



# The mod 2 homology of free spectral Lie algebras

## Citation

Antolin Camarena, Omar. 2015. The mod 2 homology of free spectral Lie algebras. Doctoral dissertation, Harvard University, Graduate School of Arts & Sciences.

## Permanent link

<http://nrs.harvard.edu/urn-3:HUL.InstRepos:17467469>

## Terms of Use

This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA>

## Share Your Story

The Harvard community has made this article openly available.  
Please share how this access benefits you. [Submit a story](#).

[Accessibility](#)

The mod 2 homology of free spectral Lie algebras

A dissertation presented

by

Omar Antolín Camarena

to

The Department of Mathematics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Mathematics

Harvard University  
Cambridge, Massachusetts

April 2015

© 2015 – Omar Antolín Camarena  
All rights reserved.

The mod 2 homology of free spectral Lie algebras

Abstract

The Goodwillie derivatives of the identity functor on pointed spaces form an operad  $\partial_*(\mathrm{Id})$  in spectra. We compute the mod 2 homology of free algebras for this operad on suspension spectra of simply-connected spaces.

## CONTENTS

Acknowledgements	v
1. Introduction	1
2. Homology operations on algebras for operads	3
3. The operad $\partial_*(\text{Id})$ and the Spectral Lie operad	5
3.1. The partition complex as a space of weighted trees	6
3.2. The cooperad structure of $B(\text{Comm})$	8
4. Two kinds of Lie algebras in characteristic 2	9
5. Homology operations for $\partial_*(\text{Id})$ -algebras	11
5.1. The shifted Lie bracket	12
5.2. Behrens's unary Dyer-Lashof-like operations	14
6. Algebraic structure of homology of $\partial_*(\text{Id})$ -algebras	17
7. Homology of free $\partial_*(\text{Id})$ -algebras on simply-connected spaces	20
7.1. The free $\partial_*(\text{Id})$ -algebra on a sphere	23
7.2. The free $\partial_*(\text{Id})$ -algebra on a finite wedge of spheres	24
7.3. The free $\partial_*(\text{Id})$ -algebra on a simply-connected space	26
References	28

## ACKNOWLEDGEMENTS

I would like to thank my advisor, Jacob Lurie, for suggesting this topic, his patience, his wonderful teaching and his almost instant answers to my many questions.

I would also like to thank everyone in the Harvard math department, especially my fellow graduate students, for making the math department such a friendly and fun environment and a most efficient place to learn mathematics! I especially learned a lot from Tobias Barthel, Lukas Brantner, Gijs Heuts and Emily Riehl. I'd particularly like to thank Lukas for correcting an error I had in my thesis.

The MIT topology community was also very welcoming and helpful, and I thank them for it.

I'd also like to thank the preceptor team, particularly Janet Chen and Jameel Al-Aidroos, for all they did to help me improve my teaching and for how fun it was.

Thanks too, to Susan Gilbert for always being extraordinarily helpful.

But most of all I'd like to thank my family for putting up with everything! Eduardo stoically accepted us not spending as much time together as we both wanted. Same goes for Paola who also somehow managed to always make things run smoothly at home.

## 1. INTRODUCTION

Goodwillie calculus [8] associates to appropriate functors  $F : \text{Top}_* \rightarrow \text{Top}_*$  a tower of approximations

$$\cdots \rightarrow P_n F \rightarrow P_{n-1} F \rightarrow \cdots \rightarrow P_1 F \rightarrow P_0 F$$

that is analogous to the sequence of Taylor polynomials for functions of a real variable. The homotopy fibers  $D_n F = \text{hofib}(P_n F \rightarrow P_{n-1} F)$  are called the layers of the Goodwillie tower and are analogous to individual monomials  $f^{(n)}(0)x^n/n!$  in the Taylor expansion of a function. Goodwillie proved that these layers are of the form  $D_n F(X) = \Omega^\infty(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$  for some sequence of spectra  $\partial_n F$  where the  $n$ -th spectrum is equipped with an action of  $\Sigma_n$ .

These derivatives  $\partial_n F$  are very interesting even for  $F = \text{Id}$  and have been much studied in that case; they can be described as the Spanier-Whitehead duals of certain finite complexes. The first such description was obtained by Johnson [11]; a second description is in terms of partitions and appears in [3]. The *partition complex*  $P_n$  is the pointed simplicial set

$$N\Pi_n / (N(\Pi_n \setminus \{\hat{0}\}) \cup N(\Pi_n \setminus \{\hat{1}\})),$$

where  $\Pi_n$  is the poset of partitions of a set with  $n$  elements, ordered by refinement;  $\hat{0}$  and  $\hat{1}$  denote its least and greatest element, respectively; and  $N$  denotes, as usual, the nerve functor. We shall regard  $P_n$  as having the action of  $\Sigma_n$  induced by permutations of the  $n$ -element set.

The layers of the Goodwillie tower of the identity are given by

$$D_n(\text{Id})(X) = \Omega^\infty(\text{Map}_*(P_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}),$$

where  $\text{Map}_*$  denotes the spectrum of maps from a pointed space to a spectrum. This implies that  $\partial_n(\text{Id})$  is  $\text{Map}_*(P_n, S)$ , the Spanier-Whitehead dual of  $P_n$ .

In [6], Ching constructs an operad structure on  $\partial_*(\text{Id})$  that is easiest to describe in dual form: as a cooperad structure on  $P_*$ . That cooperad is the bar construction on the nonunital

commutative operad in spectra (given by  $\text{Comm}_n = S$  for all  $n \geq 1$ ), so that the operadic suspension of  $\partial_*(\text{Id})$  is Koszul dual to the commutative operad and we can think of  $\partial_*(\text{Id})$  as a shifted version of the Lie operad. Alternatively, one can also see a relation between  $\partial_*(\text{Id})$  and the Lie operad using what is known about the homology of the partition complex, namely that the space of  $n$ -ary operations of the Lie operad in Abelian groups is isomorphic as a  $\mathbb{Z}[\Sigma_n]$ -module to  $\text{Hom}(H_{n-2}(P_n), \text{sgn})$  —where  $\text{sgn}$  is the sign representation of  $\Sigma_n$ .

The mod  $p$  homology of the layers,

$$\mathbb{D}_n(X) := \mathbb{D}_n(\text{Id})(X) = (\partial_n(\text{Id}) \wedge X^{\wedge n})_{h\Sigma_n},$$

was studied in [3] in the case that  $X$  is a sphere. (Since the free  $\partial_*(\text{Id})$ -algebra on a space  $X$  is given by  $\bigoplus_{n \geq 0} \mathbb{D}_n(X)$ , the results in that paper can be interpreted as being about the mod  $p$  homology of the free  $\partial_*(\text{Id})$ -algebra on  $S^n$ .) In the case  $p = 2$ , for example, what Arone and Mahowald showed is that  $H_*(\mathbb{D}_n(S^m); \mathbb{F}_p)$  is only non-zero when  $n = 2^k$  is a power of 2 and in that case it is  $\Sigma^{-k}CU_*$  as a module over the Steenrod algebra, where  $CU_*$  is the free graded  $\mathbb{F}_p$ -vector space with basis given by the “completely unadmissible” words of length  $k$ :

$$\{Q^{s_1} \cdots Q^{s_k} u : s_k \geq m, s_i > 2s_{i+1}\}$$

where  $u$  is a generator of  $H_m(S^m; \mathbb{F}_p)$  and the action of the Steenrod algebra on  $CU_*$  is given by the Nishida relations.

In Behrens’ [4], he uses this computation to introduce mod 2 homology operations  $\bar{Q}^j : H_d(L) \rightarrow H_{d+j-1}(L)$  for  $j \geq d$  on  $\partial_*(\text{Id})$ -algebras. Part of an  $\partial_*(\text{Id})$ -algebra structure on  $L$  is a map  $\xi : (\partial_2(\text{Id}) \wedge L^{\wedge 2})_{h\Sigma_2} \rightarrow L$  and since  $\partial_2(\text{Id})$  is  $S^{-1}$  with a trivial  $\Sigma_2$ -action, we have  $(\partial_2(\text{Id}) \wedge L^{\wedge 2})_{h\Sigma_2} \cong \Sigma^{-1}L_{h\Sigma_2}^{\wedge 2}$ ; using that identification we set  $\bar{Q}^j = \xi_* \sigma^{-1} Q^j$  where  $Q^j : H_d(L) \rightarrow H_{d+j}(L_{h\Sigma_2}^{\wedge 2})$  is a Dyer-Lashof operation. Behrens shows the Arone–Mahowald computation can be interpreted as saying that the homology of the free  $\partial_*(\text{Id})$ -algebra on  $S^n$  has an  $\mathbb{F}_2$ -basis consisting of completely unadmissible sequences of  $\bar{Q}^j$ ’s with excess at least



$n$  applied to the fundamental class of  $S^n$ , furthermore, he computes the relations satisfied by the  $\bar{Q}^j$ 's.

In the present work we compute the mod 2 homology of the free  $\partial_*(\text{Id})$ -algebra on the suspension spectrum  $\Sigma^\infty X$  of a simply-connected space  $X$ , showing that it is roughly speaking the free module over the ring of operations  $\bar{Q}^j$ 's on the free Lie algebra on  $H_*(X)$ . It would be of interest to extend these computations to free  $\partial_*(\text{Id})$ -algebras on more general spectra or to mod  $p$  homology for odd primes  $p$ ; we leave these extensions to future work.

## 2. HOMOLOGY OPERATIONS ON ALGEBRAS FOR OPERADS

Given an operad  $\mathcal{O}$  in spectra we will denote by  $F_{\mathcal{O}}$  the free  $\mathcal{O}$ -algebra functor. This functor is a monad, and  $\mathcal{O}$ -algebras are equivalently algebras for it. If  $E$  is an  $E_\infty$ -ring spectrum, then there is an operad in  $E$ -module spectra we will denote by  $E \wedge \mathcal{O}$ , and a free  $(E \wedge \mathcal{O})$ -algebra functor  $F_{E \wedge \mathcal{O}}$  defined on  $E$ -module spectra. The  $E$ -module of  $n$ -ary operations in  $E \wedge \mathcal{O}$  is the free  $E$ -module on the spectrum  $\mathcal{O}_n$ :  $(E \wedge \mathcal{O})_n = E \wedge \mathcal{O}_n$ ; we get an operad structure on  $E \wedge \mathcal{O}$  induced from the operad structure on  $\mathcal{O}$  because the free  $E$ -module functor is symmetric monoidal. The free algebra functors are related in the expected way:  $E \wedge F_{\mathcal{O}}(X) \cong F_{E \wedge \mathcal{O}}(E \wedge X)$ . We will also make use of the functor between  $\mathcal{O}$ -algebras and  $(E \wedge \mathcal{O})$ -algebras induced by the free  $E$ -module functor,  $E \wedge -$ .

