic and two secant lines does not satisfy interpolation does not mean a priori that the general genus 2 quintic curve indeed does not satisfy interpolation. We would like to fill this gap by demonstrating that a general such curve does not satisfy the full (2^{10})-interpolation, but it satisfies the slightly weaker (2^9)-interpolation.

A general genus 2 quintic curve $C \subset \mathbb{P}^3$ lies on a smooth quadric surface $Q \subset \mathbb{P}^3$ as a $(2,3)$-curve. The normal bundle sequence for the inclusion $C \subset Q \subset \mathbb{P}^3$ reads

$$0 \longrightarrow N_{C/Q} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{Q/\mathbb{P}^3}|_C \longrightarrow 0,$$

which simplifies to

$$0 \longrightarrow \mathcal{O}_Q(2,3)|_C \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow \mathcal{O}_C(2) \longrightarrow 0.$$

The first and third member of the sequence are line bundles of degrees 12 an 10 respectively. To see that C does not satisfy (2^{10})-interpolation, we use the Jacobi inversion theorem (see [1, I.3] and [7, Chapter 2.2]) to choose a degree 10-divisor $D = \sum_{i=1}^{10} p_i$ such that $\mathcal{O}_Q(2,3)|_C(-D)$ is non-special (i.e., non-hyperelliptic) and $\mathcal{O}_C(2)(-D) \not\cong \mathcal{O}_C$. Then

$$h^0(\mathcal{O}_Q(2,3)|_C(-D)) = 1 \quad \text{and} \quad h^0(\mathcal{O}_C(2)(-D)) = 0,$$

so

$$H^0(N_{C/\mathbb{P}^3}(-D)) \cong H^0(\mathcal{O}_Q(2,3)|_C(-D))$$

is 1-dimensional which is higher than the expected 0.

If D were a general (effective) divisor of degree 9, then the residual line bundle to $\mathcal{O}_Q(2,3)|_C(-D)$ has negative degree -1, so

$$h^0(N_{C/\mathbb{P}^3}) = h^0(\mathcal{O}_Q(2,3)|_C(-D)) + h^0(\mathcal{O}_C(2)(-D)).$$

As long as D is such that $\mathcal{O}_C(2)(-D)$ is non-special, then $h^0(N_{C/\mathbb{P}^3}) = 2$ as expected.

There is an alternative explanation which is helpful in building our geometric intuition about interpolation. We already mentioned that a genus 2 quintic curve lies on a quadric. The space of quadrics $\mathbb{P}\,H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ has dimension $\binom{3+2}{2} - 1 = 9$. There is a unique quadric through 9 general points but none through 10 general points. In particular, given 10 general points we should not expect they are all contained in a genus 2 quintic curve, even though the normal bundle of such a curve has $20 = 2 \cdot 10$ sections.

8. The equivalence of strong and regular interpolation

The goal of this section is to explore the relation between regular and strong interpolation. While it may seem that the latter is strictly stronger than the former, this is not the case. In fact, we already proved the two notions are equivalent for rank 2 bundles (Corollary 3.18). We will extend this result to arbitrary rank.

Theorem 8.1. *Strong and regular interpolation are equivalent.*

In turn, the equivalence rests on the following strengthening of Proposition 3.13.

Theorem 8.2. *Let $\lambda = (\lambda_i)$ and $\lambda' = (\lambda'_i)$ be two tableaux. If $\lambda' \trianglelefteq \lambda$, then λ-interpolation implies λ'-interpolation.*

The heart of the matter lies in the following linear algebra result.

Proposition 8.3. *Consider a sequence of vector spaces $E_1, \ldots, E_m$ and a sequence of integers $\lambda_1, \ldots, \lambda_m$. If $\Lambda \subset E_1 \oplus \cdots \oplus E_m$ is a subspace satisfying*

(a) for each i the projection $\Lambda \to E_i$ has rank at least λ_i, and

(b) for each i the projection $\Lambda \to \bigoplus_{j \leq i} E_j$ has rank at least $\sum_{j \leq i} \lambda_j$,

then there exists a sequence of subspaces $\Lambda_1, \ldots, \Lambda_m \subset \Lambda$ such that

(a) $\dim \Lambda_i = \lambda_i$ for all i,

(b) all projection morphisms $\Lambda_i \to E_i$ are injective, and

(c) all Λ_i are independent.

Remark 8.4. Note that the statement of Proposition 8.3 did not impose any order on the λ_i.

Proof. We proceed by induction on m. When $m = 1$, it suffices to pick a subspace $\Lambda_1 \subset \Lambda$ of dimension λ_1. Next, assume the statement holds for some m; we study the statement for $m + 1$. Let Λ' be the image of Λ in $E_1 \oplus \cdots \oplus E_m$. In

order to use the inductive hypotheses hold, we construct spaces $\Lambda'_1, \ldots, \Lambda'_m \subset \Lambda'$ satisfying the mentioned conditions. For each $1 \leq i \leq m$, we lift Λ'_i to $\Lambda_i \subset \Lambda$ such that $\dim \Lambda_i = \lambda_i$. Since Λ'_i are linearly independent, then so are Λ_i. For each i, the projection $\Lambda_i \to E_i$ factors through $\Lambda_i \to \Lambda'_i$ which is an isomorphism. We know that $\Lambda'_i \to E_i$ is injective, hence $\Lambda_i \to E_i$ is also injective. It remains to choose $\Lambda_{m+1} \in \mathrm{Gr}(\lambda_{m+1}, \Lambda)$ satisfying

$$\Lambda_{m+1} \cap \left(\bigoplus_{i \leq m} E_i \right) = 0 \qquad and \qquad \Lambda_{m+1} \cap \left(\bigoplus_{i \leq m} \Lambda_i \right) = 0.$$

The first of these conditions implies that the Λ_{m+1} intersects trivially the kernel of the projection morphism $\bigoplus_i E_i \to E_{m+1}$, so it must be injective onto its image. Together with the fact $\Lambda_1, \ldots, \Lambda_m$ are linearly independent, the second condition tells us that by adding Λ_{m+1} the list remains linearly independent. That said, we need to argue why such a choice of Λ_{m+1} is possible. Both conditions are open, so it suffices to show they describe non-empty varieties in $\mathrm{Gr}(\lambda_{m+1}, \Lambda)$. Numerically, we need to verify

$$\dim \Lambda_{m+1} + \dim \left(\Lambda \cap \bigoplus_{i \leq m} E_i \right) \leq \dim \Lambda, \text{ and}$$

$$\dim \Lambda_{m+1} + \dim \left(\bigoplus_{i \leq m} \Lambda_i \right) \leq \dim \Lambda$$

If we identify $\Lambda \cap (E_1 \oplus \cdots \oplus E_m)$ with the kernel of the morphism $\Lambda \to E_{m+1}$, then the first statement follows from the rank inequality of the morphism in question. The second statement, equivalent to $\lambda_1 + \cdots + \lambda_{m+1} \leq \dim \Lambda$, is a consequence of the lower bound on the rank of $\Lambda \to E_1 \oplus \cdots \oplus E_{m+1}$. This completes the proof of desired claim. $\square$

This leads us to a natural criterion for interpolation.

Proposition 8.5. *A vector bundle E satisfies $\lambda = (\lambda_1, \ldots, \lambda_m)$ interpolation if and only if there exist m distinct points $p_1, \ldots, p_m$ such that: for every i the evaluation map $H^0(E) \to \bigoplus_{j \leq i} E_{p_j}$ has rank at least $\sum_{j \leq i} \lambda_i$.*

Proof. The forward direction is simple. Let $V_i \subset E_{p_i}$ be a modification datum for E such that $h^0(M(E, V_i)) = h^0(E) - \sum \lambda_i$. It follows that the morphism $H^0(E) \to \bigoplus E_{p_i}/V_i$ is surjective; taking a further quotient we deduce that

$$H^0(E) \longrightarrow \bigoplus_{j \leq i} E_{p_j}/V_j$$

is surjective for all $1 \leq i \leq m$. These statements are equivalent to the rank inequalities we desire.

Conversely, we would like to choose points $p_1, \ldots, p_m \in C$ such that

(a) for each i the rank of $H^0(E) \to E_{p_i}$ is at least λ_i, and

(b) for each i the rank of $H^0(E) \to \bigoplus_{j \leq i} E_{p_j}$ is at least $\sum_{j \leq i} \lambda_j$.

Our hypothesis implies that the open subset of $C^m \setminus \Delta$ cut out by condition (b) is non-empty. For $i = 1$ condition (a) and (b) are equivalent. Since rank is upper semi-continuous and $\lambda_1 \geq \cdots \geq \lambda_m$, it follows that the open specified by condition (a) is also non-empty, so we can choose the points $p_i \in C$ as desired.

If $E_i = E_{p_i}$ and Λ is the image of $H^0(E)$ in $\bigoplus E_i$, we can apply Proposition 8.3. For each i, the morphism $\Lambda_i \to E_i$ is injective, so a complement V_i to its image would have codimension λ_i. Since the spaces Λ_i are linearly independent, it follows that the evaluation $H^0(E) \to \bigoplus E_i/V_i$ is surjective as desired. $\square$

It is clear that Theorems 8.1 and 8.2 follow from the interpolation criterion in Proposition 8.5. There are two much simpler corollaries worth spelling out.

