Multi-Item Vickrey-Dutch Auction for Unit-Demand Preferences

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Multi-Item Vickrey-Dutch Auction for
Unit-Demand Preferences

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Multi-Item Vickrey-Dutch Auction for Unit Demand Preferences

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Abstract

We consider an economy with one seller and $m$ selfish buyers. The seller has $n$ indivisible heterogeneous items to sell and each buyer wants at most one of those items. Buyers have private, independent and known value on the items. We propose an exact Vickrey-Dutch auction, where prices of appropriate items are decreased by unity in each iteration. This auction converges to Vickrey payoff point exactly if valuations are integer. We then introduce an approximate Vickrey-Dutch auction in which prices of appropriate items are decreased by a positive bid decrement $\epsilon$. This auction converges to the Vickrey payoff point approximately. The terminating conditions of both the auctions are related to a concept called Universal Competitive Equilibrium and we show its relationship to Vickrey payoff point in our model. Using simulation, we show that such Vickrey-Dutch auctions have significant advantage in communication complexity over other existing auctions.
1 Introduction

In this paper, we study the design of mechanisms for selling multiple indivisible heterogeneous items to a set of buyers when buyers have unit demand preference, i.e. demand at most one item. We assume that buyers have value on individual items and their utility on an item is the difference between their value and price offered. Also, we assume values of buyers are private information to them and are independent of the values of other buyers. For such a setting, we are interested in designing descending price iterative auction mechanisms. An iterative auction can be described as a price adjustment procedure which takes as input buyers’ preferences over items in each iteration and calculates the items whose prices need to be adjusted based on that. Most of the literature in microeconomics and mechanism design has focused on designing ascending price auction mechanisms (Demange et al., 1986; Bikhchandani et al., 2001; Parkes, 2001; Gul and Stacchetti, 2000; Ausubel, 2002b). We see little research in designing mechanisms which are iterative and descending price. Since price in descending price auctions starts from a high value and is lowered continuously, it requires participation from fewer buyers to allocate items. This results in significant saving in communication from buyers to the seller and increases the speed of the auction.

Traditionally, in the descending price auction for a single item (called Dutch auction), the seller starts the auction from a very high price (when supply is more than demand) and continuously lowers the price. The first buyer to accept a price wins the auction at that price. The use of such auctions is very popular (because of its speed) for selling flowers in Holland (thus the name Dutch auction), fish in Israel and tobacco in Canada. This type of descending price auction is strategically equivalent to a first-price sealed-bid auction (under independent private values assumption). Thus, bidding one’s value is not a dominant strategy (or equilibrium) in such auctions. In particular, a symmetric equilibrium for every buyer in such auctions is to accept a price slightly lower than value. But we can also consider descending price auctions that implement the outcome and price of the Vickrey auction (Vickrey, 1961) ¹. Vickrey, in his seminal paper (Vickrey, 1961), comments on the single item version of this descending price auction. As he points out, the Dutch auction can be modified to run till a second buyer accepts an offer and the first buyer to accept an offer wins but pays the price at which the second offer is accepted. Quoting Vickrey (page 23):

“On the other hand the Dutch auction scheme is capable of being modified with advantage to a second-bid price basis, making it logically equivalent to the second-price sealed-bid procedure . . . .”

¹In a Vickrey auction for a single item, buyers submit sealed bids and the highest bidder wins and pays the bid price of the second highest bidder. Bidding the true value is a dominant strategy for a participant in a Vickrey auction.
Then, he goes on to describe an apparatus that is commonly used to implement the Dutch auction and how the same apparatus can be modified to implement this second-price auction. Quoting Vickrey (page 23):

“There would be no particular difficulty in modifying the apparatus so that the first button pushed would merely preselect the signal to be flashed, but there would be no overt indication until the second button is pushed, whereupon the register would stop, indicating the price, and the signal would flash, indicating the purchaser.”

Vickrey argues that the apparatus helps in speeding up the pace of the auction and this modified design makes it strategyproof. In this work, we generalize this idea to the multi-item case when buyers have unit demand preferences. In particular, we design iterative descending price auctions in which buyers have simple strategies in an \textit{ex post Nash equilibrium}. We show experimentally that such descending price auctions have significantly lower communication complexity than existing ascending price auctions for this setting, notably the auction in Demange et al. (1986).

Our descending price auctions are different from the traditional Dutch auction. For a single item, our descending price auctions are the same as the modified Dutch auction proposed by Vickrey (1961) and implement the outcome and price of Vickrey auction (second-price sealed-bid). Similarly, for the multi-item case when buyers have unit demand preference, our auctions implement the outcomes and prices of a \textit{generalized Vickrey auction} (MacKie-Mason and Varian, 1994). Since, our auctions implement a generalized Vickrey auction by a descending price-adjustment process, we call them \textit{Vickrey-Dutch auction}. We design two such auctions. One of the auctions, called \textit{exact Vickrey-Dutch auction}, works if valuations are integer. Exact Vickrey-Dutch auction converges to the minimum competitive equilibrium price. Leonard (1983) has shown that the minimum competitive equilibrium price gives each buyer utility equal to their \textit{marginal product} and thus corresponds to the Vickrey payoff point. In our second Vickrey-Dutch auction, called \textit{approximate Vickrey-Dutch auction}, we remove this integer valuation restriction and the auction converges approximately to the minimum competitive equilibrium price.

It is interesting to observe that in a traditional Dutch auction for a single item, if buyers behave \textit{truthfully}, i.e. accept an offer when the price reaches their value, then the final price of the Dutch auction is the \textit{maximum competitive equilibrium} price. Mishra and Garg (2002) generalized this idea to multi-item unit demand case. They propose a descending price auction which converges to the maximum competitive equilibrium price if buyers behave truthfully. Our work can be described as a generalization of the single-item Vickrey-Dutch auction for the multi-item unit demand preference setting.
1.1 Previous Work

In our setting, when buyers have value on items and demand at most one of the items, the questions we seek to answer is, which item should be assigned to which buyer to maximize the total value of assigned buyers (efficiency) and what should be the payment of each buyer so that we can implement an algorithm/mechanism to achieve efficiency (incentive compatibility). The immediate answer to these questions is the famous Vickrey-Clarke-Groves (VCG) class of mechanisms due to Vickrey (1961), Clarke (1971) and Groves (1973). The canonical one-shot VCG mechanism asks each buyer his valuation function, allocates items efficiently and asks each buyer to pay enough so that his net utility is his marginal product. Many researchers seek to implement the VCG outcome using an iterative auction procedure.

Demange et al. (1986) designed two ascending price auctions for our setting. In one of the auctions, called the exact auction, buyers are asked to report their demand sets. Based on the demand sets submitted, the seller calculates a minimal overdemanded set of items and increases their prices by unity. A set of items are overdemanded if the number of buyers demanding items only from that set is greater than the number of items in that set. A minimal overdemanded set of items is such that none of its proper subset of items are overdemanded. This exact auction was shown to converge to minimum competitive equilibrium price if valuations are integer. Demange et al. (1986) also propose an approximate version of their ascending auction. In the approximate auction, buyers bid on items they are interested in by increasing the price of those items by $\epsilon$. The auction stops when there is no bidding for sufficient time. This auction converges to the minimum competitive equilibrium price as $\epsilon$ tends to zero.

The identification of a minimal overdemanded set of items in the exact auction of Demange et al. (1986) is a computationally challenging problem. Sankaran (1994) modifies this auction by suggesting that an appropriate overdemanded set (which need not be minimal) can be found in every iteration with easy computation, and prices of those items can be increased. This modified auction also converges exactly to the minimum competitive equilibrium price and has an interpretation as a variant of Hungarian method or primal-dual algorithm for the assignment problem. Bertsekas (1992) has proposed different primal-dual auction algorithms for the assignment problem. But, his work ignores the incentive compatibility issue, and assumes a straightforward agent bidding strategy.

Crawford and Knoer (1981) study the unit demand preference setting under the context of labor markets. Their setting has two sides of the market: firms who offer salaries for jobs and agents who accept jobs with their salaries. They discuss a salary-adjustment process which converges to an equilibrium salary favored by one side of the market and they remark
that the process can be reversed so that it converges to an equilibrium salary favored by the other side of the market.