*Remark 2.1.* To achieve the structure describe above one can work in a symmetric monoidal category of spectra, such as EKMM  $S$ -modules [12], taking “spectrum” to mean  $S$ -algebra and “ $E_\infty$ -ring spectrum” to mean commutative  $S$ -algebra. This is the same framework used in [6] to put an operad structure on the derivatives of the identity.

Also, we will only consider cofibrant operads for which the notion algebra is homotopy invariant, meaning that we can think of the homotopy type of the free  $\mathcal{O}$ -algebra as being given by

$$F_{\mathcal{O}}(X) = \bigvee_{n \geq 0} (\mathcal{O}_n \wedge X^n)_{h\Sigma_n},$$

and that we will think of an  $\mathcal{O}$ -algebra structure on  $A$  as providing maps  $(\mathcal{O}_n \wedge A^{\wedge n})_{h\Sigma_n} \rightarrow A$ .

Every class  $\alpha \in E_m \left( F_{\mathcal{O}} \left( \bigvee_{i=1}^k S^{d_i} \right) \right)$  in the  $E$ -homology of the free  $\mathcal{O}$ -algebra on a wedge of  $k$  spheres gives a  $k$ -ary homology operation defined on the  $E$ -homology of any  $\mathcal{O}$ -algebra  $A$ , defined as follows:

Given  $x_i \in E_{d_i}(A)$  ( $i = 1, \dots, k$ ), we can represent each  $x_i$  by a map of spectra  $S^{d_i} \rightarrow E \wedge A$ , and thus the whole collection of them can be described by a single map of spectra  $\bar{x} : \bigvee_{i=1}^k S^{d_i} \rightarrow E \wedge A$ . Since  $E \wedge A$  is an  $(E \wedge \mathcal{O})$ -algebra,  $\bar{x}$  has an adjoint  $\tilde{x}$  which is a map of  $(E \wedge \mathcal{O})$ -algebras to  $E \wedge A$  from the free  $(E \wedge \mathcal{O})$ -algebra on  $\bigvee_{i=1}^k S^{d_i}$ , namely  $F_{E \wedge \mathcal{O}}(\bigvee_{i=1}^k \Sigma^{d_i} E) = E \wedge F_{\mathcal{O}}(\bigvee_{i=1}^k S^{d_i})$ .

The homology operation corresponding to  $\alpha$ , is  $\alpha_* : \bigotimes_{i=1}^k E_{d_i}(A) \rightarrow E_m(A)$  defined by setting  $\alpha_*(x_1 \otimes \dots \otimes x_k)$  to be represented by the map

$$S^m \xrightarrow{\alpha} E \wedge F_{\mathcal{O}} \left( \bigvee_{i=1}^k S^{d_i} \right) \xrightarrow{\tilde{x}} E \wedge A.$$

Notice that an analogous construction gives operations on the stable homotopy of  $(E \wedge \mathcal{O})$ -algebras and given an  $\mathcal{O}$ -algebra  $A$ , the operations on the  $E$ -homology of  $A$  coincide with those produced on the homotopy of the  $(E \wedge \mathcal{O})$ -algebra  $E \wedge A$ .

To get a useful theory of homology operations for  $\mathcal{O}$ -algebras, besides computing those homology groups, the various  $E_m(\bigvee_{i=1}^k S^{d_i})$ , one must organize the operations: find a relatively small collection of operations that generate all others and find a generating set of relations for the operations. This has been carried out for  $\mathrm{HF}_p$ -homology of algebras for the  $E_n$ -operads, due to May in the case  $n = \infty$ , and due to F. Cohen in the case  $1 \leq n < \infty$ ; see [7].

Homology operations with field coefficients are simpler to study, because of the following result:

**Proposition 2.2.** *Let  $\mathcal{O}$  be a operad in spectra. The homology with coefficients in a field  $k$  of the free  $\mathcal{O}$ -algebra on a spectrum  $X$  is a functor of the homology of  $X$ .*

*Proof.* Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{S}p & \xrightarrow{Hk \wedge -} & Hk\text{-Mod} & \xrightarrow{\pi} & D(k) & \xrightarrow{\cong} & \text{GrVect}_k \\
F_{\mathcal{O}} \downarrow & & \downarrow F_{Hk \wedge \mathcal{O}} & & \downarrow \hat{F} & & \downarrow \tilde{F} \\
\mathcal{S}p & \xrightarrow{Hk \wedge -} & Hk\text{-Mod} & \xrightarrow{\pi} & D(k) & \xrightarrow{\cong} & \text{GrVect}_k
\end{array}$$

Here  $\mathcal{S}p$  denotes the category of spectra,  $Hk\text{-Mod}$  denotes the category of  $Hk$ -module spectra,  $D(k)$  is the homotopy category of  $Hk\text{-Mod}$  or, equivalently, the unbounded derived category of vector spaces over  $k$  and  $\text{GrVect}_k$  is the category of graded vector spaces over  $k$ .

The functor  $\pi$  is the projection from  $Hk\text{-Mod}$  to its homotopy category; this functor preserves coproducts but when the characteristic of  $k$  is not 0, it does not send homotopy quotients by the action of  $\Sigma_n$  to quotients by the action of  $\Sigma_n$ , so the induced monad  $\hat{F}$  is no longer the free algebra functor for an operad. Finally, when  $k$  is a field there is an equivalence  $D(k) \cong \text{GrVect}_k$ , allowing us to define the monad  $\tilde{F}$  so that the last square commutes.  $\square$

### 3. THE OPERAD $\partial_*(\text{Id})$ AND THE SPECTRAL LIE OPERAD

Recall that the suspension of an operad  $\mathcal{O}$  is an operad  $\mathbf{s}\mathcal{O}$  defined so that:

- $\mathbf{s}\mathcal{O}$ -algebra structures on  $\Sigma A$  correspond to  $\mathcal{O}$ -algebra structures on  $A$ ,
- the free algebra functors satisfy  $F_{\mathbf{s}\mathcal{O}}(\Sigma X) = \Sigma F_{\mathcal{O}}(X)$ , and
- as a symmetric sequence,  $(\Sigma \mathcal{O})_n$  is given by  $(S^{-1})^{\wedge n} \wedge \Sigma \mathcal{O}_n$  with  $\Sigma_n$  acting diagonally, permuting the smash factors on the left and acting on  $\Sigma \mathcal{O}_n$  via the suspension of the action on  $\mathcal{O}_n$  (that is, it acts trivially on the suspension coordinate of  $\Sigma \mathcal{O}_n$ ).

By the *spectral Lie operad* we mean the desuspension  $\mathbf{s}^{-1}\partial_*(\text{Id})$  of the operad formed by the Goodwillie derivatives of the identity. It is the spectral Lie operad which is most closely analogous to Lie algebras and some of our formulas would be simpler for it, but we will stick to the language of the  $\partial_*(\text{Id})$ -operad and  $\partial_*(\text{Id})$ -algebras to make using the available

literature easier. As a symmetric sequence, the spectral Lie operad is given by the derivatives of the functor  $\Omega\Sigma : \text{Top}_* \rightarrow \text{Top}_*$  (see [8, Section 8]).

As we said before, the easiest way to describe the operad structure of  $\partial_*(\text{Id})$  is to describe a cooperad structure on the bar construction of the nonunital commutative operad, and obtain the operad structure of  $\partial_*(\text{Id})$  by taking Spanier-Whitehead duals. To describe Ching's cooperad structure, we need to explain how to think of the points of  $|P_n|$  as trees.

**3.1. The partition complex as a space of weighted trees.** Before we describe how to assign trees to simplices of  $P_n$  or to points of  $|P_n|$ , let us fix some conventions for trees:

- A *tree* is, as usual in graph theory, a finite connected graph without cycles.
- We will only deal with *rooted trees*, that is, trees with a distinguished vertex, called the *root*, that is incident to only one edge<sup>1</sup>. That edge is called the *root edge*.
- In a rooted tree any vertex other than the root which is incident to a unique edge is called a *leaf*, and the edge it is incident to is a leaf edge.
- All other edges are called *internal edges*.
- We will orient rooted tree towards the root, so that every non-root vertex has a unique *outgoing edge*, and every non-leaf vertex has one or more *incoming edges*.
- A *leaf labeling* of a rooted tree with set of labels  $A$  is a bijection between  $A$  and the leaves of the tree.

Now, a  $k$ -simplex in  $P_n$  is given by a chain of partitions  $\hat{0} = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_k = \hat{1}$ , this gives us a rooted tree with leaves labeled by  $\{1, 2, \dots, n\}$ , of with  $k + 1$  “levels”: a level for the root and for each  $\lambda_i$  a level whose vertices correspond to the blocks of  $\lambda_i$ . The vertex corresponding to a block of some  $\lambda_i$  has a single outgoing edge connecting it to the vertex for the unique block of  $\lambda_{i+1}$  containing it. A picture makes this construction clear, see Figure 1.

To points of the geometric realization  $|P_n|$  we will associate *weighted trees* below, but first some definitions:

---

<sup>1</sup>That the root is required to have valence 1 is not standard in graph theory.

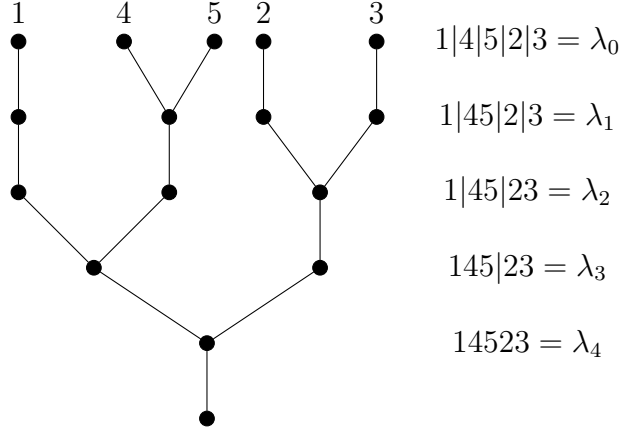


FIGURE 1. The tree associated to a simplex of  $P_n$ .

