Structures on Forms of K-Theory

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Abstract

In the early 1970s, Morava studied forms of topological K-theory and observed that they have interesting number theoretic connections. Until very recently, forms of K-theory have not been studied in greater depth and integrated into the modern theory of topological modular forms. In this dissertation, some expected structured ring spectra and locality results are established on forms of K-theory. Forms of algebraic structures are usually classified by Galois cohomology. Based on the structured ring spectra and locality results established, a criterion is given for distinguishing homotopy equivalence classes of forms of K-theory via a computation in the second homotopy group of the spectrum.

Contents

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1 Introduction

1.1 Historical Background

A principal open problem in algebraic topology is to uncover patterns in and compute the stable homotopy groups of spheres. The contemporary approach of chromatic homotopy theory seeks to capture periodic phenomena in layers enumerated by a non-negative integer n known as the height, analogous to the constituent frequencies of light. Complex K-theory is the exemplar of a height one cohomology theory, or equivalently, its representing object, a spectrum. Its p-adic completion K_p lies at the bottom of a family of spectra known as Morava E-theory E_n: K_p is a first Morava E-theory E_1. Morava E-theory E_n is central to the study of the nth monochromatic layer, and the fundamental work of Goerss, Hopkins, and Miller [3] establishes it as a highly structured commutative, or E_∞-, ring spectrum, which enables various sophisticated constructions.

In 1973, Morava, motivated by classical questions about integrality properties of characteristic classes of smooth manifolds, studied forms of p-complete K-theory cohomology $K_p^*(-)$ [8]. Generally speaking, a form of an algebraic structure over a base is an object of the same type such that the two objects become isomorphic after extension of their common base:

Definition 1.1. Given a category $\mathcal{C}$ with pullbacks, an object S in $\mathcal{C}$, and an object $S' \to S$ in the overcategory $\mathcal{C}_{/S}$, an *S'/S-form of an object $X \to S$* in $\mathcal{C}_{/S}$ is another object $Y \to S$ in $\mathcal{C}_{/S}$ such that the pullbacks $X \times_S S' \to S'$ and $Y \times_S S' \to S'$ are isomorphic.

The analogous definition with pushouts instead of pullbacks can be made in the opposite category $\mathcal{C}^{\text{op}}$. We shall work in the latter context.

The concept of forms occurs in many algebraic areas of mathematics as it is related to descent theory.

Definition 1.2. A $W(\overline{\mathbf{F}_p})/\mathbf{Z}_p$-*form of p-complete K-theory cohomology $K_p^*(-)$ is a* multiplicative cohomology theory $R^*(-)$ taking values in the category of $\mathbf{Z}_p$-modules, with the property that

$$K_p^*(-) \otimes_{\mathbf{Z}_p} W(\overline{\mathbf{F}_p}) \cong R^*(-) \otimes_{\mathbf{Z}_p} W(\overline{\mathbf{F}_p})$$

as multiplicative cohomology theories (although not canonically).

Here $W(\overline{\mathbf{F}_p})$ is the ring of p-typical Witt vectors of $\overline{\mathbf{F}_p}$, obtained by adjoining to $\mathbf{Z}_p$ all primitive m^{th} roots of unity, where m is coprime to p, and taking its p-completion. Recall that $\mathbf{Z}_p = W(\mathbf{F}_p)$. Morava showed:

Theorem 1.3 (Morava [8]). *There is a bijection*

$$\{\text{isomorphism classes of } W(\overline{\mathbf{F}_p})/\mathbf{Z}_p\text{-forms of } K_p^*(-)\} \xrightarrow{\cong} \mathbf{Z}_p^\times.$$

He observed that forms of K-theory have interesting number theoretic connections, for example to the Ramanujan tau function or the quartic residue character.

Example 1.4 (Morava [8]). The Ramanujan tau function is the multiplicative function $\tau : \mathbf{N} \to \mathbf{Z}$ defined by the generating function

$$\sum_{n \geq 1} \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24} = \Delta(z),$$

where $\Delta(z)$ is the discriminant modular form and $q = \exp(2\pi i z)$, $\Im z > 0$.