For combinatorial allocation problems (CAP), where buyers may demand bundles of items, several ascending price auctions have been proposed. Parkes and Ungar (2002) have proposed ascending price auctions which maintain non-anonymous and non-linear prices and converge to prices which give buyers utility equal to their marginal products. They introduce a concept called universal competitive equilibrium (UCE) and show that the only equilibrium prices and allocations that can give enough information about VCG payments are UCE prices. Their auction terminates in a UCE price, from which VCG payments are calculated. Other notable ascending auctions for different CAP instances include auctions by Gul and Stacchetti (2000), Ausubel (2002a), Ausubel and Milgrom (2002) and de Vries, Schummer and Vohra (2003).

In the descending price auction literature, we find very little previous work in our setting. As discussed earlier, Mishra and Garg (2002) propose a descending price auction which converges to maximum competitive equilibrium (approximately) if buyers bid honestly. To our knowledge, there is no work which proposes an incentive compatible descending price auction for the multi-item setting.

We show significant advantages of our descending price auctions over other auctions in terms of communication complexity. The issue of communication complexity in auction design is discussed in Shoham and Tennenholtz (2001). In their paper, Shoham and Tennenholtz discuss the use of single-item auctions to rationally compute different functions with minimum communication complexity.

1.2 Our Contribution

We propose two descending price auctions that converge to an efficient allocation in which buyers pay their VCG payments. One of the descending price auctions is an exact auction which works if buyers valuations are integers and converges exactly to the minimum competitive equilibrium. We call this the exact Vickrey-Dutch auction (EVDA). EVDA has a bid decrement of unity. In our second descending price auction, we relax the integer valuation assumption and use a finite bid decrement $\epsilon > 0$. This auction converges to the minimum competitive equilibrium price approximately. We call this the approximate Vickrey-Dutch auction (AVDA).

Both the auctions work on a concept called Universal Competitive Equilibrium (UCE). The concept of UCE was introduced in Parkes and Ungar (2002). A competitive equilibrium price and an efficient allocation can give enough information about Vickrey payoff point iff it is a UCE price. UCE provides a useful constructive method to adjust to minimum
competitive equilibrium price. On the basis of this result, we search for UCE prices in our auctions. We show that the only anonymous UCE price in our setting is the minimum CE price. So, our search terminates at prices which give buyers their marginal product and thus behaving truthfully is an ex post equilibrium for buyers. In both our auctions, at every iteration, we find all items which are universally unallocated ($\epsilon$-universally unallocated for AVDA) and lower their prices. The concept of universally (un)allocated items is closely related to UCE.

The approximate version of our descending price auction (AVDA) has several advantages over EVDA for practical implementation because: (i) it allows the use of a flexible bid decrement, (ii) a simpler bidding strategy is an equilibrium strategy for buyers.

Compared with the exact ascending price auction in Demange et al. (1986) and its modification in Sankaran (1994), EVDA is more in tune with the computationally efficient method of Sankaran (1994). The amount of computation done by a seller in EVDA in every iteration is significantly lower than in the exact version of the ascending price auction of Demange et al. (1986).

However, the main advantage offered by our auctions is the communication benefits. We use simulation to show that AVDA performs better than the approximate ascending price auction of Demange et al. (1986) in terms of communication complexity. The savings in communication complexity in AVDA is significant if the competition in the economy is high.

The rest of the paper is organized as follows. In Section 2, we define our model and discuss some preliminaries. Section 3 proposes an exact Vickrey-Dutch auction and shows how it exactly converges to VCG price. In Section 4, we introduce an approximate version of Vickrey-Dutch auction which converges to VCG prices approximately. Section 5 discusses connection between universal competitive equilibrium price, minimum competitive equilibrium price and our Vickrey-Dutch auctions. In Section 6, we describe our simulation results. We conclude with some future research directions in Section 7. The proofs of propositions and lemmas are provided in an appendix at the end of the paper.

2 The Model and Preliminaries

A seller is selling a set of $n$ heterogeneous items. Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be the set of $n$ items. Let $B = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be the set of buyers interested in buying at most one item from the set $A$. We denote $A_{-j} = A \setminus \{\alpha_j\}$ and $B_{-i} = B \setminus \{\beta_i\}$. We will denote $v_{ij}$ as the valuation of buyer $\beta_i$ on $\alpha_j \in A$. We let $p_j$ to denote the price of item $\alpha_j$. The vector of prices on all items is denoted as $p$. The utility or payoff of buyer $\beta_i$ at price $p$ on an item $\alpha_j$ is defined as $u_{ij}(p) = v_{ij} - p_j$. The utility of not buying any item is zero. Since, we consider unit demand preferences, the utility on a set of items $S$ at price $p$ can be defined as $\max_{\alpha_j \in S} u_{ij}(p)$. 
2.1 Competitive Equilibrium

An allocation is an assignment of buyers in \( B \) to items in \( A \). An allocation \( T \) is feasible if no item in \( A \) is assigned more than once and no buyer in \( B \) is assigned more than once. A feasible allocation may have unassigned items and unassigned buyers. We let \( \beta_i \in T \) to denote that \( \beta_i \) is assigned in allocation \( T \). The total value to the system due to an allocation \( T \) is denoted as \( V_T = \sum_{\beta_i \in T} v_{ij} \), where \( \beta_i \) is assigned \( \alpha_j \) in \( T \). An allocation \( T \) is efficient if it is feasible and there does not exist another feasible allocation \( T' \) such that \( V_T < V_{T'} \).

Before defining competitive equilibrium, we define the demand set of a buyer. The demand set of buyer \( \beta_i \), at price \( p \) is defined as

\[
D_i(p) = \{ \alpha_j \in A | u_{ij}(p) \geq u_{ik}(p) \quad \forall \alpha_k \in A \text{ and } u_{ij}(p) \geq 0 \}.
\]

The demand set of a buyer at a price contains all the items which are utility maximizing at that price. If all items in \( A \) give non-positive utility, then the demand set contains the \( \emptyset \) set, which gives zero payoff at all prices.

**Definition 1 (Competitive Equilibrium Price)** A price \( p \) is a competitive equilibrium price vector if there exists a feasible allocation \( T \) with every \( \beta_i \in B \) is assigned \( \alpha_j \) in \( T \) such that \( \alpha_j \in D_i(p) \) and \( p_j = 0 \) for all items not assigned in \( T \).

The tuple \((p, T)\) is called a competitive equilibrium (CE). We say that \( p \) supports the allocation \( T \). Shapley and Shubik (1972), show that the set of CE prices form a complete lattice when buyers demand at most one item. This means that there is a unique minimum and a unique maximum CE price. Also, Gul and Stacchetti (1999) show that if \((p, T)\) is a CE, then \( T \) is an efficient allocation. Leonard (1983) show that the minimum CE prices in our setting correspond to the VCG prices.

Realize that price \( p \) is a vector. Comparing two vectors is not always possible. For example, if \( p^1 = (3, 4), p^2 = (4, 3) \), we can not say which is greater. A minimum (maximum) CE price is a CE price in which the sum of prices of all the items is minimum (maximum) over all CE prices. Due to the lattice property, such minimum and maximum CE prices are unique. In the example, if \( p^1 \) and \( p^2 \) are CE prices then by the lattice property \( p^3 = (3, 3) \) and \( p^4 = (4, 4) \) are also CE prices and \( p^4 \) is greater than both \( p^1 \) and \( p^2 \) while \( p^3 \) is smaller than both \( p^1 \) and \( p^2 \).

3 Exact Vickrey-Dutch Auction

In this section, we introduce a Vickrey-Dutch auction which converges to minimum CE price exactly. We call it the exact Vickrey-Dutch auction (EVDA). In describing EVDA, we
assume that valuations of buyers are integer. First, we introduce some concepts. In every iteration of EVDA, the seller asks the buyers their demand sets and calculates a provisional allocation of buyers to items. This can be done by solving a restricted linear program which maximizes the number of items provisionally allocated and in which buyers can only be allocated items from their demand set and no item can be assigned to more than one buyer. An unallocated buyer at any iteration of EVDA is a buyer which is not provisionally allocated. With this, we define a universally allocated item. It should be noted that an item whose price is equal to zero can be thought to be allocated to a dummy buyer and demanded by another unallocated dummy buyer who value all items at zero.