- A *weighting* on a rooted tree is an assignment of a non-negative real *weight*<sup>2</sup> to each edge of the tree in such a way that for each leaf  $v$ , the sum of the weights of the edges on the path from  $v$  to the root is 1.
- A *weighted tree* is a rooted tree equipped with a weighting.

Now, every point of  $|P_n|$  can be described by giving a  $k$ -simplex of  $P_n$  and a point  $x$  in the topological  $k$ -simplex,  $|\Delta^k| = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : x_i \geq 0, \sum x_i = 1\}$ . Given this data we construct a weighting on the tree associated above to the  $k$ -simplex of  $P_n$ :

- First, declare that all edges going from the level for  $\lambda_i$  to the the next lowest level have weight  $x_i$ .
- Then, for each vertex with only one incoming edge and one outgoing edge, merge the two edges, adding the weights.

*Remark 3.1.* Notice that the trees we associated to simplices of  $P_n$  *have* a height, that is, all of their leaves are at the same distance from the root. This is not the case for the weighted trees associated to points of  $|P_n|$ , because of the merging of edges. On the other hand, the weighted trees we constructed never have vertices of valence 2 (vertices with a single incoming edge and a single outgoing edge), but the trees corresponding to simplices might have them.

<sup>2</sup>This is called *length* in [6].

The above construction gives a homeomorphism [6, Proposition 4.13] between  $|P_n|$  and the space  $B(\text{Comm})(\underline{n})$  of weighted trees with leaves labeled by  $\underline{n} := \{1, 2, \dots, n\}$ . For an arbitrary finite set  $A$  of leaf-labels, the space  $B(\text{Comm})(A)$  consists of certain weighted trees subject to certain identifications:

- The points of  $B(\text{Comm})(A)$  are weighted trees with leaves labeled by  $A$ . The trees are required to have no vertices of valence 2.
- The identifications are as follows:
  - If the root edge or any leaf edge of  $T$  has weight 0, then  $T$  is identified with the basepoint.
  - If an internal edge  $e$  of  $T$  has weight 0,  $T$  is identified with  $T/e$ , the weighted tree obtained by contracting the edge  $e$ .

*Remark 3.2.* Ching describes the bar construction  $B(\mathcal{O})$  on an arbitrary reduced operad  $\mathcal{O}$  in pointed spaces in terms of weighted trees; the above is what the description simplifies to in the case of the bar construction on the nonunital commutative operad, whose spaces of operations are given by  $\text{Comm}(A) = S^0$  for  $A \neq \emptyset$ .

**3.2. The cooperad structure of  $B(\text{Comm})$ .** Given (unweighted) trees  $U$  and  $V$  with leaves labeled by  $A$  and  $B$  respectively, and given  $a \in A$ , we can construct a new tree  $U \cup_a V$ , called the *grafting* of  $V$  onto  $U$ , by identifying the root edge of  $V$  with the leaf edge of  $U$  at the leaf labeled by  $a$ . The leaves of  $U \cup_a V$  are naturally labeled by  $A \cup_a B := (A \setminus \{a\}) \cup B$ . Again, a picture makes this clearer, see Figure 2.

**Definition 3.3** ([6, Definition 4.16]). Given two sets of leaf-labels  $A$  and  $B$ , and a label  $a \in A$  the cooperadic structure map

$$\circ_a : B(\text{Comm})(A \cup_a B) \rightarrow B(\text{Comm})(A) \wedge B(\text{Comm})(B)$$

on a  $(A \cup_a B)$ -labeled weighted tree  $T$  is defined as follows:

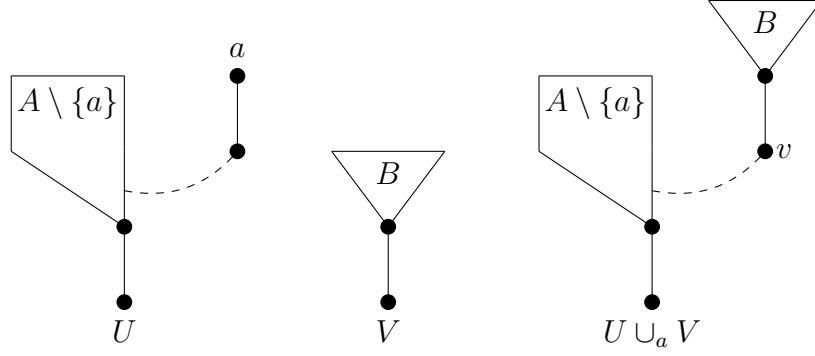


FIGURE 2. Grafting of trees.

- If the underlying unweighted tree of  $T$  can be obtained as  $U \cup_a V$  for some (necessarily unique) unweighted trees  $U$  and  $V$  with leaves labeled by  $A$  and  $B$  respectively, we will give certain weightings to  $U$  and  $V$  and declare that  $\circ_a(T) = U \wedge V$ .

Let  $v$  be the vertex of  $T$  that serves as root for the copy of  $V$  sitting inside  $T$  (as indicated in Figure 2). All paths to the root of  $T$  starting from a leaf labeled by an element of  $B$  pass through  $v$ . Since in  $T$  the total weight of each path from a leaf to the root is 1, all paths from a leaf labeled by an element of  $B$  to  $v$  must have the same total weight, say  $\omega$ .

The weightings of  $U$  and  $V$  are defined as follows:

- Every edge in  $U$  except the leaf edge at  $a$  is given the same weight as in  $T$ ; the leaf edge at  $a$  is given weight  $\omega$ .
- Each edges of  $V$  is given weight  $w/\omega$  where  $w$  is the weight that edge has in  $T$ .
- If the underlying unweighted tree of  $T$  cannot be obtained by grafting a  $B$ -labeled tree onto an  $A$ -labeled one,  $\circ_a(T)$  is the basepoint.

*Remark 3.4.* Again we gave Ching's definition only in the special case of interest here.

#### 4. TWO KINDS OF LIE ALGEBRAS IN CHARACTERISTIC 2

In this section we collect a few definitions about (graded) Lie algebras we will need later. We will actually need to use two different notions of Lie algebras. The usual definition of

Lie algebra in characteristic 0 is equivalent to being an algebra for an operad  $\mathbf{Lie}$  in Abelian groups. One can take algebras for that operad in the category of  $R$ -modules or of graded  $R$ -modules for a commutative ring  $R$  and this gives one possible definition of graded Lie algebra. Since  $\partial_*(\text{Id})$  is the suspension of the spectral version of the Lie operad, we are also interested in algebras for the suspension  $\mathbf{sLie}$ .

Spelling out the structure we see that a graded  $\mathbf{Lie}$ -algebra  $L$  over a commutative ring  $R$  is a graded module equipped with a binary operation  $[-, -] : L_i \otimes L_j \rightarrow L_{i+j}$  satisfying:

- anti-symmetry,  $[x, y] = -(-1)^{|x||y|}[y, x]$ , and
- the Jacobi identity,

$$(-1)^{|z||x|}[x, [y, z]] + (-1)^{|y||z|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

(Where  $x$  and  $y$  are homogeneous elements of degrees  $|x|$  and  $|y|$ .)

For  $\mathbf{sLie}$ -algebras things are only slightly different:

- The bracket has degree  $-1$ :  $[-, -] : L_i \otimes L_j \rightarrow L_{i+j-1}$ .
- Anti-symmetry becomes graded commutativity:

$$[x, y] = (-1)^{|x||y|}[y, x].$$

- The Jacobi identity stays the same!

All the signs in the above formulas come from the Koszul sign rule, that is, from the signs in the symmetry isomorphism of the category of graded  $R$ -modules. Since we will work over  $R = \mathbb{F}_2$  we need not worry about signs, but we mention them to point out that for an element  $x$  of even degree in a Lie-algebra (or of odd degree in a  $\mathbf{sLie}$ -algebra), the definitions imply that  $2[x, x] = 0$ , but they don't actually imply  $[x, x] = 0$  if 2 is not invertible in  $R$ .

If  $R$  has characteristic 2, while  $[x, x]$  may not be 0, we do have that any brackets involving it are 0: by the Jacobi identity,

$$[[x, x], y] = [[x, y], x] + [[y, x], x] = 2[[x, y], x] = 0.$$



As an example showing  $[x, x]$  can be nonzero, the free Lie-algebra over  $\mathbb{F}_2$  on one generator  $x$  in an even degree is easily seen to have basis  $\{x, [x, x]\}$ .

Given a graded associative  $R$ -algebra  $A$ , the graded commutator  $[x, y] = xy - (-1)^{|x||y|}yx$  gives  $A$  the structure of a Lie-algebra, but all the algebras produced this way necessarily have  $[x, x] = 0$  for  $|x|$  even. This means that if a Lie-algebra over an  $R$  of characteristic 2 has some nonzero  $[x, x]$  with  $|x|$  even, it cannot be faithfully represented by commutators, and thus does not inject into its universal enveloping algebra. This substantially changes the theory of Lie algebras requiring an embedding into the universal enveloping algebra and so at least one other definition of Lie algebra in characteristic 2 is sometimes used, one that forces an injection into a Lie algebra of commutators.

In the case of  $R = \mathbb{F}_2$  this other kind of Lie algebra simply adds the requirement that  $[x, x] = 0$  for all homogeneous  $x$ . We will call this kind of Lie algebra a Lie<sup>ti</sup>-algebra —the ti stands for *totally isotropic*. A definition for all rings  $R$ , due to Moore, just forces the representation as a commutator Lie algebra to exist:

**Definition 4.1.** A graded Lie<sup>ti</sup>-algebra (resp. sLie<sup>ti</sup>-algebra) over  $R$  is graded  $R$ -module  $L$  with a bracket  $L_i \otimes L_j \rightarrow L_{i+j}$  (resp.  $L_{i+j-1}$ ) and a monomorphism  $L \rightarrow A$  to some graded associative algebra so that the bracket goes to the graded commutator  $xy - (-1)^{|x||y|}yx$  (resp.  $xy + (-1)^{|x||y|}yx$ ).

Our main interest in these algebras is that the *basic products* appearing in Hilton's theorem about the loop space of a wedge of spheres [10] form a basis (called a Hall basis) for a totally isotropic Lie algebra, see the discussion in section 7.