Corollary 8.6 (Replacing λ). *If $\lambda' \trianglelefteq \lambda$, then $(\lambda', \mu) \trianglelefteq (\lambda, \mu)$. In particular, (λ, μ)-interpolation implies (λ', μ)-interpolation.*

Corollary 8.7 (Breaking λ). *If $a = b + c$ is a sum of non-negative integers and λ is any tableau, then (a, λ)-interpolation implies (b, c, λ)-interpolation.*

Remark 8.8. Unlike strong interpolation, weak interpolation is not equivalent its regular counterpart (see Remark 3.12). The precise statement is that a vector bundle E satisfying weak interpolation also satisfies regular interpolation if and only if $h^0(E) \equiv 0, 1 \pmod{\operatorname{rank} E}$.

9. Elementary modifications in arbitrary dimension

Elementary modifications of vector bundles are a classical topic. Most sources, including our earlier discussion in Section 2, focus on reduced divisors over curves. We will use the present section to relax these hypotheses. The resulting general theory is very useful when dealing with multiple modifications as well as modifications for families of curves and vector bundles.

Let X be a variety and E a vector bundle on it. Given an effective Cartier divisor $D \subset X$ and a subbundle $F \subset E|_U$ defined over an open U containing the support of D, we consider the composition

$$E \longrightarrow E|_D \longrightarrow (E/F)|_D$$

of the restriction to D followed by a quotient. Both parts are surjective, hence so is the composition. We will call the kernel of the composition the *(elementary) modification of E at D along F* and denote it by $E[D \to F]$. Our notation is inspired by the fact that sections of $E[D \to F]$ can be identified with sections of E which point along F when restricted to D:

$$\mathrm{H}^0(E[D \to F]) = \{\sigma \in \mathrm{H}^0(E) \mid \sigma|_D \in \mathrm{H}^0(F|_D)\}.$$

The defining exact sequence of a modification $E[D \to F]$ is

(9.1) $$0 \longrightarrow E[D \to F] \longrightarrow E \longrightarrow (E/F)|_D \longrightarrow 0.$$

The inclusion $E[D \to F] \to E$ becomes an isomorphism when restricted to the complement $X \setminus \mathrm{Supp}(D)$. This is true since the cokernel $(E/F)|_D$ is supported on D.

There is a second sequence, which can also be very handy:

(9.2) $$0 \longrightarrow E(-D) \longrightarrow E[D \to F] \longrightarrow F|_D \longrightarrow 0.$$

This is a consequence of the Snake Lemma applied to the following diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & E & = & E & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F|_D & \longrightarrow & E|_D & \longrightarrow & (E/F)|_D & \longrightarrow & 0
\end{array}
$$

The third useful sequence

(9.3) $\qquad 0 \longrightarrow F \longrightarrow E[D \to F]|_U \longrightarrow (E|_U/F)(-D) \longrightarrow 0$

is a corollary of the following diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & E|_U & = & E|_U & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (E|_U/F)(-D) & \longrightarrow & E|_U/F & \longrightarrow & (E|_U/F)|_D & \longrightarrow & 0
\end{array}
$$

Note that sequences (9.1) and (9.2) are valid over the entire variety X, while (9.3) makes sense only over the open U.

Remark 9.4. The inclusion $F \to E[D \to F]|_U$ in (9.3) splits if the inclusion $F \to E|_U$ splits. We can then write

$$
E[D \to F]|_U = F \oplus (E|_U/F)(-D).
$$

In particular, if U is affine, then both inclusions split.

Remark 9.5. The modification $E[D \to F]$ only depends on the restriction of F to D. Put differently, if F and F' are subbundles of E such that $F|_D = F'|_D$, then

$$
E[D \to F] = E[D \to F'].
$$

For example, if the support of D is an irreducible variety $Y \subset X$ and $D = nY$, then $E[D \to F]$ only depends on F in an n-th order neighborhood of Y.

Under the hypotheses we made, elementary modifications are vector bundles.

Proposition 9.6. *If $F \subset E|_U$ is a subbundle and D is a Cartier divisor on C, then $E[D \to F]$ is a vector bundle.*

Proof. Since $E[D \to F]|_{X \backslash \mathrm{Supp}(D)} \cong E|_{X \backslash \mathrm{Supp}(D)}$, we can pass to an open neighborhood of D. For example, take the locus U where F is defined.

Note that $E[D \to F]$ is finitely presented and it suffices to show it is flat. We will use the local criterion of flatness. Let $\mathcal{A}$ be a coherent sheaf over C. If we apply $\mathcal{T}\mathrm{or}_\bullet(-, \mathcal{A})$ to sequence (9.3), then $\mathcal{T}\mathrm{or}_1(E[D \to F], \mathcal{A})$ sits between $\mathcal{T}\mathrm{or}_1(F, \mathcal{A}) = 0$ and $\mathcal{T}\mathrm{or}_1((E/F)(-D), \mathcal{A}) = 0$, so it must also be zero. This proves that $E[D \to F]$ is flat, hence locally free. $\square$

Example 9.7. Taking $F = 0$ and $F = E$ gives rise to two basic examples:

$$E[D \to 0] = E(-D), \qquad E[D \to E] = E.$$

Similarly, when $D = \emptyset$, then

$$E[D \to F] = E$$

is independent of F.

Consider a morphism or varieties $f \colon Y \to X$. Let D be an effective Cartier divisor on X such that its support contains no component of the image of f. Under this hypothesis, the pullback divisor f^*D is well-defined and modifications respect pullbacks.

Proposition 9.8. *Let $f \colon Y \to X$ be a morphism of varieties and D an effective divisor on X such that its support does not contain any component of the image of f. If E is a vector bundle on X and $F \subset E$ a subbundle, then there is a natural isomorphism*

$$f^*E[D \to F] \cong (f^*E)[f^*D \to f^*F].$$

Proof. This follows by pulling back the defining sequence (9.1). □

Remark 9.9. Any vector bundle can be decomposed as a gluing of bundles over affine opens which themselves intersect in affines. The compatibility of modifications and pullbacks, open embeddings in particular, allows us to reduce various statements about modifications over general varieties to statements about affine varieties. We will use this technique in several of the arguments that follow.

Remark 9.10. Suppose we have a vector bundle E over a variety X and a collection of subbundles $F_i \in E$ indexed by $i \in I$. Stating that $\{F_i\}$ are linearly independent means that for all $x \in X$ the fibers $\{F_i|_x\}$ are linearly independent in $E|_x$ as vector spaces.

There is an alternative formulation of this statement. The individual inclusions $F_i \to E$ induce a morphism

$$\varphi \colon \bigoplus_{i \in I} F_i \longrightarrow E.$$

Then $\{F_i\}$ are linearly independent if and only if φ is injective and has locally free cokernel, that is, $\bigoplus_{i \in I} F_i \subset E$ is a subbundle. This restatement is convenient since it allows us to deal with linear independence in a global fashion.

There are correspondences between certain classes of subbundles of E and $E[D \to F]$. If X is a curve this is true more generally without any restrictions on the subbundles in consideration (see Section 10).

To state our result, consider a subbundle $F \subset E|_U$ and an effective divisor D on X whose support is contained in U. We define four sets of subbundles of E:

$$S_1(E, F, D) = \{G \subset E \text{ subbundle} \mid G|_V \subset F|_V \text{ for some neighborhood } V \subset U \text{ of } D\},$$

$$S_2(E, F, D) = \{G \subset E \text{ subbundle} \mid F|_V \subset G|_V \text{ for some neighborhood } V \subset U \text{ of } D\},$$

$$S_3(E, F, D) = \{G \subset E \text{ subbundle} \mid G|_D \text{ and } F|_D \text{ are linearly independent}\}, \text{ and}$$

$$S(E, F, D) = S_1(E, F) \cup S_2(E, F, D) \cup S_3(E, F, D).$$

Direct inspection shows that

$$S_1(E, F, D) \cap S_2(E, F, D) = \left\{ G \subset E \text{ subbundle} \left| \begin{array}{l} G|_V = F|_V \text{ for some} \\ \text{neighborhood } V \subset U \text{ of } D \end{array} \right. \right\},$$

$$S_1(E, F, D) \cap S_3(E, F, D) = \begin{cases} \{G \subset E \text{ subbundle}\} & \text{if } D = \emptyset, \\ \{0\} & \text{otherwise, and} \end{cases}$$

$$S_2(E, F, D) \cap S_3(E, F, D) = \begin{cases} \{G \subset E \text{ subbundle}\} & \text{if } F|_D = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular, if $D \cap X' \neq \emptyset$ for every irreducible component $X' \subset X$, then

$$S_1(E, F, D) \cap S_2(E, F, D) = \begin{cases} \{\overline{F}\} & \text{if } F \text{ extends to } \overline{F} \text{ defined over } X, \\ \emptyset & \text{otherwise, and} \end{cases}$$

$$S_1(E, F, D) \cap S_3(E, F, D) = \{0\}.$$

Proposition 9.11. *Let $F \subset E|_U$ be a subbundle and D an effective divisor on X whose support is contained in U. Note that we can also treat F as a subbundle of the modification*