There is a $W(\overline{\mathbf{F}_{11}})/\mathbf{Z}_{11}$-form $R^*(-)$ of $K_{11}^*(-)$ whose associated topological index

$$t\text{-ind}\, R^* : MU_* \to \mathbf{Z} \subset \mathbf{Z}_{(11)}$$

satisfies

$$t\text{-ind}\, R^*(\mathbf{CP}(n-1)) = \tau(n) \text{ and } t\text{-ind}\, R^*(M \times N) = t\text{-ind}\, R^*(M) \cdot t\text{-ind}\, R^*(N).$$

Nowadays, there is a theory that makes Example 1.4 conceivable, if no less surprising: the theory of topological modular forms, which is a universal object mapping to elliptic cohomology theories—even periodic multiplicative cohomology theories $E^*(-)$ whose formal group law is isomorphic to the formal group law of a elliptic curve over $E^0(*)$—and whose coefficient ring is closely related to the classical ring of modular forms. The theory of topological modular forms is intricately connected to the height two layer in chromatic homotopy theory. From a modern perspective, the $W(\overline{\mathbf{F}_{11}})/\mathbf{Z}_{11}$-form $R^*(-)$ of $K_{11}^*(-)$ in Example 1.4 is an elliptic cohomology theory for the modular group $\Gamma_0(11)$ of level 11.

Morava's results were proven when E_∞-ring spectra, which enable a strengthening of his results, were in their infancy, and before the advent of elliptic cohomology. Forms of K-theory were not studied in greater depth until 2009, when they began to be studied by Behrens and Lawson [2], Lawson and Naumann [6], and Hill and Lawson [4] in connection with the modern theories of topological modular and automorphic forms.

In the present dissertation, I study forms of K-theory in the contemporary framework of structured ring spectra (in a suitable Bousfield localized category)—specifically forms of the p-adically complete K-theory spectrum K_p—in an attempt to bring new

insights to the chromatic picture.

1.2 Modern Framework

In the years since Morava proved his results, there have been developments in homotopy theory that enable one to formulate a strengthening of Morava's results. In this subsection, we briefly recall three key components of this framework: E_∞-ring spectra, Morava K-theory $K(n)$, and a power operation ψ for $K(1)$-local E_∞-ring spectra which we shall subsequently require.

E_∞-ring spectra can be motivated as follows. Recall that ring spectra are ring objects when regarded as objects in the homotopy category of spectra. For sophisticated applications such as the construction of homotopy fixed points under a group action or quotient spectra, however, one does not wish to forget all information when passing to the homotopy category of spectra, nor is it feasible to restrict consideration to ring objects in the original category. A compromise suitable for describing the multiplicative structure on K-theory and other spectra of interest is to not only require the spectrum to be a ring spectrum, but simultaneously remember the homotopies that make the relevant diagrams commute and higher homotopies between these homotopies; these are known as coherence conditions. This notion is precisely defined using the notion of an algebra over the E_∞-operad [7], and spectra with this data are called E_∞-ring spectra.

It is common to study the category of ring spectra via its fields, namely the ring spectra E such that every module spectrum over E is free, that is, it is a wedge of suspensions of E. The fields in the category of ring spectra form a family known as Morava K-theory $K(n)$, which correspond to a height n cohomology theory for each non-negative integer n and a fixed prime p. Given $K(n)$, we can consider the operation of Bousfield localization with respect to it. It turns out that $K(n)$-localization has several nice properties: for instance, it does not affect the nth monochromatic layer,

and results about the $K(n)$-local category for different n and p can be combined to give information about the stable homotopy category. The first Morava K-theory $K(1)$ is closely related to the p-complete K-theory spectrum K_p as follows. The group $\mathbf{Z}_p^\times$ of p-adic units acts on K_p via the Adams operations ψ^λ, $\lambda \in \mathbf{Z}_p^\times$. For $p \geq 3$, the maximal finite subgroup μ of $\mathbf{Z}_p^\times$ is the group of $(p-1)^{\text{th}}$ roots of unity, and $K(1)$ can be obtained as K_p^{Ad}/p, where K_p^{Ad} is the Adams summand of K_p, its homotopy fixed points by the action of μ. In general, K_p/p is a wedge of $p-1$ copies of suspensions of $K(1)$.

A $K(1)$-local E_∞-ring spectrum R possesses operations

$$\theta, \psi : \pi_0(R) \to \pi_0(R)$$

such that ψ is a ring homomorphism and θ imparts the structure of a θ-algebra on $\pi_0(R)$: $\psi(x) = x^p + p\theta(x)$. We sketch the construction of these two operations here, following [5].