## 5. HOMOLOGY OPERATIONS FOR $\partial_*(\text{Id})$ -ALGEBRAS

Throughout this section  $L$  will denote an algebra for the operad  $\partial_*(\text{Id})$ . So in particular,  $L$  is a spectrum equipped with structure maps  $\xi_n : \mathbb{D}_n(L) \rightarrow L$  where  $\mathbb{D}_n(L) = (\partial_n(\text{Id}) \wedge L^{\wedge n})_{h\Sigma_n}$ . There is a more traditional way to describe the structure of an algebra for an operad: by giving maps  $\alpha_n : \partial_n(\text{Id}) \wedge L^{\wedge n} \rightarrow L$  that are  $\Sigma_n$ -equivariant for the

trivial action on the codomain and the diagonal action on the domain. The relation between these two styles of definition is captured in the following commutative diagram:

$$\begin{array}{ccccc}
\partial_n(\text{Id}) \wedge L^{\wedge n} & \xrightarrow{\alpha_n} & L & & \\
\downarrow & & \downarrow & \searrow & \\
(\partial_n(\text{Id}) \wedge L^{\wedge n})_{h\Sigma_n} & \xrightarrow{(\alpha_n)_{h\Sigma_n}} & L \wedge \Sigma_+^\infty B\Sigma_n & \xrightarrow{\quad} & L, \\
& \searrow \xi_n & & & 
\end{array}$$

where the unlabeled horizontal map is  $L \wedge \Sigma_+^\infty (-)$  applied to  $B\Sigma_n \rightarrow *$ .

where the vertical maps are the canonical maps  $Y^{\wedge n} \rightarrow Y_{h\Sigma_n}^{\wedge n}$  and the unlabeled horizontal map is  $L \wedge \Sigma_+^\infty (-)$  applied to  $B\Sigma_n \rightarrow *$ .

In this section we will describe some operations on  $H_*(L; \mathbb{F}_2)$  that will turn out to generate all others, and whose definition will only require the map

$$\xi = \xi_2 : (\partial_2(\text{Id}) \wedge L^{\wedge 2})_{h\Sigma_2} \rightarrow L.$$

Notice there is only one unweighted tree with two leaves, and it has an interval's worth of weightings; the identifications make  $B(\text{Comm})(2)$  homeomorphic to  $S^1$ , so  $\partial_2(\text{Id}) = S^{-1}$  with a trivial  $\Sigma_2$ -action. This implies that  $(\partial_2(\text{Id}) \wedge L^{\wedge 2})_{h\Sigma_2} \cong \Sigma^{-1} L_{h\Sigma_2}^{\wedge 2}$ .

**5.1. The shifted Lie bracket.** We'll start by describing the Lie bracket. Here we remind the reader that  $\partial_*(\text{Id})$  is not really analogous to the Lie operad, but rather is analogous to its operadic desuspension.

**Definition 5.1.** The *shifted Lie bracket* on the homology of an  $\partial_*(\text{Id})$ -algebra  $L$  is the map  $[\cdot, \cdot] : H_i(L) \otimes H_j(L) \rightarrow H_{i+j-1}(L)$  given by the fundamental class  $S^{-1} \rightarrow \partial_2(\text{Id})$ , that is, it is the map induced on homology by the suspension of the structure map  $\alpha_2 : \Sigma^{-1} L^{\wedge 2} \rightarrow L$ .

This operation really gives a **sLie**-algebra:

**Proposition 5.2.** *Given any  $\partial_*(\text{Id})$ -algebra  $L$ , the shifted Lie bracket on  $H_*(L)$  gives  $H_*(L)$  the structure of a **sLie**-algebra.*

*Remark 5.3.* The following proof, that works directly with spectra before taking homology, shows that  $H_*(L; \mathbb{F}_p)$  is a **sLie**-algebra over  $\mathbb{F}_p$ .

*Proof.* We've already proved symmetry, when we computed  $\partial_2(\text{Id})$  and saw it had the trivial  $\Sigma_2$ -action.

To prove the Jacobi identity, we will show that  $1 + \sigma_* + \sigma_*^2 : \partial_3(\text{Id}) \rightarrow \partial_3(\text{Id})$  is null-homotopic where  $\sigma = (123) \in \Sigma_3$ . We can work with  $\Sigma^\infty P_3$ , before taking Spanier-Whitehead duals.

Now,  $P_3$  consists of:

- a 1-simplex, corresponding to the chain  $\hat{0} < \hat{1}$ , connecting the basepoint  $\hat{0} = \hat{1}$  with itself, and
- three 2-simplices, say  $\tau_1, \tau_2, \tau_3$ , each filling in the above circle, corresponding to the three chains  $\hat{0} < (23|1) < \hat{1}$ ,  $\hat{0} < (13|2) < \hat{1}$ , and  $\hat{0} < (12|3) < \hat{1}$ , respectively.

The 3-cycle  $\sigma$  permutes those three 2-simplices cyclically. We can compute  $1 + \sigma + \sigma^2 : \Sigma^\infty P_3 \rightarrow \Sigma^\infty P_3$  as the composite:

$$\Sigma^\infty P_3 \xrightarrow{\Delta} \bigvee^3 \Sigma^\infty P_3 \xrightarrow{1 \vee \sigma \vee \sigma^2} \bigvee^3 \Sigma^\infty P_3 \xrightarrow{\nabla} \Sigma^\infty P_3.$$

Non-equivariantly we have an equivalence  $S^2 \vee S^2 \xrightarrow{\cong} P_3$ , where we will think of the first  $S^2$  as mapping to  $P_3$  by sending the northern hemisphere to  $\tau_1$ , and the southern hemisphere to  $\tau_2$ ; we'll abbreviate this map  $S^2 \rightarrow P_3$  as  $\tau_{12}$  and use similar notation for other maps. We'll think of the second wedge summand  $S^2$  as corresponding to the map  $\tau_{23}$ .

We can think of a map  $\bigvee^n \Sigma^\infty P_3 \rightarrow \bigvee^m \Sigma^\infty P_3$  as given by an  $n \times m$  matrix of maps  $\Sigma^\infty P_3 \rightarrow \Sigma^\infty P_3$ , and each such map as given by a  $2 \times 2$  matrix of maps  $\Sigma^\infty S^2 \rightarrow \Sigma^\infty S^2$ .

The matrices of  $\Delta$  and  $\nabla$  are just the  $3 \times 1$  and  $1 \times 3$  matrices each of whose entries is  $I$ , the  $2 \times 2$  identity matrix. Once we have the  $2 \times 2$  matrix  $S$  representing  $\sigma : \Sigma^\infty P_3 \rightarrow \Sigma^\infty P_3$ , the matrix of  $1 \vee \sigma \vee \sigma^2$  is given by  $3 \times 3$  diagonal matrix with  $I, S, S^2$  along the diagonal.

To compute the matrix  $S$ , notice that  $\sigma \circ \tau_{12} = \tau_{23}$  and  $\sigma \circ \tau_{23} = \tau_{31}$ . The map  $\Sigma^\infty \tau_{13}$  is given by  $\Sigma^\infty \tau_{12} + \Sigma^\infty \tau_{23}$ , and  $\tau_{31}$  differs from  $\tau_{13}$  by the reflection swapping the hemispheres of  $S^2$ , which has degree  $-1$ . So,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

This means the composite map  $1 + \sigma + \sigma^2$  has matrix:

$$\begin{pmatrix} I & I & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S^2 \end{pmatrix} \begin{pmatrix} I \\ I \\ I \end{pmatrix} = I + S + S^2,$$

which is readily computed to be 0. □

**5.2. Behrens's unary Dyer-Lashof-like operations.** In [4, Chapter 1], Behrens interprets Arone and Mahowald's calculation [3] of  $H_*(\mathbb{D}_n(X))$  for a sphere  $X$  in the case  $p = 2$  in terms of unary homology operations for the layer of the Goodwillie tower of a reduced finitary homotopy functor  $F : \text{Top}_* \rightarrow \text{Top}_*$ . The Arone-Ching chain rule [1] gives the symmetric sequence of derivatives of  $F$ ,  $\partial_*(F)$ , the structure of a bimodule for  $\partial_*(\text{Id})$ . Behrens's operations only use the left module structure and could be defined on the mod 2 homology of any symmetric sequence which is a left module over  $\partial_*(\text{Id})$ . In particular, regarding an  $\partial_*(\text{Id})$ -algebra as a symmetric sequence concentrated in degree 0, we get unary operations on the mod 2 homology of an  $\partial_*(\text{Id})$ -algebra:

**Definition 5.4** (adapted from [4, Section 1.5]). Let  $L$  be a spectrum equipped with the structure of an  $\partial_*(\text{Id})$ -algebra. We define homology operations

$$\bar{Q}^j : H_d(L) \rightarrow H_{d+j-1}(L),$$

as follows: for  $x \in H_d(L)$ , we set  $\bar{Q}^j x := \xi_* \sigma^{-1} Q^j x$  where

- $\xi : \Sigma^{-1} L_{h\Sigma_2}^{\wedge 2} \cong \mathbb{D}_2(L) \rightarrow L$  is part of the  $\partial_*(\text{Id})$ -algebra structure of  $L$ ,
- $\sigma^{-1} : H_{d+j}(L_{h\Sigma_2}^{\wedge 2}) \rightarrow H_{d+j-1}(\mathbb{D}_2(L))$  is the (de)suspension isomorphism, and
- $Q^j : H_d(L) \rightarrow H_{d+j}(L_{h\Sigma_2}^{\wedge 2})$  is a Dyer-Lashof operation.

Note that  $\bar{Q}^j$  has degree  $j - 1$  but the notation for it uses “ $j$ ” because it is named after  $Q^j$ . Also notice that if  $j < d$  and  $x \in H_d(L)$ , we have  $\bar{Q}^j x = 0$  simply because  $Q^j x = 0$ .

*Remark 5.5.* By modifying the setting of the definition of the  $\bar{Q}^j$ , we’ve introduced a potential ambiguity! For a free  $\partial_*(\text{Id})$ -algebra  $L = F_{\partial_*(\text{Id})}(X)$  on some spectrum  $X$ , there are two different ways in which we could mean  $\bar{Q}^j x$  for  $x \in H_*(L)$ : using definition 5.4, or using Behrens’ original definition for the functor  $\text{Id}$ . Let us explain what that definition is and show it agrees with our definition in this case.