$E[D \to F]|_U$ by sequence (9.3). Then there are bijections

$$\varphi_1 \colon S_1(E, F, D) \longrightarrow S_1(E[D \to F], F, D),$$

$$\varphi_2 \colon S_2(E, F, D) \longrightarrow S_2(E[D \to F], F, D),$$

$$\varphi_3 \colon S_3(E, F, D) \longrightarrow S_3(E[D \to F], F, D), \text{ and}$$

$$\varphi \colon S(E, F, D) \longrightarrow S(E[D \to F], F, D),$$

such that

(a) *$\varphi|_{S_i(E,F,D)} = \varphi_i$ for $i = 1, 2, 3$,*

(b) *φ is compatible with pullbacks,*

(c) *given $G_1, G_2 \in S(E, F, D)$, then $G_1 \subset G_2$ in a neighborhood of D if and only if $\varphi(G_1) \subset \varphi(G_2)$ in a neighborhood of D,*

(d) *given $\{G_i \mid i \in I\} \subset S_1(E, F, D)$, then $\{G_i\}$ are linearly independent along D if and only if $\{\varphi(G_i)\}$ are linearly independent along D,*

(e) *given $\{G_i \mid i \in I\} \subset S_3(E, F, D)$, then $\{F\} \cup \{G_i\}$ are linearly independent along D if and only if $\{F\} \cup \{\varphi(G_i)\}$ are linearly independent along D, and*

(f) *if $D = \emptyset$, then $E[D \to F] \cong E$ induces an identification of $S(E, F, D)$ and $S(E[D \to F], F, D)$ such that φ becomes the identity map.*

Proof. Without loss of generality, assume that X is connected. If not, we can use the morphisms φ_i and φ defined for each connected component to assemble their global versions.

Let us start by considering φ_1. We will bootstrap our way up by first constructing a simpler bijection $\varphi_1' \colon S_1(E|_U, D) \to S_1(E[D \to F]|_U, D)$. Note that F sits in both $E|_U$ and $E[D \to F]|_U$, so we can send $G \subset F \subset E|_U$ to $G' = G \subset F \subset E[D \to F]|_U$. To verify that this map sends subbundles to subbundles, we observe that given $F \subset E|_U$ is a subbundle, then $G \subset E|_U$ is a subbundle if and only if $G \subset F$ is a subbundle. An analogous statement is true for the inclusions

$G' \subset F \subset E[D \to F]|_U$. To define φ_1 in terms of φ_1', it suffices to note that φ_1' is compatible with open embeddings (more generally, it is compatible with pullbacks) and $E[D \to F]|_{X\backslash \text{Supp}(D)} \cong E|_{X\backslash \text{Supp}(D)}$. Simply put, the image $G' = \varphi_1(G)$ is glued from $\varphi_1'(G|_U)$ and $G|_{X\backslash \text{Supp}(D)}$ along $U \backslash \text{Supp}(D)$. It is easy to see that both φ_1' and φ_1 are bijections.

The compatibility with pullbacks follows from the fact sequences (9.1) to (9.3) are preserved by pullbacks as long as the hypotheses of Proposition 9.8 are satisfied.

The second and third morphisms, while still induced by the inclusion $E[D \to F] \to E$, are a little more interesting. For example, the main issue with φ_2 is that given a subbundle $G' \subset E[D \to F]$, its image in E is no longer a subbundle. One way to solve the problem is via saturation, but this makes it hard to understand the resulting subbundle $G \subset E$.

Consider the second morphism φ_2. Given a subbundle $G \subset E$ which contains F, we can show that the inclusion $G \to E$ lifts to an inclusion $G[D \to F] \to E[D \to F]$. We have constructed the following diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G[D \to F] & \longrightarrow & G & \longrightarrow & (G/F)|_D & \longrightarrow & 0 \\
& & \iota \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E[D \to F] & \longrightarrow & E & \longrightarrow & (E/F)|_D & \longrightarrow & 0
\end{array}
$$

All vertical maps are injective, and the Snake Lemma produces the short exact sequence

$$0 \longrightarrow \text{Coker}\,\iota \longrightarrow E/G \longrightarrow (E/G)|_D \longrightarrow 0,$$

which identifies

$$\text{Coker}\,\iota \simeq (E/G)(-D).$$

It follows that the inclusion $G[D \to F] \to E[D \to F]$ is a subbundle since we identified its cokernel with $(E/G)(-D)$. The morphism φ_2 can thus be defined by sending G to $G[D \to F]$.

To construct the backward direction of φ_2, we start with $G' \subset E[D \to F]$ which contains F. Similarly to (9.2), there is a morphism $E \to E[D \to F](D)$. We define $G = \varphi_2^{-1}(G')$ as the kernel of the composition

$$E \longrightarrow E[D \to F](D) \longrightarrow E[D \to F](D)/G'(D),$$

which is a subbundle of E. Following our construction, it is easy to check that φ_2 is a bijection and it is compatible with pullbacks.

We proceed to construct φ_3. Let us start by constructing the forward direction first. Take the composition

$$G \longrightarrow E \longrightarrow (E/F)|_D,$$

where $G \subset E$ is such that $G|_D$ and $F|_D$ are linearly independent. By first restricting to D, we can identify its image $((F+G)/F)|_D$ with $G|_D$, so we obtain a morphism

$$G(-D) = \mathrm{Ker}(G \to G|_D) \longrightarrow E[D \to F].$$

Sending G to $G' = G(-D)$ furnishes the forward direction of the bijection. To see that $G(-D) \subset E[D \to F]$ is a subbundle, observe that we have constructed the following diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G(-D) & \longrightarrow & G & \longrightarrow & (F|_D + G|_D)/F|_D & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E[D \to F] & \longrightarrow & E & \longrightarrow & (E/F)|_D & \longrightarrow & 0
\end{array}
$$

Applying the Snake Lemma, we obtain the short exact sequence

$$0 \longrightarrow \operatorname{Coker} \iota \longrightarrow E/G \longrightarrow (E/(F+G))|_D \longrightarrow 0,$$

which identifies

$$\operatorname{Coker} \iota \simeq (E/G)[D \to (F+G)/G].$$

In particular, the cokernel of the first vertical map is a vector bundle by Proposition 9.6. A similar analysis constructs the backward direction of the second map which sends $G' \subset E[D \to F]$ to $G = G'(D) \subset E$. Again, all diagrams are preserved by appropriate pullbacks (see Proposition 9.8).

To construct φ it suffices to note that φ_i agree on all pairwise intersections of their domains. This also ensures part (a) is true. On a similar note, part (f) follows immediately from the constructions of φ_i.

Next, we focus on part (c). Without loss of generality, we may replace X with an irreducible component which intersects D non-trivially. Since E and $E[D \to F]$ are isomorphic over $X \setminus \operatorname{Supp}(D)$, in particular, they are isomorphic over the generic point $\eta \in X$. What is more interesting is that if we identify $E|_\eta$ and $E[D \to F]|_\eta$, then φ_1, φ_2, φ_3, and φ become the identity map. Since containment is a closed property, part (c) follows immediately from the observations we made.

We are left to demonstrate parts (d) and (e) of our claim. Consider a subset $\{G_i \mid i \in I\} \subset S_1(E, F, D)$. Assume that $\{G_i\}$ are linearly independent in a neighborhood $V \subset U$ of D. We can replace X with V so $\{G_i\}$ are linearly independent everywhere. Note that F sits in both E and $E[D \to F]$ as a subbundle. Furthermore, all subbundles of F remain unchanged by φ. Since all G_i are contained in F, part (d) follows immediately.

Finally, consider a subset $\{G_i \mid i \in I\} \subset S_3(E, F, D)$ such that $\{F\} \cup \{G_i\}$ are linearly independent in a neighborhood V of D. Recall that $\varphi_3(G_i) = G_i(-D) \subset E[D \to F]$. After replacing X with V, we have a subbundle $F \oplus G \to E$ where

$G = \bigoplus_i G_i$. This inclusion lifts to a morphism $G(-D) \to E[D \to F]$ which fits in the following diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F \oplus G(-D) & \longrightarrow & F \oplus G & \longrightarrow & G|_D & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E[F \to D] & \longrightarrow & E & \longrightarrow & (E/F)|_D & \longrightarrow & 0
\end{array}
$$

All vertical morphisms are injective, so the Snake Lemma identifies the cokernel of $F \oplus G(-D) \to E[D \to F]$ with

$$
(E/(F+G))(-D)
$$

which is a vector bundle. We have thus shown that $\{F\} \cup \{\varphi(G_i)\}$ are linearly independent in $E[D \to F]$. The backward implication has an analogous proof, so we will omit that. □

Our discussion so far has only handled single modifications. This is insufficient for our purposes, and we would like to be handle more than one modification at a time. If the underlying variety X is a curve, there is a recursive definition which utilizes the curve-to-projective extension theorem [10, I.6.8] and works in full generality (see Section 10). In higher dimensions, one needs to be much more careful. The following notions formalize multi-modifications. Later, we will relate these to the recursive definition for curves.

Definition 9.12. Let $\{F_i \subset E \mid i \in I\}$ be a collection of subbundles. We will say that $\{F_i\}$ is *tree-like* at a point $x \in X$ if for all $I' \subset I$ either

(a) the set of subspaces $\{F_i|_x \mid i \in I'\}$ is linearly independent in E_x, or

(b) there is a distinct pair $i, j \in I'$ and an open $U \subset X$ containing $x \in X$ such that $F_i|_U \subset F_j|_U$.