**Definition 2 (Universally Allocated Item)** An item \( \alpha_j \in A \) is universally allocated if one of the following conditions hold:

- Its price is zero.
- \( \alpha_j \) is provisionally allocated to \( \beta_i \in B \), and it can be allocated to some buyer without changing the set of provisionally allocated items when \( \beta_i \) is removed from the economy.

We provide two examples before providing a simple algorithm to find all universally allocated items in an iteration of EVDA. We give the examples in Figures 1(a) and 1(b). In both the figures, a solid line between a buyer and an item means that the buyer is provisionally allocated to the item. If a buyer has an item in his demand set but is not provisionally allocated to that item, then we give a dashed edge between that buyer and item. So, each of the figures represent all the information available to a seller in an iteration to determine the set of universally allocated items.

![Figure 1: Universally Allocated Items](image)

In Figure 1(a), only item 3 is universally allocated. This is because if we remove buyer 3 (provisionally allocated to item 3), we can provisionally allocate buyer 4 to item 3 without
changing the total set of provisionally allocated items. In case of item 1, if we remove buyer 1, the only buyer that demands item 1 is buyer 2. This means we cannot allocate item 1 without reducing the total set of provisionally allocated items. Similar argument for item 2 shows that it is not universally allocated.

In Figure 1(b), all 3 items are universally allocated. If we remove buyer 3 (allocated to item 3), we can allocate buyer 4 (unallocated) to item 3 without changing the total set of allocated items. If we remove buyer 1 (allocated to item 1), we can allocate buyer 4 to item 3 and buyer 3 to item 1 without changing the total set of allocated items. Finally, if we remove buyer 2 (allocated to item 2), we can allocate buyer 4 to item 3, buyer 3 to item 1 and buyer 1 to item 2 without changing the total set of allocated items. Realize, how item 1, which is demanded by an unallocated buyer (4), is a universally allocated item and is the starting point for finding out other universally allocated items. We use this idea to describe an algorithm to find out the total set of universally allocated items.

Let $U_P$ denote the set of all items which have either zero price or are in the demand set of some unallocated buyer. These are all universally allocated, but there may be additional items as well. Realize here if an item is in the demand set of an unallocated buyer, then it has to be provisionally allocated. Else, we can provisionally allocate the unallocated buyer to this item. We grow the set of universally allocated items (denoted as $U$) as follows:

<table>
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<th>Algorithm: UAI</th>
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<tr>
<td><strong>Step 0:</strong> $U = U_P$.</td>
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<tr>
<td><strong>Step 1:</strong> Let $T$ be the set of buyers provisionally allocated to items in $U$.</td>
</tr>
<tr>
<td><strong>Step 2:</strong> If $T = \emptyset$, STOP. Else, find the set of items, $S$, demanded by buyers in $T$.</td>
</tr>
<tr>
<td><strong>Step 3:</strong> If $S \subseteq U$, STOP. Else, $U = U \cup S$. Repeat from Step 1.</td>
</tr>
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It is straightforward to prove that the algorithm finds the set of all universally allocated items. Now, we can describe EVDA. EVDA goes through iterations and in every iteration the prices of universally unallocated items are lowered by unity.

<table>
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<tr>
<th>EVDA</th>
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<tr>
<td><strong>Step 0:</strong> Start from a price, $p \geq p^{\min}$.</td>
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<tr>
<td><strong>Step 1:</strong> Ask for the demand sets of all the buyers at the current price $p$.</td>
</tr>
<tr>
<td><strong>Step 2:</strong> Based on the demand sets of buyers, allocate as many buyers as possible such that allocated buyers get items from their demand sets and no item in $A$ is allocated more than once.</td>
</tr>
<tr>
<td><strong>Step 3:</strong> Find the universally allocated set of items, $U$.</td>
</tr>
<tr>
<td><strong>Step 4:</strong> If $U = A$ (the set of all items), STOP. Else, set $p_{j} = p_{j} - 1 \forall \alpha_{j} \in (A \setminus U)$. Repeat from Step 1.</td>
</tr>
</tbody>
</table>
Henceforth, we will refer to the bidding strategy of buyers submitting their true demand sets in each iteration as straightforward bidding. Unless mentioned otherwise, it is assumed buyers follow straightforward bidding. Let \( p \) denote the final price in EVDA and \( p^{\min} \) the minimum CE price. If valuations are integer, \( p^{\min} \) is an integer vector (for explanations, see (Ausubel, 2002a)). So, by decreasing the price by unity for long enough, we will never skip the minimum CE price of an item in an iteration of EVDA. The following proposition proves that once the price reaches the minimum CE price for an item, it can not decrease further.

**Proposition 1** The final price \( p \) in EVDA is greater than or equal to \( p^{\min} \).

Using Proposition 1, we prove the convergence of EVDA to minimum CE price.

**Theorem 1** EVDA converges to \( p^{\min} \).

**Proof:** From Proposition 1, \( p \geq p^{\min} \). Assume for contradiction, \( p \neq p^{\min} \). Let \( S = \{ \alpha_j : p_j > p_j^{\min} \} \). By definition of \( S \), every item in \( S \) has positive price and should be demanded by a set of buyers, \( T \) at the end of EVDA (else they will be not be universally allocated). Since all the items in \( S \) are universally allocated, by the definition of universally allocated items, \( |T| > |S| \). This means at \( p^{\min} \), buyers in \( T \) will demand items from \( S \) only. So, \( S \) is overdemandend at \( p^{\min} \), which is a contradiction from Demange et al. (1986). 

Since the minimum CE price gives every buyer his VCG payoff, EVDA gives every buyer their VCG payoff and allocates items efficiently if buyers follow straightforward bidding. It has been shown in Gul and Stacchetti (2000) that if an iterative mechanism converges to minimum CE and the minimum CE gives buyers their VCG payoff, then reporting demand sets at every iteration of the mechanism is an ex post Nash equilibrium \(^2\). For completeness, we provide a proof of the result for EVDA. For this, we follow Bikhchandani and Ostroy (2002) and define some concepts first. Let \( F \) be the set of all possible valuation functions over items with unit demand preference. A strategy for a buyer \( \beta_i \) with valuation function \( v_i \in F \) in EVDA is defined as a valuation function \( \hat{v}_i \in F \) such that \( \beta_i \) reports his demand sets as if his valuation function is \( \hat{v}_i \). Denote \( v_{-i} = (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m) \in F_{-i} \). Define \( \pi_i(\hat{v}_i, v_{-i}|v_i) \) as the payoff of buyer \( \beta_i \) by following strategy \( \hat{v}_i \), when other buyers follow \( v_{-i} \) and his actual valuation is \( v_i \). Thus, \( \pi_i(v_i, v_{-i}|v_i) \) is the payoff by truthfully reporting demand sets.

\(^2\)Gul and Stacchetti (2000) prove this for their ascending-price auction mechanism for a generic demand preference called Gross Substitutes (GS). But their proof technique is independent of the mechanism being implemented. Since, GS demand preferences hold in our setting, we can directly use Gul and Stacchetti’s results here.
Definition 3 (Ex Post Equilibrium) Straightforward bidding is an ex post equilibrium in EVDA if for all \( \beta_i \in B \),

\[
\pi_i(v_i, v_{-i}|v_i) \geq \pi_i(\hat{v}_i, v_{-i}|v_i) \quad \forall \hat{v}_i \in F, \forall v_i \in F, \forall v_{-i} \in F_{-i}.
\] (2)

Theorem 2 Straightforward bidding is an ex post equilibrium for buyers in EVDA.

Proof: Let buyer \( \beta_i \) follow some other strategy and his allocation be \( \alpha_j \) whereas other buyers truthfully report their demand sets at every iteration. Let the final price of EVDA be \( \hat{p} \).

Consider the following valuation function for \( \beta_i \): \( \hat{v}_{ij} = \hat{p}_j + 1 \) and \( \hat{v}_{ik} = 0 \ \forall \alpha_k \neq \alpha_j \). In an economy with \( \beta_i \) having valuation function \( \hat{v}_i \) and other buyers having valuation functions \( v_{-i} \), \( \hat{p} \) is a CE price. So, if we implement a VCG mechanism in this economy, \( \beta_i \) will be assigned \( \alpha_j \). Let the allocation of \( \beta_i \) in a VCG mechanism in the original economy be \( \alpha_k \).