Let $\mathfrak{S}_p$ be the symmetric group on p elements, let $B\mathfrak{S}_{p+}$ denote the unreduced suspension spectrum of its classifying space, and consider its $K(1)$-localization $L_{K(1)}B\mathfrak{S}_{p+}$. There are two natural maps

$$L_{K(1)}B\mathfrak{S}_{p+} \to L_{K(1)}S^0,$$

where S^0 is the sphere spectrum. One map arises from the projection to a point $\mathfrak{S}_p \to \{e\}$; we shall denote it by ϵ. The other map comes from the transfer map; we shall denote it by Tr. One can show that the map

$$L_{K(1)}B\mathfrak{S}_{p+} \xrightarrow{(\epsilon, \text{Tr})} L_{K(1)}S^0 \times L_{K(1)}S^0$$

is a weak equivalence.

Define maps

$$\psi', \theta' : L_{K(1)}S^0 \to L_{K(1)}B\mathfrak{S}_{p+}$$

in the $K(1)$-local stable homotopy category by requiring that

$$\epsilon(\psi') = 1, \qquad\qquad \epsilon(\theta') = 0,$$

$$\mathrm{Tr}(\psi') = 0, \qquad\qquad \mathrm{Tr}(\theta') = -(p-1)! \ .$$

The inclusion $\{e\} \to \mathfrak{S}_p$ gives rise to a map

$$e : L_{K(1)}S^0 \simeq L_{K(1)}B\{e\}_+ \to L_{K(1)}B\mathfrak{S}_{p+}$$

and $\epsilon \circ e = 1$. Moreover, from the Mackey double coset formula, $\mathrm{Tr} \circ e = p!$. It follows that

$$e = \psi' - p\theta'. \tag{1.1}$$

Let R be a $K(1)$-local E_∞-ring spectrum and let $x \in \pi_0(R)$. The E_∞-ring structure associates to x a map

$$P(x) : L_{K(1)}B\mathfrak{S}_{p+} \to R$$

such that the homotopy class of $P(x) \circ e$ (precomposed by the localization $S^0 \to L_{K(1)}S^0$) is x^p. One defines operations

$$\psi, \theta : \pi_0(R) \to \pi_0(R)$$

by taking $\psi(x)$ and $\theta(x)$ to be the homotopy classes of $P(x) \circ \psi'$ and $P(x) \circ \theta'$ respectively. The equation (1.1) then gives us the relation $x^p = \psi(x) - p\theta(x)$.

In Subsection 2.2, we shall use ψ on a related spectrum to extend ψ to a map $\pi_n(R) \to \pi_n(R)$ for all $n \in \mathbf{N}$.

1.3 Statement of the Main Results

In terms of modern concepts, forms of the p-complete K-theory spectrum K_p are defined as follows. One works in the closed symmetric monoidal category of p-complete ring spectra. Denote by S_p^0 the p-complete sphere spectrum, the unit of this category, and by $S_{p\infty}^0$ the unique spectrum satisfying $\pi_*(S_{p\infty}^0) \cong \pi_*(S_p^0) \otimes_{\mathbf{Z}_p} W(\overline{\mathbf{F}_p})$ on the level of homotopy.

Definition 1.5. An $S_{p\infty}^0/S_p^0$-*form of* K_p is a p-complete ring spectrum R such that

$$(K_p \wedge_{S_p^0} S_{p\infty}^0)_p^\wedge \simeq (R \wedge_{S_p^0} S_{p\infty}^0)_p^\wedge.$$

Note that this definition is dual to that given in Definition 1.1.

