Self-Dual Strings of Six-Dimensional SCFTs

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Self-dual strings of six-dimensional SCFTs

A dissertation presented
by
Guglielmo Paul Lockhart
to
The Department of Physics
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Physics

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Self-dual strings of six-dimensional SCFTs

Abstract

In this thesis we aim to characterize the self-dual strings of six-dimensional superconformal field theories (SCFTs) and develop a variety of techniques to compute their elliptic genera, which provide information about their spectra.

All known $\mathcal{N} = (1, 0)$ supersymmetric SCFTs in 6d can be obtained by compactifying F-theory on an elliptic Calabi-Yau threefold with singular, noncompact base $B$. Resolving the singularity in $B$ and wrapping D3 branes on the exceptional curves leads to strings on the tensor branch of the SCFT. The strings become tensionless at the superconformal fixed point and appear to play an important role in the dynamics of the SCFT.

For a number of SCFTs we identify two-dimensional $\mathcal{N} = (0, 4)$ quiver gauge theories describing bound states of self-dual strings and obtain their elliptic genera by localization. We achieve this for 6d SCFTs describing M5 or D5 branes at a singularity, as well as $n$ M5 branes probing an M9 plane (corresponding to $n$ small $E_8$ instantons).

More generally, we relate the elliptic genera of strings of 6d SCFTs to the counting of supersymmetric (BPS) particles that arise upon compactification to 5d. This enables us to compute elliptic genera by topological string techniques. Following this approach we obtain information about the strings associated to SCFTs with a single tensor multiplet and minimal gauge group.

Additionally, for SCFTs whose strings arise from M2 branes suspended between M5 or M9 branes we find a quantum mechanical picture in which M5 branes or M9 planes are represented by domain wall operators or states; in this context, elliptic genera are expressed in terms of suitable correlators which we compute.
Finally, we find a non-perturbative completion of the topological string partition function that can be employed to compute $S^5$ partition functions of 5d SCFTs and superconformal indices of 6d SCFTs. In the latter case, we argue that the superconformal index can be written in terms of elliptic genera of the self-dual strings, providing further evidence of the important role these strings play in the dynamics of 6d SCFTs.
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Citations to previously published work

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Ai miei genitori
Thomas e Francesca.
Chapter 1

Introduction

The main topics of this dissertation are six-dimensional superconformal theories (SCFTs) and their self-dual strings. In particular, we study the properties of the strings after moving slightly away from the superconformal fixed point and use this knowledge to gain a better understanding of the six-dimensional theories.

Evidence for the existence of interacting SCFTs in six dimensions has accumulated over time. In Nahm’s 1978 paper [1] superconformal algebras were found to exist up to six dimensions, but their place in physics was ultimately found in the wake of the second superstring revolution, when it was realized that they capture the infrared dynamics in a variety of contexts within string or M-theory where gravity is decoupled [2–7]. In particular, the $\mathcal{N} = (2,0)$ maximally supersymmetric six-dimensional SCFTs were found to describe M5 branes in the context of M-theory, as well as compactifications of Type IIB string theory on an ADE singularity; also, certain $\mathcal{N} = (1,0)$ SCFTs were found to describe the theories of small $E_8$ instantons within $E_8 \times E_8$ heterotic string theory, which are also realized in terms of M5 branes near an M9 plane of Hořava-Witten theory, as well as D5 branes probing ADE singularities in Type IIB string theory. By now a wide collection of six-dimensional SCFTs has been discovered, and a possibly complete classification of them was constructed in [8–10] within the context of F-theory.
Beside being interesting in their own right, it is worth remarking that the $\mathcal{N} = (2,0)$ SCFTs have played a tremendous role in recent years in the study of superconformal field theories in lower dimensions. For instance, an extensive class of four-dimensional $\mathcal{N} = 2$ SCFTs, which is known as class $\mathcal{S}$, was discovered in [11] by considering a stack of $N$ M5 branes compactified on punctured Riemann surfaces; an intricate network of dualities relates different weakly or strongly coupled SCFTs which arise at degenerate limits of the Riemann surface, and significant insights can be gained about them by exploiting basic properties of the $\mathcal{N} = (2,0)$ theory. A similar story holds for three– and two–dimensional theories that arise by compactifying the $\mathcal{N} = (2,0)$ SCFT respectively on three– or four–dimensional manifolds [12,13]. Furthermore, starting with the work of Alday, Gaiotto and Tachikawa [14] it was realized that conformal invariance in 6d implies a number of remarkable dualities between $d$– and $(6 – d)$–dimensional theories on curved manifolds. A better understanding of $\mathcal{N} = (1,0)$ theories may also lead to similar developments in the context of theories with four supercharges.

Intense efforts have gone into studying the $\mathcal{N} = (2,0)$ theory in the past two decades; yet, this theory, along with the $\mathcal{N} = (1,0)$ SCFTs, remains poorly understood. For example, there is evidence that the number of degrees of freedom of the theory of $N$ M5 branes grows as $N^3$ [15–17], but such a behavior is difficult to understand from the point of view of the SCFT. Also, among the degrees of freedom of six-dimensional SCFTs are self-dual two-form fields. No appropriate Lagrangian is known for them except in the Abelian case, and even more significantly, self-duality implies that their coupling is necessarily of order one, casting serious doubts on whether a conventional perturbative scheme can be devised for studying them. Moreover, from the brane constructions it appears that the degrees of freedom of the SCFT include strings that couple to the self-dual two forms. These arise, for instance, from M2 branes suspended between M5 branes in the $A_N$ $\mathcal{N} = (2,0)$ theory, and appear to become tensionless at the superconformal point where all M5 are on top of each other.

In this thesis we study the strings associated to a number of interesting six-dimensional
SCFTs, including the theories of M5 branes probing A or D type singularities, D5 branes probing ADE singularities, and M5 branes in the vicinity of an M9 plane, as well as the theories with a single tensor multiplet and minimal gauge group. More precisely, we move to the tensor branch of these theories, on which the strings acquire a small tension and become more tractable, and develop a number of approaches to study their properties. In particular, in several cases we determine a collection of two-dimensional ultraviolet (UV) quiver gauge theories associated to bound configurations of strings, which flow in the IR to CFTs describing their dynamics. In other cases where a quiver gauge theory description is not available we are still able to study the strings by resorting to various dual realizations of the SCFTs. One of the main results in this thesis is the computation of the strings’ elliptic genera, which capture quantitative information about their spectrum.

An important question is how the features of the six-dimensional SCFTs are encoded in the strings. We find quite generally that the BPS index of the five-dimensional theories that are obtained by compactifying the 6d theories on a circle can be expressed as a sum over elliptic genera of various bound states of the strings. Among other things, this makes modular properties of these BPS indices [18] manifest. Furthermore, we find that the six-dimensional superconformal index (that is, its partition function on $S^5 \times S^1$) can also be expressed in terms of elliptic genera.

The UV description we find for the strings already reveals non-trivial information about protected quantities associated to the self-dual strings. An important next step would be to identify the two-dimensional CFTs that govern the dynamics of the strings in the infrared (IR). In particular, this would allow one to study scattering of the strings and determine to what extent six-dimensional dynamics is encoded in terms of them.

In the remainder of the introduction we provide an overview of the various themes that underlie the work presented in the main body of the thesis. We begin by summarizing basic properties of superconformal theories in six dimensions and sketching the classification of $\mathcal{N} = (1,0)$ theories within F-theory. Next, we discuss dual realizations of various classes of $\mathcal{N} = (1,0)$ theories within string theory. We then explain how self-dual strings arise in these different contexts, how
they can be described in terms of quiver gauge theories and how their elliptic genera are computed. Following this, we discuss the connection between elliptic genera, counting of BPS states in five dimensions, and topological strings. We conclude by discussing the relation between the elliptic genera of the self-dual strings and the superconformal index of six-dimensional SCFTs.

1.1 Superconformal field theories in six dimensions

In this section we give a brief summary of some well-known properties of six-dimensional superconformal theories. In six dimensions minimal supersymmetry, which is conventionally denoted as $\mathcal{N} = (1,0)$, consists of eight supercharges arranged in a pair of spinors of a given chirality satisfying a symplectic Majorana-Weyl condition. The R-symmetry group in this case is given by $Sp(1) \simeq SU(2)$. One can also consider theories with extended supersymmetry: a theory with $\mathcal{N} = (n_L, n_R)$ supersymmetry has respectively $n_L$ and $n_R$ chiral and anti-chiral supersymmetries and $Sp(n_L) \times Sp(n_R)$ R-symmetry group. However, only when $\mathcal{N} = (1,0)$, $(2,0)$ or $(1,1)$ the supersymmetry algebra makes it possible to construct theories without gravity. Let us next discuss the allowed massless field content. In the $\mathcal{N} = (1,0)$ case one finds three types of massless multiplets:

- Tensor multiplets, which are composed of a self-dual symmetric tensor field $B_{\mu\nu}$, a doublet of fermions $\chi^A$ of negative chirality, and a real scalar $\phi$;

- Vector multiplets, composed of a gauge field $A_\mu$ and a doublet of chiral fermions $\lambda^A$ of positive chirality;

- Hypermultiplets, composed of four real scalars $\varphi^i$, which can be arranged in a complex doublet under the R-symmetry group, and two fermions of negative chirality.

It is worth noting that in six dimensions vector multiplets do not include scalars, so there is no equivalent of a Coulomb branch. On the other hand, one can have a Higgs branch parametrized by vacuum expectation values of the scalars $\varphi^i$ of the hypermultiplets, as well as a ‘tensor branch’ parametrized by the vev of the real scalar $\phi$ in the tensor multiplet.
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The multiplet structure of $\mathcal{N} = (1, 1)$ and $(2, 0)$ theories is much simpler. In the $(1,1)$ case the only allowed multiplets are vector multiplets composed of $\mathcal{N} = (1, 0)$ vector and hypermultiplets. The IR limit of this class of theories is just super-Yang Mills theory which in six dimensions is free. Much more interesting is the $\mathcal{N} = (2, 0)$ case, in which one finds tensor multiplets which are composed of a $\mathcal{N} = (1, 0)$ tensor multiplet and a hypermultiplet. In the IR one finds strongly coupled theories that admit an ADE classification, which we review in the next section.

The main feature that sets six-dimensional SCFTs apart from other superconformal theories is the presence of self-dual two-form fields $B_{\mu \nu}$. Self-duality implies that the coupling is of order one, which makes it unlikely that six-dimensional superconformal theories may be formulated in a conventional perturbative scheme. Moreover, the existence of self-dual tensor fields suggests the existence of string-like objects that couple to them via an interaction of the form

$$\int_{\Sigma} B_{\mu \nu} + \ldots,$$

where $\Sigma$ is the string worldsheet and additional terms are required by supersymmetry. For the theory to be superconformal, these strings would need to have zero tension, which makes it even more difficult to make sense of them. The situation becomes more tractable on the tensor branch, however, where one does find string-like supersymmetric (BPS) objects whose tension is set by the vev of the scalar in the tensor multiplet $\langle \phi \rangle$. In this thesis, we develop various methods to study these BPS strings and in various cases we succeed at finding a UV description for them, which we expect to flow in the infrared to a CFT that captures their dynamics. Such a worldsheet theory will necessarily be very different in nature from the one describing ordinary superstrings. An obvious difference is that the strings we discuss do not propagate in ten dimensions and do not include a graviton among their massless excitations. Furthermore, we will find that they can form bound states at threshold, which makes their dynamics even more exotic. While we do manage to find a UV description for them, their IR CFT is still not known. Understanding it would make it possible to study them in much more detail, and may clarify whether it is possible to formulate six-dimensional SCFTs as theories of interacting strings.
1.2 Classification of $\mathcal{N} = (1,0)$ SCFTs from F-theory

Before reviewing the classification of $\mathcal{N} = (1, 0)$ theories in the context of F-theory, which was recently achieved in [8–10] (see also [19]), it is instructive to briefly summarize the ADE classification of $\mathcal{N} = (2, 0)$ theories. Six-dimensional SCFTs with $\mathcal{N} = (2, 0)$ supersymmetry were discovered by considering Type IIB string theory compactified on a K3 surface [2], which leads to a six-dimensional supergravity theory with sixteen chiral supercharges. Within the family of K3 surfaces, which consists of all non-trivial Calabi-Yau manifolds in two complex dimensions, one encounters surfaces with local singularities of the form $\mathbb{C}^2/\Gamma$, where $\Gamma$ is a discrete subgroup of a $SU(2)$ factor of the isometry group $SO(4) \simeq SU(2) \times SU(2)$ of $\mathbb{C}^2$. The six-dimensional SCFT is obtained by considering a singular Calabi-Yau of this kind, in the limit that the volume of the K3 surface is taken to infinity, which has the effect of decoupling gravity in the theory living in the six transverse dimensions. Only a subset of the discrete subgroups of $SU(2)$ is allowed in K3 compactification, but if in fact one does not insist on coupling the SCFT to gravity it is possible to consider Type IIB string theory on an arbitrary local singularity of the form $\mathbb{C}^2/\Gamma$. The discrete subgroups of SU(2) admit an ADE classification which is easily seen by resolving the singularity. This involves blowing up the singular point a number of times, leading to a nonsingular geometry with a number of exceptional genus zero curves ($\mathbb{P}^1$s). These curves have self-intersection $-2$ and intersect each other according to the Cartan matrices of type $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, or $E_8$, as shown in Figure 1.1.

Resolving the singularity corresponds to moving to the tensor branch of the 6d SCFT. In fact, for each exceptional $\mathbb{P}^1$ one obtains a $(2, 0)$ abelian tensor multiplet. The self-dual two forms $B_i$ in the tensor multiplets arise by decomposing the self-dual Ramond-Ramond four-form $C^+$ field of Type IIB string theory as $C^+ = \sum_i B_i \wedge S_i$, where $S_i$ is a self-dual harmonic two-form which is supported on the $i$-th exceptional $\mathbb{P}^1$. Since in the geometry under consideration the Hodge star operators in 10, 6 and 4 dimensions are related by $\star_{10} = *_{6} *_{4}$, self-duality of $B_i$ follows from self-duality of $S_i$ and $C^+$. The objects that are charged under $C^+$ are D3 branes, and wrapping them
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Figure 1.1: Exceptional $\mathbb{P}^1$s intersecting according to ADE diagrams

on the exceptional $\mathbb{P}^1$s gives rise to strings coupled to the self-dual two-forms $B_i$. Furthermore, the tension of a string is proportional to the volume of the $\mathbb{P}^1$ it is associated with, and is zero at the superconformal fixed point corresponding to the singular limit of the local geometry.

The ADE-type superconformal theories obtained from Type IIB compactification are believed to be the only consistent six-dimensional superconformal theories with $\mathcal{N} = (2,0)$ supersymmetry. A complete classification of theories with $\mathcal{N} = (1,0)$ supersymmetry is a much more laborious task, since the lower amount of supersymmetry poses weaker restrictions on the allowed theories. It turns out that F-theory [20], which can be viewed as a generalization of the standard Type IIB string theory to include situations in which the axio-dilaton field is allowed to vary and potentially become singular, provides a very natural framework for attempting this classification.

In fact, the set of theories obtained in [8–10] from F-theory includes all presently known examples of 6d $\mathcal{N} = (1,0)$ SCFTs, and may well be a complete list of consistent six-dimensional superconformal theories with eight supercharges.

The key realization of F-theory is that the $SL(2,\mathbb{Z})$ symmetry of Type IIB string theory,
which is non-perturbative in nature since it encompasses S-duality as a subgroup, admits a very natural geometric interpretation. To see this, one notes that the two scalar fields in Type IIB supergravity, namely the dilaton $\phi$ and axion $\chi$, can be combined into a complex scalar $\tau_F = \chi + i e^{-\phi}$ which transforms under $SL(2, \mathbb{Z})$ as the modulus of a torus:
\[
\tau_F \to \frac{a \tau_F + b}{c \tau_F + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}).
\]
Furthermore, since D7 branes are sources of magnetic charge for the axion, going around a D7 brane results in a monodromy of the axio-dilaton:
\[
\tau_F \to \tau_F + 1,
\]
which means that in the vicinity of the D7 brane the axio-dilaton has the following profile
\[
\tau_F(z) \sim \frac{1}{2\pi i} \log(z),
\]
which is allowed since that $\tau_F \to \tau_F + 1$ lies within the $SL(2, \mathbb{Z})$ symmetry of the theory. In fact one can also consider the possibility of the axio-dilaton undergoing more general $SL(2, \mathbb{Z})$ transformations, which would signal the presence of more exotic, non-perturbative versions of the D7 branes. Such a scenario is admissible provided that one freezes some fields to zero. For instance, the Ramond and Neveu-Schwarz two-form fields of Type IIB string theory transform as a $SL(2, \mathbb{Z})$ doublet and turning them on would in general not be consistent with allowing $\tau_F$ to undergo monodromies. Likewise, one is usually not allowed to include the D1 or F1 (fundamental) strings that couple to them. On the other hand, the four-form $C^+$ and the D3 branes that couple to it are singlets of $SL(2, \mathbb{Z})$ and should be included; in fact, they are key ingredients in obtaining six-dimensional SCFTs from F-theory, as we will see shortly.

These configurations can be captured very elegantly in terms of an elliptic fibration over ten-dimensional spacetime, in which the torus fiber, whose modulus is given by $\tau_F$, is fibered non-trivially and can become singular at complex codimension–1 loci, signaling the presence of D7
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branes. For instance, one can consider compactification of Type IIB string theory with varying
dilaton on $\mathbb{R}^{10-d} \times B$, where one allows the F-theory torus to be fibered over the $d$-dimensional
manifold $B$. It can be argued that taking the total $(d + 2)$-dimensional space $T^2 \hookrightarrow X \to B$ to
be a manifold of special holonomy leads upon compactification to a supersymmetric theory in the
remaining $10 - d$ dimensions.

In particular, compactification of F-theory on an elliptic Calabi-Yau threefold (which we
take to be compact for the moment) leads to six-dimensional supergravity theories with $\mathcal{N} = (1,0)$
supersymmetry. The field content of the theory can be read off from the geometry [21, 22]: beside
the supergravity multiplet, one finds

$$T = h^{1,1}(B) - 1$$

tensor multiplets that arise as in the $\mathcal{N} = (2,0)$ case from expanding the four-form $C^+$ in terms of
the $h^{1,1}(B) - 1$ self-dual two-forms in the base (the remaining two-form in the base is the Kähler
form which is anti-self-dual, and leads to an anti-self-dual two-form in 6d that belongs to the gravity
multiplet). Furthermore, the number of vector multiplets is given by

$$V = h^{1,1}(X) - h^{1,1}(B) - 1,$$

where the $-1$ arises because the cohomology of $X$ also includes an element associated to the elliptic
fiber, which does not contribute a multiplet upon compactification to six-dimensions. Finally, the
6d theory also includes hypermultiplets; one can distinguish between the ones which are neutral
and those which are charged under the gauge fields. The number of neutral multiplets is given by

$$H_0 = h^{2,1}(X) + 1,$$

while the number of charged hypermultiplets can be read off from the equation that guarantees
cancelation of the gravitational anomaly in six-dimensions [23, 24]:

$$H_{ch} = 273 - 29T + V - H_0.$$
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While enumerating the fields is straightforward, working out the structure of the gauge group and matter content is in general more subtle. As far as the gauge group structure is concerned, one finds that to each divisor in the base over which the elliptic fiber degenerates is associated a simple gauge group, whose type is determined by the type of singularity of the fiber. The classification of singular fibers that arise in the context of F-theory was obtained long ago by Kodaira [25], and corresponds to various types of exotic D7 branes that carry different gauge groups.

Analogously to the case of $\mathcal{N} = (2,0)$ SCFTs, gravity can be decoupled by taking the base of the elliptic fibration $B$ to be non-compact, leading to 6d superconformal field theories with $\mathcal{N} = (1,0)$ supersymmetry. The number of tensor multiplets is then given by $h^{1,1}(B)$, since the class corresponding to the anti-self-dual two-form is no longer present; wrapping D3 branes over the corresponding cycles in the base gives rise, as discussed in the context of $\mathcal{N} = (2,0)$ SCFTs, to self-dual strings on the tensor branch. The nature of the gauge groups that arise from degenerating the elliptic fiber over compact cycles in the base can still be determined by analyzing the nature of the singularity; furthermore, it is now possible to have global symmetries arising by degenerating the fiber over non-compact directions in the base, which corresponds in Type IIB language to having non-compact D7 branes.

We are now in a position to discuss the classification of the 6d SCFTs that arise in the context of F-theory. The strategy followed in [8–10] draws inspiration from the classification of allowed bases for 6d supergravity theories pursued in [26]. In both cases, it proves extremely helpful to start from the simpler problem of identifying a set of basic theories for which it is not possible to Higgs the gauge group, along with their geometric realization in F-theory. One is then faced with the task of constructing more complex theories by ‘gluing’ the corresponding basic geometries in an appropriate sense, as well as identifying all allowed choices of gauge groups (i.e. singularity types) which reduce to the minimal choices upon Higgsing. Moreover, one can determine the allowed

$^1$In addition to this, monodromies or the presence of a non-trivial Mordell-Weyl group can complicate the identification of the gauge groups or lead to additional U(1) gauge symmetries. These subtleties, although important, are largely tangential to the work presented in this thesis, so we will not dwell on them further.
hypermultiplet structures by requiring the gauge symmetries not to be anomalous \[10\].

We first give an overview of the various basic theories as well as the gluing procedure, and then provide some examples of Higgsable theories that will be relevant in the following chapters. Most of the basic theories involve a single tensor multiplet and can be obtained very easily starting from elliptic fibrations over the Hirzebruch surfaces $\mathbb{F}_n$, $n = 1, \ldots, 12$, which can be described as fibrations of a $\mathbb{P}^1$ over a $\mathbb{P}^1$ base\(^2\). The second degree homology of $\mathbb{F}_n$ is spanned by two divisors $F$ and $D$. The first divisor corresponds to the fiber $\mathbb{P}^1$ and has self-intersection $F \cdot F = 0$; the other divisor is transverse to the fiber and satisfies $D \cdot D = -n$ and $D \cdot F = 1$. Furthermore, in order for the total space to be a Calabi-Yau threefold, one finds that the elliptic fiber is generally fibered non-trivially over the base. The next step is to decompactify the Hirzebruch surface by taking the size of the fiber $\mathbb{P}^1$ to be very large and zoom onto the divisor $D$. In this limit, the base geometry is the line bundle $\mathcal{O}(-n) \to \mathbb{P}^1$, where $\mathcal{O}(-n)$ is the $n$-th tensor power of the tautological bundle $\mathcal{O}(-1) \to \mathbb{P}^1$. Let us discuss the various theories that arise:

- In the $n = 1$ case, one finds that the elliptic fiber is non-degenerate over the base $\mathbb{P}^1$, so the resulting theory has no gauge symmetry; however, one finds that the elliptic fiber does degenerate along the non-compact direction over twelve points on the base. This leads to a global $E_8$ symmetry of the 6d SCFT. The six-dimensional theory under consideration is the exceptional string theory (E-string theory for short); we will review it in more detail in Section 1.3. It is also worth remarking that one can describe the same geometry as a fibration of the anti-canonical line bundle $\mathcal{O}(-K)$ over the half-K3 surface $T^2 \hookrightarrow \frac{1}{2}K3 \to \mathbb{P}^1$, which is simply the elliptic fibration over the base $\mathbb{P}^1$ described above. This alternative description of the $n = 1$ geometry has also played an important role in understanding the E-string theory \[28\].

- In the $n = 2$ case one also finds no gauge group; in fact, the total space is the direct product

\[2\] The reader may find it helpful to consult, for example, the textbook \[27\] for a very accessible introduction to the geometric concepts that appear in the following discussion.
\( T^2 \times (\mathcal{O}(-2) \to \mathbb{P}^1) \). In other words, this geometry is equivalent to Type IIB string theory on \( \mathbb{C}^2/\mathbb{Z}_2 \), which gives rise to the \( \mathcal{N} = (2,0) \) theory of type \( A_1 \).

- In the remaining cases \( n = 3, \ldots, 12 \), one finds that the elliptic fiber necessarily degenerates over the base \( \mathbb{P}^1 \). In the cases \( n = 9, 10, 11 \) one finds that in order to obtain a non-singular base it is necessary to blow up additional singularities in the base, which eventually leads to a \( n = 12 \) curve intersecting respectively three, two, or one \( n = 1 \) curves. In the cases \( n = 3, \ldots, 8 \) and \( n = 12 \) however one finds geometries which give rise to 6d SCFTs with a single tensor multiplet and non-trivial gauge group, as listed in the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( SU(3) )</td>
</tr>
<tr>
<td>4</td>
<td>( SO(8) )</td>
</tr>
<tr>
<td>5</td>
<td>( F_4 )</td>
</tr>
<tr>
<td>6</td>
<td>( E_6 )</td>
</tr>
<tr>
<td>7</td>
<td>( E_7 )</td>
</tr>
<tr>
<td>8</td>
<td>( E_7 )</td>
</tr>
<tr>
<td>12</td>
<td>( E_8 )</td>
</tr>
</tbody>
</table>

In the \( n = 7 \) case one also finds a half-hypermultiplet in the fundamental representation of \( E_7 \) (which by itself cannot be used to Higgs the gauge group).

It is also worth remarking that the geometries corresponding to \( n = 2, 3, 4, 6, 8, \) and 12 admit a simple realization as \( \mathbb{Z}_n \) orbifolds of \( T^2 \times \mathbb{C}^2 \), which act on the complex coordinates \( (\lambda, z_1, z_2) \) of this geometry as

\[
\begin{pmatrix}
\lambda \\
z_1 \\
z_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\omega^{-2} & & \\
& \omega & \\
& & \omega
\end{pmatrix}
\begin{pmatrix}
\lambda \\
z_1 \\
z_2
\end{pmatrix},
\tag{1.2.2}
\]

where \( \omega = \exp(2\pi i/n) \).

The geometries with \( n > 3 \) are examples of the non-Higgsable clusters (NHC) defined in [26]. In addition to the NHCs with a single tensor multiplet discussed here, three additional NHCs exist which have more than one tensor multiplet; we display them in Figure 1.2, where we indicate the self-intersections of the \( \mathbb{P}^1 \)'s in the geometry, as well as the gauge groups over them and bifundamental matter between the gauge groups.

Any 6d SCFT with \( \mathcal{N} = (1,0) \) supersymmetry that arises in F-theory can be decomposed in terms of the building blocks described above. The next step in the classification is to describe
Figure 1.2: Non-Higgsable clusters with more than one tensor multiplet.

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how these basic geometries can be glued to construct more complicated SCFTs. We can distinguish between two cases:

- One can glue together \( n = 2 \) curves according to ADE-type diagrams (see Figure 1.1). This gives rise to all the \( \mathcal{N} = (2,0) \) SCFTs. It is also possible to attach a \( n = 1 \) curve at the end of a linear chain of \( n = 2 \) curves, which gives rise to the SCFT of several small \( E_8 \) instantons.

- The NHCs can be glued among each other by intersecting them with \( n = 1 \) curves to form trees of intersecting \( \mathbb{P}^1 \)s. Each \( n = 1 \) curve can intersect at most two other curves to which the gauge groups \( G_1, G_2 \) are associated, and it must be possible to have the following embedding of the corresponding gauge algebras: \( \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{e}_8 \). Furthermore, at certain nodes it is also possible to attach a \( n = 1 \) curve followed by a string of \( n = 2 \) curves. Geometric considerations lead to a set of rules for how such trees of intersecting \( \mathbb{P}^1 \)s can be constructed and to a classification of SCFTs with minimal gauge group [10]. It is also worth mentioning that the minimal six-dimensional SCFTs can also have global symmetries originating from the elliptic fiber being singular over non-compact fibers on top of \( n = 1 \) curves.

From a given minimal theory, it is then possible to construct Higgsable theories by making the elliptic fiber more singular. Again, geometric considerations (as well as cancelation of gauge anomalies, which helps in determining the hypermultiplet content) make it possible to determine the allowed set of Higgsable theories that reduce to any given minimal theory, as explained in [10]. This leads to a
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Figure 1.3: F-theory realization of the theory of $M$ M5 branes at an $A_{N-1}$ singularity. We indicate non-compact cycles supporting $I_N$-type singularities by open lines.

Figure 1.4: Chain of alternating $n = 4$ and $n = 1$ curves, corresponding to the theory of M5 branes probing a D type singularity.

In this thesis we study the self-dual strings that arise in a number of 6d SCFTs. In particular, we focus on the theories that have the following F-theory realizations:

- Theories corresponding to $M$ M5 branes probing an $A_{N-1}$ singularity, which can be viewed as orbifolds of the $(2, 0)$ theory of M5 branes and have $SU(N)^{M-1}$ gauge group in six dimensions. They can be realized in F-theory in terms of a linear chain of $n = 2$ curves with $I_N$ singularities on them, as shown in Figure 1.3. We study this class of theories extensively in Chapters 2 and 3.

- Minimal theories corresponding to a base with a single two-cycle and minimal degeneration of the elliptic fiber, which we study using topological string techniques in Chapter 4.
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Figure 1.5: F-theory realization in terms of intersecting $n = 2$ curves of the theories of D5 branes probing ADE singularities.

- Theories obtained by compactifying F-theory on a chain of alternating $n = 1$ and $n = 4$ curves, on top of which one has respectively singularities of $I_{2p}^{n_8}$ or $I_{p}^{*}$ type leading to $Sp(p)$ and $SO(8+2p)$ gauge groups. These theories generically have $SO(8+2p) \times SO(8+2p)$ global symmetry and are pictured in Figure 1.4. They also admit a realization in terms of M5 branes probing a $D_p$-type singularity. We will study them in Chapters 4 and 5.

- Theories of $p$ D5 branes probing ADE type singularities, which are realized within F-theory in terms of an ADE configuration of $n = 2$ curves with singular fibers of Kodaira type $I_p$ over them. These are pictured in Figure 1.5, and will be studied in Chapter 5. Note that the case of $p$ D5 branes probing an $A_M$ singularity is equivalent to that of $M + 1$ M5 branes probing an $A_{p-1}$ singularity, as in Figure 1.3. Indeed, these theories are related by T-duality as we discuss later.
• Theories with global $E_8$ symmetry obtained by attaching a $n = 1$ curve to a linear chain of $n = 2$ curves (see Figure 1.6). These geometries correspond to the theories of small $E_8$ instantons, which we study in Chapters 5 and 6.

In the next section we discuss these theories in more detail and provide various dual string theory realizations for them.

1.3 Dual realizations of 6d SCFTs

A number of the 6d SCFTs discussed in Section 1.2 have alternative descriptions in the context of string theory or M-theory, where they admit interesting interpretations, the most famous example being the $\mathcal{N} = (2,0)$ $A_N$ theory which describes the low-energy dynamics of a stack of $N + 1$ M5 branes. In this thesis we make extensive use of these dual descriptions, which will enable us to study these theories and their self-dual strings from different vantage points.

Type IIB realizations. It is natural to ask whether a theory that is realized within F-theory also admits a weakly coupled Type IIB realization. This is the case if it is possible to tune the Type IIB string coupling $g_s = e^\phi$ to zero. Since in F-theory $e^{-\phi}$ has a geometric interpretation as the imaginary part of the complex modulus $\tau_F = \chi + ie^{-\phi}$ of the elliptic fiber, the appropriate way to phrase the question is whether one can tune the complex structure of Calabi-Yau threefold $X$ so that $\tau_F \rightarrow i\infty$ away from the D7 branes. This is known in the F-theory literature as Sen’s limit [29]. In our context, an added subtlety is that the strings originate from wrapped $D3$ branes.
that for $n \geq 2$ curves sit within the worldvolume of D7 branes. In general, this implies that $\tau_F$ is frozen at some nonzero value on top of the D3 branes and it is not possible to have a weak coupling description.

There are however three exceptions. In the $n = 1$ case the elliptic fiber degenerates at twelve isolated points on the base $\mathbb{P}^1$. One finds, as in [30], that taking the $\tau_F \to i\infty$ limit forces one to tune the positions of some D7 branes. Two pairs of singular fibers merge to give O7$^-$ orientifold planes; additionally, one finds eight D7 branes (see Figure 1.7).

On the other hand, as we saw in Section 1.2, one can view the local geometry on top of a $n = 2$ or 4 curve as the resolution of the orbifold

$$
\begin{pmatrix}
\lambda \\
z_1 \\
z_2
\end{pmatrix} \rightarrow
\begin{pmatrix}
\omega^{-2} & & \\
& \omega & \\
& & \omega
\end{pmatrix}
\begin{pmatrix}
\lambda \\
z_1 \\
z_2
\end{pmatrix},
$$

(1.3.3)

where $\omega^2 = 1$ in the $n = 2$ case and $-1$ for the $n = 4$ case.

In the $n = 2$ case, since the orbifold does not act on the torus fiber it is possible to tune its complex modulus at will. One arrives in the weak coupling limit at Type IIB string theory on an ADE singularity, which of course corresponds to the $\mathcal{N} = (2, 0)$ SCFTs. Introducing elliptic fibers of type $I_N, I_{2N}, \ldots$ as in Figure 1.5 corresponds to wrapping D7 branes on the various $\mathbb{P}^1$s. In the singular limit, one can view this class of theories as corresponding to $N$ D5 branes probing an ADE singularity [7]. In this picture, the self-dual strings are captured in terms of D1 branes within the D5 worldvolume.

In the $n = 4$ case the action of the orbifold (1.3.3) on the elliptic fiber is equivalent to inverting $\tau_F \to -\tau_F$. This $\mathbb{Z}_2$ action is an involution of the elliptic fiber for any value of $\tau_F$, and therefore it does not force one to freeze of the modulus. As discussed in Chapter 4, the resulting Type IIB configuration consists of an O7$^-$ orbifold and four D7 branes on top of a $n = 2$ curve.

**Type IIA realizations.** In this thesis we also exploit a dual Type IIA realization for the
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Figure 1.7: Sen’s limit for the $n = 1$ curve. The stars mark the positions of O7$^-$ planes, while dots mark the positions of D7 branes.

Theories consisting of linear chains of $\mathbb{P}^1$s with wrapped D7 branes and possibly orientifold planes$^3$. The dual setup to the theory of $N$ D5 branes probing an $A_M$ singularity consists of $N$ D6 branes suspended between $M + 1$ NS5 branes (Figure 1.8); the one dual to the theory of Figure 1.4 also involves orientifold six-planes of alternating charge $O6^+/O6^-$ (Figure 1.9). One can also obtain this configuration starting with an O6$^-$ brane and $(4+N)$ D6 branes and placing NS5 branes on top of them. Next, one can space the NS5 branes within the O6$^-$/D6 worldvolume. In this setup, NS5 branes can split into pairs of $\frac{1}{2}$NS5 branes of fractional charge [32], and between these one finds an O6$^+$ plane along with $N$ D6 branes.

In either case, the self-dual strings arise from D2 branes suspended between the NS5 branes, whose separation corresponds to the tension of the strings. In Type IIA string theory it is also possible to realize the small $E_8$ instanton theory of Figure 1.6 in terms of eight D8 branes and one $O8^-$ plane and a number of parallel NS5 branes; again, self-dual strings arise from suspended D2 branes (Figure 1.10). Only a $SO(16)$ subgroup of $E_8$ is visible perturbatively, but the full $E_8$

$^3$This dual setup can be reached by replacing this geometry with multi-charge Taub-NUT space and T-dualizing along the Taub-NUT circle. One can also follow this duality in the case of D5 branes probing a $D_N$ singularity which leads to a Type IIA configuration involving D6 branes, NS5 branes and orientifold planes [31].
symmetry can be realized non-perturbatively by tuning the position of one D8 brane [33].

\[ N \text{ D6} \quad N \text{ D6} \quad N \text{ D6} \quad N \text{ D6} \quad N \text{ D6} \]

\[ \text{D2} \quad \ldots \quad \text{D2} \]

\[ \text{NS5} \quad \text{NS5} \quad \text{NS5} \quad \text{NS5} \]

Figure 1.8: Type IIA brane setup for the SCFT of \( N \) D5 branes probing an A-type singularity.

\[ \frac{1}{2} \text{NS5} \quad \frac{1}{2} \text{NS5} \quad \frac{1}{2} \text{NS5} \quad \frac{1}{2} \text{NS5} \quad \frac{1}{2} \text{NS5} \]

\[ \text{D2} \quad \ldots \quad \text{D2} \quad \text{D2} \]

\[ \text{O6}^- \quad \text{O6}^+ \text{ N D6} \quad \text{O6}^- \quad \text{O6}^+ \text{ N D6} \quad \text{O6}^- \quad \text{O6}^+ \text{ N D6} \quad \text{O6}^- \quad \text{O6}^+ \text{ N D6} \]

\[ \text{(N+4) D6} \quad \text{(N+4) D6} \quad \text{(N+4) D6} \quad \text{(N+4) D6} \]

Figure 1.9: Type IIA brane configuration for the SCFT of Figure 1.4.

**M-theory realizations.** The setup of Figure 1.8 lifts to an M-theory configuration of \( M + 1 \) parallel M5 branes probing an \( A_{N-1} \) singularity. In this context, the self-dual strings correspond to M2 branes stretched between M5 branes, whose separation again gives the string tensions. The setup of Figure 1.9 can similarly be realized in terms of M5 branes probing a D-type singularity and splitting into half-M5 branes of fractional charge. A similar picture holds for M5...
branes probing E-type singularities, as discussed in [9]. It is worth noting that the $D_N$ $(2,0)$ theory can also be realized in M-theory in terms of M5 branes at a $\mathbb{R}^5/\mathbb{Z}_2$ orbifold [5].

The small $E_8$ instanton theory also admits an elegant realization within M-theory in the context of Hořava-Witten theory [34]. Shortly after the discovery of M-theory, Hořava and Witten found that $E_8 \times E_8$ heterotic string theory is equivalent to M-theory on a $(S^1 \times \mathbb{R}^{10})/\mathbb{Z}_2$ orbifold, which can be equivalently be thought as $\mathbb{R}^{10}$ times an interval whose length is proportional to the heterotic string coupling. Gravitational anomaly cancelation requires the introduction of additional degrees of freedom localized at the two endpoints of the interval, which are sometimes referred to as M9 planes. Specifically, it was found in [34] that one needs to introduce an $E_8$ gauge bundle at each end. A small $E_8$ instanton in this context is an instanton on, say, the left end of the interval; it was argued in [3] that in the limit of zero size it is possible to trade 29 hypermultiplets with a tensor multiplet (which is consistent with Equation (1.2.1)) and move to the tensor branch of the theory, which corresponds to nucleating a M5 brane from the M9 boundary. Repeating this process for a total of $N$ small instantons leads to $N$ parallel M5 branes near the M9 plane, which is equivalent to the F-theory configuration of a chain of $N - 1$ $n = 2$ curves attached to a single $n = 1$ curve, as in Figure 1.6. Stretched M2 branes again lead to self-dual strings. This M-theory configuration is depicted in Figure 1.11.
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Figure 1.11: M5 branes near the M9 plane, corresponding to the theory of small $E_8$ instantons on the tensor branch. Moving the M9 plane infinitely far away from the M5 branes, one recovers the $A_N$ (2,0) theory.

Finally, in the case of M5 branes probing an $A_N$ singularity, upon circle compactification to 5d one can follow another duality to a system of $(p, q)$ five-branes in Type IIB string theory, or equivalently \[35\] M-theory on a toric Calabi-Yau threefold. We will discuss this realization in due course in Chapters 2 and 3, where it proves very useful in computations.

1.4 Self-dual strings and 2d quiver gauge theories

In principle, one would wish to derive the worldsheet theory on the self-dual strings by dimensional reduction of the worldvolume theory of the D3 branes along the lines of \[36\]. This should lead to a two-dimensional sigma model on the string worldsheet. On the D3 branes that wrap a two-cycle in $B$ lives the four-dimensional $U(N) \mathcal{N} = 4$ SYM theory, twisted in order to preserve part of the supersymmetry. However, one encounters the problem that the coupling of
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the 4d SYM theory $\tau_{4d} = \theta + i \frac{8\pi^2}{g_{4d}^2}$, which is identified with the axio-dilaton coupling: $\tau_{4d} = \tau_F$, is generally frozen at a nonzero value due to the presence of D7 branes. Furthermore, seven-branes transverse to the two-cycle give rise to defects in the D3 worldvolume theory around which the gauge coupling transforms by an element of the $SL(2,\mathbb{Z})$ S-duality group of $\mathcal{N} = 4$ SYM [30]. This complicates the study of theories involving curves with $n \neq 1, 2, 4$. Nevertheless, as we explain in Section 1.5, in these cases one can still use topological string techniques to obtain information about the supersymmetric states of the worldsheet theory. In the case of $n = 1, 2, 4$ curves, on the other hand, one can resort to the Type IIB/IIA realizations of the 6d SCFT to obtain a UV description for the self-dual strings more straightforwardly. In the weakly coupled string theory descriptions discussed in the previous section the strings are captured either by D1 branes probing a singularity, or by D2 branes stretched between NS5 branes. In the former case, one can follow the Douglas-Moore construction [37] along the lines of [38] to find a two-dimensional quiver gauge theory; in the latter, one obtains a two-dimensional quiver gauge theory by dimensional reduction of the D2 worldvolume theory. These theories are characterized by having a gauge group composed of several factors: $G = \prod_{i=1}^r G_i$, where the $G_i$ are all of the form $U(n_i), O(n_i), \text{or } Sp(n_i)$. The factor $G_i$ is associated to strings wrapped on the $i$th two-cycle in the F-theory base, and $n_i$ is the number of strings of that kind (equivalently, the number of D1 or D2 branes); the non-abelian nature of the gauge groups reflects the fact that the self-dual strings form bound states, which is one of the striking differences between these strings and the ordinary strings of ten-dimensional string theory.

Beside gauge fields, the quiver gauge theories also contain matter that is charged under the various gauge groups. Also, the various gauge and global symmetries in six dimensions give rise to global symmetries on the string worldsheet. One finds in general that the self-dual strings are described by quiver theories with $\mathcal{N} = (0, 4)$ supersymmetry\(^4\) ($\mathcal{N} = (4, 4)$ for the $\mathcal{N} = (2, 0)$ theories in six dimensions), which is the appropriate number of supercharges for half-BPS objects in a six-

\(^4\)Supercharges in two dimensions can be assigned a definite chirality as in six dimensions. By $\mathcal{N} = (0, 4)$ we intend that the 2d theory has four right-moving supercharges.
dimensional theory with eight supercharges. In fact, it is convenient to express the field content of these theories in terms of multiplets of a $\mathcal{N} = (0, 2)$ subalgebra of the $\mathcal{N} = (0, 4)$ superalgebra. These include vector multiplets as well as matter multiplets of two kinds: chiral and Fermi [39]. The on-shell field content of these multiplets is described in the following table:

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>Field content</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector</td>
<td>$A_\mu, \chi_-$</td>
</tr>
<tr>
<td>Fermi</td>
<td>$\psi_-$</td>
</tr>
<tr>
<td>Chiral</td>
<td>$X, \psi_+$</td>
</tr>
</tbody>
</table>

Here, $\chi_-$, $\psi_-$ are left-moving fermions, $\psi_+$ is a right moving fermion, $X$ is a complex scalar, and $A_\mu$ is a gauge boson.

In Chapters 3, 4, and 5 we derive 2d quiver gauge theories for the self-dual strings of the theories of Figures 1.3–1.6 and discuss their properties in detail. Here we content ourselves with displaying their quiver diagrams: the quivers for the SCFTs of D5 branes at ADE singularities are given in Figure 1.12 (the $A_N$-type quiver also describes orbifolds of M-strings, that is, self-dual strings for the theory of parallel M5 branes probing an $A_N$ singularity); the quiver for multiple small $E_8$ instantons is given in Figure 1.13<sup>5</sup>; finally, the quiver for the theory of M5 branes at a D-type singularity is given in Figure 1.14. From these quivers one can read off the field content of the corresponding 2d theories, as described in Section 6.3 of this thesis. The round nodes correspond to multiplets including the gauge fields, while edges correspond to matter charged under the groups they connect to; also, square nodes indicate global symmetry groups the matter multiplets are charged under.

**Elliptic genera.** The 2d quivers discussed above correspond to UV theories that flow in the IR to CFTs describing bound states of the self-dual strings; the IR CFT itself is not known in general, except in the case of a single E-string in the theory of one small $E_8$ instanton, where it is given by two free complex bosons and a level 1 $E_8$ current algebra [28], and for a single $n = 2$ string where one has two free fermions and two free bosons. Nevertheless, one can obtain information about

<sup>5</sup>The quiver theory for a single $E_8$ instanton was first obtained in [40].
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Figure 1.12: Quiver diagrams for the self-dual strings of D5 branes probing ADE singularities.

Figure 1.13: Quiver diagram for the theories of small $E_8$ instantons.
the IR CFT by computing quantities in the UV which are invariant under renormalization group flow. In the context of 2d theories with at least $\mathcal{N} = (0,2)$ supersymmetry, one such quantity which captures important details of the IR CFT is the $(0,2)$ elliptic genus, which can be viewed as a refined version of the Witten index and has the following definition:

$$Z_{\text{2d CFT}} = \text{Tr}_R (\mathbf{-1})^F e^{2\pi i \tau H_L} e^{2\pi i \tau H_R} \prod_i x_i^{K_i},$$

where the trace is taken with periodic (Ramond) boundary conditions for the fermions on the supersymmetric (right-moving) side and $F$ is fermion number. The elliptic genus also depends on a number of additional parameters $x_i = e^{2\pi i \mu_i}$; these are fugacities for the $U(1)$s in the Cartan subgroup of the global symmetry group, which we take to have $K_i$ as generators. As for the Witten index, it can be argued that the Ramond boundary conditions on the fermions and the insertion of $(-1)^F$ project out from the trace any contributions from states for which $H_R \neq 0$. This implies that the elliptic genus is a holomorphic function of $\tau$, which counts states of the string that are ground states with respect to the right-moving side of the Hamiltonian but can be arbitrary excited states on the left-moving side.

Furthermore, being a torus partition function the elliptic genus is expected to have good modular properties. In fact, it follows from spectral duality that the elliptic genus is a Jacobi form
of weight zero with modular parameter $\tau$ and several elliptic parameters $\mu_i$. The good modular properties of the elliptic genus significantly constrain its form; they will play an important role in several parts of this thesis, and especially in Chapter 6.

The elliptic genus can be viewed as a torus partition function for the 2d CFT with twisted boundary conditions; in the context of 6d SCFTs, it is obtained by compactifying the 6d theory on a $T^2$ and considering multiple self-dual strings wrapped on it. For all the theories we consider the global symmetry group includes the $SO(4)$ group of isometries of the $\mathbb{R}^4$ transverse to the torus. The elliptic genera, among other parameters, depend on fugacities $\epsilon_1, \epsilon_2$ associated to the $U(1) \times U(1)$ Cartan subgroup of $SO(4)$, and capture information about the spin of the string states in the transverse four dimensions. Furthermore, they depend on the various Coulomb branch parameters and masses of the 6d theory compactified on a torus, which are realized as global symmetries of the strings.

**Localization computation.** In recent years, significant progress in the localization of path integrals of supersymmetric theories has led to a prescription for computing elliptic genera of two-dimensional quiver gauge theories [41–43]. Localization can be used to reduce the elliptic genus path integral to a finite-dimensional integral over holonomies $z_i$ of the gauge connection around cycles of the torus; these are valued in $\mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau$ is the lattice generated by the vectors $1$ and $\tau$ in $\mathbb{C}$. This leads to the following universal expression for elliptic genera:

$$Z_{2d} = \int dz_i \prod_{\Phi} Z_{1\text{-loop}}^{\Phi}(\tau, \mu_i),$$

where the factors in the integrand are one-loop determinants for the various multiplets $\Phi$ that appear in the quiver gauge theory and are given explicitly in [43]. Once the field content and the charges of the fields are determined, it is straightforward to write down the various factors in the integrand. However, choosing the correct contour integral is quite a non-trivial task. Fortunately,

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6See for example Appendix A for a review of basic properties of Jacobi forms.
the correct choice of contour was determined in [42,43] by carefully regularizing the path integral. The correct choice of contour corresponds to a prescription for picking residues of the integral which was developed in the 1990s by Jeffrey and Kirwan [44]. By applying this prescription, the elliptic genus can be written as a sum over the residues evaluated at a subset of the poles of the integrand. In general, the prescription tends to get quite involved for higher gauge rank (corresponding to higher numbers of strings), and computations quickly become technically challenging. Indeed for several classes of theories we study we content ourselves with studying elliptic genera for small numbers of strings. Remarkably, for the strings associated to the SCFTs of D5 branes probing ADE singularities we are able to compute elliptic genera for arbitrary numbers of strings and find elegant combinatorial formulas for them.

We employ localization to compute elliptic genera for the theories of Figures 1.3–1.6 in Chapters 3, 4, and 5, and in particular we study the features of the elliptic genera of M-strings in great detail in Chapters 2 and 3. As remarked above, one of the salient features of six-dimensional self-dual strings is that they form bound states; if this were not the case, it would be possible to reconstruct the elliptic genera for a number of strings from the one for a single string [45]. By studying the explicit form of elliptic genera, we indeed find that bound configurations exist. It would be very interesting to find whether other relations exist between the elliptic genera of a single string and multiple strings. We expand on this point in Section 2.4.1, where we do find a nontrivial relation between the elliptic genera of one and two M-strings.

**Elliptic genera from M2 brane quantum mechanics.** As we discussed in Section 1.3, some notable six-dimensional SCFTs admit realizations in terms of parallel M5 branes and possibly M9 planes. In these cases, the strings arise from suspended M2 branes, which we wrap on a torus in order to compute elliptic genera. By taking the limit where this torus shrinks to zero size one should obtain a one-dimensional quantum mechanical system describing the stretched M2 branes, with the remaining direction playing the role of time, and furthermore conformal invariance suggests that...
elliptic genera should be related to observables of this quantum mechanics. Reduction of the M2 brane worldvolume theory on a $T^2$ was studied in [46–49], where it was argued that the ground states are labeled by Young diagrams whose size corresponds to the number of M2 branes. Furthermore, the energy of a BPS configuration is given by the total number of M2 branes; in other words:

$$H |\mu\rangle = |\mu| \cdot |\mu\rangle.$$  

From the point of view of the M2 branes, the M5 branes can be thought of as domain walls, which upon dimensional reduction should lead to operators $\hat{D}^{M5}$ with matrix elements $D^{M5}_{\mu \nu} = \langle \mu | \hat{D}^{M5} | \nu \rangle$. On the other hand, M9 branes can be thought as boundary states $|M9\rangle$ and $|M9\rangle$ with components $D^{M9,L}_{\mu} = \langle M9 | \mu \rangle$, $D^{M9,R}_{\mu} = \langle \mu | M9 \rangle$. For example, we expect the elliptic genera of M-strings to be given by a correlator with two M5 domain wall operators inserted, as in Figure 1.15.

Likewise, the elliptic genera of E-strings, that is, M2 branes suspended between an M9 plane and an M5 brane, should be given by the correlator displayed in Figure 1.16.

In this thesis we find evidence for this picture. First of all, in Chapter 2 we find an expression for the M5 brane domain wall operator $\hat{D}^{M5}$, in terms of which we can obtain the elliptic genera of arbitrary bound states of strings in the theory of $N$ parallel M5 branes:

$$\sum_{n_1, \ldots, n_{N-1}} e^{-\sum_i n_i t_i} Z_{n_1, \ldots, n_{N-1}} = \langle 0 | \hat{D} e^{2\pi i H_1} \hat{D} e^{2\pi i H_2} \hat{D} \cdots e^{2\pi i H_{N-1}} \hat{D} |0\rangle,$$

where $Z_{n_1, \ldots, n_{N-1}}$ is the elliptic genus for a collection of $(n_1, \ldots, n_{N-1})$ M2 branes suspended between $N$ parallel M5 branes, and $t_i$ is the separation between the $i$th and $(i+1)$st M5 brane.
In Chapter 6 we also find evidence that the M9 branes can be interpreted as states in this quantum mechanical system. We find the components of these states corresponding to Young diagrams of size one or two by requiring that they lead to the correct expressions for the elliptic genera of one or two E-strings. For one string the computation is trivial, whereas for two strings it is already quite involved; nevertheless, by exploiting modularity and comparing with the partial results for E-string elliptic genera that were available at the time of this work, we are able to uniquely fix the two E-string components of $\langle M9 \rangle$ and conjecture an exact formula for the elliptic genus of two E-strings. This result has been verified in [40], where a quiver gauge theory description for the strings of a single small $E_8$ instanton was found that is consistent with our results. This picture also passes another non-trivial check: from knowledge of $|M9\rangle$ we can compute the following correlator up to two strings:

$$\langle M9|e^{-H_1}|M9\rangle,$$

which as depicted in Figure 1.17 corresponds to M2 branes suspended between M9 walls – in other words, $E_8 \times E_8$ heterotic strings. Indeed, we find that we can easily reproduce the elliptic genus for a single heterotic string from this correlator. For two heterotic strings, naively computing the elliptic genus from the quantum mechanical prescription does not produce the correct result; however, by symmetrizing the answer with respect to some of its arguments we indeed find a formula that correctly reproduces the elliptic genus of two heterotic strings; this is quite a nontrivial result but
calls for a better understanding of this quantum-mechanical picture. The authors of [50] have found similar results at the level of three strings.

1.5 F-theory/M-theory duality, 5d BPS index and topological strings

The aim of this section is to discuss compactification of six-dimensional SCFTs on a circle, which among other things will allow us to relate the computation of elliptic genera in 6d to instanton calculus in 5d and topological string theory. Dimensional reduction in this context is of a highly unusual nature: it is known that a six-dimensional theory involving tensor multiplets can be described upon compactification on a circle in terms of vector fields associated to some gauge group \(G\); however (as follows straightforwardly from 6d conformal invariance [51]) the coupling of the 5d gauge theory is related to the compactification radius as follows:

\[ g_{YM} \propto R^{1/2}. \]

This is opposite to what one may have expected from naive dimensional reduction, namely \(g_{YM} \propto R^{-1/2}\). Furthermore, it turns out that Kaluza-Klein modes arising from compactification, whose masses are proportional to \(R^{-1}\), can be identified with the 5d BPS particles that carry instanton charge, suggesting that these five-dimensional theories at finite coupling should be identified with
the six-dimensional theories on a circle of radius $R \sim g_{YM}^2$ (in the case of the $A_\text{n (2,0)}$ SCFT this has been discussed for instance in [52–54]).

In fact, this discussion can be framed in the context of duality between F-theory and M-theory on two closely-related Calabi-Yau threefolds, leading respectively to 6d and 5d theories. This is a generalization of T-duality between Type IIB string theory on a circle and Type IIA string theory on the dual circle, or equivalently M-theory on a torus. We can consider F-theory on the following geometry:

$$X \times S^1 \times \mathbb{R}^5,$$

where $X$ is elliptically fibered over a base $B$, or equivalently Type IIB string theory on

$$B \times S^1 \times \mathbb{R}^5$$

with a varying axio-dilaton profile. One can still perform T-duality along the $S^1$, with the added subtlety that on the M-theory side the resulting $T^2$ turns out to be fibered non-trivially over the base. One finds the following equivalence:

$$\text{F-theory on } X \times S^1 \times \mathbb{R}^5 \quad \leftrightarrow \quad \text{M-theory on } \widetilde{X} \times \mathbb{R}^5,$$

where $\widetilde{X}$ is described by the same fibration of a $T^2$ over $B$, but the volume of the $T^2$ is now proportional to the inverse of the $S^1$ radius on the F-theory side as a consequence of T-duality. As we discuss later, on the M-theory side the volume of the torus fiber is indeed given by $4\pi^2/g_{YM}^2$. In other words, we have the following picture:

\footnote{However, the results of [55] seem to be in contrast with this picture, and presently the validity of this statement in non-BPS sectors is not settled.}
We will find it very useful to study the self-dual strings from these four different perspectives, each one of which has its own advantages.

This duality also implies relations between the BPS objects of the various theories. On the F-theory side, one can consider D3 branes which wrap curves $C$ in the base $B$; these, as discussed in Section 1.2, lead to strings of the 6d SCFT. Depending on whether or not the D3 branes wrap the 6d circle, after T-duality and the lift to M-theory they become M2 branes wrapped on the base or M5 branes wrapped on the four-cycle formed by $C$ and the elliptic fiber inside of $\tilde{X}$. From the point of view of the 5d theory, these branes correspond respectively to BPS particles that are electrically charged under the gauge group or BPS strings which are magnetically charged. In this thesis we focus on the strings that lead to electric BPS states in 5d, though the present discussion makes it clear that the ones leading to magnetic BPS strings can be studied along similar lines (see [56,57]). One of the main objectives of this thesis is to obtain information about the spectrum of the 6d strings by computing their elliptic genera; we now wish to relate it to the counting of BPS states of the five-dimensional theory. In order to do this we compactify F-theory on a further circle, and in going around the circle we twist the transverse $\mathbb{R}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ in the worldvolume of the SCFT by rotating the two copies of $\mathbb{C}$ respectively by an amount $\epsilon_1$ or $-\epsilon_2$. The effect of this twist is effectively to compactify this $\mathbb{R}^4$ by introducing a potential centered at its origin for the BPS strings wrapped on the $T^2$; this has the benefit of removing bosonic zero modes associated to the motion of the BPS strings in $\mathbb{R}^4$.

Likewise, on the 5d theory side we end up with the twisted geometry $S^1 \times_{\epsilon_1,\epsilon_2} (\mathbb{C} \times \mathbb{C})$. In
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In this context, one can define a Witten index that captures the spectrum of BPS particles as well as their charges under various $U(1)$ symmetries. This index can be interpreted as a five-dimensional lift of Nekrasov's instanton partition which was first considered in [58, 59]. The twisting by $\epsilon_1, \epsilon_2$ is equivalent to turning on the so-called omega background in four dimensions. The 5d Nekrasov partition function can be computed using localization techniques as an expansion in instanton charge that takes the following form:

$$Z_{5d}^{BPS} = Z_{pert} \cdot Z_{inst}, \quad (1.5.4)$$

where $Z_{pert}$ is a contribution to the index coming from the particles that are light at weak coupling, whereas $Z_{inst}$ comes from particles that carry instanton charge (given by $k = \frac{1}{8\pi^2} \int_{R^4} \text{Tr}(F \wedge F) \in Z_+$), and takes the following form:

$$Z_{inst} = 1 + \sum_{k=1}^{\infty} e^{-4\pi^2 k/g_{YM}^2} Z_{k \text{inst}}. \quad (1.5.5)$$

We expect a relation between the 5d BPS index and the spectrum of states on the strings for the following reason: a bound state of $n$ strings can wrap on the 6d circle, and to each state on the worldsheet will be associated a BPS state in five dimensions; more precisely, one obtains a tower of Kaluza-Klein excitations corresponding to the string having momentum along the 6d circle. As remarked above, since momentum is quantized in units of $R^{-1} \propto 1/g_{YM}^2$, we see that the instanton charge can be identified with KK momentum along the 6d circle. Furthermore, in computing the BPS index one imposes periodic boundary condition on the fermions along the 5d circle; a similar condition is imposed on the worldsheet fermions in computing the elliptic genus of bound states of strings, and indeed we find the following relation for 5d theories that have a 6d origin:

$$Z_{5d}^{BPS}(\tau, \epsilon_1, \epsilon_2, \vec{a}, \vec{\mu}) = Z_0(\tau, \epsilon_1, \epsilon_2, \vec{\mu}) \cdot \left( 1 + \sum_{\vec{n} \in (Z_+)^{\oplus r}} e^{-\vec{n} \cdot \vec{a}} Z_{\vec{n}}(\tau, \epsilon_1, \epsilon_2, \vec{\mu}) \right). \quad (1.5.6)$$

Here, $Z_{\vec{n}}$ is the elliptic genus of a bound state of $n_1$ strings associated to $C_1$, $n_2$ curves associated to $C_2$, and so on. Also, $\vec{a}$ are the subset of Coulomb branch parameters of the 5d theory which have an origin in 6d as the vevs of tensor multiplet scalars, or equivalently tensions of the strings associated
to different curves $\tilde{C}$ in the F-theory base $B$. Furthermore, $\tau = 4\pi^2/g_{YM}^2$, and $\bar{\mu}$ are fugacities associated to additional global symmetries in the theory. Finally, $Z_0$ is a simple factor associated to BPS particles that do not come from wrapped strings. These BPS states arise in M-theory from M2 branes that do not wrap cycles in the base$^8$, and their contribution to the BPS index can be easily computed and expressed in terms of products of Jacobi theta functions. The relation (1.5.6) has interesting consequences: on one hand, by employing instanton calculus techniques, it provides an indirect way to compute elliptic genera of the self-dual strings and match against explicit computations when these are available. We will resort to this fact at various points in this thesis. On the other hand, writing the 5d BPS index in terms of modular forms makes its modular properties (which were studied in [18] in the context of the 5d maximally supersymmetric $U(N)$ theory) manifest. This also makes it straightforward to express it as a strong coupling expansion around $4\pi^2/g_{YM}^2 = 0$. This has also been exploited in [60] to show that the partition function of the maximally-supersymmetric $U(N)$ theory on $\mathbb{P}^2 \times S^1$ can be interpreted as a six-dimensional superconformal index for the $\mathcal{N} = (2,0)$ theory of $N + 1$ M5 branes.

Directly related to the problem of 5d BPS counting is (A-model) topological string theory [61] on a (not necessarily elliptic) Calabi-Yau threefold. The study of topological string theory has led to a deeper understanding of wide range of topics in both physics and mathematics$^9$. We will mostly be interested in its interpretation in terms of counting of BPS configurations of M2 branes [65,66] wrapped on two-cycles of a Calabi-Yau threefold. In fact, the M-theory realization of (refined) topological string theory on a noncompact threefold is precisely the one we have arrived at in the discussion above, namely M-theory on $\tilde{X} \times S^1 \times \mathbb{C}^2_{\epsilon_1, \epsilon_2}$, and one finds that

$$Z_{5d}^{BPS} = Z_{\text{top}}(\tilde{X}),$$

where the topological string partition function is a product of factors associated to M2 branes

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$^8$Their description in F-theory is more subtle.

$^9$See for example [27,62–64] for overviews of topological string theory and examples of its applications.
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wrapped around curves in $\widetilde{X}$:

$$Z_{\text{top}} = \prod \prod \prod_{j_1, j_2, k_L = -j_L, k_R = -j_R}^{j_L, j_R} \prod_{m_1, m_2 = 1}^{\infty} \left( 1 - t^{k_L + k_R + m_1} - \frac{1}{2} q^{k_L - k_R + m_2} - \frac{1}{2} e^{\epsilon_1} \omega \right)^{(-1)^{2(j_L + j_R) + 1} N^\beta_{j_L, j_R}}. \tag{1.5.7}$$

Here $N^\beta_{j_L, j_R}$ is the degeneracy of BPS M2 branes wrapping a two-cycle $\beta \in H_2(\widetilde{X}, \mathbb{Z})$ that have $SO(4) = SU(2)_L \times SU(2)_R$ spins given by $(j_L, j_R)$. Also, $\omega$ is the Kähler form on $\widetilde{X}$ and $q = e^{\epsilon_1}, t = e^{-\epsilon_2}$. Therefore, we also find that

$$Z_{\text{top}}(\widetilde{X}, \epsilon_1, \epsilon_2) = Z_0(\tau, \epsilon_1, \epsilon_2, \bar{\mu}) \cdot \left( 1 + \sum_{\vec{n} \in (\mathbb{Z}_+)^{4g}} e^{-\vec{n} \cdot \vec{a}} Z_{\vec{n}}(\tau, \epsilon_1, \epsilon_2, \bar{\mu}) \right). \tag{1.5.8}$$

Note that the various fugacities on the right hand side appear on the left hand side as Kähler parameters, that is, as volumes of the various two-cycles; in particular, $\vec{a}$ are the volumes of two-cycles in the base $B$, and $\tau = 4\pi^2 / g_Y^2 m$ is the volume of the elliptic fiber. Modularity of the topological string partition function involving $SL(2, \mathbb{Z})$ transformations of Kähler parameters of elliptic Calabi-Yaus is manifest from (1.5.8).

The relation given in Equation (1.5.8) proves extremely useful in computing the elliptic genera of self-dual strings, since very powerful techniques are available to compute $Z_{\text{top}}(\widetilde{X})$. For instance, as we discuss in Chapters 2 and 3, the Calabi-Yau geometries corresponding to the multiple M5 branes at an A-type orbifold are non-compact toric varieties, and therefore $Z_{\text{top}}$ can be computed exactly using the refined topological vertex [67]. The computation can be performed in different ways, associated to different choices of preferred direction for the topological vertex. One way leads directly to Nekrasov’s instanton expansion (1.5.4); the other leads to the elliptic genus expansion (1.5.8), and provides a very efficient way to compute the elliptic genera. Additionally, in Chapter 4 we employ mirror symmetry and B-model topological strings as in [68] to study the 6d theories with a single tensor multiplet and compute their elliptic genera as an expansion in exponentiated Kähler parameters. This includes the theories for which a quiver gauge theory description is not available.
1.6 $S^5$ partition functions and 6d index

Up to this point our discussion has been focused on the tensor branch of six-dimensional SCFTs; one may raise the legitimate question whether the BPS objects appearing on the tensor branch, such as the self-dual strings that are the subject of this thesis, have anything to say about the superconformal fixed point. In Chapter 7 we argue that they indeed do, at least in an indirect way. More specifically, we find that one can use elliptic genera of the self-dual strings that arise on the tensor branch to compute superconformal indices of 6d SCFTs. The superconformal index is a Witten index that captures information about the spectrum of short multiplets of the superconformal theory in radial quantization. It can be expressed as a trace over the Hilbert space of the theory on $S^5$, or in other words as a partition function on $S^5 \times S^1$. For a $\mathcal{N} = (1,0)$ SCFT it has the following definition:

$$I_{1,0} = \text{Tr}(-1)^F q_1^{J_{12}-R} q_2^{J_{34}-R} q^{J_{56}-R} M_i^{F_i},$$

where $R$ is the Cartan generator for the $Sp(1) \simeq SU(2)$ R-symmetry, $J_{ij}$ are rotation generators of $SO(6)$ acting on $S^5$, and $F_i$ are charges associated to flavor symmetries. For $\mathcal{N} = (2,0)$ theories, the superconformal index can be viewed as an extension of the $(1,0)$ index by introducing the additional flavor symmetry $R_2 - R_1$, where $R_1, R_2$ are the Cartan generators of $Sp(2) \simeq SO(5)$:

$$I_{(2,0)} = \text{Tr}(-1)^F q_1^{J_{12}-R_1} q_2^{J_{34}-R_1} q^{J_{56}-R_1} Q_m^{R_2-R_1}. $$

Because of the 6d/5d relation discussed in the last section, the computation of partition functions on $S^5 \times S^1$ is directly related to the computation of the partition function of 5d superconformal theories on a five-sphere. In recent years, thanks to significant progress in computing partition functions of superconformal theories on different backgrounds by localization, it has become evident that these partition functions in general can be expressed in terms of contributions of BPS particles that arise on the Coulomb branch (see [69] and references therein). In Chapter 7 we find that a similar picture holds for 5d theories on a five-sphere, which can be written as an integral
over the Coulomb branch of three factors of the 5d BPS index, or equivalently the topological string partition function:

\[
Z_{SSS}(m_j, \tau_1, \tau_2) = \int dt_i \frac{Z^{\text{top}}(t_i, m_j; \tau_1, \tau_2)}{Z^{\text{top}}(t_i/\tau_1, m_j/\tau_1; -1/\tau_1, \tau_2/\tau_1) \cdot Z^{\text{top}}(t_i/\tau_2, m_j/\tau_2; \tau_1/\tau_2, -1/\tau_2)},
\]

where \(\tau_1, \tau_2\) are to be identified with the equivariant parameters \(\epsilon_1, \epsilon_2\) of the topological string partition function, \(t_i\) are Coulomb branch parameters of the theory and \(m_j\) are other parameters such as masses and fugacities for the global symmetries. After the completion of this work, this result has also been obtained by a rigorous localization computation in [70].

The integrand can be viewed as a non-perturbative completion of the topological string partition function, since it involves factors for which the refined topological string coupling parameters \(\epsilon_1, \epsilon_2\) are inverted; we find that it can be written in terms of a generalized version of the triple sine function [71–74].

In the case of six-dimensional SCFTs, the relation (1.5.6) implies that the superconformal index of six-dimensional SCFTs can be expressed in terms of an integral over the tensor branch (as well as over holonomies of the vector fields around the 6d circle in the case of \(\mathcal{N} = (1,0)\) SCFTs). The three factors of \(Z^{\text{top}}\) that appear in the integrand can be written in terms of elliptic genera of the self-dual strings, so we find that ultimately the 6d superconformal index is related to the spectrum of the strings appearing on the tensor branch. This relation is highly non-trivial, and it would be very interesting to shed further light on it.

### 1.7 Outline of the thesis

The remainder of this thesis is organized as follows:

- In Chapter 2 we study the self-dual strings that arise in the theory of parallel M5 branes, which we denote by M-strings. Upon compactification to five dimensions, one can introduce a deformation leading to the 5d \(\mathcal{N} = 1^*\) theory with \(U(N)\) gauge group. We discuss how this five-dimensional theory arises by compactifying M-theory on a certain toric Calabi-Yau
threefold and compute its BPS partition function using topological string techniques. We find an expansion of the BPS partition function whose coefficients can be interpreted as elliptic genera of bound states of strings which arise from M2 branes suspended between the M5 branes. We also find a description of these strings in terms of a \((0,4)\) supersymmetric sigma model whose elliptic genera we compute and match with the topological string expansion. Furthermore, we find that the elliptic genera can be interpreted as expectation values of products of domain wall operators corresponding to M5 branes in the quantum mechanics of M2 branes compactified on a torus.

- In Chapter 3 we consider a more general set of theories which describe parallel M5 branes probing an \(A_N\)-type singularity; the class of theories discussed in the previous chapter arises as a special case of this setup. We again employ topological string techniques to study these theories in detail and analyze their self-dual strings. We also find that these strings can be described in terms of two-dimensional \((0,4)\) quiver gauge theories which we derive starting from a brane construction in Type IIB string theory.

- In Chapter 4 we analyze the self-dual strings of six-dimensional theories with minimal gauge group and a single tensor multiplet. These theories are obtained by considering F-theory on an elliptic Calabi-Yau threefold with a single exceptional divisor in the base. We realize these threefolds as hypersurfaces in a toric variety and compute their BPS invariants by employing topological string techniques. From this data one can compute the elliptic genera of the strings as an expansion in the various parameters associated to the Kähler classes of the elliptic threefold. For the theory corresponding to an exceptional divisor of self-intersection \(-4\) we also find a brane construction within Type IIB string theory by taking Sen’s limit. The brane construction leads us to a \((0,4)\) quiver theory describing bound states of the self-dual strings; we compute elliptic genera for small numbers of strings by localization and match them against the topological string computations.
• In Chapter 5 we consider theories with several tensor multiplets associated to exceptional divisors of self-intersection $-1, -2, \text{ or } -4$. In particular, we obtain 2d $\mathcal{N} = (0,4)$ quiver gauge theories for the self-dual strings of the following SCFTs: parallel M5 branes probing singularities of type A or D; D5 branes probing singularities of type A, D, or E; parallel M5 branes in the proximity of an M9 wall, corresponding to the theory of several small $E_8$ instantons. We compute elliptic genera for these strings by localization.

• In Chapter 6 we study the E-string theory of a single M5 brane in the proximity of an M9 plane and interpret it in terms of M2 brane quantum mechanics. We employ known partial results about elliptic genera of E-strings to uniquely fix the state in the quantum mechanics associated to an M9 wall for one or two M2 branes. Using this result we are able to find an exact expression for the elliptic genus of two E-strings, which was previously not known. We also show that elliptic genera of one and two heterotic strings can be obtained from the quantum mechanics by taking appropriate combinations of factors corresponding to the two M9 walls.

• In Chapter 7 we formulate a non-perturbative completion of the topological string partition function which enables us to compute partition functions of five-dimensional superconformal theories on a five-sphere. In the case of five-dimensional theories obtained by dimensional reduction of a six-dimensional SCFT, we are also able to write an expression for the superconformal index of the parent 6d theory in terms of topological strings. From this, using the results of the previous chapters it follows that the six-dimensional superconformal index can be expressed in terms of elliptic genera of the self-dual strings of the 6d theory.
Chapter 2

M-strings

2.1 Introduction

The SCFTs with the maximal amount of supersymmetry in the highest dimension are the $\mathcal{N} = (2,0)$ theories in $d = 6$. Despite the unique status they enjoy, and despite the fact that they have been instrumental in constructing lower dimensional theories, they remain among the least understood theories. This is mainly related to the fact that we do not have a Lagrangian description of these theories. Moreover, if we go slightly away from the conformal point we get a theory of interacting almost tensionless strings. Clearly a deeper understanding of these strings is called for. In this chapter we focus on the $\mathcal{N} = (2,0)$ SCFT of type $A_{N-1}$ arising from $N$ coincident M5 branes, and study the M2 branes suspended between the M5 branes when we separate them, which leads to strings on their boundaries. We will call these strings ‘M-strings’, as they involve basic M-theory ingredients for their definition.

If we consider two parallel M5 branes, and consider one M2 brane suspended between them, then clearly the moduli space of the M2 brane is labeled by its transverse position on the M5 brane it ends on, which is a copy of $\mathbb{R}^4$. So at least the IR degrees of freedom on this string should correspond to the (4,4) supersymmetric sigma model on $\mathbb{R}^4$. Moreover if we consider $n$ M2 branes stretched between 2 M5 branes, one would naively expect the IR degrees of freedom to correspond
to the choice of $n$ points on $\mathbb{R}^4$, modulo the action of the permutation group on the points, i.e. to a $(4,4)$ supersymmetric sigma model on

$$\text{Sym}^n(\mathbb{R}^4) = (\mathbb{R}^4)^n/S_n.$$  

This space is singular and one can ask whether the target space is smoothed out at coincident points. If the target space is smoothed out, as in the Hilbert scheme of $n$-points on $\mathbb{R}^4$, then this would give us an effective way to compute at least supersymmetry protected quantities for this theory. However, as argued in a related context in [75] this is not necessarily the case (not even the B-field on the vanishing $\mathbb{P}^1$’s is turned on as in the orbifold points), and one expects that the relevant theory should be the one corresponding to the singular target space, which is infinitely far away from the smoothed out points. This in particular raises the question of whether at least for BPS quantities one may use the smoothed out target space to perform such computations. A surprising result we find is that this is not possible for $n > 1$. Instead we find a related sigma model with $(4,0)$ supersymmetry on the smoothed out space (the Hilbert scheme of $n$-points on $\mathbb{C}^2$) which has the same elliptic genus as the suspended M2 branes. The right-moving fermions couple, instead of the tangent bundle, to a bundle $V = E \oplus E^*$ where $E$ is the tautological bundle on the Hilbert scheme.

From the viewpoint of the M2 brane worldvolume theory, ending on an M5 brane corresponds to a boundary condition on the theory [48], as is familiar in the context of D-branes. More generally we will be considering a number $N_L$ of M2 branes suspended on an M5 brane from the left and a number $N_R$ of M2 branes suspended from the right. This can be viewed as a domain wall which separates $(N_L, N_R)$ M2 branes. In addition we need to choose a vacuum for each M2 brane, which in turn is labeled by the partition $\nu_L$ of $N_L$ for the left-vacuum and $\nu_R$ of $N_R$ for the right vacuum [46, 47, 49]. Thus the theory living on the 2d domain wall is labeled by $D_{\nu_L \nu_R}$. One main computational result of this chapter is the supersymmetric partition function of the theory $D_{\nu_L \nu_R}$ on $T^2$. More precisely we consider the elliptic genus of this theory, including twisting by maximal allowed symmetries consistent with $(2,0)$ supersymmetry as we go around the cycles of

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$T^2$. In some limit (turning off some chemical potentials, corresponding to turning off the ‘mass for the adjoint’) this computation can also be viewed, using a dual Type IIB description, as computing the elliptic genus of $A_N$ quiver $\mathcal{N} = (4,4)$ supersymmetric theories in $d = 2$, which can be obtained using localization in two dimensions [41, 42]. We check our computations against these results in this limit and find agreement; in Chapter 3 we will examine the Type IIB realization of this theory in more detail and from that we will obtain a two-dimensional quiver gauge theory whose elliptic genus computed via localization exactly reproduces the elliptic genus of M-strings with the most generic choice of twisting.

The main tool we use in the present chapter is the relation between the refined topological string partition function and the degeneracy of BPS states in 5d [65–67, 76], which we apply to the M5 brane SCFT compactified on $S^1$. Apart from the Kaluza-Klein reduction of the 6d modes, the suspended M2 branes wrapped around $S^1$ are the only other states contributing to BPS states, and we are thus able to extract the partition function, or more precisely the elliptic genus, of the strings obtained from suspended M2 branes. Reversing this, one can recover the full refined topological string partition function in terms of the elliptic genera of the M-strings. This in turn can be used to compute the index of M5 branes [70, 77], as we will argue in Chapter 7.

The remainder of this chapter is organized as follows: In Section 3.2 we review the relation between the M5 brane CFT, the $\mathcal{N} = 2^* U(N)$ supersymmetric Yang-Mills in 5d, and its toric realization. In Section 3.3 we show how to use this setup to compute its partition function on twisted $S^1 \times \mathbb{R}^4$ (including modular properties and symmetries) using refined topological strings as well as instanton calculus. Moreover we explain the relation between the partition function of the $U(N)$ theory and BPS degeneracies. We compute this using two dual theories: one is the $\mathcal{N} = 2^* U(N)$ theory, and the other is a dual $A_{N-1}$ quiver theory in six dimensions with $U(1)^{N-1}$ gauge group. This latter perspective turns out to be particularly important for our purposes. In Section 3.4 we contrast some expectations of BPS degeneracies based on generalities about suspended M2 branes with the actual results we obtain using the topological strings. We interpret
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our computations and explain what they tell us about M-strings, including their relation to elliptic
genus of quiver theories. Furthermore we interpret our results as leading to the partition function
of domain walls separating M2 branes. Moreover we discuss the fact that new additional bound
states between M-strings arise when we compactify the M5 brane theory on the circle, which cannot
be viewed as bound states before compactification. Furthermore, we show that in a particular limit
(where the mass term is turned off) the result agrees with that of the dual Type IIB description
involving the elliptic genus of (4,4) quiver theories. Finally in Section 3.5 we summarize the results
obtained in this chapter and briefly comment on extensions of the ideas presented here which will
be discussed in later chapters.

2.2 Parallel M5 branes on $S^1$ and $S^1 \times S^1$ and suspended M2 Branes

In this section we discuss some general aspects of parallel M5 branes including their twisted
compactifications on $S^1$ and $S^1 \times S^1$. The twisted compactification on $S^1$ leads to a theory with
the same IR degrees of freedom as $N = 2^*$ in 5 dimensions, where the mass of the adjoint field is
given by the twist parameter. The further compactification on the circle can be used to twist the
left-over 4 dimensions of the M5 brane. We also discuss general aspects of M2 branes suspended
between the parallel M5 branes. Furthermore we discuss various dualities which map this to related
systems, and in particular to compactifications of M-theory on elliptically fibered geometries, which
we will use in the following section to compute the partition function of M5 branes using refined
topological strings.

2.2.1 Basics of M-strings

Consider $N$ parallel and coincident M5 branes. This is believed to lead to a $(2,0)$ superconformal theory in six dimensions usually called the $(2,0) \, A_{N-1}$ theory. The choice for the terminology is because the same system is believed to arise when considering Type IIB string theory in the presence of an $A_{N-1}$ singularity. The latter viewpoint generalizes it to the $D$ and $E$ versions
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of the \( \mathcal{N} = (2, 0) \) theory.

This theory has \( Osp(2, 6|4) \) as the superconformal group whose bosonic part is

\[
Spin(2, 6) \times Spin(5) \subset Osp(2, 6|4).
\]

(2.2.1)

\( Spin_R(5) \) is the global R-symmetry group of this theory and is the double cover of \( SO(5) \) which is the rotation group of the space transverse to the M5 branes.

On the worldvolume of a single M5 brane we have the tensor multiplet of the \( (2, 0) \) theory which consists of:

- an antisymmetric 2-form \( B \) such that its field strength \( H = dB \) is self-dual, \( \ast H = H \)
- four symplectic Majorana-Weyl fermions in the \( (4, 4) \) of \( Spin(1, 5) \times Spin_R(5) \)
- five scalar fields giving the transverse fluctuations of the M5 brane.

If we compactify the six dimensional \( (2, 0) \) theory described above on a circle it gives a theory with 16 real supercharges in five dimensions, the \( \mathcal{N} = 2 \) super Yang-Mills theory in five dimensions. Since we will be discussing the M2 branes suspended between M5 branes let us fix the worldvolume and the transverse directions of the M5/M2 branes. We denote the coordinates of \( \mathbb{R}^{1,10} \) as \( X^I, I = 0, 1, 2, \cdots 10 \); then,

**The worldvolume of coincident M5 branes has coordinates**

\[ X^0, X^1, X^2, X^3, X^4, X^5. \]

The space transverse to the coincident M5 brane worldvolume is \( \mathbb{R}^5 \) and is acted upon by the R-symmetry group \( Spin_R(5) \). We can pick a direction in \( \mathbb{R}^5 \) and separate the coincident M5 branes along it. We choose the \( X^6 \) coordinate to separate the branes. This breaks the global \( Spin_R(5) \) symmetry to \( Spin_R(4) \) acting on the coordinates \( X^7, X^8, X^9, X^{10} \). It is important to note that \( Spin_R(4) \) does not act on the M5 brane worldvolume coordinates. For later convenience we denote the position of the M5 branes in the \( X^6 \) direction as \( a_i, i = 1, 2, \cdots, N \).

We can now introduce M2 branes ending on M5 branes with the boundary of the M2 brane inside the M5 brane coupling to the 2-form \( B \). We can introduce multiple M2 branes for
each pair of M5 branes extending in the $X^6$ direction. We consider the worldvolume of M2 branes such that

**The worldvolume of an M2 brane suspended between $(i, j)$ M5 branes is**

$$X^0, X^1, X^6 \text{ with } a_i \leq X^6 \leq a_j.$$  

The boundary of the M2 brane given by the coordinates $(X^0, X^1)$ is a string inside the M5 brane, which we call the M-string. The presence of this string breaks the M5 brane worldvolume Lorentz group $Spin(1, 5)$ to $Spin(1, 1) \times Spin(4)$, where $Spin(1, 1)$ is the Lorentz group on the string and $Spin(4)$ acts on the space transverse to the string inside the M5 brane.

From our choice of the worldvolume coordinates of the M5/M2 branes and the string it is easy to see that the supersymmetries preserved by the string are given by

$$\Gamma^{016} \varepsilon = \varepsilon, \quad \Gamma^{012345} \varepsilon = \varepsilon, \quad (2.2.2)$$

where $\varepsilon$ is the 32-component spinor, $\Gamma^{I_1I_2\cdots I_k} = \Gamma^{I_1} \Gamma^{I_2} \cdots \Gamma^{I_k}$ and $\Gamma^I$ are the $32 \times 32$ eleven dimensional Gamma matrices. Since in eleven dimensions $\Gamma^0 \Gamma^1 \Gamma^2 \cdots \Gamma^{10} = 1$ the above two conditions imply that

$$\Gamma^{2345} \varepsilon = \Gamma^{01} \varepsilon, \quad \Gamma^{789(10)} \varepsilon = \Gamma^{01} \varepsilon. \quad (2.2.3)$$

Hence the chirality under $Spin(1, 1)$, chirality under $Spin_R(4)$ and chirality under $Spin(4) \subset Spin(1, 5)$ of the preserved supersymmetries on the string are the same. Since the M2/M5 brane configuration breaks $\frac{1}{4}$ of the 32 supersymmetries, on the string world sheet we have a $(p, q)$ supersymmetric theory with $p + q = 8$. By taking a specific form of the eleven dimensional Gamma matrices it is easy to show that the theory on the string has $(4, 4)$ supersymmetry. It then follows from Equation (2.2.3) that preserved supercharges $Q_{-\frac{1}{2}}^{\dot{a}}$ and $Q_{+\frac{1}{2}}^{a}$, where $a, \dot{a} = 1, 2$ denote the chiral/antichiral spinor of $Spin_R(4)$ and $a, \dot{a} = 1, 2$ denote the chiral/anti-chiral spinor of $Spin(4) \subset Spin(1, 5)$, are in the representation

$$(2, 1, 2)_{+\frac{1}{2}} \oplus (1, 2, 1, 2)_{-\frac{1}{2}} \quad (2.2.4)$$
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of $\text{Spin}(4) \times \text{Spin}_R(4) \times \text{Spin}(1, 1)$. The $\pm 1$ denote the chirality with respect to $\text{Spin}(1, 1)$.

The above supercharges can be organized in terms of representations of $\text{Spin}(8) \supset \text{Spin}(4) \times \text{Spin}_R(4)$ as well and it will be useful for later purposes to do so. Consider a number of coincident M2 branes in $\mathbb{R}^{1,10}$ with worldvolume along $X^0, X^1$ and $X^6$. Then the transverse space is $\mathbb{R}^8$ and the global symmetry of the theory on the M2 branes is given by $\text{Spin}(8)$. Now introducing M5 branes, separated along $X^6$ as before and M2 branes ending on them, breaks $\text{Spin}(8)$ to $\text{Spin}(4) \times \text{Spin}_R(4)$. Notice that the preserved supercharges form a positive chirality spinor of $\text{Spin}(8)$, i.e. they are in $8_s$. The chirality for $\text{Spin}(8)$ is determined by $\Gamma_9 \equiv \Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^7\Gamma^8\Gamma^9\Gamma^{10}$ and therefore it follows from Equation (2.2.3) that

$$\Gamma_9 \epsilon = (\Gamma^{01})^2 \epsilon = \epsilon,$$  \hspace{1cm} (2.2.5)

and hence preserved supersymmetries form a positive chirality spinor of $\text{Spin}(8)$. If we denote by $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4$ and $\alpha_4 = e_3 + e_4$ the simple roots of $\text{Spin}(8)$ then the $(4,4)$ supercharges are in $8_s$ with highest weight vector\(^1\)

$$\frac{e_1 + e_2 + e_3 + e_4}{2}.$$

The weight vectors for the $(4,4)$ supercharges are given by:

\begin{align*}
(1,2,1,2)_{-\frac{1}{2}}: & \frac{e_1+e_2+e_3+e_4}{2}, \frac{e_1+e_2-e_1-e_4}{2}, \frac{-e_1+e_2+e_3+e_4}{2}, \frac{-e_1-e_2-e_3-e_4}{2},
\end{align*}

\begin{align*}
(1,2,1,1)_{-\frac{1}{2}}: & \frac{e_1-e_2+e_3-e_4}{2}, \frac{e_1-e_2-e_3+e_4}{2}, \frac{-e_1+e_2-e_3+e_4}{2}, \frac{-e_1+e_2+e_3-e_4}{2}.
\end{align*}

2.2.2 Compactification on $S^1$

Next, consider compactifying the M5 branes on a circle. Recall that

The worldvolume of M5 branes has coordinates

$$X^0, X^1, X^2, X^3, X^4, X^5$$

\(^1\)The $\text{Spin}(4) \times \text{Spin}(4)_R$ subgroup of $\text{Spin}(8)$ mentioned above corresponds to the simple roots $\{e_1 - e_2, e_1 + e_2, e_3 - e_4, e_3 + e_4\}$.
and that the M5 branes are separated in the $X^6$ direction. Now consider compactifying $X^1$ to a circle of radius $R_1$. More generally we can introduce a partial breaking of the supersymmetry by making the $\mathbb{R}^4$ transverse to the M5 branes fibered nontrivially over $S^1$. We will denote this $\mathbb{R}^4$ spanned by $(X^7, X^8, X^9, X^{10})$ as $\mathbb{R}^4_\perp$. We choose complex coordinates $(w_1, w_2)$ for $\mathbb{R}^4_\perp \simeq \mathbb{C}^2$, and consider a rotation of the two complex planes as we go around the circle:

$$U(1)_m : (w_1, w_2) \rightarrow (e^{2\pi i m} w_1, e^{-2\pi i m} w_2). \quad (2.2.8)$$

The resulting theory in 5d is a mass deformation of the maximally supersymmetric Yang-Mills theory, by giving a mass term to the adjoint, and we denote here as the five-dimensional $\mathcal{N} = 2^*$ theory, borrowing the terminology from the more familiar 4d case. The radius $R_1$ of $S^1$ is identified with the gauge coupling of the Yang-Mills theory as follows

$$R_1 = \frac{g_{YM}^2}{4\pi^2}. \quad (2.2.9)$$

The 5d theory has charged particles in its spectrum which carry instanton number which is identified with the momentum around the $S^1$:

$$\frac{k}{R_1} = -\frac{1}{8g_{YM}^2} \int d^4x \text{tr}(F \wedge F), \quad (2.2.10)$$

From the point of view of the six-dimensional theory these particles arise as M-strings wrapped around the $S^1$. If we consider $l$ M-strings wrapped around $S^1$ and carrying a momentum of $k$ units along $S^1$ the corresponding BPS mass is given by

$$M = l R_1 \delta_{ij} + \frac{k}{R_1}, \quad k, l \in \mathbb{Z}, \quad (2.2.11)$$

where $\delta_{ij}$ is the separation between the M5 branes which gives the tension of the M-string stretched between $i$ and $j$ M5 branes.

We can also ask how the twisting by $m$ around $S^1$ affects the theory as seen by the M-string wrapped around $S^1$. The $U(1)_m$ is embedded in the $SU(2)_L$ of $Spin_R(4) \subset Spin(8)$ of the M2 brane theory. It is easy to see that this choice of the $U(1)_m$ leaves the negative chirality supercharges
of Equation (2.2.6) invariant but not the positive chirality ones. Hence the resulting theory has 
broken the $(4,4) \mapsto (4,0)$ supersymmetric theory on the worldsheet in the $X^0, X^1$ directions.

As already mentioned we can view the 6d theory as coming from Type IIB theory with an $A_{N-1}$ singularity. Compactifying this on a circle and using the duality between M-theory and Type IIB we can view this as compactification of M-theory on a threefold with geometry $T^2 \times A_{N-1}$. The duality between Type IIB and M-theory identifies the Kähler class $t^e_M$ of $T^2$ with

$$t^e_M = \frac{1}{R_1}.$$ 

Moreover the twisting by the mass parameter can be viewed as blowing up a $\mathbb{P}^1$ [76]. This is the geometric analog of giving mass to the adjoint field in the brane construction [78]. The blow up parameter $t^M_m$ is identified with

$$t^M_m = \frac{m}{R_1}.$$ 

The geometry of the blow-up is a local Calabi-Yau and is given by the periodic toric diagram [76, 79] in Fig. 2.1 where we have specialized to the case of the $U(2)$ theory which corresponds to two M5 branes. There is a dual description of the same system [35] in terms of the $(p, q)$ web of 5-branes [80]. The picture is the same as the one of the toric diagram, only one has to associate the toric legs with branes of Type IIB as is shown in Fig. 2.1.

In the massless case, where one has the maximally supersymmetric gauge theory, the NS5 brane is extended along an $\mathbb{R}^6$ subspace of $\mathbb{R}^{10}$ while the D5 branes have the geometry $\mathbb{R}^5 \times S^1$ and intersect the NS5 branes transversally such that they have five dimensions in common. Then the gauge theory is living on the intersection and its rank is specified by the number of D5 branes. Note furthermore, that the compactified direction of the D5 branes is perpendicular to the NS5 brane. The gauge theory is then living on the intersection of these branes. Now let us deform the theory by introducing mass as shown in Fig. 2.1. To simplify matters we will take the gauge group to be $U(2)$ for the moment. In this case the Calabi-Yau is the canonical bundle over a surface $D$ which is an elliptic fibration over $\mathbb{P}^1$, that is locally we have $D \cong T^2 \times \mathbb{P}^1$. The torus arises from
Figure 2.1: The brane and toric geometry. The red line marks mean to identify the toric legs or branes with each other and therefore describe a compactified direction which is associated to the gauge coupling $\tau$ in the gauge theory. The length of the $(1,1)$ branes (depicted by the diagonal lines) is associated to the mass of the $\mathcal{N} = 2^*$ theory. Finally, the separation of the branes maps to the Coulomb branch parameter $t_f$ of the gauge theory.

The compactified direction of the brane system with size $t^M_c$ and the size of the $\mathbb{P}^1_f$ is the Coulomb branch parameter of the gauge theory of size $t^M_f$ that is the separation of the D5-branes in the brane picture. This is related to the separation between the M5 branes (which is proportional to the tension $\delta$ of the M2 brane string) times $R_1$:

$$t^M_f = R_1 \cdot \delta$$

Finally there is a third Kähler class coming from a singular elliptic fibre over the discriminant locus. The singular fibre is a degeneration of the $T^2$ into two spheres and thus adds another Kähler class corresponding to the size of one of the $\mathbb{P}^1$'s. This size determines the mass of the adjoint
hypermultiplet in five dimensions, i.e. it is identified with $t^M_m$.