Given a functor  $F : \text{Top}_* \rightarrow \text{Top}_*$ , part of the left  $\partial_*(\text{Id})$ -module structure on  $\partial_*(F)$  is a  $\Sigma_2 \wr \Sigma_i$ -equivariant map  $\partial_2(\text{Id}) \wedge \partial_i(F)^{\wedge 2} \rightarrow \partial_{2i}(F)$ . This induces a map

$$\begin{aligned} \psi_i : \Sigma^{-1}(\mathbb{D}_i(F)(X))_{h\Sigma_2}^{\wedge 2} &\cong (\partial_2(\text{Id}) \wedge \partial_i(F)^{\wedge 2} \wedge X^{\wedge 2i})_{h\Sigma_2 \wr \Sigma_i} \\ &\rightarrow (\partial_{2i}(F) \wedge X^{\wedge 2i})_{h\Sigma_{2i}} \\ &\cong \mathbb{D}_{2i}(F)(X), \end{aligned}$$

and for  $x \in H_d(\mathbb{D}_i(F)(X))$ , Behrens defines  $\bar{Q}^j x = (\psi_i)_* \sigma^{-1} Q^j x$ .

Given  $x \in H_d(\mathbb{D}_i(\text{Id})(X)) \subset H_*(F_{\partial_*(\text{Id})}(X))$  and a  $j \geq d$ , to show that the  $\bar{Q}^j x$  from Definition 5.4 agrees with this original version of  $\bar{Q}^j x \in H_{d+j}(\mathbb{D}_{2i}(\text{Id})(X)) \subset H_*(F_{\partial_*(\text{Id})}(X))$  we just need to unwind the definitions, the point being that both the left  $\partial_*(\text{Id})$ -module structure of  $\partial_*(\text{Id})$  and the  $\partial_*(\text{Id})$ -algebra structure of  $F_{\partial_*(\text{Id})}(X)$  come directly from the operad structure maps of  $\partial_*(\text{Id})$ .

**Definition 5.6.** Let  $\bar{\mathcal{R}}$  be the  $\mathbb{F}_2$ -algebra freely generated by  $\{\bar{Q}^j : j \geq 0\}$  subject to the following relations:

$$\bar{Q}^r \bar{Q}^s = \sum_{k=0}^{r-s-1} \binom{2s-r+1+2k}{k} \bar{Q}^{2s+1+k} \bar{Q}^{r-s-1-k}, \quad \text{if } s < r \leq 2s.$$

That relation allows one to rewrite any monomial in the  $\bar{Q}^j$  into a linear combination of  $CU$ -monomials, that is monomials  $\bar{Q}^J = \bar{Q}^{j_1} \bar{Q}^{j_2} \dots \bar{Q}^{j_k}$  where  $J = (j_1, \dots, j_k)$  is a (possibly empty, corresponding to  $1 \in \bar{\mathcal{R}}$ ) sequence of integers satisfying  $j_i > 2j_{i+1}$  for  $i = 1, \dots, k-1$ .

**Definition 5.7.** A non-negatively graded module  $M$  over  $\bar{\mathcal{R}}$  is called *allowable* if whenever  $x \in M$  is homogeneous of degree  $n$  and  $j_1 < j_2 + \dots + j_k + n$ , we have  $\bar{Q}^{j_1} \bar{Q}^{j_2} \dots \bar{Q}^{j_k} x = 0$ .

*Remark 5.8.* This notion of allowable requires more operations to vanish than required by degree considerations, that is, more than required by the condition  $\bar{Q}^j x = 0$  when  $x \in M_d$ ,  $j < d$ . Indeed, that last condition only implies  $\bar{Q}^{j_1} \bar{Q}^{j_2} \dots \bar{Q}^{j_k} x = 0$  when  $j_i < j_{i+1} + \dots + j_k + n - (k - i)$  for some  $i$ ; note the extra negative term  $-(k - i)$ . The reason for this extra vanishing required is the isomorphism in [4, Theorem 1.5.1], that in the notation used there, sends  $\sigma^k \bar{Q}^{j_1} \bar{Q}^{j_2} \dots \bar{Q}^{j_k} \iota_n \mapsto Q^{j_1} Q^{j_2} \dots Q^{j_k} \iota_n$ . The  $Q^j$  do have that vanishing property just for degree reasons.

**Proposition 5.9.** *Given an  $\partial_*(\text{Id})$ -algebra  $L$ , the action of the operations  $\bar{Q}^j$  makes  $H_{\geq 0}(L)$  into an allowable  $\bar{\mathcal{R}}$ -module.*

*Remark 5.10.* Since our goal is to compute  $H_*(F_{\partial_*(\text{Id})}(\Sigma^\infty X))$ , we are focusing here on the relations  $p(\bar{Q}^0, \bar{Q}^1, \dots)x = 0$  that hold between the  $\bar{Q}^j$  when applied to classes  $x$  in non-negative degree.

*Proof.* We will deduce that the operations act allowably and satisfy the relations in the algebra  $\bar{\mathcal{R}}$  from [4, Theorem 1.5.1]. That theorem states that if we define  $\bar{\mathcal{R}}_n$  to be the quotient of  $\bar{\mathcal{R}}$  obtained by imposing the additional relations  $\bar{Q}^{j_1} \bar{Q}^{j_2} \dots \bar{Q}^{j_k} = 0$  whenever

$j_1 < j_2 + \cdots + j_k + n$ , then

$$\bigoplus_{k \geq 0} H_*(\mathbb{D}_{2^k}(S^n)) = \bar{\mathcal{R}}_n \{\iota_n\},$$

where  $\iota_n$  is the fundamental class of  $\tilde{H}_n(S^n)$  (thought of as living in  $H_n(\mathbb{D}_1(S^n)) \cong \tilde{H}_n(S^n)$ ); and the operators  $\bar{Q}^j$  obey all the relations in the algebra  $\bar{\mathcal{R}}_n$ .

Given any class  $x \in H_n(L)$ , we can represent it by map  $x : \Sigma^n \mathbb{H}\mathbb{F}_2 \rightarrow \mathbb{H}\mathbb{F}_2 \wedge L$  of  $\mathbb{H}\mathbb{F}_2$ -module spectra. This corresponds to a map  $x^\dagger : \mathbb{H}\mathbb{F}_2 \wedge F_{\partial_*(\text{Id})}(S^n) \rightarrow \mathbb{H}\mathbb{F}_2 \wedge L$  of  $(\mathbb{H}\mathbb{F}_2 \wedge \partial_*(\text{Id}))$ -algebras. The naturality of the  $\bar{Q}^j$  operations shows that given any  $R \in \bar{\mathcal{R}}$  we have  $H_*(x^\dagger)R\iota_n = Rx$ , so that if the relation  $R\iota_n = 0$  is satisfied in  $H_*(F_{\partial_*(\text{Id})}(S^n))$ , the relation  $Rx = 0$  holds in  $H_*(L)$ .  $\square$

Notice that it also follows from theorem [4, Theorem 1.5.1], that the  $CU$ -monomials  $\bar{Q}^J$  are linearly independent.

## 6. ALGEBRAIC STRUCTURE OF HOMOLOGY OF $\partial_*(\text{Id})$ -ALGEBRAS

We can now state the algebraic structure of the homology of an  $\partial_*(\text{Id})$ -algebra:

**Definition 6.1.** An *allowable  $\bar{\mathcal{R}}$ -sLie-algebra* is a graded  $\mathbb{F}_2$ -vector space  $M$ , equipped with

- a shifted Lie bracket  $[-, -] : M_i \otimes M_j \rightarrow M_{i+j-1}$ , and
- the structure of an allowable  $\bar{\mathcal{R}}$ -module on  $M_{\geq 0}$ ,

such that

- (1)  $\bar{Q}^k x = [x, x]$  if  $x \in M_k$ , and
- (2)  $[x, \bar{Q}^k y] = 0$  for any  $x \in M_i, y \in M_j$ .

*Remark 6.2.* Notice that condition 2 only has content when  $k \geq j$ , since otherwise  $\bar{Q}^k y = 0$ .

**Theorem 6.3.** *Given any  $\partial_*(\text{Id})$ -algebra  $L$ , the operations described above give its mod 2 homology  $H_{\geq 0}(L)$  the structure of a allowable  $\bar{\mathcal{R}}$ -sLie-algebra.*

*Proof.* We've already shown that the bracket gives  $H_*(L)$  the structure of a **sLie**-algebra and of an allowable  $\bar{\mathcal{R}}$ -module in propositions 5.2 and 5.9. We will prove properties 1 and 2 from Definition 6.1 in lemmas 6.4 and 6.5 below.  $\square$

It will be convenient to recall a construction of the Dyer-Lashof operation  $Q^k : H_j(L) \rightarrow H_{j+k}(L_{h\Sigma_2}^{\wedge 2})$  for  $k \geq j$ . A class  $x \in H_j(L)$  can be represented by a map  $x : \Sigma^j \mathbb{H}\mathbb{F}_2 \rightarrow \mathbb{H}\mathbb{F}_2 \wedge L$  of  $\mathbb{H}\mathbb{F}_2$ -module spectra. Applying the second extended power functor we get a map  $x_{h\Sigma_2}^{\otimes 2} : (\Sigma^j \mathbb{H}\mathbb{F}_2)_{h\Sigma_2}^{\otimes 2} \rightarrow (\mathbb{H}\mathbb{F}_2 \wedge L)_{h\Sigma_2}^{\otimes 2}$ , where we've used  $\otimes$  for the smash product of  $\mathbb{H}\mathbb{F}_2$ -module spectra. Since the free  $\mathbb{H}\mathbb{F}_2$ -module functor is symmetric monoidal and preserves homotopy colimits,  $(\mathbb{H}\mathbb{F}_2 \wedge Y)_{h\Sigma_2}^{\otimes 2} \cong \mathbb{H}\mathbb{F}_2 \wedge Y_{h\Sigma_2}^{\wedge 2}$ ; so that we can regard  $x_{h\Sigma_2}^{\otimes 2}$  as being a map  $(\Sigma^j \mathbb{H}\mathbb{F}_2)_{h\Sigma_2}^{\otimes 2} \rightarrow \mathbb{H}\mathbb{F}_2 \wedge L_{h\Sigma_2}^{\wedge 2}$ .