The spectrum K_p has a unique E_∞-ring structure (up to contractible choice) [3] and is $K(1)$-local. In this dissertation, I demonstrate that $S_{p\infty}^0/S_p^0$-forms of K_p satisfy the same results:

Theorem 1.6. *The ring structure on every* $S_{p\infty}^0/S_p^0$-*form of* K_p *has a unique refinement to an* E_∞-*ring structure.*

Theorem 1.7. *Every* $S_{p\infty}^0/S_p^0$-*form of* K_p *is* $K(1)$-*local.*

In practice, forms of K_p which arise from topological modular forms often come equipped with an E_∞-ring structure as well as the data $\psi : \pi_2(R) \to \pi_2(R)$, where ψ is the map mentioned in Subsection 1.2, in terms of Hecke operators on modular forms. It is thus desirable to have a method to identify the form of K_p encountered. I further provide a method to distinguish homotopy equivalence classes of E_∞-$S_{p\infty}^0/S_p^0$-forms of K_p via a direct computation involving the operation ψ in the second homotopy group of the spectrum:

Theorem 1.8. *Let R be an E_∞-$S^0_{p^\infty}/S^0_p$-form of K_p. The operation $\psi : \pi_2(R) \to \pi_2(R)$ satisfies $\psi(x) = \varepsilon p x$ for some $\varepsilon \in \mathbf{Z}_p^\times$, and the value of the constant ε determines a bijection between homotopy equivalence classes of E_∞-$S^0_{p^\infty}/S^0_p$-forms of K_p and $\mathbf{Z}_p^\times$.*

2 Main Results

2.1 Preliminaries

Before proving the main results, we recall the theory involving forms of an algebraic structure.

There is the following general principle:

Principle 2.1 ([12, Chapter III, §1])**.** Let K/k be a field extension and let X be an object defined over k. If K/k is a Galois extension, then there is an injective map

$$\{k\text{-isomorphism classes of } K/k\text{-forms of } X\} \hookrightarrow H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(X)),$$

where the Galois group acts by conjugation, which is a bijection in many cases (e.g. when X is a vector space with a (p,q)-tensor or a quasiprojective variety).

Moreover, in such a situation, the K/k-forms of X can be recovered from $X \times_k K$ by taking the fixed points of or quotient by the action of the Galois group $\mathrm{Gal}(K/k)$ for different actions of $\mathrm{Gal}(K/k)$ on $X \times_k K$.

In the case of $S^0_{p^\infty}/S^0_p$-forms of K_p, the situation is as follows [2]:

There is an action of $\mathrm{Aut}(W(\overline{\mathbf{F}_p})/\mathbf{Z}_p) \cong \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$ on $(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$ which acts in the expected manner on homotopy groups. K_p can be recovered as the homotopy fixed points of the action of the Frobenius σ which topologically generates $\mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$: there is a homotopy pullback square of ring spectra

$$
\begin{array}{ccc}
K_p & \longrightarrow & (K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p \\
\downarrow & & \downarrow {\scriptstyle (\mathrm{id},\sigma)} \\
(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p & \xrightarrow{\Delta} & (K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p \times (K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p
\end{array}
$$

where Δ is the diagonal map.

By Goerss-Hopkins-Miller theory [3], the spaces of E_∞-ring automorphisms $\mathrm{Aut}_{E_\infty}(K_p)$ and $\mathrm{Aut}_{E_\infty}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)$ are homotopy discrete and

$$\pi_0(\mathrm{Aut}_{E_\infty}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)) \cong \pi_0(\mathrm{Aut}_{\mathrm{mult}}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)),$$

where $\mathrm{Aut}_{\mathrm{mult}}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)$ denotes the space of multiplicative automorphisms $f : (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge \to (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge$, that is, automorphisms such that the diagram

$$
\begin{array}{ccc}
((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge \wedge_{S_p^0} (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)_p^\wedge & \longrightarrow & (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge \\
{\scriptstyle f \wedge f} \downarrow & & \downarrow {\scriptstyle f} \\
((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge \wedge_{S_p^0} (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)_p^\wedge & \longrightarrow & (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge
\end{array}
$$

commutes up to homotopy. These multiplicative automorphisms are in turn in bijection with

$$\mathrm{Hom}_{(K_p \wedge_{S_p^0} S_{p^\infty}^0)_{p\,*}^\wedge}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_{p\,*}^\wedge((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge), (K_p \wedge_{S_p^0} S_{p^\infty}^0)_{p\,*}^\wedge).$$

One can show that

$$\mathrm{Ext}_{(K_p \wedge_{S_p^0} S_{p^\infty}^0)_{p\,*}^\wedge}^s((K_p \wedge_{S_p^0} S_{p^\infty}^0)_{p\,*}^\wedge((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge), (K_p \wedge_{S_p^0} S_{p^\infty}^0)_{p\,*}^\wedge)$$

vanishes for $s > 0$, so that the universal coefficient theorem implies that

$$\pi_0(\mathrm{Aut}_{E_\infty}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)) \cong ((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^{\wedge *}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge))^\times \cong \mathbf{Z}_p^\times \times \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p).$$