\subsection{Compactification on $S^1 \times S^1$}

We can also consider a further compactification on another $S^1$ which we take to be the $X^0$ direction. In trying to connect this geometry to topological string theory \cite{65,66,76,81} and the $\Omega$ background \cite{58}, we fiber the space-time $\mathbb{R}^4$ over this circle. In other words we twist the $\mathbb{R}^4 \times \mathbb{R}^4$ by the action of $U(1) \times U(1)$ as we go around the circle in the $X^0$ direction:

$$U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} : (z_1, z_2) \mapsto (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2),$$

$$: (w_1, w_2) \mapsto (e^{-\frac{\epsilon_1 + \epsilon_2}{2}} w_1, e^{-\frac{\epsilon_1 + \epsilon_2}{2}} w_2)$$

Equation (2.2.12)

Note that in the unrefined case where $\epsilon_1 + \epsilon_2 = 0$ to preserve the symmetry we do not need to rotate $\mathbb{R}^4_\perp$.

Again we can ask what the suspended M2 brane theory sees if it is wrapped around the $X^0, X^1$ directions. The M2 branes as well as the M5 branes will then be all at a fixed point in $\mathbb{R}^4$ and the M5 branes are extended along $T^2 \times \mathbb{R}^4$. Furthermore, the M2 branes will intersect the M5 branes along $T^2$ and be points in $\mathbb{R}^4$. This configuration is shown schematically in Fig. 2.2. As these points can be separated in $\mathbb{R}^4$ it is natural to expect that the effective worldvolume theory of $n$ M2 branes admits a description in terms of the Hilbert scheme on $n$ points on $\mathbb{R}^4$ as will be described in detail in Section 2.4.1.

The $\text{Spin}(8)$ weight vector corresponding to the $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ is $(\epsilon_1 e_1 + \epsilon_2 e_2 - \frac{\epsilon_1 + \epsilon_2}{2} (e_3 + e_4))$. For the unrefined case corresponding to $\epsilon_1 + \epsilon_2 = 0$ the above action leaves the $(4,0)$ supercharges invariant. However, for $\epsilon_1 + \epsilon_2 \neq 0$ it breaks $(4,0) \mapsto (2,0)$ with surviving supercharges corresponding to the $\text{Spin}(8)$ weights given by Equation (2.2.6),

$$\pm \frac{\epsilon_1 + \epsilon_2 + e_3 + e_4}{2}.$$  

Equation (2.2.13)

In general we will be interested in the compactification on a generic torus $T^2$ with complex structure $\tau$. In the case where the torus is rectangular $\tau$ can be identified as the ratio of the radii.
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Figure 2.2: The system of M2 and M5 branes. The M5 branes are depicted in yellow whereas the M2 brane is blue. They intersect at the torus $T^2$ which is depicted in green.

of the circle from six to five and the one from five to four as follows,

$$\tau = i \frac{R_0}{R_1}. \quad (2.2.14)$$

Upon further compactification to four dimensions the Kähler parameters get complexified
in the Type IIA setup. Moreover all the Kähler parameters of M-theory get rescaled by a factor of $R_0$ as we go to the Type IIA description,

$$t_i^M \rightarrow t_i^I = i R_0 t_i^M.$$  

These are the parameters we will be using, and in particular we get Kähler parameters which can be identified with the gauge theory parameters as follows

$$\text{Vol}_\mathbb{C}(T^2) = \tau, \quad Q_\tau = e^{2\pi i \tau},$$

$$\text{Vol}_\mathbb{C}(\mathbb{P}^1_f) = t_f, \quad Q_f = e^{2\pi i t_f},$$

$$\text{Vol}_\mathbb{C}(\mathbb{P}^1_m) = t_m = m\tau, \quad Q_m = e^{2\pi i t_m}. \quad (2.2.15)$$

Thus from the viewpoint of the original M5 branes, we have compactified on a torus with complex structure $\tau$, where the A-cycle of $T^2$ is twisted by $m$ and the B-cycle of the torus is twisted by $(\epsilon_1, \epsilon_2)$. Since we would be ultimately interested in computing the elliptic genus of the M2 branes stretched between the M5 branes and wrapped on $T^2$ and the twistings can be viewed as coupling to $U(1)$ background fields, the dependence of the amplitudes for each of the twistings will appear in the combination:

$$z = \theta_B + \tau \theta_A$$

where $(\theta_A, \theta_B)$ denote the twist parameters around the two-cycles. Thus for the mass term we have

$$(\theta_A, \theta_B) = (m, 0)$$

which is equivalent to

$$(\theta_A, \theta_B) = (0, m\tau) = (0, t_m)$$

and for the $\epsilon_i$ we have the twists

$$(\theta_A, \theta_B) = (0, \epsilon_i)$$

This suggests that we can think of all the twistings to be around the B-cycle as long as we use our Type IIA parameterization of $t_m$. For simplicity of notation later in this chapter we replace $t_m$
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with \(m\), when we discuss partition functions. We summarise the geometry of the torus \(T^2\) and its relation to the parameters of the gauge theory in Fig. 2.3.

![Diagram of a torus and its cycles](image)

**Figure 2.3**: The torus \(T^2\) and its cycles. In (a), the rectangular torus is depicted. While going around the circle with radius \(R_1\) one twists with the mass rotation, and along the circle \(R_0\) one introduces the \(\epsilon_i\) rotations. In (b), the same geometry is depicted where we use the holomorphy of the result to move the twisting along the A-cycle by \(m\) to twisting around the B-cycle by \(tm\).

Let us now come to the identification of states. From the discussion preceding Equation (2.2.11) it is clear that self-dual string solutions which wrap the whole \(T^2\) appear as instanton solutions in the four dimensional gauge theory. There will be also magnetically charged states (which will not be of relevance here and which we only include for the purpose of completeness of the discussion) which arise from the string which does not wrap the first \(S^1\). Electric-magnetic duality of the 4d \(\mathcal{N} = 4\) theory then corresponds to \(SL(2,\mathbb{Z})\) transformations of the \(T^2\). A string which wraps the \(T^2\) \(l\) times and has Kaluza-Klein momentum \(k\) then gives rise to a BPS degeneracy which can be counted with the topological string on the elliptic Calabi-Yau. Furthermore, such strings can have non-trivial charge \(q_m\) under the rotation induced by \(m\). Their degeneracies \(d(l, k, q_m)\) appear in the free energy of the topological string in the form

\[
d(l, k, q_m) Q^k Q^l Q^{q_m}. \tag{2.2.16}
\]
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The task of the following sections will be to compute these degeneracies in the presence of the $\epsilon_i$ rotations and obtain a closed formula for them in terms of the refined topological string partition function. More precisely, the partition function of M-theory in this background is by definition the partition function of the refined topological string on the corresponding Calabi-Yau threefold:

$$Z^{M\text{-theory}}((\mathbb{R}^4 \times \mathbb{R}^4) \times T^2_{\epsilon_1,\epsilon_2,m} \times \mathbb{R}) = Z^\text{refined}_{\text{top}}(\epsilon_1,\epsilon_2)(CY_{N,m,t_i^f})$$  \hspace{1cm} (2.2.17)

Moreover the degeneracy of BPS states is known to be computed by the topological vertex and its refinement [65–67,76,79], which in this case, as we will discuss in Section 4, consists mainly of the suspended M2 branes wrapped on $T^2$. We thus use this correspondence to compute the twisted elliptic genus of suspended M2 branes.

**Special values of parameters**

As already discussed, for generic values of $m,\epsilon_1,\epsilon_2$ the suspended M2 branes lead to a $(2,0)$ supersymmetric system on $T^2$. We can ask whether there are any special values of these parameters and in particular what happens to supersymmetry on the M-strings at these special values.

As already noted, in the unrefined limit where $\epsilon_1+\epsilon_2 = 0$ the supersymmetry gets enhanced to $(4,0)$. We can also ask if there are special values of $m$. For $m = \pm \frac{\epsilon_1 + \epsilon_2}{2}$ there is supersymmetry enhancement to $(2,2)$. The $Spin(8)$ holonomy is

$$(\epsilon_1,\epsilon_2,-\epsilon_1,-\epsilon_2),$$  \hspace{1cm} (2.2.18)

(up to the permutation of the last two factors) and the preserved charges are given by,

$$m = \frac{\epsilon_1 - \epsilon_2}{2} : \pm \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4}{2}, \pm \frac{\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4}{2},$$

$$m = -\frac{\epsilon_1 - \epsilon_2}{2} : \pm \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4}{2}, \pm \frac{\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4}{2}.$$  \hspace{1cm} (2.2.19)
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The consequence of this enhancement is that the elliptic genus of suspended M2 branes should be a constant independent of the moduli $\tau$ of $T^2$. There is also another limit in which the partition function simplifies and a different set of BPS states contribute. This limit is given by $m \mapsto \pm \frac{\epsilon_1 + \epsilon_2}{2}$. In this case the supersymmetry is still $(2,0)$ so a priori nothing should have simplified, except that the center of mass degree of freedom of the string acquires additional zero modes. This is because in this case the holonomy becomes

$$m = \frac{\epsilon_1 + \epsilon_2}{2} : (\epsilon_1, \epsilon_2, 0, -(\epsilon_1 + \epsilon_2)), \quad (2.2.20)$$

and a single M2 brane acquires a fermionic zero mode, due to the 0 direction in the holonomy twist (as we will review below in more detail). We can modify the computation of the elliptic genus in this limit to get a non-zero answer by computing instead a modified index

$$\text{Tr} \left( (-1)^F F_L q^{L_0} \bar{q}^{\bar{L}_0} \right), \quad (2.2.21)$$

to absorb the zero mode from this single fermion zero mode and obtain a non-trivial answer even in this limit. This is somewhat similar to what one sees in the context of topologically twisted $\mathcal{N} = 4$ Yang-Mills in 4d [82] where the $U(N)$ theory gives a vanishing partition function due to fermionic zero modes, but stripping off the $U(1)$ leads to a non-vanishing partition function for $SU(N)$ theories.

To summarize, the M-strings enjoy a $(4,4)$ supersymmetry. If we turn on generic $m, \epsilon_1, \epsilon_2$ on $T^2$ the supersymmetry is broken to $(2,0)$ and we would be computing a non-trivial elliptic genus. If $\epsilon_1 + \epsilon_2 = 0$ we have $(4,0)$ supersymmetry. If we tune $m = \pm(\epsilon_1 - \epsilon_2)/2$ the supersymmetry is enhanced to $(2,2)$ and the elliptic genus becomes a constant. If $m = \pm(\epsilon_1 + \epsilon_2)/2$ the supersymmetry is still $(2,0)$ but the partition function vanishes due to a fermionic zero mode associated to the ‘center of mass mode’. The fermionic zero mode can be eliminated in this case by insertion of suitable operators leading to a non-trivial function of $\tau$. We summarize our discussion in the Table 2.1.
Table 2.1: Enhanced symmetry configurations. The case $(2,0)^*$ is a configuration with $(2,0)$ supersymmetry but an extra fermionic zero mode which leads to the vanishing of the $U(1)$ part of the elliptic genus.

2.2.4 Quiver realization of the suspended M2 branes

There is a dual description of this system\(^2\) which generalizes it to $D$ and $E$ superconformal theories\(^2\). This corresponds to Type IIB theory in the presence of $ADE$ singularity. The duality between the $A$-series and M5 branes follows from the fact that $A_{N-1}$ singularities in Type IIB is dual to $N$ NS5 branes for Type IIA strings [83]. By lifting the NS5 branes to M-theory we see that this is equivalent to $N$ M5 branes where one of the five transverse directions to the 5-brane is compactified on $S^1$. Therefore, when we consider separated branes, the rotation symmetry is reduced from $SO(5)_R$ to $SO(3)$. Thus this realization has the slight disadvantage that not all the symmetries are manifest. In particular we cannot twist by the mass parameter as we go down on the circle from 6 to 5 dimensions.

The $ADE$ singularity is given by $\mathbb{C}^2/\Gamma$ where $\Gamma \subset SU(2)$ is one of the discrete subgroups of $SU(2)$, which are in one to one correspondence with the $ADE$ Dynkin diagrams, for which $\mathbb{C}^2$ is the two dimensional representation. The singularity $\mathbb{C}^2/\Gamma$ can be resolved to $X_\Gamma = \widetilde{\mathbb{C}^2/\Gamma}$. The resolution $X_\Gamma$ is such that $H_2(X_\Gamma, \mathbb{Z})$ generated by 2-cycles, which are topologically $\mathbb{P}^1$, can be identified with the root lattice of $ADE$ Lie algebra corresponding to $\Gamma$ such that the intersection

\(^2\)There is another dual description given by an M5 brane wrapped on $\mathbb{P}^1 \times T^2$ [28]. The tension of the string in this case is given by the size of the $\mathbb{P}^1$. This description can also be generalized to $ADE$ case by wrapping M5 brane on a chain of $\mathbb{P}^1 \times T^2$ where the chain of $\mathbb{P}^1$’s is given by the Dynkin diagram of the $ADE$ group. The $(1,0)$ tensionless string obtained by M5 brane wrapping $\frac{1}{2}K3$ is dual to M2 branes ending on M5 branes in the presence of the “end of the world” M9 branes.
number of the 2-cycles is given by the inner product on the root lattice which is determined by the
Cartan matrix $A_{ij}$, i.e. there exists a basis $\{C_1, C_2, \cdots, C_r\}$ of $H_2(X_\Gamma, \mathbb{Z})$ such that

$$C_i \cdot C_j = -A_{ij}. \quad (2.2.22)$$

As we blow down these 2-cycles to zero size we get back the singular space $\mathbb{C}^2/\Gamma$.

Consider Type IIB strings propagating on $\mathbb{R}^{1,5} \times X_\Gamma$ and let $t_i = \mu_i/g_s$, where $\mu_i$ is the size of blown up 2-cycles. The conformal limit is achieved by taking $t_i \to 0$. For $\Gamma_N = \begin{pmatrix} e^{\frac{2\pi i}{N}} & 0 \\ 0 & e^{-\frac{2\pi i}{N}} \end{pmatrix}$ the corresponding Dynkin diagram is that of $A_{N-1}$ and the corresponding Type IIB theory in the conformal limit gives the $(2,0)$ superconformal theory of $N$ coincident M5 branes. Moving away from the conformal point by turning on $t_i$ corresponds to the separation of the M5 branes along a linear direction as discussed before.

The emergence of conformal theory is signaled by the appearance of tensionless strings. In
the M-theory setup this arises by M2 branes ending on M5 branes, and the tension of the resulting
string is proportional to the separation of the corresponding M5 branes. Thus each pair of M5
branes leads to a string which become tensionless in the conformal limit. Similarly in the Type
IIB the strings arise by wrapping D3-branes over holomorphic 2-cycles $C$ of the blown-up geometry
$X_\Gamma$. Since holomorphic curves satisfy $C^2 = 2g - 2$, where $g$ is the genus of the curve $C$, and the
inner product $C^2$ is given by minus the Cartan matrix it follows that the only holomorphic curves
in the geometry are 2-spheres $C$ with $C^2 = -2$, i.e. they are in one to one correspondence with the
positive roots of $A_{N-1}$. The tension of a string coming from D3 brane wrapped on $\sum_i n_i C_i$ is given
by $\sum_i n_i t_i$ hence giving rise to strings with tensions $t_i$ for each 2-sphere $C_i$. Unwrapped D1 branes
can also be considered and they would correspond to M2 branes winding along a compactified circle
transverse to the M5 brane.

The theory describing M-strings, when they have finite tension, can be deduced using the
quiver description [37]. This in particular leads to the affine $ADE$ quiver. If we are interested in
the local behaviour of the 6 dimensional CFT, we will be mostly interested in the limit where the
transverse circle to the M5 brane is infinitely large where we would be ignoring the D1 brane. One could also consider the opposite limit where the transverse circle shrinks and consider the little string theory [84, 85], where the considerations made in this chapter will still apply \(^3\). If we ignore the D1 brane charge, this corresponds to deleting the affine root from the quiver and gives rise to the ordinary ADE quiver. This theory is equivalent to the reduction to two dimensions of the familiar \(\mathcal{N} = 2\) ADE quiver theory in four dimensions. In two dimensions this leads to a \((4, 4)\) supersymmetric quiver theory.

Note however that, as already remarked, not all the symmetries of the M5 branes are realized in this setup. This also impacts the symmetries that the M-string sees. In particular the symmetries of the 2d quiver theory (i.e. that of the \((4, 4)\) quiver theory) are given by

\[ Spin(4) \times SU(2) \]

where \(SU(2) = Spin(3) \subset Spin(4)_R = SU(2)_L \times SU(2)_R \) where \(SU(2)\) is diagonally embedded in the two \(SU(2)\)'s. The Cartan of this \(SU(2)\) can be identified with the rotation of the normal line bundle on the blown up \(\mathbb{P}^1\)'s. As already noted we cannot realize the twisting by \(m\) in this setup. This particular Cartan can be viewed as being in the \(e_3\) (or \(e_4\)) direction of \(Spin(8)\) holonomy. Thus in the setup of the most general twisting discussed in the previous section, we see that we are in the limit where \(m = (e_1 + e_2)/2\). Thus a 2-parameter subspace of the 3-parameter elliptic genus should be computable using the elliptic genus of \(\mathcal{N} = 4\) ADE quiver theories. Of course, as noted before, we would need to get rid of \(U(1)\) zero modes. In the \(D\) and \(E\) cases this gives a new way to compute the BPS degeneracies, which is not so simple in the geometric setup.

For concreteness let us focus on the \(A_{N-1}\) case. Let \(N_i\) D3 branes wrap the cycle \(C_i\), which corresponds to the simple roots forming a basis of positive root lattice of \(A_{N-1}\). Then this theory has gauge group

\[ G = \prod_{i=1}^{N-1} U(N_i), \tag{2.2.23} \]

\(^3\)This approach was followed in the recent paper [86].
with bi-fundamental matter between adjacent gauge factors. From the perspective of M-theory, this should be identified with the theory living on a collection of $N_i$ M-strings. For simplicity let us consider the case of the $A_1$ theory. This corresponds to having two M5 branes with $N_1$ M2 branes between them. The $(4,4)$ theory in this case corresponds to the pure $U(N_1)$ gauge theory \[75\]. This theory has a Coulomb branch which at least far away from the origin of the Coulomb branch gives rise to the sigma model on $\text{Sym}^{N_1} \mathbb{R}^4$, the $N_1$-fold symmetric product of $\mathbb{R}^4$. These $N_1$ points in $\mathbb{R}^4$ can be identified as the end points of the transverse $\mathbb{R}^4$ to the M2 brane in the M5 brane. This also follows from the fact that broken supercharges $Q^a_{\frac{\alpha}{2} a}$ and $Q^a_{\frac{-\alpha}{2} a}$ give rise to four left moving bosons $\partial_+ x^{\alpha a}$ and four right moving bosons $\partial_- x^{\alpha a}$ where $x^{\alpha a} = X^\mu \gamma^\mu_{\frac{\alpha}{2} a}$ such that $\mu = 2,3,4,5$ and $\gamma_\mu$ are the $\text{Spin}(4)$ gamma matrices. Modulo the resolution of the singularities when the points coincide, this can also be viewed as the Hilbert scheme of $N_1$ points on $\mathbb{R}^4$. What is the status of the theory when the points coincide is of course critical to the formation of BPS bound states, and therefore the above heuristic picture for $N_1 > 1$ is not guaranteed to be correct. In fact we will find later that our computation suggests that this picture is not accurate.

### 2.3 Topological string partition function for the $\mathcal{N} = 2^*$ theory

The $\mathcal{N} = 2^*$ $SU(N)$ gauge theories can be geometrically engineered using elliptic Calabi-Yau threefolds. These elliptic Calabi-Yau threefolds, which we will denote by $X_N$, are given by a deformation of the $A_{N-1}$ fibration over $T^2$. The geometry of $X_N$ is captured by the toric diagram shown in Fig. 2.4.

The gauge theory partition function can be obtained either from Nekrasov's instanton calculus or by calculating the topological string partition function of $X_N$. The topological string partition function can be calculated using the refined topological vertex formalism [67]. The refined topological vertex has a preferred direction which breaks the cyclic symmetry of the topological vertex. For a given toric diagram that engineers a gauge theory we need to pick an orientation for the preferred direction. It was argued that the total amplitude is independent of the choice [87].
although the form of the amplitude could have a significantly different looking form\textsuperscript{4}. The choice is not necessarily arbitrary and has, as we will see later, important physical meaning. The preferred direction usually determines the instanton directions. In other words, according to the gluing algorithm of the topological vertex we perform sums of Young diagrams along each internal edge in the toric diagram. All of these sums can be explicitly performed except the ones along the preferred directions. The preferred direction is generally chosen such that the left-over sums match with the instanton expansion of the corresponding gauge theory.

From Fig. 2.4 it is clear that there are two choices for the preferred direction. One choice is along the vertical direction which is compactified on a circle and the other choice is along the horizontal. We will calculate the partition function for both cases.

Before we begin calculating the partition functions we would like to explain our notation which will appear in later sections. We will denote by Greek letters $\lambda, \mu, \nu$ partitions of natural numbers. An empty partition will be denoted by $\emptyset$. A non-empty partition $\lambda$ is a set of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0$. The number of parts of the partition $\lambda$ will be denoted by $\ell(\lambda)$. We will denote by $\lambda^t$ the transpose of the partition $\lambda$. $\lambda^t$ is also a partition such that $\lambda_i^t =$ number of parts of $\lambda$ which are greater than $i - 1$. For example, if $\lambda = \{4, 2, 2, 1\}$ then

\textsuperscript{4}In a number of non-trivial examples this invariance is shown for high degrees of the curves and is used as a new way to derive identities involving Macdonald polynomials [88].
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\( \lambda' = \{4, 3, 1, 1\} \). The following are few functions on the set of partitions which will be of use later,

\[
|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i, \quad \|\lambda\|^2 := \sum_{i=1}^{\ell(\lambda)} \lambda_i^2, \quad \|\lambda'\|^2 = \sum_{i=1}^{\ell(\lambda')} (\lambda_i')^2.
\]  

(2.3.24)

As is well known the partition \( \lambda \) has a two dimensional representation called the Young diagram. A Young diagram corresponding to the partition \( \lambda \) is obtained by placing a box in the first quadrant with upper left hand coordinate \((i, j)\) for each \((i, j) \in \{(i, j) | i = 1, \ldots, \ell(\lambda); j = 1, \ldots, \lambda_i\}\). Thus the number of boxes in the \(i^{th}\) column of the Young diagram give \( \lambda_i \). We will not distinguish between the partition and its Young diagram so that \((i, j) \in \lambda\) makes sense and means the box in the Young diagram with coordinates \((i, j)\).

To calculate the topological string partition function we will use the refined topological vertex which is given by

\[
C_{\lambda\mu\nu}(t, q) = t^{-\frac{|\lambda|^2}{2}} q^{-\frac{\|\mu\|^2 + \|\nu\|^2}{2}} \bar{Z}_\nu(t, q) \sum_\eta \left( \frac{q}{t} \right)^{\frac{|\eta| + |\lambda| - |\mu|}{2}} s_{\lambda/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu} q^{-\rho}) ,
\]

(2.3.25)

where \( \rho = \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\} \), \( s_\nu(x_1, x_2, \ldots) \) is the Schur function labelled by a partition \( \nu \), and

\[
\bar{Z}_\nu(t, q) = \prod_{(i,j) \in \nu} \left( 1 - q^{\nu_{i,j}} t^{\nu_{i,j}-i+1} \right)^{-1}.
\]

(2.3.26)

We will also calculate gauge theory partition functions using equivariant instanton calculus where the torus action on \( \mathbb{C}^2 \) is given by \((z_1, z_2) \mapsto (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2)\). The topological string parameters \( q \) and \( t \) are related to the gauge theory parameters \( \epsilon_1 \) and \( \epsilon_2 \) as

\[
q = e^{2\pi i \epsilon_1}, \quad t = e^{-2\pi i \epsilon_2}.
\]

(2.3.27)

2.3.1 Case I: Preferred direction along the compactified circle

Let us consider the case of \( SU(2) \) in detail and then we will generalize this to \( SU(N) \). The toric diagram for the \( SU(2) \) case is shown in Fig. 2.5 below. The vertical lines in the toric diagram are glued and the preferred direction is along the vertical.
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Figure 2.5: Toric diagram of the geometry giving rise to \( SU(2) \mathcal{N} = 2^* \) theory. The preferred direction is taken to be vertical.

The refined topological string partition function in terms of the refined vertex \( C_{\mu \nu}(t, q) \) is given by

\[
Z^{(2)} = \sum_{\lambda, \mu, \sigma, \nu_1, \nu_2} (-\hat{Q})^{\nu_1 + \nu_2} (-Q_m)^{\sigma + |\mu|} (-Q)^{|\lambda|} C_{\mu \theta, \nu_1}(t^{-1}, q^{-1}) C_{\nu_1, \lambda \nu_2}(q^{-1}, t^{-1})
\]

\[
\times C_{\sigma \nu_1, \nu_2}(t^{-1}, q^{-1}) C_{\sigma, \theta \nu_2}(q^{-1}, t^{-1}),
\]

(2.3.28)

where the superscript refers to the number of M5 branes in the construction. Using standard techniques of summing up the Schur symmetric function given in Appendix B.2 we get \( Q_r = \hat{Q} Q_m \)

\[
Z^{(2)} = Z_{pert}^{(2)} Z_{inst}^{(2)},
\]

(2.3.29)

where

\[
Z_{pert}^{(2)} = \prod_{i,j=1}^{\infty} \frac{(1 - Q_m q^{-\frac{1}{2}} t_i^{-\frac{1}{2}} u_j^{-\frac{1}{2}})(1 - Q_f q^{-\frac{1}{2}} u_i^{-\frac{1}{2}} t_j^{-\frac{1}{2}})(1 - Q_f q^{-\frac{1}{2}} t_i^{-\frac{1}{2}} u_j^{-\frac{1}{2}})(1 - Q_f q^{-\frac{1}{2}} t_i^{-\frac{1}{2}} u_j^{-\frac{1}{2}})}{(1 - Q_f q^{-1} u_i)(1 - Q_f q^{-1} t_j^{-1})}
\]

(2.3.30)
and

\[
Z^{(2)}_{\text{inst}} = \sum_{\nu_1 \nu_2} Q_{|\nu_1 + |\nu_2|} \times \prod_{(i,j) \in \nu_1} (1 - Q_m q^{\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}})(1 - Q_m^{-1} q^{-\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}}) \frac{1}{(1 - q^{\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}})(1 - \nu_{i,j} - j q^{\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}})}
\times \prod_{(i,j) \in \nu_2} (1 - Q_m Q_f q^{\nu_{i,j}^f - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}})(1 - Q_m^{-1} Q_f q^{-\nu_{i,j}^f - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}}) \frac{1}{(1 - Q_f q^{\nu_{i,j}^f - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}})(1 - \nu_{i,j} - j q^{\nu_{i,j}^f - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}})}
\times \prod_{(i,j) \in \nu_1} (1 - Q_m Q_f q^{-\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j - \frac{1}{2}}})(1 - Q_m^{-1} Q_f q^\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}) \frac{1}{(1 - Q_f q^{-\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j - \frac{1}{2}}})(1 - \nu_{i,j} - j q^{-\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j - \frac{1}{2}}})).
\tag{2.3.31}
\]

In the limit \( Q_\tau \to 0 \) the partition function reduces to \( Z^{(2)}_{\text{pert}} \) which is the contribution of the holomorphic curves that do not wrap the elliptic curve.

\[
Q_m = (qt)^{\pm \frac{1}{2}} \text{ limit}
\]

An important property of the above partition function is that for \( Q_m = (qt)^{\pm \frac{1}{2}} \) the \( Q_\tau \) dependence drops out, that is the sum over \( Q_\tau \) in Equation (2.3.31) only gets contribution from the trivial partition and becomes 1. To see this consider the following factor which occurs in the sum over \( (\nu_1, \nu_2) \) in Equation (2.3.31),

\[
\prod_{(i,j) \in \nu_1} (1 - Q_m q^{\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}})(1 - Q_m^{-1} q^{-\nu_{i,j}^l - i + \frac{1}{2} t^{\nu_{i,j} - j + \frac{1}{2}}}) \tag{2.3.32}
\]

If we consider a box in the Young diagram \( \nu_1 \) which is an outer corner then its arm length and the leg length is zero, i.e. if \( (i_0, j_0) \in \nu_1 \) are the coordinates of such a box then \( \nu_1, i_0 - j_0 = 0 \) and \( \nu_{1, i_0}^l - i_0 = 0 \). Such a box always exists in a nonempty Young diagram and the contribution of such a box to the factor in Equation (2.3.32) is given by

\[
(1 - Q_m t^\frac{1}{2} q^\frac{i}{2})(1 - Q_m^{-1} t^\frac{1}{2} q^\frac{i}{2}) = 0 \text{ for } Q_m = (qt)^{\pm \frac{1}{2}} .
\tag{2.3.33}
\]
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Partition function from instanton calculus

The partition function of the $U(N)$ $\mathcal{N} = 2^*$ theory, calculated in the last section using geometric engineering and the topological vertex formalism, can also be determined using Nekrasov’s instanton calculus [58]. In the case of the massive adjoint hypermultiplet the supersymmetric partition function of the gauge theory reduces to the partition function of supersymmetric quantum mechanics on the instanton moduli space [89]. This quantum mechanical model is the reduction to one dimension of the $\mathcal{N} = (2, 2)$ two dimensional sigma model with instanton moduli space as the target space. Recall that the $\mathcal{N} = (2, 2)$ sigma model with target space $M$, a Kähler manifold, has Lagrangian given by

$$L = \int d^4 \theta \, K(\Phi, \overline{\Phi}) ,$$

where $K(\phi^i, \phi^j)$ is the Kähler potential of $M$ and $\Phi_i$ are the chiral superfields. In terms of the component fields the Lagrangian is given by

$$L = g_{ij} \partial_+ \phi^i \partial_- \phi^j + 2i g_{ij} \psi^i_+ \partial_+ \phi^j - 2i g_{ij} \psi^j_- D_+ \psi^i_+ + 2i g_{ij} \psi^i_+ D_+ \psi^j_- + R_{kj} \psi^j_+ \psi^j_- \psi^k_+ \psi^k_- ,$$

where the covariant derivatives are given by

$$D_+ \psi^i_+ = (\partial_+ \delta^i_+ + \Gamma^i_{lk} \partial_+ \phi^k) \psi^l_+ ,$$

$$D_+ \psi^j_- = (\partial_+ \delta^j_- + \Gamma^j_{lk} \partial_+ \phi^k) \psi^l_+ .$$

Reducing to one dimension we get the Lagrangian

$$L = \frac{1}{2} g_{ij} \phi^i \phi^j - i g_{ij} \psi^i_+ \psi^j_- - i g_{ij} \Gamma^k_{lj} \phi^k (\psi^i_+ \psi^j_- + \psi^j_- \psi^i_+) + R_{kj} \psi^j_+ \psi^j_- \psi^k_+ \psi^k_- .$$

Since the target space is a Kähler manifold, this Lagrangian is invariant under 4 supersymmetries given by

$$\delta \phi^i = \epsilon^+ \phi^i + \epsilon^- \psi^i_+,$n

$$\delta \psi^i_+ = i \epsilon^+ \phi^i - \epsilon^- \Gamma^i_{jl} \psi^j_- \psi^l_+ ,$$

$$\delta \psi^j_- = i \epsilon^- \phi^j + \epsilon^+ \Gamma^j_{kl} \psi^k_+ \psi^l_- .$$

\[5\] We are considering a Lorentzian worldsheet so that ($\theta^+, \bar{\theta}^+$) are positive chirality spinors and ($\theta^-, \bar{\theta}^-$) are negative chirality spinors. The fermions $\psi^i_+$ and $\psi^j_-$ are respectively of negative and positive chirality. Also $\partial_\pm = \frac{\partial x^\pm}{\sqrt{2}}$. 

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The Witten index of this quantum mechanical system is the Euler characteristic of the target space,

\[ \chi(M) = \text{Tr} (-1)^F e^{-\beta H}. \]  

(2.3.37)

One can define a more general partition function [90],

\[ Z := \text{Tr} (-1)^{F^+} (-y)^{F^+} e^{-\beta H}, \]  

(2.3.38)

which is invariant under only two of the supersymmetries \( \epsilon^- \) and \( \bar{\epsilon}^- \). Inserting \( y^{F^+} \) has the effect of changing the fermion boundary conditions so that \( \psi^i_- \) and \( \bar{\psi}^\bar{i}_- \) remain periodic but \( \psi^i_+ (\beta) = y \psi^i_+(0) \) and \( \bar{\psi}^{\bar{i}}_+ (\beta) = y \bar{\psi}^{\bar{i}}_+(0) \). This twisted partition function can be calculated using the supersymmetric localization with respect to the remaining supercharges and gives

\[ Z = \int_{\mathcal{M}} \prod_i x_i (1 - y e^{-x_i}) \frac{1}{1 - e^{-x_i}} = \sum_{i,j} (-1)^{i+j} y^j b_{i,j}(\mathcal{M}), \]  

(2.3.39)

where \( x_i \) are the roots of the curvature two form. This is the \( \chi_y \) genus of the manifold \( \mathcal{M} \), which for \( y = 1 \) gives the Euler characteristic. Thus the partition function of the \( U(N) \) \( \mathcal{N} = 2^* \) theory is the \( \chi_y \) genus of the rank \( N \) instanton moduli spaces,

\[ Z_{U(N)} = \sum_{k \geq 0} \varphi^k \chi_y(\mathcal{M}(N,k)), \]  

(2.3.40)

where \( \mathcal{M}(N,k) \) is the moduli space of rank \( N \) instantons with charge \( k \). We will determine Equation (2.3.40) using equivariant localization and see that it exactly reproduces \( Z^{(2)}/ (Z^{(1)})^2 \) of Equation (2.3.31) for \( N = 2 \).

The instanton moduli space of charge \( k \) for the \( U(1) \) case is the Hilbert scheme of \( k \) points in \( \mathbb{C}^2 \), \( \text{Hilb}^k[\mathbb{C}^2] \). \( \text{Hilb}^k[\mathbb{C}^2] \) is a \( 4k \) dimensional hyperkähler manifold which is obtained by a resolution of singularities of the \( k^{th} \) symmetric product of \( \mathbb{C}^2 \) [91]. It is also the space of polynomial ideals in \( \mathbb{C}[z_1,z_2] \) of codimension \( k \),

\[ \text{Hilb}^k[\mathbb{C}^2] = \{ I \subset \mathbb{C}[z_1,z_2] ~|~ \dim(\mathbb{C}[z_1,z_2]/I) = k \}. \]  

(2.3.41)

The tangent space at \( I \in \text{Hilb}^k[\mathbb{C}^2] \) is given by

\[ T_I(\text{Hilb}^k[\mathbb{C}^2]) \simeq \text{Hom}(I, \mathbb{C}[z_1,z_2]/I). \]  

(2.3.42)
The torus $T = U(1)_{e_1} \times U(1)_{e_2}$ action on $\mathbb{C}^2$, 

$$(z_1, z_2) \mapsto (e^{2\pi i e_1} z_1, e^{2\pi i e_2} z_2),$$

induces an action on Hilb$^k[\mathbb{C}^2]$ with a finite number of isolated fixed points labelled by the partitions of $k$. The fixed point corresponding to the partition $\lambda$ will be denoted by $I_\lambda$. The torus $T$ maps $I_\lambda$ to $I_\lambda$ and hence maps $T(I_\lambda(\text{Hilb}^k[\mathbb{C}^2]))$ to itself. The weights of the $T$ action on the tangent space at the fixed point $I_\lambda$ are given by [91]

$$\{q^{\lambda_i - i} t^{\lambda_i - j + 1}, q^{-\lambda_j + i - 1} t^{-\lambda_i + j} | (i, j) \in \lambda\}, \quad q = e^{2\pi i e_1}, \quad t = e^{-2\pi i e_2}.$$  

The $U(1)$ partition function can now be calculated,

$$Z^{U(1)} = \sum_{k \geq 0} \bar{Q}^k \chi_y \left(\text{Hilb}^k[\mathbb{C}^2]\right) = \sum_{k \geq 0} \bar{Q}^k \int_{\text{Hilb}^k[\mathbb{C}^2]} \prod_{i=1}^{2k} \frac{1 - y e^{-x_i} x_i}{1 - e^{-x_i}},$$

where $x_i$ are the Chern roots of the tangent bundle. The integral above can be calculated using equivariant localization [91] and is given by

$$Z^{U(1)} = \sum_{k \geq 0} \bar{Q}^k \sum_{p \in \{\text{fixed points}\}}\sum_{i=1}^{2k} \frac{1 - y e^{-x_{p,i}}}{1 - e^{-x_{p,i}}},$$

where for the fixed point $p$ labelled by the partition $\lambda$:

$$e^{-x_{p,i}} \in \{q^{\lambda_i - i} t^{\lambda_i - j + 1}, q^{-\lambda_j + i - 1} t^{-\lambda_i + j} | (i, j) \in \lambda\}.$$ 

Thus we get

$$Z^{U(1)} = \sum_{k \geq 0} \bar{Q}^k \prod_{|\lambda| = k, (i,j) \in \lambda} \frac{(1 - y q^{\lambda_i - i \lambda_j - j + 1})(1 - y q^{-\lambda_j + i - 1 \lambda_i + j})}{(1 - q^{\lambda_i - i \lambda_j - j + 1})(1 - q^{-\lambda_j + i - 1 \lambda_i + j})}.$$ 

For the case of $U(N)$ the instanton moduli space of charge $k \, \mathcal{M}(N, k)$ has a $T = U(1)_{e_1} \times U(1)_{e_2} \times U(1)^N$ action with a finite number of fixed points labelled by a set $(\nu_1, \nu_2, \ldots, \nu_N)$ of $N$ partitions such that $|\nu_1| + |\nu_2| + \cdots + |\nu_N| = k$. The weights of the $T$ action on the tangent space
above the fixed point labelled by \((\nu_1, \nu_2, \ldots, \nu_N)\) is given by [92]

\[
\sum_i e^{-x_{i,j}} = \sum_{\alpha, \beta=1}^N e^{2\pi i (a_\alpha - a_\beta)} \left( \sum_{(i,j) \in \nu_\alpha} q^{-\nu_{i,j}^\alpha + i - (\nu_{i,j}^\alpha + j)} + \sum_{(i,j) \in \nu_\beta} q^{\nu_{i,j}^\beta - i + 1 - \nu_{i,j}^\beta - j} \right).
\]

Using these weights we can write down the \(U(N)\) partition function,

\[
Z^{U(N)} = \sum_{k \geq 0} \tilde{Q}^k \chi_y(\mathcal{M}(N, k))
\]

\[
= \sum_{k \geq 0} \tilde{Q}^k \sum_{|\nu_1| + \cdots + |\nu_N| = k} \prod_{\alpha, \beta=1}^N Z_{\nu_\alpha, \nu_\beta},
\]

(2.3.49)

where \((Q_{\alpha\beta} = e^{2\pi i (a_\alpha - a_\beta)})\),

\[
Z_{\nu_\alpha, \nu_\beta} := \prod_{(i,j) \in \nu_\alpha} \frac{(1 - y Q_{\alpha\beta} q^{-\nu_{i,j}^\alpha + i - (\nu_{i,j}^\alpha + j)})}{(1 - Q_{\alpha\beta} q^{\nu_{i,j}^\alpha - i - 1 + \nu_{i,j}^\beta + j})} \prod_{(i,j) \in \nu_\beta} \frac{(1 - y Q_{\alpha\beta} q^{\nu_{i,j}^\beta + i - 1 + \nu_{i,j}^\beta + j})}{(1 - Q_{\alpha\beta} q^{-\nu_{i,j}^\beta - i + 1 + \nu_{i,j}^\beta + j})}.\]

For the case of \(U(2)\) gauge group the above partition function becomes \((Q_{12} = Q_f^{-1})\),

\[
Z_{U(2)} = \sum_{\nu_1, \nu_2} \tilde{Q}^{|
u_1| + |
u_2|} W(\nu_1, \nu_2) W(\nu_2, \nu_1),
\]

(2.3.50)

where

\[
W(\nu_1, \nu_2) = \prod_{(i,j) \in \nu_1} \frac{(1 - y q^{-\nu_{i,j}^1 + i - (\nu_{i,j}^1 + j)})(1 - y q^{\nu_{i,j}^1 - i + 1 - \nu_{i,j}^1 + j})}{(1 - q^{-\nu_{i,j}^1 + i - (\nu_{i,j}^1 + j)})(1 - q^{\nu_{i,j}^1 - i + 1 - \nu_{i,j}^1 + j})} \times \frac{(1 - y Q_f^{-1} q^{-\nu_{i,j}^2 + i - (\nu_{i,j}^2 + j)})(1 - y Q_f q^{\nu_{i,j}^2 - i + 1 - \nu_{i,j}^2 + j})}{(1 - Q_f^{-1} q^{-\nu_{i,j}^2 + i - (\nu_{i,j}^2 + j)})(1 - Q_f q^{\nu_{i,j}^2 - i + 1 - \nu_{i,j}^2 + j})}.
\]

(2.3.51)

The above partition function is precisely \(\tilde{Z}^{(2)}\) of Equation (2.3.31) with,

\[
Q_\tau = \tilde{Q} y^2, \quad Q_m = y \sqrt{\frac{q}{t}}.
\]

(2.3.52)

For \(y = 1\) we get the generating function for the Euler characteristic of \(\mathcal{M}(N, k)\),

\[
Z_{U(N)}|_{y=1} = \left( \prod_{k=1}^\infty (1 - Q_f^k)^{-1} \right)^N
\]

(2.3.53)

which is also the partition function of the \(U(N)\) \(\mathcal{N} = 4\) SYM.
2.3.2 Case II: Preferred direction along the horizontal

This choice of the preferred direction leads to a very interesting representation of the partition function. First, it makes the modular properties with respect to the elliptic fiber manifest. We can perform all the sums along this direction to get infinite products. These products can be recast in terms of Jacobi theta functions. Second, as we will discuss later this choice is the natural one to understand the world volume theory of M2 branes. For a related discussion see also [93].

For the $SU(N)$ theory the geometry is made of $N$ building blocks depicted in Fig. 2.6.

\[
\begin{align*}
W_{\nu_m\nu_{m+1}}(\tau, m, \epsilon_1, \epsilon_2) &= f_{\nu_{m+1}}(t, q) f_{\nu_m}(q, t) Q_m^{-\frac{|\nu_m| + |\nu_{m+1}|}{2}} \\
&\times \sum_{\lambda, \mu} (-Q_m)^{|\lambda|} (-Q_{1-m}^1)^{|\mu|} C_{\lambda^{t, \nu_{m+1}}} t^{\lambda^{t, \nu_m}} (q^{-1}, t^{-1}) C_{\lambda^{t, \nu_{m}}} (t^{-1}, q^{-1}). 
\end{align*}
\] (2.3.54)

Figure 2.6: The toric diagram of the building block of the $X_N$ geometry with the preferred direction along the horizontal direction.

We want to outline the computation of these blocks using the refined topological vertex. Further details can be found in Appendix B.2. According to the gluing rules this open topological string amplitude is given by\(^6\)

\[W_{\nu_m\nu_{m+1}}(\tau, m, \epsilon_1, \epsilon_2) = f_{\nu_{m+1}}(t, q) f_{\nu_m}(q, t) Q_m^{-\frac{|\nu_m| + |\nu_{m+1}|}{2}} \times \sum_{\lambda, \mu} (-Q_m)^{|\lambda|} (-Q_{1-m}^1)^{|\mu|} C_{\lambda^{t, \nu_{m+1}}} t^{\lambda^{t, \nu_m}} (q^{-1}, t^{-1}) C_{\lambda^{t, \nu_{m}}} (t^{-1}, q^{-1}). \] (2.3.54)

\(^6\)Since we are considering an open topological string amplitude we can consider the branes with some framing. In this case we have taken the branes to be with framing 1 which is the reason for the prefactor in Equation (2.3.54) which is given in terms of the framing factor $f_\mu(t, q) = q^{ln|t|} - ln|q|$. This framing will not affect the calculation of the closed topological string partition function since the framing factor will cancel when the two open string amplitudes are glued together because of the identity $f_\mu(t, q) f_{\mu'}(q, t) = 1$. 

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Chapter 2: M-strings

After some algebra in Schur functions, the building blocks take the form

\[
W_{\nu_m \nu_{m+1}}(Q_\tau, Q, t, q) = t^{|\nu_m|+1/2} q^{-|\nu_m|/2} \bar{Z}_{\nu_m}^{-1}(q^{-1}, t^{-1}) \bar{Z}_{\nu_{m+1}}(t^{-1}, q^{-1}) Q_m^{-|\nu_m|+|\nu_{m+1}|/2} \prod_{k=1}^\infty \prod_{i,j=1}^\infty \left(1 - Q_\tau^k Q_m^{-1} q^{-\nu_{m,i} - j + \nu_{m+1,j} + 1} t^{-\nu_{m,i} + j - \nu_{m+1,j} - 1/2} \right) \left(1 - Q_\tau^k Q_m q^{-\nu_{m,i} - j + \nu_{m+1,j} + 1} t^{-\nu_{m,i} + j - \nu_{m+1,j} - 1/2} \right). \tag{2.3.55}
\]

Let us define the normalized building block

\[
D_{\nu_m \nu_{m+1}}(\tau, m, \epsilon_1, \epsilon_2) = \frac{W_{\nu_m \nu_{m+1}}(\tau, m, \epsilon_1, \epsilon_2)}{W_0(\tau, m, \epsilon_1, \epsilon_2)}. \tag{2.3.56}
\]

The factor in the denominator is the closed topological string partition function of the geometry shown in Fig. 2.6. Simplifying Equation (2.3.56) using the identities given in Appendix B.1 we get

\[
D_{\nu_m \nu_{m+1}}(\tau, m, \epsilon_1, \epsilon_2) = t^{-|\nu_m|+1/2} q^{-|\nu_m|/2} Q_m^{-|\nu_m|+|\nu_{m+1}|/2} \prod_{k=1}^\infty \prod_{(i,j) \in \nu_m} \left(1 - Q_\tau^k Q_m^{-1} q^{-\nu_{m,i} - j + \nu_{m+1,j} + 1} t^{-\nu_{m,i} + j - \nu_{m+1,j} - 1/2} \right) \left(1 - Q_\tau^k Q_m q^{-\nu_{m,i} - j + \nu_{m+1,j} + 1} t^{-\nu_{m,i} + j - \nu_{m+1,j} - 1/2} \right) \prod_{(i,j) \in \nu_{m+1}} \left(1 - Q_\tau^k Q_m^{-1} q^{-\nu_{m,i} - j + \nu_{m+1,j} + 1} t^{-\nu_{m,i} + j - \nu_{m+1,j} - 1/2} \right) \left(1 - Q_\tau^k Q_m q^{-\nu_{m,i} - j + \nu_{m+1,j} + 1} t^{-\nu_{m,i} + j - \nu_{m+1,j} - 1/2} \right). \tag{2.3.57}
\]

In the unrefined case \(\epsilon_2 = -\epsilon_1 = -\epsilon\) we have \(q = t\) and the above open string amplitude can be written as

\[
D_{\nu_m \nu_{m+1}}(\tau, m, \epsilon, -\epsilon) = \prod_{(i,j) \in \nu_m} \theta_1(\tau; \alpha_{ij}) \prod_{(i,j) \in \nu_{m+1}} \theta_1(\tau; \beta_{ij}) \prod_{(i,j) \in \nu_m} \theta_1(\tau; \gamma_{ij}) \prod_{(i,j) \in \nu_{m+1}} \theta_1(\tau; \delta_{ij}). \tag{2.3.58}
\]

with

\[
e^{2\pi i \alpha_{ij}} = Q_m^{-1} q^{-\nu_{m,i} + j + \nu_{m+1,j} + 1}, \quad e^{2\pi i \beta_{ij}} = Q_m q^{-\nu_{m,i} + j + \nu_{m+1,j} - 1},
\]

\[
e^{2\pi i \epsilon_{ij}} = Q_m q^{\nu_{m,i} - j + \nu_{m+1,j} - 1}, \quad e^{2\pi i \gamma_{ij}} = Q_m^{-1} q^{\nu_{m,i} + j + \nu_{m+1,j} + 1},
\]

\[
e^{2\pi i \delta_{ij}} = Q_m^{-1} q^{\nu_{m,i} + j + \nu_{m+1,j} + 1}, \quad e^{2\pi i \delta_{ij}} = Q_m q^{-\nu_{m,i} - j + \nu_{m+1,j} - 1}, \tag{2.3.59}
\]

where

\[
\theta_1(\tau; z) = -i e^{i\pi \tau / 4} e^{i\pi z} \prod_{k=1}^\infty \left(1 - e^{2\pi i k \tau} \right) \left(1 - e^{2\pi i z} e^{2\pi i k \tau} \right) \left(1 - e^{-2\pi i z} e^{-2\pi i (k-1) \tau} \right).
\]

Recall that the theta function can be written in terms of the Eisenstein series as

\[
\theta_1(\tau; z) = \eta^3(\tau) (2\pi i z) \exp \left( \sum_{k=1}^\infty \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau) (2\pi i z)^{2k} \right). \tag{2.3.60}
\]
In the above equation $E_{2k}(\tau)$ is the weight $2k$ Eisenstein series defined as
\begin{equation}
E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{2.3.61}
\end{equation}
where $B_{2k}$ are the Bernoulli numbers$^7$ and $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor sum function. Under modular transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ the Eisenstein series $E_{2k}(\tau)$ transforms in the following way:
\begin{equation}
E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = \begin{cases} (c\tau + d)^2 E_2(\tau) - i\pi c(c\tau + d), & k = 1, \\ (c\tau + d)^{2k} E_{2k}(\tau), & k > 1. \end{cases} \tag{2.3.62}
\end{equation}
Thus the Eisenstein series $E_2(\tau)$ is not a modular form. $E_2(\tau)$ can be made modular form by adding a non-holomorphic term to it. Define
\begin{equation}
\tilde{E}_2(\tau, \tilde{\tau}) = E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}, \tag{2.3.63}
\end{equation}
then $\tilde{E}_2(\tau)$ transforms as modular form,
\begin{equation}
\tilde{E}_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau). \tag{2.3.64}
\end{equation}
Thus if we replace $E_2(\tau)$ with $\tilde{E}_2(\tau, \tilde{\tau})$ in the $\theta_1(\tau; z)$ of Equation (2.3.58) then under the transformation,
\begin{equation}
(\tau, m, \epsilon_1, \epsilon_2) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{m}{c\tau + d}, \frac{\epsilon_1}{c\tau + d}, \frac{\epsilon_2}{c\tau + d}\right), \tag{2.3.65}
\end{equation}
the open topological string amplitude $W_{\nu_m \nu_{m+1}}(\tau, \tilde{\tau}, m, \epsilon, -\epsilon)$ is invariant but no longer holomorphic. The open string amplitude satisfies a holomorphic anomaly equation:
\begin{equation}
\frac{\partial D_{\nu_m \nu_{m+1}}(\tau, m, \epsilon)}{\partial \tilde{E}_2} = \frac{1}{24} \left( \sum_{(i,j) \in \nu_m} (\beta_{ij}^2 - \alpha_{ij}^2) + \sum_{(i,j) \in \nu_{m+1}} (\delta_{ij}^2 - \gamma_{ij}^2) \right) D_{\nu_m \nu_{m+1}}(\tau, m, \epsilon),
\end{equation}
$^7$The Bernoulli numbers can be obtained from the following generating function: $\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$
where,

\[
\sum_{(i,j)\in \nu_m} (\beta_{ij}^2 - \alpha_{ij}^2) + \sum_{(i,j)\in \nu_{m+1}} (\delta_{ij}^2 - \gamma_{ij}^2) = \\
\sum_{(i,j)\in \nu_m} ((m + (\nu_{m,i} - j + \nu'_{m+1,j} - i + 1)e)^2 - ((\nu_{m,i} - j + \nu'_{m,j} - i + 1)e)^2 \\
+ \sum_{(i,j)\in \nu_{m+1}} ((m - (\nu_{m+1,i} - j + \nu'_{m,j} - i + 1)e)^2 - ((\nu_{m+1,i} - j + \nu'_{m+1,j} - i + 1)e)^2. \tag{2.3.66}
\]

**U(2) partition function**

Using the open topological string amplitude we can now determine the closed topological string partition function for $X_N$ taking the preferred direction to be horizontal.

![Toric diagram](image)

Figure 2.7: (a) Toric diagram of the geometry giving rise to $U(2)$ $N = 2^*$ theory. The preferred direction is taken to be horizontal. (b) The partition function can be obtained by gluing open topological string amplitudes.

As shown in Fig. 2.7 we can glue two copies of $D_{\theta,\nu}$ to obtain the $U(2)$ partition function,

\[
\tilde{Z}^{(2)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \sum_\nu (-Q_f)^{[\nu]} D_{\theta,\nu}(\tau, m, \epsilon_1, \epsilon_2) D_{\nu',\theta}(\tau, m, -\epsilon_2, -\epsilon_1), \tag{2.3.67}
\]

where $Q_f = e^{2\pi i t_f}$ is the parameter of the fiber $\mathbb{P}^1$, the compact part of the geometry is a $\mathbb{P}^1$ bundle over $T^2$, whose local geometry is $\mathcal{O}(0) \oplus \mathcal{O}(-2)$. Using Equation (2.3.56) and Equation (6.3.18)
the $U(2)$ partition function is given by

$$Z^{(2)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \left( Z^{(1)}(\tau, m, \epsilon_1, \epsilon_2) \right)^2 \times \sum_{\nu} (-Q_f)^{\nu} \prod_{(i,j)\in \nu} \frac{\theta_1(\tau; z_{ij}) \theta_1(\tau; v_{ij})}{\theta_1(\tau; w_{ij}) \theta_1(\tau; u_{ij})}$$

(2.3.68)

with the following definitions for the arguments of the theta functions:

$$e^{2\pi i z_{ij}} = Q_m^{-1} q^{\nu_i-j+1/2} t^{-i+1/2}, \quad e^{2\pi i v_{ij}} = Q_m^{-1} t^{i-1/2} q^{-\nu_i+j-1/2},$$

$$e^{2\pi i w_{ij}} = q^{\nu_i-j+1} t^{\nu_i-j-i}, \quad e^{2\pi i u_{ij}} = q^{\nu_i-j} t^{\nu_i-j-i+1}.$$  

(2.3.69)

Notice that for $Q_m = \left( \frac{q}{2} \right)^{\frac{1}{2}}$, i.e. $m = \pm \frac{\epsilon_1 + \epsilon_2}{2}$ we have either $z_{11} = 0$ or $v_{11} = 0$ and since the $(1,1)$ box is present in every non-trivial Young diagram the sum over $\nu$ in $\hat{Z}^{(2)}$ will only get a non-trivial contribution for $\nu = 0$, and therefore reduces to

$$\hat{Z}^{(2)}\left( \tau, m = \pm \frac{\epsilon_1 + \epsilon_2}{2}, t_f, \epsilon_1, \epsilon_2 \right) = 1.$$  

(2.3.70)

However, notice that if

$$\hat{Z}_{U(2)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \sum_{k \geq 0} (-Q_f)^k \hat{Z}_k(\tau, m, \epsilon_1, \epsilon_2)$$  

(2.3.71)

then

$$\lim_{m \to \pm \frac{\epsilon_1 + \epsilon_2}{2}} \frac{\hat{Z}_k(\tau, m, \epsilon_1, \epsilon_2)}{\hat{Z}_1(\tau, m, \epsilon_1, \epsilon_2)} \neq 0.$$  

(2.3.72)

Thus the vanishing of $\hat{Z}_k(\tau, m, \epsilon_1, \epsilon_2)$ for $m = \pm \frac{\epsilon_1 + \epsilon_2}{2}$ is entirely due to the fact that $\hat{Z}_k(\tau, m, \epsilon_1, \epsilon_2)$ has $\hat{Z}_1(\tau, m, \epsilon_1, \epsilon_2)$ as a factor. From Equation (2.3.68) and Equation (2.3.69) it is easy to see that

$$\frac{\hat{Z}_k(\tau, m, \epsilon_1, \epsilon_2)}{\hat{Z}_1(\tau, m, \epsilon_1, \epsilon_2)} = \sum_{|\nu| = k} \frac{\prod_{(i,j)\in \nu, (i,j)\neq (1,1)} \theta_1(\tau; z_{ij}) \theta_1(\tau; v_{ij})}{\prod_{(i,j)\in \nu, (i,j)\neq (1,1)} \theta_1(\tau; w_{ij}) \theta_1(\tau; u_{ij})}.$$  

(2.3.73)

From Equation (2.3.69) it also is clear that the above expression does not vanish. Below we list

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Modular properties of the partition function

The theta function factor in the \( U(2) \) partition function Equation (2.3.68) can be written as follows:

\[
\prod_{(i,j) \in \lambda} \frac{\theta_1(\tau; z_{ij}) \theta_1(\tau; v_{ij})}{\theta_1(\tau; w_{ij}) \theta_1(\tau; u_{ij})} = \left( \prod_{(i,j) \in \lambda} \frac{z_{ij} v_{ij}}{w_{ij} u_{ij}} \right) \exp \left( \sum_{k \geq 1} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau) f^{(2k)}(m, \epsilon_1, \epsilon_2) \right),
\]

where

\[
f^{(2k)}(m, \epsilon_1, \epsilon_2) = \sum_{(i,j) \in \nu} (2\pi i z_{ij})^{2k} + (2\pi i v_{ij})^{2k} - (2\pi i w_{ij})^{2k} - (2\pi i u_{ij})^{2k}.
\]

Thus we get

\[
\tilde{Z}^{(2)}(\tau, m, t_F, \epsilon_1, \epsilon_2) = \sum_{\nu} (-Q_f)^{|\nu|} \left( \prod_{(i,j) \in \nu} \frac{z_{ij} v_{ij}}{w_{ij} u_{ij}} \right) \times \exp \left( \sum_{k \geq 1} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau) f^{(2k)}(m, \epsilon_1, \epsilon_2) \right).
\]

Because of the presence of (holomorphic) \( E_2(\tau) \) the partition function is not invariant under the modular transformation,

\[
(\tau, m, t_F, \epsilon_1, \epsilon_2) \mapsto \left( \frac{a \tau + b}{c \tau + d}, \frac{m}{c \epsilon_1 + d}, \frac{t_F}{c \epsilon_2 + d}, \frac{\epsilon_1}{c \epsilon_1 + d}, \frac{\epsilon_2}{c \epsilon_2 + d} \right), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}).
\]

As before, if we replace \( E_2(\tau) \) with \( \tilde{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \Im(\tau)} \), then since the factor \( f^{(2)}(m, \epsilon_1, \epsilon_2) E_2(\tau, \bar{\tau}) \) is modular invariant (but not holomorphic) \( \tilde{Z}^{(2)}(\tau, m, t_F, \epsilon_1, \epsilon_2) \) is modular invariant as well (but not
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holomorphic). The modified partition function $\tilde{Z}^{(2)}(\tau, \bar{\tau}, m, t_f, \epsilon_1, \epsilon_2)$ now satisfies a holomorphic anomaly equation:

$$\frac{\partial \tilde{Z}^{(2)}(\tau, \bar{\tau}, m, t_f, \epsilon_1, \epsilon_2)}{\partial \bar{E}_2(\tau, \bar{\tau})} = \frac{1}{12} D_{m, \epsilon_1, \epsilon_2, t_f} \tilde{Z}^{(2)}(\tau, \bar{\tau}, m, \epsilon_1, \epsilon_2),$$

$$D_{m, \epsilon_1, \epsilon_2, t_f} := \epsilon_1 \epsilon_2 \frac{\partial^2}{\partial T^2_f} + (m^2 - (\epsilon_+ / 2)^2) \frac{\partial}{\partial T_f}, \quad (2.3.77)$$

where $\epsilon_+ = \epsilon_1 + \epsilon_2$ and $T_f = 2\pi i t_f$. This can be interpreted, as has been discussed in the context of elliptic Calabi-Yau threefolds [94–96], as the holomorphic anomaly of topological strings [97].

Here we are in the unusual situation to have also fixed the ‘holomorphic ambiguity’ to all orders in the genus expansion, as we have the full expansion of the topological string amplitude. Equation (2.3.77) is a refined version of the holomorphic anomaly equations and it would be interesting to relate it to the results of [98,99].

**U(N) partition function**

As shown in Fig. 2.8 we can calculate the partition function of the $U(N)$ theory by gluing $N$ open string amplitudes $W_{\nu_a \nu_{a+1}}$.

![Diagram](image)

Figure 2.8: (a) The toric diagram of the elliptic Calabi-Yau threefold $X_N$. (b) Partition function from gluing $W_{\nu_a \nu_{a+1}}$.

The $U(N)$ partition function in terms of the open string amplitude is given by

$$Z^{(N)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \ldots, \nu_{N-1}} \left( \prod_{a=1}^{N-1} (-Q_{f_a})^{[\nu_a]} \right) \times W_{\emptyset \nu_1}(\epsilon_1, \epsilon_2) W_{\nu_1 \nu_2}(\epsilon_2, -\epsilon_1) W_{\nu_2 \nu_3}(\epsilon_1, \epsilon_2) \cdots W_{\nu_{N-1} \emptyset}.$$
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If we separate the $U(1)$ piece by defining

$$Z^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2) = \left(Z^{(1)}(\tau, m, \epsilon_1, \epsilon_2)\right)^N \mathcal{Z}^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2), \quad (2.3.78)$$

then

$$\mathcal{Z}^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \ldots, \nu_{N-1}} \left(\prod_{a=1}^{N-1} (-Q_{f_a})^{\nu_a}\right) \prod_{a=1}^{N-1} \prod_{(i,j) \in \nu_a} \frac{\theta_1(\tau; z_{ij}^a) \theta_1(\tau; v_{ij}^a)}{\theta(\tau; w_{ij}^a) \theta(\tau; u_{ij}^a)}, \quad (2.3.79)$$

where $(\nu_0 = \nu_N = 0)$

$$e^{2\pi i z_{ij}^a} = Q_m^{-1} q^{\nu_a,j+i+j-i} t^{\nu_{a+1},j-i}, \quad e^{2\pi i v_{ij}^a} = Q_m^{-1} t^{\nu_a,1+j+i} q^{\nu_a,i+j}, \quad (2.3.80)$$

Notice that for $m = \pm \frac{1}{2}$ we have $z_{1,\nu_{a,1}}^{(N-1)} = -(\nu_{N,\nu_{a-1,1}}^{(N-1)}) = 0$ since $\nu_N = \emptyset$ and therefore

$$\mathcal{Z}^{(N)}(\tau, m = \pm \frac{1}{2}, t_{f_a}, \epsilon_1, \epsilon_2) = 1. \quad (2.3.81)$$

Using the relation between theta function and Eisenstein series given in Equation (2.3.60), we can write $\mathcal{Z}^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2)$ as

$$\mathcal{Z}^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \ldots, \nu_{N-1}} (-Q_{f_1})^{\nu_1} \cdots (-Q_{f_{N-1}})^{\nu_{N-1}} \prod_{a=1}^{N-1} \prod_{(i,j) \in \nu_a} \left(\frac{z_{ij}^a v_{ij}^a}{w_{ij}^a u_{ij}^a}\right) \times \exp \left(\sum_{k\geq 1} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau) f_{\nu_1,\ldots,\nu_{N-1}}^{(2k)}(m, \epsilon_1, \epsilon_2)\right)$$

$$f_{\nu_1,\ldots,\nu_{N-1}}^{(2k)}(m, \epsilon_1, \epsilon_2) = \sum_{a=1}^{N-1} \sum_{(i,j) \in \nu_a} \left(2\pi i z_{ij}^aight)^{2k} + \left(2\pi i v_{ij}^a\right)^{2k} - \left(2\pi i w_{ij}^a\right)^{2k} - \left(2\pi i u_{ij}^a\right)^{2k} \quad (2.3.82)$$

which shows that if we replace $E_2(\tau)$ with the non-holomorphic modular form $E_2(\tau, \bar{\tau})$ the partition function becomes modular invariant under the transformation

$$(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{m}{c\tau + d}, t_{f_a}, \frac{\epsilon_1}{c\tau + d}, \frac{\epsilon_2}{c\tau + d}\right), \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}).$$

**Partition function from instanton calculus**

In Section 2.3.1 we calculated the partition function of the $\mathcal{N} = 2^*$ theory using Nekrasov’s instanton calculus and observed that it agreed completely with the refined topological string partition function of the elliptic threefold $X_N$. A change of the preferred direction gave a different
representation of the same refined partition function. In this section we will see that this different representation of the same partition function can also be calculated using instanton calculus for a different gauge theory. This fact reflects the fiber-base duality between the $\mathcal{N} = 2^*$ theory and a quiver theory [100].

To see this, let us first consider the case of $U(2) \mathcal{N} = 2^*$ theory. The web diagram is shown in Fig. 2.9(a) below.

![Figure 2.9: (a) The theory on the compactified vertical branes is $U(2) \rightarrow U(1)^2 \mathcal{N} = 2^*$ theory and the theory on the horizontal brane is a $U(1)$ theory. (b) In the limit the circle is decompactified the $U(1)$ theory on the horizontal brane becomes $U(1)$ with $N_f = 2$.](image)

If we decompactify the circle we get the usual kind of web diagram as shown in Fig. 2.9(b). If we consider the theory on the horizontal branes then the web diagram of Fig. 2.9(b) corresponds to a $U(1)$ theory with two fundamental hypermultiplets. The partition function of the $U(1)$ theory with $N_f = 2$ can be derived using equivariant instanton calculus [58], and is given by

$$Z = \sum_{\nu} \varphi^{||\nu||^2} \prod_{(i,j) \in \nu} \frac{(1 - e^{2\pi i m_1} q^{-j+1} t^{i-1})(1 - e^{2\pi i m_2} q^{-j+1} t^{i-1})}{(1 - q^{\nu_1-j+1} t^{\nu_1-i+1})(1 - q^{\nu_1-j+1} t^{\nu_1-i+1})}$$

$$= \sum_{\nu} \left( -\varphi e^{2\pi im_1} \sqrt{\frac{q}{t}} \right)^{||\nu||^2} \prod_{(i,j) \in \nu} \frac{(1 - e^{-2\pi i m_1} q^{i-1} t^{j-1})(1 - e^{-2\pi i m_2} q^{i-1} t^{j-1})}{(1 - q^{\nu_i-j+1} t^{\nu_j-i+1})(1 - q^{\nu_i-j+1} t^{\nu_j-i+1})}$$

$$= \sum_{\nu} \left( -\overline{\varphi} \right)^{||\nu||} \prod_{(i,j) \in \nu} \frac{(e^{-i \pi m_1} q^{-j+1} t^{-i+1} - e^{i \pi m_1} q^{-j+1} t^{-i+1})}{(q^{\nu_1-j+1} t^{-\nu_1-i+1} - q^{\nu_1-j+1} t^{-\nu_1-i+1})}$$

$$\times \frac{(e^{i \pi m_2} q^{-j+1} t^{-i+1} - e^{-i \pi m_2} q^{-j+1} t^{-i+1})}{(q^{\nu_2-j+1} t^{-\nu_2-i+1} - q^{\nu_2-j+1} t^{-\nu_2-i+1})},$$

where $\overline{\varphi} = \varphi e^{i \pi (m_1 + m_2)} \sqrt{\frac{q}{t}}$. The above partition function can be seen to be identical to
\( \hat{Z}^{(2)} \) (given in Equation (2.3.68)) in the limit \( Q_r \to 0 \) if we make the following identification of parameters\(^8\):

\[
\bar{\phi} = Q_f, \quad m_1 = m - \frac{\epsilon_1 + \epsilon_2}{2}, \quad m_2 = -m - \frac{\epsilon_1 + \epsilon_2}{2}.
\]  

(2.3.84)

Notice that the above partition function becomes trivial for \( m_1 = 0 \) or \( m_2 = 0 \) which corresponds precisely to \( m = \pm \frac{\epsilon_1 + \epsilon_2}{2} \) as expected from the results of Section 2.

Equation (2.3.83) is the topological string partition function of the geometry shown Fig. 2.10(a). In Equation (3.3.34) taking \( m_1 \) and \( m_2 \), to be related to \( m \) with opposite sign essentially means that in the geometry one of the exceptional curves giving the fundamental hypermultiplet has undergone a flop transition and the geometry has become the one shown in Fig. 2.10(b).

Note however that in order for it to come from six dimensions, which corresponds to making the toric geometry periodic, one must impose a restriction on the parameters of the \( U(1) \) gauge theory. In particular we are at the origin of Coulomb branch where the vev of the scalar field is set to zero. The only left over parameters are the coupling constant of the gauge theory which is related to \( t_f = 1/g^2 \) which in turn is proportional to the separation of M5 branes, as well as the masses which have the same value (up to sign discussed above).

The instanton partition function for supersymmetric gauge theories can be calculated using an appropriate topological index on the moduli space of instantons; this is essentially due to supersymmetric localization of the partition function with respect to one of the preserved supercharges. The details of the index computation depend on the details of the gauge theory such as the

---

\(^8\)To see that the two partition functions are the same, apart from parameter identification, one needs the fact that for a fixed \( i \) the set \( \{ j - 1 \mid (i, j) \in \nu \} \) is the same as the set \( \{ \nu_i - j \mid (i, j) \in \nu \} \).
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gauge group and the matter content. The matter fields of the theory are sections of a vector bundle on the instanton moduli space. In the case of a fundamental hypermultiplet the vector bundle on the instanton moduli space is the tautological bundle $E$. The tautological bundle $E$ over $\text{Hilb}^k[\mathbb{C}^2]$ has rank $k$. The fiber of $E$ over the point $I$ (which is a codimension $k$ ideal in $\mathbb{C}[z_1, z_2]$) is given by $\mathbb{C}[z_1, z_2]/I$. If we consider $N_f$ hupermultiplets we have $U(N_f)$ global symmetry and $N_f$ copies of the bundle $E$. The appropriate bundle is

$$E \otimes \mathbb{C}^{N_f} \cong \underbrace{E \oplus E \oplus \cdots \oplus E}_{N_f \text{-copies}},$$

where $\mathbb{C}^{N_f}$ is the fundamental representation of $U(N_f)$. This group acts on the above bundle on $\mathbb{C}^{N_f}$ and, therefore, the Cartan of $U(N_f)$ acts on the $i$-th copy of $E$ by scaling by $e^{2\pi i m_i}$ where $m_i (i = 1, \cdots, N_f)$ is the mass of the $i$-th hypermultiplet.

The case we are considering is $N_f = 2$, so we have $E \oplus E$ and the action of $U(1)_{m_1} \times U(1)_{m_2} \subset U(2)$ is given by $(e^{2\pi i m_1}, e^{2\pi i m_2})$ on the two factors. However, due to flop transition on the curve giving the hypermultiplet of mass $m_2$ the relevant bundle for us is $E \oplus E^*$ which comes from lifting the action on $\mathbb{C}^2$. As mentioned before the fiber of $E$ over a point $I \in \text{Hilb}^k[\mathbb{C}^2]$ is given by $\mathbb{C}[z_1, z_2]/I$ which is a $k$ dimensional vector space. The fixed points of $\text{Hilb}^k[\mathbb{C}^2]$ under the $U(1)_{m_1} \times U(1)_{m_2} \times U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ action are labelled by partitions of $k$ and correspond to monomial ideals$^9$. The ideal corresponding to the partition $\nu$ of $k$ is given by

$$I_{\nu} = \bigoplus_{(i,j) \notin \nu} \mathbb{C} z_1^{i-1} z_2^{j-1}. 	ag{2.3.85}$$

Thus the fiber of $E$ over the fixed point $I_{\nu}$ is given by

$$E|_{I_{\nu}} = \mathbb{C}[z_1, z_2]/I_{\nu} = \bigoplus_{(i,j) \in \nu} \mathbb{C} z_1^{i-1} z_2^{j-1}, \tag{2.3.86}$$

which shows that the fiber $E$ above the fixed point $I_{\nu}$ decomposes in terms of one-dimensional vector spaces spanned by basis vectors $z_1^{i-1} z_2^{j-1}$. Equation (2.3.86) also gives us the weights of the

$^9$The fixed points under $U(1)_{m_1} \times U(1)_{m_2} \times U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ are the same as fixed points under $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ since $U(1)_{m_1} \times U(1)_{m_2}$ does not act on $\text{Hilb}^k[\mathbb{C}^2]$. 

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$U(1)_{m_1} \times U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ action on $E|_{I_{\nu}}$ which are

\[
\text{weights of } E|_{I_{\nu}} = \{ e^{2\pi i m_1} q^{-i+1} t^{j-1} | (i, j) \in \nu \}. \tag{2.3.87}
\]

Similarly the weights of $E^*$ at $I_{\nu}$ under $U(1)_{m_2} \times U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ are given by

\[
\text{weights of } E^*|_{I_{\nu}} = \{ e^{-2\pi i m_2} q^{i-1} t^{-j+1} | (i, j) \in \nu \}. \tag{2.3.88}
\]

Using the relation between $m_1, m_2$ and $m$ given in Equation (3.3.34) we see that, under $U(1)_m \times U(1)_{\epsilon_1} \times U(1)_{\epsilon_2},$

\[
\text{weights of } E|_{I_{\nu}} = \{ e^{2\pi i m} q^{-i+1} t^{j-1} | (i, j) \in \nu \},
\]

\[
\text{weights of } E^*|_{I_{\nu}} = \{ e^{2\pi i m} q^{i-1} t^{-j+1} | (i, j) \in \nu \}. \tag{2.3.89}
\]

Equation (2.3.83) can now be expressed in terms of the equivariant holomorphic Euler characteristic of $E \oplus E^*$ as follows:

\[
Z = \sum_{k \geq 0} \hat{\varphi}^k \chi_y(\text{Hilb}^k[\mathbb{C}^2], E \oplus E^*), \tag{2.3.90}
\]

where

\[
\chi_y(\mathcal{M}, V) = \sum_{p,q \geq 0} (-1)^{p+q} y^p h^{p,q}(\mathcal{M}, V) \tag{2.3.91}
\]

and

\[
h^{p,q}(\mathcal{M}, V) = h^{0,q}(\mathcal{M}, \wedge^p T^* \mathcal{M} \otimes V). \tag{2.3.92}
\]

As $\chi_y(\mathcal{M}, V)$ is the index of a twisted Dirac operator, it can be given in terms of the Chern classes of $V$ and the ones of $T_{\mathcal{M}}^{10},$

\[
\chi_y(\mathcal{M}, V) = \int_{\mathcal{M}} \text{ch}(V_y) \text{Td}(\mathcal{M}), \quad V_y = \sum_{n \geq 0} (-y)^n \wedge^n V. \tag{2.3.93}
\]

\[\text{This can also be written as } \text{Tr}(\text{e}^{-\beta H}) \text{ for a supersymmetric quantum mechanical system which is the dimensional reduction of (2,0) } d = 2 \text{ theory.}\]
Using equivariant localization,
\[
\chi_y(\text{Hilb}^k[\mathbb{C}^2], E \oplus E^*) = \int_{\text{Hilb}^k[\mathbb{C}^2]} \text{ch}((E \oplus E^*)_y) \text{Td}(\text{Hilb}^k[\mathbb{C}^2])
\]
\[
= \sum_{p \in \{\text{fixed points}\}} \frac{\prod_{i=1}^{2k}(1 - ye^{-\tilde{x}_{p,i}})}{\prod_{i=1}^{2k}(1 - e^{-x_{p,i}})},
\]
where \(\tilde{x}_{p,i}\) and \(x_{p,i}\) are weights at the fixed point \(p\) of the \(U(1)_m \times U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}\) action for \(E \oplus E^*\) and \(T_{\text{Hilb}^k[\mathbb{C}^2]}\) respectively. The Chern character of the tautological bundle \(E\) was calculated in [101] and it was shown that
\[
\text{ch}(T_{\text{Hilb}^k[\mathbb{C}^2]}) = \text{ch}(E \oplus E^*).\]
However as equivariant bundles \(T_{\text{Hilb}^k[\mathbb{C}^2]}\) has different weights than \(E \oplus E^*\). Using the weights given in Equation (2.3.89) we get
\[
\chi_y(\text{Hilb}^k[\mathbb{C}^2], E \oplus E^*) = \sum_{|\nu| = k} \prod_{(i,j) \in \nu} (1 - y Q^{-1}_m q^{\nu_{i-1} - 1 - i} t^{-j+i} + \frac{1}{2}) (1 - y Q^{-1}_m q^{-i+1} t^{-j+\frac{1}{2}})
\]
\[
(1 - q^{j_{i-1} - 1} t^{j_i} - t^{-j_i - 1})(1 - q^{j_{i-1} - 1} t^{j_i} - t^{-j_i - 1}).
\]
After some simplification Equation (2.3.90) then becomes
\[
Z = \sum_{|\nu| = k} (-\hat{\varphi} y Q^{-1}_m \sqrt{\frac{\tau}{q}})^{|\nu|} \prod_{(i,j) \in \nu} \frac{y^{\frac{1}{2}} Q^{-\frac{1}{2}}_m q^{\nu_i - \frac{1}{2} - j} t^{-j + \frac{1}{2}} - y^{-\frac{1}{2}} Q^{-\frac{1}{2}}_m q^{\nu_i - \frac{1}{2} - j} t^{-j + \frac{1}{2}}}{q^{-\nu_{i-1} + \frac{1}{2}} t^{\nu_i - j} - q^{-\nu_{i-1} + \frac{1}{2}} t^{\nu_i - j}}
\]
\[
\pi \prod_{(i,j) \in \nu} \frac{y^{\frac{1}{2}} Q^{-\frac{1}{2}}_m q^{\nu_i - \frac{1}{2} - j} t^{-j + \frac{1}{2}} - y^{-\frac{1}{2}} Q^{-\frac{1}{2}}_m q^{\nu_i - \frac{1}{2} - j} t^{-j + \frac{1}{2}}}{q^{-\nu_{i-1} + \frac{1}{2}} t^{\nu_i - j} - q^{-\nu_{i-1} + \frac{1}{2}} t^{\nu_i - j}}.
\]
This is precisely the partition function Equation (2.3.68) with \(y = 1\), \(Q_f = \hat{\varphi} y Q^{-1}_m \sqrt{\frac{\tau}{q}}\) and \(Q_r = 0\). Thus the choice of the horizontal preferred direction gives the twisted \(\chi_y\) genus of the Hilbert scheme of points on \(\mathbb{C}^2\). It is easy to generalize this to the case \(Q_r \neq 0\). From Equation (2.3.68) it is clear that we need to replace each factor in the product in Equation (2.3.96) with \(\theta_1(\tau; z)\). This is achieved by considering the elliptic genus rather than the \(\chi_y\) genus.

Now consider the following formal combination of bundles [102]:
\[
V_{Q_r,y} = \bigotimes_{k=0}^{\infty} V_{-y Q^{-1}_r} \bigotimes_{k=1}^{\infty} V_{-y^{-1} Q^{-1}_r} \bigotimes_{k=1}^{\infty} S^{k} T^{k}_{M} \bigotimes_{k=1}^{\infty} S^{k} T^{k}_{M}
\]
Then,
\[
\text{ch}(V_{Q, y}) \text{Td}(T_M) = \prod_{k=1}^{\infty} \frac{1}{\prod_{j=1}^{d} x_j^{-1}} \prod_{i=1}^{d} \frac{\theta_1(\tau; -z + \frac{x_i}{2\pi i})}{\theta_1(\tau; \frac{x_i}{2\pi i})},
\]
where \( r \) is the rank of \( V \) which in this case is equal to the dimension of \( M \), \( x_j \) are the Chern roots of the tangent bundle and \( \tilde{x}_i \) are Chern root of \( V \). In terms of theta functions we have \( (y = e^{2\pi i z}) \):
\[
\text{ch}(V_{Q, y}) \text{Td}(T_M) = y^{d/2} e^{\sum_{i=1}^{d} \frac{1}{2}(x_i - \tilde{x}_i)} \prod_{i=1}^{d} \frac{\theta_1(\tau; -z + \frac{x_i}{2\pi i})}{\theta_1(\tau; \frac{x_i}{2\pi i})}.
\]

Taking \( V = E \oplus E^* \) and \( M = \text{Hilb}^k(\mathbb{C}^2) \) we get
\[
\sum_{k \geq 0} \tilde{Q}^k \int_M \text{ch}((E \oplus E^*)_{Q, y}) \text{Td}(T_M) = \sum_{k \geq 0} (\tilde{Q} y)^k \int_M e^{\sum_{i=1}^{d} \frac{1}{2}(x_i - \tilde{x}_i)} \prod_{i=1}^{d} \frac{\theta_1(\tau; -z + \frac{x_i}{2\pi i})}{\theta_1(\tau; \frac{x_i}{2\pi i})}
\]
\[
= \sum_{\nu} (\tilde{Q} y)^{\nu} \prod_{(i,j) \in \nu} \frac{\theta_1(\tau; -z + i e_1 - j e_2 - \frac{e_2}{2} + \frac{\epsilon_{i,j}}{2})}{\theta_1(\tau; -z - i e_1 + j e_2 + \frac{e_2}{2} + \frac{\epsilon_{i,j}}{2})} \theta_1(\tau; -z + \frac{e_2}{2} + \frac{\epsilon_{i,j}}{2}) \theta_1(\tau; \frac{e_2}{2} - \frac{\epsilon_{i,j}}{2}) \theta_1(\tau; (\nu_i - \nu_j) e_1 - (\nu_i - \nu_j + 1) e_2).
\]
Comparing with Equation (2.3.68) we see that the above precisely agrees with \( \hat{Z}^{(2)} \), the topological vertex result for \( z = 0 \).

The bundle \( V = E \oplus E^* \) on \( \text{Hilb}^k[\mathbb{C}^2] \) played a crucial role in the calculation above which corresponds to the case of two M5-branes. If \( I \in \text{Hilb}^k[\mathbb{C}^2] \) then \( I \) is an ideal such that \( \dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = k \). Given such an ideal the fiber of \( V \) above \( I \) is given by \([91, 103, 104]\)
\[
V|_I = \text{Ext}^1(O, I) \otimes L^{-\frac{1}{2}} \oplus \text{Ext}^1(I, O) \otimes L^{-\frac{1}{2}},
\]
where \( L \) is a trivial line bundle and \( O = \mathbb{C}[x, y] \). \( L \) is in fact the canonical line bundle on \( \mathbb{C}^2 \) such that the weight of \( L^{-\frac{1}{2}} \) is \( e^{-2\pi i (1+\frac{i}{2})} \). The appearance of Ext groups is not unexpected since it has been shown that Ext groups count the open string states between D-branes wrapped on holomorphic submanifolds [105]. Notice that \( \text{Ext}^1(O, I) \) and \( \text{Ext}^1(I, O) \) are not dual of each other but are such that \([103, 104]\)
\[
\text{Ext}^1(I, O) = \text{Ext}^1(O, I)^* \otimes L|_I;
\]
this implies that the two factors of \( V \) in Equation (2.3.101) are dual to each other.
Now consider \((I \hookrightarrow J) \in \text{Hilb}^{k_1}[\mathbb{C}^2] \times \text{Hilb}^{k_2}[\mathbb{C}^2]\) and a bundle \(V\) on \(\text{Hilb}^{k_1}[\mathbb{C}^2] \times \text{Hilb}^{k_2}[\mathbb{C}^2]\) such that its fiber over the point \((I, J)\) is given by

\[
V|_{(I, J)} = \left( \text{Ext}^1(O, I) \oplus \text{Ext}^1(I, J) \oplus \text{Ext}^1(J, O) \right) \otimes L^{-\frac{1}{2}}.
\]  

(2.3.103)

If we take \(k_1 = 0\) or \(k_2 = 0\) then the above bundle becomes the bundle of Equation (2.3.101) on \(\text{Hilb}^\bullet[\mathbb{C}^2]\). At the fixed point \((I_\lambda, J_\mu)\) labelled by two partitions \(\lambda\) and \(\mu\) of \(k_1\) and \(k_2\) respectively, the weights of the bundle \(V\) are given by

\[
\text{weights of } V = \{Q_m q^{-i+\frac{1}{2},j\frac{1}{2}}, Q_m q^{\lambda_i,j-i+\frac{1}{2},\frac{j}{2}} \mid (i, j) \in \lambda \} \cup \{Q_m q^{-i+\frac{1}{2},j\frac{1}{2}}, Q_m q^{-j+\frac{1}{2},i\frac{1}{2}} \mid (i, j) \in \mu \},
\]

where we have included the \(U(1)_m\) weight as well in the above. Thus for each M5-brane with M2-branes ending on the left and the right we have a factor \(\text{Ext}^1(I, J) \otimes L^{-\frac{1}{2}}\) in the corresponding bundle where \(I\) is a point on \(\text{Hilb}^{k_1}[\mathbb{C}^2]\) corresponding to the M2-brane on the left and \(J\) is a point on \(\text{Hilb}^{k_2}[\mathbb{C}^2]\) corresponding to the M2-brane on the right as shown in Fig. 2.11.

\[
\begin{array}{c}
I \\
\hline \\
J
\end{array}
\]

\[
\text{Ext}^1(I, J) \otimes L^{-\frac{1}{2}}
\]

Figure 2.11: Each pair of M2 branes ending on the M5 brane from opposite sides gives rise to a factor \(\text{Ext}^1(I, J) \otimes L^{-\frac{1}{2}}\) in the corresponding bundle.

We can now generalize this construction for \(\mathcal{M}_{k_1,\ldots,k_{N-1}} = \text{Hilb}^{[k_1]}[\mathbb{C}^2] \times \cdots \times \text{Hilb}^{[k_{N-1}]}[\mathbb{C}^2]\). Let \(V\) be the bundle on \(\mathcal{M}_{k_1,\ldots,k_{N-1}}\) such that its fiber above the point \((I_1, I_2, \ldots, I_{N-1}) \in \mathcal{M}_{k_1,\ldots,k_{N-1}}\) is given by

\[
V|_{(I_1, I_2, \ldots, I_{N-1})} = \left( \oplus_{a=0}^{N-1} \text{Ext}^1(I_a, I_{a+1}) \right) \otimes L^{-\frac{1}{2}},
\]

(2.3.105)

\(^{11}\)The Young diagram convention of [103] is different from ours so we have to take the transpose of their partitions.
where $I_0 = I_N = \mathcal{O}$. At a fixed point labelled by $(\nu_1, \nu_2, \cdots, \nu_{N-1})$ the weights of $V$ are given by [103]

$$
\{Q_{m} q^{-i_1^2} t^{i_2^2} \ | (i, j) \in \nu_1 \} \cup \{Q_{m} q^{-i_1^2} t^{-j_1^2} \ | (i, j) \in \nu_{N-1} \}
$$

$$
\bigcup_{a=1}^{N-2} \left( \{Q_{m} q^\nu a_{i,j}^{-i_1^2} t^\nu a_{i,j} + j_1^2 \ | (i, j) \in \nu_a \} \cup \{Q_{m} q^{-i_1^2} t^{j_1^2} \ | (i, j) \in \nu_{a+1} \} \right) .
$$

(2.3.106)

We can now write down the partition function

$$
Z_{U(1)}^{(1)} = \sum_{k_1, \ldots, k_{N-1} \geq 0} \left( \prod_{a=1}^{N-1} Q_a^{k_a} \right) \int_{M_{k_1, \ldots, k_{N-1}}} \text{ch}(V_{Q, y}) Td(M_{k_1, \ldots, k_{N-1}}).
$$

(2.3.107)

Using Equation (2.3.106) we obtain

$$
Z_{U(1)}^{(1)} = \sum_{\nu_1, \ldots, \nu_{N-1}} \left( \prod_{a=1}^{N-1} Q_a^{[\nu_a]} \right) \prod_{a=1}^{N-1} \prod_{(i, j) \in \nu_a} \frac{\theta_1(\tau; a_{ij}) \theta_1(\tau; b_{ij})}{\theta_1(\tau; c_{ij}) \theta_1(\tau; d_{ij})}
$$

(2.3.108)

with

$$
e^{2\pi i a_{ij}} = y Q_{m} q^{-\nu a_{i,j} + i_1 - j_1^2} t^{-\nu a_{i,j} + j_1^2}, \quad e^{2\pi i b_{ij}} = y Q_{m} q^\nu a_{i,j}^{-i_1^2} t^\nu a_{i,j} + j_1^2,
$$

$$
e^{2\pi i c_{ij}} = Q^\nu a_{i,j}^{-i_1} t^\nu a_{i,j} - j_1 + 1, \quad e^{2\pi i d_{ij}} = Q^\nu a_{i,j}^{-i_1 - 1} t^{-\nu a_{i,j} + j}.
$$

(2.3.109)

The above partition function is precisely the partition function of Equation (2.3.79) if we interchange $t$ and $q$, take $y = 1$ and $\tilde{Q}_a = Q_{fa}$.

### 2.4 Elliptic genus of M-strings

In this section we interpret the results of the computations done in Section 3, using the fact that topological string partition function is the same as the BPS partition function. We start by reviewing the types of BPS states which we obtain by compactifying the M5 brane theory on $S^1$.

The BPS states can be divided to two classes: those that are coming from tensor multiplets in 6d and their tower of KK modes, and those that arise due to wrapping of the strings around the circle. The latter is the main focus for our work, but we also review how the KK tower shows up in the topological string computation, and how we can normalize the topological string partition function.
to be left entirely with the partition function of the suspended M2 branes and the corresponding wrapped M-strings. Turning our attention to wrapped string states and their elliptic genus, we first study the case with two M5 branes and a single M2 brane suspended between them. In this case we find that the theory on the resulting string is, as expected, the sigma model on $\mathbb{R}^4$, and show how our BPS formula agrees with this result. We then consider the case with two M2 branes suspended between two M5 branes. One may naively expect the theory on the string to be $\text{Sym}^2(\mathbb{R}^4)$, but it turns out not to be the case. We explain in detail the similarities and differences between our result and this expectation. Next, we explain how the elliptic genus for $n$ M2 branes suspended between two M5 branes can be related to the elliptic genus of a $(4,0)$ theory on $\text{Hilb}^n(\mathbb{R}^4)$, where however the right-moving fermions are coupled to a bundle distinct from the tangent bundle but having the same Chern character. Next, we discuss the situation where we have more than two M5 branes. In this case we find that new bound states of wrapped strings can arise due to compactification on the circle, and the BPS degeneracies do not factorize as product of pairs of bound states (this is related to the fact that $m \neq 0$). We then discuss the interpretation of our result in terms of domain walls formed between different numbers of M2 branes ending on the same M5 branes. Our computations lead to the elliptic genus of these 2d domain walls. Finally we consider special values of $m$ and in particular when $m = \pm \frac{1}{2}(\epsilon_1 + \epsilon_2)$ and confirm that our partition function agrees with the elliptic genus of $(4,4) \ A_n$ quiver theories (at least for some of the $A_1$ cases we checked).