We will use $\mathrm{TL}_X(\{F_i\})$ to denote the set of tree-like points in X. When there is no ambiguity, we may write $\mathrm{TL}(\{F_i\}) = \mathrm{TL}_X(\{F_i\})$. We will say that $\{F_i\}$ is tree-like along $Y \subset X$ if $Y \subset \mathrm{TL}(\{F_i\})$.

Remark 9.13. Note that being tree-like is a local property. Let $U \subset X$ be an open, and $x \in U$ is a point in it. Being local means that $\{F_i\}$ is tree-like at x if and only if $\{F_i|_U\}$ is tree-like at x. More strongly, being tree-like is also preserved by pullbacks.

On a similar note, since linear independence is an open property, then being tree-like is also open.

Remark 9.14. The definition of being tree-like is inspired by the following observation. Let E be a vector bundle over a variety X, and $\{F_i \subset E\}$ a collection of subbundles. In addition, we consider the inclusion graph of $\{F_i\} \cup \{E\}$. The collection $\{F_i\}$ is tree-like over X if and only if the following two conditions are satisfied:

(a) the inclusion graph is a tree, and

(b) the children of each node are linearly independent.

The definition of tree-like was crafted so we can transfer multiple subbundles and entire modification data through modifications, similarly to Proposition 9.11. To simplify the statement of the following result, set

$$S^{\mathrm{set}}(E, F, D) = \{\{F_i \subset E \text{ subbundle}\} \mid \{F\} \cup \{F_i\} \text{ is tree-like along } D\}.$$

Proposition 9.15. *Let $F \subset E|_U$ be a subbundle and D an effective divisor on X whose support is contained in U. Then there is a bijection*

$$\varphi^{\mathrm{set}} \colon S^{\mathrm{set}}(E, F, D) \longrightarrow S^{\mathrm{set}}(E[D \to F], F, D)$$

$$\{F_i\} \longmapsto \{\varphi(F_i)\}$$

such that

(a) φ^{set} *is compatible with pullbacks, and*

(b) *if $D = \emptyset$ and we identify $S^{\text{set}}(E, F, D)$ and $S^{\text{set}}(E[D \to F], F, D)$, then φ^{set} becomes*
    the identity map.

Proof. As long as we show that φ^{set} is a well-defined bijection, then parts (a) and (b) follow from Proposition 9.11.

Consider a collection of subbundles $\{F_i \subset E \mid i \in I\}$ such that $\{F\} \cup \{F_i\}$ is tree-like along D. For convenience, set $F_0 = F$ and $\bar{I} = \{0\} \sqcup I$. To verify that φ^{set} is well-defined, we first need to show that all F_i are in $S(E, F, D)$, the domain of φ. Fix an index i, and take $I' = \{0, i\}$. By the definition of being tree-like, we know that one of the following is true:

   (1) $F_i \subset F$ in a neighborhood of D,

   (2) $F \subset F_i$ in a neighborhood of D, or

   (3) F and F_i are linearly independent along D.

These cases correspond to $S_1(E, F, D)$, $S_2(E, F, D)$, and $S_3(E, F, D)$ respectively, so $F_i \in S(E, F, D)$. Next, we need to show that $\{F\} \cup \{\varphi(F_i)\}$ is tree-like along D. This follows immediately from the fact that φ respects inclusions and linear independence.

We have demonstrated that φ^{set} is a well-defined map. To conclude our proof, we need to demonstrate it is a bijection. It suffices to note we can construct an inverse $(\varphi^{\text{set}})^{-1}(\{F_i'\}) = \{\varphi^{-1}(F_i')\}$. $\qquad\square$

After establishing transfer for sets of subbundles, the next step in our bootstrapping program is to define modification data and show how to transfer them.

Definition 9.16. A *modification datum for E* is an ordered collection of triples

$$M = \{(D_i, U_i, F_i) \mid i \in I\}$$

such that for each i:

(a) D_i is an effective Cartier divisor on X,

(b) $U_i \subset X$ is an open containing the support of D_i, and

(c) $F_i \subset E|_{U_i}$ is a subbundle

In addition, we will call a datum M *tree-like* if for all subsets $I' \subset I$, there is an inclusion

$$\bigcap_{i \in I'} \mathrm{Supp}(D_i) \subset \mathrm{TL}_{U_{I'}}(\{F_i|_{U_{I'}} \mid i \in I'\}),$$

where $U_{I'} = \bigcap_{i \in I'} U_i$. Put differently, for all $x \in X$ the collection of subbundles $\{F_i \mid x \in D_i\}$ is tree-like at x.

To simplify the transfer statement for modification data, set

$$S^{\mathrm{md}}(E, F, D) = \{M = \{(D_i, U_i, F_i)\} \mid \{(D, U, F)\} \cup M \text{ is a tree-like modification datum}\}.$$

Proposition 9.17. *Let $F \subset E|_U$ be a subbundle and D an effective divisor on X whose support is contained in U. Then there is a bijection*

$$\varphi^{\mathrm{md}} : S^{\mathrm{md}}(E, F, D) \longrightarrow S^{\mathrm{md}}(E[D \to F], F, D)$$

$$\{(D_i, U_i, F_i)\} \longmapsto \{(D_i, U_i, \varphi(F_i))\}$$

such that

(a) φ^{md} *is compatible with pullbacks, and*

(b) *if $D = \varnothing$ and we identify $S^{\mathrm{md}}(E, F, D)$ and $S^{\mathrm{md}}(E[D \to F], F, D)$, then φ^{md} becomes the identity map.*

Proof. Continuing our build up, we will repeatedly refer to Proposition 9.15 in this proof. First, parts (a) and (b) follow immediately once we establish that φ^{md} is a well-defined bijection.

Fix an element $M = \{(D_i, U_i, F_i) \mid i \in I\} \in S^{\mathrm{md}}(E, F, D)$. As before, we set

$$F_0 = F, \qquad U_0 = U, \qquad D_0 = D, \qquad \bar{I} = \{0\} \cup I, \qquad \bar{M} = \{(D, U, F)\} \cup M.$$

Given a subset $I' \subset \bar{I}$, we know that the intersection $\bigcap_{i \in I'} D_i$ lies in the set of tree-like points

$$V_{I'} = \mathrm{TL}_{U_{I'}}(\{F_i \mid i \in I'\}).$$

Applying Proposition 9.15 to $\{F_i|_{V_I'} \mid i \in I'\}$, we conclude that

$$V_{I'} = \mathrm{TL}_{U_{I'}}(\{\varphi(F_i) \mid i \in I'\}).$$

We have demonstrated that $\{(D, U, F)\} \cup \{(D_i, U_i, \varphi(F_i))\}$ is a tree-like modification datum, so $\varphi^{\mathrm{md}}(M) \in S^{\mathrm{md}}(E[D \to F], F, D)$ and φ^{md} is a well-defined map. To see that it is a bijection, it suffices to note we can construct an inverse using $(\varphi^{\mathrm{set}})^{-1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now ready to provide a general definition of vector bundle modifications. The main idea is to recursively use the transfer of modification data (Proposition 9.17).

Definition 9.18. Let X be a variety, E a vector bundle over X, and M a tree-like modification datum for E. If M is empty, then we define $E[\varnothing] = E$. On the other hand, if $M = \{(D, U, F)\} \cup M'$, then

$$E[M] = E[D \to F][\varphi^{\mathrm{md}}(M')],$$

where $\varphi^{\mathrm{md}} \colon S^{\mathrm{md}}(E, F, D) \to S^{\mathrm{md}}(E[D \to F], F, D)$ is the transfer map described in Proposition 9.17. When

$$M = \{(D_1, U_1, F_1), \dots, (D_m, U_m, F_m)\},$$

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we will allow ourselves to write

$$E[M] = E[D_1 \to F_1] \cdots [D_m \to F_m].$$

After establishing the language of multi-modifications, we are ready to describe some of its basic properties. First, we note that modifications respect pullbacks. This is a direct consequence of Proposition 9.8.

Corollary 9.19. *Let $f \colon Y \to X$ be a morphism of varieties and E a vector bundle on X. If $M = \{(D_i, U_i, F_i)\}$ is a tree-like modification datum for E such that $\bigcup_i \mathrm{Supp}(D_i)$ does not contain any component of the image of f, then the pullback datum*

$$f^* M = \{(f^* D_i, f^{-1}(U_i), f^* F_i)\}$$

is tree-like, and there is a natural isomorphism

$$f^* E[M] \cong (f^* E)[f^* M].$$

Next, note that we defined a modification datum as an ordered collection of triples (see Definition 9.16). While the order plays a crucial point in our formulation, it turns out to be irrelevant for the final result $E[M]$ as long as M is a tree-like modification datum.

Proposition 9.20 (Commuting modifications)**.** *Let E be a vector bundle over a variety X, and M a tree-like modification datum. If M' is a datum obtained by reordering M, then there is a natural isomorphism $E[M] \cong E[M']$ compatible with pullbacks.*

Proof. Since any symmetric group is generated by transpositions, it suffices to consider the case

$$M = \{(D_1, U_1, F_1), (D_2, U_2, F_2)\}, \qquad M' = \{(D_2, U_2, F_2), (D_1, U_1, F_1)\}.$$

We also need to know that $\varphi^{\mathrm{md}}_{F_1,D_1} \circ \varphi^{\mathrm{md}}_{F_2,D_2} = \varphi^{\mathrm{md}}_{F_2,D_2} \circ \varphi^{\mathrm{md}}_{F_1,D_1}$ for the subset of the domain where this composition makes sense. If we assume there is an isomorphism $E[M] \cong E[M']$, this statement is automatically true if we pass to any the generic point. But subbundles which agree on all generic points must be the same, so this issue is resolved.