On the other hand, there is a short exact sequence

$$0 \to \pi_0(\mathrm{Aut}_{E_\infty}(K_p)) \to \pi_0(\mathrm{Aut}_{E_\infty}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)) \to \pi_0(\mathrm{Aut}_{E_\infty/K_p}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)) \to 0$$

where $\mathrm{Aut}_{E_\infty/K_p}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)$ are the E_∞-ring automorphisms of $(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge$ which fix K_p. It is known that $\pi_0(\mathrm{Aut}_{E_\infty}(K_p))$ is the Morava stabilizer group $\mathbf{S}_1 \cong \mathbf{Z}_p^\times$ whose elements are Adams operations ψ^λ, where $\lambda \in \mathbf{Z}_p^\times$. It follows that the short exact sequence splits, so there is an isomorphism

$$\pi_0(\mathrm{Aut}_{E_\infty}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)) \cong \pi_0(\mathrm{Aut}_{E_\infty}(K_p)) \times \pi_0(\mathrm{Aut}_{E_\infty/K_p}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge))$$

and

$$\pi_0(\mathrm{Aut}_{E_\infty/K_p}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)) \cong \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p).$$

Moreover, the action of the Frobenius is by an E_∞-ring map.

Denote the subgroup $\pi_0(\mathrm{Aut}_{E_\infty/K_p}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)) \subset \pi_0(\mathrm{Aut}_{E_\infty}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge))$ by Gal_{K_p}. We shall write $((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)^{h\,\mathrm{Gal}_{K_p}}$ to denote the above homotopy pullback and call it a homotopy fixed point spectrum.

In general, the different $S_{p^\infty}^0/S_p^0$-forms R of K_p can be recovered up to homotopy as homotopy pullbacks for twisted versions of σ, where one replaces σ by $\sigma_R = \sigma \circ \psi^\lambda$ for some choice of Adams operation $\psi^\lambda \in \pi_0(\mathrm{Aut}_{E_\infty}(K_p))$:

Lemma 2.2 (Behrens-Lawson [2]). *There is a homotopy pullback square*

$$
\begin{array}{ccc}
R & \longrightarrow & (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge \\
\downarrow & & \downarrow {\scriptstyle (\mathrm{id},\sigma \circ \psi^\lambda)=(\mathrm{id},\sigma_R)} \\
(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge & \overset{\Delta}{\longrightarrow} & (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge \times (K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge
\end{array}
$$

Moreover, the action of $\sigma_R = \sigma \circ \psi^\lambda$ is by an E_∞-ring map.

Each σ_R topologically generates a subgroup $\mathrm{Gal}_R \subset \pi_0(\mathrm{Aut}_{E_\infty}((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge))$ which is isomorphic to $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, mirroring the situation in Principle 2.1. As before, we shall write $((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)^{h\,\mathrm{Gal}_R}$ to denote the homotopy pullback defined in this manner and call it a homotopy fixed point spectrum.

2.2 Proofs of the Main Results

We now proceed to prove the results mentioned in Subsection 1.3. For the convenience of the reader, we recall the results here.

Theorem 2.3. *The ring structure on every $S_{p^\infty}^0 / S_p^0$-form of K_p has a unique refinement to an E_∞-ring structure.*

The proof uses the following result.

Definition 2.4. An E_∞-R-algebra R' is *étale* if

(a) the map $\pi_0(R) \to \pi_0(R')$ is étale in the sense of ordinary commutative algebra, and

(b) the natural map $\pi_0(R') \otimes_{\pi_0(R)} \pi_*(R) \to \pi_*(R')$ is an isomorphism.