Now,  $(\Sigma^j \mathbb{H}\mathbb{F}_2)^{\otimes 2}$  has trivial  $\Sigma_2$ -action (since this is about  $\mathbb{H}\mathbb{F}_2$ -module spectra, we need not worry about signs in the symmetry of  $\otimes$ ), so  $(\Sigma^j \mathbb{H}\mathbb{F}_2)_{h\Sigma_2}^{\otimes 2} \cong \Sigma^{2j} \mathbb{H}\mathbb{F}_2 \wedge \Sigma_+^\infty B\Sigma_2$ . Let  $q_{k-j} : \Sigma^{k-j} \mathbb{H}\mathbb{F}_2 \rightarrow \mathbb{H}\mathbb{F}_2 \wedge \Sigma_+^\infty B\Sigma_2$  pick out the unique non-zero class of degree  $k-j$  in  $H_*(B\Sigma_2)$ ; then  $Q^k x$  is represented by

$$\begin{aligned} \Sigma^{j+k} \mathbb{H}\mathbb{F}_2 &\xrightarrow{q_{k-j} \otimes \text{id}_{\Sigma^{2j} \mathbb{H}\mathbb{F}_2}} (\mathbb{H}\mathbb{F}_2 \wedge \Sigma_+^\infty B\Sigma_2) \otimes \Sigma^{2j} \mathbb{H}\mathbb{F}_2 \\ &\cong (\Sigma^j \mathbb{H}\mathbb{F}_2)_{h\Sigma_2}^{\otimes 2} \xrightarrow{x_{h\Sigma_2}^{\otimes 2}} \mathbb{H}\mathbb{F}_2 \wedge L_{h\Sigma_2}^{\wedge 2}. \end{aligned}$$

**Lemma 6.4.** *For any  $\partial_*(\text{Id})$ -algebra  $L$  and  $x \in H_k(L)$ , we have  $\bar{Q}^k x = [x, x]$ .*

*Proof.* This follows easily by unwinding the definitions: if  $x$  is represented by a map  $x : \Sigma^k \mathbb{H}\mathbb{F}_2 \rightarrow \mathbb{H}\mathbb{F}_2 \wedge L$ , both sides are represented by the desuspension of some composite

$$\Sigma^k \mathbb{H}\mathbb{F}_2 \otimes \Sigma^k \mathbb{H}\mathbb{F}_2 \rightarrow (\Sigma^k \mathbb{H}\mathbb{F}_2)_{h\Sigma_2}^{\otimes 2} \xrightarrow{x_{h\Sigma_2}^{\otimes 2}} \mathbb{H}\mathbb{F}_2 \wedge L_{h\Sigma_2}^{\wedge 2} \xrightarrow{\mathbb{H}\mathbb{F}_2 \wedge \Sigma \xi} \mathbb{H}\mathbb{F}_2 \wedge L,$$

where  $\xi : \Sigma^{-1} L_{h\Sigma_2}^{\wedge 2} \rightarrow L$  is the structure map. For  $[x, x]$  the first map is taken to be the quotient map, while for  $\bar{Q}^k x$  it is  $q_0 \otimes \text{id}_{\Sigma^{2k} \mathbb{H}\mathbb{F}_2}$ , which agrees with the quotient map.  $\square$

**Lemma 6.5.** *For an  $\partial_*(\text{Id})$ -algebra  $L$  and  $x \in H_i(L)$ ,  $y \in H_j(L)$  we have  $[x, \bar{Q}^k y] = 0$ .*



*Proof.* For  $k < j$ ,  $\bar{Q}^k y = 0$ . For  $k = j$ , by Lemma 6.4,  $[x, \bar{Q}^k y] = [x, [y, y]]$  and this is 0 as explained in section 4.

To analyze the case  $k > j$ , we begin by unwinding the definitions in terms of representing maps  $x : \Sigma^i \mathbb{H}\mathbb{F}_2 \rightarrow \mathbb{H}\mathbb{F}_2 \wedge L$  and  $y : \Sigma^j \mathbb{H}\mathbb{F}_2 \rightarrow \mathbb{H}\mathbb{F}_2 \wedge L$ . To make the next diagram fit on the page, we introduce some temporary notation:  $[i] := \Sigma^i \mathbb{H}\mathbb{F}_2$ ,  $\bar{L} := \mathbb{H}\mathbb{F}_2 \wedge L$ ,  $\overline{B\Sigma_2} := \mathbb{H}\mathbb{F}_2 \wedge \Sigma_+^\infty B\Sigma_2$  and  $\partial_n := \partial_n(\text{Id})$ . Then  $[x, \bar{Q}^k y] \in H_{i+j+k-2}(L)$  is represented by the the composite from the top left corner to the bottom right corner in the following commutative diagram:

$$\begin{array}{ccc}
[i + j + k - 2] & & \\
\downarrow \text{id}_{[i-1]} \otimes \Sigma^{-1} q_{k-j} \otimes \text{id}_{[2j]} & & \\
\Sigma^{-1}[i] \otimes \Sigma^{-1}(\overline{B\Sigma_2} \otimes [2j]) & & \\
\cong \downarrow & & \\
\partial_2 \wedge (\partial_1 \wedge [i]) \otimes (\partial_2 \wedge [j])_{h\Sigma_2}^{\otimes 2} & \xrightarrow{\theta_{[i],[j]}} & (\partial_3 \wedge [i + 2j])_{h(\Sigma_1 \times \Sigma_2)} \\
(\partial_1 \wedge x) \otimes (\partial_2 \wedge y)_{h\Sigma_2}^{\otimes 2} \downarrow & & \downarrow (\partial_3 \wedge x \otimes y^{\otimes 2})_{h(\Sigma_1 \times \Sigma_2)} \\
\partial_2 \wedge (\partial_1 \wedge \bar{L}) \otimes (\partial_2 \wedge \bar{L})_{h\Sigma_2}^{\otimes 2} & \xrightarrow{\theta_{\bar{L}, \bar{L}}} & (\partial_3 \wedge \bar{L}^{\wedge 3})_{h(\Sigma_1 \times \Sigma_2)} \\
\text{id} \otimes (\mathbb{H}\mathbb{F}_2 \wedge \xi) \downarrow & & \downarrow \xi'_3 \\
\partial_2 \wedge \bar{L} \otimes \bar{L} & \xrightarrow{\mathbb{H}\mathbb{F}_2 \wedge \alpha_2} & \bar{L}.
\end{array}$$

The horizontal arrows whose labels involve  $\theta$  are defined using the structure map  $\theta : \partial_2 \wedge \partial_1 \wedge \partial_2 \rightarrow \partial_3$ , namely,

$$\theta_{X,Y} : \partial_2 \wedge (\partial_1 \wedge X) \otimes (\partial_2 \wedge Y)_{h\Sigma_2}^{\otimes 2} \rightarrow (\partial_3 \wedge X \otimes Y^{\otimes 2})_{h(\Sigma_1 \times \Sigma_2)}$$

is given by  $(\theta \wedge \text{id}_{X \otimes Y^{\otimes 2}})_{h(\Sigma_1 \times \Sigma_2)}$ .

The arrow labeled  $\xi'_3$  is  $\mathbb{H}\mathbb{F}_2$  smashed with the composite

$$(\partial_3 \wedge L^{\wedge 3})_{h(\Sigma_1 \times \Sigma_2)} \xrightarrow{(\alpha_3)_{h(\Sigma_1 \times \Sigma_2)}} L \wedge \Sigma_+^\infty B\Sigma_2 \rightarrow L,$$

and that the bottom square commutes follows from the definition of algebra for an operad.

To conclude the proof, we will show that  $(\partial_3 \wedge [i + 2j])_{h(\Sigma_1 \times \Sigma_2)}$  is concentrated in degree  $i + 2j - 2$ , which means the composite from the top of the diagram to that point must be null if  $k \neq j$ . Now, that spectrum is equivalent to  $H\mathbb{F}_2 \wedge \Sigma^{i+2j}(\partial_3)_{h(\Sigma_1 \times \Sigma_2)}$  because the  $(\Sigma_1 \times \Sigma_2)$ -action on  $[i + 2j]$  is trivial. So we need to describe  $\partial_3$  as a  $(\Sigma_1 \times \Sigma_2)$ -spectrum. Recall the description of  $P_3 = B(\text{Comm})(\underline{3})$  from the proof of Proposition 5.2: it consists of three 2-dimensional disks with their boundaries identified, one for each of the three partitions  $(12|3)$ ,  $(13|2)$ ,  $(23|1)$ . The  $(\Sigma_1 \times \Sigma_2)$ -action fixes one of the disks and swaps the other two, so that  $P_3$  is equivariantly equivalent to  $\Sigma^2 \Sigma_+^\infty \Sigma_2$ , the double suspension of the regular representation of  $\Sigma_2$ . Then  $\partial_3$  is  $\Sigma^{-2} \Sigma_+^\infty \Sigma_2$  and  $(\partial_3)_{h(\Sigma_1 \times \Sigma_2)} \cong S^{-2}$ , as required.  $\square$

*Remark 6.6.* Lukas Brantner has written a similar proof of this lemma that will appear in [5]. His argument analyzes the structure map  $\theta$  showing it is the double desuspension of the transfer map  $\Sigma_+^\infty B\Sigma_2 \rightarrow S$  and thus vanishes on mod 2 homology. I am grateful to him for sharing his proof with me at a time when I was still confused about the "bottom operation" and thought this result only held for  $k > j$ .

## 7. HOMOLOGY OF FREE $\partial_*(\text{Id})$ -ALGEBRAS ON SIMPLY-CONNECTED SPACES

Now we can state our main result:

**Theorem 7.1.** *Given a simply-connected space  $X$ , the mod 2 homology of the free  $\partial_*(\text{Id})$ -algebra on  $\Sigma^\infty X$  is the free allowable  $\bar{\mathcal{R}}$ -sLie-algebra  $\mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}_*(X))$  on the reduced homology  $\tilde{H}_*(X)$ .*

*More precisely, the canonical map  $\mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}_*(X)) \rightarrow H_*(F_{\partial_*(\text{Id})}(\Sigma^\infty X))$  is an isomorphism.*

We will prove Theorem 7.1 in special cases of increasing generality in the next few sections, but first we will give a convenient construction of the free allowable  $\bar{\mathcal{R}}$ -sLie-algebra. This will involve the notion of *basic products*, that we now recall:

**Definition 7.2.** The *basic products* on a set of *letters*  $x_1, \dots, x_n$  are defined and ordered recursively as follows:

The basic products of weight 1 are  $x_1, x_2, \dots, x_k$ , in that order.