### 2.4.1 Sigma model, BPS states and elliptic genus

Beginning with the case of the single M5 brane, let us try to understand in detail the physical meaning of the partition functions calculated in the last section. We are interested in identifying the BPS particles and understanding their multiplicities. These have two sources: BPS states which come from particles in six dimensions, and BPS states which come from BPS strings in six dimensions wrapped around the $S^1$. For a single M5 brane the only 5d BPS states come from particles in six dimensions [77] as we review next.
Single M5 brane and $Z^{(1)}$

Consider the case of a single M5 brane wrapped on the circle. As mentioned in the last section the six dimensional theory on the M5 brane has a tensor multiplet which consists of a self-dual two form field, four symplectic Majorana-Weyl spinors and five real scalar fields. Compactification of the M5 brane on the $S^1$ then gives the Kaluza-Klein modes of the tensor multiplet in five dimensions. There will be two kinds of multiplets in five dimensions, massless and massive. The little group of massless particles in five dimensions is $Spin(3) = SU(2)$ and the little group of massive particle in five dimensions is $Spin(4) = SU(2)_L \times SU(2)_R$. If the radius of the circle is $R$ then there will be a massive multiplet, $\Phi_k$, for each $k \in \mathbb{Z}$, $k \neq 0$ with mass $\frac{|k|}{R}$. For each $k$ the fields in the massive multiplet (which is actually the $(2,0)$ multiplet of the $\mathcal{N} = 2$ supersymmetry algebra in five dimensions) are in the following representation:

$$SU(2)_L \times SU(2)_R : \quad \Phi_k = (1,0) \oplus 4 \left( \frac{1}{2},0 \right) \oplus 5 (0,0).$$

The massless multiplet, $\Phi_0$, (which is the five dimensional vector multiplet containing the massless vector field) is given by

$$Spin(3) : \quad \Phi_0 = (1) \oplus 4 \left( \frac{1}{2} \right) \oplus 5 (0).$$

Now consider the topological string partition function corresponding to the geometry $X_1$ with toric diagram given by Fig. 2.4 for $N = 1$. This geometry is dual to the brane web in which we have a single M5 brane wrapped on a circle with a mass deformation $m$. In the limit $m \to 0$ we get the $\mathcal{N} = 2$ SYM in five dimensions. The topological string partition function for this geometry is given by [88],

$$Z^{(1)} = M(t,q) \prod_{k=1}^{\infty} (1 - Q_k^t)^{-1} \prod_{i,j=1}^{\infty} \frac{(1 - Q_t^{i-\frac{1}{2}} Q_m q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})(1 - Q_t^{i-\frac{1}{2}} Q_1 q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})}{(1 - Q_t^{i} q^{i} t^{j})(1 - Q_t^{i} q^{i+1} t^{j})},$$

(2.4.112)

where $M(t,q) = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^{-1}$, $Q_m = e^{2\pi im}$ and $Q_1 Q_m = Q_\tau = e^{2\pi i \tau}$ with $\tau = \frac{1}{R}$.
Recall that in M-theory compactification on a Calabi-Yau threefold the massive particles in five dimensions, classified by the little group $SU(2)_L \times SU(2)_R$, come from M2 branes wrapping the holomorphic curves. The topological string partition function of a Calabi-Yau threefold contains the information about the spin content of these BPS particles [65,66]. The refined topological string partition function can be written in terms of BPS multiplicities as follows [65, 66, 76]:

$$Z(\omega, q, t) = \prod_{\Sigma \in H_2(X, \mathbb{Z})} \prod_{j_L, j_R} \prod_{k_L = -j_L, k_R = -j_R}^{+j_L} \prod_{i,j=1}^{+j_R} \left( 1 - q^{k_L - k_R + i - \frac{1}{2}} t^{k_L + k_R + j - \frac{1}{2}} e^{-f_{\Sigma} \omega} \right)^{(-1)^{j_L + j_R} N_{j_L, j_R}(\Sigma)}$$

$$= \exp \left( -\sum_{n=1}^{\infty} \frac{F(n\omega, q^n, t^n)}{n(q^\frac{n}{2} - q^{-\frac{n}{2}})(t^\frac{n}{2} - t^{-\frac{n}{2}})} \right),$$

where

$$F(\omega, q, t) = \sum_{\Sigma \in H_2(X, \mathbb{Z})} \sum_{j_L, j_R} \sum_{k_L, k_R} e^{-f_{\Sigma} \omega} (-1)^{2j_L + 2j_R} N_{j_L, j_R}(\Sigma) \mathrm{Tr}_{j_L}(q^\frac{j_L}{2}) \mathrm{Tr}_{j_R}(q^\frac{j_R}{2}), \quad (2.4.113)$$

$N_{j_L, j_R}(\Sigma)$ is the number of particles with charge $\Sigma$ and $SU(2)_L \times SU(2)_R$ representation $(j_L, j_R)$, and $\omega$ is the complexified Kähler class of $X$. Comparing Equation (2.4.112) and Equation (2.4.113) we see that

$$F_{U(1)} + \overline{F_{U(1)}} = Q_m + Q_m^{-1} - \sqrt{\frac{q}{t}} - \sqrt{\frac{t}{q}} + \sum_{k \in \mathbb{Z}, k \neq 0} Q_k^\pm [(Q_m + Q_m^{-1}) - (\sqrt{q}t + \frac{1}{\sqrt{q}t})] \quad \text{massless}$$

$$\sum_{k \in \mathbb{Z}, k \neq 0} Q_k^\pm [(Q_m + Q_m^{-1}) - (\sqrt{q}t + \frac{1}{\sqrt{q}t})] \quad \text{massive}$$

In the limit $m \to 0$ we see that the massless and the massive multiplets are in the following representation of the $SU(2)_L \times SU(2)_R$,

$$\text{massless : } \left( 0, \frac{1}{2} \right) \oplus 2(0,0), \quad (2.4.14)$$

$$\text{massive : } \left( \frac{1}{2}, 0 \right) \oplus 2(0,0).$$

This, however, is not the complete story. We need to take into account the universal half-hypermultiplet associated with the position of the particle to get the full $Spin(4)$ content [66].
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Tensoring the above with the half hypermultiplet \((\frac{1}{2}, 0) \oplus 2(0, 0)\) we get:

**massless** :
\[
\left( \left( \frac{1}{2}, 0 \right) \oplus 2(0, 0) \right) \otimes \left( \left( \frac{1}{2}, 0 \right) \oplus 2(0, 0) \right)
\]
\[
= \left( \frac{1}{2}, 0 \right) \oplus 2 \left( \frac{1}{2}, 0 \right) \oplus \left( \frac{1}{2}, 0 \right) \oplus 4 \left( 0, 0 \right)
\]
\[
\rightarrow (1) \oplus 4 \left( \frac{1}{2}, 0 \right) \oplus 5 (0, 0),
\]

**massive** :
\[
\left( \left( \frac{1}{2}, 0 \right) \oplus 2(0, 0) \right) \otimes \left( \left( \frac{1}{2}, 0 \right) \oplus 2(0, 0) \right)
\]
\[
= (1, 0) \oplus 4 \left( \frac{1}{2}, 0 \right) \oplus 5 (0, 0).
\]

This is precisely the spin content of the massless and the massive modes of the tensor multiplet on the circle given in Equation (2.4.110) and Equation (2.4.111).

One M2 brane suspended between two M5 branes

Recall the partition function we obtained for two M5 branes:

\[
Z^{(2)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \left( Z^{(1)}(\tau, m, \epsilon_1, \epsilon_2) \right)^2
\]
\[
\times \sum_{\nu} (-Q_f)^{|\nu|} \prod_{(i,j) \in V} \frac{\theta_1(\tau; z_{ij}) \theta_1(\tau; v_{ij})}{\theta_1(\tau; u_{ij})} \tilde{Z}^{(2)}(\tau, m, t_f, \epsilon_1, \epsilon_2)
\]

with the following definitions for the arguments of the \(\theta\)-functions

\[
e^{2\pi i z_{ij}} = Q_m^{-1} q^{\nu_i+j+1/2} t^{-i+1/2}, \quad e^{2\pi i v_{ij}} = Q_m^{-1} t^{-i+1/2} q^{-\nu_i+j+1/2},
\]
\[
e^{2\pi i w_{ij}} = q^{\nu_i-j+1} t^{\nu_j-i}, \quad e^{2\pi i u_{ij}} = q^{\nu_i-j} t^{\nu_j-i+1},
\]

where \(q = e^{2\pi i \epsilon_1}, t = e^{-2\pi i \epsilon_2}, Q_m = e^{2\pi i m}\). We can interpret these contributions as follows: the \((Z^{(1)})^2\) term captures the KK tower of states of the two tensor multiplets we have in 6d, extending our discussion for the case of single M5 brane in the previous section; the term \(\tilde{Z}^{(2)}\) we interpret as the contribution of BPS states coming from M-strings which are wrapped around the circle; moreover the term \(|\nu|\) counts the number of M-strings wrapping the circle. In particular we write

\[
\tilde{Z}^{(2)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \sum_n (-Q_f)^n \tilde{Z}^{(2)}_n(\tau, m, \epsilon_1, \epsilon_2)
\]

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where $\tilde{Z}_n^{(2)}$ corresponds to the partition function of $n$ suspended and wrapped M2 branes between two M5 branes:

$$\tilde{Z}_n^{(2)} = \sum_{|\nu| = n} \prod_{(i,j) \in \nu} \frac{\theta_1(\tau; z_{ij}) \theta_1(\tau; v_{ij})}{\theta_1(\tau; w_{ij}) \theta_1(\tau; u_{ij})}.$$

Considering the case of a single M2 brane, corresponding to $n = 1$, we have

$$\tilde{Z}_1^{(2)} = \frac{\theta_1(\tau; -m + (\epsilon_1 + \epsilon_2)/2) \theta_1(\tau; m + (\epsilon_1 + \epsilon_2)/2)}{\theta_1(\tau; \epsilon_1) \theta_1(\tau; \epsilon_2)}$$

$$= \prod_{k=1}^{\infty} \frac{(1 - Q^k q^{1/2} t^{-1/2})(1 - Q^k t^{-1/2} q^{1/2})(1 - Q t^{-1})(1 - Q^n t^{-1})}{(1 - Q^k q)(1 - Q^k t^{-1} q^{-1})(1 - Q t)(1 - Q^n t)}$$

where by $\pm 1$ we mean to include one factor with each of the two signs. This expression can be manifestly identified with the partition function of a sigma model on $\mathbb{R}^4$ where the four fermionic oscillators in the numerator are twisted according to the $Spin(4)_R$ twist parameters and the four bosonic oscillators in the denominator are twisted according to the $Spin(4)$ parameter, and represent the four directions parallel to the M5 brane but perpendicular to the M2 brane. In particular the twists in the Cartan of $Spin(4) \times Spin(4)_R$ are

$$(\epsilon_1, \epsilon_2, -m + (\epsilon_1 + \epsilon_2)/2, m + (\epsilon_1 + \epsilon_2)/2).$$

**Two M2 branes suspended between two M5 branes**

To extract the contribution from two M2 branes suspended between two M5 branes we simply take the term $|\nu| = 2$ in Equation (2.4.115). The most naive expectation, as already discussed, is that this should correspond to sigma model on $Sym^2 \mathbb{R}^4$. Indeed this term has some similarities to this expression, namely the ratio of the four theta functions can be interpreted as that of eight fermionic oscillators and eight bosonic oscillators. However, the structure of the sum of the partitions and the corresponding charges of the fermions are not what one may expect based on the symmetric product structure. Namely, if one studies the elliptic genus of the 2-fold symmetric product of $\mathbb{R}^4$ one finds

$$Z_{Sym^2(\mathbb{R}^4)} = \frac{1}{2} \left( \left( 1 \right) + \left( \frac{1}{g} \right) \right) + \left( 1 \right) + \left( \frac{1}{g} \right)$$

where $g^2 = t$. This expression is the result of summing over the partitions of $\mathbb{R}^4$ and corresponds to the partition function of a sigma model on $\mathbb{R}^4$ with the appropriate twists.
\[ \frac{1}{2} \left[ \hat{Z}_1^{(2)}(\tau, \epsilon_1, \epsilon_2, m)^2 + \hat{Z}_1^{(2)}(2\tau, 2\epsilon_1, 2\epsilon_2, 2m) \right. \\
+ \hat{Z}_1^{(2)}(\tau/2, \epsilon_1, \epsilon_2, m) + \hat{Z}_1^{(2)}((\tau + 1)/2, \epsilon_1, \epsilon_2, m) \right] \]

where \( g \) is the order two twist given by exchanging the two \( \mathbb{R}^4 \)'s. However, we find
\[ \hat{Z}_2^{(2)} \neq Z_{\text{Sym}^2(\mathbb{R}^4)}! \]

The closest we can make this to look like the symmetric product partition function is
\[
\hat{Z}_2^{(2)} = \frac{1}{2} \left( \left( \begin{array}{cccc} 1 & g & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{array} \right) + \left( \begin{array}{cccc} 0 & g & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{array} \right) \right) Y
\]
\[
= \frac{1}{2} \left( \hat{Z}_1^{(2)}(\tau, \epsilon_1, \epsilon_2, m)^2 - \hat{Z}_1^{(2)}(2\tau, 2\epsilon_1, 2\epsilon_2, 2m) \right)
+ (\hat{Z}_1^{(2)}(\tau/2, \epsilon_1, \epsilon_2, m) - \hat{Z}_1^{(2)}((\tau + 1)/2, \epsilon_1, \epsilon_2, m)) Y(\tau, \epsilon_1, \epsilon_2)
\] (2.4.118)

with
\[
Y(Q, q, t) = \frac{\theta_2(\tau; 0)\theta_2(\tau; \epsilon_1 + \epsilon_2) - \theta_3(\tau; 0)\theta_3(\tau; 1 + \epsilon_2) \theta_4(\tau; \epsilon_1 + \epsilon_2) \theta_4(\tau; \epsilon_1 \epsilon_2)}{\theta_2(\tau; \epsilon_1)\theta_2(\tau; \epsilon_2) \theta_3(\tau; \epsilon_1)\theta_3(\tau; \epsilon_2) \theta_4(\tau; \epsilon_1)\theta_4(\tau; \epsilon_2)} - 1
\]

where
\[
\theta_2(\tau; z) = -\theta_1(\tau; z - 1/2),
\]
\[
\theta_3(\tau; z) = -\exp(\pi i(-z + \tau/4))\theta_1(\tau; z - 1/2 - \tau/2),
\]
\[
\theta_4(\tau; z) = i \exp(\pi i(-z + \tau/4))\theta_1(\tau; z - \tau/2).
\]

Note that this structure is not entirely a trivial rewriting of \( \hat{Z}_2^{(2)} \) because \( Y \) does not depend on \( m \).

So in particular the \( m \) dependence of \( \hat{Z}_2^{(2)} \) is captured by the symmetric product Hilbert space. It would be interesting to see if this structure can be better understood.

Despite the fact that we could not write \( \hat{Z}_2^{(2)} \) in terms of the Hilb\(^2(\mathbb{R}^4) \) it turns out there is a way to relate it to the sigma model on the same space but a bundle different from the tangent bundle. We explain this in the next section, for the general \( \hat{Z}_n^{(2)} \) case.
Many M2 branes suspended between two M5 branes and $E \oplus E^*$ (4,0) sigma model

As we have seen the description of the theory on $n$ suspended M2 branes between two M5 branes is close to a sigma model on $\text{Hilb}^n(\mathbb{R}^4)$. Whatever this theory is, it should have (4,4) supersymmetry. However, if one is only considering the (2,0) elliptic genus of this theory, any deformation of this theory which preserves (2,0) supersymmetry will yield the same elliptic genus. In this section we show that there is a sigma model with (2,0) supersymmetry whose elliptic genus exactly reproduces the result we have found.

The sigma model we propose is on $\text{Hilb}^n(\mathbb{R}^4)$, but it is a (4,0) model. As is familiar in the context of heterotic string vacua this means that even though the left-moving fermions are coupled to the tangent bundle, the right-moving fermions are coupled to a different bundle. In fact the bundle we find is $V = E \oplus E^*$ where $E$ is the $n$-dimensional complex tautological bundle over the $\text{Hilb}^n(\mathbb{R}^4)$ space already discussed. Note that $V$ has the same dimension as the tangent bundle. In fact more is true: It turns out that they are equivalent K-theoretically \cite{101}\textsuperscript{12}. Therefore it is conceivable that $V$ is continuously deformable to the tangent bundle to $\text{Hilb}^n(\mathbb{R}^4)$.

Before explaining the potential implications of this statement let us see why the elliptic genus for this sigma model gives the same answer as we have found. As already discussed in section 3, if we view the case of two M5 branes as an elliptic version of the 5d supersymmetric $U(1)$ gauge theory with two fundamental matter multiplets of mass $m$, we saw that the contribution to elliptic genus is obtained by counting the index of the bundle

$$E_{Q_m} = \bigotimes_{k=0}^{\infty} \bigotimes_{Q_m^{-1} Q_k^r} (E \oplus E^*)^* \otimes \bigotimes_{Q_m^r} (E \oplus E^*) \otimes \bigotimes_{k=1}^{\infty} S_{Q_k^r} T_M^* \otimes \bigotimes_{k=1}^{\infty} S_{Q_k^r} T_M,$$

(2.4.119)

but this is exactly the elliptic genus of the (4,0) theory coupled to $E \oplus E^*$ bundle (where the twisting by $Q_m$ is identified with right-moving fermion number, and the action of $\epsilon_{1,2}$ is inherited from their action on $\mathbb{R}^4$).

\textsuperscript{12}This observation should be relevant in verifying the anomaly cancellation of the 2d theory on the M-string (see \cite{106} for a related discussion).
Now we attempt to demystify the appearance of a \((4,0)\) theory. The first question is why does it have half the supersymmetry expected? This is explained by noting that counting the BPS states for a 5d \(\mathcal{N} = 1\) supersymmetric gauge theory will preserve only 4 supercharges, which is consistent with a \((4,0)\) theory. Indeed in this context the above description of the index computation directly follows from the instanton calculus. This is related to the fact that even if we turn off \(m = \epsilon_i = 0\) the theory is not really the compactification of the M5 brane theory on untwisted \(S^1\) due to the fact that the geometry is not quite the product structure. To make it the product structure we need to change the geometry which in the brane description corresponds to lifting the horizontal brane off the two vertical branes (see Fig. 2.12).

![Figure 2.12: The \(l\)-deformation of the brane system. In the first step the mass of the adjoint hypermultiplet is sent to zero. Then the NS5 brane is removed from the D4 branes leaving a system with 16 preserved supercharges as \(l \to \infty\).](image)

It is conceivable that the separation of the horizontal brane from the vertical branes is a deformation which deforms the \(E \oplus E^*\) bundle to \(T(\text{Hilb}^n(\mathbb{R}^4))\), in the limit of infinite separation. One may then ask, if this is indeed the case, why we were not able to compute the elliptic genus using directly the tangent bundle of the Hilbert scheme? The natural answer is that the only way to get a non-trivial answer is to turn on \(m, \epsilon_i\) in which case the action of these is not the same between the tangent bundle and the \(E \oplus E^*\) bundle. Of course we already knew that the actions have to be different because even for a single M2 brane, the fermions transform in the spinor representation of the \(\text{Spin}(4)\) which is different from the boson. So even then we do not expect the action of these twistings to be the canonical action on the tangent bundle. The surprise is that the answer is not the same as one would obtain by considering the symmetric product theory which would have had
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a (4, 4) supersymmetry.

Multiple M2 branes suspended between multiple M5 branes

For the case of \( N \) M5 branes the degeneracy of BPS states is captured by the partition function of Equation (2.3.79),

\[
\tilde{Z}^{(N)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \ldots, \nu_N} \left( \prod_{a=1}^{N-1} (-Q_{f_a}^{\nu_a}) \right) \prod_{a=1}^{N-1} \prod_{(i,j) \in \nu_a} \frac{\theta_1(\tau; \nu_a^a) \theta_1(\tau; \nu_a^b)}{\theta(\tau; \nu_a^a) \theta(\tau; \nu_a^b)} \tag{2.4.120}
\]

with

\[
e^{2\pi i z_{ij}^a} = Q_{m}^{-1} q^{\nu_{a, j} - j + \frac{1}{2}} \ell^{\nu_{a+1, j} - i + \frac{1}{2}}, \quad e^{2\pi i \nu_{a}^a} = Q_{m}^{-1} q^{-\nu_{a, j} + i + \frac{1}{2}} q^{-\nu_{a, i} + j - \frac{1}{2}}, \quad e^{2\pi i v_{ij}^a} = q^{\nu_{a, j} - j + 1} \ell^{\nu_{a, j} - i}, \quad e^{2\pi i w_{ij}^a} = q^{\nu_{a, i} - j - 1} \ell^{\nu_{a, i} - i + 1}. \tag{2.4.121}
\]

This is the refined topological string partition function of the Calabi-Yau threefold \( X_N \).

In Section 2.3.2 we showed that this partition function is also given by the index of a twisted Dirac operator coupled to the bundle \( V_{Q_{\tau, \nu}} \). If \( V \) were the holomorphic tangent bundle of \( M \), we would be calculating the index of the Dirac operator on the loop space of \( M \), i.e. the (2, 2) elliptic genus of \( M \) [107]. However, as we have seen in the last section \( V = E \oplus E^\ast \) is not the holomorphic tangent bundle and therefore the above partition function does not give the (2, 2) elliptic genus.

To put the calculation of Section 2.3.2 in a physical perspective recall that the theory on the string which is the intersection of M2 branes and the M5 branes is a (2, 0) two dimensional theory [39] with target space the Hilbert scheme of points on \( \mathbb{C}^2 \) or product of such spaces for \( N > 2 \). The partition function of this theory on \( T^2 \) is the (2, 0) elliptic genus where the left handed fermions are sections of the tangent bundle and the right handed fermions are sections of the bundle \( E \oplus E^\ast \) for \( N = 2 \). The (2, 0) elliptic genus is given by

\[
Z(\tau, y) = \text{Tr} (-1)^F y^{J_R} q^{L_0} q^{-\frac{1}{2}}, \tag{2.4.122}
\]

where \( J_R \) is the conserved charge associated with the right \( U(1) \) symmetry. As in Section 2.3.2, if we denote by \( \bar{x}_i \) and \( x_j \) the roots respectively of the Chern polynomial of \( E \oplus E^\ast \) and \( TM \),
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then \([108]^{13}\),

\[
Z(\tau,y) = \int_\mathcal{M} \prod_{j=1}^d \frac{x_j \theta_1(\tau; -m + \bar{x}_j)}{\theta_1(\tau, x_j)}
\]

\[
= y^{-\frac{d}{2}} \int_\mathcal{M} \text{ch}(E_{Q_r, Q_m}) \text{Td}(\mathcal{M})
\]

\[
= \sum_{k,n} Q_m^k Q_n^{m-\frac{d}{2}} \chi(\mathcal{M}, V_{k,n}), \tag{2.4.123}
\]

where \(d = \dim_\mathbb{C} \mathcal{M}\) and

\[
V_{Q_r, y} = \bigotimes_{k=0}^\infty (E \oplus E^*) \bigotimes_{k=1}^\infty (E \oplus E^*)^* \bigotimes_{k=1}^\infty S_{Q_r^k T_\mathcal{M}}^* \bigotimes_{k=1}^\infty S_{Q_m^k T_\mathcal{M}}
\]

\[
= \bigoplus_{k,n} Q_m^k Q_n V_{k,n}.
\]

This is precisely the calculation carried out in Equation (2.3.107) for the target space \(\text{Hilb}^{k_1}[\mathbb{C}^2] \times \text{Hilb}^{k_2}[\mathbb{C}^2] \times \cdots \times \text{Hilb}^{k_{N-1}}[\mathbb{C}^2]\) and bundle \(V\) described in Section 2.3.2, generalizing the holomorphic Euler characteristic \(\chi_y(\mathcal{M}, E \oplus E^*)\) which appears in the calculation of the partition function of the \(U(1)\) gauge theory with fundamental hypermultiplets.

**Factorisation from topological string theory**

The BPS states for more general \(A_{N-1}\) for \(N > 2\) corresponding to \(N\) M5 branes are interesting and teach us about the dynamics of these strings: if the M2 branes act independently one would expect that answer for \(A_{N-1}\) to be obtainable from the \(A_1\) theory by considering arbitrary pairs of M5 branes. This is certainly the case before compactification on \(S^1\). However, as we will see this is not the case after compactification on \(S^1\) and there are new bound states of M2 branes stretched between different pairs of M5 branes. In this section we review the factorization property of the topological string partition function of the so-called strip geometry. We show that the partition function ceases to factorize upon the partial compactification of this geometry. The failure of such a factorization is argued to indicate the existence of new bound states of M-strings ‘glued’ by non-trivial momentum around the circle.

\(^{13}\)Assuming \(\text{rank } E \oplus E^* = \text{rank } (T\mathcal{M})\) which will be the case for us.
The strip geometry in our setup is half of the toric geometry that engineers $SU(N)$ gauge theory with $N = 2^*$ (see for example Fig. 2.13). It is also the same geometry as when we decompactify the sixth direction to zero size by taking the limit $Q_r \to 0$. Let us review the factorisation on a specific example, for the $SU(3)$ theory. The partition function can be computed using the refined topological vertex,

$$Z_{SU(3)}(Q_{f_1}, Q_{f_2}, Q_m) = \prod_{i,j=1}^{\infty} \frac{(1 - Q_m t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})(1 - Q_{f_1} Q_m t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})(1 - Q_{f_1} Q^{-1} t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})}{(1 - Q_{f_1} t^{-i+1} q^{-j})(1 - Q_{f_1} t^{-i} q^{-j+1})} \times \prod_{i,j=1}^{\infty} \frac{(1 - Q_{f_2} Q_m t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})(1 - Q_{f_2} Q^{-1} t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})}{(1 - Q_{f_2} t^{-i+1} q^{-j})(1 - Q_{f_2} t^{-i} q^{-j+1})} \times \prod_{i,j=1}^{\infty} \frac{(1 - Q_{f_1} Q_{f_2} Q_m t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})(1 - Q_{f_1} Q_{f_2} Q^{-1} t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})}{(1 - Q_{f_1} Q_{f_2} t^{-i+1} q^{-j})(1 - Q_{f_1} Q_{f_2} t^{-i} q^{-j+1})} \times \prod_{i,j=1}^{\infty} \frac{(1 - Q_{f_1} Q_{f_2} Q_m t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})(1 - Q_{f_1} Q_{f_2} Q^{-1} t^{-i + \frac{1}{2}} q^{-j + \frac{1}{2}})}{(1 - Q_{f_1} Q_{f_2} t^{-i+1} q^{-j})(1 - Q_{f_1} Q_{f_2} t^{-i} q^{-j+1})}.$$ (2.4.124)

We previously pointed out that the partition function is invariant under the different choices of the preferred direction although the functional form may vary. The above factorized form is the result of our choice for the preferred direction along the external legs of the toric diagram. The curve counting arguments suggest that the factors appearing in the partition function can be grouped in such a way that the partition function can be written in terms of $SU(2)$ partition...
functions. The \(SU(2)\) partition function is given by

\[
Z_{SU(2)} = \prod_{i,j=1}^{\infty} \frac{(1 - Q_m t^{-i+\frac{1}{2}} q^{-j+\frac{1}{2}})(1 - Q_i Q_m t^{-i+\frac{1}{2}} q^{-j+\frac{1}{2}})(1 - Q_i Q^{-1}_m t^{-i+\frac{1}{2}} q^{-j+\frac{1}{2}})}{(1 - Q_i t^{-i+1} q^{-j})(1 - Q_i t^{-i} q^{-j+1})}.
\]  

(2.4.125)

Comparing the two partition functions above, it is easy to see that they satisfy

\[
Z_{SU(3)}(Q_{f_1}, Q_{f_2}, Q_m) = \frac{1}{(1 - Q_m)^3} Z_{SU(2)}(Q_{f_1}, Q_m) Z_{SU(2)}(Q_{f_2}, Q_m) Z_{SU(2)}(Q_{f_1}, Q_{f_2}, Q_m),
\]  

(2.4.126)

where we have used the short-hand notation

\[
\frac{1}{(1 - Q_m)^3} \equiv \prod_{i,j=1}^{\infty} \frac{1}{(1 - Q_m t^{-i+\frac{1}{2}} q^{-j+\frac{1}{2}})^3}.
\]  

(2.4.127)

A priori we do not have any reason to expect that the factorization would be lost when we partially compactify the geometry along the vertical external lines to geometrically engineer \(\mathcal{N} = 2^*\) theory with the gauge group \(SU(3)\). However, there is no choice of the preferred direction that would allow us to express the partition function in terms of products as in the corresponding strip geometry. Therefore, we are forced to compare the expansions in Kähler parameters to see whether the factorization still holds. We observe that the \(\mathcal{N} = 2^*\) with \(SU(3)\) partition function can be written (up to the pre-factor including only \(Q_m\)) as

\[
Z_{SU(3)}(Q_{f_1}, Q_{f_2}, Q_{\tau}, Q_m) = Z_{SU(2)}(Q_{f_1}, Q_{\tau}, Q_m) Z_{SU(2)}(Q_{f_2}, Q_{\tau}, Q_m) Z_{SU(2)}(Q_{f_1}, Q_{f_2}, Q_{\tau}) (1 + c(t, q) Q_{f_1} Q_{f_2} Q_{\tau} + \ldots),
\]  

(2.4.128)

where \(c(t, q)\) is a non-vanishing and the \((\ldots)\) involve higher powers in \(Q_{f_1}\), \(Q_{f_2}\) and \(Q_{\tau}\). Obviously, in the limit when \(Q_{\tau} \to 0\), the expansion reduces to 1 and we recover the factorization.

We can actually shed more light into this observation by isolating the same curves from the \(SU(3)\) partition function and the products of the \(SU(2)\) partition functions. We determine the BPS content that these curves give rise to. Let us start with the curve \(Q_{f_1} Q_{f_2} Q_{\tau}\). From the \(SU(3)\) partition function we obtain the following states

\[
\left(\frac{1}{2}, 0\right) \oplus 3 \left(0, \frac{1}{2}\right).
\]  

(2.4.129)
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On the other hand, we obtain from the products of the $SU(2)$ partition functions

\[
\begin{pmatrix} 0, \frac{1}{2} \end{pmatrix},
\]

(2.4.130)

for the same curve $Q_{f_1}Q_{f_2}Q_\tau$. Clearly, the $SU(3)$ partition function includes more states than the product of $SU(2)$’s. Let us work out another curve: $(Q_{f_1}Q_{f_2}Q_\tau)^2$. This curve is particularly interesting since there are no new states appearing for the product of $SU(2)$’s. This curve does not have any other contribution than the multi-covering contributions. However, the BPS content for the $SU(3)$ partition consists of the following states

\[
\begin{pmatrix} 1, \frac{1}{2} \end{pmatrix} \oplus 5 \begin{pmatrix} 1, 1 \end{pmatrix} \oplus 3 \begin{pmatrix} 1, 0 \end{pmatrix} \oplus \begin{pmatrix} 0, 5 \frac{1}{2} \end{pmatrix} \oplus 6 \begin{pmatrix} 0, \frac{3}{2} \end{pmatrix} \oplus 10 \begin{pmatrix} 0, \frac{1}{2} \end{pmatrix},
\]

(2.4.131)

The difference in BPS spectrum originating from the same curves continues to hold for higher degree curves. The above discussion can be extended for $SU(N)$ partition functions without any complications. We always have a smaller Hilbert space of states originating from the factor partition functions.

The lack of factorisation is rather surprising. Let us try to understand the implication of this observation from the point of view of the M-theory construction. In the Coulomb branch of the $SU(3)$ theory the three M5 branes are separated. There are M2 branes stretched between them. In the case of the uncompactified $x_6$ direction, the only bound states consist of the (13) strings, stretching between the first and the third M5 branes, in addition to the (12) and the (23) strings. However, once we compactify $x_6$, the (13) strings are not the only bound states the topological string partition function counts, there are additional states that we have found. The momentum along the circle could account for the additional bound states and the $Q_\tau$ dependence of the partition function. Using the momentum along the circle we have new junction states.

2.4.2 M5 branes as domain walls between M2 branes

The computation of the BPS partition function using the topological vertex (in the second method discussed in section 3) can be reformulated as introducing a Hilbert space whose basis is
formed from arbitrary Young diagrams $\nu$, with the identity operator $I = \sum_{\nu} |\nu\rangle\langle\nu|$, and whose ‘Hamiltonian’ is $H = |\nu|$ and an operator $D$ whose matrix elements are given by $D_{\nu,\mu}(\tau, m, \epsilon_1, \epsilon_2)$,

$$
\langle \nu' | D | \mu \rangle = D_{\nu',\mu}(\tau, m, \epsilon_1, \epsilon_2) = t^{-\frac{||\nu'||^2}{2} - \frac{||\mu||^2}{2}} Q_m^{-\frac{||\nu'|| + ||\mu||}{2}}
$$

$$
\times \prod_{k=1}^{\infty} \prod_{(i,j) \in \nu} \frac{(1 - Q_k^2 Q_m^{-1} q^{- \nu_i + j + \frac{1}{2}} t^{- \mu_j + i + \frac{1}{2}})(1 - Q_k^2 Q_m q^{\nu_i - j + \frac{1}{2}} t^{\mu_j - i + \frac{1}{2}})}{(1 - Q_k^2 q^{\nu_i - j} t^{\mu_j - i + 1})(1 - Q_k^2 q^{-\nu_i + j - 1} t^{\mu_j + i + 1})}
$$

$$
\times \prod_{(i,j) \in \mu} \frac{(1 - Q_k^2 Q_m^{-1} q^{\mu_i - j + \frac{1}{2}} t^{\mu_j - i + \frac{1}{2}})(1 - Q_k^2 Q_m q^{-\mu_i + j - \frac{1}{2}} t^{-\mu_j + i - \frac{1}{2}})}{(1 - Q_k^2 q^{\mu_i - j + 1} t^{\mu_j - i - 1})(1 - Q_k^2 q^{-\mu_i + j} t^{-\mu_j + i + 1})}, \tag{2.4.132}
$$

Let $\beta_a = 2\pi i t_{f_a}$, where $t_{f_a}$ label the fiber sizes or equivalently the separation of the M5 branes. In this language the partition function can be written as

$$
\hat{Z}^{(N)} = \frac{Z^{(N)}}{(Z^{(1)})^N} = \langle 0 | D e^{-\beta_1 H} D e^{-\beta_2 H} D \ldots e^{-\beta_{N-1} H} D | 0 \rangle
$$

where $|0\rangle$ is identified with the Young diagram of zero size. We would like to explain the physical meaning of this expression.

The idea is very simple. As already noted, $Z^{(N)}/(Z^{(1)})^N$ is computing the partition function of the M2 branes stretched between M5 branes and wrapping a $T^2$ with suitable twists along the cycles of $T^2$. So far we viewed the distance between the M5 branes as small compared to the size of the $T^2$. Since nothing depends on the relative sizes of $T^2$ and the separation of M5 branes, we can take the opposite limit in which we view the $T^2$ as small. This is shown in Fig. 2.14. In this case we get a reduction of the M2 brane theory down to one dimension, where the time dimension is punctuated by M5 branes where the M2 branes end on. It is natural to identify the Hilbert space of this one dimensional theory with the Hilbert space of the M2 brane on $T^2$. In fact it has been argued that these vacua (at least in the mass deformed version of the M2 brane theory, which is effectively what we have due to twistings along the cycles of $T^2$) should correspond to partitions of $n$ where $n$ is the number of M2 branes [46,47,49]. This is in agreement with what we have found here. Moreover here $H$ is the energy of the ground state of M2 brane, which is zero, up to the addition of the M2 brane mass, given by the tension of the M2 brane times the size of $T^2$ which we have effectively normalized to 1, times the number of the M2 branes. Thus $H = |\nu|$
Figure 2.14: Domain walls arising from M5 branes intersecting the M2 branes. If we take the size of the $T^2$ to be much smaller that the distance between the M5 branes then the M2 brane theory reduces one-dimensional quantum mechanics. Again M5 branes are depicted in yellow and M2 branes in blue.

as we have here. The effect of the M5 brane domain wall reduced on $T^2$ should be an operator acting on this one-dimensional theory. So on this effective theory we identify $D$ with this operator. Note that we are not fixing the number of M2 branes (i.e. the size of the Young diagrams) on each interval, but rather summing over them.

Another way to say this is that, from the viewpoint of the M2 brane theory, ending on M5 brane is like putting a particular boundary condition on the M2 brane theory, as is familiar in the context of D-branes. More generally we can have a number of M2 branes on one side of the M5 brane and another number of M2 branes on the other. In this way we can view M5 brane as defining a domain wall separating two different theories on the left and right of the domain wall with different number of M2 branes on the two sides. In this setup we can view $\langle \nu' | D | \mu \rangle$ as the partition function of a domain wall which interpolates between a particular vacuum of the M2 brane theory labeled by $\nu'$ on the left, and the vacuum labeled by $\mu$ on the right, compactified and
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twisted on $T^2$ with complex structure $\tau$ and twist parameters $(m, \epsilon_1, \epsilon_2)$. We can also view $D$ as an operator taking a vacuum of the left M2 brane to a vacuum of the right one.

One may expect the partition function $D_{\nu' \mu}$ to be modular. This turns out to be only true, as discussed in Section 3, in the unrefined limit. Otherwise pairs of adjacent domain walls $D_{\nu' \mu}$ need to be included for it to be modular. This can be interpreted as saying that the modular transformation acts also on the boundary data at the other end of the M2 brane and only the combined object should be invariant, which is the case. We will exploit this fact in Chapter 6 when trying to find domain wall factors for the E-string theory.

2.4.3 Special limits of $m$ and comparison with the elliptic genus of $A_{N-1}$ quiver theories

As already noted we expect that the partition function for the elliptic genus simplifies in special limits: If $m = \pm (\epsilon_1 - \epsilon_2)/2$ the supersymmetry enhances from $(2,0) \to (2,2)$ and the partition function becomes a constant. This expectation agrees with the results we have, as already noted. On the other hand if $m = \pm (\epsilon_1 + \epsilon_2)/2$, the partition function vanishes again. However in this case the vanishing has nothing to do with supersymmetry, but rather it has to do with the fact that the center of mass of the string (in the absence of M5 brane) has an extra direction to move which leads to a fermionic zero mode. Moreover, as already discussed, this is the case in which we have a description of M-strings in terms of an $A_{N-1}$ quiver theory. In particular, for two M5 branes we have an $A_1$ quiver theory which for $N$ suspended M2 branes is the pure $U(N)$ Yang-Mills theory with $(4,4)$ supersymmetry. It is equally true that the elliptic genus of this theory also vanishes due to the $U(1) \subset U(N)$. Moreover the elliptic genus of the $SU(N)$ theory does not vanish. It is therefore natural to factor out the vanishing contribution of the $U(1)$ piece from our computation and obtain the $SU(N)$ result. Recall the expression we found for the partition function of two M5 branes:
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\[ Z^{(2)}(\tau, m, t_f, \epsilon_1, \epsilon_2) = \left( Z^{(1)}(\tau, m, \epsilon_1, \epsilon_2) \right)^2 \]

\[ \times \sum_{\nu} (-Q_f)^{|\nu|} \prod_{(i,j) \in \nu} \frac{\theta_1(\tau; z_{ij}) \theta_1(\tau; v_{ij})}{\theta_1(\tau; w_{ij}) \theta_1(\tau; u_{ij})} \]

(2.4.133)

with the following definitions for the arguments of the \( \theta \)-functions

\[ e^{2\pi i z_{ij}} = Q_m^{-1} q^\nu_i - j^{1/2} t^{-i+1/2}, \quad e^{2\pi i v_{ij}} = Q_m^{-1} q^{-\nu_i + j^{1/2}}, \]

\[ e^{2\pi i w_{ij}} = q^{\nu_i - j^{1/2}}, \quad e^{2\pi i u_{ij}} = q^{\nu_i - j^{1/2} + 1}. \]

(2.4.134)

We can extract the contribution for two suspended M2 branes by considering the \(|\nu| = 2\) term and we take the limit where \( m = (\epsilon_1 + \epsilon_2)/2 \), i.e. \( Q_m = e^{i\pi(\epsilon_1 + \epsilon_2)} \) to obtain

\[ \tilde{Z}_2^{(2)} = \frac{\theta_1(\tau; 0) \theta_1(\tau; -\epsilon_1 - \epsilon_2) \theta_1(\tau; \epsilon_1) \theta_1(\tau; -\epsilon_1 - 2\epsilon_2)}{\theta_1(\tau; \epsilon_1 - \epsilon_2) \theta_1(\tau; -\epsilon_2) \theta_1(\tau; -\epsilon_1 - 2\epsilon_2)} \]

(2.4.135)

\[ + \frac{\theta_1(\tau; 0) \theta_1(\tau; \epsilon_1) \theta_1(\tau; -2\epsilon_1 - \epsilon_2) \theta_1(\tau; -\epsilon_1 - \epsilon_2)}{\theta_1(\tau; \epsilon_1) \theta_1(\tau; 2\epsilon_1) \theta_1(\tau; -\epsilon_1 - 2\epsilon_2)}. \]

(2.4.136)

where the first term is the contribution of \( \nu = (1, 1) \) and the second term is the contribution of \( \nu = (2) \). As is manifest, the above expression vanishes due to the fermion zero mode from the \( U(1) \) part of the partition function which in this limit becomes

\[ \tilde{Z}_1^{(2)} \to \frac{\theta_1(\tau; 0) \theta_1(\tau; \epsilon_1 + \epsilon_2)}{\theta_1(\tau; \epsilon_1) \theta_1(\tau; \epsilon_2)}. \]

Dividing\(^4\) both terms of the above expression by \( \tilde{Z}_1^{(2)} \) we obtain an expression which should be the elliptic genus of the \( SU(2) \) partition function:

\[ \frac{\tilde{Z}_2^{(2)}}{\tilde{Z}_1^{(2)}} = \frac{\theta_1(\tau; \epsilon_2) \theta_1(\tau; \epsilon_1 + 2\epsilon_2)}{\theta_1(\tau; \epsilon_2 - \epsilon_1) \theta_1(\tau; 2\epsilon_2)} + \frac{\theta_1(\tau; \epsilon_1) \theta_1(\tau; 2\epsilon_1 + \epsilon_2)}{\theta_1(\tau; 2\epsilon_1) \theta_1(\tau; \epsilon_1 - \epsilon_2)}. \]

(2.4.137)

Notice that in the limit \( \epsilon_1 \to 0 \) the above expression becomes a constant, independent of the modulus. This is consistent with the fact that in the limit \( m = \epsilon_2, \epsilon_1 = 0 \) we have a \((2, 2)\) supersymmetric index, and so it should be independent of the modulus of \( T^2 \).

\(^4\)We can also consider instead an insertion of \( F_L \) in the elliptic genus to absorb the extra fermionic zero mode of the \( U(1) \) which has the effect of replacing the vanishing theta function with its derivative.
Equation (2.3.73) gives our prediction for the $SU(N)$ elliptic genus ($m = \pm \frac{a + 1}{2}$),

\[
\frac{\hat{Z}^{(2)}_N}{\hat{Z}^{(2)}_1} = \sum_{|\nu| = N} \frac{\prod_{(i,j) \in \nu, (i,j) \neq (1,1)} \theta_1(\tau; z_{ij}) \theta_1(\tau; v_{ij})}{\prod_{(i,j) \in \nu, (i,j) \neq (0,0)} \theta_1(\tau; w_{ij}) \theta_1(\tau; u_{ij})}, \tag{2.4.138}
\]

where we identify

\[
z_{ij} = (\nu_i - j)\epsilon_1 + (i - 1)\epsilon_2, \quad v_{ij} = -(\nu_i - j + 1)\epsilon_1 - i \epsilon_2,
\]

\[
w_{ij} = (\nu_i - j + 1)\epsilon_1 - (\nu_j - i)\epsilon_2, \quad u_{ij} = (\nu_i - j)\epsilon_1 - (\nu_j - i + 1)\epsilon_2. \tag{2.4.139}
\]

For $N = 3$ the above gives,

\[
\frac{\hat{Z}^{(2)}_3}{\hat{Z}^{(2)}_1} = \frac{\theta(\tau; 2\epsilon_1)\theta(\tau; 3\epsilon_1 + \epsilon_2)\theta(\tau; \epsilon_1)\theta(\tau; 2\epsilon_1 + \epsilon_2)}{\theta(\tau; 3\epsilon_1)\theta(\tau; 2\epsilon_1 - \epsilon_2)\theta(\tau; 2\epsilon_1)\theta(\tau; \epsilon_1 - \epsilon_2)} + \frac{\theta(\tau; \epsilon_1)\theta(\tau; 2\epsilon_1 + \epsilon_2)\theta(\tau; \epsilon_2)\theta(\tau; \epsilon_1 + 2\epsilon_2)}{\theta(\tau; 2\epsilon_1 - \epsilon_2)\theta(\tau; \epsilon_1 - 2\epsilon_2)\theta(\tau; \epsilon_1)\theta(\tau; \epsilon_1 + 2\epsilon_2)} + \frac{\theta(\tau; \epsilon_2)\theta(\tau; \epsilon_1 + 2\epsilon_2)\theta(\tau; 2\epsilon_2)\theta(\tau; \epsilon_1 + 3\epsilon_2)}{\theta(\tau; \epsilon_1 - 2\epsilon_2)\theta(\tau; 3\epsilon_2)\theta(\tau; \epsilon_1 - \epsilon_2)\theta(\tau; 2\epsilon_2)}. \tag{2.4.140}
\]

These predictions are to be compared with the elliptic genus of (4,4) supersymmetric Yang-Mills for $SU(N)$ theory as computed using the results [41,42]. The elliptic genus result for $U(N)$ as computed in [41] reads as follows:

\[
\mathcal{I}^{(N)} = \sum_{|\nu| = N} \prod_{(i,j) \in \nu, (i,j) \neq (0,0)} \frac{\theta_1(\tau; \epsilon_1(i_2 - i_1) + \epsilon_2(j_2 - j_1))}{\theta_1(\tau; \epsilon_1(1 + i_2 - i_1) + \epsilon_2(j_2 - j_1))} \\
\times \prod_{(i,j) \in \nu, (i,j) \neq (1,1)} \frac{\theta_1(\epsilon_1(1 + i_2 - i_1) + \epsilon_2(1 + j_2 - j_1))}{\theta_1(\epsilon_1(i_2 - i_1) + \epsilon_2(1 + j_2 - j_1))}. \tag{2.4.141}
\]

The expressions look slightly different but can be shown to be identical. This gives a satisfactory confirmation of the overall picture we have and the connection between BPS degeneracies computed by the topological strings and the elliptic genus of M-strings.

### 2.5 Discussion of results

In the present chapter we have shown how to compute the supersymmetric partition function for M-strings on $T^2$. We have found a number of interesting structures and insights about

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15This was computed and communicated to us for the $SU(2)$ case by K. Hori and for the $SU(N)$ case by A. Gadde and S. Gukov.
the nature of M-strings. In particular we have seen the similarities and the differences between the theory seen by \( N \) copies of M-strings as compared to the N-fold symmetric product of \( \mathbb{R}^4 \). We have computed the partition function of domain walls induced by M5 branes which separates a number of M2 branes. We have also seen how M-strings can form new bound states when they wind around a circle.

The \((4,0)\) theory was obtained by a duality involving the sigma model of instanton moduli spaces of \( U(1)^{N-1} \) quiver \( A_{N-1} \) gauge theories in six dimensions. We will get a better handle on it in the next chapter, where we provide a very convenient construction of a UV quiver gauge theory which flows to it in the IR.

The M2 branes wrapped on \( T^2 \) lead to a one-dimensional theory. We have also studied the partition function of M-strings from this viewpoint. In this context, the domain walls become operators acting on the Hilbert space, where we view the one dimension as time. In this way, the partition function of M-strings gets translated to a computation in a quantum mechanical theory, where the Hilbert space is identified with Young diagrams and the Hamiltonian is the number of boxes. It would be interesting to better understand these domain wall theories \[48\]. In particular it would be useful to further develop what these domain wall theories are and how they couple to the ABJM theory \[109\]. The viewpoint of domain walls turns out to have further applications. In particular, in Chapter 6 we will see how exceptional strings (E-strings), which can be viewed \[6,110\] as M2 branes stretched between M5 branes and M9 planes of the Hořava-Witten theory, also appear to admit a similar decomposition, where we use two types of domain wall operators: one induced by M5 branes which we have studied here, and another one induced from M9 planes (see in particular \[111\]). We will also see in Chapter 6 how a combination of domain wall factors coming from the two M9 planes reproduces in a very nontrivial fashion the elliptic genus of \( E_8 \times E_8 \) heterotic string theory. Finally, we will see in Chapter 7 how the techniques developed there for computing superconformal partition functions on \( S^5 \) or \( S^5 \times S^1 \) relate the superconformal index for \( N \) coincident M5 branes to the computation of elliptic genera of their strings that was
achieved in the present chapter.
Chapter 3

Orbifolds of M-strings

3.1 Introduction

In this chapter we apply the methods developed in the study of M-strings to investigate a class of $(1,0)$ superconformal theories corresponding to $M$ M5 branes probing a transverse $A_{N-1}$ singularity. This type of system is known to be dual to Type IIB strings with $N$ D5-branes probing transverse $A_{M-1}$ singularity. This theory has a deformation away from the conformal fixed point where the M5 branes are separated in the extra transverse direction. The separation between adjacent branes correspond to vevs of scalars in the $M-1$ tensor multiplets. Analogously to the M-string case, on this tensor branch one finds strings that couple to the self-dual two-form fields in the tensor multiplets. We will see that these strings capture the full supersymmetric partition function of the theory on $T^2$ with arbitrary twists preserving supersymmetry.

The main finding presented in this chapter is that these strings are described in terms of at two-dimensional $(4,0)$ supersymmetric quiver gauge theory, whose elliptic genus captures the partition function of the bulk six-dimensional theory. This result also applies to the case of M-strings studied in Chapter 2 as a special case (setting $N = 1$). This quiver gauge theory gives a UV description of the strings, which in the IR flows to a $(4,0)$ theory. In subsequent chapters we will be able to determine the quiver gauge theories describing the strings of a variety of different
(1, 0) SCFTs.

In the present context, the presence of a transverse $A_{N-1}$ singularity suggests that the six-dimensional theory displays $SU(N)$ gauge symmetry. This would be the case if there were no M5 branes. However in the presence of $M$ M5 branes, the gauge symmetry turns out to enhance to $SU(N)^{M-1}$ of an affine $A_{M-1}$ quiver gauge theory with bifundamental matter fields with an extra $SU(N) \times SU(N)$ being a global symmetry. If we restrict to the diagonal $SU(N)$ of the global symmetry, one can derive the quiver description by going to a dual Type IIB setup with $N$ D5-branes probing a transverse $A_{M-1}$ singularity\textsuperscript{1} [7]. In the M-theory setup we have M2 branes stretched between parallel M5 branes which lead to M-strings. Placing the M5 branes in the presence of $A_{N-1}$ singularity can be interpreted as follows: it corresponds to placing $N$ copies of M-strings and modding out by a $\mathbb{Z}_N$ action which permutes them but at the same time acts by a $\mathbb{Z}_N$ subgroup of the global $SO(4)\perp$ symmetry which the strings enjoy. In order to understand the properties of the strings associated to these theories we will need to study how this orbifold action is perceived by the M-strings.

To this end we study further compactification of this theory on $S^1$ and $S^1 \times S^1$. As we go down to five dimensions on an $S^1$, we can turn on $(N-1)(M-1)$ Wilson lines of $SU(N)^{M-1}$ and the $N-1$ fugacities from the global $SU(N)$ symmetry, giving a total of $(N-1)M$ parameters. In addition the theory depends on the vacuum expectation value of the $M-1$ tensor multiplets as well as the radius of the circle. Moreover as we go around the circle we can act by a supersymmetry-preserving transverse rotation, leading to a mass parameter for the bifundamental fields. Altogether, this gives $NM+1$ parameters. In other words, we end up with an $\mathcal{N}=1$ supersymmetric gauge theory in 5d, which depends on these parameters (which partly specify the Coulomb branch of the theory and partly the coupling parameters). One can then compute the supersymmetric partition function of these theories, either using the topological vertex formalism

\textsuperscript{1}The absence of $U(1)^{M-1}$s in the gauge factor is because they are anomalous and are higgsed by the hypermultiplets corresponding to the $A_{M-1}$ hyperkähler moduli.
Chapter 3: Orbifolds of M-strings

or the instanton calculus (which corresponds to the twisted partition function on a further compactification on $S^1$). As is well known [65, 66, 76] these capture BPS degeneracies of the theory, which can be interpreted as arising from the M-strings as we saw in Chapter 2. In particular the computation of the partition function of the resulting 5d theory is equivalent to computation of the elliptic genus of the corresponding strings, which in turn can be interpreted as the elliptic genus of the $\mathbb{Z}_N$ orbifold of $N$ M-strings, which ends up being given by a $(4,0)$ supersymmetric quiver gauge theory.

This chapter is organized as follows: in Section 2 we introduce the basic setup of M5 branes and M-strings in the presence of $A_{N-1}$ singularities, and we discuss the further compactifications on $S^1$ and $S^1 \times S^1$ and the interpretation of this system in various duality frames; we also present the quiver description of such M-strings using the Type IIB setup. In Section 3 we show how refined topological strings can be used to compute the partition function of this theory. We find that the basic building block of this computation can be interpreted in terms of the amplitudes of a collection of M2 branes which end on two sides of M5 branes, in the presence of $A_{N-1}$ singularity. In other words the presence of M5 branes can be viewed as a domain wall which acts as an operator on the states of the M2 branes on the left, to give the states of M2 branes on the right. We also discuss the modular properties of the partition functions of the theory with respect to the elliptic modulus of the $T^2$ compactification. Furthermore, we show how these results can be directly obtained from the quiver $(4,0)$ gauge theory. In Section 4 we end with some concluding remarks.

Results related to the ones presented in this chapter have been obtained independently in [112].

3.2 Geometric setup

The strings of the orbifold theories we analyze in this chapter arise from M2 branes ending on parallel M5 branes in the presence of $A_{N-1}$ singularities. In this section we clarify the details of the geometry behind this construction and discuss twisted compactifications on $S^1$ and $S^1 \times S^1$. We
then proceed to describe various dual descriptions of this system. In particular, by compactifying
the M5 branes on $S^1$ with twisted boundary conditions we end up with a theory in five dimensions
with the same degrees of freedom as a quiver version of the $\mathcal{N} = 2^*$ theory. This theory has further
realizations in terms of a $(p, q)$-fivebrane web in Type IIB string theory as well as compactifications
of M-theory on certain non-compact Calabi-Yau manifolds.

In Section 3.2.1 we present the basic setup for this class of theories, including how the
M-strings fit in this picture, and what their global symmetries are. In Section 3.2.2 we discuss
compactification on a circle and twisting around the circle to introduce a mass parameter. In
Section 3.2.3 we discuss the various duality frames: first, we provide a dual Type IIA description
involving D4-branes probing $A_{N-1}$ singularities and its T-dual IIB description involving a web
of $(p, q)$-fivebranes as well as the corresponding toric description characterizing M-theory on local
Calabi-Yau three-folds. Then, we provide yet another dual Type IIB description involving D5-
branes probing $A_{M-1}$ singularities. In Section 3.2.4 we consider further compactification on $S^1$
which allows us to introduce the $\Omega$ background. We also recall the refined topological string
description of the partition function and its connection with BPS degeneracies. In Section 3.2.5 we
provide the quiver description for the orbifold of M-strings giving a $(4, 0)$ supersymmetric system
which is deduced from the Type IIB dual description of Section 3.2.3. In that section we point out
the interpretation of the quiver theory as a gauge system whose Higgs branch describes the moduli
space of instantons on $SU(N)^{M-1}$, where the fermions are coupled to suitable bundles.

3.2.1 Basics of the setup

Consider $M$ parallel and coincident M5 branes in the presence of an $A_{N-1}$ singularity in
the transverse directions. That is, the M5 branes fill a subspace $\mathbb{R}^6$ of $\mathbb{R}^{1,10}$, whereas the transverse
space is of the form

$$\mathbb{R} \times A_{N-1}, \text{ with } A_{N-1} \equiv \mathbb{C}^2/\Gamma_N, \quad \Gamma_N = \left\{ \begin{pmatrix} e^\frac{2\pi i}{N} & 0 \\ 0 & e^{-\frac{2\pi i}{N}} \end{pmatrix} | i = 1, \cdots, N - 1 \right\}. \quad (3.2.1)$$
Chapter 3: Orbifolds of M-strings

The space on which M-theory is compactified is then \( \mathbb{R}^6 \times (\mathbb{R}^{A_{N-1}})_\perp \), where the subscripts are used to distinguish directions parallel or transverse to the worldvolume of the M5 branes. The resulting theory living on the M5 branes then has \((1,0)\) supersymmetry. The massless representations of this supersymmetry are then labeled by their \( Spin(4) \sim SU(2)^{||}_L \times SU(2)^{||}_R \) representations. Scalars arise from hypermultiplets as well as from the tensor multiplets.

We choose coordinates \( X^I, \ I = 0, 1, 2, \cdots, 10 \), and parametrize the worldvolume of the M5 branes by \( X^0, X^1, X^2, X^3, X^4, X^5 \). We take the transverse \( \mathbb{R}^4 \), which we mod out by the orbifold group \( \Gamma_N \), to be parametrized by \( X^7, X^8, X^9, X^{10} \) which we also sometimes denote by \( \mathbb{R}^4_\perp \). Next, we separate the M5 branes along the \( X^6 \) direction and denote their position in the \( X^6 \) direction by \( a_i, i = 1, 2, \cdots, M \). Thus, before orbifolding, rotations of \( \mathbb{R}^4_\perp \) will lead to a \( Spin_R(4) \sim SU(2)^{\perp}_L \times SU(2)^{\perp}_R \) R-symmetry on the M5 brane worldvolume theory. Following the approach of Chapter 2, one can introduce M2 branes ending on M5 branes with boundary coupling to the anti-symmetric 2-form field, whose worldvolume is along the \( X^0, X^1, X^6 \) directions. Altogether we have the following setup:

\[
\begin{array}{c|cccccccccc}
 & X^0 & X^1 & X^2 & X^3 & X^4 & X^5 & X^6 & X^7 & X^8 & X^{10} \\
\mathbb{C}^2/\Gamma_N & - & - & - & - & - & - & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \{a_i\} & - & - & - & - \\
M2 & \times & \times & - & - & - & - & \times & - & - & - \\
\end{array}
\]

(3.2.2)

The boundary of an M2 brane inside an M5 brane is spanned by \((X^0, X^1)\) and is a string inside the M5 brane. The presence of the string breaks the \( Spin(1,5) \) Lorentz symmetry of the M5 brane to \( Spin(1,1) \times Spin(4) \), \( Spin(1,1) \) being the Lorentz group on the string. As shown in the previous chapter, the chiralities of the preserved supersymmetries on the M-string under \( Spin(1,1), Spin_R(4) \) and \( Spin(4) \subset Spin(1,5) \) are equal. Thus before the \( \Gamma_N \) orbifold action the preserved supersymmetries organize themselves into four left-moving and four right-moving supercharges whose eigenvalues under

\[
Spin(4) \sim SU(2)^{||}_L \times SU(2)^{||}_R, \quad Spin_R(4) \sim SU(2)^{\perp}_L \times SU(2)^{\perp}_R.
\]

(3.2.3)
and are given in Table 3.1.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_3^{</td>
<td></td>
</tr>
<tr>
<td>+  +  -  -</td>
<td>+  -  -  +</td>
</tr>
</tbody>
</table>

Table 3.1: Preserved supersymmetries on the string before $\mathbb{Z}_N$ orbifold action. The table shows the Cartan eigenvalues of $SO(8)$ where it is implicit that all signs are multiplied by $\frac{1}{2}$. The two columns of the table correspond to the left-moving and right-moving supercharges on the worldsheet of the M-string.

Note that these supercharges form a positive chirality spinor of $Spin(8)$, namely $\mathbf{8}_s$. It is now easy to include the action of the orbifold group. For this we note that supercharges transform under the action of the orbifold group as

$$Q_s \mapsto \exp(2\pi i \zeta)Q_s,$$

where $\zeta = (0, 0, \zeta_1, \zeta_2)$ parametrizes the orbifold action which in our case is given by

$$(w_1, w_2) \in \mathbb{C}^2 \cong \mathbb{R}_+^4 \Rightarrow \Gamma_N : (w_1, w_2) \mapsto (e^{2\pi i \zeta_1} w_1, e^{2\pi i \zeta_2} w_2),$$

with $\zeta_1 = \frac{1}{N}$ and $\zeta_2 = -\frac{1}{N}$. Therefore, we see that only the left-moving supercharges survive as they are the only ones which are invariant under the action (3.2.5). This shows that the worldvolume supersymmetry is reduced from $(4, 4)$ to $(4, 0)$ by the orbifolding.

### 3.2.2 Compactification on $S^1$ and mass rotation

Next, we consider compactifying $X^1$ to a circle of radius $R_1$. Recall that the transverse $\mathbb{R}^4$ is parametrized by $X^7, X^8, X^9, X^{10}$ and is modded out by the orbifold group $\Gamma_N$ to give an
\(A_{N-1}\) singularity. Resolving this singularity gives rise to an ALE space with metric

\[
ds^2 = V^{-1}(dt + \vec{A} \cdot d\vec{x})^2 + V d\vec{x}^2
\]

\[
V = \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i|}
\]

\[-\nabla V = \nabla \times \vec{A}.
\]

(3.2.6)

The second homology of this space is generated by two-cycles \(C_i, i = 1, \ldots, N-1\) whose intersection numbers produce the Cartan matrix of \(A_{N-1}\). This space can be equivalently viewed as a limit of the multi-centred Taub-Nut space \(TN_N\) defined by the same equations as above with the modification that \(V\) gets replaced by

\[
V = \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i|} + \frac{1}{\lambda^2}.
\]

(3.2.7)

The underlying geometry is then a circle fibration over \(\mathbb{R}^3\) such that the circle shrinks to zero size at the points \(\vec{x}_i \in \mathbb{R}^3\) and approaches an asymptotic value \(\lambda\) at infinity. In the limit \(\lambda \to \infty\) one then regains the ALE space (3.2.6). However, for our purposes, when we talk about the \(A_{N-1}\) singularity we will always keep the circle at infinity finite and therefore will consider \(TN_N\).

Let us next come to the isometries of the space \(TN_N\). Generically, the isometry group is just \(U(1)_f\), corresponding to rotation of the circle fiber. Furthermore, for configurations where all centers are aligned along a line there is another \(U(1)\) isometry which corresponds to rotations preserving this axis, which we will denote by \(U(1)_b\).\(^2\) The situation is analogous to the isometries of \(A_{N-1}\) ALE space discussed in [113]. We want to describe both \(U(1)\)’s explicitly by choosing complex coordinates. To this end, we recall that the singular limit of this space corresponds locally around the origin to the algebraic equation

\[
X^N + YZ = 0
\]

(3.2.8)

in \(\mathbb{C}^3\). We can parametrize solutions by \(Y = w_1^N\), \(Z = w_2^N\) and \(X = w_1 w_2\). Note that these equations are preserved when blowing up the singularity and are therefore a valid description of

\(^2\)For \(N = 1\) this isometry gets enhanced to \(SU(2)\) and thus the full isometry group of \(TN_1\) is \(U(1)_f \times SU(2)_b\).
TN\textsubscript{N} around the origin. The two isometries discussed above then have the following representations in this picture:

\begin{align*}
  U(1)\text{\textsubscript{f}} : (w_1, w_2) &\mapsto (e^{2\pi i\alpha} w_1, e^{-2\pi i\alpha} w_2) \\
  U(1)\text{\textsubscript{b}} : (w_1, w_2) &\mapsto (e^{2\pi i\alpha} w_1, e^{2\pi i\alpha} w_2). \quad (3.2.9)
\end{align*}

Having identified the isometries of the space transverse to the M5 branes we next consider compactification of the coordinate \(X^{1}\) on a circle with radius \(R_1\). We can fiber \(TN\textsubscript{N}\) non-trivially over this \(S^1\) as follows: as we go around the circle we use the isometry \(U(1)\text{\textsubscript{f}}\) to rotate \((w_1, w_2)\),

\[ U(1)\text{\textsubscript{m}} \equiv U(1)\text{\textsubscript{f}} : (w_1, w_2) \mapsto (e^{2\pi im} w_1, e^{-2\pi im} w_2). \quad (3.2.10) \]

Note that the supercharges that are invariant under this rotation are precisely the left-moving supercharges that survive the orbifold action (3.2.5). For \(N = 1\), the resulting theory in 5d is an \(\mathcal{N} = 2^*\) theory with \(SU(M)\) gauge group and adjoint hypermultiplet with mass \(m\) which was studied in the previous chapter. For general \(N\) the theory is an affine \(\hat{A}_{N-1}\) quiver gauge theory with an \(SU(M)\) gauge group at each node and with bi-fundamental matter between adjacent nodes. We depict this in Figure 3.1. There are \(N\) different gauge couplings, one for each node in the quiver, and their sum is related to the radius of the circle along the \(X^{1}\) direction through

\[ \tau = \sum_{i=1}^{N} \tau_i = \sum_{i=1}^{N} \frac{4\pi^2}{g^2_{YM,i}} = \frac{1}{R_1}, \quad (3.2.11) \]

where we take the \(\tau_i\) to be the couplings of the individual nodes. Furthermore, the hypermultiplets which form the bi-fundamental matter fields will each have mass \(m\). To complete the count of parameters note that there are also \(N(M - 1)\) Coulomb branch parameters. Together with the mass parameter and the couplings we thus see that the gauge theory depends altogether on \(N(M - 1) + N + 1 = NM + 1\) parameters.

### 3.2.3 Different duality frames

In this section we present different realizations within Type II string theory of the M-theory setup discussed above. The goal will be to derive on the one hand a Type IIB \((p, q)\)-brane
Figure 3.1: Compactification of the theory of M M5-branes on a circle in the presence of an $A_{n-1}$ singularity leads to the 5d quiver gauge theory depicted here.

web construction for the 5d gauge theory which will allow us to lift the brane setup to a M-theory compactification on a non-compact Calabi-Yau threefold. On the other hand we will derive another Type IIB description in terms of D5-branes in the presence of $A_{M-1}$ singularity which will serve two purposes. First of all, it will give rise to a dual 6d gauge theory description of the original M-theory setup, and secondly and most importantly it will allow us to find a 2d quiver gauge theory description for the self-dual strings.

**Type IIB $p,q$-brane web and M-theory on toric Calabi-Yau**

Let us start with the derivation of the Type IIB $(p,q)$-fivebrane web setup through a chain of dualities. As a first step we compactify the original M-theory geometry along the $X^1$ circle. We obtain Type IIA theory with the following brane setup:

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{R}$</th>
<th>$X^0$</th>
<th>$X^2$</th>
<th>$X^3$</th>
<th>$X^4$</th>
<th>$X^5$</th>
<th>$X^6$</th>
<th>$X^7$</th>
<th>$X^8$</th>
<th>$X^9$</th>
<th>$X^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M $D4$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>${a_i}$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>k $F1$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
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</tr>
</tbody>
</table>
That is, we have \( k \) fundamental strings stretching between D4-branes, in a transverse Taub-NUT background of charge \( N \). We denote the separations between the D4-branes by \( t_i \), while \( t \) is now related to the gauge coupling of the D4-brane worldvolume theory by

\[
\frac{g_{YM}^2}{4\pi^2} = \tau^{-1}.
\]

The presence of transverse \( A_{N-1} \)-singularity leads to a \( \mathbb{Z}_N \) orbifold [37] and this gives rise to the five-dimensional quiver gauge theory described in the previous section. Let us next discuss the reduction of the M-theory 3-form \( A^{(3)} \). Before the circle-reduction it can be given an expectation value along the three-cycles \( S^1 \times C_i \) where \( S^1 \) is the M-theory circle. These particular expectation values will reduce in the Type IIA setup to non-zero B-field flux on the \( C_i \) cycles:

\[
B = \sum_{i=1}^{N-1} \tau_i \cdot \omega_i \rightarrow \int_{C_i} B = \tau_i,
\]

where we take the \( \omega_i \) to be elements of \( H^{1,1}(\text{TN}_N, \mathbb{Z}) \) and Poincare dual to the \( C_i \).

Let us next assume that \( m \) is turned off\(^3\). Now we perform T-duality along the Taub-NUT circle. The Taub-NUT geometry turns into a collection of Type IIB NS5-branes on transverse \( S^1 \times \mathbb{R}^3 \) [83], while the D4-branes become D5-branes and the fundamental strings of Type IIA turn into Type IIB fundamental strings. We end up with the following picture:

<table>
<thead>
<tr>
<th></th>
<th>( \mathbb{R} )</th>
<th>( \mathbb{R}^4 )</th>
<th>( \mathbb{R} )</th>
<th>( S^1 )</th>
<th>( \mathbb{R}^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M ) D5</td>
<td>( x ) ( x ) ( x ) ( x ) ( x ) { ( a_i ) }</td>
<td>( x ) ( - ) ( - ) ( - ) ( - ) ( - )</td>
<td>( x ) ( x ) ( x ) ( x ) ( x ) ( x )</td>
<td>( x ) ( x ) ( x ) ( x ) ( x ) ( x )</td>
<td></td>
</tr>
<tr>
<td>( k ) F1</td>
<td>( x ) ( - ) ( - ) ( - ) ( - ) ( x )</td>
<td>( - ) ( - ) ( - ) ( - ) ( - ) ( - )</td>
<td>( x ) ( - ) ( - ) ( - ) ( - ) ( - ) ( - )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( N ) NS5</td>
<td>( x ) ( x ) ( x ) ( x )</td>
<td>( x )</td>
<td>( x )</td>
<td>( x )</td>
<td>( x )</td>
</tr>
</tbody>
</table>

Now the \( X^7 \) radius is \( 1/\lambda \), the inverse of the asymptotic radius of the \( \text{TN}_N \) circle. It is argued in [114] that the integral of the B-field on \( C_i \),

\[
\int_{C_i} B = \tau_i,
\]

\(^3\)The mass parameter, which had entered as a twist along \( X^1 \) of the transverse \( \text{TN}_N \), now has the following interpretation: upon compactifying on \( X^1 \), we get a new gauge field \( A_m \) from the metric: \( A_m = g_{1\theta} = m \, d\theta \), where \( \theta \) parametrizes the Taub-NUT fiber. Thus we find that there is a nonzero Wilson line along the Taub-Nut fiber: \( f_\theta A_m = 2\pi i m \).
translates after T-duality to the separation between the NS5-branes along the $X^7$ direction. This is still valid in the singular limit we are considering where the centers of $T_{N}$ are brought together while leaving the B-flux finite. The D4-branes translate on the Type IIB side to D5-branes wrapping the $X^7$ circle and sitting at the origin of $\mathbb{R}^3$. The resulting brane picture is depicted in Figure 3.2. This brane picture describes the subset of parameters in the gauge theory where the Cartan expectation values for all $SU(M)$ gauge factors are the same and the mass is set to zero. This corresponds to a $N + M - 1$ dimensional subspace of the full parameter space. To get the full picture after turning on non-zero mass one has to introduce $(1, 1)$ branes. These will connect D5-branes which end on NS5-branes from different sides as shown in Figure 3.3. The most general setup of $(p, q)$-branes now depends on $NM + 1$ parameters and thus reproduces correctly the gauge theory counting.

We complete this chain of dualities by simply recalling the picture of [35]: Type IIB theory on $S^1$ (which we later take to be the $X^0$ circle) is the same as M-theory on $T^2$, namely a $(p, q)$-brane.
corresponds to the \((p, q)\)-cycle of the M-theory \(T^2\) shrinking over the \((X^6, X^7)\) base. This way the brane picture uplifts in M-theory to a non-compact Calabi-Yau which is elliptically fibered. For our specific brane setup it turns out that the elliptic fiber is singular and of type \(I_N\) in the Kodaira classification of elliptic fibrations [25]. The Kähler class \(t_e^{M}\) of the elliptic fiber is identified with the overall gauge coupling of the 5d quiver gauge theory and is thus the inverse of the radius of the \(X^1\) circle. That is we have

\[
t_e^{M} = \frac{1}{R_1}. \tag{3.2.12}
\]

Resolving the singularities of the elliptic fiber leads to various moduli which are identified with the Coulomb branch and mass parameters of the gauge theory.
Here we will derive a dual six dimensional gauge theory description of our original M-theory setup. To this end we start by compactifying on the Taub-NUT circle and pass to the following Type IIA description:

\[
\begin{array}{c|cccccccc}
\text{M NS5} & \mathbb{R} & S^1 & \mathbb{R}^4 & \mathbb{R}^3 & X^6 & X^7 & X^8 & X^9 \\
\text{k D2} & x & x & x & x & x & x & \{a_i\} & - & - \\
\text{N D6} & x & x & - & - & - & x & - & - & - \\
\end{array}
\]

The centers of Taub-NUT have become D6-branes, the M5 branes have become NS5-branes, and the M2 branes have become D2-branes. The separation between M5 branes simply becomes separation between the NS5-branes. The \( \tau \) parameter is the inverse of the size of the \( X^1 \) circle, multiplied by the radius \( \lambda \) of the Taub-NUT circle.

Now we can find out what happens if we perform T-duality along \( X^6 \) (which from now on we must assume to be a circle). The configuration of the branes is as follows:

\[
\begin{array}{c|cccccccc}
\text{k D1} & \mathbb{R} & S^1 & \mathbb{R}^4 & \mathbb{R}^3 & X^6 & X^7 & X^8 & X^9 \\
\text{N D5} & x & x & x & x & x & x & - & - & - \\
\end{array}
\]

In other words, the \( M \) NS5-branes of Type IIA in this picture have become \( \text{TN}_M \) geometry and the D6-branes have become D5-branes.

The theory living on the D5-branes has again an interpretation of a quiver gauge theory. This time, however, each node of the quiver is an \( SU(N) \) gauge group with bifundamental matter between adjacent nodes [7,37]. This is depicted in Figure 3.4. As explained in [7] this gauge theory comes with \( M - 1 \) tensor multiplets and a global \( SU(N) \) symmetry. Counting parameters we find \( M(N - 1) \) Coulomb branch parameters and fugacities which together with the tensor multiplet scalars, the mass parameter and the radius of compactification from six dimensionals to five give \( M(N - 1) + M - 1 + 1 + 1 = MN + 1 \) parameters. This matches with the countings from the dual five-dimensional gauge theory and the toric diagram. We would like to remark here that taking the transverse space to be ALE of type \( A_{M-1} \) would have instead resulted in a linear quiver with
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Figure 3.4: Dual six-dimensional quiver gauge theory.

SU(N) × SU(N) global symmetry, SU(N)^M−1 gauge group and a larger number of parameters. We will not consider this situation in this chapter; however, generalization of our results to that case is straightforward.

3.2.4 Compactification on T^2 and relation with topological strings

Going back to our original M-theory setup, we can also further compactify X^0 on S^1. While doing this we can introduce the Ω background by fiberizing the space R^4 over this circle. In order to preserve supersymmetry we then also have to fiber TN_N around this circle. Altogether we twist TN_N × R^4 by the action of U(1) × U(1) as we go around the circle parametrized by X^0:

\[ U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} : (z_1, z_2) \mapsto (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2), \]
\[ : (w_1, w_2) \mapsto (e^{-\frac{\epsilon_1 + \epsilon_2}{2}} w_1, e^{-\frac{\epsilon_1 + \epsilon_2}{2}} w_2) \]  

(3.2.13)

The second U(1) is nothing else than the isometry U(1)_b of TN_N.

We can now ask what the theory of the suspended M2 branes is when wrapped around
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the $X^0$ and $X^1$ directions. A sigma model description can be deduced as follows. The M2 branes as well as the M5 branes will be sitting at the fixed point of the orbifold action in $\mathbb{R}^4_{||}$ and as in the M-string setup the M5 branes are extended along $T^2 \times \mathbb{R}^4_{||}$. Also, the M2 branes will appear point-like in $\mathbb{R}^4_{||}$. However, this time their moduli space will not be the one of $U(1)$ instantons but rather that of $SU(N)$ instantons. One way to see this is from the dual Type IIB setup described in Section 3.2.3. From the Type IIB brane setup one can see that the D1-branes are instantons from the point of view of the theory living on the D5-branes. As the D1-branes are connected to the M-strings through a chain of dualities we thus see that the moduli space of $k$ strings is that of $k$ $SU(N)$ instantons. Furthermore, as the real dimension of this moduli space is $4kN$ we thus see that the string has gained more degrees of freedom compared to the M-string whose moduli were the coordinates of $\mathbb{R}^4_{||}$. From another point of view one can say that while the M-string was a point-like object on $\mathbb{R}^4_{||}$ the string now fills an extended region in $\mathbb{R}^4_{||}$ because, unlike the $U(1)$ case, the instantons can now acquire a finite size. Yet from another viewpoint one can say that in the presence of transversal $A_{N-1}$ singularity M2 branes suspended between M5 branes gain thickness (see Figure 3.5).

The task of the following sections will be to compute these degeneracies and obtain a closed formula for them in terms of the refined topological string partition function. Again, the partition function of M-theory in this background is by definition the partition function of the refined topological string on the corresponding Calabi-Yau threefold which now takes the following form:

$$Z^{M-theory}((A_{n-1} \times \mathbb{R}^4_{||}) \times T^2_{\epsilon_1,\epsilon_2,m} \times \mathbb{R}) = Z^{refined}_{top}(\epsilon_1,\epsilon_2)(CY_{N,M,m,t_i^j,\tau_i^j}). \quad (3.2.14)$$

As a main tool we will use the topological vertex and its refinement [65–67,76,79], to compute the degeneracy of BPS states. This correspondence will be used to further extract the elliptic genus of the strings. These will arise from M2 branes which wrap the torus $T^2$ and are extended along the $X^6$ direction. Having compactified on the second $S^1$ all M-theory parameters get rescaled by the
Chapter 3: Orbifolds of M-strings

Figure 3.5: M-strings versus orbifolds of M-strings. In (a) the gauge group is $U(1)$ and the corresponding instantons originating from stretched M2 branes have zero size in the $\mathbb{R}^4$ directions. In (b) we see a thickening of the M2 brane ending on the M5 brane in the case of transverse $A_{N-1}$ singularity, because instantons can now acquire a finite size.

radius $R_0$ and also get complexified due to Wilson lines along the second circle. In particular, by abuse of notation we will now denote the complex structure of $T^2$ by $\tau$. A self-dual string which has Kaluza-Klein momentum $k$ along the M-theory circle then gives rise to BPS degeneracies which will appear as the coefficient of the $k$-th power of $Q_\tau = e^{2\pi i \tau}$ in the topological string partition function of the elliptic Calabi-Yau. Furthermore, such strings can have non-trivial charge under all remaining gauge theory parameters. Their degeneracies appear in the free energy of the topological string as computed in Section 3.3.

3.2.5 Quiver theory for the self-dual strings

A 2d quiver description for the self-dual strings can be deduced from the Type IIB brane setup of $N$ D5-branes probing an $A_{M-1}$ singularity described in Section 3.2.3. Following [38] the quiver can be constructed from an orbifold of the theory on the D1-branes. Before the orbifolding
the theory living on the D1 branes is a $\mathcal{N} = (4,4)$ $U(k)$ gauge theory with one adjoint and $N$ fundamental hypermultiplets. The adjoint hypermultiplet arises from the the 1−1 strings and the $N$ hypermultiplets come from the 1−5 strings. To be more specific we have, following [115], the following massless modes on the worldvolume:

$$
\begin{align*}
\text{bosons} & \quad \text{fermions} \\
b^{A'Y} & \quad \psi^{-A'Y} \\
b^{A''\tilde{A'}} & \quad \psi^{-A''\tilde{A'}} \\
A_{-+}, A_{++} & \quad \psi^{A'A'}, \psi^{\tilde{A}'Y} \\
H^{A'} & \quad \chi^{A}, \chi^{Y},
\end{align*}
$$

(3.2.15)

where $A_{\pm\pm} = A_0 \pm A_1$. Furthermore, the indices $(A', \tilde{A}')$ represent the fundamental representations of the two $SU(2)$ groups rotating the directions $X^2, X^3, X^4, X^5$ while $(A, Y)$ are indices for the $SU(2)$’s rotating $X^6, X^7, X^8, X^9$. The scalars in the adjoint $\mathcal{N} = (4,4)$ hypermultiplet are parametrized by $b^{A'\tilde{A'}}$ while those of the vector multiplet are $b^{AY}$. The scalars of the fundamental hypermultiplets, $H^{A'}$, are doublets under $SU(2)_R \equiv SU(2)_{A'}$. The multiplet structure is then obtained by the action of the left-moving and right-moving supercharges:

$$
Q_-^{A'A'} b^{A'}_Y = \psi^{-A'Y}, \quad Q_+^{A'A'} b^{A}_Y = \psi^{A'A'}.
$$

(3.2.16)

These fields can equally well be described in the language of $\mathcal{N} = (2,2)$ chiral and twisted chiral superfields. In particular, the vector multiplet is given by the pair of superfields $(\Sigma, \Phi)$ where $\Sigma$ is a twisted chiral superfield and $\Phi$ is a chiral superfield. Furthermore, the adjoint hypermultiplet is given by the pair of chiral superfields $(B, \tilde{B})$ whereas the fundamental hypermultiplets are $(Q, \tilde{Q})$. That is, we have the following correspondence:

$$
\begin{align*}
b^{A'Y} & \leftrightarrow (\Sigma, \Phi), \quad b^{A''\tilde{A'}} & \leftrightarrow (B, \tilde{B}), \\
H^{A'} & \leftrightarrow (Q, \tilde{Q}).
\end{align*}
$$

(3.2.17)

We next consider orbifolding this theory by $\mathbb{Z}_M$. To preserve the left-moving supersymmetry and break the right-moving one we embed the orbifold group $\mathbb{Z}_M$ in $SU(2)_Y$ giving the following action
on fields with $Y$-index:

$$(b^Y, \psi_-^Y, \psi_+^Y, X^Y) \mapsto (\zeta^Y b^Y, \zeta^Y \psi_-^Y, \zeta^Y \psi_+^Y, \zeta^Y X^Y), \quad (3.2.18)$$

where $\zeta = e^{2\pi i}$ and $Y = \pm$. Note that the remaining fields are invariant under the orbifold action. The resulting theory has $N = (4,0)$ supersymmetry and its field content can equally well be described in the language of $\mathcal{N} = (2,0)$ superfields by decomposing the $\mathcal{N} = (2,2)$ superfields as follows:

$$\Sigma_{(2,2)}(\theta^+, \bar{\theta}^-) \sim \Sigma - \sqrt{2}\theta^+ \Upsilon, \quad \Phi_{(2,2)}(\theta^+, \bar{\theta}^-) \sim \Phi + \sqrt{2}\theta^+ \Lambda^\Phi,$$

$$B_{(2,2)}(\theta^+, \bar{\theta}^-) \sim B + \sqrt{2}\theta^+ \Lambda^B, \quad \bar{B}_{(2,2)}(\theta^+, \bar{\theta}^-) \sim \bar{B} + \sqrt{2}\theta^+ \bar{\Lambda}^\bar{B},$$

$$Q_{(2,2)}(\theta^+, \bar{\theta}^-) \sim Q + \sqrt{2}\theta^+ \Lambda^Q, \quad \bar{Q}_{(2,2)}(\theta^+, \bar{\theta}^-) \sim \bar{Q} + \sqrt{2}\theta^+ \bar{\Lambda}^\bar{Q}, \quad (3.2.19)$$

where $\Sigma$ and $\Phi$, $B$, $\bar{B}$, $Q$, $\bar{Q}$ are $(2,0)$ chiral superfields, $\Upsilon$ is the $(2,0)$ gauge superfield, and $\Lambda^i$ is the Fermi superfield.

The orbifolding gives rise to a quiver gauge theory with an inner quiver and an outer one. The inner quiver is the affine $\hat{A}_{M-1}$ Dynkin diagram with nodes corresponding to gauge group factors $U(k_i)$ for $i = 1, \ldots, M$ which live on the $i$th copy of D1-branes and are linked by bifundamentals between adjacent nodes. Moreover, there is also an outer $\hat{A}_{M-1}$ quiver with $SU(N)$ nodes which corresponds to the orbifold of the D5-branes. Its nodes are not connected as those modes are not visible from the viewpoint of the D1-branes. However, there are links connecting the outer with the inner quiver. In particular, there are links which connect $SU(N)_i$ nodes of the outer quiver with $U(k_i)$ nodes of the inner one. These links are $(4,0)$ hypermultiplets which are invariant under the $\mathbb{Z}_M$ orbifold action. Matter fields which are not invariant under this action still survive the orbifolding but now reach from $SU(N)_i$ nodes to $U(k_{i-1})$ and $U(k_{i+1})$ nodes. The result is the quiver depicted in Figure 3.6.

In order to connect this picture to the self-dual strings we need to turn off the D1 brane charge and instead introduce D3 branes wrapped around blow-up cycles of the resolved $A_{M-1}$ singularity. As explained in Chapter 2, in Type IIB the tension of strings arising from D3-branes
wrapping blow-up cycle $C_i$ is given by $t_i = \mu_i/g_s$ where $\mu_i$ is the size of the 2-cycle $C_i$. Taking the limit $\mu_i \to 0$ with $g_s \to 0$ decouples the D1 branes and one is left with the D3 branes. In the language of the above quiver this limit corresponds to removing the last node of the inner quiver (i.e. setting its rank to zero) as well as all links ending on it.

Figure 3.6: The quiver for the D1-D5-system. In order to obtain the self-dual strings one has to remove the last node in the inner quiver and all links ending on it. We have also included a representative set of $(2,0)$ fields corresponding to the links connecting the nodes of the quiver.

One of the goals of this chapter is to make a prediction for the elliptic genus of this quiver using the refined topological vertex which we will put to work in Section 3.3. For this we need to connect the global $U(1)$ symmetries of the quiver to the ones of the original M-theory picture. In particular, we need to identify the mass-rotation $U(1)_m$ as well as the symmetries of the $\Omega$ background, namely $U(1)_{\epsilon_1}$ and $U(1)_{\epsilon_2}$, as a subset of the symmetries of the quiver theory. To
this end, it turns out to be useful to study yet another dual brane setup which captures the field content of the quiver in a very intuitive manner. We start by recalling the Type IIA brane setup of Section 3.2.3:

<table>
<thead>
<tr>
<th></th>
<th>$S^1$</th>
<th>$S^1$</th>
<th>$\mathbb{R}^4$</th>
<th>$\mathbb{R}$</th>
<th>$\mathbb{R}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS5</td>
<td>$X^0$</td>
<td>$X^1$</td>
<td>$X^2$</td>
<td>$X^3$</td>
<td>$X^4$</td>
</tr>
<tr>
<td>$k$ D2</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>N D6</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Now perform T-duality along the circle in the $X^1$ direction. The result is the Type IIB brane setup shown in the table below and pictured in Figure 3.7.

<table>
<thead>
<tr>
<th></th>
<th>$S^1$</th>
<th>$S^1$</th>
<th>$\mathbb{R}^4$</th>
<th>$\mathbb{R}$</th>
<th>$\mathbb{R}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS5</td>
<td>$X^0$</td>
<td>$X^1$</td>
<td>$X^2$</td>
<td>$X^3$</td>
<td>$X^4$</td>
</tr>
<tr>
<td>$k$ D1</td>
<td>$x$</td>
<td>$-\</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>N D5</td>
<td>$x$</td>
<td>$-\</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Note that we have mapped the M-strings to D1 branes which are extended along the $X^6$ direction and wrap the circle parametrized by $X^0$ inside the NS5 and D5 branes. Taking the size of this circle to be very small we can reduce the theory living on the D1 branes along it and the resulting theory is a quantum mechanics living on the segments parametrized by $X^6$. It is now easy to show that the corresponding quiver diagram for this quantum mechanics is exactly the same as the one obtained from the orbifold of the D1-D5 system, depicted in Figure 3.6. To see this consider taking all D5-branes to be lying on top of each other so that the strings living on the D1-branes enjoy a full $U(k)$ gauge symmetry and $SU(N)$ flavor symmetry. Furthermore, deform the system by introducing $(N,1)$-branes connecting the D5-branes ending from different sides on the same NS5-brane. The result is depicted in Figure 3.8. Now one just has to look at the fundamental strings stretching between the D1 branes and also the ones ending on the D5 branes. One easily sees that they correspond to the links of the quiver diagram where for ease of identification we have colored the links as well as the strings.

Let us next come to the identification of the global $U(1)$ symmetries. Looking at Figure
Figure 3.7: Type IIB brane setup with $M$ NS5 branes and $N$ D5 branes. The D1 branes are parallel to the D5 branes but drawn shorter to distinguish them from the latter ones.

Figure 3.8: Type IIB brane setup after putting all D5 branes on top of each other. The theory living on the D1 branes corresponds to the quiver gauge theory discussed above.
3.8 we can identify the length of the \((N, 1)\)-branes with the mass parameter \(m\) of the M-theory setup. We can also see that the only strings acquiring mass are the ones reaching from one set of D1 branes to the neighbouring set of either D1 or D5 branes. That is, in the original quiver language the only fields getting massive by turning on non-trivial \(m\) are the ones coming from the links connecting nodes of the inner quiver and from links connecting an outer node with adjacent inner nodes. In \((2, 0)\) superfield language the first class consists of the twisted chiral multiplets \(\Sigma\), the chiral multiplets \(\Phi\) and the Fermi superfields \(\Lambda^B\) as well as \(\Lambda^\tilde{B}\). The second class is formed by the Fermi superfields \(\Lambda^Q\) and \(\Lambda^\tilde{Q}\). As it is not possible to write down a scalar mass term for these fields in the Lagrangian the mass \(m\) has to correspond to the conserved charge of a \(U(1)\)-current which is a symmetry of the theory. On the other hand from the supersymmetry transformations (3.2.16) and the field identifications (3.2.17) one can see that the fields \(B, \tilde{B}, \Lambda^\Phi, Q\) and \(\tilde{Q}\) carry either an \(A\) or a \(\tilde{A}\) index which shows that they transform nontrivially under rotations of \(\mathbb{R}_4^1\).

They will thus carry \(U(1)_{\epsilon_k}\) charge. As a clarifying example and also to set our conventions we give here the charges of the fields under the various \(U(1)\)'s for the case where \(M = 2\), that is when the D3 quiver contains only one inner node:

<table>
<thead>
<tr>
<th>(U(k))</th>
<th>(\Lambda^\Phi)</th>
<th>(B)</th>
<th>(\tilde{B})</th>
<th>(Q)</th>
<th>(\tilde{Q})</th>
<th>(\Lambda^Q)</th>
<th>(\Lambda^\tilde{Q})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U(1)_{\epsilon_1})</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(U(1)_{\epsilon_2})</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(U(1)_m)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Let us next comment on the Higgs branch of the quiver gauge theory. As the \((4, 0)\) theory contains superpotential terms coming from the faces of the quiver we have to restrict the parameters of this superpotential in order to make contact with the M-theory setup. We claim that the elliptic genus of this quiver gauge theory computed along the lines of [41–43] matches the topological vertex result of Section 3.3 where \(m\) controls the “mass-rotation” of the M-theory setup. We will return to this point in Section 3.3.4, where we will be able to perform an explicit check in the case of \(M = 2, N = 1\). Indeed as argued in [38] the Higgs branch moduli space of the quiver consists of \(M - 1\) copies of the moduli space of \(k\) \(SU(N)\) instantons and hence the quiver also contains
the sigma model description for the self-dual strings. The bosons of the sigma model will arise from the $4kN$ bosonic zero modes in the $k$ $\text{SU}(N)$ instanton background and are thus sections of the tangent bundle. Left-moving fermions are again sections of the tangent bundle whereas right-moving fermions transform as sections of a different bundle breaking supersymmetry in the right-moving sector. In the case where the inner quiver contains only one node, that is where there are only two M5 branes in the M-theory setup, this bundle is formed by the $2kN$ fermionic zero modes of the Dirac equation for an adjoint fermion in the instanton background together with their complex conjugates. For more details on this bundle and its Chern characters we refer to [116].

For the general quiver with $M - 1$ nodes the picture is more complicated. The bosons are sections of the tangent bundle of the moduli space

$$\mathcal{M}_{k_1,k_2,\cdots,k_{M-1}}^N \equiv \mathcal{M}(k_1,N) \times \mathcal{M}(k_2,N) \times \cdots \times \mathcal{M}(k_{M-1},N).$$

(3.2.20)

The right-moving fermions are sections of a bundle $V$ which is of same dimensionality as the tangent bundle. It admits a decomposition

$$V = \bigoplus_{s=0}^{M-1} V_s,$$

(3.2.21)

where the $V_s$ are bundles over $\mathcal{M}(k_s,N) \times \mathcal{M}(k_{s+1},N)$ and it is understood that $\mathcal{M}(k_0,N)$ and $\mathcal{M}(k_M,N)$ are empty spaces. The moduli space of $k_s$ instantons in $\text{SU}(N)$ gauge theory admits fixed points under the $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \times U(1)^N$ action on ADHM data which are themselves labelled by ADHM data for an $N$-tuple of $U(1)$ instantons: $(k_s^1, k_s^2, \cdots, k_s^N)$ with the property $\sum_{a=1}^N k_s^a = k_s$. The moduli space of $U(1)$ instantons is the Hilbert scheme of points on $\mathbb{C}^2$ and fixed points on $\text{Hilb}^{k_s}(\mathbb{C}^2)$ are labeled by codimension $k_s$ ideals in $\mathbb{C}[x,y]$ denoted by $I_s^a$. Thus the fixed points on $\mathcal{M}(k_s,N) \times \mathcal{M}(k_{s+1},N)$ can be identified by pairs of ideals and in this language the bundle $V_s$ restricted to these fixed points is of the form

$$V_s|_{\text{fixed points}} = \left( \bigoplus_{a=1}^N, b=1 \text{Ext}^1(I_s^a, I_{s+1}^b) \right) \otimes L^{-\frac{1}{2}},$$

(3.2.22)

where $L$ is the canonical line bundle on $\mathbb{C}^2$ and $I_0$ and $I_M$ are codimension-zero ideals. In Section 3.3.4 we will explain how an explicit description of these bundles gives another way of computing
the elliptic genus of the self-dual strings.

3.3 Topological string computation of the partition function

The goal of this section is to compute the topological partition function of M5 branes on the geometry $\mathbb{R}^4 \times T^2 \times \mathbb{R} \times A_{N-1}$ presented in Section 2. To this end we compute the refined topological string partition function of the non-compact Calabi-Yau given by the toric diagram in Figure 3.3. In computing such a partition function we have to specify a choice of preferred direction, which can be taken either to be the vertical axis or the horizontal axis. Choosing it to be the vertical axis will lead to the Nekrasov partition function for the five-dimensional gauge theory given by the quiver of Figure 3.1 (in line with the duality frame of Type IIA with D4 branes probing $A_{N-1}$ singularity), whereas the choice of the horizontal axis will lead to the Nekrasov partition function for the dual six-dimensional gauge theory of Figure 3.4 (corresponding to the duality frame involving $N$ D5-branes of Type IIB probing $A_{M-1}$ singularity). In order to extract the elliptic genus of self-dual strings we have to compute the latter partition function. We do this in steps. First, in Section 3.3.1 we study the holomorphic curves contributing to the open topological string partition function for a certain periodic strip geometry (illustrated in Figure 3.11 in the of case $N = 2$). In Section 3.3.2 we normalize the open topological string partition function for this periodic strip by the contributions of closed topological strings (that is, by the partition function of a single M5 brane on transverse $A_{N-1}$ singularity). The resulting expression, equation (3.3.44), is given an interpretation as a domain wall for the theory of M2 branes on $\mathbb{R} \times T^2$ in presence of a transverse $A_{N-1}$ singularity. In Section 3.3.3 we glue together the contributions from the $M$ different strips geometries that the toric Calabi-Yau is built out of to obtain the partition function of our system of $M$ M5 branes on transverse $A_{N-1}$ singularities, normalized by the $M$-th power of the partition function of a single M5-brane, expressed as a sum of self-dual string contributions. This is the main computational result presented in this chapter, and is given in equation (3.3.73). We also comment on the manifest modular properties of the partition function. Finally, in Section
3.3.4 we discuss other approaches for directly computing the elliptic genus of the strings: either by studying the appropriate bundles over the moduli space of instantons (3.2.20), or by computing the 2d index of the (4,0) quiver gauge theory of Section 3.2.5.