We proceed by making several reductions. First, there is a natural isomorphism $E[M] \cong E[M']$ over $X \setminus (\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2))$, so it suffices to focus on a neighborhood of $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2)$. Next, we can cover this locus by affine opens U which fall in one of the following three categories: (1) $F_1|_U \subset F_2|_U$, (2) $F_2|_U \subset F_1|_U$, or (3) F_1 and F_2 are linearly independent over U. Since cases (1) and (2) are analogous, so we will demonstrate (1) and (3). For simplicity, we can also replace X with U.

Assume that $F_1 \subset F_2$. Since we are working over an affine space, there are splittings $F_2 = F_1 \oplus F_1'$ and $E = F_2 \oplus F_2'$. Then

$$E = F_1 \oplus F_1' \oplus F_2',$$

$$E[D_1 \to F_1] = F_1 \oplus F_1'(-D_1) \oplus F_2'(-D_1), \text{and}$$

$$E[D_2 \to F_2] = F_1 \oplus F_1' \oplus F_2'(-D_2).$$

Using these splittings, we can perform the second modification to arrive at

$$
\begin{aligned}
E[M] &= E[D_1 \to F_1][D_2 \to F_2] \\
&= F_1 \oplus F_1'(-D_1) \oplus F_2'(-D_1 - D_2) \\
&= E[D_2 \to F_2][D_1 \to F_1] \\
&= E[M'].
\end{aligned}
$$

In the third case, we assume F_1 and F_2 are linearly independent which leads to a splitting $E = F_1 \oplus F_2 \oplus F$. A similar computation demonstrates that

$$E[D_1 \to F_1] = F_1 \oplus F_2(-D_1) \oplus F(-D_1), \text{and}$$

$$E[D_2 \to F_2] = F_1(-D_2) \oplus F_2 \oplus F(-D_2),$$

and

$$E[M] = F_1(-D_2) \oplus F_2(-D_1) \oplus F(-D_1 - D_2) = E[M']. \qquad \square$$

Proposition 9.21 (Commuting modifications and twists). *Let E be a vector bundle over a variety X, $F \subset E$ a subbundle, and $M = \{(D_i, U_i, F_i)\}$ a tree-like modification datum. If D is a Cartier divisor (not necessarily effective) and we define the datum $M(D) = \{(D_i, U_i, F_i(D))\}$ for $E(D)$, then $M(D)$ is tree-like and there is a natural isomorphism*

$$E[M](D) = E(D)[M(D)]$$

compatible with pullbacks.

Proof. To see that $M(D)$ is tree-like, it suffices to note that vector bundle inclusion and linear independence are preserved by twisting.

First, assume we know the desired isomorphism exists for negative effective divisors. Given a divisor D, we can always decompose it as $D = D^+ - D^-$ where D^+ and D^- are effective. Using the pair $E(D^+)$ and $M(D^+)$ with divisor $-D^+$, we deduce

$$E[M] = E(D^+ - D^+)[M(D^+ - D^+)]$$

$$\cong E(D^+)[M(D^+)](-D^+).$$

Next, we apply the same result for $E(D^+)$, $M(D^+)$ with divisor $-D^-$:

$$E[M](D) = E[M](D^+ - D^-)$$

$$\cong E(D^+)[M(D^+)](-D^-)$$

$$\cong E(D^+ - D^-)[M(D^+ - D^-)]$$

$$= E(D)[M(D)].$$

We are left to furnish an isomorphism in the case of negative effective divisors. For simplicity, replace D with its negative, so it is effective. Let U be a neighborhood of $\operatorname{Supp} D$. Note that if M is a tree-like datum, then $M' = M \cup \{(D, U, 0)\}$ is also tree-like. Since $E[D \to 0] \cong E(-D)$, then the associated morphism φ^{md} maps the datum M to $M(-D)$. Commutativity implies

$$E[M](-D) \cong E[M][D \to 0]$$

$$\cong E[M']$$

$$\cong E[D \to 0][\varphi^{\mathrm{md}}(M)]$$

$$\cong E(-D)[M(-D)],$$

which concludes out argument. $\qquad\square$

Remark 9.22. When it is clear that M is a modification datum for E, we will allow ourselves to write M instead of $M(D)$. Then the statement of Proposition 9.21 becomes

$$E[M](D) = E(D)[M],$$

so we say that modifications and twists commute.

If we focus on the case of two modifications with identical base divisors, there are two more results worth mentioning.

Proposition 9.23 (Combining modifications). *Let E be a vector bundle over a variety X. Consider a tree-like modification datum $M = \{(aD, U, F_1), (bD, U, F_2)\}$ for E, where a, b is a pair of non-negative integers.*

(a) If $F = F_1 = F_2$, then

$$E[aD \to F][bD \to F] \cong E[(a+b)D \to F].$$

(b) If F_1, F_2 are linearly independent and $a = b = 1$, then

$$E[D \to F_1][D \to F_2] \cong E[D \to F_1 + F_2](-D).$$

In addition, both isomorphisms are compatible with pullbacks.

Proof. Following Remark 9.9, we can assume X is affine. For part (a), there is a splitting $E = F \oplus E/F$, and we compute

$$
\begin{aligned}
E[aD \to F][bD \to F] &\cong (F \oplus (E/F)(-aD))[bD \to F] \\
&\cong F \oplus (E/F)(-aD)(-bD) \\
&\cong F \oplus (E/F)(-(a+b)D) \\
&\cong E[(a+b)D \to F].
\end{aligned}
$$

In part (b), consider a splitting $E = F_1 \oplus F_2 \oplus F_3$. Then

$$
\begin{aligned}
E[D \to F_1][D \to F_2] &\cong (F_1 \oplus F_2(-D) \oplus F_3(-D))[D \to F_2(-D)] \\
&\cong F_1(-D) \oplus F_2(-D) \oplus F_3(-2D) \\
&\cong (F_1 \oplus F_2 \oplus F_3(-D))(-D) \\
&\cong E[D \to F_1 + F_2](-D). \qquad \square
\end{aligned}
$$

10. Elementary modifications for curves

While Section 9 introduced vector bundle modifications in a very general set-ting, it is useful to understand the implications of this theory for curves and fami-lies of curves. The present section will explain more concretely how modifications manifest themselves in this setting, and provide several simple consequences.

A substantial part of bootstrapping the definition of multiple modifications consisted of transfer statements. It turns out that curves allow for a simpler transfer statement for subbundles which extends Proposition 9.11. In particular, this allows us to extend multi-modifications beyond tree-like data at the expense of sacrificing some of the properties we already established (e.g., commutativity).

To state our result, define

$$\overline{S}(E) = \{G \subset E \text{ subbundles}\},$$

where E is a vector bundle over a curve C.

Proposition 10.1. *Let E be a vector bundle over a curve C. Given a subbundle $F \subset E$ and a divisor D whose support is contained in the smooth locus of C, there is a bijection*

$$\overline{\varphi} \colon \overline{S}(E) \longrightarrow \overline{S}(E[D \to F]),$$

such that

(a) *$\overline{\varphi}|_{S(E,F,D)} = \varphi$ where $S(E,F,D)$ and φ are as in Proposition 9.11,*

(b) *$\overline{\varphi}$ is compatible with pullbacks,*

(c) *given $G_1, G_2 \in \overline{S}(E)$, then $G_1 \subset G_2$ in a neighborhood of D implies $\overline{\varphi}(G_1) \subset \overline{\varphi}(G_2)$ in a neighborhood of D, and*

(d) *if $D = \varnothing$, then $E[D \to F] \cong E$ induced an identification of $\overline{S}(E)$ and $\overline{S}(E[D \to F])$ such that $\overline{\varphi}$ becomes the identity map.*

Proof. We start by constructing the map $\overline{\varphi}$. Given a subbundle $G \subset E$ of rank r, we can produce a section σ of the Grassmannian bundle $\mathrm{Gr}(r, E)$ of E.

$$\mathrm{Gr}(r, E)$$
$$\Big\uparrow\!\!\Big\downarrow \sigma$$
$$C$$

The natural inclusion $E[D \to F]$ is an isomorphism over $U = C \setminus \mathrm{Supp}(D)$, so we also have an isomorphism $\mathrm{Gr}(r, E)|_U \cong \mathrm{Gr}(r, E[D \to F])|_U$. It follows that we can treat $\sigma|_U$ as a section of the second Grassmannian bundle over U. The curve-to-projective extension theorem [10, I.6.8] implies there is a unique section $\sigma' \colon C \to \mathrm{Gr}(r, E[D \to F])$ which extends $\sigma|_U$. The new section gives rise to a subbundle $\overline{\varphi}(G) = G' \subset E[D \to F]$.

For part (a), start by picking a bundle $G \in S(E, F, D)$. If we identify $E|_U$ and $E[D \to F]|_U$, then $\overline{\varphi}(G)|_U = G|_U = \varphi(G)|_U$. Since both $\varphi(G)$ and $\overline{\varphi}(G)$ are subbundles, and $U \subset C$ is dense, it follows that $\varphi(G) = \overline{\varphi}(G)$.