Theorem 2.5 (Lurie, *Higher Algebra*, Theorem 8.5.4.2)**.** *Let R be an E_∞-ring spectrum. The ∞-category of étale E_∞-R-algebras is equivalent under π_0 to the nerve of the ordinary category of étale $\pi_0(R)$-algebras.*

Proof of Theorem 2.3. It is known that K_p has a unique E_∞-ring structure. [3] First, we note that $(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge$ is an étale E_∞-K_p-algebra. $(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge$ inherits an E_∞-ring structure from the algebra map and the map

$$\mathbf{Z}_p = \pi_0(K_p) \to \pi_0((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge) = W(\overline{\mathbf{F}_p})$$

is étale by the Jacobian criterion since $W(\mathbf{F}_{p^n})$ is an unramified extension of $\mathbf{Z}_p$ for each $n \in \mathbf{N}$. By Theorem 2.5, $(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge$ has a unique E_∞-ring structure refining its ring structure because of the uniqueness of the algebra map $\pi_0(K_p) \to \pi_0((K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge)$.

As mentioned in Lemma 2.2, every $S^0_{p^\infty}/S^0_p$-form R of K_p can be recovered up to homotopy as the homotopy fixed point spectrum $((K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p)^{h\,\mathrm{Gal}_R}$. It follows that R inherits an E_∞-ring structure from $(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$. R cannot have more than one E_∞-ring structure refining its ring structure since the same would then need to be the case for $(R \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$. $\qquad\square$

Theorem 2.6. *Every $S^0_{p^\infty}/S^0_p$-form of K_p is $K(1)$-local.*

We use the following lemma.

Lemma 2.7. *$K_p/p \wedge X$ is $K(1)$-local for any spectrum X.*

Proof. We use the fact that K_p/p is a wedge of $p-1$ copies of suspensions of $K(1)$. As mentioned in Subsection 1.2, Morava K-theory is a field in the category of ring spectra, that is, every module spectrum over $K(n)$ is a wedge of suspensions of $K(n)$. In particular, $K_p/p \wedge X$ is a wedge of suspensions of $K(1)$, so it is $K(1)$-local. $\qquad\square$

Proof of Theorem 2.6. Recall that a spectrum R is E-local if every map $X \to R$ is nullhomotopic whenever X is E-acyclic, that is, $X \wedge E$ is zero.

We proceed by making a series of reductions. First, it suffices to show that $(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$ is $K(1)$-local, since every $S^0_{p^\infty}/S^0_p$-form R of K_p is obtained as a homotopy pullback involving $(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$, so that any map $X \to R$ is nullhomotopic if every map $X \to (K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$ is also nullhomotopic.

Next, since $(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$ is p-complete, it suffices to show that $(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p/p^n$ is $K(1)$-local for each $n \in \mathbf{N}$, as it will then follow from the universal property of p-completion that $(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p$ is also $K(1)$-local.

Finally, since there are fiber sequences

$$(K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p/p \to (K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p/p^n \to (K_p \wedge_{S^0_p} S^0_{p^\infty})^\wedge_p/p^{n-1},$$

it in turn suffices to show that $(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge / p$ is $K(1)$-local; $K(1)$-locality of $(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge / p^n$ for $n \geq 2$ will then follow by induction on n using the fiber sequences.

The result then follows from Lemma 2.7 since $(K_p \wedge_{S_p^0} S_{p^\infty}^0)_p^\wedge / p \simeq K_p/p \wedge S_{p^\infty}^0$. $\qquad \square$

As mentioned in Subsection 1.2, every $K(1)$-local E_∞-ring spectrum R possesses operations

$$\theta, \psi : \pi_0(R) \to \pi_0(R)$$

such that ψ is a ring homomorphism and θ imparts the structure of a θ-algebra on $\pi_0(R)$. [5] For any form R of K_p, the only ring homomorphism $\pi_0(R) = \mathbf{Z}_p \to \pi_0(R)$ is the identity homomorphism, so ψ is unable to distinguish forms of K_p. However, one can obtain an operation $\psi : \pi_2(R) \to \pi_2(R)$ that is able to distinguish forms of K_p, by considering the function spectrum R^{S^n}.

Lemma 2.8. *If R is an E_∞-ring spectrum and Z is a space, then the function spectrum R^Z is also an E_∞-ring spectrum.*

Lemma 2.9. *If a spectrum R is Y-local, then the function spectrum R^{S^n} is also Y-local.*

Proof. Suppose $X \wedge Y$ is zero. Then

$$[X, R^{S^n}] = [\Sigma^n X, R] = 0$$

since R is Y-local. $\qquad \square$

Consequently, if R is a $K(1)$-local E_∞-ring spectrum, then so is R^{S^n}. Hence one obtains operations

$$\psi : \pi_0(R^{S^n}) \to \pi_0(R^{S^n}).$$

That is, there are operations $\pi_n(R) \to \pi_n(R)$ for each $n \in \mathbf{N}$ which we shall also denote by ψ.