3.3.1 Periodic strip partition function from curve counting

The relevant geometry to compute the topological string partition function for M-strings at $A_{N-1}$ singularities is the partial compactification of the so-called strip geometry, replacing the resolved conifold geometry of the original M-stings setup. The length of the strip is determined by $N$; more specifically, $N$ is the number of external legs on each side of the strip. In Chapter 2 the refined topological vertex was employed to compute the topological string amplitudes. The recursive method used there can be employed in the present case as well; however, the computations get very cumbersome, even for the $A_1$ singularity. Instead we follow a more intuitive approach based on an observation of [88].

Let us briefly review the observation of [88]: the topological partition function for the partial compactification of the resolved conifold can be computed by counting the holomorphic maps in an infinite, but periodic, strip geometry. The Newton polygon of the resolved conifold is depicted in Figure 3.9(a) and is obviously planar. However, the Newton polygon of the partial compactification of the resolved conifold is non-planar and lives on a cylinder. In the covering space of the cylinder it can be represented as a periodic configuration. The holomorphic curves wrap the compact part of the geometry which consist of an infinite chain of $\mathbb{P}^1$'s.

A holomorphic curve $C$ satisfies $C \cdot C = 2g - 2$ for $g \geq 0$, where $g$ is the genus of the curve $C$. Two rational curves $C_1$ and $C_2$ with vanishing intersection, $C_1 \cdot C_2 = 0$, do not form a holomorphic curve $C_1 + C_2$, since $(C_1 + C_2) \cdot (C_1 + C_2) = -4$. In other words, if $C_1$ and $C_2$ are not connected $C_1 + C_2$ is not holomorphic. However, if $C_1$ and $C_2$ have the intersection number 1, $C_1 + C_2$ is a holomorphic curve of genus zero, since then $(C_1 + C_2) \cdot (C_1 + C_2) = -2$. From this discussion we can conclude that the individual $\mathbb{P}^1$'s and any connected chain of them contribute to
Figure 3.9: (a) The Newton polygon for the resolved conifold, and (b) the cover of the Newton polygon after partially compactification of the resolved conifold along the horizontal edges.

the A-model topological string partition function. We need to identify all possible such curves.

In our case, the conifold is replaced with the strip geometry and we need to consider this simple building block as one of the periods of the Newton polygon. In contrast to [88], all the external legs are labelled by Young diagrams, and the resulting open topological string partition function corresponds to the “domain walls” for self-dual strings. It turns out that the detailed understanding of the strip geometry with two external legs is enough to construct the partition function for the infinite strip. The partition function for such a strip, shown in Figure 3.10, is given by

\[
Z_{\mu_1 \mu_2}^{\nu_1 \nu_2} = q^{-\frac{\|\mu_1\|^2 + \|\mu_2\|^2}{2}} t^{-\frac{\|\nu_1\|^2 + \|\nu_2\|^2}{2}} \tilde{Z}_{\mu_1^2}(t^{-1}, q^{-1}) \tilde{Z}_{\nu_2^2}(t^{-1}, q^{-1}) \tilde{Z}_{\nu_1}(q^{-1}, t^{-1}) \tilde{Z}_{\nu_2}(q^{-1}, t^{-1}) \times \prod_{i,j=1}^{\infty} \frac{(1 - Q_A t^{\mu_2, i-j+1/2} q^{\nu_2, j-i+1/2})(1 - Q_B t^{\nu_2, i-j+1/2} q^{\mu_2, j-i+1/2})}{(1 - Q_A Q_B t^{\mu_2, i-j} q^{\mu_2, j-i+1})} \times \frac{(1 - Q_C t^{\mu_1, i-j+1/2} q^{\nu_1, j-i+1/2})(1 - Q_A Q_B Q_C t^{\mu_2, i-j+1/2} q^{\nu_2, j-i+1/2})}{(1 - Q_B Q_C t^{\nu_2, i-j+1} q^{\nu_2, j-i})},
\]

(3.3.23)

where we have used \(\|\mu\|^2 = \sum_{i=1}^{\ell_\mu} \mu_i^2\) and the specialization of the Macdonald polynomial

\[
\tilde{Z}_\nu(t^{-1}, q^{-1}) = \prod_{(i,j) \in \nu} \left(1 - q^{i-j} t^{i-j-1}\right)^{-1}.
\]

The contributions coming from \(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \mapsto \mathbb{P}^1\) curves are easy to determine and all have the same form. Likewise, one can easily find the contribution of the curves of type \(\mathcal{O}(-2) \oplus\)
Figure 3.10: The basic building block to compute the topological string partition function for the periodic strip. The small double lines (blue) denote the choice of the preferred direction of the refined topological vertex.

\[ O(0) \mapsto \mathbb{P}^1 \] (labelled by \( \mu_1 \) and \( \mu_2 \)) and \( O(0) \oplus O(-2) \mapsto \mathbb{P}^1 \) (labelled by \( \nu_1 \) and \( \nu_2 \)). Before spelling out the partition function relevant for the \( A_{N-1} \) singularity, let us demonstrate our derivation for the \( A_1 \) singularity depicted in Figure 3.11; the generalization will be obvious.

\[
O(-1) \oplus O(-1) \mapsto \mathbb{P}^1
\]

The curves which belong to this class are labelled by \((\mu_a, \nu_b)\) or \((\nu_a, \mu_b)^4\). We need to take into account all holomorphic curves from a partition \( \mu_a (\nu_a) \) to another partition \( \nu_b (\mu_b) \). Its contribution to the partition function has the following form for \((\mu_a, \nu_b)\)

\[
\prod_{i,j=1}^{\infty} (1 - Q_{ab} \ell^{\mu_a,-j+1/2}q^{\nu_b,i-1/2}),
\]

(3.3.24)

where \( Q_{ab} \) denotes the corresponding Kähler parameter of the class spanned between the two

\(^4\)The first partition is always taken to be lower than the second partition in the toric diagram.
Figure 3.11: The (periodic) toric diagram with a basic strip of “length” $N = 2$ used in computing the M5 brane partition function in the presence of a transverse $A_1$ singularity. The small double lines (blue) denote the choice of the preferred direction of the refined topological vertex.

partitions; we will later give explicit expressions for them. Note that we get infinitely many contributions since the strip is periodic for each pair partitions $(\mu_a, \nu_b)$ and $(\nu_a, \mu_b)$. In the case of the $A_1$ singularity, the following curves contribute:

$$
\begin{align*}
(\mu_1, \nu_1; Q_1), & \quad (\mu_1, \nu_2; Q_1 Q_2), \quad (\mu_2, \nu_1; Q_2 Q_1), \quad (\mu_2, \nu_2; Q_2), \\
(\nu_1, \mu_1; Q_1^{-1} Q_2), & \quad (\nu_1, \mu_2; Q_2^{-1} Q_1), \quad (\nu_2, \mu_1; Q_1^{-1} Q_2), \quad (\nu_2, \mu_2; Q_2^{-1} Q_1).
\end{align*}
$$

(3.3.25)

Here, $Q_\tau \equiv Q_{\tau_1} Q_{\tau_2}$\(^5\) is the Kähler parameter associated to the elliptic fiber. Therefore we have the

\(^5\)Let us make a remark about our notation. In the present case, the geometry possesses more Kähler parameters than in the case considered in Chapter 2. In that case (where $N = 1$) we have $Q_1 \equiv Q_m$, which corresponds to the adjoint mass of the 5d $N = 2^*$ theory. When $N \neq 1$, the $Q_i$’s are indirectly related to the bifundamental hypermultiplet masses. We also have $N$ parameters $Q_{\tau_1}, \ldots, Q_{\tau_N}$ which are related to the gauge theory coupling constants of the corresponding nodes of the quiver; they satisfy $Q_{\tau_1} \cdots Q_{\tau_N} = Q_\tau$. 

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following infinite products
\[
\prod_{k=1}^{\infty} \prod_{i,j=1}^{\infty} \left( 1 - Q_1 Q_2^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right) \left( 1 - Q_1 Q_2^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right) \\
\times \left( 1 - Q_2 Q_1^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right) \left( 1 - Q_2 Q_1^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right) \\
\times \left( 1 - Q_1 Q_2^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right) \left( 1 - Q_2 Q_1^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right) \\
\times \left( 1 - Q_2 Q_1^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right) \left( 1 - Q_2 Q_1^{k-1} \mu_{i,j} q_{i,j}^{i+1/2} \right), \tag{3.3.26}
\]
where we have included the factors $Q_{k}^{k-1}$ reflecting the periodicity of the Newton polygon, i.e., all the other curves in addition to the initial ones listed in Eq.(3.3.25).

\[
\mathcal{O}(-2) \oplus \mathcal{O}(0) \mapsto \mathbb{P}^1
\]

As mentioned before these are the curves labelled by $(\mu_a, \mu_b)$ and their contributions to the partition function can be obtained from Eq.(3.3.23):
\[
\prod_{i,j=1}^{\infty} \frac{1}{(1 - Q_{ab} \mu_{i,j} q_{i,j}^{i+1/2})}, \tag{3.3.27}
\]
with the appropriate Kähler factors $Q_{ab}$. This class includes the following curves
\[
(\mu_1, \mu_1; Q_{\tau}), \quad (\mu_1, \mu_2; Q_{\tau}'), \quad (\mu_2, \mu_1; Q_{\lambda}), \quad (\mu_2, \mu_2; Q_{\lambda}). \tag{3.3.28}
\]
We can immediately determine the corresponding amplitudes for these curves:
\[
\prod_{k=1}^{\infty} \left( 1 - Q_1^{k} \mu_{i,j} q_{i,j}^{i+1/2} \right)^{-1} \left( 1 - Q_2^{k} \mu_{i,j} q_{i,j}^{i+1/2} \right)^{-1} \\
\times \left( 1 - Q_1^{k} \mu_{i,j} q_{i,j}^{i+1/2} \right)^{-1} \left( 1 - Q_2^{k} \mu_{i,j} q_{i,j}^{i+1/2} \right)^{-1}, \tag{3.3.29}
\]
\[
\mathcal{O}(0) \oplus \mathcal{O}(-2) \mapsto \mathbb{P}^1
\]
These curves are labeled by $(\nu_a, \nu_b)$ and their contribution is close to the ones of the
\( \mathcal{O}(-2) \oplus \mathcal{O}(0) \mapsto \mathbb{P}^1 \) curves, except for a slightly different dependence on \( q, t \):
\[
\prod_{i,j=1}^{\infty} \frac{1}{1 - Q_{ab} t^{\nu_{a,i} - j + 1} q^{\nu_{b,j} - i}}. \tag{3.3.30}
\]

Let us again list the curves in the present case:
\[
(\nu_1, \nu_1; Q_\tau), \quad \quad (\nu_1, \nu_2; Q_{\tau_2}), \quad \quad (\nu_2, \nu_1; Q_{\tau_1}), \quad \quad (\nu_2, \nu_2; Q_\tau). \tag{3.3.31}
\]

The partition function will also include the following factors:
\[
\prod_{k=1}^{\infty} \left( 1 - Q_k^\tau t^{\nu_{a,i} - j + 1} q^{\nu_{b,j} - i} \right)^{-1} \left( 1 - Q_{\tau_2}^k t^{\nu_{a,i} - j + 1} q^{\nu_{b,j} - i} \right)^{-1} \\
\times \left( 1 - Q_{\tau_1}^k t^{\nu_{a,i} - j + 1} q^{\nu_{b,j} - i} \right)^{-1} \left( 1 - Q_{\tau}^k t^{\nu_{a,i} - j + 1} q^{\nu_{b,j} - i} \right)^{-1}. \tag{3.3.32}
\]

All the factors we obtain using the above approach match the explicit calculation we performed following the methods of Chapter 2 up to factors of \( \eta(\tau)^{-1} \).

\[ A_{N-1} \text{ Singularity} \]

From the above discussion the generalization for the \( A_{N-1} \) singularity is immediate. The numerator will have the form
\[
\prod_{a,b=1}^{\infty} \left( 1 - Q_{ab}^k t^{\nu_{a,i} - j + 1/2} q^{\nu_{b,j} - i + 1/2} \right) \left( 1 - Q_{\bar{a} \bar{b}}^k t^{\nu_{a,i} - j + 1/2} q^{\nu_{b,j} - i + 1/2} \right), \tag{3.3.33}
\]
where
\[
Q_{ab} \bar{Q}_{ba} = Q_\tau, \tag{3.3.34}
\]
and we define \( Q_\tau \equiv \prod_{i=1}^{N} Q_{\tau_i} \) for the \( A_{N-1} \) singularity. The last equality has a simple explanation: \( Q_{ab} \) and \( \bar{Q}_{ba} \) are defined on the basic strip, c.f. Figure 3.11. The geometry we are interested in is the partial compactification of this basic geometry. The parameter \( Q_{ab} \) measures the distance between partitions \( \mu_a \) and \( \nu_b \), and \( \bar{Q}_{ba} \) measures the distance between partitions \( \nu_b \) and \( \mu_a \). Together they add up to the circumference of the cylinder the Newton polygon is wrapped on. We will label the
Figure 3.12: The pictorial representation of the Kähler parameters used in the partition function.

Kähler classes for the \((\nu_{a+1}, \nu_a)\) and \((\mu_{a+1}, \mu_a)\) curves by \(Q_{\nu_a}\), for \(a = 1, \ldots, N - 1\). The class for curves \((\nu_1, \nu_N)\) and \((\mu_1, \mu_N)\) is denoted by \(Q_{\nu_N}\) (depicted in Figure 3.11). With this definition, \(Q_{ab}\) can be written as

\[
Q_{ab} = \begin{cases} 
Q_a \prod_{j=b}^{N} Q_{\tau_j}, & (\text{mod } Q_\tau) \quad \text{for } a = 1, \\
Q_a \prod_{i=1}^{a-1} Q_{\nu_i} \prod_{j=b}^{N} Q_{\tau_j}, & (\text{mod } Q_\tau) \quad \text{for } a \neq 1,
\end{cases} 
\]

(3.3.35)

where \(\text{(mod } Q_\tau\text{)}\) means that any \(Q_\tau\) appearing in the definition of \(Q_{ab}\) is set to 1. As an example, we put \(Q_{ab}\)'s for \(N = 4\) in a matrix:

\[
Q_{ab} = \begin{pmatrix}
Q_1 & Q_1 Q_{\tau_2} Q_{\tau_3} Q_{\tau_4} & Q_1 Q_{\tau_3} Q_{\tau_4} & Q_1 Q_{\tau_4} \\
Q_2 Q_{\tau_1} & Q_2 & Q_2 Q_{\tau_1} Q_{\tau_3} Q_{\tau_4} & Q_2 Q_{\tau_1} Q_{\tau_4} \\
Q_3 Q_{\tau_1} Q_{\tau_2} & Q_3 Q_{\tau_2} & Q_3 & Q_3 Q_{\tau_1} Q_{\tau_2} Q_{\tau_4} \\
Q_4 Q_{\tau_1} Q_{\tau_2} Q_{\tau_3} & Q_4 Q_{\tau_2} Q_{\tau_3} & Q_4 Q_{\tau_3} & Q_4
\end{pmatrix}.
\]

(3.3.36)

Using Eq. (3.3.34), the numerator can be written in terms of \(Q_{ab}\) only.
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\[ \prod_{a,b=1}^{N} \prod_{i,j,k=1}^{\infty} (1 - Q_{\tau}^{-1} Q_{ab}^{-1} Q_{a,b}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}) (1 - Q_{\tau}^{-1} Q_{ab}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}). \] (3.3.37)

The denominator of the partition function for the $A_{N-1}$ singularity has the form

\[ \prod_{a,b=1}^{N} \prod_{i,j,k=1}^{\infty} (1 - Q_{\tau}^{-1} Q_{ab}^{-1} Q_{a,b}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}) (1 - Q_{\tau}^{-1} Q_{ab}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}) \] (3.3.38)

where the Kähler parameters $Q_{ab}$ are defined as follows:

\[ \tilde{Q}_{ab} = \begin{cases} 
\prod_{i=b}^{a-1} Q_{\tau_{i}}, & \text{for } a > b, \\
Q_{\tau}, & \text{for } a = b, \\
Q_{\tau}/ \prod_{i=a}^{b-1} Q_{\tau_{i}}, & \text{for } a < b,
\end{cases} \] (3.3.39)

and $\tilde{Q}_{ab}'$'s are defined by replacing $Q_{\tau_{i}}$'s by $Q_{\tau_{i}}'$. There is a simple relation between $Q_{ab}$ and $\tilde{Q}_{ab}$:

\[ \tilde{Q}_{ab} = \frac{Q_{a}}{Q_{b}} \tilde{Q}_{ab}. \] (3.3.40)

We prefer to use this somewhat redundant notation, since it will prove convenient later in our discussion. The constraints we need to impose on the Kähler parameters of different strips will be more transparent in this convention. Combining the contributions from numerator and denominator, we see that the partition function for the self-dual strings is constructed out of the following infinite products:

\[ \prod_{a,b=1}^{N} \prod_{i,j,k=1}^{\infty} (1 - Q_{\tau}^{-1} Q_{ab}^{-1} Q_{a,b}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}) (1 - Q_{\tau}^{-1} Q_{ab}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}) \] (3.3.41)

The partition function of the $U(1)$ partition function that lives on a single M5 brane can be computed from the above expression by setting all the Young diagrams to be trivial, or in other words considering the closed amplitude

\[ Z_{U(1)} = \prod_{a,b=1}^{N} \prod_{i,j,k=1}^{\infty} (1 - Q_{\tau}^{-1} Q_{ab}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}) (1 - Q_{\tau}^{-1} Q_{ab}^{-1} q^{\nu_{a,b,i}^{j} + 1/2 q^{\nu_{a,b,j}^{i} - i + 1/2}}) \] (3.3.42)

where the all the prefactors are trivial.
3.3.2 Domain wall partition function

We are interested in the partition functions of M-strings on $A_{N-1}$ singularity. Therefore, the contributions from single M5 branes, $Z_{U(1)}$’s, need to be factored out. We normalize the open topological partition function by the closed one using the following identity:

$$\prod_{i,j=1}^{\infty} \frac{1 - Q q^{-j} \mu_1^{i} - i + 1}{1 - Q q^{-j} t^{-i + 1}} = \prod_{(i,j) \in \nu} \left(1 - Q q^{\mu_i - j} \mu_j^{i + 1}\right) \prod_{(i,j) \in \mu} \left(1 - Q q^{-\mu_i + j - 1} t^{-\nu_j^{i + 1}}\right). \quad (3.3.43)$$

As in Chapter 2, we can define the partition function of a domain wall from the normalized topological string partition function:

$$D_{\nu_1 \cdots \nu_N}^{\mu_1 \cdots \mu_N}(Q, a, b; \bar{Q}_{ab}, \bar{Q}_{ab}'; \epsilon_1, \epsilon_2) \equiv N \prod_{a=1}^{N} q^{-\frac{\|\nu_a\|^2}{2}} \bar{Z}_{\mu_a}(t^{-1}, q^{-1}) t^{-\frac{\|\nu_a\|^2}{2}} \bar{Z}_{\nu_a}(q^{-1}, t^{-1})$$

$$\times \prod_{a,b=1}^{N} \prod_{k=1}^{\infty} \prod_{(i,j) \in \mu_a} \left(1 - Q^{k-1} Q_{ab} t^{\mu_a,i,j} q^{-\nu_{a,j}^{i+1}/2}\right) \left(1 - Q^{k-1} Q_{ab'} t^{\mu_a,j} q^{\nu_{a,j+1}/2}\right)$$

$$\times \prod_{(i,j) \in \nu_b} \left(1 - Q^{k-1} Q_{ab} t^{-\nu_{b,i}^{j+1}/2} q^{\mu_{b,j}^{i+1}/2}\right) \left(1 - Q^{k-1} Q_{ab'} t^{-\nu_{b,j}^{i+1}/2} q^{\mu_{b,j+1}/2}\right). \quad (3.3.44)$$

Note that in this expression we have restored the factors of $\bar{Z}_{\mu}(t^{-1}, q^{-1})$ that were left out of the previous discussion. The domain wall defined in Chapter 2 is just the special case of the new domain wall for $N = 1$. For general $N$, we give the following interpretation for the quantity defined in (3.3.44): the ground states of the theory of $k$ M2 branes on flat transverse space on $T^2$, taking the size of $T^2$ to be much smaller than the length between the M5 branes (when the scalars are massed up), are labeled by a single Young diagram of size $k$ \cite{46–49}. Here we have a situation where the transverse space to the M2 branes has an $A_{N-1}$ singularity. To describe the M2 branes in this geometry we need to place $N$ copies of them before orbifolding the flat transverse space. In particular this implies that the configuration of ground states of M2 branes is characterized by $N$ Young diagrams with total number of boxes $k$. From this viewpoint the low energy modes of this system are given by a quantum mechanical system, where the Hilbert space is formed by an $N$-tuple of Young diagrams

$$\bar{\mu} = (\mu_1, \cdots, \mu_N), \quad (3.3.45)$$
with the identity operator $I = \sum_{\mu} |\vec{\mu}\rangle\langle\vec{\mu}|$ and Hamiltonian $H = |\vec{\mu}|$. The Hamiltonian can again be interpreted as M2 brane mass where the size of $T^2$ times the tension of the M2 brane have been normalized to 1 and $|\vec{\mu}|$ is their number. The domain wall arises by having the M2 branes ending on M5 branes on either side of it. Thus we can view the M5 brane as an operator acting from the left vacua, labeled by $N$ partitions to right vacua, again labeled by $N$ partitions. In other words, Equation (3.3.44) gives the matrix elements of this operator for this quantum mechanical system.

### 3.3.3 Partition function of M5 branes on transverse $A_{N-1}$ singularity

In this section we assemble the contributions from the different strips that compose the toric geometry derived from Figure 3.3, and we arrive at an expression for the refined topological string partition function corresponding to it (Equation (3.3.73)). As discussed in Section 3.2.4, this is also the partition function of the system of $M$ M5 branes on transverse $A_{N-1}$ singularity. More precisely, the partition function we compute is normalized by the contributions of the BPS states that do not arise from M2 branes stretching between the M5 branes (although these factors can be easily restored); our final expression is organized as a sum of contributions from different numbers of self-dual strings wrapping the torus in the worldvolume of the M5 branes.

In Chapter 2, the normalized topological string partition function was recast in terms of the Jacobi theta function

$$\theta_1(\tau; z) = -ie^{i\pi \tau/4}e^{iz} \prod_{k=1}^{\infty} (1-e^{2\pi i k \tau})(1-e^{2\pi i k \tau}e^{2\pi iz})(1-e^{2\pi i (k-1) \tau}e^{-2\pi iz}). \tag{3.3.46}$$

We will show that in the present, more general setup, the partition function can still be expressed in terms of theta functions. We first need to glue the building blocks together, as in Figure 3.13.

The topological string partition functions in the presence of $M$ parallel M5 branes can be
computed using the domain walls:

\[
Z_{M}^{\Delta N-1} = \sum_{\{\bar{\mu}^{(p)}\}_{p=1}^{M-1}} \left( \prod_{a=1}^{M-1} \prod_{\mu^{(1)}_{a} \ldots \mu^{(N)}_{a}} \left( -Q_{f,a}^{(s)} \right)^{[\mu^{(s)}_{a}]} \right) \times D^{\emptyset, \emptyset}_{\mu^{(1)}_{1} \ldots \mu^{(N)}_{1}}(Q_{\tau}, Q_{ab}^{(1)}, Q_{ab}^{(1)}; \epsilon_{1}, \epsilon_{2}) \\
\times D^{\mu^{(1)}_{2} \ldots \mu^{(N)}_{2}}(Q_{\tau}, Q_{ab}^{(2)}, Q_{ab}^{(2)}; \epsilon_{1}, \epsilon_{2}) \times \ldots \\
\times D^{\mu^{(M-1)}_{M-1} \ldots \mu^{(M-1)}_{N}}(Q_{\tau}, Q_{ab}^{(M)}, Q_{ab}^{(M)}; \epsilon_{1}, \epsilon_{2}),
\]  

(3.3.47)

where \(\{\bar{\mu}^{(p)}\}\) denotes \(N\)-tuples of Young diagrams associated with gluing of the \(p^{th}\) and \((p+1)^{st}\) domain walls. The Kähler parameters \(Q_{ab}^{(p)}, \tilde{Q}_{ab}^{(p)}\) and \(\bar{Q}_{ab}^{(p)}\) are associated to the \(p^{th}\) domain wall whereas \(Q_{f,a}\) denote the Kähler parameters along the fiber directions between the \(p^{th}\) and the \((p+1)^{st}\) domain wall. Let us focus on the contributions from the holomorphic maps which involve factors depending on \(\bar{\mu}^{(p)}\). These are
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\[
\prod_{a,b=1}^{N} \prod_{k=1}^{\infty} \prod_{(i,j) \in \mu_{b}^{(p)}} \left( 1 - Q_{r}^{k-1} Q_{ab}^{(p)} t_{r}^{\mu_{a,i} - \mu_{b,j} + j + 1/2} q_{r}^{-\mu_{a,i} + (p-1) \cdot t_{r} - i + 1/2} \right) \left( 1 - Q_{r}^{k-1} Q_{ab}^{(p)} t_{r}^{\mu_{a,i} - \mu_{b,j} + j + 1/2} q_{r}^{-\mu_{a,i} + (p-1) \cdot t_{r} - i + 1/2} \right)
\]

from the \( p^{th} \) domain wall

\[
\times \prod_{a,b=1}^{N} \prod_{k=1}^{\infty} \prod_{(i,j) \in \mu_{a}^{(p)}} \left( 1 - Q_{r}^{k-1} Q_{ab}^{(p+1)} t_{r}^{\mu_{a,i} - \mu_{b,j} + j + 1/2} q_{r}^{-\mu_{a,i} + (p+1) \cdot t_{r} - i + 1/2} \right) \left( 1 - Q_{r}^{k-1} Q_{ab}^{(p+1)} t_{r}^{\mu_{a,i} - \mu_{b,j} + j + 1/2} q_{r}^{-\mu_{a,i} + (p+1) \cdot t_{r} - i + 1/2} \right)
\]

from the \((p + 1)^{st}\) domain wall (3.3.48)

and arise from the factors of

\[
D_{\mu_{1}^{(p-1)} \ldots \mu_{N}^{(p-1)}}^{(p-1)} \times D_{\mu_{1}^{(p+1)} \ldots \mu_{N}^{(p+1)}}^{(p+1)}
\]

appearing in (3.3.47).

To compute the topological string partition function of the toric geometry we are required to glue together the \( M \) vertical strips along the horizontal edges. This imposes the constraint

\[
\bar{Q}_{ab}^{(p+1)} = \bar{Q}_{ab}^{(p)} \quad (3.3.49)
\]

which is easy to see from the toric diagrams. This is equivalent to imposing

\[
Q_{\tau_{a}}^{(p+1)} = Q_{\tau_{a}}^{(p)} \quad (3.3.50)
\]

After imposing the gluing restrictions, using the fact that \( a \) and \( b \) are dummy variables, we see that
the $\tilde{\mu}^{(p)}$-dependent terms can be written as

\[
\prod_{a,b=1}^{N} \prod_{(i,j) \in \mu_a^{(p)}} \prod_{k=1}^{\infty} (1 - Q_{a,b}^{(p)} t^{-\mu_{a,i}^{(p)} - j - 1/2} q^{-\mu_{b,j}^{(p)} - i - 1/2}) (1 - Q_{b,a}^{(p)} t^{-\mu_{a,i}^{(p)} - j - 1/2} q^{-\mu_{b,j}^{(p)} - i - 1/2})
\]

\[
\times (1 - Q_{a,b}^{(p+1)} t^{-\mu_{a,i}^{(p)} - j - 1/2} q^{-\mu_{b,j}^{(p)} - i - 1/2}) (1 - Q_{b,a}^{(p+1)} t^{-\mu_{a,i}^{(p)} - j - 1/2} q^{-\mu_{b,j}^{(p)} - i - 1/2})
\]

\[
\times (1 - Q_{a,b}^{(p)} t^{-\mu_{a,i}^{(p)} - j - 1/2} q^{-\mu_{b,j}^{(p)} - i - 1/2}) (1 - Q_{b,a}^{(p)} t^{-\mu_{a,i}^{(p)} - j - 1/2} q^{-\mu_{b,j}^{(p)} - i - 1/2})
\]

(3.3.51)

It is clear that numerator can be rewritten in terms of two theta functions:

\[
\text{Numerator} = A^{(p)} \prod_{a,b=1}^{N} \prod_{(i,j) \in \mu_a^{(p)}} \theta_1(\tau; z_{ab}^{(p)}(i,j)) \theta_1(\tau; w_{ab}^{(p)}(i,j)),
\]

(3.3.52)

where we have defined the arguments of the theta functions $z_{ab}^{(p)}(i,j)$ and $w_{ab}^{(p)}(i,j)$ as

\[
e^{2\pi i z_{ab}^{(p)}(i,j)} = (Q_{ab}^{(p+1)})^{-1} t^{-\mu_{a,i}^{(p)} - j + 1/2} q^{-\mu_{b,j}^{(p)} - i - 1/2},
\]

(3.3.53)

\[
e^{2\pi i w_{ab}^{(p)}(i,j)} = (Q_{ba}^{(p)})^{-1} t^{-\mu_{a,i}^{(p)} + j + 1/2} q^{-\mu_{b,j}^{(p)} - i + 1/2},
\]

(3.3.54)

and

\[
A^{(p)} = \prod_{a=1}^{N} \left[ -e^{\pi i \tau/2} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}) \right]^{-N|\mu_a^{(p)}|} \prod_{b=1}^{N} \prod_{(i,j) \in \mu_a^{(p)}} e^{-\pi i (z_{ab}^{(p)}(i,j) + w_{ab}^{(p)}(i,j))}.
\]

(3.3.55)

Let us now turn to the factors in the denominator of equation (3.3.51). First of all, the factors for which $a = b$ combine with the prefactors

\[
q^{-|\mu_a^{(p)}| t^{1/2}} \prod_{(i,j) \in \mu_a^{(p)}} \frac{1}{1 - q^{\mu_{a,j}^{(p)}} t^{-\mu_{a,j}^{(p)}} q^{\mu_{a,j}^{(p)}}}
\]

(3.3.56)

to give

\[
\left( -\sqrt{\frac{t}{q}} \right)^{\sum_a |\mu_a^{(p)}|} \prod_{a=1}^{N} \prod_{k=1}^{\infty} \prod_{(i,j) \in \mu_a^{(p)}} \left( 1 - Q_{\tau}^{k} t^{\mu_{a,i}^{(p)} - j - \mu_{a,j}^{(p)} - i + 1} \right) \left( 1 - Q_{\tau}^{k} t^{-\mu_{a,i}^{(p)} - j + 1} q^{\mu_{a,j}^{(p)} - i} \right)
\]

\[
\times \left( 1 - Q_{\tau}^{k} t^{\mu_{a,i}^{(p)} - j - 1} q^{\mu_{a,j}^{(p)} - i} \right) \left( 1 - Q_{\tau}^{k} t^{-\mu_{a,i}^{(p)} - j + 1} q^{\mu_{a,j}^{(p)} - i} \right)
\]

(3.3.57)
When \( a \neq b \), we will need the following identity which follows from the definition of \( \tilde{Q}_{ab} \):

\[
\tilde{Q}_{ab} Q_{ba} = Q_\tau. \tag{3.3.58}
\]

This allows us to write the denominator terms for \( a \neq b \) as

\[
\prod_{a \neq b} \prod_{k=1}^N \prod_{(i,j) \in \mu_a^{(p)}} \frac{1}{1 - Q_\tau^k \left( \tilde{Q}_{ba}^{(p)} \right)^{-1} t^{\mu_{a,i}^{(p)} - j^a q_{b,j}^{(p)}, t - i + 1} \left( 1 - Q_\tau^k \left( \tilde{Q}_{ba}^{(p)} \right)^{-1} t^{-\mu_{a,i}^{(p)} + j - 1} - q^{-\mu_{b,j}^{(p)}, t + i} \right)}
\times \frac{1}{(1 - Q_\tau^{k-1} \tilde{Q}_{ab}^{(p)} t^{\mu_{a,i}^{(p)} - j + 1} q_{b,j}^{(p)}, t - i)(1 - Q_\tau^{k-1} \tilde{Q}_{ab}^{(p)} t^{-\mu_{a,i}^{(p)} + j} - q^{-\mu_{b,j}^{(p)}, t + i - 1})}. \tag{3.3.59}
\]

If we now define a new variable \( \tilde{Q}_{ab}^{(p)} \) by

\[
\tilde{Q}_{ab}^{(p)} = \begin{cases} 
1, & \text{for } a = b \\
\tilde{Q}_{ab}^{(p)}, & \text{for } a \neq b
\end{cases}
\]

we can write the product of equations (3.3.57) and (3.3.59) as

\[
B^{(p)} = \prod_{a,b=1}^N \prod_{(i,j) \in \mu_a^{(p)}} \theta_1(\tau; u_{ab}^{(p)}(i,j)) \theta_1(\tau; v_{ab}^{(s)}(i,j)), \tag{3.3.61}
\]

where

\[
e^{2\pi i u_{ab}^{(p)}(i,j)} \equiv \left( \tilde{Q}_{ba}^{(p)} \right)^{-1} t^{-\mu_{a,i}^{(p)} - j^a q_{b,j}^{(p)}, t + i - 1}
\]

\[
e^{2\pi i v_{ab}^{(p)}(i,j)} \equiv \left( \tilde{Q}_{ab}^{(p)} \right)^{-1} t^{-\mu_{a,i}^{(p)} + j} - q^{-\mu_{b,j}^{(p)}, t + i} \tag{3.3.62}
\]

and

\[
B^{(p)} = \prod_{a=1}^N \left( -\sqrt{q} \right)^{[\mu_a^{(p)}]} \left[ -e^{\pi i \tau/2} \prod_{k=1}^\infty (1 - e^{2\pi i k \tau})^2 \right]^{N[\mu_a^{(p)}]} \prod_{b=1}^N \prod_{(i,j) \in \mu_a^{(p)}} e^{\pi i u_{ab}^{(p)}(i,j)+\pi i v_{ab}^{(p)}(i,j)}. \tag{3.3.63}
\]

Therefore equation (3.3.47) simplifies to the following expression:

\[
Z_M^{-1} = \sum_{\{\tilde{\mu}^{(p)}\}} M^{-1} \prod_{s=1}^{M-1} C^{(s)} \prod_{a=1}^N \left( -Q_{f,a}^{(s)} \right)^{[\mu_a^{(s)}]} \prod_{(i,j) \in \mu_a^{(s)}} \prod_{b=1}^N \prod_{(i,j) \in \mu_a^{(s)}} \theta_1(\tau; z_{ab}^{(s)}(i,j)) \theta_1(\tau; u_{ab}^{(s)}(i,j)) \theta_1(\tau; v_{ab}^{(s)}(i,j)).
\]

It remains to simplify the prefactor

\[
C^{(p)} = \prod_{a=1}^N \left( -\sqrt{q} \right)^{[\mu_a^{(p)}]} \prod_{b=1}^N \prod_{(i,j) \in \mu_a^{(p)}} e^{-\pi i z_{ab}^{(p)}(i,j) - w_{ab}^{(p)}(i,j)} e^{\pi i u_{ab}^{(p)}(i,j)+v_{ab}^{(p)}(i,j)}. \tag{3.3.64}
\]
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First of all, we have
\[
\prod_{a,b=1}^{N} \prod_{(i,j) \in \mu_a^{(p)}} e^{-\pi i u_{ab}^{(p)}(i,j) - v_{ab}^{(p)}(i,j)} = \left( \prod_{a,b=1}^{N} \prod_{(i,j) \in \mu_a^{(p)}} Q_{ab}^{(p+1)} Q_{ba}^{(p)} q^{\rho_{b,j}^{(p+1),t} - \rho_{b,j}^{(p-1),t}} \right)^{1/2}. \tag{3.3.65}
\]

We can simplify this expression by noting that
\[
Q_{ab}^{(p+1)} Q_{ba}^{(p)} = \left\{ \begin{array}{ll}
Q_b^{(p)} Q_b^{(p+1)}, & \text{for } a = b, \\
Q_b^{(p)} Q_b^{(p+1)}, & \text{for } a \neq b,
\end{array} \right. \tag{3.3.66}
\]

so that
\[
\left( \prod_{a,b=1}^{N} \prod_{(i,j) \in \mu_a^{(p)}} Q_{ab}^{(p+1)} Q_{ba}^{(p)} \right)^{1/2} = \prod_{a=1}^{N} Q^{-N-1|\mu_a^{(p)}|}_a \prod_{b=1}^{N} \left( Q_b^{(p+1)} Q_b^{(p)} \right)^{\frac{1}{2}|\mu_a^{(p)}|}. \tag{3.3.67}
\]

Furthermore, it turns out that the \(q\)-dependent terms in equation (3.3.67) all cancel. To see this, let us isolate the factors associated to the \(p^{th}\) and \((p+1)^{st}\) four-cycles:
\[
\prod_{(i,j) \in \mu_a^{(p)}} q^{\rho_{b,j}^{(p+1),t} - \rho_{b,j}^{(p-1),t}} \prod_{(i,j) \in \mu_a^{(p+1)}} q^{\rho_{b,j}^{(p+2),t} - \rho_{b,j}^{(p),t}}. \tag{3.3.68}
\]

Using the identity \(\sum_{(i,j) \in \nu} \mu_j^t = \sum_{(i,j) \in \mu} \nu_j^t\), this simplifies to
\[
\prod_{(i,j) \in \mu_a^{(p)}} q^{-\mu_{b,j}^{(p-1),t}} \prod_{(i,j) \in \mu_a^{(p+1)}} q^{\mu_{b,j}^{(p+2),t}}. \tag{3.3.69}
\]

By applying this identity for each four-cycle, we can cancel all \(q\)-dependent factors in equation (3.3.67) against each other.

Likewise, one can show that
\[
\prod_{a,b=1}^{N} \prod_{(i,j) \in \mu_a^{(p)}} e^{\pi i u_{ab}^{(p)}(i,j) + \pi i v_{ab}^{(p)}(i,j)} = \prod_{a=1}^{N} \left( \frac{1}{q} \right)^{-N|\mu_a^{(p)}|} Q^{-N-1|\mu_a^{(p)}|}_a, \tag{3.3.70}
\]

and therefore
\[
C^{(p)} = \prod_{a=1}^{N} -\left( \frac{1}{q} \right)^{\frac{N-1}{2}} \prod_{b=1}^{N} \left( Q_b^{(p+1)} Q_b^{(p)} \right)^{1/2}^{\frac{1}{2}|\mu_a^{(p)}|}. \tag{3.3.71}
\]

Finally, we define
\[
\overline{Q}_{f,a}^{(p)} := e^{2\pi i f_{f,a}^{(p)}} = \left( \frac{q}{t} \right)^{\frac{N-1}{2}} Q_{f,a}^{(p)} \prod_{b=1}^{N} \left( Q_b^{(p+1)} Q_b^{(p)} \right)^{1/2} = \left( \frac{q}{t} \right)^{\frac{N-1}{2}} Q_{f,a}^{(p)} Q_m^{N}, \tag{3.3.71}
\]

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where we have set
\[ Q_m = e^{2\pi i m} = \left( \prod_{b=1}^{N} Q_b^{(p)} \right)^{\frac{1}{N}}. \quad (3.3.72) \]

Here we note that \( m \) corresponds to the physical mass parameter introduced in Section 3.2.1, and that its definition is in fact independent of the label \( p \). We obtain a very compact final expression for the partition function of \( M \) M5 branes on transverse \( A_{N-1} \) singularity:
\[
Z_{M}^{A_{N-1}} = \sum_{(\mu(p))_{p=1}^{M-1}} \prod_{s=1}^{M-1} \prod_{a=1}^{N} (Q_{f,a}^{(s)})^{\mu_a^{(s)}} \prod_{(i,j) \in \mu_a^{(s)}}^{(i,j)} \theta_1(\tau; z_{ab}^{(s)}(i,j)) \theta_1(\tau; w_{ab}^{(s)}(i,j)). \quad (3.3.73)
\]

Recall that the partition function \( Z_{M}^{A_{N-1}} \) is normalized by the partition functions of single M5 branes, \( Z_{U(1)}^{(p)} \). The Kähler parameters of each domain wall are different, therefore the overall normalization is by \( \prod_{p=1}^{M} Z_{U(1)}^{(p)} \). For convenience, let us collect here the following definitions which were given in the previous discussion:

\[
e^{2\pi i z_{ab}^{(p)}(i,j)} \equiv (Q_{ab}^{(p+1)})^{-1} i^{-\mu_{a,i}^{(p)} + j + 1/2} q^{-\mu_{b,j}^{(p+1)} i + 1/2},
\]
\[
e^{2\pi i w_{ab}^{(p)}(i,j)} \equiv (Q_{ba}^{(p)})^{-1} i^{-\mu_{a,i}^{(p)} - j + 1/2} q^{-\mu_{b,j}^{(p)} i + 1/2},
\]
\[
e^{2\pi i u_{ab}^{(p)}(i,j)} \equiv (Q_{ba}^{(p)})^{-1} i^{-\mu_{a,i}^{(p)} - j} q^{-\mu_{b,j}^{(p)} i + 1},
\]
\[
e^{2\pi i v_{ab}^{(p)}(i,j)} \equiv (Q_{ab}^{(s)})^{-1} i^{-\mu_{a,i}^{(p)} + j} q^{-\mu_{b,j}^{(p)} i + 1}.
\]

The Kähler parameters appearing here can all be expressed in terms of the parameters \( Q_m = e^{2\pi i m} \), \( Q_{\tau}^{(p)} = e^{2\pi i \tau} \), and \( Q_{\tau} = e^{2\pi i \tau} \) which have the interpretation of mass-rotation, \( SU(N) \) fugacities and elliptic parameter of the two-dimensional quiver theory discussed in section 3.2.5. Finally, we define the parameters
\[
\tilde{Q}_{f}^{(p)} = e^{2\pi i \tau} = \left( \prod_{a=1}^{N} \tilde{Q}_{f,a}^{(p)} \right)^{\frac{1}{N}}, \quad (3.3.74)
\]
which we identify with the tension of the self-dual strings, or equivalently the distances between the M5 branes. From the factor \( \prod_{a=1}^{N} (\tilde{Q}_{f,a}^{(p)})^{\mu_a^{(p)}} \) in the partition function (3.3.73) we can extract an overall factor
\[
(\tilde{Q}_{f}^{(p)})^{\sum_{a=1}^{N} \mu_a^{(p)}}, \quad (3.3.75)
\]
which in the quantum mechanical framework introduced in Section 3.3.2 is associated to the prop-
agator between the $p$-th and $(p + 1)$-st domain wall. The remaining factor will depend on the
individual sizes of partitions $\mu^{(p)}_1, \cdots, \mu^{(p)}_N$. These factors, which we henceforth denote by $R_{\tilde{\mu}^{(p)}}$, should combine with the product over Jacobi theta functions in such a way that the partition function (3.3.73) displays the expected modular properties. Let us discuss this in more detail.

The Jacobi theta function acquires a non-trivial phase under the modular transformation
\[
\theta_1(-1/\tau; z/\tau) = -i(-i\tau)^{1/2} \exp(\pi iz^2/\tau) \theta_1(\tau; z).
\]

This modular anomaly can be traced back to the appearance of the second Eisenstein series $E_2(\tau)$ in the following expression for $\theta_1(\tau, z)$:
\[
\theta_1(\tau, z) = \eta(\tau)^3 z \exp \left[ \sum_{k \geq 1} \frac{B_{2k}}{(2k)(2k!)} E_{2k}(\tau) z^{2k} \right].
\]

As discussed in Chapter 2, the modular anomaly can be traded for a holomorphic anomaly: this is achieved by replacing $E_2(\tau)$ by its modular completion $\tilde{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \Im(\bar{\tau})}$ in each occurrence of the theta function, at the cost of introducing a mild dependence of the partition function on the anti-holomorphic parameter $\bar{\tau}$. This leads to a modified partition function which has the following modular behavior:
\[
Z_{M}^{A_{N-1}}(\tau, \bar{\tau}; t^{(p)}_f; m, \tau_a, \epsilon_1, \epsilon_2) = Z_{M}^{A_{N-1}}(-1/\tau, -1/\bar{\tau}; t^{(p)}_f; m/\tau, \tau_a/\tau, \epsilon_1/\tau, \epsilon_2/\tau)
\]
and satisfies a holomorphic anomaly equation. This equation relates derivatives of the partition function with respect to $\bar{\tau}$ to derivatives with respect to $t^{(p)}_f$. For this to be true it is critical that for each summand corresponding to a choice of partitions $\{ \mu^{(p)}_a \}$ the coefficient of $\bar{\tau}$ is a function of the combinations $\sum_{a=1}^N |\mu^{(p)}_a|$ only. For this highly non-trivial statement to hold the residual factors $R_{\tilde{\mu}^{(p)}}$ must combine with the product over the theta functions appropriately.
### 3.3.4 Direct computation of the elliptic genus

From Equation (3.3.73) we can extract the elliptic genus for self-dual strings arising from suspended M2 branes between $M$ M5 branes in the presence of an $A_{N-1}$ singularity:

$$\text{Ell}(N \hookrightarrow \tilde{k}) = \sum_{a \mid \mu_a(p) = k_p} \prod_{s=1}^{M-1} R_{\mu(s)} \prod_{a,b=1}^{N} \prod_{(i,j) \in \mu_a(s)} \theta_1(\tau; z_{ab}^{(s)}(i,j)) \theta_1(\tau; w_{ab}^{(s)}(i,j)) \theta_1(\tau; u_{ab}^{(s)}(i,j)) \theta_1(\tau; v_{ab}^{(s)}(i,j)), \tag{3.3.78}$$

where $k_p$, for $p = 1, \ldots, M - 1$, is the number of M2 branes suspended between the $p$-th and $(p + 1)$-st M5 brane. An alternative method to computing this elliptic genus would be through a detailed understanding of the bundles over the instanton moduli space (3.2.20) of the 2d quiver gauge theory in which the fermions and bosons transform (see [102] as well as the discussion in Chapter 2). In the present context, we will content ourself with sketching this approach. The bosons are sections of the tangent bundle of $\mathcal{M}_{m_1, \ldots, m_{M-1}}$, whereas the fermions are sections of the bundle $V$ discussed in Section 3.2.5. The weights of these bundles at the fixed points were worked out in [103], and following [102] one can use them to compute the elliptic genus by employing the Hirzebruch-Riemann-Roch theorem as follows:

$$\text{Ell}(N \hookrightarrow \tilde{k}) = \int_{\mathcal{M}_{m_1, \ldots, m_{M-1}}} \text{ch}(E_{Q_r}) \text{Td}(\mathcal{T}\mathcal{M}_{m_1, \ldots, m_{M-1}}), \tag{3.3.79}$$

where $\mathcal{T}\mathcal{M}$ is the tangent bundle, and the bundle $E_{Q_r}$ is given by

$$E_{Q_r} = \bigotimes_{l=0}^{\infty} \bigwedge Q_r^{l-1} V \bigotimes_{l=1}^{\infty} V^* \otimes \bigotimes_{l=1}^{\infty} S_{Q_r^l} \mathcal{T}\mathcal{M}^* \otimes \bigotimes_{l=1}^{\infty} S_{Q_r^l} \mathcal{T}\mathcal{M}, \tag{3.3.80}$$

where for brevity we have suppressed the dependence of the bundle $E_{Q_r}$ on the different parameters. The fugacities on which the elliptic genus depends can be obtained from the quiver description of Section 3.2.5 as follows. For each node of the inner quiver we get $4N$ fugacities from the bifundamental fields $\tilde{Q}, Q, \Lambda^Q$ and $\Lambda^{\tilde{Q}}$, which are multiplied by $M - 1$ as there are $M - 1$ inner nodes. Furthermore, we have $3(M - 1)$ parameters from the fields $\Lambda^\Phi$, $B$, $\tilde{B}$ associated to each inner node, and $4(M - 2)$ fugacities from the bifundamentals $\Sigma, \Lambda^B, \Lambda^{\tilde{B}}$ and $\Lambda^\Phi$ of the inner quiver. Thus we have a total of $4N(M - 1) + 3(M - 1) + 4(M - 2)$ parameters. However, there will be
constraints from superpotentials and gauge anomalies. Including all these constraints should reduce
the number of independent parameters to NM - M + 3.

Having directly computed the elliptic genus then makes it possible to reconstruct the
partition function of M5 branes in the presence of an $A_{N-1}$ singularity as follows:

$$Z_{M5} = \left( \prod_{p=1}^{N} Z_{U(1)}^{(p)} \right) \left( \prod_{p=1}^{N} Z_{U(1)}^{(p)} \right) \left( \sum_{\tilde{k}}^{M-1} (\tilde{Q}_f^{(s)})^{k_s} \text{Ell}(N, \tilde{k}) \right),$$

(3.3.81)

where $Z_{U(1)}$ is the contribution of a single M5 brane to the partition function and does not contain
any contributions from BPS string states. Yet another way to compute the elliptic genus of the
self-dual strings would be by directly applying the techniques developed in [41–43] to the 2d quiver
gauge theory described in Section 3.2.5. Here we will illustrate how this works in the case of $k$ M2
branes suspended between two M5 branes on transverse TN$_1$ space, that is for $M = 2, N = 1$. This
generalizes the result of Chapter 2 to arbitrary mass. On the one hand, the partition function for
$k$ M-strings is given by

$$Z_{M-strings}^k(\tau, m, \epsilon_1, \epsilon_2) = \sum_{[\nu]=k} \prod_{(i,j) \in \nu} \frac{\theta_1(\tau; Q_m^{-1} q_{\nu_i - j + 1/2} t^{i-1/2}) \theta_1(\tau; Q_m^{-1} q_{\nu_i + j - 1/2} t^{i-1/2})}{\theta_1(\tau; q_{\nu_i - j + 1} t^{i-1}) \theta_1(\tau; q_{\nu_i + j} t^{i+1})}.$$  

(3.3.82)

On the other hand, the elliptic genus of the 2d affine $A_1$ quiver gauge theory with one node removed
and $U(k)$ gauge group (see Figure 3.14) is given by

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (k) at (2,0) {$k$};
\node (1') at (4,0) {1};
\node (Q_1) at (1,1) {$Q_1$};
\node (Q_1') at (1,-1) {$\tilde{Q}_1$};
\node (L_2^\phi) at (3,2) {$\Lambda_2^\phi$};
\node (L_2^\tilde{\phi}) at (3,-2) {$\tilde{\Lambda}_2^\phi$};
\draw[->] (1) to (k);
\draw[->] (k) to (1') node[midway,above] (TextNode) {};\end{tikzpicture}
\end{center}

Figure 3.14: The quiver for two M5 branes in presence of transverse TN$_1$. 

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\[ \text{Ell}(k; \tau, m, \epsilon_1, \epsilon_2) = \frac{1}{k!} \int \frac{dz_\alpha}{2\pi iz_\alpha} \prod_{\alpha, \beta=1}^{k} \left( \frac{\theta_1(\tau; z_\alpha/z_\beta)\theta_1(\tau; Q_\tau t q^{-1}z_\alpha/z_\beta)}{\theta_1(\tau; qz_\alpha/z_\beta)\theta_1(\tau; t^{-1}z_\alpha/z_\beta)} \right) \times \prod_{\alpha=1}^{k} \left( \frac{\theta_1(\tau; Q_m z_\alpha)\theta_1(\tau; Q_m z_\alpha^{-1})}{\theta_1(\tau; \sqrt{q/t} z_\alpha)\theta_1(\tau; \sqrt{q/t} z_\alpha)} \right) = \sum_{\nu} \left( \prod_{(i_1,j_1) \in \nu} \frac{\theta_1(\tau; q^{j_1-j_2+1} t^{-(i_1-i_2+1)})\theta_1(\tau; q^{j_1-j_2} t^{-(i_1-i_2)})}{\theta_1(\tau; q^{j_1-j_2+1} t^{-(i_1-i_2+1)})\theta_1(\tau; q^{j_1-j_2+1} t^{-(i_1-i_2)})} \right) \times \prod_{(i,j) \in \nu} \frac{\theta_1(\tau; Q_m^{-1} q^{j-1/2} t^{-(i-j+1/2)})\theta_1(\tau; Q_m^{-1} q^{j-1/2} t^{-(i-j+1/2)})}{\theta_1(\tau; q^{j-1} t^{i-1})\theta_1(\tau; q^{j-1} t^{i-1})}, \quad (3.3.83) \]

where it is understood that each occurrence of \( \theta_1(\tau; 1) \) in the previous equation is to be replaced by \( -\partial_z \theta_1(\tau; z) |_{z=1} \). The two expressions are superficially different, but one can show that for each Young diagram the product over pairs of boxes of Equation (3.3.83) simplifies to the product over individual boxes of the same Young diagram in Equation (3.3.82). Analogously, we predict that the elliptic genus of 2d affine \( A_{M-1} \) quiver theories with \( N \) flavors will coincide with the partition function of M-strings for a system of \( M \) parallel M5 branes on transverse \( T_N \) space.

### 3.4 Discussion of results

In this chapter we have shown that the partition function of \( M \) parallel M5 branes in the presence of transverse \( A_{N-1} \) singularity compactified on \( T^2 \) can be computed for arbitrary supersymmetry preserving twists using the corresponding strings, obtained by stretched M2 branes suspended between M5 branes and wrapping \( T^2 \). Moreover we have shown that their worldvolume theory is given by a 2d quiver gauge theory which can be used to effectively compute the partition function of this theory. In a way, this is similar in spirit to quantum field theories where the partition functions can be computed using the particle contributions to amplitudes. Here the analog of particles are the strings and they indeed do yield the partition function for the \((1,0)\) superconformal theory at least when compactified on \( T^2 \). Note that as a special case of our computation we can also compute in this way the partition function of 6d \((2,0)\) A-type theory. Furthermore, since

\[ ^6 \text{We are grateful to A. Gadde for communicating this result to us.} \]
we can use this building block to compute the superconformal index of the 6d theory \([70, 77]\) following the methods that will be presented in Chapter 7, we have thus effectively related the superconformal index in 6d to the computation of elliptic genera on the collection of 2d theories living on the resulting strings. This reinforces the picture that these 6d theories are indeed a theory of interacting strings.
Chapter 4

Theories with a single tensor multiplet

4.1 Introduction

In the previous chapters we have studied in great detail the strings associated to the theory of M5 branes probing singularities of type $A_N$ and showed how their elliptic genera encode the SCFT’s BPS partition function on $\mathbb{R}^4 \times T^2$. In this chapter we aim at obtaining a similar understanding for a larger class of (1, 0) 6d SCFTs. We focus on minimal SCFTs that have only one string charge (i.e. a one dimensional tensor branch), and are non-Higgsable. These theories are labelled by an integer $2 \leq n \leq 12$ (excluding $n = 9, 10, 11$) and are realized within F-theory as elliptic fibrations over a base $O(-n) \to \mathbb{P}^1$ [26]. It is also natural to include the $n = 1$ case here although strictly speaking it is not part of the non-Higgsable family. The cases $n = 2, 3, 4, 6, 8, 12$ also arise in the F-theory context as simple orbifolds of $\mathbb{C}^2 \times T^2$ [117], where we rotate each plane of $\mathbb{C}^2$ by an $n$-th root of unity $\omega$ and compensate by rotating the elliptic fiber by $\omega^{-2}$.

The $n = 2$ case coincides with the theory of two M5 branes and was the subject of the previous chapters. The $n = 1$ case corresponds to the exceptional CFT with $E_8$ global symmetry describing an M5 brane near the M9 boundary wall [3, 6, 110, 118]; a two-dimensional quiver de-
Chapter 4: Theories with a single tensor multiplet

scribing its strings was found in [40]. In this chapter we extend this list by finding the quiver for the
$n = 4$ case. This is one of the orbifold cases for which the elliptic fiber can have arbitrary complex
modulus $\tau$, as the only symmetry required in the fiber is $\mathbb{Z}_2$, which does not fix the modulus of the
torus. To find the quiver describing the strings of this theory, we use Sen’s limit of F-theory [29],
which corresponds to taking the modulus of the torus $\tau_2 \gg 1$. Following this approach we are able
in particular to compute the elliptic genus of these strings, which we do explicitly for the first few
string numbers.

If we compactify the theory on a circle, the elliptic genus computes the BPS degeneracies of
the wrapped strings. Following the duality between F-theory and M-theory and the relation between
M-theory, BPS counting in five dimensions, and topological strings, we find that the elliptic genera
are encoded in the topological string partition function defined on the corresponding elliptically-
fibered Calabi-Yau, similar to the observation for the $n = 1$ case in [118]. Within topological string
theory the genus zero BPS invariants can be easily calculated using mirror symmetry even for high
degree $k$ in the base, which corresponds to strings wrapped $k$ times. However, since the boundary
conditions are only known to some extent [119], the higher genus theory cannot be completely
solved with the generalized holomorphic anomaly equations; on the other hand, the elliptic genus
computation provides the all genus answer. In particular we can use this relation to successfully
check our answer for the elliptic genus of the $n = 4$ strings.

For the other values of $n$, no worldsheet description of the associated strings is known. For
these cases we employ topological string techniques to obtain BPS invariants of the corresponding
geometry, which can be related to an expansion of the elliptic genus of small numbers of strings for
specific values of fugacities.

This chapter is organized as follows: In Section 4.2 we review the classification of minimal
6d SCFTs and their F-theory realization. We also review the quivers describing the worldsheet
dynamics of the strings of the $n = 1$ and $n = 2$ models. In Section 4.3 we derive the quiver for
the strings of the $n = 4$ theory by exploiting its orbifold realization. Furthermore, using the quiver
we obtain an integral expression for the elliptic genus of \( k \) strings which we evaluate explicitly for the cases \( k = 1 \) and \( k = 2 \). We then discuss how one can extract from this the BPS degeneracies associated to the strings. In Section 4.4 we construct explicitly the elliptic Calabi-Yau manifolds corresponding to \( n = 1, \ldots, 12 \) as hypersurfaces in toric ambient spaces, solve the topological string theory and calculate the genus zero BPS invariants associated to these Calabi-Yau manifolds, from which one can obtain BPS degeneracies associated to the strings.

### 4.2 Minimal SCFTs in six dimensions

Six-dimensional SCFTs can be classified in the context of F-theory by considering compactifications on an elliptically fibered Calabi-Yau threefold \( X \) with non-compact base \( B \). In the case where all fiber components are blown down the fibration \( \pi : X \rightarrow B \) can be described in terms of the Weierstrass form

\[
y^2 = x^3 + fx + g \tag{4.2.1}
\]

where \( f \) and \( g \) are sections of the line bundles \( \mathcal{O}(-4K_B) \) and \( \mathcal{O}(-6K_B) \). The discriminant locus, along which the elliptic fibers are singular, is a section of \( \mathcal{O}(-12K_B) \) and has the following form:

\[
\Delta = 4f^3 + 27g^2. \tag{4.2.2}
\]

The discriminant locus corresponds to the location of seven-branes in the system. More precisely, each component of the discriminant locus is identified with a seven-brane wrapping a divisor \( \Sigma \subset B \). Each seven-brane supports a gauge algebra \( g_\Sigma \) which is determined by the singularity type of the elliptic fiber along \( \Sigma \) [21, 22].

In the maximally Higgsed phase (that is, when all hypermultiplet vevs that can be set to non-zero value are turned on) one can classify the resulting models in terms of the base geometry \( B \) only [26]. Non-Higgsability requires that the divisor \( \Sigma \subset B \) be rigid. This implies that \( \Sigma \) must be a \( \mathbb{P}^1 \) curve with self-intersection \( -n < 0 \) for a positive number \( n \) (in the following we will refer to this as a \( (-n) \) curve), and the local geometry is the bundle \( \mathcal{O}(-n) \rightarrow \mathbb{P}^1 \). Furthermore, it
can be shown that $n$ is only allowed to take the values $1 \leq n \leq 8$ or $n = 12$ \cite{Falkiewicz:2012, Fateev:2012, Fujimori:2012}. In the $n = 1$ case, corresponding to the E-string $(1, 0)$ SCFT \cite{Falkiewicz:2012, Fateev:2012, Fujimori:2012}, the discriminant vanishes along the non-compact fiber over isolated points on the $\mathbb{P}^1$. In this case instead of a gauge symmetry one finds an $E_8$ global symmetry. In the $n = 2$ case the fiber is everywhere non-singular, and one finds the $A_1 (2, 0)$ SCFT which corresponds to the world-volume theory of M5 branes in flat space. For $n > 2$, the seven-branes wrap the compact $\mathbb{P}^1$, and therefore the 6d SCFT has non-trivial gauge symmetry. In the non-Higgsable case this gauge symmetry is completely determined by the integer $n$. We summarize the list of possibilities in the following table:

<table>
<thead>
<tr>
<th>7-brane</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_\Sigma$</td>
<td>$SU(3)$</td>
<td>$SO(8)$</td>
<td>$F_4$</td>
<td>$E_6$</td>
<td>$E_7$</td>
<td>$E_7$</td>
<td>$E_8$</td>
</tr>
<tr>
<td>Hyper</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>$\frac{1}{2}56$</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

In the $n = 7$ case, one finds that in addition to $E_7$ gauge symmetry the 6d theory also contains a half-hypermultiplet. The cases $n = 9$, $n = 10$ and $n = 11$ lead to $E_8$ gauge symmetry but additionally contain “small instantons”; these cases can be reduced to chains of the more fundamental geometries summarized in the table, as discussed in \cite{Falkiewicz:2012}.

These geometries (excluding the cases $n = 1, 5, 7$) can equivalently be realized as orbifolds of the form $(T^2 \times \mathbb{C}^2)/\mathbb{Z}_n$, $n = 2, 3, 4, 6, 8, 12$. Here, $\mathbb{Z}_n$ acts on the $z_i$, $i = 1, 2, 3$, coordinates of $T^2$ and $\mathbb{C}^2$ as

$$
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix}
\mapsto
\begin{pmatrix}
\omega^{-2} \\
\omega \\
\omega
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix},
$$

(4.2.3)

with $\omega^n = 1$. This construction will be in particular useful when we study the $n = 4$ SCFT, as it will enable us to find a weak coupling description for the corresponding model.

### 4.2.1 Strings of the $\mathcal{O}(-1) \to \mathbb{P}^1$ and $\mathcal{O}(-2) \to \mathbb{P}^1$ models

Let us next discuss the strings that appear on the tensor branch of 6d SCFTs. From the point of view of F-theory these strings arise from D3 branes which wrap the $\mathbb{P}^1$ curve in the base $B$ in the limit of small $\mathbb{P}^1$ size. Let us first analyze from this perspective the M-strings that
Chapter 4: Theories with a single tensor multiplet

arise when $n = 2$. Since in this case the orbifold acts trivially on the torus, its modulus $\tau$ can be taken to be arbitrary, and in particular one can take the weak coupling limit $\tau \to i\infty$ and study this system from the point of view of Type IIB string theory compactified on $B$. As explained in Chapters 2 and 3, the dynamics of M-strings are captured by the two-dimensional quiver gauge theory depicted in Figure 4.1. For $k$ strings, this quiver describes a two-dimensional $\mathcal{N} = (4,0)$ theory with gauge group $U(k)$ and the following $(2,0)$ field content: $Q$ and $\tilde{Q}$ are chiral multiplets in the fundamental representation of $U(k)$, while $\Lambda^Q$ and $\Lambda^\tilde{Q}$ are fundamental Fermi multiplets. Furthermore, the Fermi multiplet $\Lambda^\phi$ and vector multiplet $\Upsilon$ combine into a $(4,0)$ vector multiplet, and the adjoint chiral multiplets $B, \tilde{B}$ combine into a $(4,0)$ hypermultiplet. One intuitive way to see how this comes about is to look at the configuration of $(-2)$ curves which captures the local geometry $\mathcal{O}(-2) \to \mathbb{P}^1$ [9] and is pictured in Figure 4.2.

![Figure 4.1: The quiver describing the $\mathcal{O}(-2) \to \mathbb{P}^1$ strings.](image)

The left and right $(-2)$ curves are non-compact, whereas the curve in the middle is a compact $\mathbb{P}^1$. Choosing the elliptic fiber to be trivial leads upon circle compactification to $U(2)$ $\mathcal{N} = 4$ gauge theory; the $\mathcal{N} = 2^*$ theory is obtained by letting the elliptic fiber degenerate over
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Figure 4.2: Configuration of \((-2)\) curves that gives rise to the \(O(-2) \rightarrow \mathbb{P}^1\) local geometry. We have also indicated the degeneration of the elliptic fiber over each curve that gives rise to the M-string geometry.

each curve to an \(I_1\) singularity (that is, by wrapping a D7 brane over each curve). D3 branes wrapping the compact \((-2)\) curve give rise to the strings of the resulting 6d SCFT, and upon circle compactification their BPS degeneracies then capture the BPS particle content of the \(U(2)\) \(\mathcal{N} = 2^+\) theory. It is easy to understand how the field content of the quiver in Figure 4.1 arises from strings that end on the D3 branes: D3-D3 strings give rise to a \((4,4)\) vector multiplet in the adjoint of \(U(k)\) consisting of the \((2,0)\) multiplets \(\Upsilon, \Lambda^\phi, B, \tilde{B}\); strings stretching from the D3 branes to the D7 brane wrapping the same compact \(\mathbb{P}^1\) give rise to the chiral multiplets \(Q\) and \(\tilde{Q}\); finally, strings stretching between the D3 branes and the D7 branes that wrap the non-compact \((-2)\) curves give rise to the Fermi multiplets \(\Lambda^Q\) and \(\Lambda^{\tilde{Q}}\). Whether D3-D7 strings give rise to chiral or Fermi multiplets is determined by the number of dimensions not shared by the D3 and D7 branes (four for the D3-D7 strings leading to \(Q, \tilde{Q}\), eight for the ones leading to \(\Lambda^Q, \Lambda^{\tilde{Q}}\)).

In [40] a two-dimensional quiver gauge theory was found that describes the dynamics of E-strings, corresponding to the \(O(-1) \rightarrow \mathbb{P}^1\) case. In terms of \((2,0)\) multiplets, the theory of \(k\) E-strings was found to have the following field content: a vector multiplet \(\Upsilon\) and a Fermi multiplet \(\Lambda^\phi\) in the adjoint representation of \(O(k)\), two chiral multiplets \(B, \tilde{B}\) in the symmetric representation.
of $O(k)$, and a Fermi multiplet $\Lambda^Q$ in the bifundamental representation of $O(k)$ and of a $SO(16)$ flavor group, which enhances to $E_8$ at the superconformal point. The relevant quiver is shown in Figure 4.3.

4.3 Quiver for the $O(-4) \to \mathbb{P}^1$ model

We now turn to the strings of the $n = 4$ (1,0) SCFT in 6d and construct a quiver theory that describes their dynamics. Furthermore, we find an integral expression for the elliptic genera of these strings; we evaluate these integrals explicitly for one and two strings and present an answer in a form from which BPS degeneracies may be readily extracted. In Section 4.4.2 we compute the topological string partition function of the corresponding Calabi-Yau geometry and extract BPS invariants which can be shown to agree with the elliptic genus computations.

Recall that the six-dimensional theory is obtained by compactifying F-theory on the following orbifold geometry:

$$CY_3 = (T^2 \times \mathbb{C} \times \mathbb{C}) / \mathbb{Z}_4,$$

where the orbifold action $\mathbb{Z}_4$ on the complex coordinates $(z_1, z_2, z_3)$ of $T^2 \times \mathbb{C} \times \mathbb{C}$ is given by:

$$
\begin{pmatrix}
    z_1 \\
    z_1 \\
    z_3
\end{pmatrix}
\mapsto
\begin{pmatrix}
    \omega^{-2} & & \\
    \omega & & \\
    \omega
\end{pmatrix}
\begin{pmatrix}
    z_1 \\
    z_2 \\
    z_3
\end{pmatrix},
$$

Figure 4.3: The quiver for $O(-1) \to \mathbb{P}^1$ strings.
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and $\omega = i$. To obtain an F-theory construction in terms of a non-compact elliptic Calabi-Yau one has to first blow up the singularity at $\mathbb{C}^2/\mathbb{Z}_4$. The resulting space is described by the bundle

$$\mathcal{O}(-4) \rightarrow \mathbb{P}^1,$$

(4.3.6)

with the singular elliptic fiber $T^2/\mathbb{Z}_2$ over the $\mathbb{P}^1$ base. The resolution of this fiber leads to the $I_0^*$ fiber in the Kodaira classification of elliptic fibrations. In fact, one can obtain an infinite family of six-dimensional theories by taking the singular fiber to be of type $I_p^*$, with $p \geq 0$. Lowering $p$ corresponds in physical terms to Higgsing. This geometry can be equivalently viewed in the weak coupling limit as a Type IIB orientifold of the $\mathbb{C}^2/\mathbb{Z}_2$ singularity [29]. In this limit the singular elliptic fiber over $\mathbb{P}^1$ can be interpreted as the presence of $4 + p$ D7 branes wrapping the $\mathbb{P}^1$ together with an orientifold 7-plane. This gives rise to a $\mathcal{N} = (1, 0)$ $SO(8 + 2p)$ gauge theory in the six non-compact directions parallel to the branes. Furthermore, D3 branes wrapping the $\mathbb{P}^1$ give rise to strings in the six-dimensional theory.

In the following we study the worldsheet theory of these strings and obtain a quiver gauge theory description for it. The particular orientifold we are interested in has been studied in some detail in [120] and we will describe it here briefly. The theory we want study is Type IIB theory on $\mathbb{C}^2/\mathbb{Z}_2$, modded out by $\Omega \Pi$, where $\Omega$ is worldsheet parity and $\Pi$ acts as

$$\Pi : z_1 \rightarrow z_2, \quad z_2 \rightarrow - z_1,$$

(4.3.7)

with $z_1, z_2$ parametrizing the two complex planes in $\mathbb{C}^2/\mathbb{Z}_2$. The D7 branes wrapping the $\mathbb{P}^1$ can, in the singular limit, be thought of as D5 branes probing $\mathbb{C}^2/\mathbb{Z}_2$ together with an orientifold 5-plane at $z_1 = z_2 = 0$. Similarly, D3 branes become D1 branes whose worldvolume theory we wish to determine. We can start by performing the $\mathbb{Z}_2$ orbifold on $\mathbb{C}^2$ and successively add the $\mathbb{Z}_2$ orientifold actions. The $\mathbb{Z}_2$ orbifold leads to the M-string theory discussed in Chapter 3 and is captured by the quiver gauge theory of Figure 4.2. On top of this, we need to perform the orientifold discussed in [120]. There it was found that for $\mathbb{C}^2/\mathbb{Z}_n$ orbifolds the gauge group in the $\omega^l$-twisted ($l = 0, \cdots, n - 1$) D-brane sector is of $Sp$-type if $l$ is even and of $SO$-type if $l$ is
odd. This implies for our case that we have an orthogonal gauge group on the D5 branes in the untwisted sector and a symplectic one on the D5 branes of the ω-twisted sector. Furthermore, anomaly cancellation in six dimensions fixes the ranks of the gauge groups such that the allowed configurations are $SO(8 + 2p) \times Sp(p)$ [120]. This corresponds to having $4 + p$ D5 branes and an O5 plane at the orbifold singularity. Uplifting this to F-theory one finds that the $p = 0$ case is obtained from the $I_0^*$ fiber while the $p > 0$ cases come from $I_p^*$ fiber types.

![Figure 4.4: The local geometry that gives rise to the $SO(8 + 2p)$ 6d SCFT.](image)

In fact, in the F-theory setup the six-dimensional theory has $SO(8 + 2p)$ gauge group and two $Sp(p)$ flavor nodes. The situation here is analogous to the $O(-2) \rightarrow \mathbb{P}^1$ case: the two flavor nodes correspond to non-compact D7 branes intersecting the compact curve as shown in Figure 4.4 (see for example [9] for more details about the geometry).

From the point of view of the two-dimensional theory living on the strings the $SO(8 + 2p)$ gauge node and the $Sp(p)$ flavor nodes descend to flavor nodes. Furthermore, orientifolding implies that $(\Upsilon, \Lambda^\phi)$ transform in the symmetric (that is, adjoint) representation of $Sp(k)^1$, while $(B, \tilde{B})$ transform in the antisymmetric representation [121]. It is interesting to note that the introduction of two $Sp(p)$ nodes is also necessary from gauge anomaly cancellation in two dimensions which will be reviewed later. The resulting two-dimensional quiver is the one depicted in Figure 4.5.

\[^1\text{Orientifolding amounts to projecting the gauge group from } U(2k) \text{ to } Sp(k).\]
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Figure 4.5: The $\mathbb{C}^2/\mathbb{Z}_4$ quiver.

The various fields in the quiver have different charges with respect to the two $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \subset SO(4)$ that rotates the $X^2, X^3, X^4, X^5$ directions. We denote the fugacities by $\epsilon_1, \epsilon_2$, as they are the same parameters that appear in the Nekrasov partition function. For completeness we also present the charges of the fields of the quiver under the different $U(1)$'s and gauge groups; these charges are obtained directly by the orbifolding construction, as in Chapter 3.

<table>
<thead>
<tr>
<th>$Sp(k)$</th>
<th>$\Lambda^\Phi$</th>
<th>$B$</th>
<th>$\bar{B}$</th>
<th>$Q$</th>
<th>$\bar{Q}$</th>
<th>$\Lambda^Q$</th>
<th>$\bar{\Lambda}^Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Sp(k)$</td>
<td>symmetric</td>
<td>anti-symmetric</td>
<td>anti-symmetric</td>
<td>$\square$</td>
<td>$\square$</td>
<td>$\square$</td>
<td>$\square$</td>
</tr>
<tr>
<td>$U(1)_{\epsilon_1}$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$U(1)_{\epsilon_2}$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

We have arrived at the conclusion that the theory for $k$ strings is an $Sp(k)$ gauge theory with a $(2,0)$ vector multiplet $\Upsilon$ and a Fermi multiplet $\Lambda^\Phi$ in the adjoint (i.e. symmetric) representation, two chiral multiplets $B, \bar{B}$ in the antisymmetric representation, and two chiral multiplets $Q, \bar{Q}$, each in the bifundamental representation of $Sp(k) \times SO(8 + 2p)$. If $p > 0$ one also has Fermi multiplets $\Lambda^Q, \bar{\Lambda}^Q$ in the bifundamental of $Sp(k) \times Sp(p)$. One can pick a basis $\{e_i\}_{i=1}^k$ of the weight lattice of $Sp(k)$ in which the fundamental representation has weights $\pm e_i$ ($i = 1, \ldots, k$). In this basis, the symmetric representation has weights $e_i \pm e_j$ ($\forall i, j$), while the antisymmetric representation has weights $e_i \pm e_j$ ($i \neq j$). We also pick $\pm m_i$ to be the Cartan parameters dual to the weights of the
fundamental representation of \( SO(8 + 2p) \), and \( \pm \mu_i \) and \( \pm \tilde{\mu}_i \) to be the Cartan parameters for the two \( Sp(p) \) flavor groups.

Let us next comment on gauge anomaly cancellation in two dimensions. The contribution of chiral fermions running in the loop to the anomaly is proportional to the index of their representation \( T(R) \) defined as:

\[
\text{Tr}(T^a_R T^b_R) = T(R)^{ab}.
\] (4.3.8)

Furthermore, left-moving fermions contribute with a positive sign to the anomaly while right-moving ones contribute with a negative sign. Thus, for our particular quiver we obtain the following result:

\[
a(L) - a(R) = T_B(\text{anti-sym}) + T_{\hat{B}}(\text{anti-sym}) + (n + 4)(T_Q(\square) + T_{\bar{Q}}(\square))
\]

\[
- T_T(\text{sym}) - T_{\Lambda^\phi}(\text{sym}) - n(T_{A_Q}(\square) + T_{\Lambda_Q}(\square))
\]

\[
= 2k - 2 + 2k - 2 + (n + 4)(1 + 1) - (2k + 2) - (2k + 2) - n(1 + 1)
\]

\[
= 0,
\] (4.3.9)

where we have used the identities

\[T(\text{anti-sym}) = 2k - 2, \ T(\text{sym}) = 2k + 2, \ T(\square) = 1,\] (4.3.10)

and the fact that the fundamental fields transform in real representations and therefore only have half the number of degrees of freedom.

### 4.3.1 Localization computation

Having written down the field content of the two-dimensional theory of \( k \) strings, it is straightforward to compute its elliptic genus, following the localization computation of [42, 43]. The elliptic genus is given by a contour integral of a one-loop determinant:

\[
Z_{k \text{strings}} = \int Z_{k \text{strings}}^{1\text{-loop}}(z_i, m_j, \mu_i, \tilde{\mu}_i, \epsilon_1, \epsilon_2),
\] (4.3.11)
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where $Z_{k \text{-strings}}^{1\text{-loop}}$ is a $k$-form on the $k$ complex-dimensional space of flat $Sp(k)$ connections on $T^2$, which is a complex torus parametrized by variables $\zeta_i = \frac{1}{2\pi i} \log z_i$, and the contour of integration is determined by the Jeffrey-Kirwan prescription \[44\]. The one-loop determinant is obtained by multiplying together the contributions of all multiplets and takes the following form:

$$Z_{k \text{-strings}}^{1\text{-loop}} = Z_Y Z_{\Lambda^\phi} Z_B Z_{\bar{B}} Z_Q Z_{\bar{Q}} Z_{\Lambda Q} Z_{\bar{\Lambda} \bar{Q}},$$

where\(^2\)

$$Z_Y = \left( \prod_{i=1}^{k} \frac{d\zeta_i}{\eta^2} \frac{\theta_1'(0) \theta_1(z_i^2) \theta_1(z_i^{-2})}{\eta} \right) \left( \prod_{i<j} \frac{\theta_1(z_i^1 z_j^2)}{\eta} \right),$$

$$Z_{\Lambda^\phi} = \left( \prod_{i=1}^{k} \frac{\theta_1(dt) \theta_1(dt z_i^2) \theta_1(dt z_i^{-2})}{\eta^4} \right) \left( \prod_{i<j} \frac{\theta_1(dt z_i^1 z_j^2)}{\eta} \right),$$

$$Z_B Z_{\bar{B}} = \frac{\eta^2}{\theta_1(d) \theta_1(t)} \left( \prod_{i<j} \frac{\theta_1(d z_i^1 z_j^2)}{\eta^2} \right),$$

$$Z_Q Z_{\bar{Q}} = \prod_{i=1}^{k} \frac{\eta^4}{\theta_1(\sqrt{d} z_i M_{m_i}) \theta_1(\sqrt{d} z_i^{-1} Q_{m_i}) \theta_1(\sqrt{d} z_i^{-1} Q_{m_i})},$$

$$Z_{\Lambda Q} Z_{\bar{\Lambda} \bar{Q}} = \prod_{i=1}^{k} \frac{\eta^4}{\theta_1(z_i Q_{\mu_i}) \theta_1(z_i^{-1} Q_{\mu_i}) \theta_1(z_i Q_{\bar{\mu}_i}) \theta_1(z_i^{-1} Q_{\bar{\mu}_i})},$$

and $Q_{m_i} = e^{2\pi i m_i}$, $Q_{\mu_i} = e^{2\pi i \mu_i}$, $Q_{\bar{\mu}_i} = e^{2\pi i \bar{\mu}_i}$, $d = e^{2\pi i t_1}$, $t = e^{2\pi i t_2}$. The integral itself is then obtained by computing a sum over Jeffrey-Kirwan residues of the one-loop determinant:

$$Z_{k \text{-strings}} = \frac{1}{|\text{Weyl}[Sp(k)]|} \sum_\alpha \text{JK-res}(\alpha, q) Z_{k \text{-strings}}^{1\text{-loop}},$$

where $\alpha$ labels poles of $Z_{k \text{-strings}}^{1\text{-loop}}$ and the role of $q$ will be clarified shortly. In the following sections we compute the residue sum for one and two strings, in which case the evaluation of Jeffrey-Kirwan residues turns out to be straightforward and we do not need to resort to the full-fledged formalism.

\(^2\)In this chapter we use the following definitions for the Dedekind eta function $\eta(\tau)$ and Jacobi theta function $\theta_1(z, \tau)$:

$$\eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j), \quad \theta_1(z, \tau) = -i q^{1/8} \sqrt{z} \prod_{j=1}^{\infty} (1 - q^j)(1 - z^{-1} q^j)(1 - z^{-1} q^j),$$

where $q = e^{2\pi i \tau}$.  

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One string

For a single string, the one-loop determinant is given by a one-form:

\[
Z_{\text{1-loop}}^{\text{string}} = d\zeta \frac{2\pi i \eta^2 \theta(z^2) \theta(z^{-2}) \theta(dt) \theta(dt^2) \theta(dz^2) \theta(dtz^{-2})}{\theta(d) \theta(i) \eta^3} 
\times \prod_{i=1}^{p} \frac{\theta(Q_{\mu_i} z) \theta(Q_{-1\mu_i} z) \theta(Q_{\bar{\mu_i}} z) \theta(Q_{-1\bar{\mu_i}} z)}{\eta^4} 
\times \prod_{i=1}^{4+p} \frac{\theta(\sqrt{dt} Q_{m_i} z) \theta(\sqrt{dt} Q_{-1m_i} z) \theta(\sqrt{dt} Q_{m_i}^{-1} z) \theta(\sqrt{dt} Q_{-1m_i}^{-1} z)}{\eta^4}. \tag{4.3.19}
\]

One first needs to identify the singular loci of the integrand. Each of the theta functions in the second line of (4.3.19) determines a (zero-dimensional) singular hyperplane within the one complex dimensional space \( \mathfrak{M}_1^{\text{string}} \) spanned by \( \zeta = \log z \), for a total of \( 4 \cdot (4 + p) \) distinct singular points at

\[
\pm \zeta + \frac{\epsilon_1 + \epsilon_2}{2} \pm m_i = 0, \ i = 1, \ldots, 4 + p. \tag{4.3.20}
\]

To determine which poles contribute to the residue sum, one needs to consider the normal vectors to the singular hyperplanes. In this case, the normal vector is simply \( \pm \partial \zeta \), where the sign is the one multiplying \( \zeta \) in (4.3.20). The data that enters the Jeffrey-Kirwan residue computation corresponds of two quantities: the position of the pole in the \( \zeta \) plane and a choice of a vector \( q \in T\mathfrak{M}_1^{\text{string}} \). In this case, we can choose either \( q = \pm \partial \zeta \); let us pick \( q = -\partial \zeta \). For two-dimensional theories, it can be argued that once the sum over residues is performed the answer is independent of the choice of \( q \). Next, one picks the poles satisfying the property that \( q \) lies within the one-dimensional cone spanned by the vector normal to the corresponding hyperplane. In this trivial example one finds that only the following poles contribute to the integral:

\[
-\zeta + \frac{\epsilon_1 + \epsilon_2}{2} \pm m_i = 0. \tag{4.3.21}
\]

Evaluating the Jeffrey-Kirwan residues in this situation corresponds to summing over the ordinary residues at these poles. Summing over the eight residues and dividing by \( \text{Weyl}[\text{Sp}(1)] = \mathbb{Z}_2 \) leads

\[3\text{Since } \zeta \sim \zeta + 1 \sim \zeta + \tau, \text{ each theta function leads to a single pole.} \]
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to the following answer:

\[
Z_{1\text{\_string}} = \frac{1}{2} \frac{\eta^2}{\theta(d)\theta(t)} \sum_{i=1}^{4+p} \left[ \frac{\theta(d^2 Q_{m_i}^2)\theta(d^2 t^2 Q_{m_i}^2)}{\eta^2} \prod_{j \neq i} \prod_{s=\pm 1} \frac{\eta^2}{\theta(Q_{m_i} Q_{m_j}^s)\theta(d^2 Q_{m_i} Q_{m_j}^s)} \times \prod_{j=1}^p \prod_{s=\pm 1} \frac{\theta(\sqrt{d^2 t^2 Q_{m_i} Q_{m_j}^s})\theta(\sqrt{d^2 t^2 Q_{m_i} Q_{m_j}^s})}{\eta^2} + (Q_{m_i} \to Q_{m_i}^{-1}) \right].
\]

(4.3.22)

Note some features of this expression: the existence of theta functions in the denominator which depend on $SO(8 + 2p)$ fugacities suggests that the $SO(8 + 2p)$ continues to be carried by some bosonic degrees of freedom in the IR. Also, the fact that the expressions include a mixture of $\epsilon_i$ (captured by $t, d$) and $m_i$ suggests a non-trivial structure for the theory which makes it unlikely to correspond to a free theory in the IR. It would be interesting to identify the non-trivial $(4, 0)$ CFT whose elliptic genus is given by the above expression. Perhaps ideas similar to the ones employed in [122] can be used to do this.

In Section 4.3.3 we will explain how to extract from this expression the BPS degeneracies corresponding to a single string; for the $p = 0$ case, one finds a precise match with the BPS invariants of the geometry that engineers the $O(-4) \to \mathbb{P}^1$ SCFT, to be discussed in Section 4.4.

Two strings

The computation for two strings proceeds analogously; first, one should identify the hyperplanes in the two-dimensional space $\mathfrak{M}_{2\text{\_strings}}$ along which the denominator of $Z_{2\text{\_strings}}^{1\text{-loop}}$ vanishes. There are $8(p+5)$ such hyperplanes:

\[
\pm \zeta_j + \frac{\epsilon_1 + \epsilon_2}{2} \pm m_i = 0, \quad i = 1, \ldots, 4 + p, \quad j = 1, 2; \quad (4.3.23)
\]

\[
\pm \zeta_1 \pm \zeta_2 + \epsilon_1 = 0; \quad (4.3.24)
\]

\[
\pm \zeta_1 \pm \zeta_2 + \epsilon_2 = 0, \quad (4.3.25)
\]

where $\zeta_i = \log(z_i)$. For concreteness, let us focus from now on to the case where $p = 0$, keeping in mind that the computation for arbitrary $p$ proceeds analogously. We display the vectors normal to
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the hyperplanes, as well as our choice of $q \in TM^{2 \text{strings}}$, in Figure 4.6. The next step is to identify

![Figure 4.6: Singular hyperplane configuration for the two-string elliptic genus. The vectors normal to the singular hyperplanes are displayed, along with the multiplicity with which they occur. Our choice of $q$ is also displayed here.](image)

the points at which hyperplanes intersect. The computation of Jeffrey-Kirwan residues is simplified by the fact that for generic values of $m, \epsilon_1, \epsilon_2$ at most two hyperplanes intersect at the same time. The poles whose residues contribute to the elliptic genus are those for which $q$ lies within the cone spanned by the vectors normal to the corresponding hyperplanes. For example, since $q$ lies in the cone spanned by $-\partial \zeta_1$ and $-\partial \zeta_1 - \partial \zeta_2$, but not in the one spanned by $-\partial \zeta_1$ and $-\partial \zeta_1 + \partial \zeta_2$, the residue evaluated at

$$-\zeta_1 + \frac{\epsilon_1 + \epsilon_2}{2} + m_1 = 0; \quad -\zeta_1 - \zeta_2 + \epsilon_1 = 0,$$

will contribute, while the one at

$$-\zeta_1 + \frac{\epsilon_1 + \epsilon_2}{2} + m_1 = 0; \quad -\zeta_1 + \zeta_2 + \epsilon_1 = 0$$

will not. Following this prescription, one arrives at the following list of poles whose residues contribute to the computation:
The prescription outlined above also picks up some additional poles, but they do not contribute to the elliptic genus since the numerator of $Z^1_{\text{loops}}$ turns out to vanish for them. Therefore, the elliptic genus of two strings is obtained by summing over the residues that correspond the 112 poles listed in Equations (4.3.28)–(4.3.32). In practice, one can exploit $Sp(2)$ Weyl symmetry to show that the residues of poles (4.3.29) and (4.3.30) are identical to the ones of (4.3.28). For the same reason, one can set $j < i$ in (4.3.32) and multiply the corresponding 24 residues by a factor of 2.

After these considerations, we are ready to write down the elliptic genus of two strings:

$$Z^{2_{\text{strings}}} = \frac{1}{8} \left[ 3 \sum_{\alpha_{i,j,s}} \text{Res}_{\alpha_{i,j,s}} Z^{1-\text{str.}}_2 + \sum_{\beta_{i,j,s}} \text{Res}_{\beta_{i,j,s}} Z^{1-\text{str.}}_2 + 2 \sum_{\gamma_{i,j,s_1,s_2}} \text{Res}_{\gamma_{i,j,s_1,s_2}} Z^{1-\text{str.}}_2 \right],$$

where we have divided by an overall factor of $8 = |\text{Res}[Sp(2)]|$, and the residues have the following

$$\alpha_{i,j,s}: \quad \zeta_1 = \frac{\epsilon_1 + \epsilon_2}{2} + sm_i, \quad \zeta_2 = \zeta_1 + \epsilon_j; \quad (i = 1, \ldots, 4, j = 1, 2, s = \pm 1) \quad (4.3.28)$$

$$\alpha'_{i,j,s}: \quad \zeta_2 = \frac{\epsilon_1 + \epsilon_2}{2} + sm_i, \quad \zeta_1 = \zeta_2 + \epsilon_j; \quad (i = 1, \ldots, 4, j = 1, 2, s = \pm 1) \quad (4.3.29)$$

$$\alpha''_{i,j,s}: \quad -\zeta_2 = \frac{\epsilon_1 + \epsilon_2}{2} + sm_i, \quad \zeta_1 = -\zeta_2 + \epsilon_j; \quad (i = 1, \ldots, 4, j = 1, 2, s = \pm 1) \quad (4.3.30)$$

$$\beta_{i,j,s}: \quad \zeta_1 = \frac{\epsilon_1 + \epsilon_2}{2} + sm_i, \quad \zeta_2 = -\zeta_1 + \epsilon_j; \quad (i = 1, \ldots, 4, j = 1, 2, s = \pm 1) \quad (4.3.31)$$

$$\gamma_{i,j,s_1,s_2}: \quad \zeta_1 = \frac{\epsilon_1 + \epsilon_2}{2} + s_{1m_i}, \quad \zeta_2 = \frac{\epsilon_1 + \epsilon_2}{2} + s_{2m_j}. \quad (i = 1, \ldots, 4, j \neq i, s_1 = \pm 1, s_2 = \pm 1) \quad (4.3.32)$$
explicit form:

\[
\text{Res}_{\alpha_{1,2}} Z_{2}^{1\text{-loop}} = \frac{\theta_{1}(d^{2}t Q_{m_{1}}^{2s})\theta_{1}(d^{3}t Q_{m_{1}}^{2s})\theta_{1}(d^{3}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})}{\theta_{1}(d)\theta_{1}(t)\theta_{1}(d^{2})\theta_{1}(t/d)} \times \prod_{j \neq i} \prod_{r = \pm 1} \frac{\theta_{1}(Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(d Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(dt Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(d^{2}t Q_{m_{1}}^{s} Q_{m_{1}}^{s})}{\eta^{4}}; \tag{4.3.34}
\]

\[
\text{Res}_{\beta_{1,3}} Z_{2}^{1\text{-loop}} = \frac{\theta_{1}(d^{2}t)\theta_{1}(t Q_{m_{1}}^{2s})\theta_{1}(t/d Q_{m_{1}}^{2s})\theta_{1}(d^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})}{\theta_{1}(d)\theta_{1}(t)\theta_{1}(d^{2})\theta_{1}(t/d)\theta_{1}(Q_{m_{1}}^{2s})} \times \prod_{j \neq i} \prod_{r = \pm 1} \frac{\theta_{1}(Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(d Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(t Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(dt Q_{m_{1}}^{s} Q_{m_{1}}^{s})}{\eta^{4}}; \tag{4.3.35}
\]

\[
\text{Res}_{\beta_{1,2}} Z_{2}^{1\text{-loop}} = \frac{\theta_{1}(d^{2}t^{2})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})}{\theta_{1}(d^{2})\theta_{1}(t^{2})\theta_{1}(d^{2})\theta_{1}(d/t)\theta_{1}(Q_{m_{1}}^{2s})} \times \prod_{j \neq i} \prod_{r = \pm 1} \frac{\theta_{1}(Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(d Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(t Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(dt Q_{m_{1}}^{s} Q_{m_{1}}^{s})}{\eta^{4}}; \tag{4.3.36}
\]

\[
\text{Res}_{\gamma_{1,2}} Z_{2}^{1\text{-loop}} = \frac{\theta_{1}(d^{2}t^{2})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})}{\theta_{1}(d^{2})\theta_{1}(t^{2})\theta_{1}(d^{2})\theta_{1}(d/t)\theta_{1}(Q_{m_{1}}^{2s})} \times \prod_{j \neq i} \prod_{r = \pm 1} \frac{\theta_{1}(Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(d Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(t Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(dt Q_{m_{1}}^{s} Q_{m_{1}}^{s})}{\eta^{4}}; \tag{4.3.37}
\]

\[
\text{Res}_{\alpha_{1,3}} Z_{2}^{1\text{-loop}} = \frac{\theta_{1}(d^{2}t^{2})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})\theta_{1}(d^{2}t^{2} Q_{m_{1}}^{2s})}{\theta_{1}(d^{2})\theta_{1}(t^{2})\theta_{1}(d^{2})\theta_{1}(d/t)\theta_{1}(Q_{m_{1}}^{2s})} \times \prod_{j \neq i} \prod_{r = \pm 1} \frac{\theta_{1}(Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(d Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(t Q_{m_{1}}^{s} Q_{m_{1}}^{s})\theta_{1}(dt Q_{m_{1}}^{s} Q_{m_{1}}^{s})}{\eta^{4}}; \tag{4.3.38}
\]

After summing over the 56 residues, one is left with a weight zero meromorphic elliptic function with modular parameter \(\tau\) and six elliptic parameters \((\epsilon_{1}, \epsilon_{2}\text{ and the } SO(8)\text{ fugacities } (m_{1}, \ldots, m_{4}))\).