Note that it makes sense to consider the pullback by a morphism $f \colon C' \to C$ only if the pullback divisor f^*D is well-defined. This happens exactly when no component of C' is contracted to a point which lies in the support of the divisor D on C (see Proposition 9.8). In particular, the condition is always satisfied for finite morphisms f. Once we understand this limitation, running through the section extension definition of $\overline{\varphi}$, it is clear that $\overline{\varphi}$ is compatible with pullbacks.

Finally, the proofs of (c) and (d) are identical to the arguments we gave in Proposition 9.11. $\qquad\square$

Remark 10.2. Note that $\overline{\varphi}$ satisfies all properties φ does except it does not preserve linear dependence and independence. To illustrate the point, take $C = \mathbb{A}^1$ with a

coordinate x on it, $p = 0$ is the origin, and $E = \mathcal{O}_C \oplus \mathcal{O}_C$. Set

$$F = \langle(1,1)\rangle, \qquad G_1 = \langle(1,0)\rangle, \qquad G_2 = \langle(0,1)\rangle,$$

$$F' = \langle(1,0)\rangle, \qquad G_1' = \langle(1,x)\rangle, \qquad G_2' = \langle(1,-x)\rangle.$$

Then G_1 and G_2 are linearly independent in E, while $\overline{\varphi}(G_1)$ and $\overline{\varphi}(G_2)$ coincide over p in $E[p \to F]$. On the other hand, G_1' and G_2' are linearly dependent at p, but their transfers $\overline{\varphi}(G_1'), \overline{\varphi}(G_2') \subset E[p \to F']$ are linearly independent at p.

In summary, it is possible to modify curves along modification data which are not tree-like, but we need to be careful about switching the order of modifications. Unless otherwise stated, all modifications will be tree-like.

Finally, we present a result which relates the Euler characteristics of a modified bundle and the original one.

Proposition 10.3 (The Euler characteristic of modifications). *Let E be a vector bundle over a curve C.*

(a) *If $D_1, \ldots, D_m$ are effective divisors, and $F_1, \ldots, F_m \subset E$ are subbundles, then*

$$\chi(E[D_1 \to F_1] \cdots [D_m \to F_m]) = \chi(E) - \sum_{i=1}^{m} \deg(D_i)\,\mathrm{rank}(E/F_i).$$

(b) *If D is a any divisor, then*

$$\chi(E(D)) = \chi(E) + \mathrm{rank}(E)\deg(D).$$

Proof. Note that the general statement of part (a) follows by applying the $m = 1$ case several times. When $m = 1$, we take Euler characteristics of the sequence (9.1) and note that

$$\chi((E/F_1)|_{D_1}) = \deg(D_1)\,\mathrm{rank}(E/F_1).$$

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Similarly to the proof of Proposition 9.21, we can reduce (b) to the case of a negative effective divisor which is subsumed by part (a). $\qquad\square$

Remark 10.4. The theory of modifications over general varieties (Section 9) is certainly more complicated than the statements we presented for curves. Dimensions greater than one become very useful if we deal with families of curves and vector bundles. The fact that constructing modifications preserve pullbacks allows us to treat a modification over the total space of a family of curves as a family of modifications over the individual curves.

We will demonstrate this point through a simple example. Let C be a smooth curve, E a vector bundle over C, and $F \subset E$ a subbundle. We consider the family of curves

$$\mathrm{pr}_2 \colon \mathcal{C} = C \times B \longrightarrow B$$

where $B = C$. Given $b \in B$, we will use $i_b \colon C \to \mathcal{C}$ to denote the inclusion of the fiber over the point b. Choose a point $p_0 \in C$, and construct the divisors

$$D_0 = \{p_0\} \times B, \qquad D_1 = \Delta_C, \qquad D = D_0 + D_1.$$

If

$$E' = (\mathrm{pr}_1^* E)[D \to \mathrm{pr}_1^* F]$$

is the global modification, then restricting to a fiber over b gives

$$i_b^* E' = E[i_b^* D \to F] = E[(b + p_0) \to F].$$

This shows that varying the modification divisor in a family produces modifications which also fit in a family. Furthermore, we know that $E[2p_0 \to F]$ is the "limit modification" as b approaches p_0. This is a very simple example to

illustrate the power of modifications over higher dimensional varieties. In general, understanding limits of multiple modifications can be very tricky and being tree-like is the right condition to back our intuitive notion of limits.

11. Interpolation and short exact sequences

The goal of this section is to further develop the properties of interpolation, and in particular its behavior with respect to short exact sequences. We start by recalling the notion of (regular) interpolation from Definitions 3.1 and 3.3.

Definition 11.1. Let E be a rank n vector bundle over a curve C. We say that E satisfies *interpolation* if it is nonspecial, and for every $d \geq 1$, there exists a collection of d points $p_1, \ldots, p_d \in C_{\text{sm}}$ such that

$$h^0(E(-\sum p_i)) = \max\{0,\ h^0(E) - dn\}.$$

There are a number of observations which allow us to verify interpolation more easily.

Remark 11.2. By the upper semi-continuity of h^0, the existence of d points satisfying the equality above implies that a general collection of d points (in one component of C_{sm}^d) satisfies this condition (see Theorem 5.8).

Remark 11.3. In fact, we do not need to check the interpolation condition for every positive integer d. It suffices to verify that the statement holds for $\lfloor h^0(E)/n \rfloor$ and $\lceil h^0(E)/n \rceil$. The first value implies the statement holds for all $d \leq \lfloor h^0(E)/n \rfloor$ and the second for all values $d \geq \lceil h^0(E)/n \rceil$.

We have arrived at a convenient rephrasing of Definition 11.1. Let $h^0(E) = n \cdot d + r$ where $0 \leq r < n$. Consider the following two statements.

(a) There exist points $p_1, \ldots, p_d \in C_{\text{sm}}$ such that

$$h^0(E(-\sum p_i)) = r.$$

(b) There exist points $p_1, \ldots, p_{d+1} \in C_{\text{sm}}$ such that

$$h^0(E(-\sum p_i)) = 0.$$

Assume E has no higher cohomology. If $r = 0$, then interpolation for E is equivalent to (a). In the cases when $r > 0$, interpolation is equivalent to (a) and (b) together.

Remark 11.4. It is also possible to use the language of divisors to characterize interpolation. Consider a vector bundle E which satisfies interpolation. Given an integer $d \geq 1$ and a general effective divisor D (general in one of the components of $\mathrm{Sym}^d C$), then either $h^1(E(-D)) = 0$ (when $\deg D \leq h^0(E)/\mathrm{rank}(E)$) or $h^0(E(-D)) = 0$ (when $\deg D \geq h^0(E)/\mathrm{rank}(E)$). Conversely, if this disjunction holds for all d, then we can deduce interpolation. We have arrived at the following restatement of Definition 11.1.

Proposition 11.5. *A nonspecial vector bundle E satisfies interpolation if and only if for every $d \geq 1$, a general effective Cartier divisor D of degree d satisfies*

$$h^0(E(-D)) = 0 \qquad \text{or} \qquad h^1(E(-D)) = 0.$$

There is a further simplification worth mentioning. Note that we do not need to verify the vector bundle is nonspecial before applying this result.

Proposition 11.6. *A vector bundle E of rank n satisfies interpolation if and only if*

(a) *a general effective divisor D of degree $\lceil h^0(E)/n \rceil$ satisfies $h^0(E(-D)) = 0$, and*

(b) *a general effective divisor D of degree $\lfloor h^0(E)/n \rfloor$ satisfies $h^1(E(-D)) = 0$.*

Furthermore, if $\chi(E) \geq 0$, we can replace $h^0(E)$ with $\chi(E)$ in $\lceil h^0(E)/n \rceil$ and $\lfloor h^0(E)/n \rfloor$.

Proof. To conclude that E is nonspecial, we note that $h^1(E(-D)) = 0$ for some effective divisor of non-negative degree $\lfloor h^0(E)/n \rfloor$. The first part is a direct consequence of Proposition 11.5 and Remark 11.3. For the second part, it suffices to note the same argument implies that $h^1(E) = 0$ as long as $\chi(E) \geq 0$. $\qquad \square$

Note that a general effective divisor of degree d refers to a general point in one of the components of $\mathrm{Sym}^d C$. This is an important distinction to keep in mind when C is reducible.

Characterizing line bundles which satisfy interpolation is particularly simple and worth elaborating on.

Proposition 11.7. *A line bundle satisfies interpolation if and only if it is nonspecial.*

Proof. One direction is implied by the definition of interpolation. For the converse, consider a nonspecial line bundle L. We proceed to choose $m = \mathrm{h}^0(L)$ points $p_i \in C_{\mathrm{sm}}$ as follows. First, pick p_1 such that $\mathrm{h}^0(L(-p_1)) = \mathrm{h}^0(L) - 1$. If $m \geq 2$, we choose a second point p_2 such that $\mathrm{h}^0(L(-p_1 - p_2)) = \mathrm{h}^0(L) - 2$, and so on. This demonstrates that L satisfies interpolation. $\square$

In addition, interpolation is preserved by positive twists. This is not always the case for negative twists, and we will investigate this topic later on (see Proposition 11.19).