From Leonard (1983), the VCG price is \( p_{\text{min}} \). So, from the incentive compatibility of VCG mechanism we get \( v_{ik} - p_{k_{\text{min}}} = v_{ij} - \hat{p}_j \), where \( \hat{p}_{\text{min}} \) is the minimum CE price (and hence VCG price) of the modified economy. Due to the lattice nature of CE prices, we can also write \( v_{ij} - \hat{p}_{i_{\text{min}}} \geq v_{ij} - \hat{p}_j \). This gives us \( v_{ik} - p_{k_{\text{min}}} \geq v_{ij} - \hat{p}_j \). By Theorem 1, EVDA converges to minimum CE price if buyers follow straightforward bidding. So, following some other strategy weakly lowers the payoff of \( \beta_j \).

\[\blacksquare\]

3.1 An Example

There are two buyers \((\beta_1, \beta_2)\) and two items \((\alpha_1, \alpha_2)\). Valuations are: \( v_{11} = 8, v_{12} = 4, v_{21} = 6, v_{22} = 3 \). Starting price of the auction is \((8, 8)\).

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Price</th>
<th>(D_1())</th>
<th>(D_2())</th>
<th>Provisional Allocations</th>
<th>Universally Allocated Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((8,8))</td>
<td>{(\alpha_1)}</td>
<td>-</td>
<td>(\beta_1 \rightarrow \alpha_1)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>2</td>
<td>((7,7))</td>
<td>{(\alpha_1)}</td>
<td>-</td>
<td>(\beta_1 \rightarrow \alpha_1)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>3</td>
<td>((6,6))</td>
<td>{(\alpha_1)}</td>
<td>{(\alpha_1)}</td>
<td>(\beta_1 \rightarrow \alpha_1)</td>
<td>{(\alpha_1)}</td>
</tr>
<tr>
<td>4</td>
<td>((6,5))</td>
<td>{(\alpha_1)}</td>
<td>{(\alpha_1)}</td>
<td>(\beta_1 \rightarrow \alpha_1)</td>
<td>{(\alpha_1)}</td>
</tr>
<tr>
<td>5</td>
<td>((6,4))</td>
<td>{(\alpha_1)}</td>
<td>{(\alpha_1)}</td>
<td>(\beta_1 \rightarrow \alpha_1)</td>
<td>{(\alpha_1)}</td>
</tr>
<tr>
<td>6</td>
<td>((6,3))</td>
<td>{(\alpha_1)}</td>
<td>{(\alpha_1, \alpha_2)}</td>
<td>(\beta_1 \rightarrow \alpha_1, \beta_2 \rightarrow \alpha_2)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>7</td>
<td>((5,2))</td>
<td>{(\alpha_1)}</td>
<td>{(\alpha_1, \alpha_2)}</td>
<td>(\beta_1 \rightarrow \alpha_1, \beta_2 \rightarrow \alpha_2)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>8</td>
<td>((4,1))</td>
<td>{(\alpha_1)}</td>
<td>{(\alpha_1, \alpha_2)}</td>
<td>(\beta_1 \rightarrow \alpha_1, \beta_2 \rightarrow \alpha_2)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>9</td>
<td>((3,0))</td>
<td>{(\alpha_1)}</td>
<td>{(\alpha_1, \alpha_2)}</td>
<td>(\beta_1 \rightarrow \alpha_1, \beta_2 \rightarrow \alpha_2)</td>
<td>{(\alpha_1, \alpha_2)}</td>
</tr>
</tbody>
</table>
EVDA converges to the minimum CE price (3, 0). It is interesting to note that the price trajectory never touches the maximum CE price \( (p_{\text{max}} = (7, 3)) \). We plot the CE price space, price trajectory of EVDA and price trajectory of exact ascending price auction (Demange et al., 1986; Sankaran, 1994) (henceforth, the exact DGS auction) in Figure 2. It is important to observe that EVDA travels through a significant portion of the entire CE price space before reaching the minimum CE price, whereas the exact DGS auction touches only one CE price and that is the minimum CE price. As we will see later, this has implications on the communication complexity.

![Figure 2: Plot of Price Trajectory and CE Price Space](image)

### 3.2 A Note on an Alternate Vickrey-Dutch Auction

A Vickrey-Dutch auction can also be designed by taking an approach similar to the exact ascending price auction in Demange et al. (1986). A set of items is *weakly underdemanded* if the number of buyers demanding items from that set is less than or equal to the number of items in the set. A *minimal* weakly underdemanded set of items is such that it does not contain a weakly underdemanded set of items. At every iteration of the auction, we can search for a *minimal weakly underdemanded set* of items and lower their prices by unity. This continues till the prices of items either reach zero or there is no weakly underdemanded set of items. We can show that such a process converges to the minimum CE price. Just as Sankaran (1994) argued for his extension of the exact DGS auction, we believe that EVDA is better than this form of auction in two aspects:

- Calculation of a minimal weakly underdemanded set is a computationally difficult task. EVDA gets around this by lowering prices of items which are not universally allocated, which are much simpler to determine.
• EVDA lowers the prices of an underdemanded set of items which are maximal in some sense, which can result in faster convergence to minimum CE price and less communication overheads from buyers to seller, specially for larger economies.

4 An Approximate Vickrey-Dutch Auction

We propose an approximate version of the exact Vickrey-Dutch auction which uses some finite bid decrement $\epsilon > 0$ and does not require valuations to be integers. We call this the approximate Vickrey-Dutch auction (AVDA). We show that AVDA converges to minimum CE price approximately. There is a simple bidding strategy which buyers can follow in AVDA and this constitutes an approximate ex post equilibrium for buyers. AVDA is more suitable for practical implementation than EVDA because it allows the use of flexible bid decrement and a simpler bidding strategy is an approximate ex post equilibrium for buyers.

AVDA goes through iterations. In every iteration, a price is maintained on every item. The seller allocates items to buyers provisionally in every iteration and asks the buyers to report items which give them weakly better utility than their provisional allocation. Based on this, the seller calculates the $\epsilon$-universally allocated set of items in each iteration and lowers the prices of items not in this set by $\epsilon$.

To describe AVDA, we describe some concepts first. Denote the maximum utility of buyer $\beta_i$ on any item at an iteration $t$ of AVDA as $\pi^t_i$.

**Definition 4 (Improvement Set)** If a buyer $\beta_i$ is not provisionally allocated, then improvement set of $\beta_i$ at an iteration $t$ consists of all items which give him at least $\max(0, \pi^t_i - \epsilon)$ utility. If $\beta_i$ is provisionally allocated to some item $\alpha_j \in A$, then improvement set of $\beta_i$ consists of all items which give him weakly more utility than $\alpha_j$ in that iteration.

Analogous to the definition of demand set, we can define the $\epsilon$-demand set of a buyer at a price as all items which give utility within $\epsilon$ of the maximum utility. Later, we will show that if a buyer is not allocated to any item, then his improvement set is exactly his $\epsilon$-demand set. But if a buyer is allocated to an item, his improvement set is a subset of his $\epsilon$-demand set.

Buyers are asked to submit their improvement sets at the current iteration price. Based on the submitted improvement sets, the seller provisionally allocates a buyer if any of the items in the improvement set is unallocated (i.e. not provisionally allocated to any other

---

3From the definition of universally allocated items, the set of all items which are not universally allocated are also weakly underdemanded, but they need not be minimal. In our example in Section 3.1, in iteration 1, $\{\alpha_1, \alpha_2\}$ is not universally allocated and weakly underdemanded. But $\{\alpha_1\}$ and $\{\alpha_2\}$ are also weakly underdemanded and thus minimal.
buyer). The provisional allocation of a buyer can be changed to an item in his improvement set only. Similarly, an unallocated buyer can only be provisionally allocated to an item in his improvement set. If \( \alpha_j \) is in the improvement set of \( \beta_i \), then \( \beta_i \) is said to \( \epsilon \)-demand \( \alpha_j \).