In Section 4.4.2 we will check the validity of our answer in the unrefined limit \(\epsilon_{2} = -\epsilon_{1}\) by verifying that it exactly reproduces the genus 0 BPS invariants of the Calabi-Yau threefold that engineers the six-dimensional theory under consideration. If \(\epsilon_{1}, \epsilon_{2}\) are left arbitrary, Equation (4.3.33) can be used to compute arbitrary genus refined BPS invariants of this geometry.
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For higher numbers of strings the computation of the elliptic genus from Equation (4.3.18) proceeds analogously, but for simplicity and clarity of exposition we limit our discussion to the cases of one and two strings.

4.3.2 Modular anomaly

In this section we study the behavior of the elliptic genus under $SL(2, \mathbb{Z})$ transformations

$$\gamma: (t_b, \tau, m_i, \mu_i, \epsilon_i) \rightarrow \left( t_b, \frac{a \tau + b}{c \tau + d}, m_i \frac{\mu_i}{c \tau + d}, \frac{\epsilon_i}{c \tau + d} \right), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}). \quad (4.3.39)$$

The modular properties of the elliptic genus can be best understood starting from the integral expression (4.3.11), where the integration variables $\zeta_i$ also transform as elliptic parameters: $\zeta_i \rightarrow \zeta_i/(c \tau + d)$. Each $(2,0)$ multiplet contributes to the integrand a factor of the form $\theta_1(e^{2\pi i \zeta_i}, \tau + 1)$. Each $(2,0)$ multiplet contributes to the integrand a factor of the form $\theta_1(e^{2\pi i \zeta_i}, \tau + 1)$.

Recall that under $S$ and $T$ transformations

$$\frac{\theta_1(e^{2\pi i \zeta_i}, \tau + 1)}{\eta(\tau + 1)} = e^{\pi i/4} \frac{\theta_1(e^{2\pi i \zeta_i}, \tau)}{\eta(\tau)}, \quad (4.3.40)$$

$$\frac{\theta_1(e^{2\pi i \zeta_i}, 1/(\tau))}{\eta(1/\tau)} = e^{-\pi i/4} \frac{\theta_1(e^{2\pi i \zeta_i}, \tau)}{\eta(\tau)}. \quad (4.3.41)$$

Using this, one can easily check that the elliptic genus of $k$ strings is invariant under $\tau \rightarrow \tau + 1$:

$$Z_{k \text{strings}}(\tau + 1) = Z_{k \text{strings}}(\tau); \quad (4.3.42)$$

on the other hand, under $\tau \rightarrow -1/\tau$ one can show that the integrand picks up a $\zeta_i$-independent phase:

$$\frac{Z_{k \text{strings}}(-1/\tau)}{Z_{k \text{strings}}(\tau)} = \exp \left[ -\frac{\pi i}{\tau}\left( \epsilon_1 \epsilon_2 \left( 4k^2 - 2k \right) - (\epsilon_1 + \epsilon_2)^2 k(2 + p) \right. \right.$$

$$\left. - 4k \sum_{j=1}^{4+p} m_j^2 + 2k \sum_{j=1}^{p} \left( \mu_j^2 + \tilde{\mu}_j^2 \right) \right]. \quad (4.3.43)$$

In other words, $Z_{k \text{strings}}$ transforms as a modular function up to an anomalous phase factor. The origin of this factor, as we have seen in the previous chapters, can be seen by considering the
following representation of the theta function:

\[ \theta(z, \tau) = \eta(\tau)^3(2\pi \zeta) \exp \left( \sum_{k \geq 1} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi i \zeta)^{2k} \right). \]  

(4.3.44)

In particular, under modular transformation \( E_2(\tau) \) transforms anomalously:

\[ E_2 \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^2 E_2(\tau) - \frac{6ic}{\pi} (c \tau + d). \]  

(4.3.45)

In other words, the phase factors appearing in Equation (4.3.43) are completely determined by the \( E_2(\tau) \)-dependence of the integrand, and in lieu of (4.3.43) we might as well have written:

\[ \partial_{E_2} Z_{k \text{ strings}} = -\frac{1}{24} (2\pi)^2 \left( \epsilon_1 \epsilon_2 \left( 4k^2 - 2k \right) - (\epsilon_1 + \epsilon_2)^2 k(2 + p) \right. \]
\[ \left. - 4k \sum_{j=1}^{4+p} m_j^2 + 2k \sum_{j=1}^{p} (\bar{\mu}_j^2 + \bar{\mu}_j^2) \right) Z_{k \text{ strings}}. \]  

(4.3.46)

This expression is very similar to the modular anomaly equations for \( M \)-strings and \( E \)-strings found in [94, 119, 123, 124] and discussed in Chapters 2 and 6. In the \( E \)-string \((\mathcal{O}(-1) \to \mathbb{P}^1)\) case, one has:

\[ \frac{1}{Z_E} \partial_{E_2} Z^d_E = -\frac{1}{24} (2\pi)^2 (\epsilon_1 \epsilon_2 (k^2 + k) - k(\epsilon_1 + \epsilon_2)^2 + k \sum_i m_i^2), \]

while in the \( M \)-string \((\mathcal{O}(-2) \to \mathbb{P}^1)\) case, one finds:

\[ \frac{1}{Z_M^k} \partial_{E_2} Z^d_M = -\frac{1}{12} (2\pi)^2 (\epsilon_1 \epsilon_2 k^2 - \frac{k}{4} (\epsilon_1 + \epsilon_2)^2 + k m^2). \]

In all these cases the elliptic genera of the strings capture part of the topological string partition function of the corresponding Calabi-Yau \( X \):

\[ Z^{\text{top}}(X) = Z_0(X) \cdot \left( 1 + \sum_{k=1}^{\infty} Q^k Z_{k \text{ strings}}(X) \right). \]  

(4.3.47)

The Calabi-Yau \( X \) is elliptically fibered, and the topological string partition function is expected to be invariant under modular transformations (4.3.43). However, this is in contradiction with the fact that \( Z_{k \text{ strings}} \) is only invariant up to a phase. The resolution to this apparent contradiction
is well known: in the topological string expression the second Eisenstein series \( E_2(\tau) \) should be replaced by its modular completion
\[
\tilde{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{6i}{\pi(\tau - \bar{\tau})},
\]
which under the \( SL(2, \mathbb{Z}) \) action transforms as follows:
\[
\tilde{E}_2 \left( \frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = (c\tau + d)^2 \tilde{E}_2(\tau, \bar{\tau}).
\]
This implies that the topological string partition function is a well defined modular function of \( \tau \), but no longer depends holomorphically on it:
\[
\partial_\tau Z_{\text{top}}(X) = \frac{6i}{\pi(\tau - \bar{\tau})^2} \partial_\tau \tilde{E}_2 Z_{\text{top}}(X) \neq 0.
\]

### 4.3.3 Refined BPS invariants

Let us now explain how to extract refined BPS invariants from the elliptic genera of \( k \) strings \( Z_k \), again specializing to the case \( p = 0 \) where the global symmetry on the worldsheet is just \( SO(8) \). In order to proceed note that the full partition function of the topological string is given by
\[
Z^{\text{top}} = e^F = Z_0 \left( 1 + \sum_{k=1}^{\infty} Q^k Z_k \right),
\]
where \( Z_k \) is the elliptic genus of \( k \) strings and \( Q \) is a combination of exponentiated Kähler moduli of the elliptic Calabi-Yau geometry to be determined later. Furthermore, we perform the following change of basis which replaces the mass parameters \( Q_{m_i} \) by the parameters \( Q_i \) corresponding to a choice of simple roots of \( SO(8) \):
\[
Q_{m_1} = Q_1 Q_c \sqrt{Q_2} \sqrt{Q_3}, \quad Q_{m_2} = Q_c \sqrt{Q_2} \sqrt{Q_3}, \quad Q_{m_3} = \sqrt{Q_2} \sqrt{Q_3}, \quad Q_{m_4} = \sqrt{Q_3} \sqrt{Q_2}.
\]

In addition to these parameters, let us also define the parameter \( Q_4 \) corresponding to the affine node of the extended Dynkin diagram of type \( D_4 \) as shown in Figure 4.7. The elliptic genera \( Z_k \) can then be expanded in positive powers of \( Q_1, Q_2, Q_3, Q_4 \) and \( Q_c \) upon replacing \( Q_\tau = e^{2\pi i \tau} \) by
the following combination:

\[ Q_T = Q_1 Q_2 Q_3 Q_4 Q_c^2, \]  

(4.3.53)

where the powers are determined by the Coxeter labels of the nodes in Figure 4.7.

Taking the logarithm of Equation (4.3.51), the free energy \( F \) can be expanded as:

\[ F = \log Z = \log(Z_0) + Z_1 Q + \left( -\frac{1}{2} Z_1^2 + Z_2 \right) Q^2 + \left( \frac{Z_3^3}{3} - Z_1 Z_2 + Z_3 \right) Q^3 + O(Q^4). \]  

(4.3.54)

In order to make contact with the computation of the Calabi-Yau BPS invariants from Section 4.4 we identify \( Q \) with the combination

\[ Q = Q_b Q_4 \frac{Q_1 Q_2 Q_3 Q_c^2}{Q_1^2 Q_2 Q_3 Q_c^3}, \]  

(4.3.55)

where \( Q_b = e^{-t_b} \) and \( t_b \) is the Kähler class of the base of the elliptic fibration. The refined BPS invariants are encoded in the free energy \( F \) as follows [66, 76]:

\[ F = \sum_{m=1}^{\infty} \sum_{n, n'=n, \beta \in H_1(M, \mathbb{Z})} \frac{n_{\beta}^{j_L, j_R} (-1)^{2(j_L + j_R)} \sqrt{d m t^m} \left( \sum_{n=-j_L}^{j_L} (d/t)^{m n} \right) \left( \sum_{n=-j_R}^{j_R} (d t)^{m n} \right) c^m(\beta, \mathbb{L})}{(1 - d^m)(1 - t^m)}, \]  

(4.3.56)

where \( t \) denote the Kähler moduli of the Calabi-Yau. The above free energy encodes BPS degeneracies \( n_{\beta}^{j_L, j_R} \) of short multiplets of the five-dimensional quantum field theory arising from circle compactification of the six-dimensional SCFT. In this context the labels \( j_L \) and \( j_R \) refer to the spins of
the two $SU(2)$ subgroups of the little group $SO(4)$ in the decomposition $SO(4) = SU(2)_L \times SU(2)_R$ and $\beta$ labels the string charge as well as the various flavor charges. Denoting by $t_f$ the collection of the Kähler moduli of the resolved elliptic fiber and making use of the expansion

$$F(\epsilon_1, \epsilon_2, t) = F_0(\epsilon_1, \epsilon_2, t_f) + \sum_{i=1}^{\infty} F_i(\epsilon_1, \epsilon_2, t_f)Q_i^i,$$  

(4.3.57)

we find

$$F_1(\epsilon_i, m_i, \tau) = Z_1(\epsilon_i, m_i, \tau)$$

$$F_2(\epsilon_i, m_i, \tau) = Z_2(\epsilon_i, m_i, \tau) - \frac{1}{2} Z_1(\epsilon_i, m_i, \tau)^2$$

$$\vdots$$

(4.3.58)

Since for $F_1$ there is no multi-wrapping, we can set $m = 1$ in Equation (4.3.56) and extract the invariants $n^{jL,jR}_\beta$ immediately from the expression (4.3.22) for the elliptic genus of one string. Let us specify $\beta$ in terms of the following basis of $H^2(X, \mathbb{Z})$: $J_b, J_1, J_2, J_3, J_4, J_c$; that is, we write:

$$n_{\beta}^{jL,jR} = n_{b,1,2,3,4,c}^{jL,jR}.$$  

(4.3.59)

In the following tables we present a sample of invariants for some specific choices of low degree curves.
Analogously, we can extract all refined invariants for two strings, that is for base wrapping number \( n_b = 2 \). For example:

\[
\begin{array}{cccccccc}
2j_L \backslash 2j_R & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\( n_{1,3,3,1,1,4}^{J_L, J_R} \)

\[
\begin{array}{cccccccc}
2j_L \backslash 2j_R & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\( n_{2,1,1,0,0,1}^{J_L, J_R} \)

\[
\begin{array}{cccccccc}
2j_L \backslash 2j_R & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\( n_{2,2,2,0,0,1}^{J_L, J_R} \)

In order to extract unrefined invariants from these one has to sum over the right-moving spin of the multiplets as follows:

\[
n_{\beta}^{j_L} = \sum_{j_R} (-1)^{2j_R} (2j_R + 1) n_{\beta}^{j_L, J_R}.
\] (4.3.60)

Furthermore, in order compare with the genus expansion of the topological string, the \( SU(2)_L \) representations have to be organized into

\[
I_L^n = \left[ \left( \frac{1}{2} \right) + 2(0) \right] \otimes n,
\] (4.3.61)

and the BPS invariants \( n_{\beta}^{g} \) can be obtained by comparing the two sides of the identity

\[
\sum n_{\beta}^{J_L, J_R} (-1)^{2j_R} (2j_R + 1) [j_L] = \sum_{g} n_{\beta}^{g} I_L^n.
\] (4.3.62)

The expansion coefficients in \( I_L^n = \sum_{j} c_j^{2n}[j/2] \) can be found for example in [99]. Using these results we can compute unrefined invariants. For \( n_b = 1 \), for example, for the curves considered above one
Chapter 4: Theories with a single tensor multiplet

has:

\[
\begin{align*}
   n_{1,2,2,1,1,3}^0 &= -272 & n_{1,2,2,1,1,3}^1 &= 16, \\
   n_{1,2,2,1,1,4}^0 &= -534 & n_{1,2,2,1,1,4}^1 &= 32, \\
   n_{1,3,3,1,1,3}^0 &= -108 & n_{1,3,3,1,1,3}^1 &= 0, \\
   n_{1,3,3,1,1,4}^0 &= -582 & n_{1,3,3,1,1,4}^1 &= 36,
\end{align*}
\]

(4.3.63)

and for the \( n_b = 2 \) curves considered above we obtain

\[
\begin{align*}
   n_{2,1,1,0,0,1}^0 &= -16 & n_{2,1,1,0,0,1}^1 &= 0, \\
   n_{2,2,2,0,0,1}^0 &= -18 & n_{2,2,2,0,0,1}^1 &= 0.
\end{align*}
\]

(4.3.64)

For these classes all invariants with \( g \geq 2 \) vanish.

Following the procedure outlined above we have extracted an extensive list of BPS invariants corresponding to one and two strings. The genus zero invariants can be computed independently by employing the mirror symmetry and topological string techniques presented in the next section. When comparing the elliptic genus results to the topological string computation presented in Section 4.4.2 we find a perfect agreement.

### 4.4 The Calabi-Yau geometries with elliptic singularities

In this section we construct the local elliptic Calabi-Yau geometries corresponding to the different minimal 6d SCFTs. Our strategy will be to first find a minimal compact elliptic Calabi-Yau 3-fold with the right type of elliptic fiber degeneration over the rigid divisor \( \Sigma \) in the base \( B \) and subsequently take the local limit by decompactifying the normal direction to \( \Sigma \) in \( B \). The resulting space will be the non-compact Calabi-Yau threefold.
Non-compact Calabi-Yau geometries played an important role in the development of topological string theory, which can frequently be completely solved on these geometries, by well understood relations to matrix models, integrable models and gauge theories. One wide class of examples are the non-compact toric Calabi-Yau spaces; another one with some overlap to the first consists of the local (almost) Fano varieties \( \mathcal{O}(-K_S) \rightarrow S \), where \( S \) is an (almost) Fano variety. That is, one considers a rigid divisor \( S \) in the Calabi-Yau 3-fold and decompactifies the normal direction to \( S \). In these cases local mirror symmetry leads to a mirror curve. In the present case, instead, we decompactify the normal direction to a rigid divisor \( \Sigma \) in the base \( B \) of an elliptic Calabi-Yau 3-fold. Then, the mirror geometry does not reduce to a curve. In cases where there is an orbifold description one can describe the local mirror geometry as a non-compact Landau-Ginzburg model.

### 4.4.1 The local geometries

The new local geometries we consider arise in Calabi-Yau threefolds, where we zoom close to the elliptic singularity of an elliptic fibration over a divisor \( \Sigma \) in a two-dimensional base \( B \). The divisor \( \Sigma \) corresponds to the seven-brane locus in F-theory with gauge symmetry \( g_\Sigma \) and the exceptional divisors that resolve the elliptic singularity intersect with the negative Cartan matrix \( C_{g_\Sigma} \) of the affine Lie algebra \( \hat{g}_\Sigma \) associated to \( g_\Sigma \).

We consider non-Higgsable singularities; in other words, the divisor \( \Sigma \) has to be rigid and the Calabi-Yau space has no complex structure deformations which could resolve the singularity. The simplest example for a non compact threefold of this type are elliptic fibrations over \( B = (\mathcal{O}(-n) \rightarrow \mathbb{P}^1) \). We will start with a compact threefold \( M_3 \) constructed as elliptic fibration over a Hirzebruch surface \( B = \mathbb{F}_n \) [22]. The \((-n)\) section of the rational fibration of \( \mathbb{F}_n \) is then the rigid gauge symmetry divisor \( \Sigma \), with \( \mathcal{O}(-n) \) as its normal bundle.

This setup allows us to solve the topological string using mirror symmetry with normalizable intersections and instanton actions. By decompactifying the normal direction we can easily decouple six-dimensional gravity.
Hirzebruch surfaces as base

Let us recall that the Hirzebruch surfaces \( F_n \) are rational \( \mathbb{P}^1 \) fibrations over \( \mathbb{P}^1 \), where \( n \) parametrizes the twisting of the fiber. They can be constructed torically or as gauged linear sigma models with four chiral fields \( \Phi_i, i = 1, \ldots, 4 \) and two \( U(1) \)'s under which the fields have charges \( l_i^{(1)} \) and \( l_i^{(2)} \). We also use the description in terms of a toric fan in which each field \( \Phi_i \) corresponds to a primitive vector \( \nu_i \) spanning the fan in the integer lattice \( \mathbb{Z}^2 \) and summarize the base data as follows:

\[
\begin{array}{c|cc|cc}
Div & \nu_i^a & l^{(1)} & l^{(2)} \\
\hline
D_0 = K & 1 & 0 & 0 & -2 & n - 2 \\
D_1 = S & 1 & 0 & 1 & 1 & 0 \\
D_2 = F & 1 & 1 & 0 & 0 & 1 \\
D_3 = S' & 1 & 0 & -1 & 1 & -n \\
D_4 = F & 1 & -1 & -n & 0 & 1 \\
\end{array}
\]

(4.4.65)

Here we added the inner point \( \nu_0 = (0, 0) \) and promoted the points \( \nu_i \) to \( \nu_i^a = (1, \nu_i) \in \mathbb{Z}^3 \). This is useful for describing the non-compact Calabi-Yau as the anti-canonical bundle over \( F_n \) (\( \Phi_0 \) is the noncompact direction), but it could be omitted for the discussion of the compact Hirzebruch surface. Each point \( \nu_i \) corresponds to a toric divisor \( D_i = \{ \Phi_i = 0 \} \) and within the surface the homological relations between these divisors are \( S = S' + nF \). The nonvanishing intersections are \( S^2 = n, FS = 1 \) and \( (S')^2 = -n \); therefore, \( S' \) becomes the gauge theory divisor.

Geometrically the \( l^{(k)}, k = 1, 2 \) represent curve classes \( [C_k] \) and the intersection with the toric divisors \( D_i \) is given by

\[
[C_k] \cdot [D_i] = l_i^{(k)}.
\]

(4.4.66)

The \( l^{(k)} \) are also called Mori vectors; in the present example, \( k = 1 \) represents the base \( \mathbb{P}^1 \) while \( k = 2 \) represents the fiber \( \mathbb{P}^1 \) of \( F_n \).
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The elliptic fiber types

Next, we want to construct the relevant Calabi-Yau spaces as elliptic fibrations over the Hirzebruch surfaces $\mathbb{F}_n$. From these we will finally obtain the local $\mathcal{O}(-n) \to \mathbb{P}^1$ models by taking the size of the $\mathbb{P}^1$ fiber specified by the class $F$ of the Hirzebruch surface $\mathbb{F}_n$ to infinity. However, it turns out that there are multiple ways to realize the elliptic fiber singularity of the appropriate type leading to different Mordell-Weyl groups. In this chapter we will be interested in a rank one Mordell-Weyl group, that is an elliptic fibration with a single section. In the following we describe how to achieve this desired fibration structure.

Generically, for $n > 2$, the situation is such that the discriminant vanishes on the base $\mathbb{P}^1$ called $S'$ (we therefore have $\Sigma = S'$) and on isolated points of the fiber $\mathbb{P}^1$ denoted by $F$. Let us describe the non-compact geometry which arises when we take the size of $F$ to be infinite. We will denote curves on which the discriminant vanishes point-wise by $(-1)$-curves borrowing the terminology from E-strings. In fact, the analogy goes even further in that a subset of the $E_8$ Weyl symmetry of E-strings can act on sections of this elliptic fibration. The exact subset is determined by the elliptic fiber singularity type at the intersection points of the non-compact limit of $F$ with $S'$. We will denote the corresponding Kodaira group by $g_E$ which should not be confused with $g_\Sigma$ discussed in Section 4.2 which labels the fiber degeneration on $S'$. A consistency condition is that $g_\Sigma$ should be a subgroup of $g_E$, that is $g_\Sigma \subset g_E$. In order to restrict to one section for each model we only consider the fiber type $g_E = E_8$ in this chapter which leads to the following schematic picture of curve configurations:

For other choices of $g_E$ (simplest cases are $g_E = E_n$, $n = 3, \ldots, 8$, where $\{E_n\}_{n=3}^8 = A_1 \times A_2, A_4, D_4, E_6, E_7, E_8$) the subset of $E_8$ which becomes the Mordell-Weyl group of the elliptic Calabi-Yau is determined by the commutant of the Weyl group $g_E$ in the Weyl group of $E_8$. Note that gluing together the two non-compact $(-1)$-curves gives back the compact Calabi-Yau $M_3$ with base $B = \mathbb{F}_n$. $M_3$ can equivalently be viewed as a K3 fibration over the $(-n)$-curve as the elliptic fibration over $F$ has second Chern class 24 due to the two $E_8$-type degenerations of the elliptic fiber
shown in the above figure. For pure 6d gauge theories the Euler number of $M_3$ is given purely in terms of group theory data as \[ \chi_{\mathcal{P}G}(M_3) = -2C(g_E) \int_B c_1^2(B) - \text{rank}(g_{\Sigma}) C(g_{\Sigma}) \int_{\Sigma} c_1(\Sigma). \] (4.4.67)

Since we are interested in having a single section, we take $g_E = E_8$. In this case the generic elliptic fiber can be given as a degree 6 hypersurface in the weighted projective space $\mathbb{P}^2(1,2,3)$. For the different models labeled by $n$ the corresponding dual Coxeter number $C(g_{\Sigma})$ and the Euler numbers of the minimal compact Calabi-Yau manifolds are given in Table 4.1.

<table>
<thead>
<tr>
<th>7-brane</th>
<th>$n=1,2$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{\Sigma}$</td>
<td>$E_{1,2}$</td>
<td>$A_2$</td>
<td>$D_4$</td>
<td>$F_4$</td>
<td>$E_6$</td>
<td>$E_7^{\text{HM}}$</td>
<td>$E_8^{(1)}$</td>
<td>$E_8^{(2)}$</td>
<td>$E_8^{(3)}$</td>
<td>$E_8$</td>
<td></td>
</tr>
<tr>
<td>$C_g$</td>
<td>-</td>
<td>3</td>
<td>8</td>
<td>12</td>
<td>12</td>
<td>18</td>
<td>18</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$-\chi(M_3)$</td>
<td>480</td>
<td>492</td>
<td>528</td>
<td>576</td>
<td>624</td>
<td>676</td>
<td>732</td>
<td>780</td>
<td>840</td>
<td>900</td>
<td>960</td>
</tr>
<tr>
<td>$h_{11}(M_3)-1$</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>13</td>
<td>12</td>
<td>11</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 4.1: Table of Coxeter numbers $C_g$ and Euler numbers $\chi(M_3)$ for the different minimal SCFT Calabi-Yau threefolds.

Note that, compared to the local geometry associated to the 6d SCFT, the compact geometry leads to a larger number of hypermultiplets and one additional vector multiplet.

In the table, we also list the $E_8$ cases with a non-zero number $n_I = 12 - n$ of small instantons as $E_8^{(n_I)}$. Each instanton corresponds to an additional tensor multiplet in the 6d theory.
Each of the latter contains one additional modulus, so $h_{11}$ increases by $n_I$. The 6d anomaly cancellation condition \cite{23,126} moreover enforces the relation $\#_{HM} - \#_{VM} = 273 - 29n_I$, which implies $\chi(M_3) = \chi_{pG}(M_3) - 2C_{Es,E}n_I$. This yields the Hodge numbers $E_8^{(1,2,3)}$ in Table 4.1. The corresponding toric hypersurfaces are specified in Section 4.4.2.

**Tate’s algorithm for elliptic fiber singularities and toric constructions**

If the generic fiber is the elliptic curve $X_6(3,2,1)$, the elliptically fibered Calabi-Yau threefold over a base $B$ for this fiber type takes the Tate form

$$
y^2 + x^3 + a_6(u)z^6 + a_4(u)xz^4 + a_3(u)yz^3 + a_2(u)z^2x^2 + a_1(u)zxy = 0, \tag{4.4.68}
$$

where coordinates on the base $B$ are denoted generically by $u$.

The construction of gauge singularities inside an elliptically fibered Calabi-Yau $n$-fold with the necessary toric data to solve the topological string proceeds as follows \cite{127,128}. One constructs reflexive polyhedra such that the Calabi-Yau is given by the anti-canonical hypersurface $W_\Delta(Y) = 0$ in the corresponding toric variety, with $W_\Delta(Y)$ being in a generic Tate form (4.4.68). Then one chooses the divisor $\Sigma$ in $B$, restricts the coefficients of $W_\Delta(Y)$ so that at $\Sigma$ one has the suitable Tate singularity \cite{129}, constructs the Newton polytope $\Delta_r$ to the restricted polyhedron and its dual $\Delta_r^*$, and finally resolves all non-toric divisors by modifying $\Delta_r$, without changing the singularity at $\Sigma$.

For instance, if $B$ is a $\mathbb{P}^1$ fibration over $\Sigma$, one splits the coordinates of the Tate form into $\{Y_k\} = \{z, x, y, u_1, u_2, w, v\}$, where $u_i$ are coordinates of $\Sigma$ and $w, v$ are homogeneous coordinates of the $\mathbb{P}^1$ fiber. For example, for $w = 0$ the whole $\Sigma$ becomes a gauge divisor and by setting the coefficients of the monomials in $a_i(u, v, w, z)$ to zero (that is, choosing a specialization of the complex structure moduli), one can put the $a_i(u, v, w, z)$ in the following form:

$$
a_1 = \alpha_1 w^{[a_0]}, \quad a_2 = \alpha_2 w^{[a_2]}, \quad a_3 = \alpha_3 w^{[a_3]}, \quad a_4 = \alpha_4 w^{[a_4]}, \quad a_6 = \alpha_6 w^{[a_6]}, \tag{4.4.69}
$$

where $\alpha_i(u, v, w, z)$ are of order zero in $w$. Choosing this leading behavior at $w = 0$ leads by Tate’s
algorithm to singular fibers and hence results in a gauge group along \( \Sigma \). The association of the leading powers of \([a_i]\) with the singularity is given by Tate’s algorithm \([129]\). The discussion applies to the Hirzebruch surfaces as bases \( B \) which can be viewed as a \( \mathbb{P}^1 \) fibration over \( \Sigma = \mathbb{P}^1 \). As already mentioned the divisor \( \Sigma \) becomes \( S' \) in this case.

In the following sections we construct the minimal compact Calabi-Yau threefolds with the prescribed local geometries as hypersurfaces in a toric ambient space. Minimal means that they have just one additional modulus, whose decompactification leads to the local geometry. The cases of main interest have the Euler number \((4.4.67)\) as indicated in Table 4.1.

### 4.4.2 Solution of the topological string on the toric hypersurface Calabi-Yau spaces

Generically the Calabi-Yau geometries under consideration can be described as anticanonical hypersurfaces \( H \) given by

\[
W_\Delta(Y) = \sum_{\nu_i \in \Delta} a_i \sum_{\nu_k' \in \Delta^*} Y_k^{(\nu_i, \nu_k')} + 1 = 0
\]

in \( \mathbb{P}_{\Delta^*} \), where \((\Delta, \Delta^*)\) are reflexive polyhedra.

We denote by \( \nu_i^* \in \mathbb{Z}^4 \) the relevant points of \( \Delta^* \) whose complex hull in \( \mathbb{R}^4 \) is \( \Delta^* \) and by \( l^{(k)} \) the charges or Mori vectors, which fulfill

\[
\sum_i l^{(k)}_i \tilde{\nu}_i^* = 0 ,
\]

where \( \tilde{\nu}_i^* = (1, \nu_i^*) \). The Mori vectors span the Mori cone, which is dual to the Kähler cone. The possible choices of Mori cones constitute the secondary fan whose data are encoded in the possible star triangulations of \( \Delta^* \). Some of them are redundant, because the Calabi-Yau manifold still has the same Mori cones. Others correspond truly to different topological phases of the gauged linear sigma model. According to the theorem of C.T.C. Wall \([130]\) the topological type of the Calabi-Yau threefold \( M_3 \) is fixed by the independent Hodge numbers, which for an \( SU(3) \) holonomy manifold are \( h_{11} \) and \( h_{21} \), the triple intersection numbers \( J_i \cdot J_j \cdot J_k \) and evaluation of the second Chern class.
on the basis $J_i$ of divisors dual to the basis of the Kähler cone. Only the $J_i \cdot J_j \cdot J_k$ change in a non-trivial way in the transitions. Given the $l^{(k)}$ and the C.T.C. Wall topological data one can use toric mirror symmetry [131] to predict the genus zero BPS numbers for all toric hypersurfaces following [68].

$\mathcal{O}(-n) \to \mathbb{P}^1$ geometries with $n = 1, 2$

The cases $n = 1, 2$ have only Kodaira fibers of type $I_1$ over codimension one in the base and hence no gauge theory divisor.

To fix the notation used in the following sections we review the $n = 1$ case, which is of particular interest as the local geometry is the $1/2$K3 on which F-theory compactification yields the E-string theory. The refined BPS spectrum of the E-string has an interpretation as the refined stable pair invariants on the local geometry. The data associated to this geometry is summarized by the following table:

<table>
<thead>
<tr>
<th>Div.</th>
<th>$\tilde{\nu}^*_i$</th>
<th>$l_I^{(e)}$</th>
<th>$l_I^{(f)}$</th>
<th>$l_I^{(b)}$</th>
<th>$l_{II}^{(e)}$</th>
<th>$l_{II}^{(b)}$</th>
<th>$l_{II}^{(-b)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0$</td>
<td>1 0 0 0 0 0</td>
<td>-6 0 0</td>
<td>-6 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_1$</td>
<td>1 -1 0 0 0 0</td>
<td>2 0 0</td>
<td>2 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_2$</td>
<td>1 0 -1 0 0 0</td>
<td>3 0 0</td>
<td>3 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S'$</td>
<td>1 2 3 0 -1</td>
<td>1 1 -1</td>
<td>-1 0 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td>1 2 3 0 0</td>
<td>1 -2 -1</td>
<td>0 -3 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>1 2 3 -1 -1</td>
<td>0 0 1</td>
<td>1 1 -1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>1 2 3 0 1</td>
<td>0 1 0</td>
<td>0 1 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>1 2 3 1 0</td>
<td>0 0 1</td>
<td>1 1 -1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The polyhedron $\Delta^*$ has two star triangulations, denoted by subscripts $I$ and $II$, which lead to different Mori cones in the secondary fan. Such different Mori or Kähler cones can be understood as different geometrical phases of the 2d sigma model, which can have non-geometrical phases as well [39].
We give the C.T.C. Wall data for the first phase, namely phase $I$, which corresponds to the E-string geometry. Both phases have $h_{21} = 243$ and $h_{11} = 3$, and hence Euler number $\chi = -480$. The topological data in the phase marked with $I$ in (4.4.72) are encoded in

$$R_I = 8J_e^3 + 3J_e^2J_f + J_eJ_f^2 + 2J_eJ_b + J_eJ_fJ_b,$$  \hspace{1cm} (4.4.73)

whose coefficients are the classical triple intersection numbers $c_{ijk} = \int J_iJ_jJ_k$. The evaluation of the second Chern class is $\int c_2J_e = 92$, $\int c_2J_f = 36$, and $\int c_2J_b = 24$. We can see from (4.4.66) that $l_f^{(b)}$ corresponds to the section $[C_b] = [S']$ of the base in $F_1$, the $(-1)$ curve, while $l_f^{(f)}$ corresponds to the fiber $[C_f] = [F]$ in $F_1$, a $(0)$ curve. Over the $(-1)$ curve one has a $\frac{1}{2}$-K3, which is the divisor $J_f$ dual to $[C_f]$ in $M_3$, while over $[C_f]$ one has an elliptically fibered K3, which is the divisor $J_b$ dual to $[C_b]$ in $M$. According to Oguiso’s criterion [132] we see that the latter is a fibration of the geometry $M$ as the K3 does not intersect $J_b^2 = 0$ and $\int c_2J_b = 24$. The class $[C_e]$ represents the elliptic fiber.

The E-string partition function has the structure $Z = \exp(\lambda^{2g-2}F(g)(Q_\tau, Q_b))$ where the free energies have the form $F^{(g)} = \sum_{n=0}^{\infty} \tilde{F}^{(g)}_n(Q_\tau)Q_b^n$. Here $Q_\tau = \exp(2\pi i\tau)$ with $\tau$ the modular parameter and $\tilde{F}^{(g)}_n(Q_\tau) = \frac{Q_n^{\frac{g}{2}}}{\eta(Q_\tau)^{2n}}P^{(g)}_n$ with $P^{(g)}_n(E_2, E_4, E_6)$ an almost holomorphic modular form of weight $2g - 6n - 2$, e.g. $P^{(g)}_1(E_4) = E_4$ etc. One has hence to redefine the Kähler parameters so that $Q = Q_bQ^{\frac{1}{2}}$ and

$$F^{(g)} = \sum_{n=0}^{\infty} F^{(g)}_n(Q_\tau)Q^n.$$ \hspace{1cm} (4.4.74)

In the above formula $F^{(g)}_n(Q_\tau)$ are truly $SL(2, \mathbb{Z})$ invariant coefficients. The analogous redefinition has been made in (4.3.55) for the $D_4$ string. The analysis of the monodromies of $M_3$ that yield an $SL(2, \mathbb{Z})$ action on $\tau$ and a non-trivial decoupling limit fix the combination $Q$. This was discussed in detail in [133] in a similar context and applies to geometries discussed below.

The second phase is obtained by flopping the base $[C_b]$ out of the half K3, which becomes thereby an elliptic pencil, the del Pezzo surface with degree one. The latter can be obtained by
eight blow ups of $\mathbb{P}^2$ and is called therefore $d_8\mathbb{P}^2$. Note that the intersections are

$$\mathcal{R}_{II} = 8J_{e'}^3 + 3J_{e'}^2J_h + J_{e'}J_h^2 + 9J_{e'}^2J_{-b} + 3J_{e'}J_hJ_{-b} + J_h^2J_{-b} + 9J_{e'}J_{-b}^2 + 3J_hJ_{-b}^2 + 9J_{-b}^3,$$

while the evaluation of the second chern class is given by $\int c_2J_i = \{92, 36, 102\}$. The transformation of the basis $l^{(e)}_I = l^{(e)}_I + l^{(b)}_I$, $l^{(f)}_I = l^{(f)}_I + l^{(b)}_I$ and $l^{(-b)}_I = -l^{(b)}_I$ gives already almost the intersection ring $\mathcal{R}_{II}$ except that one gets $8J_{-b}^3$ instead of $9J_{-b}^3$, i.e. in a coordinate independent formulation one observes that the triple intersection of the divisors dual to the rational curve that gets flopped increases by +1. This can be argued in general in various ways, see e.g. [117].

The case $n = 2$ has only one phase:

<table>
<thead>
<tr>
<th>Div.</th>
<th>$\bar{\nu}_i^*$</th>
<th>$l_I^{(e)}$</th>
<th>$l_I^{(f)}$</th>
<th>$l_I^{(b)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0$</td>
<td>1 0 0 0 0</td>
<td>-6 0 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_1$</td>
<td>1 -1 0 0 0</td>
<td>2 0 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_2$</td>
<td>1 0 -1 0 0</td>
<td>3 0 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S'$</td>
<td>1 2 3 0 -1</td>
<td>0 1 -2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td>1 2 3 0 0</td>
<td>1 -2 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>1 2 3 -1 -1</td>
<td>0 0 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>1 2 3 0 1</td>
<td>0 1 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>1 2 3 1 0</td>
<td>0 0 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In this phase one has a $K3$ and an elliptic fibration and the intersection ring is in general

$$\mathcal{R} = 8J_e^3 + 4J_e^2J_f + 2J_eJ_f^2 + 2J_e^2J_b + J_eJ_fJ_b,$$  \hfill (4.4.76)

with

$$\int c_2J_e = 92, \quad \int c_2J_f = 48, \quad \int c_2J_b = 24.$$  \hfill (4.4.77)

The $n = 2$ geometry corresponds to the $A_1$ $\mathcal{N} = (2, 0)$ SCFT; by making the elliptic fiber singular over the $(-2)$ curve, one obtains the M-string geometry which was studied in detail in Chapters 2 and 3.
\( \mathcal{O}(-3) \to \mathbb{P}^1 \) geometry with \( \hat{A}_2 \) resolution

The easiest example with a non-Higgsable gauge symmetry is the \( A_2 \) case, which has the following polyhedron \( \Delta^* \):

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{Div.} & \nu_i^* & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l_{T2} & l^{(1)}_{p2} & l^{(2)}_{p2} & l^{(3)}_{p2} & l_{de} \\
\hline
D_0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -6 & -3 & 0 & 0 \\
D_1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 2 & 1 & 1 & 0 & 0 \\
D_2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 3 & 3 & 0 & 0 & 0 \\
D_3 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 2 & 0 & -3 & 0 & 0 & 0 \\
D_4 & 1 & 2 & 0 & -1 & 1 & -3 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 \\
S' & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\
K & 2 & 3 & 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 & 0 & 0 & -3 & 0 \\
F & 2 & 3 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -3 & 0 \\
S & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
F & 2 & 3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \frac{1}{3} \\
\hline
\end{array}
\]

We study the basis which is appropriate to exhibit the curve classes that exhibit the affine \( \hat{A}_2 \) singularity over the divisor \( S' \), which are depicted in the figure below:

\[
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]

This basis corresponds to the following choice of vectors:

\[
\begin{align*}
l_{A_2}^1 &= 4l^{(1)} + l^{(2)} + l^{(5)} = (-4, 1, 3, -2, 1, 1, 0, 0, 0, 0), \\
l_{A_2}^2 &= l^{(1)} + l^{(2)} + l^{(5)} = (-1, 1, 0, 1, -2, 1, 0, 0, 0, 0), \\
l_{A_2}^3 &= l^{(1)} + l^{(2)} + l^{(4)} + l^{(5)} = (-1, 0, 0, 1, 1, -2, 1, 0, 0, 0), \\
l_b &= -(l^{(1)} + l^{(5)}) \quad \text{ (4.4.79)} \]
\[
l_{de} &= 5l^{(1)} + \frac{8}{3}l^{(2)} + l^{(3)} + \frac{10}{3}l^{(4)} + \frac{8}{3}l^{(5)} = (-5, \frac{14}{3}, \frac{7}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{8}{3}, 0, 1, 0))
\end{align*}
\]
Chapter 4: Theories with a single tensor multiplet

Note that we have flopped the \( \mathbb{P}^1 \) represented by the vector \( l^{(1)} + l^{(5)} \) in the Mori cone in order to arrive at the appropriate \( \mathbb{P}^1 \) base for the affine \( \hat{A}_2 \) singularity. The \( \mathbb{P}^1 \) base is represented by the Mori vector \( l_b \), which intersects the three rational components of the degenerate elliptic curve with \((-1)\). The decompactification direction can be specified as a rational element in \( H_2(M_3) \) so that the intersection form of the compact two-dimensional part becomes

\[
\frac{1}{3^3} l_b \sum_{i,j=1}^{3} C_{ij} J_i^{(i)} J_j^{(j)}. \tag{4.4.80}
\]

We note that the Coxeter labels \( a^i \) have the property that for \( C_{ij} \) the affine Cartan matrix

\[
\sum_{j=0}^{r} a^j C_{ij} = 0. \tag{4.4.81}
\]

A first check on our identification is therefore that the \( \mathbb{P}^1 \) curve classes called \( l^{(i)}_{D_4} \) add up to the class of the elliptic fiber with the Coxeter lables indicated at the affine Dynkin diagram, i.e.

\[
l_{T^2} = 1 l^{(1)}_{A_2} + 1 l^{(2)}_{A_2} + 1 l^{(3)}_{A_2}. \]

This is geometrically required, because the curve class of the elliptic fiber has self intersection 0. We list in the following some of the BPS invariants \( n_{d_0,d_1,d_2,d_3}^{(0)} \), where the degree in the decompactified direction is zero. The number \( d_0 \) corresponds to the base wrapping number and therefore indicates the string charge in the 6d SCFT whereas the other numbers correspond to the flavor fugacity charges. The numbers \( n_{d_0,d_1,d_2,d_3}^{(0)} \) are symmetric in \( d_1^{A_2}, d_2^{A_2}, d_3^{A_2} \). Since the emphasis of this chapter is on the strings of the 6d SCFTs, we focus on the BPS invariants corresponding to \( n_b \geq 1 \). For example, the following tables display BPS invariants corresponding to \( n_b = 1, 2 \) and small values of \( d_1^{A_2} \).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
d_1^{A_2} \\
\hline
0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 1 & 3 & 5 & 7 & 9 & 11 \\
1 & 3 & 4 & 8 & 12 & 16 & 20 \\
2 & 5 & 8 & 9 & 15 & 21 & 27 \\
3 & 7 & 12 & 15 & 16 & 24 & 32 \\
4 & 9 & 16 & 21 & 24 & 25 & 35 \\
5 & 11 & 20 & 27 & 32 & 35 & 36 \\
\hline
\end{array}
\]

\[
n_{d_0=1,d_1,d_2,d_3}^{(0)}
\]
Chapter 4: Theories with a single tensor multiplet

<table>
<thead>
<tr>
<th>$d^1_{A_2} \setminus d^2_{A_2}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>36</td>
<td>60</td>
<td>84</td>
<td>108</td>
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<td>2</td>
<td>36</td>
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<td>252</td>
</tr>
<tr>
<td>4</td>
<td>84</td>
<td>144</td>
<td>180</td>
<td>208</td>
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<tr>
<td>5</td>
<td>108</td>
<td>192</td>
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<td>320</td>
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</table>

$n_{d^1_{A_2}=1,d^2_{A_2}}^{(0)}$

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<td>2</td>
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<td>288</td>
<td>465</td>
<td>651</td>
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<td>456</td>
<td>735</td>
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<td>4</td>
<td>465</td>
<td>735</td>
<td>954</td>
<td>1371</td>
</tr>
<tr>
<td>4</td>
<td>651</td>
<td>1080</td>
<td>1371</td>
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</table>

$n_{d^1_{A_2}=1,d^2_{A_2}}^{(0)}$

<table>
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<tr>
<td>3</td>
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$n_{d^1_{A_2}=2,d^2_{A_2}}^{(0)}$

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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
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<td>0</td>
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<td>0</td>
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<td>-70</td>
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<td>-10</td>
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$n_{d^1_{A_2}=2,d^2_{A_2}}^{(0)}$

<table>
<thead>
<tr>
<th>$d^1_{A_2} \setminus d^2_{A_2}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
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<td>-1432</td>
<td>-4280</td>
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<tr>
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<td>-850</td>
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<td>-6084</td>
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<tr>
<td>4</td>
<td>-1432</td>
<td>-3164</td>
<td>-6084</td>
<td>-13000</td>
<td>-29526</td>
</tr>
<tr>
<td>5</td>
<td>-4280</td>
<td>-9720</td>
<td>-16960</td>
<td>-29526</td>
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</tbody>
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$n_{d^1_{A_2}=2,d^2_{A_2}}^{(0)}$
Chapter 4: Theories with a single tensor multiplet

<table>
<thead>
<tr>
<th>$d^1_{A_2}$ \ $d^2_{A_2}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>27</td>
<td>286</td>
<td>1651</td>
</tr>
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<td>800</td>
<td>1998</td>
<td>6260</td>
<td>21070</td>
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<td>5</td>
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<td>5184</td>
<td>11473</td>
<td>26880</td>
<td>70362</td>
<td>191424</td>
</tr>
</tbody>
</table>

\[ n_{d_0=3,d_1^1,d_2^2}^{(0)} \]

$O(-4) \to \mathbb{P}^1$ geometry with $\tilde{D}_4$ resolution

In this section we describe the elliptic Calabi-Yau which has base $B = \mathbb{F}_4$ and corresponds to the two-dimensional quiver studied in Section 4.3. Taking the local limit by sending the size of the $\mathbb{P}^1$ fiber of $\mathbb{F}_4$ to infinity one arrives at a local Calabi-Yau which has an affine $\tilde{D}_4$ Kodaira singularity over the $(-4)$ curve. The singularity in the elliptic fiber is resolved by sphere configurations with the affine $D_4$ intersection numbers and multiplicities as depicted in the following diagram:

![Diagram](image)

The toric data are given by a reflexive polyhedron $\Delta^*$, whose points $\nu$ are the first entries in the table below.
The Calabi-Yau hypersurface has \( \chi(M_3) = -528 \) and \( h_{11} = 7 \). The polyhedron \( \Delta^* \) has 30 star triangulations and the \( l^{(1)}, \ldots, l^{(7)} \) are generators of a simple geometrical Mori cone, which are needed to solve the topological string on the global model. Note that the evaluation of the Kähler classes against the second Chern class are:

\[
\{ \int c_2 J_i \} = \{ 620, 204, 140, 24, 72, 452, 616 \}.
\]

It turns out that \( J_4 \) appears only linearly in the intersection ring and \( \int c_2 J_4 = 24 \). Oguiso’s criterion implies that \( M_3 \) is a \( K3 \) fibration whose base \( \mathbb{P}^1 \) is represented by \( l^{(4)} \). This \( \mathbb{P}^1 \) is also the base of the local surface \( B = \mathcal{O}(-4) \to \mathbb{P}^1 \) and we will henceforth denote it by \( l_b \). Since \( J_5 \) appears only quadratically in the intersection ring, \( l^{(5)} \) represents the base of the elliptic fibration. Also, \( l_{T^2} \) represents the elliptic fiber class and the \( l_i^{(4)} D_4 \) correspond to the simple roots of the affine \( D_4 \).

Finally, \( l_{de} \) is the class that one can take large to zoom in on the local surface geometry. The
relation to the nef classes in the global model are

\[ l_{T^2} = 6l^{(1)} + 2l^{(2)} + l^{(3)} + 4l^{(6)} + 6l^{(7)}, \quad l^{(1)}_D = l^{(6)}, \quad l^{(2)}_D = 2l^{(1)} + l^{(6)} + 2l^{(7)}, \]  
\[ l^{(3)}_D = 2l^{(1)} + l^{(6)}, \quad l^{(4)}_D = l^{(3)} + l^{(6)}, \quad l^{(c)}_D = l^{(1)} + l^{(2)} + 2l^{(7)}. \]  
(4.4.83)

A first check on these identifications is that the \( P^1 \) curve classes called \( I^{(i)}_D \) add up to the class of the elliptic fiber with Coxeter labels \( a^i \) indicated in the affine Dynkin diagram, i.e.

\[ l_{T^2} = 1l^{(1)}_D + 1l^{(2)}_D + 1l^{(3)}_D + 1l^{(4)}_D + 2l^{(c)}_D, \]  
(4.4.84)

which is obviously the case. Another check is that after transforming to this basis the intersection form takes on a very simple appearance and is symmetric in \( l^{(i)}_D \) for \( i = 1, \ldots, 4 \):

\[ R = 9J_{T^2}^3 + 3J_4J_{T^2}^2 + 6J_{de}J_{T^2}^2 + 4J_{de}^2J_{T^2} + J_4J_{de}J_{T^2} - \sum_{i=1}^{4} (J_{D^4}^{(i)})^3 - \frac{1}{2}J_4 \sum_{i=1}^{4} (J_{D^4}^{(i)})^2. \]  
(4.4.85)

The curve whose volume has to be scaled to infinity to decouple the \( O(4) \to P^1 \) geometry from the compact manifold is the Kähler class dual to the Mori cone vector \( l^{(5)} \). This decompactifies the base of the Cababi-Yau threefold by scaling the fiber of the Hirzebruch surface to infinity. The class can be further modified to \( l_{de} \) above to make the intersections even simpler.

Let us next come to the evaluation of BPS numbers. We denote the charge associated to the class dual to \( l^{(4)} \), which is the base, by \( n_b \), the one dual to \( l_{de} \) by \( n_{de} \), the ones dual to \( l^{(i)}_D \) by \( n_1, n_2, n_3, n_4, n_c \) and the one dual to \( l_{T^2} \) by \( n_e \). From the viewpoint of the strings of the 6d SCFT, \( n_b \) denotes the string charge; \( n_i, i = 1, \ldots, 4, n_c \) correspond to the flavor fugacity charges and \( n_e \) is the exponent of \( Q_\tau \) in an expansion of the elliptic genus \( Z_{n_b} \). We consider by definition of the local limit only \( n_{de} = 0 \). Due to the relation (4.4.84) the class \( l_{T^2} \) is not an independent class and therefore when labelling BPS states we can omit the dependence on \( n_e \). The genus zero invariants are then given by \( n_{n_b,n_1,n_2,n_3,n_4,n_c}^{(0)} \) and are fully symmetric in \( n_1, \ldots, n_4 \). Let us first focus on the BPS states associated to a single string (that is, those corresponding to \( n_b = 1 \)). For \( n_e = 0 \) and \( n_c = 1 \) we get:
Chapter 4: Theories with a single tensor multiplet

\[
\begin{array}{cccccc}
 n_1 \backslash n_2 & 0 & 1 & 2 & 3 & 4 & 5 \\
 0 & -4 & -6 & -6 & -10 & -14 & -18 \\
 1 & -6 & -8 & -6 & -10 & -14 & -18 \\
 2 & -6 & -6 & 0 & 0 & 0 & 0 \\
 3 & -10 & -10 & 0 & 0 & 0 & 0 \\
 4 & -14 & -14 & 0 & 0 & 0 & 0 \\
 5 & -18 & -18 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[n_{1,n_1,n_2,0,0,1}^{(0)}\]

For \(n_e = 0\) and \(n_c = 2\) we get:

\[
\begin{array}{cccccc}
 n_1 \backslash n_2 & 0 & 1 & 2 & 3 & 4 & 5 \\
 0 & -6 & -10 & -12 & -12 & -18 & -24 \\
 1 & -10 & -16 & -18 & -16 & -24 & -32 \\
 2 & -12 & -18 & -18 & -12 & -18 & -24 \\
 3 & -12 & -16 & -12 & 0 & 0 & 0 \\
 4 & -18 & -24 & -18 & 0 & 0 & 0 \\
 5 & -24 & -32 & -24 & 0 & 0 & 0 \\
\end{array}
\]

\[n_{1,n_1,n_2,0,0,2}^{(0)}\]

For \(n_e = 1\) one finds:

\[
\begin{array}{cccc}
 n_1 \backslash n_2 & 0 & 1 & 2 \\
 0 & -80 & -78 & -96 \\
 1 & -78 & -48 & 32 \\
 2 & -96 & -32 & - \\
 3 & -144 & - & - \\
\end{array}
\]

\[n_{1,n_1+1,n_2+1,1,1,2}^{(0)}\]

Let us now consider the two-string sector. For \(n_b = 2\), \(n_e = 0\) and \(n_c = 1\) we get:

\[
\begin{array}{cccccc}
 n_1 \backslash n_2 & 0 & 1 & 2 & 3 & 4 & 5 \\
 0 & -6 & -10 & -12 & -30 & -98 & -306 \\
 1 & -10 & -16 & -18 & -40 & -112 & -324 \\
 2 & -12 & -18 & -18 & -30 & -42 & -54 \\
 3 & -30 & -40 & -30 & -50 & -70 & -90 \\
 4 & -98 & -112 & -42 & -70 & -98 & -126 \\
 5 & -306 & -324 & -54 & -90 & -126 & -162 \\
\end{array}
\]

\[n_{2,n_1,n_2,0,0,1}^{(0)}\]

For \(n_b = 1,\ldots,4\) and \(n_c = 0\) we also find the following invariants:

\[
\begin{array}{cccccccccccc}
 n_b \backslash n_1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 1 & -2 & -2 & -4 & -6 & -8 & -10 & -12 & -14 & -16 & -18 & -20 \\
 2 & 0 & 0 & 0 & -6 & -32 & -110 & -288 & -644 & -1280 & -2340 & 4000 \\
 3 & 0 & 0 & 0 & 0 & -8 & -110 & -756 & -3556 & -13072 & -40338 & -109120 \\
 4 & 0 & 0 & 0 & 0 & 0 & -10 & -288 & -3556 & -27264 & -153324 & -690400 \\
\end{array}
\]

\[n_{n_b,n_1,0,0,0,0}^{(0)}\]
Remarkably, and this is the main non-trivial test of the chapter, all the invariants listed
above can be reproduced from the elliptic genus computation in Section 4.3!

$\mathcal{O}(-5) \to \mathbb{P}^1$ geometry with $\hat{F}_4$ resolution

This elliptic singularity corresponds now to a non-simply laced Lie algebra. Unlike in
the simply laced case, the Coxeter labels differ from the dual Coxeter labels. We indicate both
Coxeter/dual Coxeter labels in the following diagram:

The toric polyhedron has 25 star triangulations; in the next table we present the polyhedron
together with the simplest choice of Mori vectors:

<table>
<thead>
<tr>
<th>$D$</th>
<th>$\nu_i^*$</th>
<th>$l^{(1)}$</th>
<th>$l^{(2)}$</th>
<th>$l^{(3)}$</th>
<th>$l^{(4)}$</th>
<th>$l^{(5)}$</th>
<th>$l^{(6)}$</th>
<th>$l^{(7)}$</th>
<th>$l^{(4)}_{\hat{F}_4}$</th>
<th>$l^{(3)}_{\hat{F}_4}$</th>
<th>$l^{(3)}_{\hat{F}_4}$</th>
<th>$l^{(1)}_{\hat{F}_4}$</th>
<th>$l^{(6)}_{\hat{F}_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0$</td>
<td>0 0 0 0</td>
<td>-2 0 0 0 0 0 0 0</td>
<td>0 0 0 0 -2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_1$</td>
<td>-1 0 0 0</td>
<td>0 -2 0 0 0 0 0</td>
<td>1 0 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_2$</td>
<td>0 -1 0 0</td>
<td>1 0 0 0 0 0 0</td>
<td>0 0 0 0 0 0</td>
<td>0 0 0 0 0 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_3$</td>
<td>0 1 0 -1</td>
<td>0 3 0 0 0 0 1</td>
<td>-2 -2 1 0 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_4$</td>
<td>1 2 0 -2</td>
<td>2 0 0 0 0 0 -2</td>
<td>1 1 -2 0 0 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S'$</td>
<td>2 3 0 -1</td>
<td>0 1 -2 0 1 0 0</td>
<td>0 0 -2 1 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S''$</td>
<td>2 3 0 -2</td>
<td>1 -2 1 -1 0 0 0</td>
<td>0 0 1 -2 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S'''$</td>
<td>2 3 0 -3</td>
<td>-2 0 0 -1 0 1 0</td>
<td>0 1 0 1 -2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td>2 3 0 0</td>
<td>0 0 1 0 -2 0 0</td>
<td>0 0 1 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>2 3 -1 -5</td>
<td>0 0 0 1 0 0 0</td>
<td>0 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>2 3 0 1</td>
<td>0 0 0 0 1 0 0</td>
<td>0 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F$</td>
<td>2 3 1 0</td>
<td>0 0 0 1 0 0 0</td>
<td>0 0 0 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4.4.86)
Evaluation of the second Chern class on the Kähler forms yields

\[ \{ \int c_2 J_i \} = \{ 336, 240, 164, 24, 84, 692, 708 \} . \]

The intersection ring has the property that \( J_4 \) appears only linearly so that \( l^{(4)} \) represents the base of a K3 fibration and the base of the local surface \( B \), we therefore have \( l^{(4)} = l_b \). \( J_5 \) appears only quadratically in the intersection ring and \( l^{(5)} \) represents the base of an elliptic fibration. As before, it is the normal direction to the base of the local surface and gets decompactified.

We find the following basis, which reflect the curves that represent the Cartan elements of the affine \( \hat{F}_4 \).

\[
\begin{align*}
    l^{(0)}_{\hat{F}_4} &= l^{(3)}_{\hat{F}_4},
    l^{(1)}_{\hat{F}_4} &= 2l^{(7)}_{\hat{F}_4} + l^{(2)}_{\hat{F}_4} + l^{(6)}_{\hat{F}_4},
    l^{(2)}_{\hat{F}_4} &= l^{(1)}_{\hat{F}_4},
    l^{(3)}_{\hat{F}_4} &= l^{(6)}_{\hat{F}_4},
    l^{(4)}_{\hat{F}_4} &= l^{(7)}_{\hat{F}_4}.  
\end{align*}
\]  

(4.4.87)

We see that in this basis

\[ l_T^2 = l^{(0)}_{\hat{F}_4} + 2l^{(1)}_{\hat{F}_4} + 3l^{(2)}_{\hat{F}_4} + 4l^{(3)}_{\hat{F}_4} + 2l^{(4)}_{\hat{F}_4}, \]

(4.4.88)

as expected (the vector corresponding to a given node is multiplied by the Coxeter label of that node). Note that the height in the last coordinate of \( \nu_i^* \) is the dual Coxeter number of the Dynkin diagram of \( F_4 \). This is a consequence of the \( F \)-theory realization of the \( G \) bundle moduli space of the heterotic string on an elliptically fibered K3 over the same \( \mathbb{P}^1 \) base as \( \mathbb{P}(a_0, \ldots, a_r) \) [134] and will hold for all models below. From these data, which we have calculated up to high multi-degree, it is possible to calculate genus zero BPS invariants analogously to the \( n \leq 4 \) cases.

\( \mathcal{O}(-6) \to \mathbb{P}^1 \) geometry with \( \hat{E}_6 \) resolution
In this base the hypersurface CY has Euler number $\chi(M_3) = -624$ and $h_{11} = 9$. The polyhedron $\Delta^*$ has 199 star triangulations. Again we choose a simple one

\[
\begin{array}{c|cccccccc}
D & \nu_i^* & l^{(1)} & l^{(2)} & l^{(de)} & l^{(4)} & l^{(6)} & l^{(7)} & l^{(9)} \\
\hline
D_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
D_1 & 0 & -1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
D_2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
D_3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
D_4 & 0 & 1 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 \\
D_5 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\
D_6 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
F & 2 & 3 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S' & 2 & 3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
S'' & 2 & 3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
S''' & 2 & 3 & 0 & -1 & 1 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
K & 2 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
S & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

which has the property that

\[
\{ \int c_2J_i \} = \{ 276, 360, 96, 188, 24, 764, 1524, 728, 800 \}.
\]

By Oguiso’s criterion $J_b$ represents the volume of the base of a K3 fibration while $J_{de}$ represents the volume of the base of an elliptic fibration. We have calculated the genus zero BPS invariants up to multi-degree 21.
The cases with $\hat{E}_7$ resolution

Here we discuss the cases $O(-n) \to \mathbb{P}^1$ with $n = 7$ and $n = 8$. Let us start with $n = 8$, that is, the pure $E_7$ gauge theory case. Listed below is the Mori cone that corresponds to a simple of a total of 420 star triangulations of $\Delta^*$.

\[
\begin{array}{cccccccccc}
D & \nu_i^* & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l^{(6)} & l^{(7)} & l^{(8)} & l^{(9)} & l^{(10)} \\
D_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
D_1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
D_2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
D_3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
D_4 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
D_5 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
D_6 & 1 & 2 & 0 & -3 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\
S' & 2 & 3 & 0 & -4 & -3 & 0 & -2 & 0 & 0 & 1 & 0 \\
S'' & 2 & 3 & 0 & -3 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\
S''' & 2 & 3 & 0 & -2 & 0 & 1 & 0 & 1 & -2 & 0 & 0 \\
S'''' & 2 & 3 & 0 & -1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
F & 2 & 3 & -1 & -8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
K & 2 & 3 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\
S & 2 & 3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
F & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]
Chapter 4: Theories with a single tensor multiplet

By standard toric methods we calculate $\chi(M_3) = -732$ and $h_{11} = 10$, i.e. $h_{21} = 376$. For this choice of Mori cone one has:

$$\{ \int_{M_3} c_2 J_i \} = \{ 560, 456, 24, 120, 236, 348, 1724, 1764, 884, 848 \}.$$  

In the intersection ring $J_3$ appears only linearly, while $J_4$ appears only quadratically indicating a K3 and an elliptic fibration structure respectively.

The case with an additional $\frac{1}{2}56$ hypermultiplet is obtained by replacing the point $v^*_F = (2, 3, -1, -8)$ with $v^*_F = (2, 3, -1, -7)$. The Hodge numbers change to $h_{21} = 348$, $h_{11} = 10$, and hence $\chi(M_3) = -676$. The only change in the Mori generators is the modified element

$$l^{(3)} = (0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 1),$$

which leads to

$$\{ \int_{M_3} c_2 J_i \} = \{ 524, 408, 24, 108, 212, 312, 1592, 1608, 806, 792 \}.$$  

While the intersection ring is modified, the fibration structure with respect to the classes $J_3$ and $J_4$ is maintained. For the $\hat{E}_7$ geometries, we have computed genus zero BPS invariants up to multi-degree 21.

The cases with $\hat{E}_8$ resolution

The Calabi-Yau hypersurface has $h_{11} = 11$ and $\chi(M_3) = -960$ and is the case with maximal absolute value of the Euler number within the class of toric hype-surfaces. From the 588 star triangulations of $\Delta^*$ we choose one leading to simple Mori cone with desired fibration structure:
The evaluation of the Kähler forms against the second Chern class is:

$$\{ \int_{M_3} c_2 J_i \} = \{1496, 948, 168, 332, 492, 648, 800, 24, 1092, 2228, 3360\}.$$  

We have a K3 fibration over the $\mathbb{P}^1$ represented by $J_8$ and an elliptic fibration over the base whose volume is given by $J_3$. As in all models with K3 fibrations, the BPS numbers depend only on the square of the curve classes in the K3, whose intersection form in the Picard lattice is given by coefficients of the ring that is linear in $J_8$, and can be obtained by a heterotic one-loop calculation.

The genus zero BPS states are available to multi-degree 18.
Chapter 4: Theories with a single tensor multiplet

The geometry can be modified by successively adding tensor multiplets corresponding to small instantons. This can be done by blow-ups of \((-1)\) curves in the base. In the particular case of \(E_8\) gauge symmetry this corresponds to the blowing up Hirzebruch surfaces \(F_n, n = 11, 10, 9\). We can construct explicitly models in which all divisors have toric representatives, by modifying \(\Delta^*\) in the following way:

- \(n = 11\) (one small instanton case): \(\nu_F\) gets replaced by two points:

  \[\nu_F \rightarrow \{\{2, 3, -1, -11\}, \{0, 0, 0, -1\}\} .\]

  Thus \(h_{11} = 12\), and in accord with the 6d anomaly condition we find \(\chi(M_3) = -900\);

- \(n = 10\): one must replace \(\nu_F \rightarrow \{\{2, 3, -1, -10\}, \{0, 0, 0, -1\}, \{2, 3, 1, -1\}\} ;\)

- \(n = 9\): here, \(\nu_F \rightarrow \{\{2, 3, -1, -9\}, \{0, 0, 0, -1\}, \{2, 3, 1, -1\}, \{2, 3, 1, -2\}\} .\)

4.5 Discussion of results

In this chapter we have made further progress in studying the strings of six-dimensional conformal field theories by identifying an appropriate geometric realization for them in terms of hypersurfaces in a toric Calabi-Yau geometry and computing the topological string partition function on this class of geometries. From this we extracted BPS invariants of the five-dimensional theory that one obtains by moving to the tensor branch of the six-dimensional theory and compactifying on a circle. This information can be used to reconstruct the elliptic genus of the self-dual strings of the six-dimensional theory as an expansion in its various fugacities.

We were able to say more in the case of the \(\mathcal{O}(-n) \rightarrow \mathbb{P}^1\) geometries with \(n = 1, 2, 4\), for which the modulus of the torus is not fixed and it was possible to obtain a Type IIB description by taking Sen’s limit. The \(n = 1\) and \(n = 2\) cases correspond respectively to the E-string and M-string; for these theories we were able to exactly reproduce the elliptic genera of the strings from two-dimensional quiver gauge theories that describe their dynamics. In the \(n = 4\) case,
by resorting to the Type IIB description of this theory in terms of branes and orientifold planes we derived a corresponding two-dimensional quiver, and by computing its elliptic genus we found perfect agreement with topological string computations in the case of one and two strings.

In the next chapter we further extend these results by constructing two-dimensional quivers for SCFTs with several tensor multiplets, by taking the quivers discussed in this chapter as building blocks and combining them appropriately.
Chapter 5

Theories with several tensor multiplets

5.1 Introduction

In the previous chapters we have discussed how under favorable circumstances one can find two-dimensional quiver gauge theories with $\mathcal{N} = (0,4)$ supersymmetry which flow in the IR to the CFTs describing the self-dual strings of six-dimensional SCFTs. In this chapter we extend the list of theories for which such a description is available to include three additional classes of SCFTs:

- The theory of M5 branes probing a singularity of D type, which corresponds in F-theory to a linear chain of alternating $(-1)$ and $(-4)$ curves which support respectively gauge group $SO(8 + 2p)$ and $Sp(p)$.

- The theory of $N$ small $E_8$ instantons, or equivalently $N$ M5 branes probing the M9 plane of Hořava-Witten theory [6,110]. This corresponds to a single $(-1)$ curve linked to a chain of $N - 1$ curves of self-intersection $-2$. Upon circle compactification, this theory admits deformation by a parameter corresponding to the mass of a 5d anti-symmetric hypermultiplet.
We focus on the case where this parameter is turned off.

- The theory of $N$ D5 branes probing an ADE singularity. This corresponds in F-theory to an ADE configuration of $-2$ curves supporting gauge groups of $SU$ type.

We compute elliptic genera of strings via localization [41,43]; we do this for certain specific bound states of strings for the first two classes of theories, and for arbitrary bound states of strings for the third class.

The organization of this chapter is as follows: In Section 2 we review some of the building blocks for the $(0,4)$ supersymmetric 2d quiver gauge theories which will be needed for the description of the worldsheet degrees of freedom of the tensionless strings. In Section 3 we discuss the 2d quiver for the strings of the theory of M5 branes probing A- or D- type singularities. In Section 4 we study the quiver for the strings of the theory of $N$ small $E_8$ instantons. In Section 5 we discuss the quiver for strings of the theory of D5 branes probing an ADE singularity.

### 5.2 Quiver gauge theories in two dimensions

We are interested in computing the elliptic genera of the strings that arise on tensor branches of 6d $(1,0)$ SCFTs with several tensor multiplets, along the lines of Chapter 4. In this chapter we aim to obtain 2d quiver gauge theories for a variety of 6d SCFTs that arise within M- and F-theory. These will generally consist of $(0,4)$ quiver gauge theories with gauge group $\prod_{i=1}^N G_i(k_i)$, where $G_i$ is the gauge group associated to $k_i$ strings of the $i$th type, and will capture the dynamics of a bound state of such a collection of strings. The gauge groups arising in the theories discussed in this chapter are either unitary, symplectic, or orthogonal.

In what follows we summarize our notation for the following sections. In drawing quiver diagrams, we will denote $(0,2)$ Fermi multiplets by dashed lines and $(0,4)$ hypermultiplets by a solid line. The various fields of the quiver gauge theory can be organized in terms of four combinations of $(0,2)$ multiplets:
Chapter 5: Theories with several tensor multiplets

1. To each gauge node $i$ corresponds the following field content valued in representations of $G_i$ (corresponding to $n_i$ strings of the $i$th kind): a vector multiplet $\Upsilon_i$; a Fermi multiplet $\Lambda^\phi_i$; and two chiral multiplets $B_i, \tilde{B}_i$.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$(0,2)$ field content</th>
<th>$U(1)_{c_1}$</th>
<th>$U(1)_{c_2}$</th>
<th>$U(1)_m$</th>
<th>$U(1)_R$</th>
<th>$G_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Upsilon_i$ (vector)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>adj.</td>
<td></td>
</tr>
<tr>
<td>$\Lambda^\phi_i$ (Fermi)</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>adj.</td>
<td></td>
</tr>
<tr>
<td>$B_i$ (chiral)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{R}$</td>
<td></td>
</tr>
<tr>
<td>$\tilde{B}_i$ (chiral)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{R}$</td>
<td></td>
</tr>
</tbody>
</table>

The representation $\mathbb{R}$ is the adjoint representation whenever the gauge group is unitary, symmetric whenever the gauge node is orthogonal, and anti-symmetric if the gauge group is symplectic.

2. Between each pair of nodes $i, j$ such that the corresponding element of the adjacency matrix of the underlying quiver $M_{ij}$ is non-zero one has the following bifundamental fields of $G_i \times G_j$: two Fermi multiplets $\Lambda^B_{ij}, \Lambda^{\tilde{B}}_{ij}$, and chiral multiplets $\Phi_{ij}, \Sigma_{ij}$ forming a twisted $(0, 4)$ hypermultiplet.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$(0,2)$ field content</th>
<th>$U(1)_{c_1}$</th>
<th>$U(1)_{c_2}$</th>
<th>$U(1)_m$</th>
<th>$U(1)_R$</th>
<th>$G_i \times G_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda^B_{ij}$ (Fermi)</td>
<td>1/2</td>
<td>-1/2</td>
<td>1</td>
<td>0</td>
<td>$\boxtimes \boxtimes$</td>
<td></td>
</tr>
<tr>
<td>$\Lambda^{\tilde{B}}_{ij}$ (Fermi)</td>
<td>-1/2</td>
<td>1/2</td>
<td>1</td>
<td>0</td>
<td>$\boxtimes \boxtimes$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{ij}$ (chiral)</td>
<td>-1/2</td>
<td>-1/2</td>
<td>1</td>
<td>1</td>
<td>$\boxtimes \boxtimes$</td>
<td></td>
</tr>
<tr>
<td>$\Phi_{ij}$ (chiral)</td>
<td>-1/2</td>
<td>-1/2</td>
<td>-1</td>
<td>1</td>
<td>$\boxtimes \boxtimes$</td>
<td></td>
</tr>
</tbody>
</table>

3. Between each gauge node $i$ and the corresponding global symmetry node one has a link corresponding to two chiral multiplets, $Q, \tilde{Q}$, charged under $G_i \times F_i$, where $F_i$ is the global symmetry group at that node, which we depict by a square in the quiver.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$(0,2)$ field content</th>
<th>$U(1)_{c_1}$</th>
<th>$U(1)_{c_2}$</th>
<th>$U(1)_m$</th>
<th>$U(1)_R$</th>
<th>$G_i \times F_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_i$ (chiral)</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>$\boxtimes \boxtimes$</td>
<td></td>
</tr>
<tr>
<td>$\tilde{Q}_i$ (chiral)</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>$\boxtimes \boxtimes$</td>
<td></td>
</tr>
</tbody>
</table>

4. Between each gauge node $i$ and any successive node $j$ one has a Fermi multiplet $\Lambda^Q_{ij}$; between the same gauge node $i$ and any preceding node $j$ one has a Fermi multiplet $\Lambda^{\tilde{Q}}_{ji}$.
5.3 Partition functions of M5 branes probing ADE Singularities

In this section we consider 6d $(1,0)$ SCFTs which arise from M5 branes probing singularities of type A and D, and obtain the 2d quiver gauge theory describing the self-dual strings that arise on the tensor branch of the corresponding 6d theory.

5.3.1 M5 branes probing an $A_{N-1}$ singularity

Consider a setup where $M$ parallel M5 branes span directions $X^0, \ldots, X^5$ and are separated along the $X^6$ direction in 11d spacetime. Taking the transverse space of the M5 branes to be $\mathbb{R} \times \mathbb{C}^2/\mathbb{Z}_N$ and blowing up the singular locus gives rise to a $(1,0)$ 6d SCFT on the tensor branch which enjoys a $SU(N) \times SU(N)$ flavor symmetry. The $A_{N-1}$ singularity can be thought of as a limit of Taub-NUT space with charge $N$; this space has a canonical circle fibration over $\mathbb{R}^3$, and compactifying M-theory along this circle one arrives at a system of $N$ parallel D6-branes stretched between NS5 branes. The dynamics of the strings that arise in this system are captured by the two-dimensional quiver theory of Figure 5.1:

The quiver corresponds to a 2d $\mathcal{N} = (0,4)$ theory obtained from a $\mathbb{Z}_M$ orbifold of a $\mathcal{N} = (4,4)$ supersymmetric Yang-Mills theory. As such, each gauge node contains a $(0,4)$ vector multiplet together with an adjoint $(0,4)$ hypermultiplet and the bifundamental fields between the gauge nodes consist of $(0,4)$ Fermi and twisted hypermultiplets. Furthermore, between each gauge node and the adjacent flavor nodes one has $(0,2)$ Fermi multiplets in the fundamental representation of the gauge group and between each gauge node and the corresponding flavor node a fundamental $(0,4)$ hypermultiplet. The exact field content is described in [135]. Following the rules of [43] and the charge tables of Section 6.3 one can straightforwardly write down an expression for the elliptic
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Figure 5.1: Non-critical strings in M5 branes probing $A_{N-1}$ singularities.

genus for any configuration of strings, corresponding to different choices of the ranks of the gauge groups in the 2d quiver.

We can also relax the condition that all nodes should have the same $SU(N)$ flavor symmetry. In particular, we can consider the situation where the $i$th flavour node has $SU(N_i)$ symmetry together with the convexity condition

$$N_i \geq \frac{1}{2}(N_{i-1} + N_{i+1}), \quad (5.3.1)$$

which ensures that the parent 6d theory is not anomalous. In case of equality, all $N_i$ are ordered along a linear function and gauge anomaly cancellation is automatically satisfied. However, if $N_i$ is strictly greater than $\frac{1}{2}(N_{i-1} + N_{i+1})$ the net number of right moving fermions is greater than the number of left-moving ones and the theory will be anomalous. To cure this, we introduce for each gauge node a fourth flavor node with left-moving fermions to compensate for the excess of the right-moving ones. The corresponding quiver is the one depicted in Figure 5.2. Again the elliptic genus can be computed straightforwardly using the charge table of Section 6.3, however one has to be careful with the charge assignments of the new vertical Fermi multiplets: these are not charged under $U(1)_m, U(1)_{\epsilon_+}$ or $U(1)_{\epsilon_-}$. The origin of the different flavor groups can be explained from the brane construction that corresponds to this theory: one has $N_1, \ldots, N_{M+2}$ D6 branes separated by NS5 branes. The difference between the number of D6 branes on the two sides of an NS5 brane
must equal the negative of the cosmological constant in that region [136]. So, for instance, if we have an NS5 brane with \( N_{i-1} \) D6 branes on the left and \( N_i \) on the right we must have cosmological constant \( N_{i-1} - N_i \) there. At the next NS5 brane, however, we must have cosmological constant \( N_i - N_{i+1} \). This is achieved by placing \( (N_i - N_{i+1}) - (N_{i-1} - N_i) = 2N_i - (N_{i-1} + N_{i+1}) \) D8 branes between the two NS5 branes [137], which has the effect of changing the cosmological constant as required. This leads to the two-dimensional quiver considered above.

5.3.2 M5 branes probing a \( D_{p+4} \) singularity

A \( D_{p+4} \) singularity gives rise to 7d SYM theory with gauge group \( SO(2p + 8) \) for \( p \geq 0 \). We can place \( N \) M5 branes at the singularity and separate them along the remaining direction in seven dimensions. Each M5 brane actually splits into two fractional branes, which gives rise to parallel domain walls in the 7d theory [9]. Reducing along this direction leads to a 6d (1,0) SCFT with \( SO(2p + 8) \times SO(2p + 8) \) global symmetry. Following [9] we can obtain a Type IIA description by replacing the \( D_{p+4} \) singularity with the corresponding \( D_{p+4} \) ALF space and taking the circle fiber to be the M-theory circle. This results in a stack of \( p + 4 \) parallel D6 branes on top of an \( O6_- \) plane, together with their mirrors. Furthermore, one has \( 2N \) fractional NS5 branes [32]
which are of codimension 1 with respect to the orientifold plane. Whenever an $O6_\pm$ plane meets
an NS5-brane, it turns into an $O6_{\mp}$ plane. A system of $p + 4$ D6-branes parallel to an $O6_+$ plane
leads to an $Sp(p)$ gauge theory, and therefore one obtains alternating $SO(2p + 8)$ and $Sp(p)$
gauge groups in 6d. On top of this, the NS5 branes contribute a total of $2N - 1$ tensor multiplets.

Furthermore, M2 branes suspended between M5 branes in the M-theory picture become
D2 branes suspended between NS5 branes in Type IIA. The brane setup we have arrived at is
pictured in Figure 5.3. Upon reduction along the $X^6$ direction, the D2 branes give rise to the
self-dual strings that arise on the tensor branch of the 6d SCFT. The resulting two-dimensional
quiver theory is depicted in Figure 5.4. One can easily check that gauge anomalies correctly cancel
out for this theory. However, one finds that $U(1)_m$ is anomalous. The reason for this anomaly can
be traced back to geometry: the $D$-type singularity transverse to the M5 branes has only $SU(2)_R$
symmetry which is the $SU(2) \subset SO(4)$ which commutes with the action of the binary extension of
the dihedral group. This $SU(2)_R$ is the R-symmetry group of 6d SCFT. The situation has to be
contrasted with the $A$-type singularity where the surviving isometry of the space is $U(1)_m \times SU(2)_R$.
Therefore, we see that $U(1)_m$ is not present for the $D$-type theory and hence the elliptic genus of
its strings should not be refined with respect to it.

Having an explicit description of the two-dimensional quiver theory makes it possible
to compute the corresponding elliptic genus. In the simplest case of a single tensor multiplet
 corresponding to a $(−1)$ curve, this corresponds to the E-string elliptic genus which was studied in
detail in [40] (although in the present setup one must identify the fugacities associated to the two
$SO(8)$ subgroups of the $SO(16)$ flavor symmetry group). For the sake of illustration, let us also
consider the non-Higgsable $(−1)(−4)(−1)$ theory with three tensor multiplets, gauge group $SO(8)_g$
and flavor group $SO(8)_L \times SO(8)_R$. Let us denote by $m^{(g)}_\ell$, $\ell = 1, \ldots, 4$ the fugacities associated to
$SO(8)_g$ and by $m^{(L)}_\ell$ (or $m^{(R)}_\ell$) the ones associated to $SO(8)_L$ ($SO(8)_R$). From the previous discussion,
one can write down the elliptic genus for any bound state of the strings associated to this theory.
For instance, if one considers the bound state of one string coupled to the first $(−1)$ tensor multiplet
Figure 5.3: Type IIA brane setup corresponding to M5 branes probing $D_{p+4}$ a singularity. The fundamental strings depicted as blue or red wavy lines in this Figure give rise to fields in the 2d quiver theory.

Figure 5.4: Non-critical strings in M5 branes probing $D_{p+4}$ singularities.

and one string coupled to the $(-4)$ multiplet, one finds:

$$
\mathcal{I}_{(-1)(-4)} = \frac{1}{4} \int \, du \eta^2 \sum_{i=1}^{4} \left( \frac{\eta^2}{\theta_1(e_1) \theta_1(e_2)} \right) \left( \prod_{i=1}^{4} \frac{\theta_i(m_i^{(L)})}{\eta} \right) \left( \prod_{i=1}^{4} \frac{\theta_i(m_i^{(g)})}{\eta} \right) \\
\times \left( \frac{\theta_1(2u)^2 \theta_1(2e_+ \theta_1(2u + 2e_+) \theta_1(-2u + 2e_+)}{\eta^3 \theta_1(e_1) \theta_1(e_2)} \right) \\
\times \left( \prod_{i=1}^{4} \frac{\theta_i(e_+ - u) \theta_i(e_+ + u)}{\theta_i(-e_+ - u) \theta_i(-e_+ + u)} \right). 
$$

(5.3.2)
The contour integral can be performed by using the Jeffrey-Kirwan prescription for computing residues [43]. Similarly, one can compute the elliptic genus for other bound states of strings.

5.4 Multiple M5 branes probing an M9 wall

In this section we study the $\mathcal{N} = (1,0)$ six-dimensional theory of $N$ small $E_8$ instantons [6,110]; upon moving to the tensor branch, this becomes the theory of $N$ parallel M5 branes in the proximity of the M9 boundary wall of M-theory. The strings originate from M2 branes that are suspended between neighboring M5 branes or between the M5 branes and the M9 plane (see Figure 6.2). Upon circle reduction to five dimensions with an $E_8$ background Wilson line (which breaks $E_8$ global symmetry to $SO(16)$), the theory of $N$ small instantons reduces to the $Sp(N)$ theory with 8 fundamental and 1 antisymmetric hypermultiplets [6]. The instanton calculus for this five-dimensional theory provides a way to check elliptic genus computations and will be exploited in Section 5.4.2.

5.4.1 Two-dimensional quiver

In order to derive a quiver description for the theory of the strings, it again proves useful to switch to an equivalent brane configuration within string theory. Let us begin by discussing $N = 1$ theory of a single small $E_8$ instanton, whose associated two-dimensional quiver gauge theory has been worked out in [40]. The quiver was derived from a Type I’ brane configuration, which arises as follows: upon reduction of M-theory on a circle, the M9 plane is replaced by eight D8 branes on top of an O8$^-$ orientifold plane (which has D8 brane charge $-8$); the M5 brane, on the other hand, becomes an NS5 brane. Furthermore, M2 branes are replaced by D2 branes that stretch between the NS5 brane and the D8-O8$^-$ system. By studying the two-dimensional reduction of the worldvolume theory of the D2 branes in the limit of small separation between the NS5 and eight-branes, one arrives at the two-dimensional quiver gauge theory of [40]. The (0,4) quiver gauge theory for $n$
strings has gauge group $O(n)$ and the following field content: a vector multiplet in the adjoint (antisymmetric) representation of $O(n)$, a hypermultiplet in the symmetric representation, and eight Fermi multiplets in the bifundamental representation of $O(n) \times SO(16)$. Elliptic genera for this theory have been computed in [40] for up to four strings and shown to agree with results from the instanton calculus for the five-dimensional $Sp(1)$ theory with eight fundamental hypermultiplets.

The generalization of the Type I' brane setup to the case of $N$ small instantons is illustrated in Figure 5.6 and is again obtained by reducing the above M9-M5 setup on a circle. The brane setup is a combination of the E-string and M-string brane configurations without the D6 branes which are usually present for the M-string system. As we will see this becomes crucial when we look at the quiver-gauge theory governing the dynamics of the strings to which we now turn.

The brane setup implies a simple quiver gauge theory governing the dynamics of the strings. The first $n_1$ D2 branes ending on the D8-O8 system correspond to a $O(n_1)$ gauge node; from
the D2-D8 strings one finds eight bifundamental Fermi multiplets charged under $O(n_1) \times SO(16)$. Furthermore, there is a symmetric hyper at the $O(n_1)$ node as already observed in [40]. All other gauge nodes corresponding to the the D2 branes suspended between NS5 branes have unitary gauge groups with bi-fundamental matter between them familiar from the orbifolds of M-strings [135]. Finally, one also obtains $O(n_1) \times U(n_2)$ bifundamentals from strings ending on the $n_1$ and $n_2$ D2 branes. These bifundamental fields consist of a (0,4) hyper and a (0,4) Fermi multiplet, as is the case for M-strings. The resulting quiver is illustrated in Figure 5.7.

We comment on the global symmetries of this quiver gauge theory, and compare them with the symmetries that we expect for the infrared CFT on these strings. Our (0,4) gauge theory has $SO(4) = SU(2) \times SU(2)$ R-symmetry. The first $SU(2)$ is part of the $SO(4)$ symmetry which rotates $\mathbb{R}^4$ along the worldvolume of NS5-branes, transverse to the strings. The second $SU(2) \sim SO(3)$ rotates the $\mathbb{R}^3$ space transverse to the NS5-branes and D2-branes. The infrared (or equivalently

Figure 5.6: Brane configuration for the theory of $N$ small $E_8$ instantons.
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strong coupling) limit of the 2d gauge theory is realized by going to the M-theory regime of the type I' theory. Then the space $\mathbb{R}^3$ transverse to NS5-D2 is replaced by $\mathbb{R}^3 \times S^1$, including the M-theory circle, and becomes $\mathbb{R}^4$ in the strong coupling limit. So in the IR, we expect the $SO(3)$ symmetry to enhance to $SO(4)$. Any analysis from our gauge theory, such as the elliptic genus calculus below, will be missing the extra Cartan charges of the enhanced $SO(4)$. Let us denote by $\epsilon_{1,2}$ the chemical potentials for the rotations on $\mathbb{R}^4$, as in the previous sections. Apart from rotating $\mathbb{R}^4$ along the 5-brane, there will be an extra rotation on $\mathbb{R}^4$ transverse to the M5-brane, with $\epsilon_+ \equiv \frac{\epsilon_1 + \epsilon_2}{2}$. Let us denote by $m$ the chemical potential for the missing Cartan of the enhanced IR symmetry. Then the $\mathbb{R}^3$ part in the type I' setting is rotated by $m + \epsilon_+$, while the rotation by $m - \epsilon_+$ is invisible on $\mathbb{R}^3 \times S^1$. Thus, our UV gauge theory will be computing the elliptic genus only at $m = \epsilon_+^{1,2}$. At $N = 1$, it is known that the 6d SCFT engineered by a single M5 and M9 brane does not see the extra Cartan of $SO(4)$ (conjugate to $m$) at all. In other words, all the states in the Hilbert space of the 6d SCFT are completely neutral under this $U(1)$ charge (while the full M-theory would see the charged states decoupled from the 6d SCFT). One way to see this is from the 5 dimensional $Sp(N)$ gauge theory obtained after circle compactification. Namely, the parameter $m$ above corresponds to the mass parameter for the $Sp(N)$ antisymmetric hypermultiplet in the resulting 5d theory. At $N = 1$, the antisymmetric representation is neutral in $Sp(1)$ and the corresponding hypermultiplet decouples. This implies that the extra $U(1)$ for $m$ decouples from the 6d CFT at $N = 1$, and this has been tested from the instanton partition function in [139]. This is the reason why the 2d gauge theory above provided maximally refined elliptic genera at $N = 1$ in [40], since the restriction $m = \epsilon_+$ loses no information about the 6d SCFT. However, the parameter $m$ appears in the 6d CFT spectrum for $N \geq 2$, which was checked from the $Sp(N)$ instanton calculus [139].

Below we present sample computations for the elliptic genera corresponding to the lowest

---

1. We have also found a $\mathcal{N} = (0, 2)$ quiver gauge theory that appears to give the correct elliptic genera for arbitrary values of $m$, and we conjecture flows to a $(0, 4)$ CFT in IR. This quiver is discussed in Appendix A of [138].

2. The parameter $m$ is the one appearing in the instanton calculus of the Nekrasov partition function and should not be confused with the fugacity of $U(1)_m$ in the 2d gauge theory. With respect to the 2d fugacity it is shifted by $\epsilon_+$. 

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charge sectors, namely \((n_1, n_2) = (1, 1), (n_1, n_2) = (1, 2), \) and \((n_1, n_2) = (2, 1)\) for the \(N = 2\) quiver.

\[
\begin{array}{cccc}
\text{SO}(16) & - & \text{O}(n_1) & \text{U}(n_2) & \text{U}(n_3) & -
\end{array}
\]

Figure 5.7: Quiver for the theory of \(N\) small \(E_8\) instantons.

**Charge sector** \((n_1, n_2) = (1, 1)\)

Combining the one-loop determinants, the zero-mode integral \(I_{(1,1)}\) is given by

\[
- \oint \frac{du}{2} \left( \frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^4 \left( \frac{\eta^4 \theta_1(\epsilon_1 + \epsilon_2)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^2 \left( \prod_{i=1}^{8} \frac{\theta_1(m_i + a_i)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right) \left( \frac{\theta_1(\epsilon_1)\theta_1(-\epsilon_2)}{\eta^4 \theta_1(-\epsilon_1 - \epsilon_2)} \right)
\]

where \(a_i = (0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2})\). Repeated signs \(\pm\) in the arguments mean that both factors are multiplied:

\(\theta_1(-\epsilon_+ \pm (a_i - u)) \equiv \theta_1(-\epsilon_+ + a_i - u)\theta_1(-\epsilon_+ - a_i + u)\). The contour integral given by the JK-Res is done with \(\eta = e_1\). Then the only nonzero JK-Res comes from the pole \(u = a_i + \epsilon_+\). The result is

\[
I_{(1,1)} = - \frac{\eta^5 \theta_1(\epsilon_1 + \epsilon_2)}{2 \theta_1(\epsilon_1)\theta_1(\epsilon_2)} \left( \frac{\eta^4 \theta_1(\epsilon_1 + \epsilon_2)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^2 \left( \prod_{i=1}^{8} \frac{\theta_1(m_i + a_i)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right) \left( \frac{\theta_1(\epsilon_1)\theta_1(-\epsilon_2)}{\eta^4 \theta_1(-\epsilon_1 - \epsilon_2)} \right)
\]

\[
= - \frac{\eta^5}{2 \theta_1(\epsilon_1)\theta_1(\epsilon_2)} \sum_{i=1}^{4} \left( \frac{\theta_1(m_i)}{\eta} \right)^2 = I_{(1,0)}.
\]

This is the elliptic genus of the single E-string, i.e. with charge \((n_1, n_2) = (1, 0)\) [40, 118].

**Charge sector** \((n_1, n_2) = (1, 2)\)

The zero-mode integral is given by

\[
I_{(1,2)} = - \oint \frac{du_1 du_2}{4} \left( \frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^4 \left( \frac{\eta^4 \theta_1(\epsilon_1 + \epsilon_2)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^2 \left( \prod_{i=1}^{8} \frac{\theta_1(m_i + a_i)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right) \times \left( \frac{\theta_1(\pm(u_1-u_2))\theta_1(\epsilon_1+\epsilon_2 \pm (u_1-u_2))\theta_1(\epsilon_1)\theta_1(\epsilon_2)}{\theta_1(\epsilon_1 \pm (u_1-u_2))\theta_1(\epsilon_2 \pm (u_1-u_2))\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)
\]

\[
= - \frac{\eta^5}{2 \theta_1(\epsilon_1)\theta_1(\epsilon_2)} \sum_{i=1}^{8} \left( \frac{\theta_1(m_i)}{\eta} \right)^2 = I_{(1,0)}.
\]

If we choose \(\eta = e_1 + \epsilon \epsilon_2\) in which \(\epsilon \ll 1\), nonzero JK-Res can only come from the following poles.
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- \((\epsilon_{1,2} - u_1 + u_2, -\epsilon_+ - a_i + u_1) = (0, 0)\)
- \((\epsilon_{1,2} + u_1 - u_2, -\epsilon_+ - a_i + u_2) = (0, 0)\)
- \((-\epsilon_+ - a_i + u_1, -\epsilon_+ - a_i + u_2) = (0, 0)\).