Proposition 11.8. *If E satisfies interpolation, and D is any effective Cartier divisor, then $E(D)$ satisfies interpolation.*

Proof. We need to show that for every degree d, there exists a divisor D' of degree d such that either $\mathrm{h}^0(E(D - D')) = 0$ or $\mathrm{h}^1(E(D - D')) = 0$. If $d > \deg D$, take $D' = D + D''$ such that $\mathrm{h}^0(E(-D'')) = 0$ or $\mathrm{h}^1(E(-D'')) = 0$ from the interpolation of E. If $d = \deg D$, take $D' = D$ and note that E is nonspecial. Since $\mathrm{h}^1(E(D - D')) = 0$ is an open condition in D', it follows that there exists some $D' = D_0$ supported on C_{sm} such that $\mathrm{h}^1(E(D - D_0)) = 0$. The interesting case is $d < \deg D$. If we choose an effective divisor $D' \leq D_0$, then $\mathrm{h}^1(E(D - D')) = 0$ follows from $\mathrm{h}^1(E(D - D_0)) = 0$. $\square$

Our next goal is to investigate the behavior of interpolation in exact sequences. In particular, we would like to develop tools which allow us to prove interpolation by inducting on rank or other features of a vector bundle. For the remainder of this section, we will assume that C is an irreducible curve.

Consider a short exact sequence

(11.9) $$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

of vector bundles over a curve C. Given a Cartier divisor D, we twist back by D and consider associated the long exact sequence in cohomology.

(11.10)
$$0 \longrightarrow H^0(F(-D)) \longrightarrow H^0(G(-D)) \longrightarrow H^0(H(-D)) \xrightarrow{\delta_D}$$
$$\xrightarrow{\delta_D} H^1(F(-D)) \longrightarrow H^1(G(-D)) \longrightarrow H^1(H(-D)) \longrightarrow 0$$

We will use $\delta_D \colon H^0(H(-D)) \to H^1(F(-D))$ to denote the only non-trivial connecting homomorphism. Our first result allows us to transfer interpolation from the edges F and H to the middle term G.

Proposition 11.11. *Let F, G, and H be as above. If F and H satisfy interpolation, then G satisfies interpolation if and only if*

(a) $h^0(F) / \operatorname{rank}(F) \leq \lfloor h^0(H) / \operatorname{rank}(H) \rfloor + 1$, and

(b) *for every $d \geq 1$ and a general effective divisor D of degree d, the boundary map δ_D has maximal rank (i.e., it is either injective or surjective).*

Proof. First, assume that G satisfies interpolation. By Proposition 11.5, this means that for a general effective D, either $h^0(G(-D)) = 0$ or $h^1(G(-D)) = 0$. Using sequence (11.10), the first case implies that $h^0(F(-D)) = 0$ and δ_D is injective, and the second that $h^1(H(-D)) = 0$ and δ_D is surjective. In particular, we have demonstrated condition (b) stating that δ_D has maximal rank.

Condition (a) is a little more interesting. Its negative asserts there exists an integer d such that

(11.12) $$\frac{h^0(H)}{\operatorname{rank}(H)} < d < \frac{h^0(F)}{\operatorname{rank}(F)}.$$

For contradiction, assume such an integer d exists. Let D be a general effective divisor of degree d and consider the associated sequence (11.10). The first side of the inequality implies

$$h^0(H(-D)) = 0 \quad \text{and} \quad h^1(H(-D)) > 0,$$

while the second side implies

$$h^0(F(-D)) > 0 \quad \text{and} \quad h^1(F(-D)) = 0.$$

Then

$$h^0(G(-D)) = h^0(F(-D)) > 0 \quad \text{and} \quad h^1(G(-D)) = h^1(H(-D)) > 0,$$

which contradicts our hypothesis that G satisfies interpolation by Proposition 11.5.

The reverse direction also follows by inspecting sequence (11.10). Let us pick an integer $d \geq 1$ and consider a general effective divisor D of degree d. We are given that F and H satisfy interpolation, so the argument can be split in four cases.

Case 1, $h^0(F(-D)) = 0$ **and** $h^0(H(-D)) = 0$**.:**

Since $H^0(G(-D))$ sits between two zeros, it must also be zero.

Case 2, $h^1(F(-D)) = 0$ **and** $h^1(H(-D)) = 0$**.:**

Since $H^1(G(-D))$ sits between two zeros, it must also be zero.

Case 3, $h^0(F(-D)) = 0$ **and** $h^1(H(-D)) = 0$**.:**

   If δ_D is injective, then $h^0(G(-D)) = 0$. Otherwise, if δ_D is surjective, then $h^1(G(-D)) = 0$.

Case 4, $h^1(F(-D)) = 0$ **and** $h^0(H(-D)) = 0$**.:**

   If $h^0(F(-D)) > 0$ and $h^1(H(-D)) > 0$, then d satisfies Equation (11.12), so condition (a) is violated. If either of these is zero, we fall back to one of the first three cases.

Finally, G is nonspecial since $H^1(G)$ sits between $H^1(F) = 0$ and $H^1(H) = 0$ in sequence (11.10) for $D = 0$. This proves that G satisfies interpolation, so the converse implication is complete. $\qquad\square$

Remark 11.13. The forward direction of Proposition 11.11 holds without the irreducibility hypothesis on C, that is, if G satisfies interpolation, then statements (a) and (b) are true.

To construct a counterexample for the converse, consider the curve C obtained by gluing two rational components C_1 and C_2 in a single point. Let F be the line bundle obtained by gluing $\mathcal{O}_{C_1}$ and $\mathcal{O}_{C_2}(2)$, and let H be the line bundle obtained by gluing $\mathcal{O}_{C_1}(2)$ and $\mathcal{O}_{C_1}$. Next, we will take $G = F \oplus H$. Both F and H are nonspecial line bundles, so they satisfy interpolation. Condition (a) is automatically satisfied since $h^0(F) = h^0(H)$ by the symmetry between F and H. Since G is the direct of F and H, it follows that all boundary maps δ_D are zero. To show they are of maximal rank, we need to know that either the source $h^0(H(-D))$ or the target $h^1(F(-D))$ is zero. Again, this is true by the symmetry between F and H and the fact both of them satisfy interpolation. Finally, to see that G does not satisfy interpolation note that there exists no degree 3 divisor D such that $h^0(G(-D)) = 0$ or $h^1(G(-D)) = 0$.

Specializing F to a line bundle yields the following useful result.

Corollary 11.14. *Let F, G, and H be as above, and F is a nonspecial line bundle. If H satisfies interpolation, then G satisfies interpolation if and only if*

(a) $\mathrm{rank}(H)(\mathrm{h}^0(F) - 1) \le \mathrm{h}^0(H)$, *and*

(b) *for every $d \ge 1$ and a general divisor D of degree d, the boundary map δ_D has maximal rank.*

Proof. Other than simplifying the inequality in condition (a), this result follows by noting that a line bundle satisfies interpolation if and only if it is nonspecial (Proposition 11.7). □

Let us return to the short exact sequence of vector bundles (11.9). Defining a positive modification of G takes more effort than the (negative) elementary modifications we have been working with since Section 2. Without introducing any new notation, we will construct positive modification at $p \in C_{\mathrm{sm}}$ first by twisting up to arrive at $G(np)$, and then applying an elementary modification to get

$$G(np)[np \to F] = G(np)[np \to F(np)].$$

We could have also started with the elementary modification $G[np \to F]$ and then twisted up to obtain $G[np \to F](np)$. The two results are naturally isomorphic by Proposition 9.21, so we will avoid stressing the distinction for the sake of convenience.

The reason we call $G(np)[np \to F]$ a positive modification is the existence of a natural morphism $G \to G(np)[np \to F]$. To construct this map, start by observing that both G and $G(np)[np \to F]$ admit injective maps into $G(np)$. The cokernel of the latter $G(np) \to H(np)_{np}$ factors through the cokernel of the

former $G(np) \to G(np)_{np}$, so we have an inclusion $G \to G(np)[np \to F]$.

A very similar argument shows that there is a natural inclusion $F(np) \to G(np)[np \to F]$. The Snake Lemma provides an isomorphism between the cokernel of this morphism and

$$H = \mathrm{Ker}(H(np) \longrightarrow H(np)_{np}).$$

The following diagram with exact rows summarizes our observations.

(11.15)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F(np) & \longrightarrow & G(np)[np \to F] & \longrightarrow & H & \longrightarrow & 0
\end{array}
$$

Remark 11.16. If the inclusion $F \to G$ splits (e.g., if we work in an affine neighborhood of p), then the positive modification is

$$G(np)[np \to F] = F(np) \oplus H.$$

The existence and exactness of diagram (11.15) become immediate.

Proposition 11.17. *Consider diagram* (11.15). *If*

(a) *F, G, and H satisfy interpolation,*

(b) *F is a line bundle,*

(c) *the point $p \in C_{\mathrm{sm}}$ is general, and*

(d) $\mathrm{rank}(H)(h^0(F) + n - 1) \leq h^0(H)$,

then $G(np)[np \to F]$ satisfies interpolation.