Now, we define the concept of \( \epsilon \)-universally allocated item, analogous to the concept of universally allocated, introduced in the previous section. We assume that an item whose price is less than \( \epsilon \) can be allocated to some dummy buyer who values all items at \( \epsilon \).

**Definition 5 (\( \epsilon \)-Universally Allocated Item)** An item \( \alpha_j \in A \) is \( \epsilon \)-universally allocated if any of the following conditions hold:

- Its price is less than \( \epsilon \).
- \( \alpha_j \) is provisionally allocated to \( \beta_i \in B \), and it can be allocated to some buyer without changing the set of provisionally allocated items when \( \beta_i \) is removed from the economy.

Using a similar algorithm to the one described for finding universally allocated items, a seller can calculate the set of \( \epsilon \)-universally allocated items. Let \( U^\epsilon_p \) denote the set of all items whose prices are less than \( \epsilon \) or which are \( \epsilon \)-demanded by an unallocated buyer. The set of \( \epsilon \)-universally allocated items (denoted as \( U^\epsilon \)) is grown as follows:

**Algorithm: \( \epsilon \)-UAI**

**Step 0:** \( U^\epsilon = U^\epsilon_p \).

**Step 1:** Let \( T \) be the set of buyers provisionally allocated to items in \( U^\epsilon \).

**Step 2:** If \( T = \emptyset \), STOP. Else, find the set of items, \( S \), \( \epsilon \)-demanded by buyers in \( T \).

**Step 3:** If \( S \subseteq U^\epsilon \), STOP. Else, \( U^\epsilon = U^\epsilon \cup S \). Repeat from Step 1.

It is straightforward to prove that the algorithm finds the set of all \( \epsilon \)-universally allocated items. Now, we describe AVDA. At every iteration in AVDA, the list of unallocated items (not provisionally allocated to any buyer) and \( \epsilon \)-universally allocated items are updated.

**AVDA**

**Step 0:** Start from a price \( p \geq p^{\text{min}} + w\epsilon \), where \( w = \min(m, n) \).

**Step 1:** For each buyer \( \beta_i \in B \) (in any arbitrary order):

- Ask for the improvement set of \( \beta_i \) at current price \( p \).
- If \( \beta_i \) \( \epsilon \)-demands an unallocated item \( \alpha_j \), then \( \beta_i \) is provisionally allocated to \( \alpha_j \).
  - In that case, \( \alpha_j \) becomes provisionally allocated.
  - If \( \beta_i \) was previously allocated to some item then that item becomes unallocated.

**Step 2:** Find the set of \( \epsilon \)-universally allocated items, \( U^\epsilon \).

**Step 3:** If \( U^\epsilon = A \), STOP. Else, \( p_j = p_j - \epsilon \forall \alpha_j \in A \setminus U^\epsilon \). Repeat from Step 1.
Since AVDA is very different from EVDA and auctions in Demange et al. (1986), we summarize some of its features.

- Once a buyer is provisionally allocated to some item, he remains provisionally allocated throughout AVDA to some item.

- Every buyer (including the buyers who are provisionally allocated) submits his improvement set in every iteration. This is necessary because the utilities of buyers on items increase in each iteration and buyers should be allowed to switch their provisional allocation.

- An item cannot be provisionally allocated to more than one buyer. Also, the price of an allocated item can fall if it is not $\epsilon$-universally allocated.

- The order in which buyers are asked to submit their improvement sets in every iteration can be arbitrary and does not effect our results for AVDA.

- The provisional allocation of a buyer is changed only if an item in his improvement set is unallocated.

Now, we prove some important properties of AVDA. We give an LP formulation of our problem because these properties have a connection to LP theory. Let $x_{ij} = 1$ if buyer $\beta_i$ is allocated to item $\alpha_j$ and $x_{ij} = 0$, otherwise. The primal (P) of our problem is:

$$\max_{x_{ij}} \sum_{\alpha_i \in A} \sum_{\beta_j \in B} v_{ij} x_{ij}$$

s.t.

$$\sum_{\alpha_j \in A} x_{ij} \leq 1 \quad \forall \beta_i \in B \quad (P)$$

$$\sum_{\beta_i \in B} x_{ij} \leq 1 \quad \forall \alpha_j \in A$$

$$x_{ij} \geq 0 \quad \forall \alpha_j \in A, \forall \beta_i \in B$$

The dual of this problem is (DP):

$$\min_{p_j, \pi_i} \sum_{\alpha_j \in A} p_j + \sum_{\beta_i \in B} \pi_i$$

15
s.t.

\[ p_j + \pi_i \geq v_{ij} \quad \forall \alpha_j \in A, \forall \beta_i \in B \]
\[ p_j \geq 0 \quad \forall \alpha_j \in A, \pi_i \geq 0 \quad \forall \beta_i \in B \]  \hspace{1cm} \text{(DP)}

The complementary slackness (CS) conditions can be written as:

**CS-1** If \( \sum_{\alpha_j \in A} x_{ij} < 1 \), then \( \pi_i = 0 \).

**CS-2** If \( x_{ij} = 1 \), then \( p_j + \pi_i = v_{ij} \).

**CS-3** If \( \sum_{\beta_i \in B} x_{ij} < 1 \), then \( p_j = 0 \).

By duality theory, a feasible solution to \( P \) and \( \text{DP} \) satisfying **CS-1**, **CS-2** and **CS-3**, are optimal solutions. In AVDA, the dual variables \( \pi \) can be interpreted as the maximum utility vector of buyers and \( p \) can be interpreted as the price vector. In AVDA, we try to look for primal and dual feasible solutions which satisfy the CS conditions \textit{approximately}. We prove this fact in Propositions 2 and 4.

Henceforth, we denote the strategy of buyers truthfully submitting their improvement sets as a \textit{straightforward} bidding strategy. Unless mentioned otherwise, we assume buyers follow the straightforward bidding strategy.

**Proposition 2 (CS-2,CS-3)** Let \( p^t \) be the price at an instant \( t \) in AVDA and \( p \) be the final price in AVDA. The following properties hold:

- **CS-2**: If a buyer \( \beta_i \) is allocated \( \alpha_j \) at \( t \), then \( v_{ij} - p^t_j + \epsilon > v_{ik} - p^t_k \quad \forall \alpha_k \in A \).
- **CS-3**: If an item \( \alpha_j \) is unallocated at the end of AVDA, then \( p_j < \epsilon \).

Using Proposition 2, we give a tight upper bound on the final prices of items in AVDA. We will denote \( p^{\text{min}} \) as the minimum CE price and \( \text{min}(m,n) \) as \( w \).

**Proposition 3** Let \( p \) be the final price in AVDA. \( p_j < p_j^{\text{min}} + w \quad \forall \alpha_j \in A \).

Using Proposition 3, we prove that **CS-1** holds approximately at the end of AVDA.

**Proposition 4 (CS-1)** If a buyer \( \beta_i \) has at least \( \epsilon \) utility on some item at the end of AVDA, then he is allocated some item.

Using Propositions 2 and 4, we give a tight lower bound on prices at the end of AVDA.

**Proposition 5** Let \( p \) be the final price in AVDA. \( p_j > p_j^{\text{min}} - w \quad \forall \alpha_j \in A \).
Using Propositions 3 and 5, the following theorem is immediate.

**Theorem 3** If \( \epsilon \to 0 \) then \( p = p^{\text{min}} \), where \( p \) is the final price in AVDA. In particular, \( p_j^{\text{min}} - w\epsilon < p_j < p_j^{\text{min}} + w\epsilon \).

Theorem 3 also shows that if AVDA is run more than once with the same buyers, items and buyers follow the straightforward bidding strategy then the final price of any item in two separate runs will be within \( 2w\epsilon \). At this point, we want to comment on the efficiency of AVDA.

**Theorem 4** Let \( V^{A\text{VDA}} \) be the total value of buyers on their assigned items in AVDA and \( V \) be the total value of buyers on their assigned items in an efficient allocation. \( V^{A\text{VDA}} \geq V - (m + 1)\epsilon \).