Actually evaluating the residues, it turns out that all these poles yield vanishing residues, so that

\[ I_{(1,2)} = 0. \] (5.4.6)

**Charge sector** \((n_1, n_2) = (2, 1)\)

Now the gauge theory comes with \(O(2) \times U(1)\) gauge group. In the elliptic genus calculus, there are seven disconnected sectors of \(O(2)\) flat connections [40]. In six sectors, the flat connections are discrete, while in one sector it comes with one complex parameter.

In the first sector with continuous parameter, which we label by superscript 0, one has to do the following rank 2 contour integral for the elliptic genus:

\[
I^0_{(2,1)} = \int \frac{du_1 du_2}{2} \frac{\eta^7 \theta_1(\epsilon_1 + \epsilon_2) \eta^3 \theta_1(\epsilon_1 + \epsilon_2)}{\theta_1(\epsilon_{1,2}) \theta_1(\epsilon_{1,2} + 2u_1)} \left( \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u_1)}{\eta^2} \right) \\
\times \frac{\theta_1(\epsilon_+ \pm u_1 + u_2)}{\theta_1(\epsilon_+ \pm u_1 + u_2)} \theta_1(\epsilon_+ \pm u_1 + u_2) \theta_1(\epsilon_+ \pm u_1 - u_2)\] (5.4.7)

If we choose \(\eta = \epsilon_1 + \epsilon \epsilon_2\) in which \(\epsilon \ll 1\), nonzero JK-res can only appear from the following poles.

- \((-\epsilon_+ - u_1 + u_2, \epsilon_{1,2} + 2u_1) = (0, 0) \rightarrow (u_1, u_2) = (-\frac{\epsilon_{1,2}}{2} + a_i, \frac{\epsilon_{1,2}}{2} + a_i).\) Its residue is zero.
- \((-\epsilon_+ - u_1 - u_2, \epsilon_+ + u_1 + u_2) = (0, 0) \rightarrow (u_1, u_2) = (\epsilon_+ + a_i, a_i)\)
- \((-\epsilon_+ + u_1 + u_2, \epsilon_{1,2} + 2u_1) = (0, 0) \rightarrow (u_1, u_2) = (-\frac{\epsilon_{1,2}}{2} + a_i, \epsilon_{1,2} + \frac{\epsilon_{1,2}}{2} - a_i)\)
Collecting all the residues, $I^0_{(2,1)}$ is given by

$$I^0_{(2,1)} = \frac{\eta^{-12}}{4\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \sum_{i=1}^{4} \left[ \frac{\prod_{l=1}^{8} \theta_i(m_{l} \pm \epsilon_{+})}{\theta_1(2\epsilon_1 + \epsilon_2)\theta_1(\epsilon_1 + 2\epsilon_2)} - \frac{\theta_1(\epsilon_1 + \epsilon_2)\prod_{l=1}^{8} \theta_i(m_{l} \pm \frac{\epsilon_{+}}{2})}{\theta_1(2\epsilon_1 + \epsilon_2)\theta_1(\epsilon_1 - \epsilon_2)} + (\epsilon_1 \leftrightarrow \epsilon_2) \right].$$

On the second line, we used the following identity

$$\sum_{i=1}^{4} \prod_{l=1}^{8} \theta_i(m_{l})^2 = \sum_{i=1}^{4} \left[ \frac{\theta_1(\epsilon_{1,2})\prod_{l=1}^{8} \theta_i(m_{l} \pm \epsilon_{+})}{\theta_1(2\epsilon_1 + \epsilon_2)\theta_1(\epsilon_1 + 2\epsilon_2)} - \frac{\theta_1(\epsilon_2)\theta_1(\epsilon_1 + \epsilon_2)\prod_{l=1}^{8} \theta_i(m_{l} \pm \frac{\epsilon_{+}}{2})}{\theta_1(2\epsilon_1 + \epsilon_2)\theta_1(\epsilon_1 - \epsilon_2)} + (\epsilon_1 \leftrightarrow \epsilon_2) \right],$$

which we checked in an expansion in $e^{2\pi i \tau}$, for the first 5 terms up to $(e^{2\pi i \tau})^{5/2}$ order.

The contributions from the other six sectors are given by

$$I^m_{(2,1)} = \int \frac{du}{4} \frac{\eta^4 \theta_1(a_1 + a_2)\theta_1(\epsilon_1 + \epsilon_2 + a_1 + a_2) \eta^2 \theta_1(\epsilon_1 + \epsilon_2)}{\theta_1(\epsilon_{1,2})^2\theta_1(\epsilon_{1,2} + a_1 + a_2)\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \times \left( \prod_{l=1}^{8} \frac{\theta_1(m_l + a_1)\theta_1(m_l + a_2)}{\eta^2} \right) \frac{\theta_1(+\epsilon_{-} \pm a_1 \mp u)\theta_1(+\epsilon_{-} \pm a_2 \mp u)}{\theta_1(-\epsilon_{+} \pm a_1 \mp u)\theta_1(-\epsilon_{+} \pm a_2 \mp u)},$$

where we take the discrete $O(2)$ holonomies $(a_1, a_2) = (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (0, \frac{\tau}{2}), (\frac{1}{2}, \frac{1+\tau}{2}), (0, \frac{1+\tau}{2}), (\frac{1}{2}, \frac{\tau}{2})$ for $m = 1, 2, \cdots, 6$, respectively. JK-res with $\eta = e_1$ can be nonzero only at the pole $u = a_1 + \epsilon_{+}$ or $u = a_2 + \epsilon_{+}$, yielding the following result:

$$I^m_{(2,1)} = \frac{\eta^4}{2\theta_1(\epsilon_{1,2})^2} \prod_{l=1}^{8} \frac{\theta_i(m_{l})\theta_j(m_{l})}{\eta^2}$$

where $(i, j) = (1, 2), (4, 3), (1, 4), (2, 3), (1, 3), (2, 4)$ for $m = 1, 2, \cdots, 6$. Combining (5.4.7) and (5.4.10), one obtains

$$I_{(2,1)} = I^0_{(2,1)} + \sum_{m=1}^{6} I^m_{(2,1)} = (I_{(1,1)})^2 = (I_{(1,0)})^2,$$

which exhibits a factorization structure.
In the next subsection, we will show that all the results above are in complete agreement with the 5 dimensional $Sp(2)$ instanton calculus of [139]. Before that, let us first try to interpret these rather simple results that we have found at $m = \epsilon_+$.

The strings made of $n_1$ and $n_2$ D2-branes in Fig. 6, winding a circle, contribute to the elliptic genus as both $n_1 + n_2$ multi-particle states, and also through various threshold bound states with lower particle numbers. There could be various kinds of bound states. Generally, $m_1 (\leq n_1)$ of the $n_1$ strings and $m_2 (\leq n_2)$ of the $n_2$ strings may form bounds. One can first deduce that the index is zero at $m = \epsilon_+$ in the sector which contains bound states with charges $(0, m_2)$. This is because the $(0, m_2)$ bounds are basically M-strings in a maximally supersymmetric theory. Note that the M-strings are half-BPS states of the 6d $(2, 0)$ theory, so will see 8 broken supercharges as Goldstone fermions. This is in contrast to the strings in 6d QFTs preserving $(1, 0)$ SUSY only. The extra fermionic zero modes for M-strings provide the factor

$$\sin \pi (m + \epsilon_+) \sin \pi (m - \epsilon_+)$$

(5.4.12)

to the elliptic genus [93]. Thus, M-strings which are unbound to E-strings (i.e. at $m_1 = 0$) will contribute a 0 factor to the elliptic genus at $m = \pm \epsilon_+$.

With this understood, let us start by considering the sector with $(n_1, n_2) = (1, 1)$. At $m = \epsilon_+$, there is no contribution from the two particle states $(1, 0) + (0, 1)$ due to the above reasoning. So one should only obtain a single particle contribution in the $(1, 1)$ sector. This is consistent with our finding $I_{(1,1)} = I_{(1,0)}$. A slightly surprising fact from our finding is that the single particle bound with charges $(1, 1)$ behaves exactly the same as a single E-string with charge $(1, 0)$, at least at $m = \epsilon_+$. Although the $(1, 1)$ bound look like one long E-string suspended between the M9-plane and the second M5-brane, it penetrates through the first M5-brane so in principle there could be extra contributions to the BPS degeneracies from the intersection. For instance, in the case of M-strings, it is known that the charge $(1, 0)$ M-string and the single particle bound part of the $(1, 1)$ M-string exhibit different spectra (at general chemical potentials, with $m \neq \epsilon_+$) [93]. So we interpret that $I_{(1,1)} = I_{(1,0)}$ implies some simplification of the $(1, 1)$ elliptic genus at $m = \epsilon_+$. 212
Other results also have nontrivial implications on the E-/M-string bound state elliptic genera at $m = \epsilon_+$. For $(n_1, n_2) = (1, 2)$, $I_{(1,2)} = 0$ implies that there are no $(1, 2)$ bound states captured by the elliptic genus at $m = \epsilon_+$, since we know that $(1, 1) + (0, 1)$, $(1, 0) + (0, 2)$, or $(1,0) + 2(0,1)$ multi-particles cannot contribute to the elliptic genus at $m = \epsilon_+$. As we will consider from the 5d $Sp(2)$ instanton calculus, this feature generalizes to higher string numbers: the $(m_1, m_2)$ bounds with $m_1 < m_2$ do not contribute to the elliptic genus at $m = \epsilon_+$.

Finally, $I_{(2,1)} = I_{(1,0)}^2$ can also be understood with the above observations. Namely, with a $(0, m_2)$ particle yielding a factor of zero in the elliptic genus, the nonzero contribution can come from $(1, 1) + (1,0)$ two particle states. But since we already know that these contributions give equal elliptic genera namely that of a single E-string, we can naturally understand this relation. (So our finding implies that the $(2, 1)$ bound does not contribute to the index at $m = \epsilon_+$.)

Based on the above observations, we propose that

$$I_{(n_1, n_2)} = \begin{cases} 0 & \text{if } n_1 < n_2 \\ I_{(n_2,0)}I_{(n_1-n_2,0)} & \text{if } n_1 \geq n_2 \end{cases} \quad (5.4.13)$$

Namely, at $m = \epsilon_+$, the $(n_1, n_2)$ string elliptic genus factorizes to two E-string elliptic genera. Although we have shown this result for only a few charges from the 2d gauge theories, we shall confirm such factorizations to a much higher order in $n_1, n_2$ from the 5d $Sp(2)$ instanton calculus below.

### 5.4.2 Five dimensional $Sp(2)$ instanton calculus

In this subsection, we shall consider the circle compactification of the $(1, 0)$ theory on 2 M5 and one M9, and consider the string spectra from the instanton calculus of the resulting 5d gauge theory.

Let us consider the six-dimensional conformal field theory living on two M5 branes probing the M9 plane. The space transverse to the two M5-branes is $\mathbb{R}^4 \times \mathbb{R}^+$, where the latter $\mathbb{R}^+$ is obtained
by the $\mathbb{Z}_2$ action of M9. This space has $SO(4) = SU(2) \times SU(2)$ symmetry. The first $SU(2)$ is the superconformal R-symmetry, and the second $SU(2)$ is a flavor symmetry. The full flavor symmetry is thus $SU(2) \times E_8$.

We compactify this system on a circle, with an $E_8$ Wilson line which breaks $E_8$ into $SO(16)$. Then at low energy, one obtains a 5 dimensional $\mathcal{N} = 1$ $Sp(N)$ gauge theory with $N_f = 8$ fundamental and one anti-symmetric hypermultiplet. The 8 masses $\tilde{m}_i$ for the fundamental hypermultiplets and the mass $m$ for the antisymmetric hypermultiplet are in 1-1 correspondence to the chemical potentials of the $E_8 \times SU(2)$ flavor symmetries. The precise relations that we use are given in [40, 139]. $m$ is simply uplifting to the $SU(2)$ flavor chemical potential, while the masses $\tilde{m}_i$ are related to the $E_8$ chemical potentials $m_i$ by [40]

$$\tilde{m}_i = m_i \quad \text{(for } i = 1, \cdots, 7), \quad \tilde{m}_8 = m_8 - \tau .$$ (5.4.14)

The chemical potentials $\tilde{\alpha}_1, \tilde{\alpha}_2$ for the $Sp(2)$ electric charges are related to those $\alpha_{1,2}$ for the string winding numbers $n_1 - n_2, n_2$ by

$$\tilde{\alpha}_1 = \alpha_1 + \frac{\tau}{2} - m_8 , \quad \tilde{\alpha}_2 = \alpha_2 + \frac{\tau}{2} - m_8 .$$ (5.4.15)

We chose the convention for $\alpha_1, \alpha_2$ in a way that they correspond to the distances from the M5-branes to the M9-plane.

The elliptic genera of the previous subsection are related to the instanton partition function for this 5d $Sp(2)$ Yang-Mills, as follows. Let us first define the fugacities

$$s = e^{2\pi i \epsilon_+}, \quad u = e^{2\pi i \epsilon_-}, \quad v = e^{2\pi i m}, \quad \tilde{w}_i = e^{2\pi i \tilde{\alpha}_i}, \quad \tilde{y}_i = e^{2\pi i \tilde{m}_i}$$ (5.4.16)

where $\tilde{w}_i$ satisfy $\tilde{w}_2 < \tilde{w}_1 < 1$ to probe the sectors with $n_1 > 0$ and $n_2 > 0$. One should first
consider the perturbative partition function $Z^{Sp(2)}_{\text{pert}}$, given by

$$Z^{Sp(2)}_{\text{pert}} = PE \left[ \frac{s(s + s^{-1})}{(1 - su)(su - 1)} \left[ \bar{w}_1 \bar{w}_2 + \bar{w}_2/\bar{w}_1 + \bar{w}_2^2 \right] \right. $$

$$+ \frac{s(v + v^{-1})}{(1 - su)(1 - su^{-1})} \left[ \bar{w}_1 \bar{w}_2 + \bar{w}_2/\bar{w}_1 \right]$$

$$\left. + \sum_{i=1}^{8} s(\tilde{y}_i + \tilde{y}_i^{-1}) \left[ \bar{w}_1 + \bar{w}_2 \right] \right],$$

(5.4.17)

where $PE[f]$ is defined as

$$PE[f(s, u, v, \bar{w}_1, 2, \tilde{y}_i, q)] = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f(s^n, u^n, v^n, \bar{w}_{1,2}^n, \tilde{y}_i^n, q^n) \right].$$

(5.4.18)

The instanton partition function is given by

$$Z^{Sp(2)}_{\text{inst}} = \frac{Z^{Sp(2)}_{\text{ADHM}}}{Z^{Sp(2)}_{\text{extra}}}$$

(5.4.20)

where $Z^{Sp(2)}_{\text{ADHM}}$ is the index computed from the ADHM quantum mechanics for $Sp(2)$ instantons. See [139] for the computation of $Z^{Sp(2)}_{\text{ADHM}}$, which uses the quantum mechanical version of the contour integral formula using Jeffrey-Kirwan residues.

Let us consider the full index $Z^{Sp(2)} = Z^{Sp(2)}_{\text{pert}} Z^{Sp(2)}_{\text{inst}}$. $Z^{Sp(2)}$ is very complicated in general. However, setting $m = \epsilon_+$, $Z^{Sp(2)}$ simplifies a lot and reduces to

$$Z^{Sp(2)}(\bar{w}_1, \bar{w}_2, q, s, u, \tilde{y}_i, v = s) = Z^{Sp(1)}(\bar{w}_1, q, s, u, \tilde{y}_i) Z^{Sp(1)}(\bar{w}_2, q, s, u, \tilde{y}_i),$$

(5.4.21)

where $Z^{Sp(1)}$ is the partition function for the 5d $Sp(1)$ gauge theory obtained by compactifying the rank 1 6d SCFT for one M5 and M9. ($Z^{Sp(1)}$ is computed by the same procedure as explained in...
the previous paragraph. See [139] for the details.) This factorization was checked up to $q^2$ and $\tilde{w}_{1,2}^5$ order.

Now we would like to connect the above findings to the 2d calculus of the previous subsection. The two indices are essentially the same, but the latter captures the contributions only with positive winding numbers (or 5d electric charges). On the other hand, the former also captures some $Sp(2)$ neutral states’ contribution with instanton number only. The missing part in the 2d calculus can be supplemented by multiplying a $U(1)$ instanton partition function factor for 5d maximal SYM, for each M5-brane [123]. This factor is given by

$$Z^{U(1)} = \text{PE} \left[ \frac{s(-u - u^{-1} + v + v^{-1}) q^2}{(1 - su)(1 - su^{-1})} \frac{1}{1 - q^2} \right].$$

Therefore, setting $v = s$, the E-string elliptic genera $I_{(n,0)}$ are given by

$$Z^{U(1)} \sum_{n=0}^{\infty} w^n I_{(n,0)}(q, s, u, y_i) = Z^{Sp(1)}(\tilde{w}, q, s, u, \tilde{y}_i)$$

where $I_{(0,0)} \equiv 1$. The coefficients $I_{(n,0)}$ computed from the two different approaches (2d gauge theory and instanton calculus) were shown to agree with each other [40], for $n \leq 4$. The elliptic genera $I_{(n_1, n_2)}$ for the $(n_1, n_2)$ strings can be computed from the instanton calculus by

$$\left(Z^{U(1)}\right)^2 \sum_{n_1, n_2=0}^{\infty} w_1^{n_1-n_2} w_2^{n_2} I_{(n_1, n_2)}(q, s, u, y_i) = Z^{Sp(2)}(\tilde{w}_1, \tilde{w}_2, q, s, u, \tilde{y}_i).$$

One can show that the right hand side of (5.4.24), computed up to $q^2$ and $\tilde{w}_{1,2}^5$ orders from the instanton calculus, completely agrees with $I_{(n_1, n_2)}$ computed in the previous subsection for $(n_1, n_2) = (1, 1), (1, 2), (2, 1)$. In particular, our proposal (5.4.13) is justified from the factorization (5.4.21) of the $Sp(2)$ instanton partition function at $m = \epsilon_+$. We show this result pictorially in Figure 5.8.

### 5.5 D5 branes probing ADE singularities

In this section we study a third class of theories that arises from F-theory compactified on an elliptic Calabi-Yau threefold $X$ defined as follows: we take the base to be the blown-up
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In the massless case the strings can recombine and one arrives at a configuration of a single E-string from the first M5 brane and two E-strings from the second.

ALE singularity of ADE type, and over each blown up $\mathbb{P}^1$ we let the elliptic fiber have Kodaira degeneration $I_N$. Equivalently, we can interpret this setup as Type IIB string theory with $N$ D5 branes probing an ALF singularity of ADE type. We soon will be interested in decompactifying the circle at infinity of ALF and recover the ALE singularity. As follows from the Douglas-Moore construction [37], the resulting $\mathcal{N} = (1, 0)$ six dimensional theory is captured by an affine quiver of ADE type, with the following field content: to a node $i$ of the affine quiver with Coxeter label $d_i$ is associated a gauge group $SU(d_i N)$, and to each edge is associated bifundamental matter. Furthermore, to each node of the quiver corresponds an abelian tensor multiplet. Naively one would expect to have gauge groups $U(d_i N)$, but one finds that in fact the abelian factor $\prod_i U(1)$ is Higgsed via a Green-Schwarz mechanism (apart from a decoupled global $U(1)$ factor) [7]. We now take the $\text{ALF} \to \text{ALE}$ limit, so that the the D5 brane associated to the affine node becomes noncompact and gives rise to a global (as opposed to gauge) $SU(N)$ symmetry. In fact, in the $A_r$ case, one can actually associate distinct $SU(N)$ global symmetries to the two non-compact

Figure 5.8: String charge sector $(3, 2)$ for the configuration of two M5 branes probing an M9 wall.
half-$\mathbb{P}^1$s that arise from the affine node.

Of even more interest to us is the quiver gauge theory associated to the self-dual strings of the theory, which arise in F-theory as D3 branes wrapping the blown-up $\mathbb{P}^1$s, or equivalently in Type IIB string theory as D1 branes probing the singularity. The resulting two-dimensional quiver theory was derived for the $A_N$ singularity by following a suitable generalization of Douglas-Moore construction [38], and is pictured in Figure 5.9 (this quiver is equivalent to the one depicted in Figure 5.1). Note that there is no restriction on the ranks of the gauge groups; the rank at any node is equal to the number of D3 branes (self-dual strings) coupled to the corresponding tensor multiplet.

\[
\begin{align*}
SU(N) & \quad SU(N) & \quad SU(N) & \quad SU(N) & \quad SU(N) \\
U(n_1) & \rightarrow U(n_2) & \rightarrow U(n_3) & \rightarrow \cdots & \rightarrow U(n_r)
\end{align*}
\]

Figure 5.9: Quiver gauge theory for the non-critical strings of $N$ D5 branes probing an $A_r$ singularity. Note that the global symmetry are obtained from the affine $\widehat{A}_r$ quiver by ‘opening up’ the quiver at the affine node.

To avoid clutter, in this section, we work with the exponentiated variables,

\[ t = e^{-\ell_1}, \quad d = e^{-\ell_2}, \quad c = e^{-m}. \quad (5.5.25) \]

We also use exponentiated fugacity variables for gauge and global symmetries. With that, let us consider an $A_r$ singularity probed by a single D5 brane. The elliptic genus for this case, in the $(n_1, \ldots, n_r)$ sector corresponding to $n_1$ strings associated to the first node, $n_2$ for the second node,
and so on, is computed by a contour integral which we write schematically as

\[ I_{n_1, \ldots, n_n} = \phi \left( \prod_{i=1}^{r} \prod_{a=1}^{n_i} \frac{dz_a^{(i)}}{2\pi i z_a} \right) \left( \prod_{a=1}^{n_i} \frac{\theta_1 \left( \frac{q^{1/2} c}{z_a^{(i)}} \right)}{\theta_1 \left( \frac{q^{1/2} c}{z_b^{(i)}} \right)} \right) \left( \prod_{a=1}^{n_i} \frac{\theta_1 \left( q^{1/2} c \sqrt{\frac{d_a\cdot a}{z_b^{(i)}}} \right)}{\theta_1 \left( q^{1/2} c \sqrt{\frac{d_b\cdot b}{z_b^{(i)}}} \right)} \right) \]

\times \ \left( \theta_1 \left( q^{1/2} c z_a^{(i)} \right) \theta_1 \left( q^{1/2} c \frac{1}{z_a^{(i)}} \right) \right) \left( \frac{\theta_1 \left( \sqrt{td} z_a^{(i)} \right)}{\theta_1 \left( \sqrt{td} z_b^{(i)} \right)} \right).

\text{(5.5.26)}

Here we have used that the index of the chiral multiplet and Fermi multiplet of R-charge \( s \) is \( \theta(q^{1/2} a)^{-1} \) and \( \theta(q^{1/2} a) \) respectively. It is implied in the expression above that one should take products of theta functions for all combinations of \( z_a^{(i)} \) that appear in the arguments. So for instance one should take \( \theta_1 \left( q^{1/2} c \sqrt{\frac{d_a\cdot a}{z_b^{(i)}}} \right) / \eta \) to mean:

\[ \prod_{i=1}^{r} \prod_{a=1}^{n_i} \prod_{b=1}^{n_{i+1}} \frac{\theta_1 \left( q^{1/2} c \frac{1}{z_a^{(i)}} \right)}{\eta}. \]

Also, the factors in the integrand of the form \( \theta_1 \left( z_a^{(i)} / z_a^{(i)} \right) = \theta_1 (1) \) should be replaced with \( \theta_1 (1) = \eta^3 \). The same considerations apply for equations (5.5.29) and (5.5.30). The poles of this integral are classified by a sequence of \( r \) Young diagrams \( Y_i \) with \( n_i \) boxes respectively and are located at

\[ z_a^{(i)} = t x^{(i)} + \frac{1}{2} d^{(i)} + \frac{1}{2} \]

\text{(5.5.27)}

where \((x^{(i)}, y^{(i)})\) are the coordinates of \( t \)-th box in \( Y_i \). Evaluating the residues, we get

\[ I = \sum_{\{Y_i\}} \prod_{i=1}^{r} \prod_{s^{(i)} \in Y_i} \left( \frac{\theta_1 (t x^{(i)} + 1 d^{(i)} + 1)}{\theta_1 (t x^{(i)} + 1 d^{(i)} + 1)} \right) \left( \frac{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)}{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)} \right) \left( \frac{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)}{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)} \right) \left( \frac{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)}{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)} \right) \left( \frac{\theta_1 (t x^{(i)} + 1 d^{(i)} + 1)}{\theta_1 (t x^{(i)} + 1 d^{(i)} + 1)} \right) \right) \cdot \left( \frac{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)}{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)} \right) \left( \frac{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)}{\theta_1 (q^{1/2} c t^{x^{(i)}+1} + 1)} \right) \left( \frac{\theta_1 (t x^{(i)} + 1 d^{(i)} + 1)}{\theta_1 (t x^{(i)} + 1 d^{(i)} + 1)} \right) \right). \]

\text{(5.5.28)}

We have used the following notation: \( \Delta x^{(i)} = x_1^{(i)} - x_2^{(i)} \) and \( \Delta x^{(i,j)} = x_1^{(i)} - x_2^{(j)} \), and similarly for \( y \).

This construction generalizes straightforwardly to \( N \) D5 branes probing singularities of type \( D_r \) and \( E_6, E_7, E_8 \). Their two-dimensional quivers are pictured in Figures 5.10–5.13. There is
an important difference from the case with A type singularity, the symmetry $U(1)_m$ is anomalous.
Hence, we will not be able to refine the index with the corresponding fugacity $c$. The Lagrangian
of the $(0,4)$ theory is chiral. This is apparent from the fact that $\Lambda^Q$ and $\Lambda^{\tilde{Q}}$ transform oppositely.
The same is the case with the fields $\Lambda^B, \Lambda^{\tilde{B}}, \Sigma$ and $\Phi$. Although, for $D$ and $E$ type quivers there
is no preferred orientation of, say, the $\Phi$ arrow. But as it will soon become clear, the elliptic genus
of the theory doesn’t depend on the choice of this orientation.

Let us label the gauge nodes by the index $i$, $i$ takes the values from 1 to $r + 1$. The
rank of the affine node is taken to be zero, $n_{r+1} = 0$. Let $z_{\alpha}^{(i)}$, $\alpha = 1, \ldots, n_i$ be the fugacity
corresponding to the $U(n_i)$ gauge group. The flavor nodes are also labeled by the same index $i$.
Let $a_{\ell}^{(i)}$, $\ell = 1, \ldots, d_i N$ be the fugacity corresponding to the flavor symmetry group $SU(d_i N)$ where
d$_i$ is the Coxeter label of the node $i$. These fugacities obey $\prod a_{\ell}^{(i)} = 1$. Let $M_{ij}$ be the adjacency
The index of the 2d theory is then given as

$$\mathcal{I} = \oint \prod_i \left( \frac{1}{n_i!} \prod_{\alpha} \frac{dz_{\alpha}^{(i)}}{2\pi i z_{\alpha}^{(i)}} \right) \left( \frac{\theta_1(\frac{z_{\alpha}^{(i)}}{z_{\beta}^{(i)}})}{\theta_1(\frac{z_{\alpha}^{(i)}}{z_{\beta}^{(i)}})} \frac{\eta^2}{\theta_1(\sqrt{a_i} \frac{z_{\alpha}^{(i)}}{z_{\beta}^{(i)}}) \theta_1(\sqrt{a_i} \frac{z_{\alpha}^{(i)}}{z_{\beta}^{(i)}})} \right) \left( \frac{\theta_1(\frac{1}{q^{1/2}} M_{ij} z_{\alpha}^{(i)}}{z_{\beta}^{(i)}}) \theta_1(\frac{1}{q^{1/2}} M_{ij} z_{\alpha}^{(i)}}{z_{\beta}^{(i)}}) \right) \left( \frac{\theta_1(\frac{1}{q^{1/2}} M_{ij}^T z_{\alpha}^{(i)}}{z_{\beta}^{(i)}}) \theta_1(\frac{1}{q^{1/2}} M_{ij}^T z_{\alpha}^{(i)}}{z_{\beta}^{(i)}}) \right) \eta^2. \right) \right.$$

(5.5.29)
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Figure 5.12: Quiver gauge theory for the non-critical strings of $N$ D5 branes probing an $E_7$ singularity.

Figure 5.13: Quiver gauge theory for the non-critical strings of $N$ D5 branes probing an $E_8$ singularity.
Using the identity $\theta_1(q x) = q^{1/2} z \theta_1(x^{-1})$ we get (up to a prefactor)

$$I = \int \left( \prod_i \frac{1}{n_i!} \prod_{\alpha \sigma} \frac{dz_{\alpha}^{(i)}}{2\pi iz_{\alpha}^{(i)}} \right) \left( \frac{\theta_1(z_{\alpha}^{(i)})}{\theta_1(dz_{\alpha}^{(i)})} \right) \left( \frac{\theta_1(z_{\alpha}^{(i)})}{\theta_1(dz_{\alpha}^{(i)})} \right) \left( \frac{\theta_1(\sqrt{\epsilon} z_{\alpha}^{(i)})}{\eta^2} \right).$$

As it is clear from the expression, the elliptic genus depends only on the combination $M_{ij} + M_{ij}^T \equiv A_{ij}$, the undirected adjacency matrix of the $D$ or $E$ quiver as promised. The poles of this integral can be shown to come from the two terms in the first line. They are classified by the Young diagrams $Y^{(i)}_{\ell}$, $\ell = 1, \ldots, d_i$. The number of boxes $n^{(i)}_{\ell}$ in $Y^{(i)}_{\ell}$ obey $\sum_{\ell=1}^{d_i} n^{(i)}_{\ell} = n_i$, and the poles are at

$$z_{\alpha}^{(i)} = a_{\ell}^{(i)} t^{1+\epsilon^{(i)}} d^{1+y^{(i)}}.$$

Evaluating the residues we get

$$I = \sum_{\{Y^{(i)}_{\ell}\}} \prod_i \prod_{s^{(i)}_{1}, s^{(i)}_{2}} \prod_{i,j} \theta_1(\alpha_{ij}^{(i)} t^{x_{\ell}^{(i)} - x^{m(i)}} d^{y_{\ell}^{(i)} - y^{m(i)}}) \theta_1(\alpha_{ij}^{(i)} t^{x_{\ell}^{(i)} - x^{m(i)}} + 1 d^{y_{\ell}^{(i)} - y^{m(i)} + 1})$$

$$\prod_{i,j} \prod_{s^{(i)}_{1}, s^{(i)}_{2}} \left( \frac{\theta_1(q^{1/2} A_{ij}^{(i)} t^{x_{\ell}^{(i)} - x^{m(i)}} - y_{\ell}^{(i)} - y^{m(i)} + 1/2)}{\theta_1(q^{1/2} A_{ij}^{(i)} t^{x_{\ell}^{(i)} - x^{m(i)}} - y_{\ell}^{(i)} - y^{m(i)} + 1/2)} \right)$$

$$\prod_{i} \prod_{s^{(i)}} \left( \frac{\prod_j \prod_{m} \theta_1(q^{1/2} A_{ij}^{(i)} t^{x_{\ell}^{(i)} + 1 d^{y_{\ell}^{(i)} + 1}}) \theta_1(\alpha_{ij}^{(i)} t^{x_{\ell}^{(i)} d^{y_{\ell}^{(i)}}})}{\prod_{m} \theta_1(\alpha_{ij}^{(i)} t^{x_{\ell}^{(i)} + 1 d^{y_{\ell}^{(i)} + 1}}) \theta_1(\alpha_{ij}^{(i)} t^{x_{\ell}^{(i)} d^{y_{\ell}^{(i)}}})} \right).$$

5.6 Discussion of results

In this chapter we have constructed $\mathcal{N} = (0,4)$ quiver gauge theories that describe the self-dual strings for three infinite classes of six-dimensional SCFTs, corresponding to M5 branes transverse to an A or D type singularity, M5 branes probing a M9 wall, and D5 branes transverse to ADE singularities. In all these cases, the quiver description allows one to compute elliptic genera of bound states of the strings; notably, in the case of D5 branes at an ADE singularity we found
explicit expressions for the elliptic genera of arbitrary numbers of strings, which take an elegant combinatorial form as sums over collections of Young diagrams.
Chapter 6

Exceptional and heterotic strings
from stretched M2 branes

6.1 Introduction

The main focus of this chapter is E-strings, which arise [3, 6, 110] from an M5 brane probing the Hořava-Witten M9 plane; the E-strings are identified with M2 branes stretched between the M5 brane and the M9 plane. In this context string dualities [118] relate this system to topological strings on a Calabi-Yau threefold in the vicinity of the $\frac{1}{2}K3$ surface. The topological string partition function for this theory has been studied in [28, 94, 118, 119, 140–144] and, even though major progress had already been made, at the time the results in this chapter were obtained the results were still incomplete\(^1\). Our main aim is to build on these partial results to obtain explicit formulas for the elliptic genus of two E-strings, $Z^{E\text{-str}}_2$; the answer we propose passes highly non-trivial checks\(^2\). We find that, as in the case of M-strings, two E-strings have a rather non-trivial

\(^1\)In a later work, the authors of [40] showed how one may find elliptic genera for an arbitrary number of E-strings (and therefore compute entire topological string partition function) starting from a two-dimensional quiver along the same lines presented in Chapter 3 for M-strings. Recently it was also found that E-string elliptic genera can be computed starting from a ‘Tao’ web of $(p, q)$ fivebranes [145].

\(^2\)The result for one E-string is much simpler and was already studied in [94, 118].
bound state structure, unlike fundamental strings which do not form bound states. The lack of the bound states for fundamental strings is reflected in the fact that the partition function for \( n \) fundamental strings is simply an order \( n \) Hecke transform of the one for a single string. We find that this is not the case for two E-strings.

This raises the following question: We know that an M5 brane placed between the two M9 planes of M-theory gives rise to \( E_L \)-strings from the left plane as well as \( E_R \)-strings from the right plane. On the other hand, we also know that an \( E_8 \times E_8 \) heterotic string can be identified with an M2 brane stretched between the two M9 planes [34]. Thus, \( n \) pairs of E-strings can recombine to give \( n \) heterotic strings (\( H \)):

\[
nE_L + nE_R \rightarrow nH .
\]

At first glance this is puzzling, as it is not obvious how the lack of bound states of two heterotic strings is compatible with the existence of a bound state structure for two E-strings. The answer to this puzzle is provided by the presence of the M5 brane which serves as a ‘glue’ for the M2 branes. One may also wonder whether it is possible to recover the partition function of heterotic strings from that of E-strings. The fact that E-strings recombine to give heterotic strings strongly suggests
that this should be possible, and in this chapter we indeed show that this can be done at least up to $n = 2$ E-strings. The basic idea is to view the theory of $n$ M2 branes on $\mathbb{R} \times T^2$, in the limit where the area of the $T^2$ (on which the elliptic genus does not depend) is small, as a quantum mechanical system on $\mathbb{R}$. Under this reduction the states in the Hilbert space of $n$ M2 branes are labelled by Young diagrams of size $n$ [49,123], and M5 branes as well as M9 planes intersecting the M2 branes on $T^2$ can be interpreted as operators or states in this quantum mechanical system. We call them domain wall operators/states due to their interpretation in the worldvolume theory of M2 branes.

In Chapter 2 we computed the contribution of M5 brane domain walls to this quantum mechanical system. Here, using low genus results from topological strings for up to two E-strings, and using the known M5 brane domain wall, we determine the exact M9 domain wall wave function for up to two M2 branes. We then deduce a closed formula for the elliptic genus of two E-strings, which from the viewpoint of topological string theory provides an all-genus A-model amplitude for up to two E-strings. We also test our M9 domain wall expressions by checking whether the left and right walls combine correctly into the elliptic genus of up to two heterotic strings, and remarkably we find that they do (up to taking into account a symmetrization which the heterotic string enjoys, and which is broken in the E-string background by the M5 brane). After the completion of this work, the authors of [50] showed that the ideas discussed in this chapter correctly extend to the case of three E-strings.

The organization of this chapter is as follows: In Section 6.2 we present the M2-M5-M9 configurations corresponding to the heterotic, E- and M-strings. In Section 6.3 we review the computation of the M-string elliptic genus in terms of M5 domain wall operators and the resulting partition function for two M5 branes. In Section 6.4 we obtain the elliptic genus of heterotic strings by using the Hecke transform. We then proceed in Section 6.5 to outline the series of string dualities which relate the E-string theory to the topological string on the half-K3 Calabi-Yau threefold. Finally, in Section 6.6 we determine the M9 domain wall operator for up to two strings and use it to compute the elliptic genus of E- and heterotic strings.
6.2 M2 branes on $T^2 \times \mathbb{R}$ and boundary conditions

In this section we review possible boundary conditions for M2 branes together with the preserved supersymmetries. To do this we consider M-theory on $T^2 \times \mathbb{R}^9$ and take the M2 branes to wrap the $T^2$ and extend along one of the directions of $\mathbb{R}^9$, so that their worldvolume is given by $T^2 \times \mathbb{R}$. We choose coordinates $X^I$, $I = 0, 1, 2, \cdots, 10$ and parametrize the torus by $X^0$, $X^1$ and take the direction along which the M2 branes are extended to be $X^6$. We obtain different boundary conditions by letting the M2 branes end on M5 branes or M9 planes. This can be done in various combinations which we describe here.

M9-M9

Here the relevant setup is the one of Hořava and Witten [34]. We compactify M-theory on $T^2 \times \mathbb{R}^8 \times S^1/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts as an orbifold action,

$$X^6 \mapsto -X^6,$$

(6.2.1)

together with a suitable action on the fields. At the two fixed points of the orbifold action, $X^6 = 0$ and $X^6 = \pi$, one has two fixed planes which we denote as M9 planes and are here of the topology $T^2 \times \mathbb{R}^8$; the situation is illustrated in Figure 6.1.

In the limit where the size of $S_1/\mathbb{Z}_2$ goes to zero, the M2 branes give rise to heterotic strings charged under an $E_8 \times E_8$ current algebra, with each $E_8$ coming from one M9 plane [34]. Next, we want to look at the preserved supersymmetries on these strings. Each brane type projects out half of the 32 supercharges as follows:

$$M9 : \Gamma^6 \epsilon = \epsilon, \quad M2 : \Gamma^{016} \epsilon = \epsilon,$$

(6.2.2)

and thus we see that the wordsheet theory on the strings is chiral and carries $(8,0)$ supersymmetry. We can break this supersymmetry down to $(4,0)$ and $(2,0)$ by introducing a twisted background, i.e. turning on fugacities when going along the cycles of the $T^2$. The way this works is as follows. As explained in [123] viewing the torus as $S^1 \times S^1$ we twist the $\mathbb{R}^4_{2345} \times \mathbb{R}^4_{78910}$ by the action of the
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Figure 6.1: An M2 branes suspended between M9 planes corresponding to the heterotic string. The worldvolume of the M2 branes and M9 planes share a common $T^2$ which is suppressed in the picture. The directions orthogonal to the torus are represented as the separation $X^6$ and the quaternionic subspaces $X^{2345}$ and $X^{78910}$.

Cartan subalgebra of the $SO(8)$ R-symmetry parametrized by $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \times U(1)_{\epsilon_3} \times U(1)_{\epsilon_4}$ as we go around the cycles of the torus:

$$\prod_{i=1}^{4} U(1)_{\epsilon_i} : \ (z_1, z_2) \mapsto (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2), \quad (6.2.3)$$

$$\ : \ (w_1, w_2) \mapsto (e^{2\pi i \epsilon_3} w_1, e^{2\pi i \epsilon_4} w_2), \quad (6.2.4)$$

where we impose the relation

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0 \quad (6.2.5)$$

in order to preserve supersymmetry. For generic values of the $\epsilon_i$ only a $(2,0)$ subset of the supercharges is preserved which enhances to $(4,0)$ for the locus given by $\epsilon_2 = -\epsilon_1$ and $\epsilon_3 = -\epsilon_4$ or permutations of these.
In this chapter we are interested in the computation of the elliptic genus of \(n\) heterotic strings wrapping the \(T^2\), which is given by

\[
\text{Tr}_R(-1)^F q^{H_L} q^{H_R} \prod_a x_a^{K_a},
\]

where the \(K_a\) denote the Cartan generators associated with general supersymmetry preserving \(SO_R(8)\) spacetime twists and \(E_8 \times E_8\) fugacities. We will denote this quantity by

\[
Z_{\text{Het}}^n(\tau, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \bar{m}_{E_8 \times E_8}),
\]

where \(\tau\) denotes the complex structure of the torus.

**M9-M5**

This setup leads to the theory of E-strings [6,110] which is a six-dimensional superconformal field theory with \((1, 0)\) supersymmetry. This theory arises from a system of M9 and M5 branes with M2 branes suspended between them [110]. To be more specific, we take an M9 plane as before along the coordinates \(X^0, \cdots, X^5, X^7, \cdots, X^{10}\) and an M5 brane along the directions \(X^0, \cdots, X^5\) and separate them along the \(X^6\) direction. We depict this in Figure 6.2.

Each of the branes projects out half of the 32 supercharges and the surviving supercharges satisfy the condition

\[
M_9 : \Gamma^6 \epsilon = \epsilon, \quad M_5 : \Gamma^{012345} \epsilon = \epsilon, \quad M_2 : \Gamma^{016} \epsilon = \epsilon.
\]

Thus the worldvolume theory of the E-string being the intersection of the M2 brane and the M5 brane has \((4, 0)\) supersymmetry. As the M2 brane is ending only on one of the M9 planes the string is charged under one \(E_8\) current algebra. One can now again consider a twisted background by introducing boundary conditions labelled by Cartan generators of \(SO_R(8)\) along cycles of the \(T^2\). Generic twists will break the supersymmetry down to \((2, 0)\), while setting \(\epsilon_1 = -\epsilon_2\) gives an
Figure 6.2: An M2 brane suspended between an M9 and an M5 brane corresponding to the E-string. The worldvolume of the branes share a common $T^2$ which is suppressed in the picture. The directions orthogonal to the torus are represented as the separation $X^6$ and the quaternionic subspaces $X^{2345}$ and $X^{78910}$.

enhancement to $(4,0)$. In this chapter we are interested in the computation and properties of the elliptic genus of $n$ E-strings with various fugacities turned on, namely

$$Z_n^{E-str}(\tau, \epsilon_1, \epsilon_2, \bar{m}_E).$$

(6.2.9)

Here it is important to note that the E-string elliptic genus does not depend on $\epsilon_3$ and $\epsilon_4$. The reason is that the six-dimensional E-string theory only enjoys a $SU(2)$ R-symmetry which can be identified with $SU(2)_L$ in the decomposition

$$Spin(4)_{78910} = SU(2)_L \times SU(2)_R,$$

(6.2.10)

while the $U(1)$ symmetry associated to $\epsilon_3 - \epsilon_4$ lies in $SU(2)_R$. 

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M5-M5

This configuration leads to the six-dimensional $A_{N-1} (2, 0)$ superconformal field theory [2]. Specializing to two M5 branes we obtain the $A_1$ theory that we describe here. The compactification of this theory on $T^2$ gives rise to $\mathcal{N} = 4$ SYM in four dimensions. Taking the M5 branes to be extended along $X^{012345}$ and the M2 branes along $X^{016}$ we obtain the schematic picture shown in Figure 6.3.

![Diagram](image_url)

**Figure 6.3:** An M2 brane suspended between two M5 branes corresponding to the M-string. The worldvolume of the branes share a common $T^2$ which is suppressed in the picture. The M5 branes are extended along the $X^{012345}$ directions.

We denote by M-strings the strings arising from M2 branes stretching between the M5 branes. The supercharges preserved by M-strings obey the constraints

$$M2 : \Gamma^{016} \epsilon = \epsilon, \quad M5 : \Gamma^{012345} \epsilon = \epsilon,$$

(6.2.11)

which lead to $(4, 4)$ supersymmetry on the string worldsheet. Turning on fugacities $\epsilon_i$ breaks this
down to (4, 0), (2, 2), or (2, 0) SUSY as discussed in [123]. The theory one arrives at is five-dimensional $\mathcal{N} = 2^*$ SYM compactified on a circle. The mass $m$ of the adjoint hypermultiplet is related to the $\epsilon_3$ and $\epsilon_4$ parameters as follows:

$$
\epsilon_3 = -m - \frac{\epsilon_1 + \epsilon_2}{2}, \quad \epsilon_4 = m - \frac{\epsilon_1 + \epsilon_2}{2}.
$$

(6.2.12)

Note that the condition $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$ is automatically satisfied. The elliptic genus of $n$ M-strings, with generic fugacities, was computed in Chapter 2; we denote this elliptic genus by

$$
Z_n^{M-str}(\tau, \epsilon_1, \epsilon_2, m).
$$

(6.2.13)

In the next section we will review its computation and its connection to the $\mathcal{N} = 2^*$ partition function.

### 6.3 Review of M-strings

M-strings arise naturally in the context of $A_{N-1}$ (2, 0) theories and capture the spectrum of BPS states which arise when deforming away from the CFT point [123]. It was shown there that the 5d BPS index one obtains from performing a compactification of these theories on $T^2$ with general twists can be written in terms of a sum of elliptic genera of different numbers of M-strings. In particular, for two M5 branes we have the relation

$$
Z_{5d}^{\mathcal{N}=2^* SU(2)} = \sum_n Q^n Z_n^{M-str}(\tau, \epsilon_1, \epsilon_2, m),
$$

(6.3.14)

where

$$
Q = e^{2\pi i t},
$$

(6.3.15)

t being the Coulomb branch parameter of the gauge theory. Since the Coulomb branch parameter is the separation between the M5 branes along an interval $I$, one sees that $t$ is also the tension of the self-dual strings of the (2, 0) theory, i.e. the M-strings. In the expansion (6.3.14) the M2 branes wrapped on $T^2 \times I$ play the role of instantons whose moduli space gives rise to a path integral.
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representation of the elliptic genus, as discussed in Chapters 2 and 3. This leads to the computation of the partition function of the 5d $\mathcal{N} = 2^*$ SU(2) theory in terms of the elliptic genera of M-strings.

Let us next review how $Z_n^{\text{M-str}}$ is computed. As shown in Chapter 2 it can be decomposed into “domain wall” contributions as follows:

$$Z_n^{\text{M-str}} = \sum_{|\nu| = n} \mathcal{D}_{\nu}^{M5} \mathcal{D}_{\nu}^{M5},$$

(6.3.16)

where $\nu$ are Young tableaux with $n$ boxes and $\emptyset$ denotes the empty Young tableau. The $\mathcal{D}_{\mu\nu}^{M5}$ factors can be interpreted as domain walls of ABJM theory which interpolate between two different vacua. More precisely, it is known that ABJM theory [109] on $T^2 \times \mathbb{R}$ has vacua labeled by Young tableaux [49]. Thus reducing the theory on $T^2$ gives rise to a quantum mechanics whose Hilbert space is labelled by Young tableaux, where the euclidean time flows along $X^6$. We can then define the $D_{\mu\nu}^{M5}$ as matrix elements in this quantum mechanics with an insertion of an M5 brane defect operator as follows:

$$D_{\mu\nu}^{M5} = \langle \nu | \hat{D}_{\mu\nu}^{M5} | \mu \rangle.$$  

(6.3.17)

A pictorial representation of this operator is shown in Figure 6.4.

![Figure 6.4: A M2 domain wall with a M5 defect.](image)

This operator was computed in Chapter 2 by means of the refined topological vertex [67], and the
result is given by

\[ D^\nu_M(\tau, m, \epsilon_1, \epsilon_2) = t^{-\|\mu\|^2/2} q^{-\|\nu\|^2/2} Q^{|\nu|+|\mu|}/2 \]

\[ \times \prod_{\nu \in \mu} \prod_{i,j} \left( 1 - Q^k Q^{-1}_m q^{-\nu_i + j - \frac{1}{2} t^{-\mu_j + i - \frac{1}{2}}} \right) \left( 1 - Q^k Q^{-1}_m q^{-\nu_i - j + \frac{1}{2} t^\mu_j + i + \frac{1}{2}} \right) \]

\[ \times \prod_{i,j} \left( 1 - Q^k Q^{-1}_m q^{\mu_i - \frac{1}{2} t^{-\mu_j + i - \frac{1}{2}}} \right) \left( 1 - Q^k Q^{-1}_m q^{\mu_i + j - \frac{1}{2} t^\mu_j + i - \frac{1}{2}} \right), \]

(6.3.18)

where we have defined

\[ q = e^{2\pi i \epsilon_1}, \quad t = e^{-2\pi i \epsilon_2}, \quad Q_m = e^{2\pi i m}, \quad Q_\tau = e^{2\pi i \tau}. \]

We now specialize to the cases of interest for us, namely \( D^M_{\nu \nu} \) and \( D^M_{\nu \nu} \), and introduce the notation

\[ \xi_+(\tau; z) = \prod_{k \geq 1} (1 - Q^k e^{2\pi i z}), \quad \xi_-(\tau; z) = \prod_{k \geq 1} (1 - Q^k e^{-2\pi i z}). \]

(6.3.20)

The functions \( \xi_-(\tau; z) \) and \( \xi_+(\tau; z) \) are quantum dilogarithms which can be thought of as “half theta functions”, as they combine nicely into a theta function

\[ -ie^{-i\pi z} e^{\pi i \tau} \eta(\tau) \xi_-(\tau; z) \xi_+(\tau; z) = \theta_1(\tau; z), \]

(6.3.21)

so that the product \( D^M_{\nu \nu} D^M_{\nu \nu} \) is a modular function in \( \tau \).

Up to prefactors, one finds\(^3\)

\[ D^M_{\nu \nu} = \prod_{\nu \in \mu} \frac{\theta_1(\tau; -m + \epsilon_1 (\nu_i - j + 1/2) - \epsilon_2 (-i + 1/2)) \eta(\tau)^{-1}}{\xi_- (\tau; \epsilon_1 (\nu_i - j) - \epsilon_2 (\nu_j^t - i + 1)) \xi_+ (\tau; \epsilon_1 (\nu_i - j + 1) - \epsilon_2 (\nu_j^t + i))}, \]

(6.3.22)

\[ D^M_{\nu \nu} = \prod_{\nu \in \mu} \frac{\theta_1 (\tau; -m - \epsilon_1 (\nu_i - j + 1/2) + \epsilon_2 (-i + 1/2)) \eta(\tau)^{-1}}{\xi_- (\tau; \epsilon_1 (\nu_i - j + 1) - \epsilon_2 (\nu_j^t - i)) \xi_+ (\tau; \epsilon_1 (\nu_i - j) - \epsilon_2 (\nu_j^t - i + 1))}. \]

(6.3.23)

Note that \( D^M_{\nu \nu} \) and \( D^M_{\nu \nu} \) get exchanged under the map\(^4\)

\[ m \mapsto -m, \quad \xi_\pm \mapsto \xi_\mp. \]

(6.3.24)

\(^3\)We will ignore prefactors here as well as in the rest of this chapter, since once two domain walls are glued together all surviving prefactors can be removed by a redefinition of \( Q \), as in Chapter 2.

\(^4\)This also leads to an overall \((-1)^{|\nu|}\) factor multiplying \( D_{\nu \nu}, D_{\nu \nu} \) which can always be absorbed by shifting \( Q \to -Q \).
Indeed one can immediately see, using the above building blocks, that the partition function of two M5 branes (6.3.14) has the following form:

\[
Z^{5d \mathcal{N}=2} SU(2) = \sum_{\nu} \prod_{(i,j) \in \nu} \frac{\theta_1(\tau; z_{ij})\theta_1(\tau; v_{ij})}{\theta_1(\tau; w_{ij})\theta_1(\tau; u_{ij})},
\]  

(6.3.25)

where

\[
e^{2\pi iz_{ij}} = Q_m^{-1} q^{\nu_i-j+1/2} t^{-i+1/2}, \quad e^{2\pi iv_{ij}} = Q_m^{-1} t^{i-1/2} q^{-\nu_i+j-1/2},
\]

\[
e^{2\pi i w_{ij}} = q^{\nu_i-j+1} t^{u_{ij}-i}, \quad e^{2\pi i u_{ij}} = q^{\nu_i-j} t^{v_{ij}-i+1}.
\]

(6.3.26)

One can clearly see from the expression in (6.3.25) that the elliptic genus of \(n\) M-strings receives contributions from \(4n\) bosons as well as from \(4n\) fermions coming from the theta functions in the denominator and numerator respectively. These have the interpretation of coordinates on the target space which is the moduli space of \(n\) \(U(1)\) instantons on \(\mathbb{R}^4\) in the sigma model description of M-strings. The elliptic genus can be computed by localization on the target space, which in this case is the Hilbert scheme of \(n\) points on \(\mathbb{C}^2\), namely \(\text{Hilb}^n[\mathbb{C}^2]\). Localization is done with respect to a \(U(1)^2\) action with generators \(\epsilon_1\) and \(\epsilon_2\) and the path integral turns into a sum over the fixed points of this action which are labelled by Young tableaux. The coefficients of \(\epsilon_1\) and \(\epsilon_2\) in the theta functions in the numerator are the weights of the \(U(1)^2\) action on the fermions while those in the denominator are the corresponding ones for the bosons. The different weights reflect the fact that, while the bosons are sections of the tangent bundle, the right-moving fermions transform as sections of the tautological bundle and therefore supersymmetry in the right-moving sector is broken.

We would also like to remark here that the elliptic genera \(Z_n^{\text{M-str}}\) satisfy a holomorphic anomaly equation derived in Chapter 2. To see this note that the sum in (6.3.25) is not modular invariant, as under \(SL(2,\mathbb{Z})\) transformations each summand transforms with a different phase factor:

\[
Z_n^{\text{M-str}} \left( -\frac{1}{\tau}, \frac{\epsilon_1}{\tau}, \frac{\epsilon_2}{\tau}, \frac{m}{\tau} \right) = e^{2\pi i (\epsilon_1 \epsilon_2 n^2 + (m^2 - (\epsilon_+ / 2)^2) n)} Z_n^{\text{M-str}} (\tau, \epsilon_1, \epsilon_2, m).
\]

(6.3.27)
To compensate for this phase factor and make the full partition function a modular function, one needs to make the theta function nonholomorphic. This is done by using its expansion in terms of Eisenstein series

\[
\theta_1(\tau; z) = \eta(\tau)^3(2\pi z) \exp \left( \sum_{k \geq 1} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi iz)^{2k} \right), \tag{6.3.28}
\]

and making the replacement

\[
E_2(\tau) \to \tilde{E}_2(\tau, \bar{\tau}) \equiv E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}. \tag{6.3.29}
\]

From now on we will often suppress the dependence on \(\tau\) in the modular forms we use, as well as in \(\xi_\pm\). Using these modified theta functions one can check easily using (6.3.25) that the elliptic genus of \(n\) M-strings, which is no longer holomorphic, satisfies the following holomorphic anomaly equation:

\[
\frac{\partial Z_{\text{M-str}}^n}{\partial \tilde{E}_2} = -\frac{(2\pi)^2}{12} \left( \epsilon_1 \epsilon_2 n^2 + (m^2 - (\epsilon_+/2)^2)n \right) Z_{\text{M-str}}^n, \tag{6.3.30}
\]

where \(\epsilon_+ = \epsilon_1 + \epsilon_2\). Said differently, we are trading here the modular anomaly of (6.3.27) with the holomorphic anomaly of (6.3.30). More generally, whenever we encounter in this chapter a Jacobi form with a modular anomaly

\[
Z_n(-\frac{1}{\tau}, \frac{z_1}{\tau}, \cdots, \frac{z_k}{\tau}) = e^{\frac{z_1}{\tau} a_n(z_1, \cdots, z_k)} Z_n(\tau, z_1, \cdots, z_k), \tag{6.3.31}
\]

we replace it with a non-holomorphic but modular Jacobi form with a holomorphic anomaly

\[
\frac{\partial Z_n(\tau, \bar{\tau}, z_1, \cdots, z_k)}{\partial \tilde{E}_2} = -\frac{(2\pi)^2}{24} \alpha_n(z_1, \cdots, z_k) Z_n(\tau, \bar{\tau}, z_1, \cdots, z_k). \tag{6.3.32}
\]

Thus the concepts modular and holomorphic anomaly are interchangeable and whenever we are talking about one of them one should keep in mind that an analogous statement holds for the other.

It is instructive to pause here and consider a slight modification of the above M-string setup. To this end we look at a geometry which arises by taking the trace of a single domain wall as shown in
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Figure 6.5: A M2 domain wall with a M5 defect where the $X^6$ direction is taken to be circular.

Figure 6.5. This configuration describes a six-dimensional supersymmetric $U(1)$ gauge theory \cite{76} with an adjoint hypermultiplet of mass $m$ whose partition function is given by a sum over elliptic genera which correspond to the domain-wall traces as follows

$$Z_{6d \ U(1)} = \sum_n \sum_{|\nu|=n} D_{\nu}^{M5}. \quad (6.3.33)$$

The fundamental objects of this theory are known as “little strings” \cite{146}. From the point of view of the six-dimensional $U(1)$ gauge theory which arises in the weak coupling limit of Type IIA string theory the string is a solitonic object whose moduli space is that of $U(1)$ instantons. Note that in the compactification of the six-dimensional theory on $T^2$ the string wrapping the torus becomes a $U(1)$ gauge instanton in four dimensions. Using the explicit definition of the M5 domain wall formula (6.3.18) we obtain

$$\sum_{|\nu|=n} D_{\nu}^{M5} = \prod_{(i,j)\in\nu} \frac{\theta_1(\bar{m} + \epsilon_1(\nu_i - j) - \epsilon_2(\nu_j^I - i + 1))}{\theta_1(\epsilon_1(\nu_i - j) - \epsilon_2(\nu_j^I - i + 1))} \times \frac{\theta_1(-\bar{m} + \epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu_j^I - i + 1))}{\theta_1(\epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu_j^I - i))}, \quad (6.3.34)$$

where we have redefined the R-symmetry generator $U(1)_m$ to be

$$\bar{m} = m + \frac{\epsilon_1 - \epsilon_2}{2}. \quad (6.3.35)$$

\footnote{We know that this statement holds for the single variable holomorphic Jacobi forms we consider here, and we also expect it to be true for the class of meromorphic or multivariable Jacobi forms that are used in this chapter.}
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Analogously to the M-string case, the expression (6.3.34) can be interpreted as the elliptic genus of a sigma model with target space \( \text{Hilb}^n[\mathbb{C}^2] \). But while in (6.3.25) the fermions and bosons transformed as sections of different bundles one can clearly see upon inspection of the \( U(1)^2 \) weights that in (6.3.34) they both are sections of the tangent bundle. Therefore, supersymmetry in the right-moving sector is unbroken. Remarkably, in contrast to the M-string setup, the partition function (6.3.33) has another representation which is fundamentally different from the one given in (6.3.34). This fact was first noticed for the unrefined case in [76] where the authors rewrote the result in terms of the symmetric product elliptic genus of [45]. The underlying reason for this is the equivalence of the Hilbert scheme of points with the resolution of the singular space of the \( n \)-fold symmetric product of \( \mathbb{R}^4 \):

\[
\text{Hilb}^n[\mathbb{C}^2] = \text{Res} (\text{Sym}^n(\mathbb{R}^4)) .
\] (6.3.36)

Using instead of the Hilbert scheme the orbifold \( \text{Sym}^n(\mathbb{R}^4) \) as the target space of the sigma model one arrives at an equivalent formula for the partition function (6.3.33):

\[
Z^{6d \ U(1)} = \sum_{n=1}^{\infty} Q^n \chi(\text{Sym}^n(\mathbb{R}^4)) = \prod_{n=1, k=0}^{\infty} \prod_{p_1, p_2, p_3} \frac{1}{1 - Q^n Q^{k} q^{p_1} t^{p_2} Q_{m}^{p_3} c(kn, p_1, p_2, p_3)}, \] (6.3.37)

where in the above \( \chi(\text{Sym}^n(\mathbb{R}^4)) \) is the elliptic genus of \( n \) strings and the expansion of the elliptic genus of one string is taken to be

\[
\chi(\mathbb{R}^4) = \sum_{k \geq 0, p_1, p_2, p_3} c(k, p_1, p_2, p_3) Q^{k} q^{p_1} t^{p_2} Q_{m}^{p_3}. \] (6.3.38)

Thus in some sense the \( n \)-string result is fully determined in terms of the 1-string result, which is a reflection of the fact that the \( n \)-string sector is obtained by winding single strings multiple times around the different cycles of \( T^2 \). It is remarkable that such a fundamentally different representation of the elliptic genus exists as the individual terms appearing in the two expansions (6.3.33) and (6.3.37) have a completely different pole structure in \( \epsilon_1, \epsilon_2 \) and \( m \) as can be seen from (6.3.34). Note that in the case of the \( A_1 \ (2,0) \) theory a symmetric product representation for the M-string elliptic genus does not exist; this hints at the existence of bound states of M-strings.
6.4 Heterotic strings from orbifolding

Let us recall in this section the basics of the $E_8 \times E_8$ heterotic string for the geometry considered in Section 6.2. To start, we note that the Hilbert space of $n$ heterotic strings wrapping the $T^2$ is the symmetric product of the Hilbert space of a single heterotic string, as heterotic strings do not form bound states. Said differently, at the level of the free energy the $n$-heterotic string result is the same as the $n$-times wound single heterotic string. This can be used to compute the $Z_{\text{Het}}^n$ purely from the knowledge of $Z_{\text{Het}}^1$. To proceed we thus just have to know the result for one heterotic string, which we discuss next. From now on we will be working in lightcone gauge.

For generic twist parameters $\epsilon_i$, $i = 1, \cdots, 4$, we have $(2,0)$ SUSY on the worldsheet. Thus there are four chiral multiplets with twisted boundary conditions coming from $R^{4}_{2345}$ and $R^{4}_{78910}$. When computing the elliptic genus the supersymmetric side contributes just a factor of 1, as bosonic and fermionic degrees of freedom cancel out, but the non-supersymmetric side depends on all 8 spacetime bosons which organize themselves into 4 complex bosons with twisted boundary conditions:

$$Z_{\text{bosons}}(\tau, \epsilon_i) = -\frac{\eta^4}{\theta(\epsilon_1)\theta(\epsilon_2)\theta(\epsilon_3)\theta(\epsilon_4)}.$$  \hspace{1cm} (6.4.39)

Furthermore, as the string is charged under the $E_8 \times E_8$ current algebra there will be also a bosonic path integral which contributes a factor of the character of $E_8 \times E_8$:

$$\chi_{E_8 \times E_8} = \frac{\Theta_{E_8}(\tau; \vec{m}_{E_8,L})\Theta_{E_8}(\tau; \vec{m}_{E_8,R})}{\eta^{16}},$$  \hspace{1cm} (6.4.40)

where we have introduced the $E_8$-Weyl invariant theta function of modular weight 4 and level 1

$$\Theta_{E_8}(\tau; \vec{m}) = \frac{1}{2} \sum_{i=1}^{4} \theta_i(\tau; \vec{m})^8.$$  \hspace{1cm} (6.4.41)

Combining the factor (6.4.40) with the contributions from the 8 spacetime bosons (6.4.39) one arrives at:

$$Z_{\text{Het}}^1 = -\frac{\Theta_{E_8}(\vec{m}_{E_8,L})\Theta_{E_8}(\vec{m}_{E_8,R})}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)}.$$  \hspace{1cm} (6.4.42)
To obtain the formula for the elliptic genus of $n$ heterotic strings we can apply the results of [45]. First of all, we expect the full partition function of heterotic strings to be:

$$Z_{\text{Het}} = \sum_{n \geq 0} Q^n Z_{\text{Het}}^n,$$

(6.4.43)

where $Q = e^{2\pi i \rho}$ with $\rho$ being the complexified Kähler parameter of the $T^2$ and $Z_{\text{Het}}^0$ is taken to be 1. We can next obtain the one-loop free energy\(^6\) by taking the logarithm of the partition function

$$F_{\text{Het}} = \log(Z_{\text{Het}}) = \sum_{n \geq 1} Q^n F_{\text{Het}}^n.$$

(6.4.44)

Now, following [45] we can express $F_{\text{Het}}^n$ in terms of $F_{\text{Het}}^1$ through the Hecke transform:

$$F_{\text{Het}}^n = T_n F_{\text{Het}}^1,$$

(6.4.45)

where the Hecke operator $T_n$ acts on a weak Jacobi form $f(\tau, z)$ of weight $k$ as

$$T_n f(\tau, z) = n^{k-1} \sum_{\frac{a d = n}{a, d > 0}} \sum_{(\mod d)} \frac{1}{d^k} f\left(\frac{a \tau + b}{d}, az\right).$$

(6.4.46)

Applying this to our setup and noting that $F_{\text{Het}}^1 = Z_{\text{Het}}^1$ has modular weight zero we obtain

$$Z_{\text{Het}}^n = \exp\left[\sum_{n \geq 0} Q^n \frac{1}{n} \sum_{\frac{a d = n}{a, d > 0}} \sum_{(\mod d)} Z_{\text{Het}}^1\left(\frac{a \tau + b}{d}, ae_i, am_i\right)\right],$$

(6.4.47)

which together with (6.4.43) allows us to compute $Z_{n\text{Het}}$. Again, in order for (6.4.47) to be modular, analogously to the M-string case one has to introduce some non-holomorphicity; the resulting holomorphic anomaly can be deduced from the modular anomaly. In the case of a single string this anomaly can be read off from (6.4.42):

$$Z_{1\text{Het}}^1\left(-\frac{1}{\tau}, \frac{\bar{e}}{\tau}, \frac{m}{\tau}\right) = \exp\left[-\pi i \left(\sum_{i=1}^{4} e_i^2 - \sum_{i=1}^{16} m_i^2\right)\right] Z_{1\text{Het}}^1(\tau, \bar{e}, m),$$

(6.4.48)

which shows that $Z_{1\text{Het}}^1$ is a weight zero Jacobi form of index 1/2 in each of the elliptic parameters $e_i$ and $m_i$. As the order $n$ Hecke operator transforms an index $m$ Jacobi form to an index $nm$ Jacobi

---

\(^6\)This free energy can also be computed through a heterotic string one-loop amplitude, hence the name. Similar computations have for example been performed in [147].
form we see that the elliptic genus for $n$ heterotic strings suffers from the anomaly

$$Z^\text{Het}_n\left(-\frac{1}{\tau}, \frac{\tilde{c}}{\tau}, \frac{\tilde{m}}{\tau}\right) = \exp\left[-\frac{\pi i}{\tau} n \left(\sum_{i=1}^{4} c_i^2 - \sum_{i=1}^{16} m_i^2\right)\right] Z^\text{Het}_n(\tau, \tilde{c}, \tilde{m}).$$

(6.4.49)

### 6.5 Review of known results for E-strings

In this section we recall a geometric setup which gives rise to the E-string theory, as well as topological string computations on this geometry that are related to the computation of the E-string elliptic genus. To do so, we first start with the F-theory realization of the six-dimensional superconformal field theory whose degrees of freedom are the E-string, as well as its M-theory dual. In a second subsection, we review the connection between the M-theory picture and topological strings and the way this connection has been exploited to compute the E-string free energy as a genus expansion.

#### 6.5.1 M- and F-theory realizations

E-strings arise in the Coulomb branch of small instantons [3] in $E_8 \times E_8$ heterotic string compactifications on K3 [6, 110]. In order to connect this to the picture of M2 branes suspended between M9 and M5 branes discussed in section 6.2 one embeds the small instanton in one of the $E_8$ factors and considers a specific limit of the K3 where its volume is sent to infinity and one zooms into the neighbourhood of the small instanton. In this limit the K3 can be replaced locally by $\mathbb{R}^4_{\tau}$, with an M5 brane sitting at the origin of $\mathbb{R}^4_{\tau}$ and wrapping $T^2 \times \mathbb{R}^4_{2345}$. Furthermore, the gauge fields of the two $E_8$ factors are on two different “end of the world” M9 planes as discussed in Section 6.2. Moving the M5 brane away from the M9 plane one gains a tensor multiplet whose scalar component parametrizes the distance and hence this phase can be interpreted as Coulomb branch. On the other hand the phase where one lets the size of the instanton grow is the Higgs branch.

These transitions have a beautiful F-theory realization obtained by compactifying F-theory on an elliptic Calabi-Yau threefold [21, 22, 117]. Introducing a tensor multiplet by moving an M5
brane away from an M9 plane translates to blowing up the base of the elliptic fibration at a point. This leads locally to the replacement of $\mathbb{C}^2$ with its blow-up which can be identified with the bundle $O(-1) \to \mathbb{P}^1$. In order for the Calabi-Yau property to be satisfied the elliptic fibration over the resulting $\mathbb{P}^1$ is chosen to be such that the resulting elliptic rational surface is the so called “half-K3” surface. Alternatively, this surface can also be described as the del Pezzo 9 surface $\mathbb{B}_9$ obtained by blowing up $\mathbb{P}^2$ at 9 points. The Calabi-Yau is then locally the anti-canonical bundle over this surface, namely

$$CY_3 = O(-K) \to \frac{1}{2}K3.$$  (6.5.50)

In this picture the exceptional strings, which in the M-theory setup come from M2 branes suspended between the M9 and the M5 brane, arise from D3 branes wrapping the base of $\frac{1}{2}K3$. These are pierced by 8 7-branes, corresponding to the deformation moduli of the elliptic fibration, and are thus expected to be charged under a $E_8$ current algebra.

Next, we employ the duality between F-theory and M-theory. This corresponds to compactifying our F-theory setup on $S^1$ to five dimensions, which is dual to M-theory on the $\frac{1}{2}K3$ Calabi-Yau manifold. The D3 branes wrapping $n$ times the base of $\frac{1}{2}K3$ and having KK-momentum $k$ along $S^1$ map in the dual picture to M2 branes which again wrap the base of $\frac{1}{2}K3$ $n$ times but are now also wrapping the elliptic fiber $k$ times. In the following section we utilize the topological string to compute the BPS degeneracies associated with these states.

6.5.2 Results from the topological string on $\frac{1}{2}$-K3

Let us now review the connection between the M-theory setup we have arrived at and the refined topological string, and how the latter can be used to compute the elliptic genus of $n$ E-strings. We have arrived through various dualities at the following setup:

$$M\text{-theory on } (S^1 \times \mathbb{C}^2)_{\epsilon_1, \epsilon_2} \times X,$$
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where

\[ X = O(-K) \rightarrow \frac{1}{2} K3 \]  \hspace{1cm} (6.5.51)

is the local Calabi-Yau threefold given by the anti-canonical bundle over the half-K3 surface discussed above. In going around the circle, one twists the two copies of \( \mathbb{C} \) in \( \mathbb{C}^2 \) respectively by \( \epsilon_1 \) and \( \epsilon_2 \), and furthermore performs a rotation of the fiber of \( X \) by \( -\frac{\epsilon_1 + \epsilon_2}{2} \) in order to preserve supersymmetry. In this setup, one can count the number of BPS configurations of M2-branes wrapping cycles in \( X \). These are precisely the states that are counted by the A-model refined topological string partition function on \( X \) [81], so the following statement holds:

\[ Z_{M-\text{theory}}(S^1 \times \mathbb{C}^2_{\epsilon_1,\epsilon_2} \times X) \equiv Z_{\text{top}}(X; \epsilon_1, \epsilon_2). \]  \hspace{1cm} (6.5.52)

Besides depending on \( \epsilon_1, \epsilon_2 \), the topological string partition function also depends on the Kähler parameters associated to the two-cycles of \( X \). The second-degree homology of the local half-K3 Calabi-Yau is given by

\[ H_2(X, \mathbb{Z}) = \Gamma^{E_8} \oplus \Gamma_{1,1}, \]  \hspace{1cm} (6.5.53)

where \( \Gamma_{1,1} \) is the two-dimensional hyperbolic lattice generated by the \( \mathbb{P}^1 \) base of \( \mathbb{B}_9 \), of area \( t \), and the torus fiber, of area \( \tau; \Gamma^{E_8} \), on the other hand, is the \( E_8 \) lattice generated by eight additional two-cycles of area \( (m_{E_8,1}, \ldots, m_{E_8,8}) \). Therefore the topological string partition function for this geometry is a function of the 12 parameters \( (\epsilon_1, \epsilon_2, t, \tau, m_{E_8}) \).

At the same time, upon compactification on \( X \), one obtains an effective 5d gauge theory on \( \mathbb{C}^2 \times S^1 \), where \( S^1 \) plays the role of the thermal circle. This is the \( Sp(1) \approx SU(2) \) gauge theory with eight fundamental hypermultiplets of \([148–150]\), obtained from the worldvolume theory of the M5 brane in the presence of the M9 plane by first reducing along a circle to obtain a 5d theory, and then further compactifying along the thermal circle. The theory has a superconformal fixed point at strong coupling where the flavor group is enhanced to affine \( E_8 \). The BPS configurations of M2 branes that are counted by the topological string partition function give rise to BPS particles of
this gauge theory. From the point of view of the 5d gauge theory, \((\epsilon_1, \epsilon_2)\) are fugacities associated to the \(U(1) \times U(1)\) Cartan subgroup of the little group \(SO(4) = SU(2) \times SU(2)\). Furthermore, \(t\) corresponds to the Coulomb branch parameter of the theory (which descends from the vev of the 6d tensor multiplet parametrizing the separation between the M5 branes and M9 planes); \(\tau\) is related to the 5d gauge coupling as follows:

\[
\tau = \frac{2\pi i}{g_{YM}^2};
\]

Finally, \(\tilde{m}_{E_8}\) are simply the masses of the eight hypermultiplets.

Since the gauge theory has its origin from the 6d theory of the M5 brane, the BPS instantons of the gauge theory are in one-to-one correspondence with the states of the E-string wrapping the torus. One is thus led to the following relation between the refined topological string (i.e. the 5d BPS index) and the E-string elliptic genus:

\[
Z_{\text{top}}(X; \epsilon_1, \epsilon_2) = \sum_{n=0}^{\infty} Q^n Z_n^{\text{str}}(\tau; \epsilon_1, \epsilon_2, \tilde{m}_{E_8});
\]

that is, the coefficient of \(Q^n = \exp(2\pi i n t)\), where \(t\) is interpreted as the string tension, counts the states coming from \(n\) E-strings wrapping the torus\(^7\). It is by exploiting this connection with topological strings that it has been possible to perform explicit computations of the E-string elliptic genus, beginning with the work of \[118\] in the context of unrefined topological strings (i.e. setting \(\epsilon_2 = -\epsilon_1 = g_s\)). More precisely, in this context one aims at computing the topological string free energy as a perturbative expansion which takes the following form:

\[
\mathcal{F} \equiv \log(Z_{\text{top}}(X; \epsilon_1, \epsilon_2)) = \sum_{n\geq0} \sum_{g\geq0} Q^n g_s^{2g-2} \mathcal{F}_{n,g}.
\]

The free energy of a single E-string is known to arbitrary genus (see the discussion in Section 6.6.3); in the case of several strings, topological string techniques have been employed to compute the free

\(^7\)Note that we take this sector to include both a single string wrapping the torus \(n\) times and configurations of \(k\) strings wrapping the torus respectively \(n_1, \ldots, n_k\) times, such that \(\sum_{j=1}^{k} n_k = n\).
energy to high genus (for instance, in [140] the free energy of up to five E-strings is computed up to $g = 5$). Recently, a similar approach was successfully developed [119] in the refined case (where $\epsilon_1, \epsilon_2$ are taken to be arbitrary), generalizing techniques that were employed in the unrefined limit. In the refined case, the free energy takes the following form:

$$F = \sum_{n \geq 0} \sum_{g \geq 0} \sum_{n \geq 0} Q^n (-\epsilon_1 \epsilon_2)^{g-1} (\epsilon_1 + \epsilon_2)^{2\ell} F_{n,g,\ell}. \quad (6.5.57)$$

One then observes that (in the case where $\tilde{m}_{E8}$ are set to zero) the free energy satisfies the following modular anomaly equation (which immediately gives the holomorphic anomaly equation upon replacing $E_2(\tau)$ with its modular completion $\tilde{E}_2(\tau, \bar{\tau})$):

$$\partial_{E_2} F_{n,g,\ell} = \frac{1}{24} \sum_{\nu=1}^{n-1} \sum_{\gamma=0}^{g-1} \sum_{\lambda=0}^{\ell} \nu(n-\nu) F_{\nu,\gamma,\lambda} F_{n-\nu, g-\gamma, \ell-\lambda} + \frac{n(n+1)}{24} F_{n,g-1,\ell} - \frac{n}{24} F_{n,g,\ell-1}. \quad (6.5.58)$$

This generalizes the modular anomaly equation that was found in the unrefined case by Hosono et al. [94]; the form of this expression was determined (up to the $n/24$ coefficient in front of the last term, which was obtained by other means) by requiring it to reduce correctly to known expressions in different limits. It is known that $n$ E-strings can form bound states and thus admit no simple description in terms of the Hecke transform of a single string as was the case for heterotic strings. This is reflected in the $F_{\nu} F_{n-\nu}$ term of the holomorphic anomaly equation (6.5.58). As has been noted in [28] (see also [151]) this term shows that bound states of $\nu$ and $n-\nu$ strings can pair up to form a configuration of $n$ E-strings. In Section 6.6 we will provide a simple new derivation of this formula using the M-theory realization of the E-string.

The modular anomaly equation allows one to fix the $E_2$-dependent part of the $(n, g, \ell)$ piece of the topological string free energy as long as the terms with lower values of $n, g$ and $\ell$ are known. It does not fix the $E_2$-independent piece; however, this is captured by a modular form of definite weight, and since the vector space of modular forms of a given weight is finitely generated, it can
be uniquely determined by fixing a finite number of coefficients in the $Q_\tau$ expansion of $F_{n,g,\ell}$. For low values of $g$ and $\ell$, these coefficients can be fixed by imposing ‘vanishing conditions’, that is, the fact that certain contributions to the topological string free energy are required to vanish [119]. Unfortunately, it is known that as one increases the values of $g$ and $\ell$ the number of coefficients that need to be fixed grows faster than the number of vanishing conditions [119], so one cannot use this method to compute the free energy to arbitrary order.

The same approach has been employed to compute the free energy for nonzero values of $\tilde{m}_{E_8}$ [119] (although again the vanishing conditions only allow one to determine the free energy for small enough values of $\ell$ and $g$). It is known [28] that the contributions to the free energy coming from the $n$-string sector can be written in terms of combinations of level $n$ characters of the affine $E_8$ algebra; furthermore, these combinations of characters can be written as polynomials in nine Jacobi forms $A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6$ which are $\tilde{m}_{E_8}$-dependent and invariant under the Weyl group of $E_8$. The subscript in $A_n, B_n$ indicates the amount by which they contribute to the $E_8$ level; for example, level 2 modular invariant combinations of characters of affine $E_8$ can be written as a linear combinations of $A_1^2, A_2, B_2$. Furthermore, $A_{1,\ldots,5}$ are weight-4 Jacobi forms that reduce to the Eisenstein series $E_4$ in the limit $\tilde{m}_{E_8} \to 0$, while $B_{2,3,4,6}$ have weight 6 and reduce to $E_6$ in the same limit. In the next section we will use the explicit results of [119] for the E-string free energy, written in terms of these Weyl-$[E_8]$-invariant Jacobi forms, as input to compute the elliptic genus for two E-strings to arbitrary powers of $\ell$ and $g$.

### 6.6 The elliptic genus of E- and heterotic strings

In this section, we provide evidence that the elliptic genera of up to two heterotic and exceptional strings can be written in terms of domain wall contributions, analogously to the M-string case. Recall that, as discussed in Section 6.3, the elliptic genus for $n$ M-strings could be
written in terms of M5 domain wall contributions as follows:

$$Z^{M\text{-str}}_n = \sum_{|\nu|=n} D_{\nu}^{M5} D^{M5}_{\nu}.$$  \hspace{1cm} (6.6.59)

This result had a very natural interpretation from the point of view of the M2 branes suspended between the M5 branes along the $X^6$ direction. From the point of view of the M2 branes, the M5 branes are codimension one operators supported at a point along the $X^6$ direction. In the limit where the area of the $T^2$ is taken to be very small, one is left with one dimensional quantum mechanics, where $X^6$ plays the role of time; furthermore, the Hilbert space of $n$ M2 branes is labeled by size $n$ Young diagrams [49, 123]. These states are eigenfunctions of the Hamiltonian $\hat{H}$, with eigenvalue given by the number of boxes in the Young diagram. Furthermore, the M5 defect operators become quantum mechanical operators that map a certain number of M2 branes to linear combinations of arbitrary numbers of M2 branes, so one can interpret the elliptic genus as expectation value of two M5 domain wall operators inserted at different times$^9$:

$$e^{-nt}Z^{M\text{-str}}_n = \langle 0 | \hat{D}^{M5} e^{-\hat{H}t} \hat{D}^{M5} | 0 \rangle.$$

Given that the difference between the E-string and the M-string is simply that in the former case the M2 branes terminate on an M9 plane, while in the latter they terminate on an M5 brane, it is natural to ask whether one can find a similar domain wall formula for the E-string elliptic genus, where instead of inserting an M5 domain wall operator on the left we take the product with an appropriate state $|\psi_{M9}\rangle = D^{M9,L}_{\nu} |\nu\rangle$ for the M9 plane at the left end of $S_1/\mathbb{Z}_2$. In this section, therefore, we seek an expression for the E-string elliptic genus of the form

$$e^{-nt}Z^{E\text{-str}}_n = \sum_{|\nu|=n} \langle 0 | \hat{D}^{M5} e^{-\hat{H}t} |\nu\rangle \langle \nu |\psi_{M9}\rangle = \sum_{|\nu|=n} D^{M9,L}_{\nu} D^{M5}_{\nu}.$$ \hspace{1cm} (6.6.60)

Building on results from topological string theory and exploiting several properties that the elliptic genus of E-strings is expected to satisfy, we are able to uniquely fix the left M9 domain walls for

$^9$Note that the definition of $t$ we employ here differs by the one employed elsewhere by a Wick rotation, and therefore is rescaled by a factor of $2\pi i$. 

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one and two strings:

\[
D^{M9,L}_{\square} = \left( \frac{\Theta_{E_8}(\bar{m}_{E_8,L})}{\eta^8} \right) \frac{\eta}{\theta_1(\epsilon_3)} \cdot \frac{1}{\xi_-(\epsilon_1)\xi_+(-\epsilon_2)}
\]  
(6.6.61)

and

\[
D^{M9,L}_{\square} = \frac{N_{\square}(\bar{m}_{E_8,L}, \epsilon_1, \epsilon_2)/\eta^{16}}{\xi_+(\epsilon_1)\xi_-(\epsilon_1 - \epsilon_2)\xi_+(2\epsilon_1)\theta_1(m + \epsilon_+/2)\theta_1(m + \epsilon_+/2 + \epsilon_1)\eta^{-2}},
\]  
(6.6.62)

\[
D^{M9,L}_{\square} = \frac{N_{\square}(\bar{m}_{E_8,R}, \epsilon_1, \epsilon_2)/\eta^{16}}{\xi_-(\epsilon_1)\xi_+(\epsilon_1 - \epsilon_2)\xi_+(-2\epsilon_2)\theta_1(m - \epsilon_+/2)\theta_1(m - \epsilon_+/2 - \epsilon_2)\eta^{-2}},
\]  
(6.6.63)

where \(N_{\square}\) and \(N_{\square}\) (explicit expressions for which can be found in Equations (6.6.102) and (6.6.103)) are certain Jacobi forms of weight 8 that depend on the \(E_8\) Wilson lines as well as \(\epsilon_1, \epsilon_2\). We also find corresponding formulas for the right M9 domain walls. Combining these domain walls with the known M5 domain walls, we are able to reproduce the known elliptic genus for a single E-string, and find a novel closed formula for the two E-string elliptic genus, which takes the following form:

\[
Z_{E-str}^2 = D^{M9,L}_{\square} D^{M5,L}_{\emptyset} + D^{M9,L}_{\square} D^{M5,L}_{\emptyset}.
\]  
(6.6.64)

Once the M9 domain walls have been computed, it is natural to ask the following question: given that the heterotic string is given by M2 branes which end on two M9 planes, is it possible to also write the elliptic genus of heterotic strings in terms of domain wall expressions, where now we take both domain walls to be of the M9 type? We find that this is indeed the case: for one heterotic string, we show that

\[
Z_{het}^1 = D^{M9,L}_{\square} (\bar{m}_{E_8,L}) D^{M9,R}_{\square} (\bar{m}_{E_8,R}).
\]  
(6.6.65)

Similarly, for two heterotic strings we find the following identity:

\[
Z_{het}^2 = D^{M9,L}_{\square} (\bar{m}_{E_8,L}) D^{M9,R}_{\square} (\bar{m}_{E_8,R}) + D^{M9,L}_{\square} (\bar{m}_{E_8,L}) D^{M9,R}_{\square} (\bar{m}_{E_8,R}) + (\ldots),
\]  
(6.6.66)

where \((\ldots)\) are two additional terms that are obtained from the first two by symmetrizing with respect to permutations of \(\epsilon_1, \epsilon_2, \epsilon_3\) and \(\epsilon_4\). This formula matches with the expression one obtains by using the Hecke transform of the one heterotic string elliptic genus, despite the fact that it has a very different appearance. In particular, each one of the terms appearing in our new expression
is manifestly modular, and is split into two factors, one which only depends on the $E_{8,L}$ degrees of
freedom, and one which only depends on the $E_{8,R}$ degrees of freedom.

While the computation of elliptic genera in this chapter is limited to the case of one and two E-
strings, we offer some evidence that our approach should work for an arbitrary number of strings by
deriving the E-string modular anomaly equation, which was recently conjectured in [119], from the
known holomorphic anomaly equations for M- and heterotic strings, assuming that these elliptic
genera can all be written in terms of M5 and M9 domain walls.

This section is divided as follows: in Section 6.6.1 we provide more details of our approach and of
the ingredients that go into it; in Section 6.6.2 we show how this leads to an alternate derivation of
the E-string modular anomaly equation. in Section 6.6.3 we compute the M9 domain wall factor
associated to a single M9 plane, and use it to reproduce known formulas for the elliptic genus for a
single E-string and a single heterotic string. In 6.6.4 we turn to the case of two strings; we derive
expressions for the corresponding M9 domain walls, and we use these to derive a closed formula for
the elliptic genus of two E-strings; furthermore, we obtain a novel expression for the elliptic genus
of two heterotic strings. Finally, in 6.7 we make some additional comments about the features of
the domain wall expressions we obtained.

6.6.1 M9 domain walls

The computation of the elliptic genus of the E-strings is not an easy task, since configu-
rations of several E-strings form bound states and therefore their elliptic genus cannot be deduced
from the elliptic genus of a single E-string by means of the Hecke transform. Here we describe
in some detail an alternative approach, which is based on the computation of M9 domain wall
factors. Fortunately, from the symmetries of the problem one can deduce several properties that
these factors must satisfy which will allow us to uniquely determine them in the case of $n = 1, 2$. 
We list the expected properties here:

- The elliptic genus for $n$ E-strings is expected to transform with modular weight 0 under the $SL(2, \mathbb{Z})$ transformation

$$
(t, m, \epsilon_1, \epsilon_2, \tau) \rightarrow \left( t, \frac{m}{c\tau + d}, \frac{\epsilon_1}{c\tau + d}, \frac{\epsilon_2}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right), \quad (6.6.67)
$$

where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \).

Since the denominator of the M5 domain wall expression (6.3.23) by itself is not modular invariant, each factor of \( \xi_\pm(z) \) that appears there must be matched by an equivalent factor of \( \xi_\pm(z) \) in the M9 domain wall in order to combine them into the Jacobi form \( \theta_1(\tau, z)/\eta(\tau) \), which has well-defined modular transformation properties.

- The E-string partition function does not depend on the mass parameter \( m = (\epsilon_4 - \epsilon_3)/2 \).

This implies that the mass-dependent factors in the numerator of the M5 domain wall must be canceled by identical factors in the denominator of the M9 domain wall.

- As discussed above, the \( \tilde{m}_{E_8} \) dependence of the $n$ E-string free energy (and therefore also the $n$ E-string elliptic genus) is captured in terms of level $n$ characters of affine $E_8$, and thus the \( \tilde{m}_{E_8} \)-dependent factors in the $n$ E-string elliptic genus can be written in terms of level $n$ combinations of the Weyl\(|E_8|\)-invariant Jacobi forms $A_{1,2,3,4,5}, B_{2,3,4,6}$ which are discussed in Appendix A.

- From the modular anomaly equation for the E-string free energy, Eq. (6.5.58), one easily derives a modular anomaly equation for the elliptic genus of $n$ E-strings in the limit $\tilde{m}_{E_8} \rightarrow 0$:

$$
\partial_{\epsilon_2} Z_n^{E-str} \bigg|_{\tilde{m}_{E_8}=0} = -\frac{(2\pi)^2}{24} [\epsilon_1 \epsilon_2 (|\nu|^2 + |\nu|) - \epsilon_+^2 |\nu|] \cdot Z_n^{E-str} \bigg|_{\tilde{m}_{E_8}=0}, \quad (6.6.68)
$$

where \( \epsilon_+ = \epsilon_1 + \epsilon_2 \). This is most easily satisfied by requiring that each summand $Z_\nu = D_\nu M^5 D_{\nu \theta}$ in Eq. (6.6.60) satisfy the same equation, so we conjecture that the following
holds:
\[
\partial_{E_2} Z_{\nu}^{E-\text{str}} \bigg|_{\tilde{m}_{E_8} = 0} = -\frac{(2\pi)^2}{24} \left[ \epsilon_1 \epsilon_2 (|\nu|^2 + |\nu|) - \epsilon_+^2 |\nu| \right] \cdot Z_{\nu}^{E-\text{str}} \bigg|_{\tilde{m}_{E_8} = 0}.
\] (6.6.69)

The Weyl-\(E_8\)-invariant Jacobi forms \(A_{1,2,3,4,5}, B_{2,3,4,6}\) satisfy the following modular anomaly equation:
\[
\begin{align*}
\partial_{E_2} A_n(\tau; \tilde{m}_{E_8}) &= -n \cdot \frac{(2\pi)^2}{24} \left( \sum_{i=1}^{8} m^2_{E_8,i} \right) A_n(\tau; \tilde{m}_{E_8}), \quad (6.6.70) \\
\partial_{E_2} B_n(\tau; \tilde{m}_{E_8}) &= -n \cdot \frac{(2\pi)^2}{24} \left( \sum_{i=1}^{8} m^2_{E_8,i} \right) B_n(\tau; \tilde{m}_{E_8}); \quad (6.6.71)
\end{align*}
\]

this leads us to guess the following form for the E-string holomorphic anomaly, for arbitrary values of \(\tilde{m}_{E_8}\):
\[
\partial_{E_2} Z_{\nu}^{E-\text{str}} = -\frac{(2\pi)^2}{24} \left[ \epsilon_1 \epsilon_2 (|\nu|^2 + |\nu|) - \epsilon_+^2 |\nu| + |\nu| \left( \sum_{i=1}^{8} m^2_{E_8,i} \right) \right] \cdot Z_{\nu}^{E-\text{str}}. \quad (6.6.72)
\]

- Finally, the elliptic genus for \(n\) E-strings is expected to be symmetric under exchange of \(\epsilon_1\) and \(\epsilon_2\). This is guaranteed by the fact that \(Z_\nu(\epsilon_1, \epsilon_2)\) and \(Z_\nu(\epsilon_2, \epsilon_1) = Z_\nu^t(\epsilon_1, \epsilon_2)\) both appear in the expression for \(Z_{|\nu|}(\epsilon_1, \epsilon_2)\).

We also make the assumption that from the \(E_8\) degrees of freedom (which for a single E-string are eight bosons compactified on the \(E_8\) lattice) one obtains a factor of \(\eta^{8n}\) in the denominator of the elliptic genus of \(n\) E-strings. From this, and from the first three properties listed above, we can immediately write down the following ansatz for the left M9 domain wall:
\[
D_{\nu}^{M9,L} = \frac{N_{L}^{L}(\tau; \tilde{m}_{E_8,L}, \epsilon_1, \epsilon_2)}{\eta(\tau)^8|\nu| B_{\nu}^{L}(\tau; \epsilon_1, \epsilon_2) F^{R}_{\nu}(\tau; \epsilon_1, \epsilon_2, m)^{1/2}},
\] (6.6.73)

where\(^{10}\)
\[
B_{\nu}^{L}(\tau; \epsilon_1, \epsilon_2) = \prod_{(i,j) \in \nu} \xi_+(\epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu_j - i))\xi_-(\epsilon_1(\nu_i - j) - \epsilon_2(\nu_j - i + 1))
\] (6.6.74)

\(^{10}\)Up to a prefactor \(t^{\frac{\nu \cdot \nu}{2}}\), which is needed to ensure that after gluing the factors of \(\xi_\pm\) combine correctly into theta functions.
and
\[ F^R_\nu(\tau; \epsilon_1, \epsilon_2) = \prod_{(i,j) \in \nu} \theta_1(-m - \epsilon_1(\nu_i - j + 1/2) + \epsilon_2(-i + 1/2))/\eta \] (6.6.75)

are obtained by requiring that they combine correctly with the bosonic (that is, denominator) and fermionic (numerator) pieces of the M5 domain wall \( D^{M5}_\nu \) (Equation (6.3.23)).