Suppose the basic products of weight less than  $k$  have been defined and ordered. A basic product of weight  $k$  is a bracket  $[w_1, w_2]$  where

- $w_1$  and  $w_2$  are basic products whose weights add up to  $k$ ,
- $w_1 < w_2$  in the order defined so far,
- if  $w_2 = [w_3, w_4]$  for some basic products  $w_3$  and  $w_4$ , then we require that  $w_3 \leq w_1$ .

Once all the products of weight  $k$  are defined, they are ordered arbitrarily among themselves and declared to be greater than all basic products of lower weight. We will assume these choices of order are fixed once and for all.

Marshall Hall proved in [9] that the basic products form a basis for the free Lie algebra on  $x_1, x_2, \dots, x_k$ . That result is for the totally isotropic, ungraded version of Lie algebra, but it clearly extends, at least for  $R = \mathbb{F}_2$  where the grading does not introduce signs, to both  $\text{Lie}^{\text{ti}}$ -algebras and  $\text{sLie}^{\text{ti}}$ -algebras: if the letters have assigned degrees  $|x_i|$ , we assign to each basic product  $w$  with  $\ell$  letters of total degree  $d$ , the degree  $|w| = d$  in the  $\text{Lie}^{\text{ti}}$  case and  $|w| = d - \ell$  in the  $\text{sLie}^{\text{ti}}$  case.

**Proposition 7.3.** *The free allowable  $\bar{\mathcal{R}}$ -sLie-algebra  $\text{s}\mathcal{L}_{\bar{\mathcal{R}}}(V)$  on a graded  $F_2$ -vector space  $V$  is the free allowable  $\bar{\mathcal{R}}$ -module on the free  $\text{sLie}^{\text{ti}}$  algebra on  $V$ , in symbols  $\mathcal{A}_{\bar{\mathcal{R}}}(\text{F}_{\text{sLie}^{\text{ti}}}(V))$ , equipped with a bracket defined as follows:*

*First, fix a basis  $\beta$  of  $V$  and consider the basis of  $\mathcal{A}_{\bar{\mathcal{R}}}(\text{F}_{\text{sLie}^{\text{ti}}}(V))$  consisting of all  $\bar{Q}^J w$  where:*

- $J = (j_1, \dots, j_k)$  is a *CU*-sequence of integers, and
- $w$  is a basic product of degree at most  $j_k$  in letters from  $\beta$ .

*Now define the bracket on  $\mathcal{A}_{\bar{\mathcal{R}}}(\text{F}_{\text{sLie}^{\text{ti}}}(V))$  on that basis as indicated below and extended bilinearly:*

- $[\bar{Q}^{J_1}w_1, \bar{Q}^{J_2}w_2] = 0$  if  $J_1 \neq \emptyset$  or  $J_2 \neq \emptyset$ .
- The bracket  $[w_1, w_2]$  of basic products is defined recursively as follows:
  - (1) If  $[w_1, w_2]$  is also a basic product, then the bracket is the basis element corresponding to  $[w_1, w_2]$ .
  - (2)  $[w_1, w_2] = \bar{Q}^{|w_1|}w_1$  if  $w_1 = w_2$ .
  - (3)  $[w_1, w_2] = [w_2, w_1]$  if  $w_1 > w_2$ .
  - (4)  $[w_1, w_2] = [w_3, [w_4, w_1]] + [w_4, [w_1, w_3]]$  if  $w_1 < w_2$  and  $w_2 = [w_3, w_4]$  with  $w_1 < w_3$ .

*Proof.* In [9], Hall defines the  $\text{Lie}^{\text{ti}}$  bracket on the linear span of the basic products as above, except that (2) is replaced with  $[w_1, w_1] = 0$ . He then proves that the recursion in the definition does terminate and that it produces a  $\text{Lie}^{\text{ti}}$ -algebra, that is, that the bracket is anti-symmetric, satisfies the Jacobi identity and  $[x, x] = 0$  for all  $x$ . A straightforward adaptation of his proof will show that the above definition also terminates and produces an allowable  $\bar{\mathcal{R}}\text{-sLie}$ -algebra. But before we explain that, let's assume the bracket does define a  $\bar{\mathcal{R}}\text{-sLie}$ -algebra and check that it is free. Let  $f : V \rightarrow E$  be a morphism of graded vector spaces where  $E$  is an allowable  $\bar{\mathcal{R}}\text{-sLie}$ -algebra. There is a unique bracket-preserving extension of  $f$  to the linear span of the basic products, and therefore a unique extension of  $f$  to a morphism of allowable  $\bar{\mathcal{R}}$ -modules  $\mathcal{A}_{\bar{\mathcal{R}}}(F_{\text{sLie}^{\text{ti}}}(V)) \rightarrow E$ . That this unique extension is also a morphism of allowable  $\bar{\mathcal{R}}\text{-sLie}$ -algebras is clear from the above definition of the bracket.

And now we check the bracket correctly produces an allowable  $\bar{\mathcal{R}}\text{-sLie}$ -algebra. First of all, notice that the degrees of the various parts of the definition are correct for a shifted bracket.

Secondly, having  $[w_1, w_1] = \bar{Q}^{|w_1|}w_1$  instead of 0 does not affect termination of the recursion at all. Both 0 and  $\bar{Q}^{|w_1|}w_1$  have the following properties: (1) they are expressions containing no further brackets, so if a term reduces to one of them that term requires no further reduction, and (2) if they appear *inside* a bracket, the term containing that bracket is 0.

This means that the process of reducing a bracket  $[x, y]$  to a linear combination of basic products by repeatedly applying the recursive definition uses exactly the same steps in both Hall's  $\text{Lie}^{\text{ti}}$  case and in our  $\bar{\mathcal{R}}\text{-sLie}$  case, the only difference being that any  $[w, w]$  that appear on their own (that is, not inside a bracket) will reduce to  $\bar{Q}^{|w|}w$  instead of 0.

Next we must check that this bracket satisfies  $[x, \bar{Q}^k y] = 0$ ,  $[x, x] = \bar{Q}^{|x|}x$ , symmetry and the Jacobi identity. All of these need only be checked on the given basis. Symmetry and that  $[x, \bar{Q}^k y] = 0$  are directly built in to the definition, as is the fact that  $[x, x] = \bar{Q}^{|x|}x$  when  $x$  is a basic product. When  $x = \bar{Q}^J w$  for  $J = (j_1, \dots, j_k)$  with  $k \geq 1$ , we have  $[x, x] = 0$  (since  $J \neq \emptyset$ ), but we also have  $|x| = j_1 + \dots + j_k + |w| - k < j_1 + \dots + j_k + |w|$  so that  $\bar{Q}^{|x|}x = \bar{Q}^{|x|}\bar{Q}^J w = 0$  is required by allowability.

Now only the Jacobi identity remains to be checked:

$$\sum_{\text{cyclic}} [\bar{Q}^{J_1} w_1, [\bar{Q}^{J_2} w_2, \bar{Q}^{J_3} w_3]] = 0.$$

If any  $J_i \neq \emptyset$ , all three terms are 0, so assume all  $J_i = \emptyset$ . This remaining case can be proved exactly as in [9, Section 3, p. 579], with one tiny change. There is only one place in that proof where the condition  $[w, w] = 0$  is used: it is at the very beginning of the argument for the Jacobi identity. The proof starts by considering the case when two of the  $w_i$  are equal, say  $w_1 = w_2$ . Then the terms  $[w_1, [w_1, w_3]]$  and  $[w_1, [w_3, w_1]]$  cancel by anti-symmetry and the remaining term is 0 since  $[w_3, [w_1, w_1]] = [w_3, 0]$ . In our case, that last term still vanishes:  $[w_3, [w_1, w_1]] = [w_3, \bar{Q}^{|w_1|}w_1] = 0$ . The rest of Hall's argument goes through verbatim.  $\square$

**7.1. The free  $\partial_*(\text{Id})$ -algebra on a sphere.** For  $X = S^n$ , Theorem 7.1 is essentially a restatement of [4, Theorem 1.5.1] using Proposition 7.3. Indeed, the free  $\text{sLie}^{\text{ti}}$ -algebra on  $\tilde{H}_*(S^n) = \mathbb{F}_2\{\iota_n\}$  is just  $\mathbb{F}_2\{\iota_n\}$  again, so that  $\text{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}_*(S^n)) = \mathcal{A}_{\bar{\mathcal{R}}}(\mathbb{F}_2\{\iota_n\})$ , which is what Behrens shows  $H_*(F_{\partial_*(\text{Id})}(S^n))$  to be.

**7.2. The free  $\partial_*(\text{Id})$ -algebra on a finite wedge of spheres.** Now we consider the case of  $X = S^{d_1} \vee S^{d_2} \vee \cdots \vee S^{d_k}$  for some integers  $d_i \geq 2$ ; in this case  $F_{\mathcal{O}}(X)$  can be computed from the results of [2], which we now summarize.

Consider a bit more generally the case  $X = \Sigma(X_1 \vee \cdots \vee X_k)$ , where the  $X_i$  are some connected spaces. In [2] there is a computation of  $\mathbb{D}_n(\text{Id})(X) = \Sigma \mathbb{D}_n(\Omega \Sigma)(X_1 \vee \cdots \vee X_k)$  that takes multi-variable Goodwillie derivatives on “both sides of the Hilton-Milnor theorem”.

The Hilton-Milnor theorem (see [13, Section XI.6]) gives a homotopy equivalence between  $\Omega \Sigma X = \Omega \Sigma(X_1 \vee \cdots \vee X_k)$  and the weak<sup>3</sup> infinite product  $\prod_w \Omega \Sigma Y_w(X_1, \dots, X_k)$ , where  $w$  runs over the basic products on  $k$  letters, and each  $Y_w$  is the functor obtained from the word  $w$  by interpreting the  $i$ -th letter as  $X_i$ , and the bracket as the smash product; so that  $Y_w(X_1, \dots, X_k) = X_1^{\wedge m_1(w)} \wedge \cdots \wedge X_k^{\wedge m_k(w)}$  with  $m_i(w)$  counting the number of occurrences of the  $i$ -th letter in  $w$ .

To describe an explicit map giving this equivalence we need to recall the definition of the Samelson products. Let  $G$  be an  $H$ -group and  $c : G \times G \rightarrow G$  be the commutator map. The composite map  $G \times \{e\} \rightarrow G \times G \xrightarrow{c} G$  is null as is the analogous map from  $\{e\} \times G$ . This means that  $G \vee G \rightarrow G \times G \xrightarrow{c} G$  is null and thus there is a pointed map  $\bar{c} : G \wedge G \rightarrow G$  (whose homotopy class is well-defined). Now given any two pointed maps  $\alpha : W \rightarrow G$  and  $\beta : Z \rightarrow G$ , we can define their Samelson product  $\langle \alpha, \beta \rangle$  as the composite  $W \wedge Z \xrightarrow{\alpha \wedge \beta} G \wedge G \xrightarrow{\bar{c}} G$ .