*Proof:* Following Bikhchandani (1999), a feasible allocation at a price satisfying conditions in Propositions 2 and 4, is called an \( \epsilon \)-CE. So, the final price of AVDA is an \( \epsilon \)-CE price. A feasible allocation \( T \) is \( \epsilon \)-efficient if there does not exist another feasible allocation \( T' \) with \( V^T < V^{T'} - \epsilon \), where \( V^T \) denotes the total utility to the system by feasible allocation \( T \). Bikhchandani (1999) show that any \( \epsilon \)-CE allocation is an \( (m + 1)\epsilon \)-efficient allocation. ■

Next, we show that a buyer cannot improve his utility by more than \( we \) by following some strategy other than the straightforward bidding strategy.

**Theorem 5** If a buyer follows a strategy other than the straightforward bidding strategy, whereas all other buyers follow the straightforward bidding strategy in AVDA, his maximum gain in utility is less than \( we \). We say straightforward bidding is a \( we \)-ex post equilibrium.

*Proof:* The proof parallels the proof of Theorem 2 except some minor modifications. Let buyer \( \beta_i \) follow some other strategy and his allocation be \( \alpha_j \) whereas other buyers truthfully report their improvement sets at every iteration. Let the final price of AVDA be \( \hat{p} \). Consider the following valuation function for \( \beta_i \): \( \hat{v}_{ij} = \hat{p}_j + 1 \) and \( \hat{v}_{ik} = 0 \ \forall \ \alpha_k \neq \alpha_j \). In an economy with \( \beta_i \) having valuation function \( \hat{v}_i \) and other buyers having valuation functions \( v_{-i} \), \( \hat{p} \) is a CE price. So, if we implement a VCG mechanism in this economy, \( \beta_i \) will be assigned \( \alpha_j \). Let the allocation of \( \beta_i \) in a VCG mechanism in the original economy be \( \alpha_k \). From Leoreanrd (1983), VCG price is \( p^{\text{min}} \). So, from the incentive compatibility of VCG mechanism we get \( v_{ik} - p_k^{\text{min}} \geq v_{ij} - \hat{p}_j^{\text{min}} \), where \( \hat{p}^{\text{min}} \) is the minimum CE price (and hence VCG price) of the modified economy. Due to the lattice nature of CE prices, we can also write \( v_{ij} - \hat{p}_i^{\text{min}} \geq v_{ij} - \hat{p}_j \). This gives us \( v_{ik} - p_k^{\text{min}} \geq v_{ij} - \hat{p}_j \). By Theorem 3, \( \hat{p}_k^{\text{min}} > p_k - w\epsilon \), where \( p \) is final price in AVDA when all the buyers follow straightforward bidding strategy.
This gives us, \( v_{ik} - p_k + w\epsilon > v_{ij} - \hat{p}_j \). So, following some other strategy can increase the utility of \( \beta_i \) by less than \( w\epsilon \).

Finally, we want to comment on the set of buyers who never communicate in AVDA. A buyer communicates with the seller in AVDA if he has a non-empty improvement set. Due to the descending price nature of AVDA, a significant number of buyers may never communicate in AVDA. The following proposition characterizes such buyers. In Section 6, we comment on this further.

**Proposition 6** Let \( S \) be the set of buyers who have utility less than or equal to \((-w\epsilon)\) on any item at \( p^{\min} \). Buyers in \( S \) never communicate in AVDA.

### 4.1 An Example
We will now illustrate AVDA with an example. There are two items \((\alpha_1, \alpha_2)\) and two buyers \((\beta_1, \beta_2)\) with valuations: \(v_{11} = 8, v_{12} = 4, v_{21} = 6, v_{22} = 3\). AVDA is started at price \((8,8)\) and \(\epsilon = 1.5\). Denote the improvement set of \( \beta_1 \) as \( U_1 \) and \( \beta_2 \) as \( U_2 \).

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Price</th>
<th>Allocation</th>
<th>( U_1 )</th>
<th>( U_2 )</th>
<th>( \epsilon )-Universally Allocated Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( (8,8) )</td>
<td>( \beta_1 \to \alpha_1 )</td>
<td>( {\alpha_1} )</td>
<td>-</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>2</td>
<td>( (6.5,6.5) )</td>
<td>( \beta_1 \to \alpha_1 )</td>
<td>-</td>
<td>-</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>3</td>
<td>( (5,5) )</td>
<td>( \beta_1 \to \alpha_1 )</td>
<td>-</td>
<td>( {\alpha_1} )</td>
<td>( {\alpha_1} )</td>
</tr>
<tr>
<td>4</td>
<td>( (5,3.5) )</td>
<td>( \beta_1 \to \alpha_1 )</td>
<td>-</td>
<td>( {\alpha_1} )</td>
<td>( {\alpha_1} )</td>
</tr>
<tr>
<td>5</td>
<td>( (5,2) )</td>
<td>( \beta_1 \to \alpha_1, \beta_2 \to \alpha_2 )</td>
<td>-</td>
<td>( {\alpha_1,\alpha_2} )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>6</td>
<td>( (3.5,0.5) )</td>
<td>( \beta_1 \to \alpha_1, \beta_2 \to \alpha_2 )</td>
<td>-</td>
<td>( {\alpha_1} )</td>
<td>( {\alpha_2} )</td>
</tr>
<tr>
<td>7</td>
<td>( (2,0.5) )</td>
<td>( \beta_1 \to \alpha_1, \beta_2 \to \alpha_2 )</td>
<td>-</td>
<td>( {\alpha_1} )</td>
<td>( {\alpha_1,\alpha_2} )</td>
</tr>
</tbody>
</table>

The minimum CE price is \( p^{\min} = (3,0) \) and the auction converges to \( (2,0.5) \). This is well within the bounds \((3 \pm 2*1.5, 0 \pm 2*1.5)\) provided in Theorem 3.

### 5 Universal Competitive Equilibrium
The price trajectory of AVDA is very different from the price trajectory of the descending price auction in Mishra and Garg (2002). While Mishra and Garg (2002) search for a CE price in their auction, we search for a particular CE price. As a result, their auction converges to the maximum CE price while AVDA, driven by a unique property of minimum CE price, converges to the minimum CE price.
Both our auctions are based on the idea of universally allocated items, which is in turn related to a concept called universal competitive equilibrium (UCE), introduced in Parkes and Ungar (2002) and is related to VCG payment and the minimum CE price. To define UCE, we need some notation. Let the economy with a set \( \bar{B} \subseteq B \) of buyers be denoted as \( \text{ASSIGN}(\bar{B}) \), i.e. in \( \text{ASSIGN}(\bar{B}) \), we can assign items in \( A \) to buyers in \( \bar{B} \) only.

**Definition 6 (Universal Competitive Equilibrium Price)** Price \( p \) is a UCE price if it is a CE price in \( \text{ASSIGN}(B) \) and also a CE price in \( \text{ASSIGN}(B-j) \) \( \forall \beta_j \in B \).

For combinatorial problem ⁴, Parkes and Ungar (2002) show that given a CE price (may be non-linear and non-anonymous) and efficient allocation, then it is sufficient that the prices are UCE price to compute VCG payments. They also prove the converse, i.e. if the VCG payments can be computed from a CE price (may be non-linear and non-anonymous) and efficient allocation, then the prices must be UCE price. Here, we first show that the only linear and anonymous CE price in our unit-demand setting which is UCE is the minimum CE price.

Denote \( A^p \) to be the set of items whose prices are positive in price vector \( p \). Denote \( L(S, p) \) to be the number of buyers demanding items from \( S \subseteq A^p \) at price \( p \). Consider the following two propositions first.

**Proposition 7** \( p_{min} \) is the only CE price at which \( L(S, p_{min}) > |S| \) \( \forall S \subseteq A^{p_{min}} \).

**Proposition 8** Price \( p \) is a CE price and \( L(S, p) > |S| \) \( \forall S \subseteq A^p \), iff \( p \) is a UCE price.

Using these two propositions the following theorem is immediate.

**Theorem 6** \( p_{min} \) is the only anonymous CE price which is a UCE price.

*Proof:* Suppose \( q \neq p_{min} \) is an anonymous CE price which is a UCE price. By Proposition 8, \( L(S, p) > |S| \) \( \forall S \subseteq A^p \). By Proposition 7, \( q = p_{min} \), which gives us a contradiction. ■

Theorem 6 along with results from Parkes and Ungar (2002) say that we can only hope to compute VCG payments from a CE price and an efficient allocation in the unit demand problem if that CE price is the minimum CE price. UCE is particularly useful because it suggests a constructive method to adjust towards minimum CE price.