Likewise, we take the right M9 domain wall to be given by
\[ D^{M9, R}_\nu = \frac{N^R_\nu(\tau; \bar{m}_{E_8, R}, \epsilon_1, \epsilon_2)}{\eta(\tau)^{8|m|} B^R_\nu(\tau; \epsilon_1, \epsilon_2) F^L_\nu(\tau; \epsilon_1, \epsilon_2, m)}, \] (6.6.76)

where
\[ B^R_\nu(\tau; \epsilon_1, \epsilon_2) = \prod_{(i,j) \in \nu} \xi_-(\tau; \epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu_j - i))\xi_+(\tau; \epsilon_1(\nu_i - j) - \epsilon_2(\nu_j - i + 1)) \] (6.6.77)

and
\[ F^L_\nu(\tau; \epsilon_1, \epsilon_2) = \prod_{(i,j) \in \nu} \theta_1(\tau; -m + \epsilon_1(\nu_i - j + 1/2) - \epsilon_2(-i + 1/2))/\eta(\tau). \] (6.6.78)

The transformation that exchanges left and right M5 domain walls leaves \( \epsilon_1 \) and \( \epsilon_2 \) fixed (see Equation (6.3.24)). Therefore it is natural to expect that
\[ N^R_\nu(\bar{m}_{E_8, R}, \epsilon_1, \epsilon_2) = N^L_\nu(\bar{m}_{E_8, R}, \epsilon_1, \epsilon_2) = N(\bar{m}_{E_8, R}, \epsilon_1, \epsilon_2), \] (6.6.79)

so the only difference between the numerators of the left and right domain walls is that they depend on the fugacities for the corresponding \( E_8 \) group. The non-trivial task is to compute the numerator factor \( N^L_\nu \); later in this section we will use the remaining properties listed above to uniquely determine it in the case of one and two E-strings.

Once the M9 domain walls have been computed, it should also be possible to express the heterotic string partition function in terms of them (now replacing every occurrence of mass parameter \( m \) with \( (\epsilon_4 - \epsilon_3)/2 \), as is more appropriate in this context):
\[ Z_{het}^n \sim \sum D^{M9, L}_n(\bar{m}_{E_8, L}) \cdot D^{M9, R}_n(\bar{m}_{E_8, R}). \] (6.6.80)

\(^{11}\)Up to a prefactor \( q^{-\frac{16|m|}{2}} \).
We will show how this works explicitly for one and two heterotic strings in the following sections. For now, let us list the properties that the elliptic genus for $n$ heterotic strings, reviewed in Section 6.4, is known to satisfy:

- The elliptic genus for $n$ heterotic strings transforms with modular weight 0 under the $SL(2,\mathbb{Z})$ modular transformation

\[
(t, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \tau) \rightarrow \left(t, \frac{\epsilon_1}{c\tau + d}, \frac{\epsilon_2}{c\tau + d}, \frac{\epsilon_3}{c\tau + d}, \frac{\epsilon_4}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right).
\]

- The elliptic genus for $n$ heterotic strings is invariant under pairwise exchange of $\epsilon_i, \epsilon_j$, for any $i, j = 1 \ldots 4$.

- The modular anomaly equation for the heterotic string is given by Eq. (6.4.49):

\[
\frac{\partial Z_{\text{Het}}^n}{\partial E_2} = n \cdot \frac{(2\pi)^2}{24} \left( \sum_{i=1}^{4} \epsilon_i^2 - \sum_{i=1}^{8} ((m_{E_8,L}^i)^2 + (m_{E_8,R}^i)^2) \right) Z_{\text{Het}}^n.
\]

6.6.2 E-string holomorphic anomaly

The purpose of this section is to demonstrate that the modular anomaly equation (6.6.72) for the E-strings can be easily derived from the modular anomaly equations for heterotic strings and M-strings, by using our ansatz for the M9 domain walls, Equation (6.6.73). To see this, note that each summand appearing in the elliptic genus of $n$ M-strings, Equation (6.6.59), has the form

\[
\frac{F^L_{\nu} \cdot F^R_{\nu}}{B^L_{\nu} \cdot B^R_{\nu}},
\]

where the explicit expressions for these factors is unimportant for the present discussion. Similarly, from our domain wall ansatz we expect each summand in the expression for $Z_{\text{Het}}^n$ to have the form

\[
\frac{N^L_{\nu}(\tau; \bar{m}_{E_8,L}, \epsilon_1, \epsilon_2)}{\eta^{8n}(B^L_{\nu} \cdot B^R_{\nu})},
\]

and each summand in $Z_{\text{Het}}^n$ to have the form

\[
\frac{N(\bar{m}_{E_8,L}, \epsilon_1, \epsilon_2) \cdot N(\bar{m}_{E_8,R}, \epsilon_1, \epsilon_2)}{\eta^{16n}(B^L \cdot B^R)(F^L \cdot F^R)}.
\]
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Note that, if we set \( \widehat{m}_{E_8,R} = \widehat{m}_{E_8,L} \), Eqn. (6.6.84) is the square root of the product between Equations (6.6.83) and (6.6.85). Therefore, we expect that

\[
\frac{1}{(2\pi)^2} \frac{1}{Z_n^{E-\text{str}}} \frac{\partial Z_n^{E-\text{str}}}{\partial E_2} = \frac{1}{(2\pi)^2} \frac{1}{2Z_n^{M-\text{str}}} \frac{\partial Z_n^{M-\text{str}}}{\partial E_2} + \frac{1}{(2\pi)^2} \left[ \frac{1}{2Z_n^{\text{het}}} \frac{\partial Z_n^{\text{het}}}{\partial E_2} \right] \bigg|_{\widehat{m}_{E_8,R} = \widehat{m}_{E_8,L}}. \tag{6.6.86}
\]

Indeed, a short calculation reveals that the right hand side is given by

\[
\text{r.h.s.} = -\frac{n}{24} \left[ \epsilon_1 \epsilon_2 n + \left( \frac{\epsilon_3 - \epsilon_4}{2} \right)^2 - \left( \frac{\epsilon_1 + \epsilon_2}{2} \right)^2 \right] + \frac{n}{48} \left( \sum_{i=1}^{4} \epsilon_i^2 - 2 \sum_{i=1}^{8} (m_{E_8,i}^L)^2 \right)
\]
\[
= -\frac{n}{24} \left[ \epsilon_1 \epsilon_2 n - \frac{\epsilon_3 \epsilon_4}{2} \right] + \frac{n}{48} \left( 2(\epsilon_1 + \epsilon_2)^2 - 2(\epsilon_1 \epsilon_2 + \epsilon_3 \epsilon_4) - 2 \sum_{i=1}^{8} (m_{E_8,i}^L)^2 \right)
\]
\[
= \frac{n}{24} \left[ (\epsilon_1 + \epsilon_2)^2 - (n+1)\epsilon_1 \epsilon_2 - \left( \sum_{i=1}^{8} (m_{E_8,i}^L)^2 \right) \right], \tag{6.6.87}
\]

which is identical to the conjectural E-string modular anomaly of Eq. (6.6.72) that was obtained using completely different techniques!

### 6.6.3 One E-string and one heterotic string

We now turn to the explicit computation of the M9 domain wall for the partition \( \nu = \boxempty \) and show that the elliptic genera for a single E-string and the one for a single heterotic string can both be written in terms of it. The elliptic genus for a single E-string is known exactly: it is simply given by the torus partition function for eight bosons compactified on an internal \( E_8 \) lattice and four spacetime bosons:

\[
Z_1^{E-\text{str}} = -\left( \frac{A_1(\widehat{m}_{E_8,L})}{\eta^8} \right) \frac{\eta^2}{\vartheta_1(\epsilon_1) \vartheta_1(\epsilon_2)}, \tag{6.6.88}
\]

where \( A_1(\widehat{m}_{E_8,L}) = \Theta_{E_8}(\tau; \widehat{m}_{E_8,L}) \) is the \( E_8 \) theta function. If we make the ansatz

\[
Z_1^{E-\text{str}} = D_\square^{M_9,L} D_\square^{M_5,R}, \tag{6.6.89}
\]
where

\[ D^{M5}_{\Box \emptyset} = \frac{\theta_4(-m - \epsilon_+ / 2) \eta^{-1}}{\xi_-(\epsilon_1)\xi_+(-\epsilon_2)}, \]  

we immediately see that

\[ D^{M9,L}_{\Box} = \left( \frac{A_1(\tilde{m}_{E_8,L})}{\eta^8} \right) \frac{\eta}{\theta_1(-m - \epsilon_+ / 2)} \cdot \frac{1}{\xi_+(\epsilon_1)\xi_-(\epsilon_2)}, \]

\[ D^{M9,R}_{\Box} = \left( \frac{A_1(\tilde{m}_{E_8,R})}{\eta^8} \right) \frac{\eta}{\theta_1(\epsilon_4)} \cdot \frac{1}{\xi_-(\epsilon_1)\xi_+(-\epsilon_2)}. \]

Recalling that under left-right exchange \( \epsilon_3 \leftrightarrow \epsilon_4, \tilde{m}_{E_8,L} \leftrightarrow \tilde{m}_{E_8,R}, \) and \( \xi_\pm \leftrightarrow \xi_\pm, \) we also find that

\[ Z_{\text{het}}^1 = \left( \frac{A_1(\tilde{m}_{E_8,L}) \times A_1(\tilde{m}_{E_8,R})}{\eta^{16}} \right) \frac{\eta^4}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)}, \]

which is precisely the elliptic genus for a single heterotic string, since

\[ A_1(\tilde{m}_{E_8,L}) \times A_1(\tilde{m}_{E_8,R}) = \Theta_{E_8 \times E_8}(\tau; \tilde{m}_{E_8,L}, \tilde{m}_{E_8,R}). \]

### 6.6.4 Two E-strings and two heterotic strings

We now turn to the discussion of domain walls for two strings. Using these domain walls we will be able to deduce an exact expression for the elliptic genus of two E-strings; we will also be able to obtain an expression for the elliptic genus of two heterotic strings which is in agreement with the orbifolding formula. Before turning to computations, we would like to highlight a remarkable fact. As discussed in Section 6.4, the heterotic strings do not form bound states, and therefore their elliptic genus can be computed by means of the Hecke transform; on the other hand, the E-strings, like the M-strings, do form bound states and therefore do not admit such a simple description. Nevertheless, we will see that the same building blocks – the M9 and M5 domain walls – can be
used to compute the elliptic genera for two E-strings as well as for two heterotic strings.

Following the approach outlined at the beginning of the section, we start by making the following ansatz for the two E-string elliptic genus:

\[ Z_2^{E-\text{str}} = D_{\boxtimes}^{M9,L} D_{\boxtimes}^{M5} + D_{\boxtimes}^{M9,L} D_{\boxtimes}^{M5}, \]  

(6.6.95)

where

\[ D_{\boxtimes}^{M5} = \frac{\theta_1(m + \epsilon_+ / 2) \eta^{-1}}{\xi_-(\epsilon_1) \xi_+(\epsilon_1 + \epsilon_2)} \]  

(6.6.96)

and

\[ D_{\boxtimes}^{M5} = \frac{\theta_1(m - \epsilon_+ / 2) \eta^{-1}}{\xi_+(\epsilon_1) \xi_-(\epsilon_1 + \epsilon_2)} \]  

(6.6.97)

This leads to the following ansatz for the M9 domain walls:

\[ D_{\boxtimes}^{M9,L} = \frac{N_{\boxtimes}(\bar{m}_{E_8,L}, \epsilon_1, \epsilon_2)/\eta^{16}}{\xi_-(\epsilon_1) \xi_+(\epsilon_1 + \epsilon_2) \xi_-(\epsilon_1 - \epsilon_2) \xi_+(\epsilon_1 + \epsilon_2) \theta_1(m + \epsilon_+/2) \theta_1(m + \epsilon_+/2 + \epsilon_1) \eta^{-2}}, \]  

(6.6.98)

\[ D_{\boxtimes}^{M9,L} = \frac{N_{\boxtimes}(\bar{m}_{E_8,L}, \epsilon_1, \epsilon_2)/\eta^{16}}{\xi_+(\epsilon_1) \xi_+(\epsilon_1 - \epsilon_2) \xi_-(\epsilon_1 - \epsilon_2) \xi_+(\epsilon_1 + \epsilon_2) \theta_1(m - \epsilon_+/2) \theta_1(m - \epsilon_+/2 - \epsilon_2) \eta^{-2}}. \]  

(6.6.99)

We expect that the two E-string elliptic genus can be written as

\[ Z_2^{E-\text{str}} = -\frac{N_{\boxtimes}(\bar{m}_{E_8,L}, \epsilon_1, \epsilon_2)/\eta^{16}}{\theta_1(\epsilon_1) \theta_1(\epsilon_2) \theta_1(\epsilon_1 - \epsilon_2) \theta_1(\epsilon_1 + \epsilon_2) \eta^{-4}} - \frac{N_{\boxtimes}(\bar{m}_{E_8,L}, \epsilon_1, \epsilon_2)/\eta^{16}}{\theta_1(\epsilon_1) \theta_1(\epsilon_2) \theta_1(\epsilon_2 - \epsilon_1) \theta_1(2\epsilon_2) \eta^{-4}}. \]  

(6.6.100)

To fix the numerator terms we exploit the following facts:

- Modular invariance of \( Z_2^{E-\text{str}} \) requires the modular weight of \( N_{\boxtimes} \) and \( N_{\boxtimes} \) to be 8 in order to cancel with the modular weight of the denominator (since \( \eta \) and \( \theta_1(z) \) have modular weight 1/2).

- The numerator terms can be written as linear combinations of the three level-two Weyl\([E_8]\)-invariant modular forms \( A_1^2, A_2, \) and \( B_2 \).

- From the modular anomaly equation for \( Z_2^{E-\text{str}} \) one obtains

\[ \frac{1}{N_{\boxtimes}} \partial_{E_8} N_{\boxtimes} = -\frac{(2\pi)^2}{12} \left[ \epsilon_1^2 + \left( \sum_{i=1}^{8} (m_{E_8,i}^L)^2 \right) \right]; \]  

(6.6.101)
the $m_{E_8,L}$ terms in this equation are consistent with the fact that $N_{\Box}$ can be expressed in terms of level 2 characters of affine $E_8$, while the $-\epsilon_1^2/12$ term indicates that $N_{\Box}$ is a function of $\epsilon_1$ and not $\epsilon_2$, and furthermore that it transforms with index 2 with respect to $\epsilon_1$ under modular transformations. Since $N_{\Box}(\tau; \tilde{m}_{E_8,L}, \epsilon_1, \epsilon_2) = N_{\Box}(\tau; \tilde{m}_{E_8,L}, \epsilon_1, \epsilon_2)$, an analogous conclusion holds for $N_{\Box}$, which is written in terms of level 2 $E_8$ characters and index two Jacobi forms with elliptic parameter $\epsilon_2$.

Concretely, this forces the numerator terms to have the following form:

$$N_{\Box}(\tau; \tilde{m}_{E_8,L}, \epsilon_1, \epsilon_2) = A_1(\tilde{m}_{E_8,L})^2 f_1(\tau; \epsilon_1) + B_2(\tilde{m}_{E_8,L}) f_2(\tau; \epsilon_1) + A_2(\tilde{m}_{E_8,L}) f_3(\tau; \epsilon_1), \quad (6.6.102)$$
$$N_{\Box}(\tau; \tilde{m}_{E_8,L}, \epsilon_1, \epsilon_2) = A_1(\tilde{m}_{E_8,L})^2 f_1(\tau; \epsilon_2) + B_2(\tilde{m}_{E_8,L}) f_2(\tau; \epsilon_2) + A_2(\tilde{m}_{E_8,L}) f_3(\tau; \epsilon_2), \quad (6.6.103)$$

where $f_1(\tau; \epsilon), f_2(\tau; \epsilon), f_3(\tau; \epsilon)$ are Jacobi forms of index 2 with elliptic parameter $\epsilon$, respectively of modular weight 0, 2 and 4.

We now resort to the following fact about Jacobi forms [152]:

The weak Jacobi forms with modular parameter $\tau$ and elliptic parameter $\epsilon$ of index $k$ and even weight $w$ form a polynomial ring which is generated by the four modular forms $E_4(\tau), E_6(\tau), \phi_{0,1}(\epsilon, \tau)$, and $\phi_{-2,1}(\epsilon, \tau)$, where

$$\phi_{-2,1}(\epsilon, \tau) = -\frac{\theta_1(\epsilon; \tau)^2}{\eta^6(\tau)} \quad \text{and} \quad \phi_{0,1}(\epsilon, \tau) = 4 \left[ \frac{\theta_2(\epsilon; \tau)^2}{\theta_2(0; \tau)^2} + \frac{\theta_3(\epsilon; \tau)^2}{\theta_3(0; \tau)^2} + \frac{\theta_4(\epsilon; \tau)^2}{\theta_4(0; \tau)^2} \right]$$

are Jacobi forms of index 1, respectively of weight $-2$ and 0.

Thus modularity implies that $f_1, f_2, f_3$ can be written as follows:

$$f_1(\epsilon) = c_{1,1} \phi_{0,1}(\epsilon)^2 + c_{1,2} E_4 \phi_{-2,1}(\epsilon)^2; \quad (6.6.104)$$
$$f_2(\epsilon) = c_{2,1} E_4 \phi_{0,1}(\epsilon) \phi_{-2,1}(\epsilon) + c_{2,2} E_6 \phi_{-2,1}(\epsilon)^2; \quad (6.6.105)$$
$$f_3(\epsilon) = c_{3,1} E_4 \phi_{0,1}(\epsilon)^2 + c_{3,2} E_6 \phi_{0,1}(\epsilon) \phi_{-2,1}(\epsilon) + c_{3,3} E_4^2(\tau) \phi_{-2,1}(\epsilon)^2. \quad (6.6.106)$$
We now can determine the numerical coefficients $c_{i,j}$ as follows: we use the results for the topological string free energy for up to $n = 2$ computed in [119] to calculate $Z_{\text{str}}^E$ as an expansion in $\epsilon_1 \cdot \epsilon_2$ and $\epsilon_1 + \epsilon_2$, and match it against our ansatz (6.6.100). We find that the terms in the free energy up to $g + \ell = 2$ are sufficient to uniquely fix all the coefficients in our expression.\textsuperscript{12} We find the following result:

\[
N_{\text{D}}(\bar{m}_{E_8,L}, \epsilon_1) = \frac{1}{576} \left[ 4A_1^2(\phi_{0,1}(\epsilon_1)^2 - E_4\phi_{-2,1}(\epsilon_1)^2) \\
+ 3A_2(E_4^2\phi_{-2,1}(\epsilon_1)^2 - E_6\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)) + 5B_2(E_6\phi_{-2,1}(\epsilon_1)^2 - E_4\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)) \right],
\]

and

\[
N_{\text{D}}(\bar{m}_{E_8,L}, \epsilon_2) = N_{\text{D}}(\bar{m}_{E_8,L}, \epsilon_2).
\]

In fact, in [119] the free energy was computed up to $g + \ell = 3$; we have checked that our domain wall expressions also match exactly with those coefficients; this provides a nontrivial check that we have found a formula for the two E-string elliptic genus which is exact to all orders in $g$ and $\ell$. Given the explicit formula for the two E-string elliptic genus, Equation (6.6.100), one can easily check that the answer is not what one would have obtained using the Hecke transform:

\[
Z_{\text{str}}^E(\tau; \bar{m}_{E_8}, \epsilon_1, \epsilon_2) \neq \frac{1}{2} \left[ Z_{1}^{\text{str}}(\tau; \bar{m}_{E_8}, \epsilon_1, \epsilon_2)^2 + Z_{1}^{\text{str}}(2\tau; 2\bar{m}_{E_8}, 2\epsilon_1, 2\epsilon_2) \\
+ Z_{1}^{\text{str}}(\tau/2; \bar{m}_{E_8}, \epsilon_1, \epsilon_2) + Z_{1}^{\text{str}}(\tau/2 + 1/2; \bar{m}_{E_8}, \epsilon_1, \epsilon_2) \right].
\]

This was to be expected, since the right hand side is not supposed to produce the right answer in contexts where the strings can form bound states.

\textsuperscript{12}In fact, the form of the numerator is constrained even further if we observe that, when we set $\bar{m}_{E_8} \to 0$, it should be given by a genuine holomorphic Jacobi form in $\epsilon_1$ or $\epsilon_2$ (not just a weakly holomorphic Jacobi form), as one expects from a unitary theory.
heterotic string partition function. Note that

\[
D^{M9,L}(\tilde{m}_{Es,L})D^{M9,R}(\tilde{m}_{Es,R})
= - \frac{N_{\square}(\tilde{m}_{Es,L}, \epsilon_1)N_{\square}(\tilde{m}_{Es,R}, \epsilon_1)}{\eta(\tau)^{24}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)\theta_1(2\epsilon_1)\theta_1(\epsilon_1 - \epsilon_2)\theta_1(\epsilon_1 - \epsilon_3)\theta_1(\epsilon_1 - \epsilon_4)}
= - \frac{N_{\square}(\tilde{m}_{Es,L}, \epsilon_1)N_{\square}(\tilde{m}_{Es,R}, \epsilon_1)}{\eta(\tau)^{24}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)\theta_1(2\epsilon_1)\theta_1(\epsilon_1 - \epsilon_2)\theta_1(\epsilon_1 - \epsilon_3)\theta_1(\epsilon_1 - \epsilon_4)}
\]

and

\[
D^{M9,L}(\tilde{m}_{Es,L})D^{M9,R}(\tilde{m}_{Es,R})
= - \frac{N_{\square}(\tilde{m}_{Es,L}, \epsilon_2)N_{\square}(\tilde{m}_{Es,R}, \epsilon_2)}{\eta(\tau)^{24}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)\theta_1(2\epsilon_2)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(\epsilon_2 - \epsilon_3)\theta_1(\epsilon_2 - \epsilon_4)}
= - \frac{N_{\square}(\tilde{m}_{Es,L}, \epsilon_2)N_{\square}(\tilde{m}_{Es,R}, \epsilon_2)}{\eta(\tau)^{24}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)\theta_1(2\epsilon_2)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(\epsilon_2 - \epsilon_3)\theta_1(\epsilon_2 - \epsilon_4)}
\]

Here we run into a puzzle: we would naively have guessed that

\[
Z_\text{het}^2 \equiv D^{M9,L}(\tilde{m}_{Es,L})D^{M9,R}(\tilde{m}_{Es,R}) + D^{M9,L}(\tilde{m}_{Es,L})D^{M9,R}(\tilde{m}_{Es,R});
\]

however, notice that this expression is not invariant under arbitrary exchanges of the four parameters \(\epsilon_1, \ldots, \epsilon_4\), as we would expect from the heterotic string! We find instead that the first summand is invariant under arbitrary permutation of \(\epsilon_2, \epsilon_3, \epsilon_4\) while the second is invariant under any permutation of \(\epsilon_1, \epsilon_3, \epsilon_4\); furthermore, the two terms are exchanged by \(\epsilon_1 \leftrightarrow \epsilon_2\). The most natural remedy for this is to symmetrize the right hand side of Equation (6.6.112). This leads to the following formula for the elliptic genus for two heterotic strings:

\[
Z_\text{het}^2 \equiv D^{M9,L}(\tilde{m}_{Es,L})D^{M9,R}(\tilde{m}_{Es,R}) + (\epsilon_1 \leftrightarrow \epsilon_2) + (\epsilon_1 \leftrightarrow \epsilon_3) + (\epsilon_1 \leftrightarrow \epsilon_4).
\]

We find by direct comparison that this expression exactly matches with the orbifold formula for the elliptic genus for two heterotic strings, despite their completely different appearance!\(^{13}\)

\(^{13}\)We have checked this result up to powers of \(Q_r^8\), with a generic choice of \(E_8 \times E_8\) Wilson lines.
In order to highlight the novel properties of formula (6.6.113), let us recall here the result of the orbifold formula of Section 6.4 specialized to the case of two heterotic strings:

\[
Z_{\text{het}}^2(\tau, \bar{\epsilon}, \bar{m}_E \times E_8) = \frac{1}{2} \left[ \left( Z_{\text{het}}^1(\tau, \bar{\epsilon}, \bar{m}_E \times E_8) \right)^2 + Z_{\text{het}}^1(2\tau, 2\bar{\epsilon}, 2\bar{m}_E \times E_8) + Z_{\text{het}}^1\left( \frac{\tau + 1}{2}, \bar{\epsilon}, \bar{m}_E \times E_8 \right) \right].
\] (6.6.114)

One can clearly see from this expression that the left and right \( E_8 \) masses are entangled in a non-trivial way. By this we mean that it is not possible to perform independent \( SL(2, \mathbb{Z}) \) transformations on the left and right degrees of freedom, since under an \( SL(2, \mathbb{Z}) \) transformation the last three terms in Equation (6.6.114) transform into each other in a nontrivial way. In contrast to this expression (6.6.113) is manifestly \( SL(2, \mathbb{Z}) \) invariant and one can perform independent modular transformations on the left and right degrees of freedom.

6.7 Discussion of results

We have derived expressions for one and two E-strings (6.6.100) as well as for one and two heterotic strings (6.6.100) in terms of domain wall building blocks. Our results have a few unusual and interesting properties on which we want to comment. First of all, observe that the well-known orbifold representation for heterotic strings given by (6.4.47) has a very simple pole structure in the parameters \( \epsilon_i \). On the other hand, the individual terms in Equation (6.6.113) have a more complicated pole structure. The agreement with the orbifold formula is a consequence of a nontrivial cancelation between the poles appearing in these terms.

We also wish to remark that it should be possible to give a direct physical interpretation to M9 domain wall expressions. This was the case for the M5 brane domain wall formula, which in [123] was shown to be equal to the open topological string partition function for a certain toric Calabi-Yau threefold. It would be interesting to determine whether the M9 domain wall formulas can also...
be related to the computation of the open topological string partition function on some specific Calabi-Yau geometry. A hint in this direction comes from the fact that the M9 expressions we computed have an integral expansion in the parameters $Q, q, t$ and $Q_m$, which is consistent with a BPS degeneracy interpretation of the expansion coefficients.

It remains to comment on the validity of our ansatz for three or more strings. One supporting argument we provided is that our domain wall picture leads to the correct holomorphic anomaly equation for the E-string from those of the M-string and heterotic string. Furthermore, we have checked that for three heterotic strings the leading term of the expected result given by the orbifold formula is reproduced correctly by our ansatz. After the completion of this work, a paper appeared [50] in which the elliptic genus for three E-strings and heterotic strings were obtained following the methods presented in this chapter.

\footnote{See also [153] for a computation of the anomaly polynomial of E-strings by using contributions from the M5 brane and M9 plane.}
Chapter 7

Non-perturbative topological strings, $S^5$ partition functions, and 6d superconformal index

7.1 Introduction

In this chapter we discuss the computation of partition functions of superconformal field theories on $S^5$ and on $S^5 \times S^1$, and explain how this is related to a non-perturbative completion of the topological string theory partition function. In the case of superconformal theories that have a six-dimensional origins, since the topological string partition function can be computed in terms of elliptic genera of the self-dual strings of the 6d SCFT, the results of the present section also imply that the six-dimensional superconformal index can be expressed in terms of elliptic genera of the self-dual strings.

Topological strings have been defined perturbatively, but it is certainly interesting to ask whether one can find a non-perturbative definition for them. In a strong sense topological strings, which capture the BPS content of the deformations of the superconformal theories, compute relevant
amplitudes for supersymmetric partition functions of superconformal theories. Thus one idea is to reverse the statement and define non-perturbative topological strings using supersymmetric partition functions.

The relation between topological string partition functions and superconformal index for $\mathcal{N} = 1$ 5d theories has been explored in [69, 154]. The aim of this chapter is to extend this relation in two directions: Given the relation between superconformal partition functions and topological strings we come up with both a definition of non-perturbative topological strings on the one hand, and also a proposal for how to use topological strings to compute certain supersymmetric partition functions. In particular, we focus on the partition function of $\mathcal{N} = 1$ superconformal theories in 5d on $S^5$ and superconformal $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (1, 0)$ theories in 6d on $S^5 \times S^1$.

The perturbative parts of the superconformal partition functions were computed for certain gauge theories on $S^5$ [155–159], and using this ingredient and the condition that the BPS content captured by topological strings behaves as the fundamental degrees of freedom of the theory, an idea advanced in [69], we propose not only a way to compute the full answer for superconformal partition functions on $S^5$, but also a non-perturbative definition for topological strings. Moreover by viewing 6d $(2, 0)$ and $(1, 0)$ superconformal theories compactified on $S^1$ as a supersymmetric system in 5d, we are able to also compute the superconformal index for a large class of $(2, 0)$ (and in particular $N$ coincident M5 branes) and $(1, 0)$ theories in 6 dimensions.

The highly non-trivial aspect of this proposal is that the full non-perturbative aspect of the topological partition function enters because we have coupling constants of topological strings inverted. In particular, roughly speaking the proposal for the non-perturbative topological string partition function $Z_{np}$ takes the form (which will be made more precise later in this chapter)

$$Z_{np}(t_i, m_j, \tau_1, \tau_2) = \frac{Z^{\text{top}}(t_i, m_j; \tau_1, \tau_2)}{Z^{\text{top}}(t_i/\tau_1, m_j/\tau_1; -1/\tau_1, \tau_2/\tau_1) \cdot Z^{\text{top}}(t_i/\tau_2, m_j/\tau_2; \tau_1/\tau_2; -1/\tau_2)}$$

where $t_i, m_j$ are normalizable and non-normalizable Kahler classes, and $\tau_1, \tau_2$ are the two couplings

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1By analytic continuation this can also be written in the form $Z_{np} = Z^{\text{top}} \cdot Z^{\text{top}} \cdot Z^{\text{top}}$. 264
of the refined topological strings. Of course to define exactly what this means we have to be more precise and we use the BPS degeneracies captured by topological strings to give a precise meaning to $Z_{np}$. Furthermore, the superconformal partition function on $S^5$ is written in terms of this composite non-perturbative $Z$ by

$$Z_{S^5}(m_j; \tau_1, \tau_2) = \int dt_i Z_{np}(t_i, m_j; \tau_1, \tau_2)$$

where $m_j$ are interpreted as mass parameters and $\tau_1, \tau_2$ can be viewed as squashing parameters for $S^5$. The relevant 5d theories we consider can be viewed as compactification of M-theory on singular loci of Calabi-Yau manifolds where some 4-cycles have shrunk [117, 149, 150, 160, 161]. For a subset of these, which geometrically engineer a gauge theory [100], $Z_{top}$ can be identified with the 5d gauge theory partition function [59], with $\tau_i = \epsilon_i$. We can also consider 6d superconformal theories: There are two classes of them, with $(2 \hookrightarrow 0)$ or $(1 \hookrightarrow 0)$ supersymmetry. A large class of these theories can be obtained as F-theory on elliptic 3-folds (in the case of $(2 \hookrightarrow 0)$ it corresponds to a constant elliptic fiber). Compactifying these theories on a circle down to 5 dimensions leads to dual descriptions involving M-theory on elliptic Calabi-Yau threefolds. Upon further compactification on $S^5$, we can use the resulting non-perturbative topological string on elliptic Calabi-Yau threefold to compute the partition function on $S^5$. This leads to the partition function of the 6d theory on $S^1 \times S^5$, i.e. it leads to the computation of the 6d superconformal index, where $m_j$ correspond to fugacities for flavor symmetry and $\tau_{1,2}$ correspond to parameters of supersymmetric rotations on $S^5$. Moreover one of the fugacities $m_i$ corresponds to the Kahler class $\tau$ of the elliptic fiber. This will correspond to the extra parameter in the superconformal $(1, 0)$ theory. For the $(2, 0)$ theory the superconformal index depends on 4 parameters. In this case the corresponding topological theory computes the partition function of $\mathcal{N} = 2^*$ gauge theory in 5 dimensions and the mass and coupling constant of the gauge theory correspond to the two additional parameters of the $(2, 0)$ 6d index (see [157] for a related discussion). Thus, we are able to compute the superconformal index for $\mathcal{N} = (2, 0)$ systems in 6 dimensions.

We can also consider Lagrangian defects of topological strings. These lead to 3d theories
living on the non-compact part of the M5 brane wrapping the Lagrangians. Upon compactification on $S^3$ these can also be viewed as a non-perturbative completion of the open topological string, which has already been considered in [162–165]. In particular the structure for the open string part has the form

$$Z_{\text{open}}^{np}(\ldots) = \frac{Z_{\text{open}}(t_i, m_j, x_k; \tau)}{Z_{\text{open}}(t_i/\tau, m_j/\tau, x_k/\tau; -1/\tau)},$$

where the $t_i, m_i$ are closed string parameters and $x_k$ label open string moduli. The corresponding partition function on $S^3$ is given by

$$Z_{S^3} = \int dx_k Z_{\text{open}}^{np}(t_i, m_j, x_k; \tau).$$

The organization of this chapter is as follows: In Section 2 we review the relation between open topological strings and the $S^3$ partition function of M5 branes wrapping Lagrangians in CY. In Section 3 we study the partition function of the $\mathcal{N} = 1$ superconformal theories in 5 dimensions. In Section 4 we propose a non-perturbative definition of topological strings which can be used for the computation of these amplitudes. In Section 5, we offer a possible explanation of our results from M-theory. In Section 6 we discuss the connection with 6d superconformal indices and in particular compute the superconformal index for a single M5 brane. In Section 7 we provide an expression for the superconformal index of several M5 branes and relate it to the computation of elliptic genera of M-strings. In Section 8 we present our conclusions. Some more technical aspects of the discussion are presented in Appendix C.

### 7.2 SCFT on squashed $S^3$ and open topological strings

One of the common themes that have emerged in the study of superconformal theories in various dimensions is the important role played by the BPS states that arise when one moves away from the superconformal fixed point (see [69] and references therein). In particular it was shown in [69] that the superconformal index in diverse dimensions is deeply related to BPS spectrum and this data can be used to fully compute the index in $\mathcal{N} = 2$ theories in $d = 3$ and $\mathcal{N} = 1$ theories in $d = 5$. 
These correspond to partition functions on \( S^2 \times S^1 \) and \( S^4 \times S^1 \) respectively. Here we are interested in computing the partition functions of these theories on \( S^3 \) and \( S^5 \), respectively. To this end, it is instructive to review the case of \( \mathcal{N} = 2 \) superconformal theories on the squashed three-sphere \( S^3_b \). This class of theories is particularly simple, since away from the superconformal point only a finite number of BPS particles appear, which are in one-to-one correspondence with the electrically charged fields of the SCFT. The full partition function for these theories has been computed exactly [166,167] and indeed we will see that it can be reinterpreted in terms of contributions coming from the BPS particles (as occurs in a similar context in [162–164]).

We can write the squashed three-sphere geometry in terms of variables \((z_1, z_2) \in \mathbb{C}^2\) as

\[
\omega_1^2 |z_1|^2 + \omega_2^2 |z_2|^2 = 1.
\]

For \( \omega_1 \neq \omega_2 \), the \( SO(4) \) isometry group of \( S^3 \) gets broken to \( U(1) \times U(1) \). The ratio of the equivariant parameters for the two rotations is \( \tau = b^2 = \omega_1/\omega_2 \).

We now recall the partition function for superconformal gauge theories on the squashed three-sphere, whose gauge and matter content are provided respectively by vector and chiral multiplets. Away from the superconformal point, many of these theories can be constructed from M-theory as the worldvolume theories of M5-branes wrapping \( S^3 \) times a Lagrangian submanifold of an appropriately chosen Calabi-Yau threefold \( X \). The geometry of \( X \) determines the BPS content of the theory, and the superconformal theory is recovered in the IR (shrinking the size of the Lagrangian to zero).

Let \( \mathfrak{g} \) be the Lie algebra of the gauge group \( G \), and \( \mathfrak{h} \) its Cartan subalgebra. Let \( h_i, i = 1, \ldots, \text{rank}(G) \) be a basis for \( \mathfrak{h} \). We denote a generic element of \( \mathfrak{h} \) by \( \phi = \sum \phi_i h_i \), and for an arbitrary weight \( \nu \) of \( \mathfrak{g} \) we write \( \phi_\nu = \langle \nu, \phi \rangle \). By localization, the computation of the partition function of the SCFT reduces to an integral over \( \mathfrak{h} \), with contributions from one-loop determinants for the chiral and vector multiplets:

\[
Z_{S^3_b} = \int d\phi \prod_{\beta \in \Delta_+} \phi_\beta^2 \cdot Z_0(\phi) \cdot Z_{\text{vect}}^{1-\text{loop}}(\phi) \cdot Z_{\text{chiral}}^{1-\text{loop}}(\phi), \quad (7.2.1)
\]
where $\Delta_+$ is the set of positive roots of $G$. The classical action can contain Chern-Simons and FI terms, and produces a factor of

$$Z_0(\phi) = e^{-\frac{\pi i}{2} k_i \phi_i^2 + 2\pi i \xi_i \phi_i}, \quad (7.2.2)$$

where $k_i$ is the CS level and $\xi_i \in \mathbb{R}$ is the FI-term. For abelian factors we can also have additional off-diagonal CS interactions as well as mixed CS terms with flavors symmetries.

If we include matter fields in a (not necessarily irreducible) representation $R$ of the gauge group $G$, for each weight in $R$ we obtain a chiral multiplet. The one-loop contribution to the partition function is

$$Z_{\text{chiral}}^{1-\text{loop}}(\phi) = \prod_{\mu \in R} \prod_{j,k \geq 0} \frac{(j + 1/2)\omega_1 + (k + 1/2)\omega_2 + i\phi_\mu}{(j - 1/2)\omega_1 + (k - 1/2)\omega_2 - i\phi_\mu}$$

$$= \prod_{\mu \in R} S_2^{-1}(i\phi_\mu + (\omega_1 + \omega_2)/2|\omega_1, \omega_2), \quad (7.2.3)$$

where the double sine function $S_2(z|\omega_1, \omega_2)$ is defined in Appendix C.1. The vector multiplet, on the other hand, contributes a factor of (taking into account the shift in spin $s = 1/2$)

$$Z_{\text{vect}}^{1-\text{loop}}(\phi) = \prod_{\beta \in \Delta} \frac{1}{i\phi_\beta} \prod_{j,k \geq 0} \frac{j\omega_1 + k\omega_2 + i\phi_\beta}{(j + 1)\omega_1 + (k + 1)\omega_2 - i\phi_\beta}$$

$$= \prod_{\beta \in \Delta} \frac{1}{i\phi_\beta} S_2(i\phi_\beta + \omega_1 + \omega_2|\omega_1, \omega_2), \quad (7.2.4)$$

where by $\Delta$ we mean the set of roots of $G$. Note that for a spin $s$ field we get a shift of

$$\left(\frac{1}{2} + s, \frac{1}{2} + s\right) \cdot (\omega_1, \omega_2).$$

Putting all the pieces together, the partition function is

$$Z_{S^5} = \int d\phi e^{-\frac{i\pi k_i \phi_i^2}{2} + 2\pi i \xi_i \phi_i} \prod_{\beta \in \Delta} S_2(i\phi_\beta + \omega_1 + \omega_2|\omega_1, \omega_2) \prod_{\mu \in R} S_2^{-1}(i\phi_\mu + (\omega_1 + \omega_2)/2|\omega_1, \omega_2).$$

Thus to each multiplet $\alpha$ corresponds a factor of $S_2(z_\alpha|\omega_1, \omega_2)^{\pm 1}$, where the argument of the double sine function depends on the data attached to the multiplet. Note that for the vector multiplet the $\prod_{\beta \in \Delta} S_2(i\phi_\beta + \omega_1 + \omega_2|\omega_1, \omega_2)$ is equal to a $q$-deformed Vandermonde. The double sine has simple
modular transformation under the $S$ transformation of $SL(2, \mathbb{Z})$. Indeed, when $\tau = \omega_1/\omega_2 \in \mathbb{H}$ the double sine function can be written in the following suggestive form (C.1.11):

\[
S_2(z_\alpha + (\omega_1 + \omega_2)/2|\omega_1, \omega_2)
= \exp \left( \frac{\pi i}{2} B_{2,2}(z_\alpha + (\omega_1 + \omega_2)/2|\omega_1, \omega_2) \right) \prod_{j=0}^{\infty} \left( 1 - e^{\zeta_\alpha + \pi i + 2\pi i(j+1/2)\tau} \right) \prod_{j=0}^{\infty} \left( 1 - e^{\zeta_\alpha + \pi i + 2\pi i(j+1/2)\hat{\tau}} \right),
\]

where we have defined $\zeta_\alpha = 2\pi iz_\alpha/\omega_2$, $\hat{\zeta}_\alpha = \zeta_\alpha/\tau$, $\hat{\tau} = -1/\tau$, and $q = \exp(2\pi i \tau)$ and $\hat{q} = \exp(-2\pi i / \tau)$. The exponential prefactors come from the $(2,2)$ multiple Bernoulli polynomial (C.1.8),

\[
B_{2,2}(z_\alpha|\omega_1, \omega_2) = \frac{z_\alpha^2}{\omega_1 \omega_2} - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} z_\alpha + \frac{\omega_1^2 + \omega_2^2 + 3\omega_1 \omega_2}{6 \omega_1 \omega_2}.
\]

Under an $S$ modular transformation that takes $\tau \to \hat{\tau}$ and $\zeta_\alpha \to \hat{\zeta}_\alpha$,

\[
S_2(z_\alpha + (\omega_1 + \omega_2)/2|\omega_1, \omega_2) \to S_2(z_\alpha + (\omega_1 + \omega_2)/2|\omega_1, \omega_2)^{-1}.
\]

On the other hand, the double sine function does not transform into itself under the $T$ transformation $\tau \to \tau + 1$, so we cannot complete this to a full $SL(2, \mathbb{Z})$ action.

We would now like to clarify the relation with BPS states and open topological string theory. For this purpose, it is convenient to strip away the prefactors from the double sine function and define

\[
S_2(z|\omega_1, \omega_2) = \exp \left( -\frac{\pi i}{2} B_{2,2}(z|\omega_1, \omega_2) \right) S_2(z|\omega_1, \omega_2).
\]

Using the building block of the double sine function we can write down the contribution of particles of charges $n_i, n_j$ under $U(1)$ gauge factors and flavor factors respectively with central terms $(x_i, m_j)$ (before gauging) and spins $s$:

\[
S_2((n_i x_i + n_j m_j) + (\frac{1}{2} + s)(\omega_1 + \omega_2)|\omega_1, \omega_2)^{(-1)^{2s}}.
\]
Thus we would get for many particles a partition function of the form:

\[ Z = e^{Q(x, m_j)} \cdot \prod_a S_2(n^a_i x_i + n^a_j m_j) + \left( \frac{1}{2} + s_a \right)(\omega_1 + \omega_2)|\omega_1, \omega_2|^{-(1)^{2s_a}} \]

where we have included the prefactor (involving the exponential of the quadratic form) which is added at the end depending on the FI terms and the CS levels (see [168] for a thorough discussion of these terms). To obtain the final partition function we have to integrate over the scalars in the \( U(1) \) vector multiplets leading to

\[ Z_{S^3} = \int dx_i Z(x_i, m_j, \tau). \]

In the next section we discuss how this can be presented in the context of 3d theories living on M5 branes wrapped on special Lagrangian 3-cycles, using open topological string amplitudes.

### 7.2.1 Topological string reformulation

We now use topological strings to reformulate this partition function (see also [164]). It is known that open topological strings captures the BPS content of M5 branes wrapped on special Lagrangian cycles of Calabi-Yau threefold [169]. For simplicity we will focus on the unrefined case here (but will extend the discussion to the refined case when considering the closed string sector).

Consider M-theory compactification on a Calabi-Yau threefold, and consider a number of M5 branes wrapping some special Lagrangian cycles. Then M2 branes ending on M5 branes constitute the BPS states of the theory. The partition function of topological strings captures this. In particular we have (up to quadratic exponential prefactor)\(^2\):

\[ Z_{\text{top}}^{\text{open}} = \prod_a \prod_{k=0}^{\infty} \left( 1 - q^{k+s_a+\frac{1}{2}} e^{2\pi i n^a_i x_i + 2\pi i n^a_j m_j} \right)^{N_{a_i, n_j, s_a} (-1)^{2s_a + 1}} \]

For our purposes it is more convenient to define a slightly shifted version of the topological string amplitude given by

\[ \tilde{Z}_{\text{top}}^{\text{open}} = \prod_a \prod_{k=0}^{\infty} \left( 1 - (-1)^{2s_a + 1} q^{k+s_a+\frac{1}{2}} e^{2\pi i n^a_i x_i + 2\pi i n^a_j m_j} \right)^{N_{a_i, n_j, s_a} (-1)^{2s_a + 1}} \]

\(^2\)We are always free to rescale the arguments of the double sine function \( z, \omega_1, \omega_2 \) by a common factor. When comparing to topological strings, we choose a gauge where \( \omega_2 = 1 \).
where \( q = \exp(2\pi i \tau) \) and \( N_{n_i,n_j,s_a} \) denote the number of BPS states with the corresponding charges as spin. We will drop the tilde in the rest of this chapter as we will be mainly discussing this shifted version. The unshifted version can be recovered by shifting the \( \tau \) back.

We now simply ask what would the partition function of this theory be if we were to put it on the squashed \( S^3 \)? Even though we have no a priori Lagrangian description of this theory we will assume, as in [69], that the BPS states can be treated as elementary degrees of freedom. Using the fact that double sine computes the corresponding term we would thus naturally get

\[
Z = e^{Q(x_i,m_i)} \cdot \prod_{n_i,n_j,s_a} S_2((n_i^a x_i + n_j^a m_j) + (\frac{1}{2} + s_a)(\omega_1 + \omega_2)|\omega_1,\omega_2)^{N_{n_i,n_j,s_a}(-1)^{2s_a+1}},
\]

where we have included the prefactor involving the quadratic classical term \( Q \) of the topological string. Using the product representation of the double sine function and the form of \( Z_{\text{open}}^{\text{top}} \) we can rewrite this entirely in terms of the topological string partition function as

\[
Z_{\text{np}}^{\text{open}} = \frac{Z_{\text{top}}^{\text{open}}(x_i,m_j;\tau)}{Z_{\text{top}}^{\text{open}}(x_i/\tau,m_j/\tau;1/\tau)}
\]

and we can view this as a non-perturbative definition of topological string. Then the partition function on squashed \( S^3 \) is given by

\[
Z_{S^3} = \int dx \hspace{1mm} e^{Q(x_i,m_i)} \cdot Z_{\text{np}}^{\text{open}} = \int dx_i \hspace{1mm} e^{Q(x_i,m_i)} \cdot \frac{Z_{\text{top}}^{\text{open}}(x_i,m_j;\tau)}{Z_{\text{top}}^{\text{open}}(x_i/\tau,m_j/\tau;1/\tau)}
\]

where by definition what we mean by \( Z_{\text{top}} \) at \(-1/\tau\) is the product expression we have given. Notice that the factor of \((-1)^s\) in the expansion, which for even \( s \) does not seem to affect the perturbative \( Z_{\text{top}} \), will be relevant under the \( \tau \to -1/\tau \), which we include in the definition of \( Z_{\text{top}} \) at \(-1/\tau\).

As we have seen, when \( \text{Im} \tau > 0 \),

\[
Z_{\text{np}}^{\text{open}}(\ldots;\tau) = Z_{\text{top}}^{\text{open}}(\ldots|\tau)/Z_{\text{top}}^{\text{open}}(\ldots;1/\tau);
\]

similarly for \( \text{Im} \tau < 0 \),

\[
Z_{\text{np}}^{\text{open}}(\ldots;\tau) = Z_{\text{top}}^{\text{open}}(\ldots|1/\tau)/Z_{\text{top}}^{\text{open}}(\ldots;\tau).
\]
But in fact the proposed non-perturbative completion of the open topological string is also valid for \( \tau \in \mathbb{R}_+ \), i.e. at \( |q| = 1 \), even though the perturbative topological string is ill-defined there.

### 7.3 Five dimensional superconformal theories

We saw in the last section that knowing the properties of BPS states of the theory on the squashed three-sphere away from the superconformal fixed point is sufficient to compute the partition function of the SCFT. We now shift our focus to superconformal theories on \( S^5 \) which can be obtained from the compactification of M-theory on a Calabi-Yau threefold. Assuming that in this case too the BPS states account for all the degrees of freedom of the SCFT, we can introduce squashing parameters for \( S^5 \) and propose an exact answer for the partition function (equation (7.3.10)), which includes all gauge theory instanton contributions. To this end all we need to know is the contribution of each individual BPS particle to the partition function and take the product of them over all BPS states, as if they are non-interacting fundamental degrees of freedom. Thus the main thing we need to do is to do a computation of the partition function on squashed \( S^5 \) for a single BPS particle.

Such a computation has been carried out in [155, 156] for certain BPS particles which appear as the perturbative part of the partition function of \( \mathcal{N} = 1 \) superconformal field theory on \( S^5 \) with non-abelian gauge group and matter in an arbitrary representation \( R \). We review this result and propose a generalization of it to particles of arbitrary spin. This is also important to us for another reason: as in the 3d case, even if the gauge theory is non-abelian, the computations can be entirely recast in terms of an integral over the abelian Coulomb branch parameters, where the non-abelian aspects are reflected by the existence of additional BPS states in the computation. This allows us to formulate our final result in term of an integral over the Coulomb branch.

In the perturbative computation, the path integral localizes on the Cartan subgroup of the gauge group, and the hyper- and vector multiplets, which correspond respectively to the matter and gauge content of the theory, contribute the following one-loop determinants evaluated on the
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Localization locus:

$$Z^{1\text{-loop}}_{\text{hyper}}(\phi) = \prod_{\mu \in R} \prod_t (t - i\phi_\mu + 3/2)^{-t^2/2 - 3t/2 - 1},$$

where $\mu$ are the weights in the representation and $\phi$ is an element of the Cartan, and

$$Z^{1\text{-loop}}_{\text{vect}}(\phi) = \prod_{\beta \in \Delta_+} \prod_{t \neq 0} [(t + i\phi_\beta)(t - i\phi_\beta)]^{t^2/2 + 3t/2 + 1},$$

where $\Delta_+$ denotes the positive roots of the gauge group.

In appendix C.3 we show that these expressions can be recast in terms of triple sine functions [71–74] as

$$Z^{1\text{-loop}}_{\text{hyper}}(\phi) = \prod_{\mu \in R} S_3^{-1}(i\phi_\mu + 3/2|1, 1, 1) \quad (7.3.7)$$

and

$$Z^{1\text{-loop}}_{\text{vect}}(\phi) = \prod_{\beta \in \Delta_+} (i\phi_\beta)^{-2} \prod_{\beta \in \Delta_+} S_3(i\phi_\beta|1, 1, 1)S_3(i\phi_\beta + 3|1, 1, 1), \quad (7.3.8)$$

up to a prefactor which can be reabsorbed into the cubic prepotential. The triple sine function is defined as a regularized infinite product over three indices:

$$S_3(z|w_1, w_2, w_3) \sim \prod_{n_1, n_2, n_3 = 0}^{\infty} (n_1 w_1 + n_2 w_2 + n_3 w_3 + z)((n_1 + 1)w_1 + (n_2 + 1)w_2 + (n_3 + 1)w_3 - z)$$

(the precise definition and several important properties of this function are collected in Appendix C.1). From this expression it is clear that the one-loop determinants for the theory on $S^5$ are evaluated at a very degenerate choice of parameters for the triple sine. In the theory on $S^3$ an interesting deformation was obtained by introducing squashing parameters $\omega_{1,2}$, and the one-loop determinants were found to be built out of factors of $S_2(z|\omega_1, \omega_2)$. In our current setup, it is also very natural to move away from this limit and consider an analogous deformation by three parameters $\omega_{1,2,3}$. That is, we conjecture that one can formulate a deformation of the theory on squashed $S^5$, which can be embedded in $\mathbb{C}^3$ as

$$\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 + \omega_3^2|z_3|^2 = 1,$$
and that each occurrence of $S_3(z|1,1,1)$ gets replaced by $S_3(z|\omega_1,\omega_2,\omega_3)$. The $SO(6)$ isometry of $S^5$ gets broken to $U(1)^{(1)} \times U(1)^{(2)} \times U(1)^{(3)}$, where $U(1)^{(i)}$ corresponds to rotation of the $z_i$-plane. The ratio of the equivariant parameters for $U(1)^{(i)}$ and $U(1)^{(j)}$ is given by $\omega_i/\omega_j$.

The hyper and vector multiplet one-loop determinants become

$$Z_{\text{pert}}^{1-\text{loop}}(\phi) = \prod_{\mu \in R} S_3^{-1}(i\phi_\mu + \omega_1/2 + \omega_2/2 + \omega_3/2|\omega_1,\omega_2,\omega_3),$$

and

$$Z_{\text{hyper}}^{1-\text{loop}}(\phi) = \prod_{\beta \in \Delta_+} (i\phi_\beta)^{-2} \prod_{\beta \in \Delta_+} S_3(i\phi_\beta|\omega_1,\omega_2,\omega_3)S_3(i\phi_\beta + \omega_1 + \omega_2 + \omega_3|\omega_1,\omega_2,\omega_3).$$

Putting all these contributions together, the perturbative contribution to the partition function (choosing units where the radius of $S^5 = 1$) is

$$Z_{S^5}^{\text{pert}} = \int_{\text{Cartan}} d\phi \left( \prod_{\beta \in \Delta_+} \phi_\beta^2 \right) Z_0(\phi) Z_{\text{hyper}}^{1-\text{loop}}(\phi) Z_{\text{vect}}^{1-\text{loop}}(\phi)$$

$$= \int_{\text{Cartan}} Z_0(\phi) \prod_{\beta \in \Delta_+} S_3(i\phi_\beta|\omega_1,\omega_2,\omega_3)S_3(i\phi_\beta + \omega_1 + \omega_2 + \omega_3|\omega_1,\omega_2,\omega_3) \cdot$$

$$\prod_{\mu \in R} S_3^{-1}(i\phi_\mu + \omega_1/2 + \omega_2/2 + \omega_3/2|\omega_1,\omega_2,\omega_3),$$

where

$$Z_0(\phi) = \exp \left[ \frac{1}{\omega_1\omega_2\omega_3} \left( \frac{4\pi^3}{g_{YM}^2} \text{Tr}\phi^2 + \frac{ik}{24\pi^2} \text{Tr}\phi^3 \right) \right],$$

which comes from the tree level Lagrangian (where we have included the effect of $\omega_i$ being turned on). Notice that this term is the exponential of a cubic polynomial $Z_0 = \exp[C(\phi,1/g_{YM}^2)]$ where $C$ captures the cubic content of the prepotential term, where we view $1/g_{YM}^2$ as a scalar in an ungauged vector multiplet.

Just as in the 3d case the non-abelian measure factors have disappeared and we can interpret the integrand as the contribution of the electric BPS states in an abelian theory, as we go away from the conformal fixed point on the Coulomb branch. However, unlike in the 3d case, here there are more BPS states than those captured by the perturbative content of the theory. In fact, the five-dimensional theory will have an infinite number of BPS states, including ones which carry
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instanton charge. Our proposal is that the full partition function on squashed $S^5$ is simply given by the contribution over all BPS states and not just the electric ones. In other words, we propose:

$$Z_{S^5} = \int d\phi \ Z_0(\phi) \prod_{\alpha \in BPS} Z_\alpha(z_\alpha|\omega_1, \omega_2, \omega_3),$$

(7.3.10)

where each $Z_\alpha$ is a contribution from a BPS particle written in terms of triple sine function (and its generalization), and $Z_0(\phi) = e^{C(\phi,m)}$ is the effective semi-classical contribution and is a polynomial of degree 3 in $\phi$ and $m$. By $Z_\alpha$ we mean the determinant contributions coming from the individual BPS states with the exponential prefactor stripped off (see the next section for more details). This proposal fits naturally with the computation in [155–157] where the main missing ingredient was the contribution of instantons to the partition function. Here we are proposing that the BPS content of the theory, which includes instanton charged states, completes the computation.

In the case where the superconformal theory comes from a Calabi-Yau threefold, $C$ can be related to the classical properties of the CY and captures the classical prepotential term, as well as genus 1 corrections which are linear in $\phi$ and $m$. In the unrefined case $C$ is simply given by

$$C(\phi, m) = \frac{1}{6\lambda^2} \int_{CY} J \wedge J \wedge J + \frac{1}{24} (\frac{1}{\lambda^2} - 1) \int_{CY} J \wedge c_2,$$

where $J(\phi, m)$ denotes the Kahler form on the CY which is parameterized by $\phi, m$ and $c_2$ is the second Chern class of the CY where the genus 0 piece can be read off from [170,171] and the genus 1 piece from [172]. In the refined case where $\tau_1 + \tau_2 \neq 0$ this becomes

$$C(\phi, m) = \frac{1}{6\tau_1 \tau_2} \int_{CY} J \wedge J \wedge J - \frac{1}{24} (\frac{\tau_1}{\tau_2} + \frac{\tau_2}{\tau_1} + \frac{1}{\tau_1 \tau_2} + 3) \int_{CY} J \wedge c_2.$$

We will choose normalizations where the Kahler class is given by $2\pi iT$. In this normalization we can write this as

$$C(T) = -2\pi i \left( \frac{C T^3}{6\tau_1 \tau_2} - \frac{c_2}{24} \cdot \frac{T}{\tau_2} + \frac{\tau_2}{\tau_1} + \frac{1}{\tau_1 \tau_2} + 3 \right)$$

We have used the unrefined case together with $SL(3, \mathbb{Z})$ invariance of the classical prepotential, up to sign, to predict this structure. One should be able to derive this directly from the definition of the refined topological string [81].
where

\[ CT^3 = C_{ijk} T^i T^j T^k, \quad c_2 \cdot T = c_2^i T^i, \]

and \( C_{ijk} \) denotes the triple intersection and \( c_2^i \) the second Chern class in this basis. In the next section we show how topological strings capture this partition function elegantly, leading on the one hand to the full partition function for \( \mathcal{N} = (1, 0) \) theories obtained by compactification of M-theory on toric CY threefold, and on the other hand to a non-perturbative definition of topological string.

### 7.4 Non-perturbative topological strings and the partition function on \( S^5 \)

Consider M-theory on Calabi-Yau threefolds. It is known that topological strings capture the BPS content of M2 branes wrapped over 2-cycles of the Calabi-Yau [65,66]. Furthermore, in the case of toric threefolds (which lead to \( \mathcal{N} = (1, 0) \) theories of interest to us here) we can consider a refinement of the BPS counting [76]. The relation between the topological string partition function and BPS state counting is given by

\[
Z^{\text{top}} = \prod_{s_1,s_2,k_i,l_j} \prod_{m,n=0}^{\infty} (1 - q^{m+s_1+\frac{1}{2}t^n-s_2+\frac{1}{2}e^{2\pi i (t_{ki}+m_j l_j)})^{(-1)^{2s_1} N_{s_1,s_2,k_i,l_j}}}
\]

Note that we have stripped off the classical terms, and below when we restore the classical pieces we will make it clear. Here \( q = \exp(2\pi i \tau_1), t = \exp(-2\pi i \tau_2) \) are the coupling constants of the refined topological string, the \( N_{s_1,s_2,k_i,l_j} \) are the BPS degeneracies, where \( (k_i,l_j) \) denotes the gauge and flavor charges of the BPS states and is an element of \( H_2 \) of the CY where the M2 brane wraps to give rise to BPS state. Here \( k_i \) corresponds to charges of normalizable Kahler classes \( t_i \), and flavor charge \( l_j \) corresponds to non-normalizable Kahler classes \( m_j \). The \( s_i \) give the \((s_1,s_2)=(J_{12},J_{34})\) content of the \( SO(4) \) rotation group in 5 dimensions. Namely, viewing \( SO(4) = SU(2)_L \times SU(2)_R \) each BPS state is given by

\[ I_L \otimes (j_i,j_r) \]
where

\[ I_L = [(\frac{1}{2}, 0) + 2(0, 0)] \]

and the \( s_i \) just capture the spin content (not including the \( I_L \) factor):

\[-ji \leq \frac{s_1 - s_2}{2} \leq ji, \quad -jr \leq \frac{s_1 + s_2}{2} \leq jr\]

It will be useful for us to slightly change the definition of topological strings (as in the open sector discussed in the 3d context) by shifting\(^4\) one of the couplings by 1:

\[
\tilde Z^{\text{top}} = \prod_{s_1, s_2, k, l} \prod_{m, n=0}^{\infty} (1 - (-1)^{2s_1 + 1} q^{m+s_1+\frac{1}{2}} t^{n-s_2+\frac{1}{2}} e^{2\pi i (t_k + m, l_j)}) (-1)^{2s_1} N_{s_1, s_2, k, l_j}
\]

\[
= Z^{\text{top}}(t, m; \tau_1 + 1, \tau_2).
\]

Since we will be mainly dealing with this object we will be calling it \( Z^{\text{top}} \) and drop the tilde. Of course one can recover the usual definition of topological string by shifting back the coupling by 1.

In order to connect this to the partition function on \( S^5 \) we need to know how each field contributes to the partition function. Consider a field with spins \((s_1, s_2)\) (coming as part of a BPS multiplet). Then we already know that when \((s_1, s_2) = 0\) the contribution is given by a shifted triple sine function:

\[
S_3^{-1}\left( z + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cdot (\omega_1, \omega_2, \omega_3) \right) |_{\omega_1, \omega_2, \omega_3}.
\]

Moreover for a vector multiplet \((0, 1/2)\) which has \((s_1, s_2) = (\pm \frac{1}{2}, \pm \frac{1}{2})\) we get

\[
S_3\left( z + [(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pm (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})] \cdot (\omega_1, \omega_2, \omega_3) \right) |_{\omega_1, \omega_2, \omega_3}.
\]

Now comes the main point. The connection to non-perturbative topological strings come

\(^4\)We can shift either \( \tau_1 \) or \( \tau_2 \) since \( 2s_1 = 2s_2 \mod 1 \). Note that this shift is equivalent to insertion of \((-1)^P\) and will be explained in Section 5.
to life thanks to a remarkable formula (equation (C.1.12)) for the triple sine function:

\[
\exp\left(-\frac{\pi i}{6} B_{3,3}(z + \Delta|\omega_1, \omega_2, \omega_3)\right) S^3_3(z + \Delta|\omega_1, \omega_2, \omega_3)
\]

\[
= \prod_{j,k=0}^\infty \frac{(1 + e^{2\pi i T + 2\pi i (j+1/2)\tau_1 - 2\pi i (k+1/2)\tau_2})}{(1 + e^{2\pi i T + 2\pi i (j+1/2)\tilde{\tau}_1 - 2\pi i (k+1/2)\tilde{\tau}_2})}.
\]

where we have shifted the argument of the triple sine by the universal term \(\Delta = (\omega_1 + \omega_2 + \omega_3)/2\), and we set \(T = z/\omega_3, \tau_1 = \omega_1/\omega_3, \tau_2 = \omega_2/\omega_3\), and also

\[
(\hat{T}, \hat{\tau}_1, \hat{\tau}_2) = (T/\tau_1, -1/\tau_1, \tau_2/\tau_1),
\]

\[
(\tilde{T}, \tilde{\tau}_1, \tilde{\tau}_2) = (T/\tau_2, \tau_1/\tau_2, -1/\tau_2).
\]

Furthermore, \(q = \exp(2\pi i \tau_1)\) and \(t = \exp(-2\pi i \tau_2)\) and similarly for the other variables. Each infinite product in this expression is convergent when \(\text{Im } \tau_1 > 0 > \text{Im } \tau_2\), but similar convergent expressions can be obtained in other regions (see Appendix C.1). The expression for the triple sine function also includes an exponential prefactor which comes from the \((3,3)\) multiple Bernoulli polynomial (C.1.9) with shifted argument,

\[
-\frac{\pi i}{6} B_{3,3}(z + (\omega_1 + \omega_2 + \omega_3)/2|\omega_1, \omega_2, \omega_3) = \frac{1}{\omega_1 \omega_2 \omega_3} \left[-\frac{\pi i}{6} z^3 + \frac{\pi i}{24} (\omega_1^2 + \omega_2^2 + \omega_3^2) z\right]
\]

\[
= -i \frac{T^3}{6 \tau_1 \tau_2} \left[\frac{1}{24} (1 + \tau_1^2 + \tau_2^2)\right].
\]

Taking \(z = z_0 = k_l t_i + l_j m_j\) for the hypermultiplets and and \(z = z_0 \pm (\omega_1 + \omega_2 + \omega_3)/2\) for the vector multiplets and choosing the gauge \(\omega_3 = 1\), one finds that the numerator in (7.4.11) gives precisely the contributions of the hyper and vector multiplets to the topological string partition function!

Similarly when \(s_1 = s_2 = s\) and \(z = z_0 + s(\omega_1 + \omega_2 + \omega_3)\) the right hand side of (7.4.11) becomes

\[
\prod_{j,k=0}^\infty \frac{(1 - (-1)^{2s+1} e^{2\pi i z_0} q^j s + 1/2 k - s + 1/2})}{(1 - (-1)^{2s+1} e^{2\pi i z_0/\tau_1} q^j s - 1/2 k - s + 1/2)} \cdot (1 - (-1)^{2s+1} e^{2\pi i z_0/\tau_2} \tilde{q}^j s + 1/2 k + s + 1/2).
\]

The numerator in this expression also captures the contribution to the topological string partition function of a BPS states with spin \((s, s)\). It is thus natural to propose that the triple sine function also gives the determinant for spin \((s, s)\) states.

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This triple product structure involving topological string contributions has a simple generalization for arbitrary spin \((s_1, s_2)\):

\[
C_{s_1, s_2}(z_0|\tau_1, \tau_2)^{-1} = \prod_{j,k=0}^{\infty} (1 - (-1)^{2s_1+1}e^{2\pi i z_0/\tau_1} q^{j-s_1+1/2} t^{k-s_2+1/2}) \times \prod_{j,k=0}^{\infty} (1 - (-1)^{2s_1+1}e^{2\pi i z_0 q^j+s_1+1/2} t^{k-s_2+1/2}) \times \prod_{j,k=0}^{\infty} (1 - (-1)^{2s_1+1}e^{2\pi i z_0/\tau_2} q^{j-s_1+1/2} t^{k+s_2+1/2}),
\]

(7.4.12)

which we propose to be giving the determinant contribution for spin \((s_1, s_2)\) states. Note that for \(s_1 \neq s_2\) this differs from the triple sine function. Taking the product over all the BPS states, which we need to do according to our proposal for the computation of the partition function over \(S^5\), we obtain

\[
Z(t_i, m_j; \tau_1, \tau_2) = Z_0 \cdot \prod_{s_1, s_2, k_i, l_j} C_{s_1, s_2}(z_0|\tau_1, \tau_2)^{(-1)^{2s_1+1}N_{s_1, s_2, k_i, l_j}},
\]

where in the above, in addition to the product over the BPS states, we have included the cubic prefactor \(Z_0 = \exp(C(t_i, m_j; \tau_1, \tau_2))\). We can rewrite this expression as follows:

\[
Z(t_i, m_j; \tau_1, \tau_2) = Z_0 \cdot \frac{Z_3(t_i, m_j; \tau_1, \tau_2)}{Z_1(t_i, m_j; \tau_1, \tau_2)} \cdot Z_2(t_i, m_j; \tau_1, \tau_2).
\]

(7.4.13)

The numerator is precisely the topological string partition function,

\[
Z_3(t_i, m_j; \tau_1, \tau_2) = Z^{\text{top}}(t_i, m_j; \tau_1, \tau_2),
\]

and we can also relate the two factors in the denominator to the topological string partition function:

\[
Z_1(t_i, m_j; \tau_1, \tau_2) = \prod_{s_1, s_2, k_i, l_j} \prod_{j,k=0}^{\infty} (1 - (-1)^{2s_1+1} e^{2\pi i z_0/\tau_1} q^{j-s_1+1/2} t^{k-s_2+1/2}) (-1)^{2s_1} N_{s_1, s_2, k_i, l_j}
\]

\[
= Z^{\text{top}}(t_i/\tau_1, m_j/\tau_2; -1/\tau_1, 1/\tau_2)
\]

and

\[
Z_2(t_i, m_j; \tau_1, \tau_2) = \prod_{s_1, s_2, k_i, l_j} \prod_{j,k=0}^{\infty} (1 - (-1)^{2s_1+1} e^{2\pi i z_0/\tau_2} q^{j+s_1+1/2} t^{k+s_2+1/2}) (-1)^{2s_1} N_{s_1, s_2, k_i, l_j}
\]

\[
= Z^{\text{top}}(t_i/\tau_2, m_j/\tau_1; -1/\tau_2, 1/\tau_1).
\]
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The prime signifies that these two factors of the topological string have $SU(2)_L$ and $SU(2)_R$ exchanged, which is equivalent to replacing $(s_1, s_2)$ with $(-s_1, s_2)$ (or equivalently $(s_1, -s_2)$) for each BPS state. In fact, not worrying about regions of convergence, we can use the identity

$$
\prod_{p=0}^{\infty} (1 - X e^{2\pi i p \gamma}) = \prod_{p=0}^{\infty} (1 - X e^{-2\pi i (p+1) \gamma})^{-1}
$$

to rewrite the product of BPS contributions simply as the product of three factors of the topological string partition function:

$$
Z_{\text{top}}(t_i; \tau_1, \tau_2) \cdot Z_{\text{top}}(t_i/\tau_1; 1/\tau_1, \tau_2/\tau_1) \cdot Z_{\text{top}}(t_i/\tau_2; \tau_1/\tau_2, 1/\tau_2).
$$

Equation (7.4.13) can be viewed as defining a non-perturbative completion of topological string, in the sense that the two additional factors are non-perturbative, as they involve at least one $\tau_i \to -1/\tau_i$. At the end of this section we will explain the analytic properties of $Z$ as a function of $\tau_i$. Just to complete our discussion, in order to compute the $S^5$ partition function we simply have to integrate this over the directions in $t_i$:

$$
Z_{S^5} = \int_{t_i} dt_i Z(t_i, m_j; \tau_1, \tau_2).
$$

7.4.1 Contribution of the massless vector multiplet

The massless vector multiplets also make a contribution to the partition function. These contributions do not depend on the moduli but depend on the squashing parameters. Therefore they can be brought out of the integrals over the Coulomb branch. These terms are given in the topological string context by powers of the MacMahon function. If we have $U(1)^r$ gauge theory this leads, as discussed in [69], to

$$(M(q, t)M(t, q))^{r/2}$$

where

$$
M(t, q) = \prod_{i,j=1}^{\infty} (1 - q^i t^j)^{-1}
$$
Figure 7.1: $\mathbb{P}^1 \times \mathbb{P}^1$ geometry corresponding to SU(2) theory on the squashed five-sphere. The non-perturbative topological string computed from this geometry is to be integrated over $a$.

If we use our prescription to compute the contribution of this factor to the partition function we get a factor of

$$(S_3(1 + \tau_1 + \tau_2|1, \tau_1, \tau_2) \cdot S_3(0|1, \tau_1, \tau_2))^{r/2} = S_3(0|1, \tau_1, \tau_2)^r.$$ 

This has a zero for each $U(1)$ reflecting the fact that we have to delete the zero mode associated to the Coulomb branch parameters and instead integrate over it, which is part of the prescription. This is equivalent to replacing $S_3$ with its derivative $S_3'$ evaluated at 0. In other words, the contributions for the massless vector multiplet to the partition function is

$$S_3'(0|1, \tau_1, \tau_2)^r \prod_{i=1}^r dT_i.$$

### 7.4.2 An example: SU(2) gauge theory

Here we present one example of how the computation is done. The case we focus on is a toric 3-fold that engineers $SU(2)$ gauge theory coming from the $O(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ geometry.

We consider the partition function of this theory on the squashed five-sphere. As discussed, we
predict the full partition function to be

\[
Z_{SU(2)}(Q_b, Q_f, \tau_1, \tau_2) := [M(q, t)M(t, q)]^{1/2}
\]

where

\[
Q_f = e^a, \quad Q_b = e^{a+1/g^2_{YM}}, \quad \text{and} \quad Z_{SU(2)}(Q_b, Q_f, \tau_1, \tau_2)
\]

is the refined topological string partition function for the \( \mathbb{P}^1 \times \mathbb{P}^1 \) geometry of Figure 7.1, which was obtained in [67] (which is the same as Nekrasov's partition function for the 5d \( SU(2) \) theory [59] with \( \epsilon_i = \tau_i \):

\[
Z_{SU(2)}(Q_b, Q_f, \tau_1, \tau_2) := [M(q, t)M(t, q)]^{1/2}
\]

\[
\cdot \sum_{\nu_1, \nu_2} (-Q_b)^{|\nu_1|+|\nu_2|} Z_{\nu_1, \nu_2}(t, q, Q_f) f_{\nu_1, \nu_2}(q, t) Z_{\nu_2, \nu_1}(q, t, Q_f),
\]

where

\[
q = \exp(2\pi i \tau_1), \quad t = \exp(-2\pi i \tau_2),
\]

\[
f_{\nu_1, \nu_2}(q, t) = (-1)^{|\nu_1|} \left( \frac{t}{q} \right)^{||\nu_1||^2 - |\nu_1|} q^{-n(\nu_1)/2} (-1)^{|\nu_2|} \left( \frac{q}{t} \right)^{||\nu_2||^2 - |\nu_2|} t^{-n(\nu_2)/2},
\]

and

\[
Z_{\nu_1, \nu_2}(t, q, Q_f) = q^{||\nu_1||^2 + ||\nu_2||^2 - |\nu_1|} \tilde{Z}_{\nu_1}(t, q) \tilde{Z}_{\nu_2}(t, q) \prod_{i,j} \left( 1 - Q_f t^{i-1-\nu_{2,j}} q^{j-\nu_{1,i}} \right)^{-1},
\]

where

\[
\tilde{Z}_{\nu}(t, q) = \prod_{s \in \nu} (1 - t^a(s)+1 q^\ell(s))^{-1}
\]

and

\[
M(t, q) = \prod_{i,j=1}^{\infty} (1 - q^{i-1} t^{j-1})^{-1}.
\]

The classical piece \( C(a, \frac{1}{g^2_{YM}}) \) is given by

\[
C(a, \frac{1}{g^2_{YM}}) = -\frac{2\pi i}{\tau_1 \tau_2} \left( \frac{a^2}{2g^2_{YM}} + \frac{a^3}{6} \right) + \frac{2\pi i}{24} \left( -2a + \frac{4}{g^2_{YM}} \right) \left( \frac{\tau_1}{\tau_2} + \frac{\tau_2}{\tau_1} + \frac{1}{\tau_1 \tau_2} + 3 \right).
\]

The partition function involves sums over Young diagrams. We use the following notation: \( \nu^t \) is the transpose of \( \nu \); \( |\nu| \) denotes the number of boxes in \( \nu \); \( \nu_i \) is the number of boxes in the \( i \)-th column of \( \nu \); \( ||\nu||^2 = \sum_i \nu_i^2 \); for a box \( s = (i, j) \) in the \( i \)-th column and \( l \)-th row of \( \nu \), \( a(s) = \nu_j^t - i \) and \( \ell(s) = \nu_i - j \); and, lastly, \( \kappa(\nu) = 2 \sum_{s \in \nu} (j - i) \). Recall that we need to shift \( \tau_1 \rightarrow \tau_1 + 1 \) in these formulas to obtain the \( Z_{SU(2)} \) appearing in the integrand.
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7.4.3 Analytic properties of $Z$

The triple sine function (as discussed in Appendix C.1) is defined only when all three $\omega_i$ are in the same half plane. If this is satisfied, the triple sine function is well defined and is an entire function which has zeroes at a lattice of points corresponding to $n_i t_i + k_j m_j = (n_1 + \frac{1}{2}) \tau_1 + (n_2 + \frac{1}{2}) \tau_2 + (n_3 + \frac{1}{2})$ (see Appendix C.1). Similarly the function $C_{s_1,s_2}(n_i t_i + k_j m_j | \tau_1, \tau_2)$ has zeros and poles at values of $n_i t_i + k_j m_j$ which can be read off from equation (C.6.24). It is natural to also expect that $C_{s_1,s_2}$ is well-defined only when all three $\omega_i$ are in the same half plane.

The non-perturbative topological string partition function is made up of an infinite product of such functions which we conjecture to exist.

7.5 A possible derivation from M-theory

In this section we propose an explanation for the triple product structure that arises when one introduces squashing parameters for $S^5$. We start by recalling in more detail the M-theory setup that computes the topological string partition function. We pick a non-compact toric Calabi-Yau threefold $X$, and take the remaining five-dimensional space to be the Taub-NUT space $TN$ times the M-theory circle $S^1$. We express Taub-NUT space in terms of complex variables $(z_1, z_2)$ and introduce a twist: as we go around $S^1$, we rotate $(z_1, z_2) \rightarrow (e^{2\pi i \tau_1} z_1, e^{2\pi i \tau_2} z_2)$ (and do a compensating twist on $X$ to keep it supersymmetric). We denote this twisted space by $(TN \times S^1)_{\tau_1, \tau_2}$. Then it is known that [81]

$$Z_{\text{top}}(X, \tau_1, \tau_2) = Z_{\text{M-theory}}(X \times TN \times S^1)_{\tau_1, \tau_2}.$$ 

The M-theory partition function counts the number of M2-branes wrapping cycles in $X$, which project to points in Taub-NUT space. When the equivariant parameters are turned on, the particles are concentrated around the origin $z_1 = z_2 = 0$.

We can also consider the open string sector of topological strings, which corresponds to adding $M5$ branes wrapping a Lagrangian submanifold $L \subset X$ and the Melvin cigar ($MC$) subspace.
of \((TN \times S^1)_{\tau_1, \tau_2}\), which has the geometry of \(S^1 \times \mathbb{C}_{\tau_1}\). Here \(S^1\) is the M-theory circle, and \(\mathbb{C}_{\tau_1}\) is the plane in \(TN\) with rotation parameter \(\tau_1\) (but we could as well have chosen our M5-branes to fill \(\mathbb{C}_{\tau_2}\)). In topological string theory, wrapping an M5 brane on \(K = MC \times L\) translates to placing a \(\tau_1\)-brane on \(L\) [169] (see [162, 173] for a discussion of the refined case). The problem of counting worldsheet instantons ending on \(L\) translates to counting the states of a gas of M2-branes which wrap two-cycles of \(X\) with boundary on \(L\); the M2 branes project to points on the Melvin cigar. Turning on equivariant parameters again forces these particles to be concentrated at the tip of the cigar, which is located at \(z_1 = z_2 = 0\). Then the M5 brane partition function in this setup is the same as the open topological string partition function:

\[
Z_{M5}(X \times (TN \times S^1)_{\tau_1, \tau_2}, K) = Z^{\text{open}}(\vec{t}, \vec{x}, \tau_2; \tau_1),
\]

where \(\vec{t}\) and \(\vec{x}\) denote, respectively, the closed and open string moduli corresponding to \(X\) and \(L\).

In other words, the open topological string theory computes the partition function of the 3d theory obtained by wrapping an M5 brane on \(L\) in the background of \(MC\). Note that for fixed \(|z| \neq 0\) on \(C\), the \(MC\) has a torus structure, where one circle corresponds to the phase in the \(z\)-plane and the other is the circle in the fiber. Moreover the twisting of the \(MC\) as we go around the \(S^1\) suggests that changing \(\tau\) changes the complex structure of this torus and it is natural to view this torus as having complex structure \(\tau\).

To obtain the partition function of the resulting theory on squashed \(S^3\) we take a second copy of the Melvin cigar, which we denote by \(\widehat{MC}\), and glue it to the first one along the common boundary (as was suggested in the topological string context in [162, 163] and discussed in detail in [164]). This operation can be visualized most clearly by regarding the squashed \(S^3\) as a torus fibration over the interval, as in Figure 7.2, and the \(T^2\) is the one we have discussed away from the tips of \(MC\) and \(\widehat{MC}\). Each Melvin cigar fills out a solid torus, and we glue the two after performing an \(S\) modular transformation which interchanges the two circles in \(\widehat{MC}\). The only subtlety is that we need to ensure that the two cigars are twisted in a compatible way. In particular the complex structure parameter as seen from the viewpoint of one tip is different from that of the other end.
Figure 7.2: Squashed $S^3$ viewed as a torus fibered over the interval. At the ends of the interval one of the two circles degenerates. On the left half of the geometry, as one goes around the red (dashed) circle, the second circle is twisted by $2\pi i \tau$. In gluing the left and right halves, one must interchange the two circles of $T^2$. On the right half, in going around the blue circle the red circle gets twisted by $-2\pi i / \tau$.

This forces us to rescale the rotation parameter for $MC$ to

$$\hat{\tau}_1 = -1/\tau_1.$$ 

Moreover the topological string has opposite orientation on the $MC$ suggesting complex conjugation of the topological string amplitude, which is equivalent to inversion of $Z$. The partition on $S^3_b$ then is just the product of the topological string factors on the two hemispheres$^5$, 

$$Z_{S^3} = \frac{Z_{\text{open}}^{\text{top}}(\vec{t}, \vec{x}, \tau_1)}{Z_{\text{top}}^{\text{open}}(\vec{t}/\tau_1, \vec{x}/\tau_1, -1/\tau_1)}.$$ 

The main lesson we extract from the open string case is that for generic choices of the rotation parameters the topological string (or, equivalently, M-theory) computation localizes at the fixed points of the equivariant action on $\mathbb{C}^2$. In discussing aspects of closed strings we will have to recall that when we have a more complicated geometry made of patches which look like $\mathbb{C}^2_{\tau_a, \tau_b} \times S^1$, we

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$^5$Here we are ignoring the $\tau_2$ dependence which we discuss later in the context of closed strings.
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Figure 7.3: Squashed $S^5$ as a $T^3$ fibration over a triangle: the cube, whose opposite faces are identified, represents the torus. At the edges of the triangles the torus collapses to a $T^2$; at the vertices it collapses to $S^1$. At each vertex we also display the correctly normalized equivariant parameters corresponding to the three circles.

would expect by localization to get a contribution of $Z_{top}$ from each patch. The main new ingredient is to find the identification of $\tau_1, \tau_2$ between the patches.

With this picture in mind, we wish to study the partition function on $S^5$. We view $S^5$ as a circle fibration over $\mathbb{CP}^2$. Moreover $\mathbb{CP}^2$ itself can be viewed as consisting of a $T^2$ over a triangle, as is familiar in the context of toric geometries (see e.g. [35]). Thus we can think about the squashed five-sphere as a $S^1 \times S^1 \times S^1 = T^3$ fibration over a triangle, where each circle in the fiber gets rotated by a different parameter $\tau_i$ (see Figure 7.3). In the interior of the triangle all three circles have finite size, but along the edges one of them shrinks to zero size, and the vertices are the points where two of the circles degenerate. We find it convenient to label by $v_i$ the vertex where the $i$-th circle of the fiber does not degenerate. We also denote by $e_{ij}$ the edge that connects $v_i$ and $v_j$. It is easy to convince oneself that the neighborhood of $v_i$ looks like $S^1_i \times \mathbb{C}_j \times \mathbb{C}_k$, where
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$i \neq j \neq k$ and each circle in the fiber corresponds to a different factor in the geometry. So from each vertex we expect a contribution of $Z_{\text{top}}^{\text{closed}}$. To figure out the appropriate parametrization at each vertex, one can start by setting the equivariant parameters to be $(\tau_1, \tau_2, 1)$ at $v_3$, so that we get a factor of $Z_{\text{top}}^{\text{closed}}(i, \tau_1, \tau_2)$. We can reach the two other vertices by moving along the edges $e_{31}$ and $e_{32}$. At $v_1$ the role of the M-theory circle is played by the first circle, so for the gluing along the edge to be consistent we are required to rescale the equivariant parameters by $1/\tau_1$. This gives us a factor of $Z_{\text{top}}^{\text{closed}}(\tilde{i}/\tau_1, 1/\tau_1, \tau_2/\tau_1)$. Similarly we learn that $v_2$ contributes a factor of $Z_{\text{top}}^{\text{closed}}(\tilde{i}/\tau_2, \tau_1/\tau_2, 1/\tau_2)$. Collecting the contributions from the three vertices, we find that M-theory on squashed $S^5$ computes

$$Z_{\text{top}}^{\Delta} = Z_{\text{top}}^{\text{closed}}(i, \tau_1, \tau_2) \cdot Z_{\text{top}}^{\text{closed}}(\tilde{i}/\tau_1, 1/\tau_1, \tau_2/\tau_1) \cdot Z_{\text{top}}^{\text{closed}}(\tilde{i}/\tau_2, \tau_1/\tau_2, 1/\tau_2).$$

As explained in Section 7.4, we can rewrite this expression in convergent form as

$$Z_{\text{top}}^{\Delta} = \frac{Z_{\text{top}}^{\text{closed}}(i, \tau_1, \tau_2)}{Z_{\text{top}}^{\text{closed}}(\tilde{i}/\tau_1, -1/\tau_1, \tau_2/\tau_1) \cdot Z_{\text{top}}^{\text{closed}}(\tilde{i}/\tau_2, \tau_1/\tau_2, -1/\tau_2)},$$

where the factors in the denominator are to be computed with the $SU(2)_L$ and $SU(2)_R$ spins exchanged.

The non-perturbative open topological string fits very nicely in this picture: the fiber over an edge $e_{ij}$ consists of two non-degenerate circles $S^1_i$ and $S^1_j$, which play inverted roles at the two vertices. This means that over each edge we have a squashed $S^3$, so we can get an open sector by wrapping an M5-brane around it and around a Lagrangian submanifold in $X$. If we do this for the $e_{13}$ edge we get a contribution of

$$Z_{\text{open}}^{e_{13}} = Z_{\text{open}}(i, \tilde{x}, \tau_1, \tau_2)/Z_{\text{open}}(\tilde{i}/\tau_1, \tilde{x}/\tau_1, -1/\tau_1, \tau_2/\tau_1).$$

If we were to choose the $e_{23}$ edge, we would obtain

$$Z_{\text{open}}^{e_{23}} = Z_{\text{open}}(i, \tilde{x}, \tau_1, \tau_2)/Z_{\text{open}}(\tilde{i}/\tau_2, \tilde{x}/\tau_2, \tau_1/\tau_2, -1/\tau_2).$$

$^6$Up to the factor of $(-1)^F$ because the corresponding $S^1$ in this case is shrinkable inside $S^5$ and gives a different spin structure compared to the usual case where $S^1$ is not contractible. This explains the origin of the shift $\tau_1 \to \tau_1 + 1$ in the previous sections.
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To make this into a rigorous derivation for arbitrary toric Calabi-Yau, we would need to have a way to compactify the full M-theory on $S^5$, which will necessarily involve some unconventional fields being turned on (similar to what was found in the 4d case [174]). It is natural to conjecture, given what we are finding, that such a setup should be consistent, at least in the case of non-compact Calabi-Yau’s. In the subset of cases where the CY engineers a gauge theory, where $Z^{\text{top}}$ is identified with the Nekrasov partition function, it should be possible to rigorously derive this result from the localization arguments in the path-integral.

7.6 Superconformal indices in six dimensions

It is natural to ask whether the techniques we have introduced can be used to compute superconformal indices in 6 dimensions. This is natural because this involves computations of the amplitudes on $S^5 \times S^1$. Moreover, compactification on $S^1$ leads to a 5 dimensional theory, of the type we have studied. Also, as in the lower dimensional case studied (such as $S^1 \times S^4$) turning on the fugacities and supersymmetric rotations of the $S^5$ should correspond to introducing squashing parameters for $S^5$.

In this section we show how this can be done. The generic case of interest is superconformal theories with $\mathcal{N} = (1, 0)$ supersymmetry. A special case of these are the $(2, 0)$ theories. We will discuss each one in turn.

7.6.1 $\mathcal{N} = (1, 0)$ superconformal index

Interacting superconformal theories with $\mathcal{N} = (1, 0)$ supersymmetry have $Sp(2)$ R-symmetry. Let $R$ denote its Cartan. The superconformal index in this case can be defined as follows [175]:

$$I_{1,0} = \text{Tr}(-1)^F q_{J_{12} - R} q_{J_{34} - R} q_{J_{56} - R} M_i^{F_i},$$

where $J_{ij}$ denote the rotation generators of $SO(6)$ acting on $S^5$, and $F_i$ are charges associated to flavor symmetries (where we have only kept the terms which appear non-trivially in the partition
function). The choice of the parameters $q_1, q_2$ is motivated from connection with the rotations in 4d, already discussed in the context of 5d theories.

The basic idea, similar to relating the 4d index to 3d partition functions [176–178], is to connect the 6d index to our 5d setup by compactifying this theory on $S^1$. The only subtlety is to identify the charges as well as the relation of the parameters in the lower dimensional theory with the higher dimensional theory. In the context of compactification of the 6d theory on a circle, we would need to enumerate the resulting 5d BPS states (including winding of 6d BPS strings around the $S^1$) and simply apply the formalism we have developed to this 5d theory. Here the 5d theory will have a tower of BPS states with a specific structure due to the fact that it is coming as a KK reduction from a one higher dimensional theory. If this theory is dual to M-theory on a CY then from the perspective of this 5d theory we can enumerate all BPS states using topological strings. Then using the three combinations of them and integrating over the scalars in the gauge multiplets yields the partition function on $S^5$, thus effectively computing the index of the 6d theory.

Note that from the perspective of the 5d BPS counting, the KK momentum should appear as a special flavor symmetry. In the context of F-theory on elliptic CY and its duality with M-theory upon compactification on $S^1$ (as we will review below), this will turn out to be the winding number over an elliptic fiber. We will denote the Kahler class of the elliptic fiber by $\tau$ and define $q = \exp(2\pi i \tau)$, where $\tau$ is the Kahler modulus of the elliptic fiber (the reason for this terminology will become clear later). Let $M_i = \exp(2\pi i m_i)$, where $m_i$ denote the non-dynamical fields (coming from non-normalizable Kahler moduli). The question is what is the relation between the 5d parameters $q, q_1, q_2, m_i$ with the parameters appearing in the 6d index $q, q_1, q_2, m_i$? A similar situation was studied in the relation between superconformal index in 4d and the partition function in 3d [176–178]. In that case the squashing parameter are rescaled by a factor of $R$, the radius of the circle. We propose a similar relation in this case. Using the fact that the Kahler class of the elliptic fiber in F-theory is related to $R$ by

$$2\pi i \tau = \frac{1}{R}$$
we are led to

\[(\tau, \tau_1, \tau_2, m_j)_{6d} = \left(\frac{-1}{\tau}, \frac{\tau_1}{\tau}, \frac{\tau_2}{\tau}, m_j / \tau\right)_{5d}\]

In computing the partition function on squashed \(S^5\) we need to integrate over the dynamical fields. Let \(t_i\) denote the scalars associated to the resulting gauge fields in 5d coming from 6d tensor multiplets, which are normalizable (corresponding to normalizable Kahler moduli of the CY). Then we obtain the formula

\[I_{(1,0)}(m_j / \tau; -1 / \tau, \tau_1 / \tau, \tau_2 / \tau) = \int dt_i \frac{Z_{\text{top}}(t_i, m_j; \tau, \tau_1, \tau_2)}{Z_{\text{top}}(\frac{t_i}{\tau_1}, \frac{m_j}{\tau}; \tau, \tau_1, \tau_2) \cdot Z_{\text{top}}(\frac{t_i}{\tau_2}, \frac{m_j}{\tau}; \tau, \tau_2, \tau_1)}.
\]

This naturally follows from our formalism. It is a general proposal regardless of whether or not we have a topological string realization of the theory: the \(Z_{\text{top}}\) factor simply denotes the BPS partition function. However the question is how to compute the BPS partition function. If we can relate it to an actual topological string then we have techniques for its computation; the most convenient one for this purpose is the F-theory construction, because of the duality between F-theory compactified on \(S^1\) and M-theory on the same space [20]. Thus in 5 dimensions we obtain the theory involving M-theory on an elliptic 3-fold. Luckily topological strings on elliptic 3-folds have very nice properties and have been studied extensively [28, 94, 96, 118, 179, 180]. The relation between 6d and 5d theories via F-theory/M-theory duality has also been studied in [181]. As an example, consider the superconformal theory associated with a small \(E_8\) instanton. In the F-theory setup, this corresponds to F-theory with vanishing \(\mathbb{P}^1\) in the base of F-theory [22, 117]. After compactification on \(S^1\), this gives an elliptic 3-fold containing \(\frac{1}{2} K3\) (obtained by the elliptic fibration over the \(\mathbb{P}^1\)). This theory has 10 Kahler classes: One elliptic fiber class \(\tau\), the base \(t_b\) and eight mass parameters \(m_i\) (to be identified with the Cartan of \(E_8\)). \(\tau\) corresponds to momentum and \(t_b\) corresponds to the winding of the 6d tensionless string along the circle [118]. The unrefined topological string for this theory was studied in [28, 94, 118, 179]. To obtain the index for this theory we have to integrate over the \(t_b\). Similarly a large class of \((1,0)\) theories can be obtained by
considering F-theory where the base contains more blow ups on $\mathbb{C}^2$ (see [182] for a recent discussion related to this). This would entail blowing up a multiple of times, each corresponding to a Kahler parameter $t_i$, which we will have to integrate over in computing the index (the corresponding $U(1)$ vector multiplet in 5d arises from the 6d tensor multiplet in the same multiplet as the blow up parameter $t_i$). A subset of such blowups are the toric ones. These are in one-to-one correspondence with 2d Young diagrams [183]. Elliptic threefolds over these spaces, in the limit of blowing down all the 2-cycles, should correspond to a $(1,0)$ conformal theory. The topological string partition functions for this class of theories seem to enjoy the following perturbative modular property under the inversion of the Kahler class of the elliptic fiber [28, 94, 96, 180]:

$$Z^{\text{top}}(t_i, m_j/\tau; -1/\tau, \tau_1/\tau, \tau_2/\tau) = Z^{\text{top}}(t_i, m_j; \tau, \tau_1, \tau_2).$$

Note the asymmetric role in the modular transformation for the dynamical fields $t_i$ versus the non-dynamical fields $m_j$ which correspond to flavor symmetries\footnote{To get this modular transformation, $t_i$ should be suitably defined by shifting the blow up parameters with a multiple of elliptic fiber [96].}. In the context of our non-perturbative completion, as we will see later in the context of the theory of M5 branes, this relation receives additional non-perturbative factors. This turns out to be rather important for simplifying the computation of the 6d index as we will discuss in Section 7.6.5.