Theorem 2.10. *Let R be an E_∞-$S^0_{p\infty}/S^0_p$-form of K_p. The operation $\psi : \pi_2(R) \to \pi_2(R)$ satisfies $\psi(x) = \varepsilon p x$ for some $\varepsilon \in \mathbf{Z}_p^\times$, and the value of the constant ε determines a bijection between homotopy equivalence classes of E_∞-$S^0_{p\infty}/S^0_p$-forms of K_p and $\mathbf{Z}_p^\times$.*

Proof. Recall from Lemma 2.2 that R can be recovered as the homotopy fixed point spectrum $((K_p \wedge_{S^0_p} S^0_{p\infty})^\wedge_p)^{h\,\mathrm{Gal}_R}$. On the level of homotopy, we obtain from the Mayer-Vietoris sequence

$$\pi_2(R) = \{x \in \pi_2((K_p \wedge_{S^0_p} S^0_{p\infty})^\wedge_p) \mid \sigma_R(x) = x\}.$$

Recall that $\sigma_R = \sigma \circ \psi^\lambda$, where σ is the Frobenius and ψ^λ is an Adams operation on K_p.

Let $y = 1 - \mathcal{L} \in \pi_2((K_p \wedge_{S^0_p} S^0_{p\infty})^\wedge_p)$ be a generator of $\pi_2(K_p) \subset \pi_2((K_p \wedge_{S^0_p} S^0_{p\infty})^\wedge_p)$, where $\mathcal{L}$ is the class of the tautological bundle on $S^2 = \mathbf{CP}^1$. Recall that $(1 - \mathcal{L})^2 = 0$. Thus

$$\psi^\lambda(y) = \psi^\lambda(1 - \mathcal{L}) = 1 - \mathcal{L}^\lambda = 1 - (1 - (1 - \mathcal{L}))^\lambda = \lambda(1 - \mathcal{L}) = \lambda y.$$

Suppose that $x = \mu y$ where $\mu \in \pi_0((K_p \wedge_{S^0_p} S^0_{p\infty})^\wedge_p)$. Then

$$\mu y = \sigma_R(\mu y) = \sigma \circ \psi^\lambda(\mu y) = \sigma(\mu \psi^\lambda(y)) = \sigma(\mu)\psi^\lambda(y) = \sigma(\mu)\lambda y.$$

Hence $\sigma(\mu) = \mu \lambda^{-1}$. Finally, we use the fact that ψ is Frobenius-linear and, on

$\pi_2(K_p)$, is induced by the Adams operation ψ^p. [10] It then follows that

$$\psi(x) = \psi(\mu y) = \sigma(\mu)\psi(y) = \mu\lambda^{-1}py = \lambda^{-1}px. \qquad \square$$

Theorem 2.10 thus provides a way to identify a form of K_p without recourse to 1-cocycles in the first Galois cohomology group.

3   Future Directions

As noted in Subsection 1.1, concrete forms of K-theory which involve the Ramanujan tau function or the quartic residue character arise from elliptic cohomology. I intend to compute the action of ψ on $\pi_2(R)$ in these cases and other examples of interest.

I wish to interpret both existing and new results about forms of K-theory in the context of topological modular forms, so as to tie them closer to current research.

By extending topological modular forms to a functorial family of objects corresponding to elliptic curves with level structure and modular forms on them and restricting to the cusps, Hill and Lawson [4] obtain E_∞-ring maps from topological modular forms with level structure to forms of K-theory. I am interested in studying the properties of these maps.

Besides considering forms of K-theory that arise from topological modular forms, I also hope to consider forms of K-theory arising in topological automorphic forms, as in [2].

Being an E_∞-ring spectrum, Morava E-theory E_n possesses power operations whose structure has been studied by Ando [1] and Rezk [10]. Rezk has calculated the structure in detail for height two [9, 11]. I hope to use the theory of power operations in Morava E-theory to develop analogous criteria to distinguish forms of higher Morava E-theory.

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