In both EVDA and AVDA we search for the UCE price, which is also the minimum CE price. An interesting feature of our auctions is once we achieve a CE price, we still keep

⁴In a combinatorial problem, buyers can have value on bundles of items and demand bundles of items.
lowering the prices of appropriate items till the minimum CE price is reached. Once a CE price is reached in EVDA, the subsequent iterations of EVDA does not change the provisional allocation and the prices in all subsequent iterations are also CE prices (see also Figure 2). We eventually reach prices that are a CE price in the economy without the buyer who is assigned to that item. This process continues by adjusting prices on universally unallocated items till we reach a CE price that is also a CE price for the economy without each buyer assigned to each item. Thus, we converge to a UCE price using EVDA. This process is mimicked in AVDA approximately.

6 Experimental Results

In this section, we compare the communication complexity of AVDA with that of the approximate ascending price auction by Demange, Gale and Sotomayor (1986) (henceforth DGS auction) and a one-shot sealed-bid auction.

We show that AVDA has much less communication complexity than the DGS auction and any sealed-bid auction (first-price or VCG based). Note that we only run simulations for AVDA and DGS auction as they are more suitable for a practical implementation than EVDA and exact ascending price auction. But we expect qualitatively similar behavior for the exact versions.

We assume that the communication from seller to buyer is inexpensive but communications from buyers to seller is costly. This assumption is fairly standard in the literature and readers are referred to Shoham and Tennenholtz (2001) for more explanation.

The experiment setup is as follows. We draw valuations of participating buyers uniformly from \([0,100]\). We can model our setting as a bipartite graph with buyers as vertices on one side and items as vertices on the other side. An edge between a buyer and an item represents that the buyer has positive value for that item. These edges are created randomly. We measure the density of the graph as the average number of items to which every buyer is connected. So, if every buyer is connected to every item, that is the highest density you can have in the graph. In our simulations, for every buyer, we randomly assign positive values to \(k\) \((0 < k \leq 1)\) fraction of items and zero values to \((1 - k)\) fraction of the items. So, \(k\) measures the density of the graph (higher \(k\) means higher density and more competition). Throughout, we assume integer valuations, i.e. valuations are an integer uniformly drawn from \([0,100]\). The bid increment in ascending price auction is 1 and bid decrement for EVDA is also 1.

The communication complexity is measured by the number of bits transferred from buyers to seller in an auction. We assume a binary encoding scheme. For the sealed-bid auction, all buyers submit their bids on all items in which they are interested. Since bids on any item are
uniformly drawn from [0, 100], the communication complexity for a buyer to communicate a
bid on a single item is $\lceil \log_2(100) \rceil = 7$. To submit information about the item on which
the bid is submitted, the communication complexity for a buyer is $\lceil \log_2(n) \rceil$, where $n$ is
the number of items. Since each buyer is not interested in $k$ fraction of items (randomly
selected), the total communication complexity in a sealed-bid auction is number of buyers
$\times$ number of items $\times 7k \times \lceil \log_2(n) \rceil$.

In both AVDA and the approximate DGS auction, buyers do not communicate any
price or value information. In each iteration of the DGS auction, the seller goes to each
buyer and asks if he is interested in any item. A buyer sends a single item he is interested
in and the price is increased on that item by the bid increment. If the buyer does not
communicate for long enough time, then it is assumed that the buyer is happy with its
current allocation (if any) or he is not interested in any item at the current price. So, the
maximum communication overhead of a buyer in each iteration is communication of one item.
No communication by a buyer means zero overhead. In AVDA, a buyer sends a set of items
(his improvement set) that he is interested in. To communicate interest in an item there is a
communication overhead of $\lceil \log_2(\text{number of items}) \rceil$. If a buyer is not interested in any
item, then he does not communicate to the seller and there is no communication overhead. A
smarter implementation of communication in AVDA is to ask every buyer to communicate
the items: (i) which were in his improvement set in the previous iteration but not there
in this iteration and (ii) which were not in his improvement set in the previous iteration
but is in this iteration. We observed that this communication of change in every iteration
drastically reduces the communication complexity in AVDA. The plot in Figure 3(a) shows
communication complexity when all the items of interest are communicated, whereas the
plot in Figure 3(b) shows communication complexity when only changes in improvement
sets are communicated.

6.1 Density

We first study the effect of density on the communication complexity. For $m = 20$ and
$n = 15$, we plot the graphs in Figure 3(a) and 3(b). The average number of items allocated
in an efficient allocation increases from around 10 (5 unallocated items) for $k = 0.1$ to 15 (no
unallocated items) for $k = 0.9$ for such data. Figure 3(a) is the plot when buyers submit their
entire improvement set in each iteration of AVDA. In general, communication complexity
increases with density. Clearly, the communication is much higher in AVDA in this case then
in Figure 3(b), when only changes in improvement sets are communicated. But AVDA still
does better than DGS in Figure 3(a). Looking at Figure 3(b), notice that AVDA does much
better than DGS auction and sealed-bid auction when density is high ($k \geq 0.3$). Clearly a
high density setting is more practical in real world. For the rest of the section, we assume buyers communicate only change in their improvement sets in AVDA.

![Figure 3: Communication Complexity with Density](image)

In DGS auction, the communication complexity decreases with a decrease in density. This is because, with a decrease in density the competition in the economy decreases, which means less bidding in DGS auction. This leads to less communication with decreasing density. The communication complexity in AVDA depends on two factors: (i) the change in improvement sets of buyers from iteration to iteration (ii) the number of iterations at which communication happens. As the density increases, the changes in improvement sets of buyers increase. In AVDA, we observed that the number of messages sent by buyers to seller per iteration increased steadily from 0.4 when $k = 0.1$ to 3.3 when $k = 0.9$. This results in more communication with the increase in $k$. But this effect is negated for higher $k$ due to lesser number of iterations (at high $k$, competition is high and minimum CE price is high resulting in lower iterations for AVDA).

As discussed earlier, bidders communicate just once in sealed-bid auctions. In almost all our plots, we see that even with one-time communication, AVDA has better communication complexity. The intuition behind this will be clear from a single item example. In the sealed-bid auction for a single item, all $m$ buyers submit bids on the item resulting in a communication complexity of $7m$ (7 comes from the binary representation of values drawn uniformly from $[0, 100]$). But in AVDA, for a single item (also described in Vickrey (1961)), only two binary communications are needed from the top two valued buyers. The same concept can be generalized for the multi-item setting. AVDA (approximately) asks values of buyers on items which are necessary to compute VCG payments, whereas sealed-bid auction
asks values of all buyers on all items. This explains the sealed-bid auction plots versus AVDA plots in all our figures. The nature of the sealed-bid auction plots in all our figures are straightforward to interpret as they depend on density, number of items and number of buyers and we skip their analysis in the remaining figures.

6.2 Increasing Number of Buyers

We keep the value of $k$ of the graph fixed at 0.5 and the number of items fixed at 20. We observe the communication complexity with increase in number of buyers in Figure 4.

![Figure 4: Communication Complexity with Number of Buyers](image)

The communication complexity of AVDA is lower than DGS auction. In particular, when the number of buyers are more than the number of items (high competition), AVDA has much smaller communication complexity than DGS auction. The communication complexity of AVDA increases with an increase in number of buyers when the number of buyers are less than the number of items. This is because more buyers lead to more items getting allocated in this case and as the number of items getting allocated increases, the communication complexity increases. But this effect is absent once the number of buyers are more than the number of items (when the number of items getting allocated becomes almost constant). This follows from proposition 6. By increasing the number of buyers after a critical point, we just increase the set $S$ in Proposition 6, which does not change communication complexity. This critical point depends on the number of items and the density of the graph.

In the DGS auction, the effect of number of buyers is quite opposite to this. As the number of buyers increase, we see more bidding as the minimum CE prices of items increase. When the number of buyers is less than the number of items, the minimum CE price is low. This results in less communication complexity in DGS auction compared to the communication complexity when the number of buyers is more than the number of items. When the number
of buyers are more than the number of items, then the competition is high resulting in high communication complexity in DGS auction. But the communication complexity increases in both these regions of the graph with increase in the number of buyers. This is because more buyers lead to more bidding and thus more communication.