More generally we can consider instead of $\mathbb{C}^2$ the $A_{n-1}$ orbifold as the base of F-theory. If we do not add any further blow ups, this gives the $A_{n-1}$, $(2,0)$ theory, which we discuss in the next section and whose self-dual strings were the subject of Chapters 2 and 3. If in addition we also blow up the points in the base we get among the various possibilities the small $E_8$ instantons in the $A_{n-1}$ geometry, as $(1,0)$ superconformal theories of the type studied in [184].

### 7.6.2 Superconformal index for $\mathcal{N} = (2,0)$ theories

The superconformal group for $\mathcal{N} = (2,0)$ theories has $Sp(4)$ R-symmetry. Let $R_1$ and $R_2$ denote the two Cartans of $Sp(4)$ in an orthogonal basis, where we view $R_2$ as the additional
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symmetry compared to the $(1,0)$ theory. Then the superconformal index can be viewed as an extension of the $I_{1,0}$ by introducing the additional flavor symmetry $R_2 - R_1$:

$$I_{(2,0)} = \text{Tr}(-1)^F q_1^{J_{12}-R_1} q_2^{J_{34}-R_1} q^{J_{56}-R_1} Q_m^{R_2-R_1}.$$

The same reasoning as in the case of $(1,0)$ superconformal theories leads to the following picture. The 5d theory we obtain by compactifying the $(2,0)$ theory is an ADE Yang-Mills theory with 16 supercharges. Turning on the fugacity $Q_m$ corresponds to turning on a mass $m$ for the adjoint field, where $Q_m = e^{2\pi i m}$ (for the identification of this with $R_2 - R_1$ generator of R-symmetry see [185]). In other words we can view the resulting theory as $\mathcal{N} = 2^*$ theory in 5d. Let $Z^{\text{top}}(t_i, \tau_1, \tau_2, m)$ capture the BPS partition function for this 5d theory where $t_i$ denotes the Cartan of ADE. Then to compute the index we have

$$I^{\text{ADE}}_{(2,0)}(-1/\tau, \tau_1/\tau, \tau_2/\tau, m/\tau) = \int dt_i Z^{\text{top}}(t_i, \tau_1, \tau_2, \tau, m),$$

where we have taken into account the relation between the 5d parameters and 6d parameters.

In order to gain insight into the mechanics of this computation we show how it works for the simplest case, namely a single M5 brane, which corresponds to $A_0$ theory and recover the result of [175]. This lends support to our general proposal and more specifically to the identification of the squashing parameters and Kahler classes with the parameters appearing in the 6d superconformal index. The case of $A_0$ theory is particularly simple because we have no integrals to perform. In that case the non-perturbative $Z$ we obtain is exactly the same as the perturbative one! This ends up being related to the modularity of the topological string partition function on elliptic threefolds. Moreover we discuss the possibility that this may be the general story for all $(1,0)$ and $(2,0)$ theories in Section 7.6.5.

### 7.6.3 Index for a single M5 brane

As discussed above the case for single M5 brane corresponds to studying topological strings for $\mathcal{N} = 2^* U(1)$ theory in 5 dimensions. This corresponds to a periodic toric geometry, where we
compactify the base of the toric plane along one direction, obtaining a cylinder. The corresponding toric diagram for this theory was introduced in [76] and extends the 4d construction of these theories in [78] to 5d. The case of $U(1)$ is shown in Figure 7.4. The class corresponding to the circle identification of the toric base is $q$ (corresponding to the elliptic fiber). The class corresponding to the mass parameter $m$, which we denoted by $Q_m$ is also shown in the figure. The refined topological vertex formalism applied to this case involves introducing the two vertices and summing over the two internal line edges with arbitrary representations, where the smaller edge is weighted by $Q_m^n$ where $n$ is the number of boxes in the Young diagram of the representation on that edge. Similarly the longer edge is weighted with $(qQ_m^{-1})^k$ where $k$ is the number of boxes in the Young diagram of the representation on that edge. The topological string partition function for this theory was

Figure 7.4: Toric diagram for the geometry that engineers the $\mathcal{N} = 2^*$ $U(1)$ theory in five dimensions. The toric plane is compactified to a cylinder, and the horizontal edges are identified with each other.
worked out in [186] (see also [187]) and the result is given by\(^8\) (after shifting \(\tau_i \to \tau_i + 1\)):

\[
Z_{U(1)} = \prod_{k=0}^{\infty} \left( \prod_{i,j=0}^{\infty} \frac{(1 + Q_m^{-1} q^{k+1/2} q_1^{i+1/2} q_2^{-j+1/2}) (1 + Q_m q^{k+1/2} q_1^{i+1/2} q_2^j)}{(1 - q^{k+1} q_1^i q_2^{-j}) (1 - q^{k+1} q_1^i q_2^j)} \right)^k,
\]

where \(Q_m = e^{2\pi i m}, q = e^{2\pi i \tau}, q_1 = e^{2\pi i \tau_1}, \) and \(q_2 = e^{2\pi i \tau_2},\) and we have included one factor of the MacMahon function which is somewhat ambiguous in the computation of the refined topological string. The refined topological string captures the Kahler moduli dependence of the amplitudes and does not fix the terms purely depending only on \(q_1, q_2.\) In fact we will need to multiply the above expression by \(1/\eta(q_1)\) for reasons that we will explain below, where \(\eta(q_1)\) is the Dedekind eta-function.

The spectrum of this theory consists of a tower of hyper multiplets of mass \(2\pi i (m + k\tau)\) (one for each integer \(k\)) and a tower of tensor multiplets with mass \(2\pi i k\tau.\) This is as expected, because the reduction of a single M5 brane on a circle leads exactly to such a multiplet, where \(2\pi i \tau\) is identified with \(1/R,\) with \(R\) the radius of the circle taking us from 6 to 5 dimensions. It is important to rewrite the above partition function in a more symmetric way: Let us redefine \(Q_m\) by

\[
Q_m \to Q_m q^{1/2}
\]

Then the partition function is totally symmetric in \((q, q_1, q_2),\) if we in addition include a factor of \(1/\eta(q_1)\) which is ambiguous for the refined topological vertex. To see this, we have to rewrite everything in terms of positive powers of \(q_2:\)

\[
Z_{U(1)} = \frac{1}{\eta(q_1)} \prod_{i,j,k=0}^{\infty} \left( \frac{(1 + Q_m^{-1} q^{k+1/2} q_1^{i+1/2} q_2^{-j+1/2}) (1 + Q_m q^{k+1/2} q_1^{i+1/2} q_2^j)}{(1 - q^{k+1} q_1^i q_2^{-j}) (1 - q^{k+1} q_1^i q_2^j)} \right)^k
\]

\[
= \frac{1}{\eta(q_1) \eta(q) \eta(q_2)} \prod_{i,j,k=0}^{\infty} \left( \frac{(1 - q^{k+1} q_1^i q_2^{-j}) (1 - q^{k+1} q_1^i q_2^j)}{(1 + Q_m^{-1} q^{k+1/2} q_1^{i+1/2} q_2^{-j+1/2}) (1 + Q_m q^{k+1/2} q_1^{i+1/2} q_2^j)} \right)^k
\]

where we delete the \(i = j = k = 0\) terms for the second term in the numerator. The manifest permutation symmetry between \(q, q_1, q_2\) is expected from the fact that in the 6d they become the

\(^8\)We thank A. Iqbal for a very helpful explanation of this result and its modular properties.
parameters associated to the three rotation planes. Note also that the way we have rewritten the numerator corresponds to the fact that a tensor multiplet in 5d is dual to the vector multiplet. This accounts for the form of the numerator which now gives a tower of vector multiplets. In dualizing from tensor multiplets to vectors we lose the zero modes associated to modes of the tensor multiplets which corresponds to rotations in only one of the three planes (where $B_{i1}^k$ has a mode only in the $z_i$ direction). This accounts for the three $\eta$’s in the denominator. The reduction of the fields of the (2,0) theory to five dimensions has also been studied in detail in [188]. The partition function can be written elegantly in terms of double elliptic gamma functions (see Appendix C.2 for a brief discussion of some of their properties):

$$G_2(z|a, b, c) = \prod_{i,j,k=0}^{\infty} (1 - Z A^i B^j C^k)(1 - Z^{-1} A^{i+1} B^{j+1} C^{k+1}),$$

where $(Z; A, B, C) = \exp(2\pi i (z; a, b, c))$. We have

$$Z_{U(1)} = \frac{1}{\eta(q)\eta(q_1)\eta(q_2)} \cdot \frac{G'_2(0|\tau, \tau_1, \tau_2)}{G_2(m + \frac{1}{2} + \frac{\tau + \tau_1 + \tau_2}{2}|\tau, \tau_1, \tau_2)}, \quad (7.6.14)$$

where we are deleting the zero mode of $G_2(0)$ as noted before. To construct the partition function of this theory on $S^5$ we simply have to consider the above topological string partition function and take three copies of it for the modes of the vector multiplet and the hypermultiplet on the $S^5$. Dropping for now the factors of $\eta$, we get:

$$\frac{G'_2(0|\tau, \tau_1, \tau_2)}{G_2(m - \frac{1}{2} + \frac{\tau + \tau_1 + \tau_2}{2}|\tau, \tau_1, \tau_2)} \cdot \frac{G'_2(0|\tau_2, \tau_1, -1/\tau_2)}{G_2((m - \frac{1}{2} + \frac{\tau_1 + \tau_2}{2})/\tau_1|\tau_1, -1/\tau_1, \tau_2/\tau_1)} \cdot \frac{G'_2(0|\tau_2, \tau_1, -1/\tau_2)}{G_2((m - \frac{1}{2} + \frac{\tau_1 + \tau_2}{2})/\tau_2|\tau_2, \tau_1, -1/\tau_2)}.$$

The non-perturbative contributions to the partition function of an arbitrary 5d theory can a priori be quite complicated, but, in fact, here we find that they cancel out! This is because elliptic gamma functions satisfy a beautiful modular property [74]:

$$G_2(z|\tau_0, \tau_1, \tau_2) = \exp\left(\frac{\pi i}{12} B_{44}(z|\tau_0, \tau_1, \tau_2, 1)\right) G_2\left(\frac{z}{\tau_0}\right) \cdot G_2\left(\frac{z}{\tau_2}\right) \cdot G_2\left(\frac{z}{\tau_2}, \frac{\tau_0}{\tau_2}, \frac{\tau_1}{\tau_2}, \frac{\tau_2}{\tau_2}, \frac{\tau_3}{\tau_2}, -1\right). \quad (7.6.15)$$
Using this, the expression above simplifies to

\[ \frac{G'_2(0) - 1/\tau, \tau_1/\tau, \tau_2/\tau}{G_2\left((m/\tau + 1/2(1 + (\tau_1 + \tau_2 - 1)/\tau)| - 1/\tau, \tau_1/\tau, \tau_2/\tau\right)} \]

It is remarkable that taking the three copies of the five-dimensional partition function led to an answer which is perturbative in \( \tau_1, \tau_2 \), and we offer an explanation of it below.

Likewise, the contributions from the \( \eta \) factors simplify. From \( \eta(\tau_1)\eta(\tau_2) \) we get, up to prefactor:

\[ \eta(\tau_1)\eta(\tau_2) \rightarrow \frac{\eta(\tau_1)\eta(\tau_2)}{\eta(-1/\tau_1)\eta(-1/\tau_2)\eta(\tau_1/\tau_2)\eta(\tau_2/\tau_1)} = 1. \]

From \( \eta(\tau) \) we get

\[ \frac{\eta(\tau)}{\eta(\tau/\tau_1)\eta(\tau/\tau_2)} = \eta(-1/\tau)\eta(-\tau_1/\tau)\eta(-\tau_2/\tau) = \eta(-1/\tau)\eta(\tau_1/\tau)\eta(\tau_2/\tau). \]

We thus end up with

\[ Z_{U(1)}^{top} = \frac{1}{\eta(-1/\tau)\eta(\tau_1/\tau)\eta(\tau_2/\tau)} \frac{G'_2(0) - 1/\tau, \tau_1/\tau, \tau_2/\tau}{G_2\left((m/\tau + 1/2 + \frac{\tau_1 + \tau_2 - 1}{2\tau}| - 1/\tau, \tau_1/\tau, \tau_2/\tau\right)}. \quad (7.6.16) \]

A glance at equations (7.6.14) and (7.6.16) reveals that the only difference between the perturbative answer and the full non-perturbative result is a rescaling of

\[ (m, \tau, \tau_1, \tau_2) \rightarrow (m/\tau, -1/\tau, \tau_1/\tau, \tau_2/\tau), \]

which is the correct map between the 5d and 6d parameters, as discussed above. We now offer an explanation of the fact that the non-perturbative completion of the \( Z^{top} \) resulted in the same function in modular transformed variables. As discussed before (and which can be verified explicitly for this example), we expect a perturbative modularity of the topological string partition functions of elliptic Calabi-Yau threefold of the form:

\[ Z^{top}(m_i, \tau, \tau_1, \tau_2) = Z^{top}(m_i/\tau, -1/\tau, \tau_1/\tau, \tau_2/\tau). \]

\[ ^9 \text{In the following manipulations we do not keep track of the cubic and quartic prefactors which arise as a result of modular transformations. It would be interesting to understand these factors in greater detail.} \]
Instead what we have found in this example is that

\[
\frac{Z^{\text{top}}(m, \tau_1, \tau_2)}{Z^{\text{top}}(\frac{m}{\tau_1}, \frac{\tau}{\tau_1}, -\frac{1}{\tau_1}, \frac{\tau}{\tau_1}) \cdot Z^{\text{top}}(\frac{m}{\tau_2}, \frac{\tau}{\tau_2}, \frac{\tau_1}{\tau_2}, -\frac{1}{\tau_2})} = Z^{\text{top}}(\frac{m}{\tau}, -\frac{1}{\tau}, \frac{\tau_1}{\tau}, \frac{\tau_2}{\tau}).
\]

Note that the additional terms in the denominator are non-perturbative in the topological string coupling constants and thus can be viewed as a non-perturbative completion of the modularity of topological strings. We will comment on the implication of this for possible simplification for the general computation of the index of all 6d theories in Section 7.6.5.

The same result could also have been derived from the relation between the triple sine and elliptic gamma functions (equation (C.2.14)), which we also report here:

\[
G_2(z|\varrho) = \exp \left( \frac{2\pi i}{4!} B_{4,4}(z|\varrho, -1) \right) \cdot \prod_{k=0}^{\infty} \exp \left( \frac{S_3(z + k + 1|\varrho)S_3(z - k|\varrho)}{\frac{3\pi i}{2}(B_{3,3}(z + k + 1|\varrho) - B_{3,3}(z - k|\varrho))} \right).
\]

Let us now denote \(e^{-2\pi i/\tau}, e^{2\pi im/\tau}, e^{2\pi i\tau_1/\tau}\) and \(e^{2\pi i\tau_2/\tau}\) respectively by \(q, q_m, q_1, q_2\). Then, using equation (C.2.17), we can write

\[
G_2(0| -1/\tau, \tau_1/\tau, \tau_2/\tau) = \exp \left( -\sum_n \frac{1}{n} \frac{1 + q^n q_1^n q_2^n}{(1 - q^n)(1 - q_1^n)(1 - q_2^n)} \right)
\]

and

\[
G_2 \left( \frac{m}{\tau} + \frac{1 + \tau_1/\tau + \tau_2/\tau - 1/\tau}{2} \left| -\frac{1}{\tau}, \frac{\tau_1}{\tau}, \frac{\tau_2}{\tau} \right. \right)^{-1} = \exp \left( \sum_n \frac{1}{n} \frac{(q q_1 q_2)^{n/2}((-q_m)^n + (-q_m)^{-n})}{(1 - q^n)(1 - q_1^n)(1 - q_2^n)} \right).
\]

Likewise,

\[
\frac{1}{\eta(-1/\tau)} = \exp \left( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{2\pi i n k}{n} \right) = \exp \left( \sum_n \frac{1}{n} \frac{q^n}{(1 - q^n)} \right),
\]

and similarly for \(\eta(\tau_1)\) and \(\eta(\tau_2)\). Writing

\[
Z_{U(1)}^{np} = \exp \left( \sum_n \frac{I(q_m^n, q^n, q_1^n, q_2^n)}{n} \right),
\]
we get
\[
I = \frac{q}{1-q} + \frac{q_1}{1-q_1} + \frac{q_2}{1-q_2} + \frac{\sqrt{qq_1q_2(-q_m - q_m^{-1}) - 1 - qq_1q_2}}{(1-q)(1-q_1)(1-q_2)}
\]
\[
= \frac{\sqrt{qq_1q_2(-q_m - q_m^{-1}) + qq_1q_2 - qq_1 - qq_2 - q_1q_2}}{(1-q)(1-q_1)(1-q_2)} - 1.
\]
Deleting the zero mode of \( G_2(0) \) correspond to deleting the \(-1\) in the above expression. The resulting expression matches exactly with the result of [175],
\[
I = \frac{x^6(z^{1/2} + z^{-1/2}) + x^{12} - x^8(y_2 + 1/y_1 + y_1/y_2)}{(1 - x^4y_1)(1 - x^4/y_2)(1 - x^4y_2/y_1)},
\]
provided that we identify
\[
x^4y_2/y_1 = q, \quad x^4/y_2 = q_2, \quad x^4y_1 = q_1, \quad -z^{1/2} = q_m,
\]
which is in accord with the transformation of the basis used in that paper compared to our setup above in writing the index.

7.6.4 \( Z_{\text{top}} = Z_{np} \) in 6d?

As we have seen in the context of computation of the superconformal index for a single M5 brane, the non-perturbative completion of \( Z_{\text{top}} \) yields again \( Z_{\text{top}} \) with modular transformed variables. This raises the question whether this is always true, namely\(^{10}\):
\[
Z_{np}(t_i, m_j, \tau, \tau_1, \tau_2) = Z_{top}(t_i/\tau, m_j/\tau, -1/\tau, \tau_1/\tau, \tau_2/\tau)\]
However, as already discussed, we expect from the perturbative modularity of \( Z_{\text{top}} \) a relation of almost this form, namely
\[
Z_{top}(t_i, m_j, \tau, \tau_1, \tau_2) = Z_{top}(t_i, m_j/\tau, -1/\tau, \tau_1/\tau, \tau_2/\tau) \bigg|_{\text{pert.}}
\]
This is almost of the naive form we expected, except that \( t_i \), the dynamical variables which we need to integrate over, are not transformed under \( \tau \to -1/\tau \). This strongly suggests that the

\(^{10}\text{We would like to thank D. Jafferis for discussions on this point.}\)
non-perturbative completion of the above equation is simply

\[ Z_{np}(t_i, m_j, \tau, \tau_1, \tau_2) = Z_{top}(t_i, m_j/\tau, -1/\tau, \tau_1/\tau, \tau_2/\tau). \]

This would be consistent with the fact that the BPS states of the elliptic 3-fold should organize according to a tower of KK modes and for each such tower the identity (7.6.15) would transform the answer back to the original form except in the modular transformed variables. This would give a dramatic simplification for the computation for the 6d case. Namely we would get (taking into account the change of parameters from 5d to 6d):

\[ I^{6d}(m_j, \tau, \tau_1, \tau_2) = \int dt_i Z_{top}(t_i, m_j, \tau, \tau_1, \tau_2). \]

where \( Z_{top} \) is the same as the 5d gauge theory partition function (including the cubic prefactor).

7.6.5 Superconformal index for \( \mathcal{N} = 1, 2 \) in \( d = 4 \)

Similarly in the above context we can consider the open string sectors. These will support 4d field theories in the following way: Consider again F-theory on elliptic 3-folds and consider \((p, q)\) 5-branes of IIB wrapped around the Lagrangian 2-cycles of the base. This of course needs to be compatible with the elliptic fibration structure of F-theory as the 5-branes transform under \( SL(2, \mathbb{Z}) \). This leads to an \( \mathcal{N} = 1, d = 4 \) theory living on the uncompactified directions of the 5-brane. To the best of our knowledge these theories have not been studied before. It would be interesting to investigate this class of theories.

Upon further compactification on a circle, where we wrap one of the directions of the brane on the circle, this will correspond to a 3d theory living on its world volume. By the duality between F-theory and M-theory, this corresponds to M5 branes wrapping Lagrangian cycles of the resulting 3-fold, which we can compactify on the \( S^3 \) and compute the partition function, as already discussed for the open string sector. The corresponding index in the 4d theory is given by

\[ \text{Tr}(-1)^F q_1^{J_{12} - r} q_2^{J_{34} - r} M_i^{F_i} \]
where \( q \) can be identified with the modular transformed elliptic fiber parameter of the 3-fold and \( q_1 \) corresponds to the direction in which we have placed the brane and \( F_i \) correspond to extra symmetries one may have (associated to non-integrated Kahler classes and positions of the brane).

Similarly if the elliptic fibration of F-theory is constant the same construction will lead to an \( \mathcal{N} = 2 \) theory. Here we will have one extra flavor symmetry (the analog of the mass in the \( N = 2^* \) theory discussed before) which will play the role of the additional parameter \( t \) that one can add to the index in the context of \( N = 2 \) theories in \( d = 4 \) [189]:

\[
\text{Tr}(-1)^F q_1^{J_{12}-r_1} q_2^{J_{34}-r_2} q^{J_{56}-R_4} M_i^{F_i}
\]

It would be interesting to study these and explore connections with the computations already done in the literature (see [190] and references therein for examples of such computations).

### 7.7 Superconformal index for \( N \) M5 branes

The aim of this section is to use the techniques developed in this chapter to give a prescription for computing the superconformal index for \( N \) M5 branes, that is, their partition function on \( S^1 \times S^5 \). This gives a highly nontrivial relation between the six-dimensional superconformal index and elliptic genera of self-dual strings. Recall that the \( (2, 0) \) superconformal index is defined as follows:

\[
I_{(2,0)}(q_1, q_2, q, Q_m) = \text{Tr}(-1)^F q_1^{J_{12}-R_1} q_2^{J_{34}-R_2} q^{J_{56}-R_4} Q_m^{R_2-R_4}.
\]

(7.7.17)

where \( J_{12}, J_{34}, J_{56} \) are the generators of \( SO(6) \) rotations acting on \( S^5 \), while \( R_1 \) and \( R_2 \) are generators of the \( Sp(4) \) R-symmetry group, and the trace is over the Hilbert space obtained from radial quantization of the \( (2, 0) \) theory. Following the discussion in the previous sections, this index is equal to the squashed five-sphere partition function of the 5d theory obtained by compactifying the \( (2,0) \) theory on a circle, and the parameters \( q, q_1, q_2, Q_m \) are related to the 5d gauge theory parameters in the following way:

\[
q = \exp(-2\pi i/\tau), \quad q_1 = \exp(2\pi i\epsilon_1/\tau), \quad q_2 = \exp(2\pi i\epsilon_2/\tau), \quad Q_m = \exp(2\pi i m/\tau).
\]
In the case of $N$ M5 branes, the partition function involves an integral over the Coulomb branch parameterized by $(t_1, \ldots, t_{N-1})$.

In addition to the classical prefactor, we need to include the three factors of the topological string partition function

$$Z^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2) = \left( Z^{(1)}_{\mathbb{R}^4 \times T^2}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2) \right)^N Z^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2), \quad (7.7.18)$$

where

$$\hat{Z}^{(N)}(\tau, m, t_{f_a}, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \ldots, \nu_{N-1}} (-Q_{f_1})^{\nu_1} \cdots (-Q_{f_{N-1}})^{\nu_{N-1}} \prod_{a=1}^{N-1} \prod_{(i,j) \in v_a} \frac{\theta_1(\tau; z_{ij}^a)\theta_1(\tau; v_{ij}^a)}{\theta_1(\tau; w_{ij}^a)\theta_1(\tau; w_{ij}^a)},$$

and $z_{ij}^a, \ldots$ are defined as in Equation (2.3.78).

Then, performing analytic continuation to rewrite all three factors of the topological string partition function in the numerator, we find the following:

$$I^{(N)}_{(2,0)} = \int dt_i e^{C(t_i, m, \epsilon_1, \epsilon_2)} \left[ Z^{(N)}_{\mathbb{R}^4 \times S^1}(\tau, m, t_i, \epsilon_1 + 1, \epsilon_2) \times Z^{(N)}_{\mathbb{R}^4 \times S^1}(\tau/\epsilon_1, m/\epsilon_1, t_i/\epsilon_1, 1/\epsilon_1 + 1, \epsilon_1/\epsilon_2) \times Z^{(N)}_{\mathbb{R}^4 \times S^1}(\tau/\epsilon_2, m/\epsilon_2, t_i/\epsilon_2, 1/\epsilon_2 + 1, 1/\epsilon_2) \right]$$

$$= \left( Z^{(1)}(-1/\tau, m/\epsilon_2/\epsilon_1/\tau, \epsilon_1/\tau + 1) \right)^N$$

$$\times \int dt_i e^{C(t_i, m, \epsilon_1, \epsilon_2)} \left[ \hat{Z}^{(N)}(\tau, m, t_i, \epsilon_1 + 1, \epsilon_2) \times \hat{Z}^{(N)}(\tau/\epsilon_1, m/\epsilon_1, t_i/\epsilon_1, 1/\epsilon_1 + 1, \epsilon_1/\epsilon_2) \times \hat{Z}^{(N)}(\tau/\epsilon_2, m/\epsilon_2, t_i/\epsilon_2, 1/\epsilon_2 + 1, 1/\epsilon_2) \right]. \quad (7.7.20)$$

From this expression it one sees that, apart from a simple prefactor, the six-dimensional superconformal index is written in terms of an integral over the tensor branch (parametrized by $t_i$, which have an interpretation as the tension of the self-dual strings). The integrand is given by three factors of the $U(N)$ partition function, which in turn is written as a sum over elliptic genera of the strings.
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The index of $N$ M5 branes has also been computed in [70] by localization of the gauge theory partition function on $S^5$ coming from Scherk-Schwarz reduction of the worldvolume theory of the M5 branes. We refer to [77] for a proof that our result agrees with the one obtained from localization. This is the case despite the fact that the authors of [70] consider a different squashing of $S^5$: while the deformation parameters we consider enter the equation defining the five-sphere,

$$\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 + \omega_3^2|z_3|^2 = 1$$

(7.7.21)

(this type of geometry is also frequently denoted as the ellipsoid), the computation of [70] is for a round sphere

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$$

(7.7.22)

with a non-trivial metric obtained by Scherk-Schwarz reduction of the 6d theory on a circle of radius $r$, which depends on three squashing parameters $a, b, c$ satisfying the constraint $a + b + c = 0$. The round sphere limit for them corresponds to setting $a = b = c = 0$. The equality of the partition functions for these two geometries is akin to the fact that the 3d partition functions for the three-dimensional squashed sphere [166] and ellipsoid [167] turn out to be identical.

More generally, for $\mathcal{N} = (1, 0)$ theories a relation akin to Equation (7.7.19) can be expected to hold between the 6d superconformal index and the elliptic genera of self-dual strings, as follows straightforwardly from the fact that the 5d BPS index admits an expansion in terms of elliptic genera. In this case, one also expects to have an integral over holonomies of the vector fields of the $(1, 0)$ theory around the 6d circle.

7.8 Discussion of results

We have provided evidence that the partition function of superconformal theories on $S^5$ and on $S^5 \times S^1$ can be computed using closed topological strings. Similarly the partition function on $S^3$ and $S^3 \times S^1$ associated to the open string theories can be computed in an analogous
manner. These computations involve in the closed string case an $SL(3, \mathbb{Z})$ action involved in inverting the coupling constants of the refined topological string, and in the open string case an $SL(2, \mathbb{Z})$ transformation. We used the connection with the partition function computation to define what this inversion precisely means and the regions of convergence of topological string coupling constant.

These results complement that in [69] which shows how one can use topological strings to compute associated partition functions on $S^4 \times S^1$ for closed topological strings and $S^2 \times S^1$ for open ones, which does not involve the inversion of the string coupling constant. Thus altogether we have a unified picture where the partition functions of a large class of superconformal theories which can be engineered in dimensions 6, 5, 4, and 3 associated to Calabi-Yau threefolds or Lagrangians in them can be computed using topological string data. This leads to computation of all supersymmetric partition functions in these dimensions on $S^d$ and $S^{d-1} \times S^1$ for the ones that can be geometrically engineered, using topological strings.

This suggests that the BPS states in a supersymmetric theory (with enough supersymmetry) go a long way in defining the superconformal fixed points they come from. It would be very interesting to see whether this can be made into a systematic method for defining the full superconformal theory.

Lastly, we have applied the techniques developed in this chapter to the study the superconformal index of several M5 branes and argued that it can be expressed in terms of a nontrivial combination of elliptic genera of the self-dual strings of the theory. Likewise, the 5d BPS index for other theories that arise by compactification of $(1,0)$ 6d SCFTs can be written in terms of elliptic genera of their self-dual strings, and this also implies that the 6d $(1,0)$ superconformal index can be written in terms of them. This gives further evidence to the claim that the self-dual strings are an important ingredient in understanding the dynamics of six-dimensional superconformal theories.
Appendix A

Modular and Jacobi forms

In this appendix we collect several definitions and results related to (quasi)modular and Jacobi forms that we make extensive use of in this thesis, in particular in Chapter 6.

A.1 Modular forms

We begin by defining an important class of holomorphic functions of the modular parameter $\tau$, the Eisenstein series, which have the following series expansion:

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$  \hspace{1cm} (A.1.1)

where $q = e^{2\pi i \tau}$, $B_{2k}$ are the Bernoulli numbers, defined as $\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}$, and $\sigma_k(n) = \sum_{d|n} d^k$. For $k > 1$, the Eisenstein series $E_{2k}$ transforms as a holomorphic modular form of weight $2k$, in the sense that

$$E_{2k} \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^{2k} E_{2k}(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$  \hspace{1cm} (A.1.2)

For $k = 1$, the Eisenstein series $E_2(\tau)$ is modular up to an anomalous term:

$$E_2 \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^2 E_2(\tau) - \frac{6i c}{\pi} (c \tau + d), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$  \hspace{1cm} (A.1.3)
Appendix A: Modular and Jacobi forms

The anomalous term can be removed by defining an alternative form of the Eisenstein series,

$$\tilde{E}_2(\tau, \tau) = E_2(\tau) - \frac{6i}{\pi(\tau - \tau)}, \quad (A.1.4)$$

at the cost of introducing a mild dependence on the anti-holomorphic parameter $\tau$. Unlike $E_2(\tau)$, $\tilde{E}_2(\tau, \tau)$ transforms as an honest weight-two modular form:

$$\tilde{E}_2 \left( \frac{a\tau + b}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \tilde{E}_2(\tau, \tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (A.1.5)$$

The space of quasi-holomorphic $SL(2, \mathbb{Z})$ modular forms, that is, modular forms which are polynomials in $\text{Im}(\tau)^{-1}$ with coefficients which are holomorphic functions of $\tau$, is a polynomial ring which is generated by

$$\tilde{E}_2(\tau, \tau), \ E_4(\tau), \ \text{and} \ E_6(\tau); \quad (A.1.6)$$

similarly,

$$E_2(\tau), \ E_4(\tau), \ \text{and} \ E_6(\tau) \quad (A.1.7)$$
generate the polynomial ring of $SL(2, \mathbb{Z})$ quasi-modular forms, defined simply by taking the holomorphic part of quasi-holomorphic modular forms.

Another function we make extensive use of is the Dedekind eta function

$$\eta(\tau) = e^{\frac{\pi i}{12}\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau}), \quad (A.1.8)$$

which, up to a phase, transforms as a weight-1/2 modular form:

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}\eta(\tau)}, \quad \eta(-1/\tau) = e^{-\pi i/2\tau^{1/2}}\eta(\tau). \quad (A.1.9)$$

A.2 Jacobi forms

We now turn to a brief discussion of Jacobi forms, which are functions of a modular parameter $\tau$ and an elliptic parameter $z$. Under a modular transformation parametrized by
Appendix A: Modular and Jacobi forms

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad \text{Jacobi forms transform as follow:}
\]

\[
\phi \left( \frac{a\tau + b}{c\tau + d}; \frac{z}{c\tau + d} \right) = (c\tau + d)^k e^{\frac{2\pi imz^2}{c\tau + d}} \phi(\tau; z);
\]  \hspace{1cm} (A.2.10)

also, under translations of \( z \) by \( Z\tau + Z \) they transform as follows:

\[
\phi(\tau; z + \lambda \tau + \mu) = e^{-2\pi im(\lambda^2\tau + 2\lambda z)} \phi(\tau; z), \quad \lambda, \mu \in \mathbb{Z}.
\]  \hspace{1cm} (A.2.11)

The two numbers \( k \) and \( m \) are referred to respectively as the weight and index of the modular form.

Jacobi forms have a Fourier expansion of the form

\[
\sum_{n,r} c(n,r) e^{2\pi in\tau} e^{2\pi irz}.
\]  \hspace{1cm} (A.2.12)

One usually requires the coefficients \( c(n,r) \) of Jacobi forms to vanish for \( r^2 > 4mn \); imposing the less strict condition that \( c(n,r) = 0 \) if \( n < 0 \) leads to a larger class of functions denoted as weak Jacobi forms.

A prominent example of Jacobi form is the Jacobi theta function

\[
\theta_1(\tau; z) = -ie^{\pi iz/6}e^{\pi iz\eta(\tau)} \prod_{k=1}^{\infty} (1 - e^{2\pi ik\tau} e^{2\pi iz})(1 - e^{2\pi i(k-1)\tau} e^{-2\pi iz}),
\]  \hspace{1cm} (A.2.13)

which has weight 1/2 and index 1/2; it satisfies the property that

\[
\partial_{E_2} \theta_1(\tau; z) = \frac{(2\pi iz)^2}{24} \theta_1(\tau; z).
\]

Closely related to \( \theta_1(z, \tau) \) are the functions

\[
\theta_2(\tau; z) = \theta_1(\tau; z + 1/2),
\]  \hspace{1cm} (A.2.14)

\[
\theta_3(\tau; z) = e^{\pi iz + \pi i\tau/4} \theta_1(\tau; z + 1/2 + \tau/2),
\]  \hspace{1cm} (A.2.15)

\[
\theta_4(\tau; z) = -ie^{\pi iz + \pi i\tau/4} \theta_1(\tau; z + \tau/2).
\]  \hspace{1cm} (A.2.16)
In Chapter 6 we make use of the following result concerning weak Jacobi forms of even weight (see for example [152]):

The weak Jacobi forms with modular parameter \( \tau \) and elliptic parameter \( \epsilon \) of index \( k \) and even weight \( w \) form a polynomial ring which is generated by the four modular forms \( E_4(\tau) \), \( E_6(\tau) \), \( \phi_{0,1}(\tau; z) \), and \( \phi_{-2,1}(\tau; z) \), where

\[
\phi_{-2,1}(\tau; z) = -\frac{\theta_1(\tau; z)^2}{\eta^6(\tau)} \quad \text{and} \quad \phi_{0,1}(\tau; z) = 4 \left[ \frac{\theta_2(\tau; z)^2}{\theta_2(\tau; 0)^2} + \frac{\theta_3(\tau; z)^2}{\theta_3(\tau; 0)^2} + \frac{\theta_4(\tau; z)^2}{\theta_4(\tau; 0)^2} \right]
\]

are Jacobi forms of index 1, respectively of weight \(-2\) and 0.

It is important to note that

\[
\partial_{E_2} \phi_{-2,1}(\tau; z) = \frac{(2\pi iz)^2}{12} \phi_{-2,1}(\tau; z), \quad \partial_{E_2} \phi_{0,1}(\tau; z) = \frac{(2\pi iz)^2}{12} \phi_{0,1}(\tau; z).
\]

Lastly, we define the \( n \)-th Hecke operator \( T_n \) by its action on a weak Jacobi form \( f(\tau, z) \) of weight \( k \) as

\[
T_n f(\tau; z) = n^{k-1} \sum_{\substack{ad=n \atop a, d > 0}} \frac{1}{d^k} \sum_{b \pmod{d}} f \left( \frac{a \tau + b}{d}; az \right).
\]

Under this transformation, weak Jacobi forms of weight \( k \) and index \( m \) are mapped to weak Jacobi forms of weight \( k \) and index \( nm \).

### A.3 Weyl invariant Jacobi forms for \( E_8 \)

We conclude this appendix by mentioning a class of multivariate Jacobi forms which are invariant under the Weyl group of \( E_8 \). These functions depend on the modular parameter \( \tau \) as well as eight parameters \( m_{E_8}^{1:8} \), and are organized in two classes:

\[
A_1, A_2, A_3, A_4, A_5 \quad \text{and} \quad B_2, B_3, B_4, B_6.
\]

In the limit \( \tilde{m}_{E_8} \to 0 \), these functions reduce to Eisenstein series:

\[
A_i(\tau; \tilde{m}_{E_8}) \to E_4(\tau); \quad B_i(\tau; \tilde{m}_{E_8}) \to E_6(\tau).
\]
Furthermore, one has:

\[
\begin{align*}
\partial_{E_2} A_n(\tau; \vec{m}_{E_8}) &= -n \cdot \frac{(2\pi)^2}{24} \left( \sum_{i=1}^{8} m_{E_8,i}^2 \right) A_n(\tau; \vec{m}_{E_8}), \\
\partial_{E_2} B_n(\tau; \vec{m}_{E_8}) &= -n \cdot \frac{(2\pi)^2}{24} \left( \sum_{i=1}^{8} m_{E_8,i}^2 \right) B_n(\tau; \vec{m}_{E_8}).
\end{align*}
\]

\begin{equation}
\tag{A.3.20}
\end{equation}

\begin{equation}
\tag{A.3.21}
\end{equation}

It is known that any Jacobi form which is given by a linear combination of characters of affine \(E_8\) representations and is invariant under the Weyl group of \(E_8\) can be written as a polynomial in \(A_{1,2,3,4,5}, B_{2,3,5}\). The superscript of these functions indicates the amount by which they contribute to the level of the affine \(E_8\) character. Thus, for example, any Weyl-invariant Jacobi form which is a combination of level 2 characters of affine \(E_8\) can be written as a linear combination of \(A_1(\tau; \vec{m}_{E_8})^2, A_2(\tau; \vec{m}_{E_8}),\) and \(B_2(\tau; \vec{m}_{E_8})\).

The simplest of these functions, \(A_1(\vec{m}_{E_8}, \tau)\), is equal to the \(E_8\) theta function

\[
\Theta_{E_8}(\tau; \vec{m}_{E_8}) = \sum_{\vec{k} \in \Gamma_{E_8}} \exp(\pi i \vec{k} \cdot \vec{k} + 2\pi i \vec{m}_{E_8} \cdot \vec{k}) = \frac{1}{2} \sum_{k=1}^{4} \prod_{\ell=1}^{8} \theta_k(\tau; m_{E_8}^\ell),
\]

\begin{equation}
\tag{A.3.22}
\end{equation}

where \(\vec{k}\) runs over the points of the \(E_8\) lattice \(\Gamma_{E_8}\). The other eight \(E_8\) Jacobi functions can be defined starting from \(A_1(\vec{m}_{E_8}, \tau)\), as discussed in more detail in [144] and [119].

Finally, it is worth mentioning the \(E_8 \times E_8\) theta function, which depends on sixteen parameters \(\vec{m}_{E_8 \times E_8}\) and is defined as

\[
\Theta_{E_8 \times E_8}(\tau; \vec{m}_{E_8 \times E_8}) = \sum_{\vec{k} \in \Gamma_{E_8 \times E_8}} \exp(\pi i \vec{k} \cdot \vec{k} + 2\pi i \vec{m}_{E_8 \times E_8} \cdot \vec{k}).
\]

\begin{equation}
\tag{A.3.23}
\end{equation}

Since \(\Gamma_{E_8 \times E_8} = \Gamma_{E_8} \oplus \Gamma_{E_8}\), one can pick a basis where

\[
\vec{m}_{E_8 \times E_8,1}, \ldots, \vec{m}_{E_8 \times E_8,8} = \vec{m}_{E_8, L}
\]

\begin{equation}
\tag{A.3.24}
\end{equation}

only have nonzero product with the first \(\Gamma_{E_8}\) factor, while

\[
\vec{m}_{E_8 \times E_8,9}, \ldots, \vec{m}_{E_8 \times E_8,16} = \vec{m}_{E_8, R}
\]

\begin{equation}
\tag{A.3.25}
\end{equation}
only have nonzero product with the second $\Gamma_{E_8}$ factor. It is thus clear that

$$\Theta_{E_8 \times E_8}(\tau; \bar{m}_{E_8 \times E_8}) = \Theta_{E_8}(\tau; \bar{m}_{E_8}, L) \Theta_{E_8}(\tau; \bar{m}_{E_8}, R) = A_1(\tau; \bar{m}_{E_8}, L) A_1(\tau; \bar{m}_{E_8}, R). \quad (A.3.26)$$
Appendix B

Appendices for Chapter 2

B.1 Useful identities

The refined topological vertex can be written in different bases of the ring of symmetric functions. In the body of Chapter 2 we use the representation of [67] which is based on the combinatorial interpretation of the vertex. Like the usual topological vertex, the refined one is labeled by three Young diagrams and can be written in terms of skew Schur functions $s_{\lambda/\eta}(x)$ and Macdonald polynomial $P_{\nu}(t^{-\rho}; q, t)$ as

$$C_{\lambda, \mu, \nu}(t, q) = \left( \frac{q}{t} \right)^{\frac{1}{2} \sum \nu \cdot \nu} \frac{t^{1/2} - q^{1/2}}{t - q} q^{\frac{1}{2} \parallel \nu \parallel^2} P_{\nu}(t^{-\rho}; q, t)$$

$$\times \sum_{\eta} \left( \frac{q}{t} \right)^{\frac{1}{2} \sum \eta \cdot \eta - \mid \eta \mid} s_{\lambda/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t} q^{-\rho}),$$

(B.1.1)

where $\rho$ is used for $\rho = \{ -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots \}$. The Macdonald polynomial with the special arguments appearing in the refined vertex can be expressed in terms of Young diagrams:

$$P_{\nu}(t^{-\rho}; q, t) = t^{\frac{1}{2} \parallel \nu \parallel^2} \tilde{Z}_{\nu}(t, q),$$

(B.1.2)

where we have defined the following function:

$$\tilde{Z}_{\nu}(t, q) = \prod_{(i,j) \in \nu} \left( 1 - q^{\nu_i - j} t^{\nu_j - i + 1} \right)^{-1}.$$

(B.1.3)
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In our computations, we have explicitly used the function $\widetilde{Z}_\nu(t, q)$. The refined vertex has the following form in terms of it:

$$C_{\lambda, \mu, \nu}(t, q) = \left( \frac{q}{t} \right)^{\frac{1}{2}} e^{\frac{1}{2} t^2 \sigma(\nu)} q^{\frac{1}{2} |\nu|^2} \sum_{\eta} \left( \frac{q}{t} \right)^{\frac{1}{2} |\eta|} s_{\lambda^\nu/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu^\rho} q^{-\rho}) \right) . \quad (B.1.4)

We have made use of the following identities in our computations of the topological string partition functions:

$$n(\lambda) \equiv \sum_{i=1}^{\ell(\lambda)} (i-1) \lambda_i = \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \lambda_i^t (\lambda_i^t - 1) = \sum_{(i,j) \in \lambda} (\lambda_j^t - i) = \frac{||\lambda||^2}{2} - \frac{|\lambda|}{2}, \quad (B.1.5)

$$n(\lambda^t) \equiv \sum_{i=1}^{\ell(\lambda^t)} (i-1) \lambda_i^t = \frac{1}{2} \sum_{i=1}^{\ell(\lambda^t)} \lambda_i (\lambda_i - 1) = \sum_{(i,j) \in \lambda} (\lambda_i - j) = \frac{||\lambda^t||^2}{2} - \frac{|\lambda|}{2}, \quad (B.1.6)

with $\ell(\lambda)$ being the number of non-zero $\lambda_i$'s. We have also used $||\lambda||^2 = \sum_{i=1}^{\ell(\lambda)} \lambda_i^2$. The hook length $h(i, j)$ and the content $c(i, j)$ are defined as

$$h(i, j) = \nu_t - j + \nu_j^t - i + 1, \quad c(i, j) = j - i, \quad (B.1.7)

which satisfy

$$\sum_{(i,j) \in \lambda} h(i, j) = n(\lambda_t) + n(\lambda) + |\lambda|, \quad (B.1.8)

$$\sum_{(i,j) \in \lambda} c(i, j) = n(\lambda^t) - n(\lambda) = \frac{1}{2} ||\lambda||^2 - \frac{1}{2} ||\lambda^t||^2 \equiv \frac{1}{2} \kappa(\lambda). \quad (B.1.9)

We also have made use of the following identities [191]:

$$\sum_{(i,j) \in \nu} \mu_j^t = \sum_{(i,j) \in \mu} \nu_j^t \quad (B.1.10)

$$\sum_{(i,j) \in \nu} \nu_j^t = ||\nu^t||^2 \quad (B.1.11)

The following sum rules are essential for vertex computations [192]:

$$\sum_{\eta} s_{\eta/\lambda}(x) s_{\eta/\mu}(y) = \prod_{i,j=1}^{\infty} (1 - x_i y_j) \sum_{\tau} s_{\mu/\tau}(x) s_{\lambda/\tau}(y) . \quad (B.1.12)

$$\sum_{\eta} s_{\eta^t/\lambda}(x) s_{\eta/\mu}(y) = \prod_{i,j=1}^{\infty} (1 + x_i y_j) \sum_{\tau} s_{\mu/\tau^t}(x) s_{\lambda^t/\tau^t}(y) . \quad (B.1.13)
We have normalized the topological string amplitudes. Both the open and closed amplitudes are infinite series in the Kähler parameters; however, their ratio is finite as a result of the following identity:

$$\prod_{i,j=1}^{\infty} \frac{1 - Q q^{\nu_i - j} t^i}{1 - Q q^{j} t^{-i+1}} = \prod_{(i,j) \in \nu} \left(1 - Q q^{\nu_i - j} t^i \right) \prod_{(i,j) \in \mu} \left(1 - Q q^{\mu_i + j} t^{-i+1} \right)$$  (B.1.14)

and its specializations

$$\prod_{i,j=1}^{\infty} \frac{1 - Q t^{\nu_j - i + \frac{1}{2}} q^{-j+\frac{1}{2}}}{1 - Q t^{-i + \frac{1}{2}} q^{-j+\frac{1}{2}}} = \prod_{(i,j) \in \nu} \left(1 - Q q^{-j + \frac{1}{2}} t^i \right)$$  (B.1.15)

$$\prod_{i,j=1}^{\infty} \frac{1 - Q q^{\nu_i - j + \frac{1}{2}} t^{-i + \frac{1}{2}}}{1 - Q q^{j - \frac{1}{2}} t^{-i + \frac{1}{2}}} = \prod_{(i,j) \in \nu} \left(1 - Q q^{j - \frac{1}{2}} t^{-i} \right)$$  (B.1.16)

$$\prod_{i,j=1}^{\infty} \frac{1 - Q t^{\nu_j - i} q^{\nu_i - j + \frac{1}{2}}}{1 - Q t^{-i} q^{-j+\frac{1}{2}}} = \prod_{(i,j) \in \nu} \left(1 - Q t^{\nu_j - i} q^{\nu_i - j + 1} \right) \left(1 - Q t^{-\nu_j - i - 1} q^{-\nu_i + j} \right).$$  (B.1.17)

We have also written the partition functions in terms of theta functions. We employ the following definitions in our computations. The first theta function and the Dedekind $\eta$-function are defined as:

$$\theta_1(\tau; z) = -i e^{i \frac{\pi \tau}{4}} e^{i \pi z} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k \tau} \right) \left(1 - e^{2\pi i (k-1) \tau} e^{2\pi i z} \right) \left(1 - e^{2\pi i k \tau} e^{-2\pi i z} \right)$$

$$= -i e^{i \frac{\pi \tau}{4}} (e^{i \pi z} - e^{-i \pi z}) \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k \tau} \right) \left(1 - e^{2\pi i k \tau} e^{2\pi i z} \right) \left(1 - e^{2\pi i k \tau} e^{-2\pi i z} \right),$$

$$\eta(\tau) = e^{i \frac{\pi \tau}{12}} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k \tau} \right).$$

They satisfy the following modular transformations:

$$\theta_1(\tau + 1; z) = \theta_1(\tau; z), \quad \theta_1 \left(\frac{-1}{\tau}; \frac{z}{\tau} \right) = -i (-i \tau)^{\frac{1}{2}} \exp \left(\frac{i \pi z^2}{\tau} \right) \theta_1(\tau; z),$$

$$\eta(\tau + 1) = e^{i \frac{\pi \tau}{12}} \eta(\tau), \quad \eta \left(-\frac{1}{\tau} \right) = \sqrt{-i \tau} \eta(\tau).$$  (B.1.18)

### B.2 Derivation of the building block $W_{\nu_m \nu_{m+1}}$

In this appendix, we present the derivation of the topological string partition function of the building blocks $W_{\nu_m \nu_{m+1}}(Q, Q_m, t, q)$ that we used to compute the partition function of the
geometry engineering the $\mathcal{N} = 2^*$ $SU(N)$ theory. The corresponding toric diagram is shown in Fig. B.1. The partition function is in the following generic form when the refined topological vertex is used to compute it:

$$G(x \hookrightarrow y \hookrightarrow w \hookrightarrow z) \equiv \sum_{\lambda, \mu, \eta_1, \eta_2} Q^{[\lambda]} \rho_{\lambda/\eta_1}(x) s_{\lambda/\eta_2}(y) s_{\mu/\eta_1}(w) s_{\mu/\eta_2}(z). \quad (B.2.19)$$

We can perform the sums twice and show that $G(x, y, w, z)$ satisfies the following recursion relation:

$$G(x, y, w, z) = \prod_{i,j} \frac{(1 + Q x_i y_j)(1 + \rho w_i z_j)}{(1 - Q \rho x_i w_j)(1 - Q \rho y_i z_j)} G(Q^x x \hookrightarrow Q^y y \hookrightarrow Q^w w \hookrightarrow Q^z z). \quad (B.2.20)$$

Note that the argument in each of the four factors scales with $Q \rho$ as follows:

$$x_i y_j \mapsto (Q \rho) x_i y_j$$

$$w_i z_j \mapsto (Q \rho) w_i z_j$$

$$x_i w_j \mapsto (Q \rho) x_i w_j$$

$$y_i z_j \mapsto (Q \rho) y_i z_j.$$ 

Switching to the conventions used in the rest of Chapter 2, we call $Q_\tau = Q \rho$ and write down the $n^{th}$ iterative step:

$$G(x, y, w, z) = \prod_{k=1}^n \prod_{i,j} \frac{(1 + Q^k x_i y_j)(1 + Q^k w_i z_j)}{(1 - Q^k x_i w_j)(1 - Q^k y_i z_j)} G(Q^n x, \rho^n y, \rho^n w, Q^n z). \quad (B.2.21)$$

Under the assumption that $Q^n, \rho^n \to 0$ as $n \to \infty$ (this assumption is the same as the one used by Macdonald [192] to prove the summation rules we are using), we need to take the following limit,

$$\lim_{n \to \infty} G(Q^n x, \rho^n y, \rho^n w, Q^n z). \quad (B.2.22)$$
It is easy to show that

\[
G(Q^n x, \rho^n y, \rho^n w, Q^n z) = \sum_{\lambda, \mu, \eta_1, \eta_2} Q^{[\lambda]} \rho^{[\mu]} s_{\lambda/\eta_1}(Q^n x) s_{\lambda/\eta_2}(y) s_{\mu/\eta_1}(w) s_{\mu/\eta_2}(Q^n z). \tag{B.2.23}
\]

The only surviving terms in the \(n \to \infty\) limit are when \(\lambda = \eta_1\) and \(\mu = \eta_2\). Therefore, the four sums reduce to two sums:

\[
\sum_{\eta_1, \eta_2} Q^{[\eta_1]} \rho^{[\eta_2]} s_{\eta_1/\eta_2}(y) s_{\eta_2/\eta_1}(z). \tag{B.2.24}
\]

The only non-zero terms in these resulting sums are those for which \(\eta_1^I > \eta_2\) and simultaneously \(\eta_2^I > \eta_1\). These two conditions require \(\eta_1^I = \eta_2\). A further reduction of the sums to a single one occurs. What is left can easily be summed up and takes the following form:

\[
G(x, y, w, z) = \prod_{k=1}^{\infty} (1 - Q^k)^{-1} \prod_{i,j=1}^{\infty} \frac{(1 + Q^k \rho^{-1} x_i y_j)(1 + Q^k \rho w_i z_j)}{(1 - Q^k x_i w_j)(1 - Q^k y_i z_j)}. \tag{B.2.25}
\]

Having performed all the sums and found a product formula we can now replace \(x, y, w\) and \(z\) with what we have in the vertex computation:

\[
x = t^p \frac{1}{2} q^{\nu_m+\frac{1}{2}}, \quad y = q^p \frac{1}{2} t^{\nu_m+\frac{1}{2}}
\]

\[
w = t^{\nu_m+\frac{1}{2}} \rho, \quad z = q^{\nu_m+\frac{1}{2}}.
\tag{B.2.26}
\]

The building block therefore takes the following form:

\[
W_{\nu_m^I \nu_m+1}(Q, Q, t, q) = t^{-\frac{1}{2} \nu_m^2} q^{-\frac{1}{2} \nu_m^2} \tilde{Z}_{\nu_m^I}(q^{-1}, t^{-1}) \tilde{Z}_{\nu_m+1}(t^{-1}, q^{-1}) Q_m^{-\frac{1}{2} \nu_m^2} \prod_{k=1}^{\infty} \left(1 - Q^k t^{\nu_m+\frac{1}{2}} \right) \left(1 - Q^k q^{\nu_m+\frac{1}{2}} \right).
\tag{B.2.27}
\]

### B.3 The \(SU(N)\) partition function in terms of theta functions

In this section we collect a few details of the computation we performed to express the \(SU(N)\) partition function in terms of theta functions. In the previous section of the appendix we demonstrated our derivation of the building blocks that we use to compute the topological
string partition function for the geometries engineering $SU(N)$ theories. Although the individual building blocks are modular in the unrefined case (after the non-holomorphic extension), the blocks are not modular in the refined case. However, factors appearing in the neighbouring building blocks combine nicely into theta functions. Let us look at the gluing along the $m$th internal leg and collect all the infinite products including $\nu_m$:

$$
\prod_{k=1}^{\infty} \prod_{(i,j) \in \nu_m} \frac{(1 - Q^k_t Q_{m-1}^{-1} q^{\nu_{m,i} - j + \frac{1}{2} t^{\nu_{m-1,j}} - i + \frac{1}{2}})}{(1 - Q^k_t q^{\nu_{m,i} - j + \frac{1}{2} t^{\nu_{m,j}} - i + \frac{1}{2}})} \left(1 - Q^k_t q^{\nu_{m,i} - j + \frac{1}{2} t^{\nu_{m+1,j}} - i + \frac{1}{2}}\right) \left(1 - Q^k_t q^{\nu_{m,i} - j + \frac{1}{2} t^{\nu_{m,j}} - i + \frac{1}{2}}\right),
$$

where we have written the factors including $\nu_m$ from $D_{\nu_{m-1}^{\nu_m}}(\tau, m, \epsilon_1, \epsilon_2)$ in the first line and the ones from $D_{\nu_m^{\nu_{m+1}}}(\tau, m, \epsilon_1, \epsilon_2)$ in the second line. It is clear from the above expression that the factors in the numerator originating from the same block can be combined into theta functions; on the other hand, the underlined factors will combine into theta functions with equivalent factors from neighboring blocks. The first Jacobi theta function $\theta_1(\tau; z)$, in addition to the infinite products in the above expansion, includes a factor of $-ie^{i\pi/2}e^{i\pi z}(1 - Q^k_t)$. We can multiply the numerator and denominator by the $\tau$-dependent pieces without anything else needed; however, $e^{i\pi z}$ requires a little bit more attention. Let us separately treat the numerator and the denominator and start with the easier one, the denominator: we will have the following factors

$$
\prod_{(i,j) \in \nu_m} \frac{1}{(qt)^{-\nu_{m,i} + j - \nu_{m,j} + i}} = q^{\frac{\|\nu_m\|^2}{t^{\|\nu_m\|^2}}},
$$

where we have made use of Equation (B.1.5) and Equation (B.1.6). These factors will cancel against equivalent terms appearing in the definition of the domain wall factors $D_{\nu_{m-1}^{\nu_m}}(\tau, m, \epsilon_1, \epsilon_2)$ and $D_{\nu_m^{\nu_{m+1}}}(\tau, m, \epsilon_1, \epsilon_2)$. In the numerator we end up with

$$
\prod_{(i,j) \in \nu_m} Q_t^{\nu_{m+1,j} - \nu_{m-1,j}},
$$

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where we take the product over the Young diagram $\nu_m$ of quantities which depend on the neighboring Young diagrams $\nu_{m-1}$ and $\nu_{m+1}$. We do not know a closed form expressions for these products. However, we can use the Equation (B.1.10) to show that these factors all disappear if we consider the $SU(N)$ partition function as a whole. After gluing all the blocks for the $SU(N)$ theory we end up with the following sum (we take $\nu_0 = \nu_N = \emptyset$):

$$
\sum_{(i,j) \in \nu_1} (-\nu_{0,j}^t + \nu_{2,j}^t) + \sum_{(i,j) \in \nu_2} (-\nu_{1,j}^t + \nu_{3,j}^t) + \ldots + \sum_{(i,j) \in \nu_{N-1}} (-\nu_{N-2,j}^t + \nu_{N,j}^t)
= \left( \sum_{(i,j) \in \nu_1} \nu_{2,j}^t - \sum_{(i,j) \in \nu_2} \nu_{1,j}^t \right) + \ldots + \left( \sum_{(i,j) \in \nu_{N-2}} \nu_{N-1,j}^t - \sum_{(i,j) \in \nu_{N-1}} \nu_{N-2,j}^t \right) = 0. \tag{B.3.31}
$$

The partition function of the $SU(N)$ theory can thus be written only in terms of theta functions without any additional factors.

### B.4 Spin content of the $SU(2)$ theory

In this section we tabulate the spin content of the BPS states which we have obtained by isolating low degree curves in the topological string free energy for the $SU(2)$ theory. Higher degree curves can in principle be computed as well but require more and more computational power.

![Figure B.2: The toric diagram for $\mathcal{N} = 2^*$ SU(2) theory with the curve classes labeled by $M, B$ and $F'$, after mass, base and fiber.](image)

The topological string free energy is the generating function for the BPS states and is a positive power expansion in the homology classes $M, B$ and $F'$. One interesting property of the
$SU(2)$ partition function is invariance under the following transformation:

\[(M, B, F') \mapsto (-M, B, F').\]  

This transformation is nothing but realisation of flop transition on the curve in the class $M$. Since the geometry is the same after flop transition (toric diagram is shown in Fig. B.3) the partition function is expected to be invariant. It is easy to see from the Equation (2.3.31) that the partition function is indeed invariant under transformation given in Equation (B.4.32). A flop transition can also be carried out with respect to the curve with parameter $E - M$.

![Toric diagram of $SU(2)$ geometry.](image)

The invariance under transformation (B.4.32) implies that the spin content of the curve $kF' + nB + rM$ is the same as that of $kF' + nB + (2k + 2n - r)M$. This also implies that curves for which $r > 2k + 2n$ are not holomorphic and do not contribute. Thus for a given $k$ and $n$ the curves which contribute are the following:

\[kF' + nB + rM, \ r = 0, \ldots, 2(k + n).\]  

(B.4.33)

If we change the basis of the second homology of our target space the symmetries and properties of the BPS content becomes more manifest. We will find the following choice natural:

\[F = F' + M\]  

(B.4.34)

\[E = B + M.\]  

(B.4.35)

Here, $F$ and $E$ refer to fiber and elliptic curves, respectively. In our computations we denoted the Kähler parameter for $F$ by $Q_f$, for $E$ by $Q_e$, and for $M$ by $Q_m$. In this new basis only the following
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curves contribute:

\[ kF + nE + rM, \quad r = -(k + n), \ldots, (k + n). \]  \hspace{1cm} \text{(B.4.36)}

As shown below, for \( k = 1, n = 0 \) we will only have \( F - M, F \) and \( F + M \), and for \( k = n = 1 \) we will only have \( F + E - 2M, F + E - M, F + E, F + E + M, F + E + 2M \). Another interesting observation based on the BPS content we computed is that the state corresponding to the curve \( C \) are included in \( C + nE \) for any positive \( n \):

\[ \mathcal{H}_C \subset \mathcal{H}_{C+nE} \]  \hspace{1cm} \text{(B.4.37)}

In the \((F, E, M)\) basis the consequence of flop invariance is that the spin content is the same for all curves belonging to the same orbit of the group \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \). The action of its generators on the collection of curves \( \{fF + eE + mM\} \) is given by

\[ r : \quad (f,e,m) \rightarrow (f,e,-m) \]  \hspace{1cm} \text{(flop transition on curve in class M)}

\[ s : \quad (f,e,m) \rightarrow (f,f+e+m, -2f-m) \]  \hspace{1cm} \text{(flop transition on curve in class E - M)}.

It is straightforward to check that \( r^2 = s^2 = id \) and that the stabilizer of a generic curve is trivial, the only exceptions being curves in the classes \((f,e,0)\) and \((f,e,-f)\), which are respectively fixed points under the action of \( r \) and \( s \).
### Appendix B: Appendices for Chapter 2

<p>| ( F - M ) | (0, 0) |
| ( F ) | (0, 1/2) |
| ( F + M ) | (0, 0) |
| ( F + E - 2M ) | (0, 1/2) |
| ( F + E - M ) | (0, 1) ( \oplus (1/2, 1/2) \oplus (0, 0) ) |
| ( F + E ) | (1/2, 1) ( \oplus (1/2, 0) \oplus 2(0, 1/2) ) |
| ( F + E + M ) | (0, 1) ( \oplus (1/2, 1/2) \oplus (0, 0) ) |
| ( F + E + 2M ) | (0, 1/2) |
| ( F + 2E - 3M ) | (0, 0) |
| ( F + 2E - 2M ) | (1/2, 1) ( \oplus (1/2, 0) \oplus 2(0, 1/2) ) |
| ( F + 2E - M ) | (1/2, 3/2) ( \oplus (1, 1) \oplus 2(0, 1) \oplus 3(1/2, 1/2) \oplus 4(0, 0) ) |
| ( F + 2E ) | (1, 3/2) ( \oplus (1, 1/2) \oplus 3(1/2, 1) \oplus (0, 3/2) \oplus 3(1/2, 0) \oplus 5(0, 1/2) ) |
| ( F + 2E + M ) | (1/2, 3/2) ( \oplus (1, 1) \oplus 2(0, 1) \oplus 3(1/2, 1/2) \oplus 4(0, 0) ) |
| ( F + 2E + 2M ) | (1/2, 1) ( \oplus (1/2, 0) \oplus 2(0, 1/2) ) |
| ( F + 2E + 3M ) | (0, 0) |
| ( F + 3E - 3M ) | (0, 1) ( \oplus (1/2, 1/2) \oplus (0, 0) ) |
| ( F + 3E - 2M ) | (1, 3/2) ( \oplus (1, 1/2) \oplus 3(1/2, 1) \oplus (0, 3/2) \oplus 3(1/2, 0) \oplus 5(0, 1/2) ) |
| ( F + 3E - M ) | (1, 2) ( \oplus (3/2, 3/2) \oplus 3(1/2, 3/2) \oplus 3(1, 1) \oplus 2(1, 0) \oplus 7(0, 1) \oplus 9(1/2, 1/2) \oplus 7(0, 0) ) |
| ( F + 3E ) | (3/2, 2) ( \oplus (3/2, 1) \oplus 3(1, 3/2) \oplus (1/2, 2) \oplus 4(1, 1/2) \oplus 9(1/2, 1) \oplus 3(0, 3/2) \oplus 8(1/2, 0) \oplus 12(0, 1/2) ) |
| ( F + 3E + M ) | (1, 2) ( \oplus (3/2, 3/2) \oplus 3(1/2, 3/2) \oplus 3(1, 1) \oplus 2(1, 0) \oplus 7(0, 1) \oplus 9(1/2, 1/2) \oplus 7(0, 0) ) |
| ( F + 3E + 2M ) | (1, 3/2) ( \oplus (1, 1/2) \oplus 3(1/2, 1) \oplus (0, 3/2) \oplus 3(1/2, 0) \oplus 5(0, 1/2) ) |
| ( F + 3E + 3M ) | (0, 1) ( \oplus (1/2, 1/2) \oplus (0, 0) ) |
| ( 2F + E - 2M ) | (0, 3/2) |
| ( 2F + E - M ) | (1/2, 3/2) ( \oplus (0, 2) \oplus (0, 1) ) |
| ( 2F + E ) | (1/2, 2) ( \oplus (1/2, 1) \oplus 2(0, 3/2) ) |
| ( 2F + E + M ) | (1/2, 3/2) ( \oplus (0, 2) \oplus (0, 1) ) |
| ( 2F + E + 2M ) | (0, 3/2) |
| ( 2F + 2E - 3M ) | (1/2, 3/2) ( \oplus (0, 2) \oplus (0, 1) ) |
| ( 2F + 2E - 2M ) | (1, 5/2) ( \oplus (1, 3/2) \oplus 3(1/2, 2) \oplus (0, 5/2) \oplus 3(1/2, 1) \oplus 4(0, 3/2) \oplus 2(0, 1/2) ) |
| ( 2F + 2E - M ) | (3/2, 5/2) ( \oplus (1, 3) \oplus 3(1, 2) \oplus 3(1/2, 5/2) \oplus 8(1/2, 3/2) \oplus 6(0, 2) \oplus 2(1, 1) \oplus 7(0, 1) \oplus 3(1/2, 1/2) \oplus (0, 0) ) |
| ( 2F + 2E ) | (3/2, 3) ( \oplus (3/2, 2) \oplus 3(1, 5/2) \oplus (1/2, 3) \oplus 4(1, 3/2) \oplus 8(1/2, 2) \oplus 3(0, 5/2) \oplus 1(1/2) \oplus 8(1/2, 1) \oplus 10(0, 3/2) \oplus (1/2) \oplus 5(0, 1/2) ) |
| ( 2F + 2E + M ) | (3/2, 5/2) ( \oplus (1, 3) \oplus 3(1, 2) \oplus 3(1/2, 5/2) \oplus 8(1/2, 3/2) \oplus 6(0, 2) \oplus 2(1, 1) \oplus 7(0, 1) \oplus 3(1/2, 1/2) \oplus (0, 0) ) |
| ( 2F + 2E + 2M ) | (1, 5/2) ( \oplus (1, 3/2) \oplus 3(1/2, 2) \oplus (0, 5/2) \oplus 3(1/2, 1) \oplus 4(0, 3/2) \oplus 2(0, 1/2) ) |
| ( 2F + 2E + 3M ) | (1/2, 3/2) ( \oplus (0, 2) \oplus (0, 1) ) |</p>
<table>
<thead>
<tr>
<th>Expression</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2F + 3E - 4M$</td>
<td>$(1/2, 2) \oplus (1/2, 1) \oplus 2(0, 3/2)$</td>
</tr>
<tr>
<td>$2F + 3E - 3M$</td>
<td>$(3/2, 5/2) \oplus (1, 3) \oplus 3(1, 2) \oplus 3(1/2, 5/2) \oplus 8(1/2, 3/2) \oplus 6(0, 2) \oplus 2(1, 1) \oplus 7(0, 1) \oplus 3(1/2, 1/2) \oplus (0, 0)$</td>
</tr>
<tr>
<td>$2F + 3E - 2M$</td>
<td>$(2, 7/2) \oplus (2, 5/2) \oplus 3(3/2, 3) \oplus (1, 7/2) \oplus 4(3/2, 2) \oplus 10(1, 5/2) \oplus 4(1/2, 3) \oplus 3(3/2, 1) \oplus 12(1, 3/2) \oplus 20(1/2, 2) \oplus 7(0, 5/2) \oplus 5(1, 1/2) \oplus 20(1/2, 1) \oplus 24(0, 3/2) \oplus 4(1/2, 0) \oplus 11(0, 1/2)$</td>
</tr>
<tr>
<td>$2F + 3E - M$</td>
<td>$(5/2, 7/2) \oplus (2, 4) \oplus 3(2, 3) \oplus 3(3/2, 7/2) \oplus 11(3/2, 5/2) \oplus 11(1, 3) \oplus 2(1/2, 7/2) \oplus 2(2, 2) \oplus 26(1, 2) \oplus 23(1/2, 5/2) \oplus 4(0, 3) \oplus 7(3/2, 3/2) \oplus 2(3/2, 1/2) \oplus 46(1/2, 3/2) \oplus 31(0, 2) \oplus 19(1, 1) \oplus 4(1, 0) \oplus 36(0, 1) \oplus 23(1/2, 1/2) \oplus 9(0, 0)$</td>
</tr>
<tr>
<td>$2F + 3E$</td>
<td>$(5/2, 4) \oplus (5/2, 3) \oplus 3(2, 7/2) \oplus (3/2, 4) \oplus 4(2, 5/2) \oplus 10(3/2, 3) \oplus 4(1, 7/2) \oplus (2, 3/2) \oplus 14(3/2, 2) \oplus 26(1, 5/2) \oplus 12(1/2, 3) \oplus (0, 7/2) \oplus 6(3/2, 1) \oplus 31(1, 3/2) \oplus 48(1/2, 2) \oplus 18(0, 5/2) \oplus 3(3/2, 0) \oplus 14(1, 1/2) \oplus 48(1/2, 1) \oplus 50(0, 3/2) \oplus 12(1/2, 0) \oplus 26(0, 1/2)$</td>
</tr>
<tr>
<td>$2F + 3E + M$</td>
<td>$(5/2, 7/2) \oplus (2, 4) \oplus 3(2, 3) \oplus 3(3/2, 7/2) \oplus 11(3/2, 5/2) \oplus 11(1, 3) \oplus 2(1/2, 7/2) \oplus 2(2, 2) \oplus 26(1, 2) \oplus 23(1/2, 5/2) \oplus 4(0, 3) \oplus 7(3/2, 3/2) \oplus 2(3/2, 1/2) \oplus 46(1/2, 3/2) \oplus 31(0, 2) \oplus 19(1, 1) \oplus 4(1, 0) \oplus 36(0, 1) \oplus 23(1/2, 1/2) \oplus 9(0, 0)$</td>
</tr>
<tr>
<td>$2F + 3E + 2M$</td>
<td>$(2, 7/2) \oplus (2, 5/2) \oplus 3(3/2, 3) \oplus (1, 7/2) \oplus 4(3/2, 2) \oplus 10(1, 5/2) \oplus 4(1/2, 3) \oplus 3(3/2, 1) \oplus 12(1, 3/2) \oplus 20(1/2, 2) \oplus 7(0, 5/2) \oplus 5(1, 1/2) \oplus 20(1/2, 1) \oplus 24(0, 3/2) \oplus 4(1/2, 0) \oplus 11(0, 1/2)$</td>
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<td>$2F + 3E + 3M$</td>
<td>$(3/2, 5/2) \oplus (1, 3) \oplus 3(1, 2) \oplus 3(1/2, 5/2) \oplus 8(1/2, 3/2) \oplus 6(0, 2) \oplus 2(1, 1) \oplus 7(0, 1) \oplus 3(1/2, 1/2) \oplus (0, 0)$</td>
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</table>
Appendix C

Multiple sine and multiple elliptic gamma hierarchies

C.1 Multiple sine hierarchy

In this appendix we provide the definition and relevant properties of the multiple sine and multiple elliptic gamma functions [71–74]. We begin by defining the multiple zeta functions

\[
\zeta_r(z, s|\omega) = \sum_{n_1, \ldots, n_r=0}^{\infty} \left( \bar{n} \cdot \omega + z \right)^{-s},
\]

for \( z \in \mathbb{C} \) and \( \Re s > r \). We adopt the notation \( \omega = (\omega_1, \ldots, \omega_r) \) and \( \bar{n} \cdot \omega = n_1 \omega_1 + \cdots + n_r \omega_r \). We require that all \( \omega_i \in \mathbb{C} \) lie within the same half of the complex plane. By analytic continuation the domain of definition of multiple zeta functions can be extended to \( s \in \mathbb{C} \).

Multiple gamma functions are defined as

\[
\Gamma_r(z|\omega) = \exp \left( \frac{\partial}{\partial s} \zeta_r(s, z|\omega) \bigg|_{s=0} \right),
\]

which we can view as a regularized infinite product,

\[
\Gamma_r(z|\omega) \sim \prod_{n_1, \ldots, n_r=0}^{\infty} \left( \bar{n} \cdot \omega + z \right)^{-1}.
\]
Finally, the multiple sine is defined as
\[ S_r(z|\omega) = \Gamma_r(z|\omega)^{-1} \Gamma_r(|\omega| - z|\omega|)^{(-1)^r}, \]  \hspace{1cm} (C.1.1)
where \(|\omega| = \omega_1 + \cdots + \omega_r\). Multiple sine functions can also be written as regularized products,
\[ S_r(z|\omega) \sim \prod_{n_1, \ldots, n_r = 0}^{\infty} (\bar{n} \cdot \omega + |\omega| - z)(\bar{n} \cdot \omega + z)^{(-1)^{r+1}}, \]  \hspace{1cm} (C.1.2)
and enjoy a number of remarkable properties:

- **Analyticity:**
  
  For \( r \) odd the multiple sine is an entire function in \( z \), with zeros at
  \[ z = \bar{n} \cdot \omega \quad (n_1, \ldots, n_r \geq 1), \]
  coming from \( \Gamma_r(z|\omega)^{-1} \), as well as zeros at
  \[ z = \bar{n} \cdot \omega \quad (n_1, \ldots, n_r \leq 0), \]
  coming from \( \Gamma_r(|\omega| - z|\omega|)^{-1} \). For even \( r \), the multiple sine is meromorphic with zeros for (\( n_1, \ldots, n_r \geq 1 \)) and poles for (\( n_1, \ldots, n_r \leq 0 \));

- **Difference equation:**
  
  \[ S_r(x + \omega_i|\omega) = S_{r-1}(x|\omega(i))^{-1} S_r(x), \]  \hspace{1cm} (C.1.3)
  where \( \omega(i) = (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_r) \);

- **Symmetries:** \( S_r(z, \omega) \) is invariant under permutations of the parameters \( \omega_i \). It also enjoys a reflection property:
  \[ S_r(z|\omega) = S_r(|\omega| - z|\omega|)^{(-1)^{r+1}}; \]  \hspace{1cm} (C.1.4)

- **Rescaling invariance:**
  
  \[ S_r(cz|c\omega) = S_r(z|\omega), \]  \hspace{1cm} (C.1.5)
  for any \( c \in \mathbb{C}; \)
Appendix C: Multiple sine and multiple elliptic gamma hierarchies

- **Integral representation:** In [74] it was shown that, when $\text{Re } \omega_j > 0$ and $0 < \text{Re } z < |\omega|$, multiple sine functions can be expressed in terms of contour integrals. In particular, the double and triple sine functions have the following representation:

$$S_2(z|\omega_1, \omega_2) = \exp\left(\frac{\pi i}{2} B_{2,2}(z|\omega) + \int_{\mathcal{R}+i0} \frac{d\ell}{\ell} \frac{e^{z\ell}}{(e^{\omega_1\ell} - 1)(e^{\omega_2\ell} - 1)}\right), \quad (C.1.6)$$

$$S_3(z|\omega_1, \omega_2, \omega_3) = \exp\left(-\frac{\pi i}{6} B_{3,3}(z|\omega) - \int_{\mathcal{R}+i0} \frac{d\ell}{\ell} \frac{e^{z\ell}}{(e^{\omega_1\ell} - 1)(e^{\omega_2\ell} - 1)(e^{\omega_3\ell} - 1)}\right), \quad (C.1.7)$$

where

$$B_{2,2}(z|\omega_1, \omega_2) = \frac{z^2}{\omega_1\omega_2} - \frac{\omega_1 + \omega_2}{\omega_1\omega_2} z + \frac{\omega_1^2 + \omega_2^2 + 3\omega_1\omega_2}{6\omega_1\omega_2}, \quad (C.1.8)$$

$$B_{3,3}(z|\omega_1, \omega_2, \omega_3) = \frac{z^3}{\omega_1\omega_2\omega_3} - \frac{3}{2} \frac{\omega_1 + \omega_2 + \omega_3}{\omega_1\omega_2\omega_3} z^2$$

$$+ \frac{\omega_1^2 + \omega_2^2 + \omega_3^2 + 3(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)}{2\omega_1\omega_2\omega_3} z$$

$$- \frac{(\omega_1 + \omega_2 + \omega_3)(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)}{4\omega_1\omega_2\omega_3}, \quad (C.1.9)$$

are members of the family of multiple Bernoulli polynomials, which are defined as follows:

$$\sum_{n=0}^{\infty} B_{r,n}(z|\omega) \frac{t^n}{n!} = \frac{t^r e^{zt}}{\prod_{j=1}^{r} (e^{\omega_j t} - 1)}, \quad (C.1.10)$$

- **Factorization:** When $\text{Im } \omega_1/\omega_2 > 0$, the double sine function can be written as the following infinite product [74]:

$$S_2(z|\omega_1, \omega_2) = \exp\left(\frac{\pi i}{2} B_{2,2}(z|\omega_1, \omega_2)\right) \cdot \frac{\prod_{j=0}^{\infty} (1 - e^{2\pi i(z/\omega_2 + j\omega_1/\omega_2)})}{\prod_{j=0}^{\infty} (1 - e^{2\pi i(z/\omega_1 + (j+1)\omega_2/\omega_1)})}. \quad (C.1.11)$$

Similarly, when $\text{Im } \omega_1/\omega_2 > 0$, $\text{Im } \omega_1/\omega_3 > 0$, and $\text{Im } \omega_3/\omega_2 > 0$, the triple sine factorizes as

$$S_3(z|\omega_1, \omega_2, \omega_3) = \exp\left(-\frac{\pi i}{6} B_{3,3}(z|\omega_1, \omega_2, \omega_3)\right) \prod_{j,k=0}^{\infty} (1 - e^{2\pi i(z/\omega_3 + j\omega_1/\omega_3 + k\omega_2/\omega_3)}) \cdot \frac{\prod_{j,k=0}^{\infty} (1 - e^{2\pi i(z/\omega_1 + (j+1)\omega_2/\omega_1) - (k+1)\omega_3/\omega_3})}{\prod_{j,k=0}^{\infty} (1 - e^{2\pi i(z/\omega_3 + j\omega_1/\omega_3 + (k+1)\omega_2/\omega_2)})}. \quad (C.1.12)$$

Similar expressions can be obtained for other regions by using the invariance of the triple sine function under exchange of $\omega_1, \omega_2, \omega_3$.  

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Appendix C: Multiple sine and multiple elliptic gamma hierarchies

C.2 Multiple elliptic gamma hierarchy

When \( \omega_j \in \mathbb{H}, j = 0, \ldots, r \), the \( r \)-th multiple elliptic gamma function is defined as

\[
G_r(z|\omega) = \prod_{j_0, \ldots, j_r = 0}^{\infty} (1 - e^{2\pi i(z+j_0\omega_0+\cdots+j_r\omega_r)})^{(-1)^r} \cdot (1 - e^{2\pi i|\omega|-z+j_0\omega_0+\cdots+j_r\omega_r}).
\] (C.2.13)

One can extend the definition to \( \omega_j \in \mathbb{C} - \mathbb{R} \) by repeated use of

\[
\prod_{p=0}^{\infty} (1 - X e^{2\pi i\omega_j}) = \prod_{p=0}^{\infty} (1 - X e^{-2\pi i(p+1)\omega_j})^{-1}.
\]

The multiple elliptic gamma function is related to the multiple sine function by the following identity, (which was proved in [74] if \( \text{Im} \omega_j > 0 \) for all \( j \), and \( 0 < \text{Im} z < \text{Im} |\omega| \)):

\[
G_r(z|\omega) = \exp\left(\frac{2\pi i}{(r+2)!} B_{r+2,r+2}(z|(-\omega,-1))\right) \cdot \prod_{k=0}^{\infty} \exp\left(\frac{\pi i}{(r+1)!}(B_{r+1,r+1}(z+k+1|\omega) - B_{r+1,r+1}(z-k|\omega))\right).
\] (C.2.14)

These functions have nice modular properties [74]. For example, if \( \text{Im} \tau_j \neq 0 \) and \( \text{Im} \tau_j/\tau_j \neq 0 \),

\[
G_2(z|\tau_0, \tau_1, \tau_2) = \exp\left(\frac{\pi i}{12} B_{44}(z|\tau_0, \tau_1, \tau_2, 1)\right) G_2\left(\frac{z}{\tau_0}, \frac{z}{\tau_1}, \frac{z}{\tau_2}\right)
\]

\[
G_2\left(\frac{z}{\tau_0}, \frac{z}{\tau_1}, \frac{z}{\tau_2}\right) \cdot G_2\left(\frac{z}{\tau_0}, \frac{z}{\tau_1}, \frac{z}{\tau_2}\right).
\] (C.2.15)

Similar formulas exist for \( r \neq 2 \). Multiple elliptic gamma functions also satisfy recursion relations, including

\[
G_r(z+1|\tau_0, \ldots, \tau_r) = G_r(z|\tau_0, \ldots, \tau_r),
\]

and

\[
G_r(z+\tau_i|\tau_0, \ldots, \tau_r) = 1/G_r(z|\tau_0, \ldots, \tau_{i-1}, -\tau_i, \tau_{i+1}, \ldots, \tau_r).
\]

Furthermore, the infinite product representation of the multiple elliptic gamma function can written in the form of a plethystic exponential. The first factor of equation (C.2.13) can be written as

\[
\exp\left((-1)^r \sum_{j_0, \ldots, j_r = 0}^{\infty} \log(1 - e^{2\pi i(z+j_0\omega_0+\cdots+j_r\omega_r)}) \right)
\]

\[
= \exp\left((-1)^{r+1} \sum_{j_0, \ldots, j_r = 0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i n(z+j_0\omega_0+\cdots+j_r\omega_r)}}{n} \right).
\]
Resumming the geometric series corresponding to $j_0, \ldots, j_r$, we get
\[
\exp \left( (-1)^{r+1} \sum_{n=1}^{\infty} \frac{I_r(q^n_1|q^n_0, \ldots, q^n_r)}{n} \right),
\]
where we defined $q_i = e^{2\pi i \omega_i}$ for $i = 0, \ldots, r$, $q_z = e^{2\pi i z}$, and
\[
I_r(q_z|q_0, \ldots, q_r) = \frac{q_z}{\prod_{i=0}^{r} (1 - q_i)}.
\]
The other infinite product in equation (C.2.13) contributes a similar term, and we find that
\[
G_r(z|\omega) = \exp \left( \sum_{n=0}^{\infty} \frac{(-1)^{r+1} I_r(q^n_z|q^n_0, \ldots, q^n_r) - I_r(q_z^n : \prod_{j=0}^{r} q^n_j|q^n_0, \ldots, q^n_r)}{n} \right).
\]
Multiple elliptic gamma functions enjoy a number of other notable properties; we refer the reader to [74] for further details.

### C.3 Triple sine formulas for hyper and vector multiplets

In this appendix we recast the one-loop hyper and vector multiplet contributions to the 5d partition function on unsquashed $S^5$ as computed in [156] in terms of triple sine functions.

### C.4 Hypermultiplets

We wish to show that the one-loop partition function
\[
Z_{\text{hyper}} = \prod_{\mu \in R} \prod_{t} (t + 3/2 - i \phi_\mu)^{-\left(1 + \frac{3}{2} t + \frac{1}{2} t^2\right)},
\]
for a hypermultiplet in the representation $R$ of the gauge group, whose weights we denote by $\mu$, is equal to
\[
\prod_{\mu} S_3(i \phi_\mu + 3/2|1,1,1)^{-1}.
\]
From the definition of triple sine, we have
\[
S_3(z|1,1,1) = \prod_{n_1,n_2,n_3 \geq 0} (n_1 + n_2 + n_3 + z)(n_1 + n_2 + n_3 + 3 - z),
\]
which can be expressed as a sum over a single integer

\[ S_3(z|1,1,1) = \prod_{t \geq 0} [(t + z)(t + 3 - z)]^{t^2/2 + 3t/2 + 1}. \]

For each weight in the representation we have

\[ S_3(i \phi_\mu + 3/2|1,1,1) = \prod_{t \geq 0} (t + 3/2 + i \phi_\mu)^{t^2/2 + 3t/2 + 1} (t + 3/2 - i \phi_\mu)^{t^2/2 + 3t/2 + 1}. \]

By taking \( t \to -t \) in the first factor, we can rewrite it as

\[ \prod_{t \leq 0} (-t + 3/2 + i \phi_\mu)^{t^2/2 - 3t/2 + 1} = \prod_{t \leq -3} (-t - 3/2 + i \phi_\mu)^{(t+3)^2/2 - 3t/2 + 1} \]

\[ = \prod_{t \leq -3} (t + 3/2 - i \phi_\mu)^{(t^2/2 + 3t/2 + 1)}, \]

up to a numerical phase. Here and in the following we will be cavalier about such numerical factors.

Putting everything together, we have

\[ S_3(i \phi_\mu + 3/2|1,1,1) = \prod_{t \geq 0} (t + 3/2 - i \phi_\mu)^{t^2/2 + 3t/2 + 1} \prod_{t \leq -3} (t + 3/2 - i \phi_\mu)^{(t^2/2 + 3t/2 + 1)} \]

\[ = \prod_{t \in \mathbb{Z}} (t + 3/2 - i \phi_\mu)^{t^2/2 + 3t/2 + 1} \]

Notice that when \( t = \{-1, -2\} \) the exponent \( t^2/2 + 3t/2 + 1 \) vanishes. So in fact we can write

\[ S_3(i \phi_\mu + 3/2|1,1,1) = \prod_{t \in \mathbb{Z}} (t + 3/2 - i \phi_\mu)^{t^2/2 + 3t/2 + 1}, \]

and indeed we find that

\[ Z_{\text{hyper}} = \prod_{\mu} S_3(i \phi_\mu + 3/2|1,1,1)^{-1}. \]

### C.5 Vector multiplets

We wish to show that the one-loop contribution from the vector multiplets,

\[ \left( \prod_{\beta > 0} (i \phi_\beta)^2 \right) \times Z_{\text{vect}} = \prod_{\beta > 0} \left( (i \phi_\beta)^2 \prod_{t \neq 0} (t^2 - (i \phi_\beta)^2)^{t^2/2 + 3t/2 + 1} \right) \]

\[ = \prod_{\beta > 0} \prod_{t \in \mathbb{Z}} [(t + i \phi_\beta)(t - i \phi_\beta)]^{t^2/2 + 3t/2 + 1} \]
Appendix C: Multiple sine and multiple elliptic gamma hierarchies

is equal to

$$\prod_{\beta>0} S_3(i\phi_\beta|1,1,1)S_3(3+i\phi_\beta|1,1,1).$$

To see this we simply shift $i\phi_\beta$ by $\frac{3}{2}$ in (C.4.18) to get

$$S_3(i\phi_\beta + 3|1,1,1) = \prod_{t \in \mathbb{Z}} (t - i\phi_\beta)^{t^2/2 + 3t/2 + 1}.$$ 

To get the other half of the answer we use

$$S_3(i\phi_\beta|1,1,1) = S_3(-i\phi_\beta + 3|1,1,1) = \prod_{t \in \mathbb{Z}} (t + i\phi_\beta)^{t^2/2 + 3t/2 + 1},$$

so that indeed

$$\prod_{\beta>0} S_3(i\phi_\beta|1,1,1)S_3(i\phi_\beta + 3|1,1,1) = \prod_{\beta>0} \prod_{t \in \mathbb{Z}} (t^2 - (i\phi_\beta)^2)^{t^2/2 + 3t/2 + 1}. $$

C.6 Zeros and poles of $C_{s_1,s_2}(z|\tau_1,\tau_2)$

In the main text we defined the following generalization to the triple sine function:

$$C_{s_1,s_2}(z|\tau_1,\tau_2) = \prod_{j,k=0}^{\infty} \frac{(1 - (-1)^{2s_1+1} \varepsilon^{2\pi iz/\tau_1} q^j - s_1 + 1/2 \varepsilon^{k-s_2+1/2})}{\prod_{j,k=0}^{\infty} (1 - (-1)^{2s_1+1} \varepsilon^{2\pi iz} q^j + s_1 + 1/2 \varepsilon^{k+s_2+1/2})}.$$  \hspace{1cm} (C.6.19)

We would like to express this in a form analogous to the definition of the triple sine function, equation (C.1.2). Assuming that this function has similar analytic properties to the triple sine function, we can read off the zeros $\alpha_i$ and poles $\beta_j$ of this function from its definition and express it as a regularized infinite product,

$$C_{s_1,s_2}(z|\tau_1,\tau_2) \sim \prod_i (z - \alpha_i) \prod_j (z - \beta_j),$$

which is valid up to an exponential prefactor. In particular, from the denominator of (C.6.19) we get

$$\prod_{j,k=0}^{\infty} (1 - (-1)^{2s_1+1} \varepsilon^{2\pi iz} q^j + s_1 + 1/2 \varepsilon^{k+s_2+1/2})$$
$\sim \prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (z + \tau_1(j + s_1 + 1/2) - \tau_2(k - s_2 + 1/2) + p + s_1 + 1/2)$

$= \prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (\xi + \tau_1j - \tau_2(k + 1) + p)$, \hspace{1cm} (C.6.20)

where

$\xi = z + \tau_1(s_1 + 1/2) + \tau_2(s_2 + 1/2) + (s_1 + 1/2))$.

Similarly, the numerator or (C.6.19) contributes a factor of

$\prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (1 - (-1)^{2s_1+1} e^{2\pi iz/\tau_1} \hat{q}^{-s_1+1/2} \hat{p}^{-s_2+1/2})$

$\sim \prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (z + \tau_1(p + s_1 + 1/2) - \tau_2(k - s_2 + 1/2) - (j - s_1 + 1/2))$

$= \prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (\xi + \tau_1p - \tau_2(k + 1) - (j + 1))$, \hspace{1cm} (C.6.21)

as well as a factor of

$\prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (1 - (-1)^{2s_1+1} e^{2\pi iz/\tau_2} \hat{q}^{s_1+1/2} \hat{p}^{s_2+1/2})$

$\sim \prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (z + \tau_1(j + s_1 + 1/2) + \tau_2(p + s_1 + 1/2) + k + s_2 + 1/2)$

$= \prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (z + \tau_1(j + s_2 + 1/2) + \tau_2(p + s_2 + 1/2) + k + s_2 + 1/2)\cdot \prod_{j,k=0}^{\infty} \prod_{p=-\infty}^{\infty} (\xi + \tau_1j + \tau_2p + k) = F_{s_1,s_2}$, \hspace{1cm} (C.6.22)

where in going from the second to the third line we used the fact that $s_1 = s_2 \mod 1$, and in the last line

$$F_{s_1,s_2} = \begin{cases} 
1 & \text{if } s_1 = s_2 \\
\prod_{j=0}^{\infty} \prod_{p=-\infty}^{\infty} \prod_{l=1}^{s_1-s_2} (\xi + \tau_1j + \tau_2p - l) & \text{if } s_1 > s_2 \\
\prod_{j=0}^{\infty} \prod_{p=-\infty}^{\infty} \prod_{l=s_1-s_2+1}^{0} (\xi + \tau_1j + \tau_2p - l)^{-1} & \text{if } s_1 < s_2
\end{cases}$$
Dividing equation (C.6.21) by (C.6.20) gives

\[
\frac{\prod_{j,k,p=0}^{\infty}(\xi - \tau_1 (p+1) - \tau_2 (k+1) - (j+1))}{\prod_{j,k,p=0}^{\infty}(\xi + \tau_1 j - \tau_2 (k+1) + p)}; \tag{C.6.23}
\]

further multiplying by factor (C.6.22) gives

\[
C_{s_1,s_2}(z|\tau_1,\tau_2) \sim F_{s_1,s_2} \cdot \prod_{m,n,p=0}^{\infty} (\xi + \tau_1 m + \tau_2 n + p)(\xi - \tau_1 (p+1) - \tau_2 (n+1) - (m+1)); \tag{C.6.24}
\]

in other words,

\[
C_{s_1,s_2}(z|\tau_1,\tau_2) \sim S_3(\xi,1,1,\tau_2) \cdot F_{s_1,s_2}.
\]
Bibliography


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