Given a basic product  $w$  there is a map  $h_w : Y_w(X_1, \dots, X_n) \rightarrow \Omega \Sigma X$  obtained from  $w$  by interpreting the  $i$ -th letter as the canonical map  $X_i \hookrightarrow X \rightarrow \Omega \Sigma X$  and interpreting the bracket as the Samelson product. Let  $\bar{h}_w : \Omega \Sigma Y_w(X_1, \dots, X_n) \rightarrow \Omega \Sigma X$  be the extension of  $h_w$  to a map of  $A_\infty$ -spaces and for any set  $B$  of basic words let  $\bar{h}_B$  be the composite  $\prod_{w \in B} \Omega \Sigma Y_w(X_1, \dots, X_n) \xrightarrow{\prod \bar{h}_w} (\Omega \Sigma X)^B \xrightarrow{\mu} \Omega \Sigma X$ . Then the Hilton-Milnor theorem can be stated as saying that the colimit of  $\bar{h}_B$  over all finite sets of basic products is an equivalence.

---

<sup>3</sup>This means the homotopy colimit of the finite products, where the maps in the colimit include a product into a larger product using the basepoint on the extra factors.

The result Arone and Kankaanrinta obtain from the Hilton-Milnor equivalence [2, Theorem 0.1] is the following equivalence of spectra:

$$\begin{aligned} & (\partial_n(\Omega\Sigma) \wedge \Sigma^\infty (X_1^{\wedge n_1} \wedge \dots \wedge X_k^{\wedge n_k}))_{h(\Sigma_{n_1} \times \dots \times \Sigma_{n_k})} \cong \\ & \bigvee_{d|\gcd(n_1, \dots, n_k)} \left( \bigvee_{w \in W(\frac{n_1}{d}, \dots, \frac{n_k}{d})} \mathbb{D}_d(\Omega\Sigma) (Y_w(X_1, \dots, X_k)) \right), \end{aligned}$$

where  $W(\frac{n_1}{d}, \dots, \frac{n_k}{d})$  is the set of basic products on  $k$ -letters involving the  $i$ -th letter exactly  $\frac{n_i}{d}$  times.

We can use this to get a nice formula for  $\Sigma^{-1}F_{\partial_*(\text{Id})}(X)$ :

$$\begin{aligned} F_{\partial_*(\Omega\Sigma)}(X_1 \vee \dots \vee X_k) &= \bigvee_n \mathbb{D}_n(\Omega\Sigma)(X_1 \vee \dots \vee X_k) \\ &= \bigvee_n (\partial_n(\Omega\Sigma) \wedge \Sigma^\infty (X_1 \vee \dots \vee X_k)^{\wedge n})_{h\Sigma_n} \\ &= \bigvee_n \left( \partial_n(\Omega\Sigma) \wedge \bigvee_{n_1 + \dots + n_k = n} \text{Ind}_{\Sigma_{n_1} \times \dots \times \Sigma_{n_k}}^{\Sigma_n} (X_1^{\wedge n_1} \wedge \dots \wedge X_k^{\wedge n_k}) \right)_{h\Sigma_n} \\ &= \bigvee_{n_1, \dots, n_k} (\partial_{n_1 + \dots + n_k}(\Omega\Sigma) \wedge (X_1^{\wedge n_1} \wedge \dots \wedge X_k^{\wedge n_k}))_{h(\Sigma_{n_1} \times \dots \times \Sigma_{n_k})} \\ &= \bigvee_{n_1, \dots, n_k} \left( \bigvee_{d|\gcd(n_1, \dots, n_k)} \left( \bigvee_{w \in W(\frac{n_1}{d}, \dots, \frac{n_k}{d})} \mathbb{D}_d(\Omega\Sigma) (Y_w(X_1, \dots, X_k)) \right) \right) \\ &= \bigvee_{m_1, \dots, m_k, d} \left( \bigvee_{w \in W(m_1, \dots, m_k)} \mathbb{D}_d(\Omega\Sigma) (Y_w(X_1, \dots, X_k)) \right) \\ &= \bigvee_{w \in W} F_{\partial_*(\Omega\Sigma)}(Y_w(X_1, \dots, X_k)), \end{aligned}$$

where the last wedge runs over all basic products in  $k$  letters, and the next to last step uses the change of variables  $m_i = \frac{n_i}{d}$ : this gives a bijection between all  $(k+1)$ -tuples  $(n_1, \dots, n_k, d)$  of positive integers with  $d \mid \gcd(n_1, \dots, n_k)$ , and all  $(k+1)$ -tuples  $(m_1, \dots, m_k, d)$  of positive integers.

This in turn tells us, for  $F_{\partial_*(\text{Id})}$ , that:

$$F_{\partial_*(\text{Id})}(\Sigma(X_1 \vee \cdots \vee X_k)) = \bigvee_{w \in W} F_{\partial_*(\text{Id})}(\Sigma Y_w(X_1, \dots, X_k)).$$

Plugging in  $X_i = S^{d_i-1}$ , for some  $d_i \geq 2$ , we get that

$$F_{\partial_*(\text{Id})}(S^{d_1} \vee \cdots \vee S^{d_k}) = \bigvee_{w \in W} F_{\partial_*(\text{Id})}(S^{|w|}),$$

so that Proposition 7.3 allows us to conclude Theorem 7.1 for the wedge  $S^{d_1} \vee \cdots \vee S^{d_k}$  from the case of single spheres.

**7.3. The free  $\partial_*(\text{Id})$ -algebra on a simply-connected space.** Bootstrapping from the previous cases to  $F_{\partial_*(\text{Id})}(X)$  for general simply-connected  $X$  is purely formal using the fact that  $H_*(F_{\partial_*(\text{Id})}(X))$  only depends on the homology of  $X$ , as shown in Proposition 2.2.

Let  $\phi_X : \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\tilde{H}(X)) \rightarrow H_*(F_{\partial_*(\text{Id})}(\Sigma^\infty X))$  be the canonical map coming from the universal property of the the free allowable  $\bar{\mathcal{R}}$ -sLie-algebra.

If  $X$  is an arbitrary wedge of spheres, each of dimension at least 2, then we can write  $X$  as a filtered colimit of finite wedges of spheres and these fall under the previous case. Since homology and the free functors we are using all commute with filtered colimits, the result also holds for such an  $X$ .

Now, for a general simply-connected  $X$ , pick an  $\mathbb{F}_2$ -basis  $\{x_j\}$  of  $\tilde{H}_*(X)$  and use it to construct an equivalence of  $\text{HF}_2$ -module spectra  $f : \bigvee_j \Sigma^{|x_j|} \text{HF}_2 \rightarrow \text{HF}_2 \wedge \Sigma^\infty X$ . The natural transformation  $\phi$  is a special case of a natural transformation  $\psi_V : \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(V)) \rightarrow \pi_*(F_{\text{HF}_2 \wedge \partial_*(\text{Id})}(V))$  for  $\text{HF}_2$ -module spectra  $V$ , in the sense that  $\phi_X = \psi_{\text{HF}_2 \wedge \Sigma^\infty X}$ . In the naturality square

$$\begin{array}{ccc} \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(\bigvee_j \Sigma^{|x_j|} \text{HF}_2)) & \xrightarrow{\psi_{(\bigvee_j \Sigma^{|x_j|} \text{HF}_2)}} & \pi_*(F_{\text{HF}_2 \wedge \partial_*(\text{Id})}(\bigvee_j \Sigma^{|x_j|} \text{HF}_2)) \\ \downarrow \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(f)) & & \downarrow \pi_*(F_{\text{HF}_2 \wedge \partial_*(\text{Id})}(f)) \\ \mathbf{s}\mathcal{L}_{\bar{\mathcal{R}}}(\pi_*(\text{HF}_2 \wedge \Sigma^\infty X)) & \xrightarrow{\psi_{(\text{HF}_2 \wedge \Sigma^\infty X)}} & \pi_*(F_{\text{HF}_2 \wedge \partial_*(\text{Id})}(\text{HF}_2 \wedge \Sigma^\infty X)), \end{array}$$



all maps are known to be isomorphisms (the vertical ones because  $f$  is an equivalence, the top one because it is  $\phi_{(\bigvee_j S^{|x_j|})}$  except the bottom one, which therefore also is an isomorphism.

## REFERENCES

- [1] G. Arone and M. Ching. Operads and chain rules for the calculus of functors. *Astérisque*, (338):vi+158, 2011.
- [2] G. Arone and M. Kankaanrinta. The homology of certain subgroups of the symmetric group with coefficients in  $\mathcal{L}ie(n)$ . *J. Pure Appl. Algebra*, 127(1):1–14, 1998.
- [3] G. Arone and M. Mahowald. The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. *Invent. Math.*, 135(3):743–788, 1999.
- [4] M. Behrens. The Goodwillie tower and the EHP sequence. *Mem. Amer. Math. Soc.*, 218(1026):xii+90, 2012.
- [5] L. Brantner. The Lubin-Tate theory of spectral Lie algebras. In preparation.
- [6] M. Ching. Bar constructions for topological operads and the goodwillie derivatives of the identity. *Geom. Topol.*, 9:833–933, 2005.
- [7] F. R. Cohen, T. J. Lada, and J. P. May. *The homology of iterated loop spaces*, volume 533. Springer-Verlag Berlin, 1976.
- [8] T. G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711 (electronic), 2003.
- [9] M. Hall. A basis for free lie rings and higher commutators in free groups. *Proceedings of the American Mathematical Society*, 1(5):575–581, 1950.
- [10] P. J. Hilton. On the homotopy groups of the union of spheres. *J. London Math. Soc.*, 30(154-172):71, 1955.
- [11] B. Johnson. The derivatives of homotopy theory. *Trans. Amer. Math. Soc.*, 347(4):1295–1321, 1995.
- [12] I. Kriz, M. Mandell, and J. May. *Rings, modules, and algebras in stable homotopy theory*. Number 47. American Mathematical Soc., 2007.
- [13] G. W. Whitehead and G. Whitehead. *Elements of homotopy theory*, volume 61. Springer-Verlag New York, 1978.