Proof. Both $F(np)$ and H satisfy interpolation (for the first, we apply Proposition 11.8), and

$$\text{rank}(H)(\text{h}^0(F(np)) - 1) = \text{rank}(H)(\text{h}^0(F) + n - 1) \leq \text{h}^0(H).$$

To apply Corollary 11.14 and conclude that $G(np)[np \to F]$ satisfies interpolation, we need to verify that the connecting homomorphism $\delta'_D \colon \text{H}^0(H(-D)) \to \text{H}^1(F(-D+np))$ has maximal rank for D a general divisor of degree d and every $d \geq 1$. On the other hand, we can present δ'_D as a composition using the connecting homomorphism $\delta_D \colon \text{H}^0(H(-D)) \to \text{H}^1(F(-D))$ which has maximal rank (Corollary 11.14).

$$
\begin{array}{ccc}
\text{H}^0(H(-D)) & \xrightarrow{\ \delta_D\ } & H^1(F(-D)) \\
\big\| & & \big\downarrow {\scriptstyle \pi} \\
\text{H}^0(H(-D)) & \xrightarrow{\ \delta'_D\ } & \text{H}^1(F(-D+np))
\end{array}
$$

Since the cokernel of $F(-D) \to F(-D+np)$ is supported at p, it has no higher cohomology, so the morphism $\pi \colon \text{H}^1(F(-D)) \to \text{H}^1(F(-D+np))$ is surjective. If δ_D is surjective, then δ'_D is automatically surjective.

The case when δ_D is injective requires a little more work. Note that the image V of δ_D is independent of the point p. Therefore, it suffices to show that the restriction of π to an arbitrary fixed subspace V has maximal rank.

Set $L = K_C \otimes F(-D)^\vee$, where K_C is the dualizing line bundle (which exists since C is lci). The dual problem asks whether the image of the natural inclusion $\text{H}^0(L(-np)) \to \text{H}^0(L)$ intersects an arbitrary fixed space $V \subset \text{H}^0(L)$ transversely. Since the inclusion has codimension n, by shrinking or enlarging V, it suffices

to answer this question when $\dim V = n$. In turn, this is equivalent to the non-vanishing of the Wronskian associated to V [3]. □

Remark 11.18. It is natural to ask whether Proposition 11.17 holds if the rank of F is greater than 1. As presented, the proof does not go through if $\operatorname{rank} F \geq 2$. One of the major obstacles is that the image of $\delta_D \colon \mathrm{H}^0(H(-D)) \to \mathrm{H}^1(F(-D))$ may be contained in $\mathrm{H}^1(F'(-D))$ for some proper subbundle $F' \subset F$.

We have already investigated interpolation and twisting up (see Proposition 11.8). The following result provides a partial converse. Note that the base curve C needs to be irreducible and $\chi(E)$ is relatively large.

Proposition 11.19. *Let E be a vector bundle on an irreducible curve C, and D an effective divisor on C. If*

(a) $E(D)$ *satisfies interpolation, and*

(b) $\chi(E) \geq \operatorname{genus}(C) \operatorname{rank}(E)$,

then E also satisfies interpolation.

Proof. Since interpolation is an open condition, we may replace D by a divisor supported on the smooth locus of C.

Induction on $\deg(D)$ reduces our claim to the case $D = p$. By Proposition 11.6, we only need to show that $\mathrm{h}^1(E(-D')) = 0$ for a general divisor D' of degree $\lfloor \chi(E)/\operatorname{rank}(E) \rfloor$, and $\mathrm{h}^0(E(-D')) = 0$ for general D' of degree $\lceil \chi(E)/\operatorname{rank}(E) \rceil$. Since the arguments are analogous, we will focus on the first case.

For convenience, set $d = \lfloor \chi(E)/\operatorname{rank}(E) \rfloor$ and $g = \operatorname{genus}(C)$. Since $d \geq g$, the Riemann-Roch theorem implies that the natural map $\operatorname{Sym}^d C \to \operatorname{Pic}^d C$ is dominant; hence, it suffices to show that $\mathrm{h}^1(E \otimes L^\vee) = 0$ for a line bundle L of degree d. Since $E(p)$ satisfies interpolation, we know that there exists a divisor D'' of degree $d + 1$ such that $\mathrm{h}^1(E(p - D'')) = 0$. Taking $L = \mathcal{O}_C(D'' - p)$ completes the argument. □

Remark 11.20. Suppose we have a family of curves $\pi\colon \mathcal{C} \to B$ whose central fiber $\mathcal{C}_0 = \pi^{-1}(0)$ is reducible but the general fiber is irreducible. If $\mathcal{E}$ is a vector bundle on $\mathcal{C}$ whose restriction $\mathcal{E}_0$ to $\mathcal{C}_0$ satisfies the hypotheses of Proposition 11.19, then the general fiber $\mathcal{C}_b = \pi^{-1}(b)$ together with $\mathcal{E}_b = \mathcal{E}|_{\mathcal{C}_b}$ also satisfy these conditions. While we cannot conclude that $\mathcal{E}_0$ satisfies interpolation, the general bundle $\mathcal{E}_b$ does satisfy interpolation, which is often sufficient.

We will close this section by presenting a result which states that interpolation is preserved by modifications if we choose appropriately general subbundles. To provide the precise statement, we need to introduce the following notion.

Definition 11.21. Let V be a vector space, and $\{W_b \subset V \mid b \in B\}$ be a collection of subspaces indexed by a set B. We will call $\{W_b\}$ *linearly general* if for each subspace $W \subset V$, there exists $b \in B$ such that W_b and W intersect transversely.

Remark 11.22. Suppose the ambient vector space has dimension n, and all members of the collection $\{W_b\}$ have dimension m. Then to conclude that $\{W_b\}$ is linearly general, it suffices to know that for all subspaces $W \subset V$ of complimentary dimension $n - m$ there exists $b \in B$ such that $W \cap W_b = 0$.

Proposition 11.23. *Let E be a vector bundle over a curve C and $p \in C_{\mathrm{sm}}$ a smooth point. Suppose we have a collection of vector bundles $\{G_b \subset E \mid b \in B\}$ indexed by a set B and $F \subset E$ is a subbundle, such that*

(a) *$F|_p \subset G_b|_p$ for all $b \in B$, and*
(b) *$\{G_b/F|_p \mid b \in B\}$ is linearly general in $E/F|_p$.*

If E and $E[p \to F]$ both satisfy interpolation, then $E[p \to G_b]$ satisfies interpolation for at least one element $b \in B$.

Proof. We can assemble two copies of sequence (9.1) into the following diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
& & & & & & 0 & & \\
& & & & & & \downarrow & & \\
& & 0 & & & & G_b/F|_p & & \\
& & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & E[p \to F] & \longrightarrow & E & \longrightarrow & E/F|_p & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & E[p \to G_b] & \longrightarrow & E & \longrightarrow & E/G_b|_p & \longrightarrow & 0 \\
& & \downarrow & & & & \downarrow & & \\
& & G_b/F|_p & & & & 0 & & \\
& & \downarrow & & & & & & \\
& & 0 & & & & & &
\end{array}
$$

Given a divisor D, we twist the entire sequence by $-D$ and take cohomology. For $A = E/F$, E/G_b, and G_b/F, there are induced isomorphisms $A(-D)|_p \cong A|_p$, so we will use the latter. We will also avoid the H^0-functor in front of skyscraper sheaves supported on a point. The operation we described leads to the following diagram.

$$
\begin{array}{ccccccccccc}
& & & & G_b/F|_p & & & & & & \\
& & & & \downarrow & & & & & & \\
0 \to \mathrm{H}^0(E[p \to F](-D)) & \to & \mathrm{H}^0(E(-D)) & \to & E/F|_p & \longrightarrow & \mathrm{H}^1(E[p \to F](-D)) & \to & \mathrm{H}^1(E(-D)) & \to & 0 \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| & & \\
0 \to \mathrm{H}^0(E[p \to G_b](-D)) & \to & \mathrm{H}^0(E(-D)) & \to & E/G_b|_p & \to & \mathrm{H}^1(E[p \to G_b](-D)) & \to & \mathrm{H}^1(E(-D)) & \to & 0
\end{array}
$$

Now that we have described the basic tools we need, we can proceed with the proof. For each $d \geq 1$, choose a divisor D_d of degree d such that

$$h^0(A(-D_d)) = 0 \qquad \text{or} \qquad h^1(A(-D_d)) = 0$$

for $A = E$ and $A = E[p \to F]$ (Proposition 11.5). Note that each value d falls in one of three cases:

(1) $h^0(E(-D_d)) = 0$,

(2) $h^1(E(-D_d)) = 0$ and $h^1(E[p \to F](-D_d)) = 0$, or

(3) $h^1(E(-D_d)) = 0$ and $h^0(E[p \to F](-D_d)) = 0$.

With the aid of the diagram above, case 1 implies $h^0(E[p \to G_b](-D_d)) = 0$, and case 2 implies $h^0(E[p \to G_b](-D_d)) = 0$.

Note that our argument so far works for all $b \in B$. The handling of case 3 requires a choice of b. Fortunately, there can be at most one value of d which satisfies this case. First observe that $H^0(E(-D_d)) \to E/F|_p$ is an inclusion, and we choose $b \in B$ so that $G_b/F|_p$ is transverse to $H^0(E(-D))$. It follows that the composition $H^0(E(-D)) \to E/F|_p \to E/G_b|_p$ has maximal rank. Injectivity and surjectivity respectively imply $h^0(E[p \to G_b](-D_d)) = 0$ and $h^1(E[p \to G_b](-D_d)) = 0$.

To summarize the argument, as long as we choose $b \in B$ to satisfy the transversality condition described above, Proposition 11.5 applies and we can deduce that $E[p \to G_b]$ satisfies interpolation. $\qquad\square$

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