6.3 Increasing Number of Items

We keep the value of $k$ of the graph fixed at 0.5 and the number of items fixed at 30. We increase the number of items and observe the communication complexity in Figure 5. Again, AVDA outperforms DGS auction, especially for high competition graph settings.

![Figure 5: Communication Complexity with Number of Items](image)

Again, there are two distinct regions in the graph:

- **Region 1** Number of items less than the number of buyers: In such a setting, the competition is high and the average minimum CE price is farther away from zero. In this region, almost all the items are allocated.

- **Region 2** Number of items greater than or equal to number of buyers: In such a setting, the competition is lower and the average minimum CE price is lower than in Region 1 (and closer to zero). In this region, some items (almost equal to the difference in the number of buyers and number of items) remain unallocated.

In DGS auction, region 1 requires more iterations and bidding to converge to minimum CE price than region 2. This explains the heavy drop in communication complexity from region 1 to region 2 in DGS auction. In region 1, as the number of items increase, the number of allocated items increase and since the average minimum CE price is reasonably high for all allocated items, the communication complexity in DGS auction increases with number
of items. But in region 2, the number of unallocated items increase with increase in number of items. Since minimum CE price of unallocated items is zero and the minimum CE price is low for all items in this region, the communication complexity for DGS auction decreases with increased number of items in region 2.

As discussed earlier, the trajectory of AVDA goes through almost the entire CE price space. Hence, the communication complexity in AVDA depends on the size of the CE price space. In region 1, the minimum CE price is high and the overall CE price space is large. As the difference between the number of buyers and items decreases, the minimum CE price falls and the CE price space increases. This leads to higher communication complexity in AVDA. But in region 2, there is little change in the improvement sets of buyers from iteration to iteration due to low competition and this effect cancels out the CE price space effect, resulting in almost constant communication complexity for AVDA.

### 6.4 Increasing Size of Economy

We keep the value of $k$ at 0.5 and the ratio of number of buyers to number of items at $\frac{4}{3}$ and increase the number of items. These numbers represent a reasonably competitive economy. We measure the communication complexity in Figure 6. Communication complexity of both AVDA and DGS auction increase with the size of economy. But the increase rate is much higher for DGS auction because of higher bidding involved due to high minimum CE price with a larger economy.

![Plot of Communication Complexity from Buyers to Seller versus Size of Economy, $k = 0.5$](image)

**Figure 6:** Communication Complexity with Number of Items
7 Conclusion and Future Work

In this work, we have designed an exact and an approximate descending price auction to converge to VCG payments in the assignment problem. We interpreted our efficient descending price auctions as a search for UCE price. Further, we showed that the only anonymous UCE price in our setting is the minimum CE price. Due to its descending price nature, only necessary buyers participate in our auctions. This leads to significantly lower communication from buyers to seller. We provided simulation results for different settings to validate this claim. Simulation results showed significant improvement in communication complexity in our descending price auctions over ascending price auction and one-shot sealed-bid auctions. Particularly, in high competition environments, our descending price auctions dominate other known auctions.

The price trajectory of DGS auctions is such that the only CE price it touches is the minimum CE price. Before reaching the minimum CE price, the demand of items is greater than the supply at all prices in ascending price auctions. This drives the prices up till demand and supply balance at minimum CE price. On the other hand, our descending price auctions initially travel through a price space where demand is less than supply (assuming the starting price is very high). This drives down the price till demand and supply balance. At this point our auction reaches a particular CE price (which may be arbitrarily higher than the minimum CE price). But our search for UCE price drives the prices down till it reaches the minimum CE price. Our descending price auctions travel through the CE price space.

We hope that the intuitions gained in designing our efficient descending price auctions will help in extending these auctions to more general settings. Our future work consists of generalizing EVDA and AVDA to various combinatorial allocation problem instances.

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Appendix

Proof (Proof of Proposition 1): Assume for contradiction, \( p \nless p^{\min} \). Denote the price vector in EVDA in iteration \( t \) as \( p^t \). Consider the last iteration, \( t \), at which \( p^t \geq p^{\min} \). Let \( S_1 = \{ \alpha_j : p^t_j = p^{\min}_j \} \). Let \( S \subseteq S_1 \) be the set of items whose prices were lowered in iteration \( t \). This means, items in \( S \) are universally unallocated in iteration \( t \). Since prices are lowered in iteration \( t \), \( p^{\min}_j > 0 \) \( \forall \alpha_j \in S \). This means all the items in \( S \) are assigned in minimum CE. Let \( T \) be the set of buyers demanding items from \( S \) in minimum CE. Clearly, \( |T| > |S| \) (else we can decrease the prices by sufficiently small amount and get another CE price).

Since \( p^t_j = p^{\min}_j \forall \alpha_j \in S_1 \) and \( p^t_j > p^{\min}_j \forall \alpha_j \in A \setminus S_1 \), buyers in \( T \) will demand items in \( S_1 \) only and demand some item from \( S \) in iteration \( t \). Since prices of items in \( S_1 \setminus S \) are not lowered, they are universally allocated in iteration \( t \). Also, \( |T| > |S| \) implies some buyer in \( T \) is unallocated or allocated to a universally allocated item in \( S_1 \setminus S \) and demands an item in \( S \) in iteration \( t \). From the definition of universally allocated items, some item in \( S \) is universally allocated in iteration \( t \). This gives us a contradiction. This shows that once the price of an item reaches \( p^{\min} \), it cannot be lowered further. ■

Proof (Proof of Proposition 2): CS-2: Consider the first instant \( \beta_i \) gets provisionally allocated to any item in AVDA. By the allocation rules of AVDA and straightforward bidding strategy, \( \pi_i + \epsilon < \pi_i^* \), where \( \pi_i \) is his utility on the allocated item and \( \pi_i^* \) is his maximum utility on any item at this instant. Let the item provisionally allocated to \( \beta_i \) at any instant be \( \alpha_j \) and the item giving maximum utility be \( \alpha_k \). The inequality is satisfied as long as the price decrease of \( \alpha_j \) is as much as that of \( \alpha_k \). If the price decrease of \( \alpha_j \) stops at some instant, this means, \( \alpha_j \) is \( \epsilon \)-universally allocated. If \( \alpha_k \) is also \( \epsilon \)-universally allocated as \( \beta_i \) \( \epsilon \)-demands \( \alpha_k \). Hence, price decrease of \( \alpha_k \) should also stop. So, the inequality will still hold.

CS-3: The proof follows from the rules of AVDA. ■

Proof (Proof of Proposition 3): Consider the following lemma.

Lemma 1 Let \( I \) be a set of items such that \( p_j \geq p^{\min}_j + \delta \forall \alpha_j \in I \) and \( \delta \geq \epsilon \). There exists an item \( \alpha_k \notin I \) such that \( p_j > p^{\min}_j + \delta - \epsilon \).

Proof: By the definition of \( I \), all the prices at the end of AVDA in \( I \) are \( \geq \epsilon \). By Proposition 2, all the items in \( I \) are allocated in AVDA. Let \( J \) be the set of buyers allocated items from \( I \) in the auction. By straightforward bidding, buyers in \( J \) will have \( \geq 0 \) utility in AVDA. By the definition of \( I \), buyers in \( J \) will have more than \( \delta \geq \epsilon \) utility in minimum CE price. By the definition of CE, buyers in \( J \) should be assigned some item in minimum CE. There are two cases:
Case 1: Buyers from $J$ are assigned items from only $I$ in minimum CE. Now, consider the instant when the last item from $I$ reaches its final price in AVDA. Since $p_j \geq \delta \geq \epsilon \forall \alpha_j \in I$, these items are provisionally allocated and $\epsilon$-universally allocated (by Proposition 2). By the definition of $\epsilon$-universally allocated items, there exists a buyer $\beta_i \notin J$ which $\epsilon$-demands an item $\alpha_j \in I$. Since $p_j \geq p_j^{\min} + \delta \geq \epsilon$, $\beta_i$ should have $\geq \epsilon$ utility on some item in minimum CE price. By the definition of CE, $\beta_i$ should be assigned some item $\alpha_k$ in CE. By definition of $J$, $\alpha_k \notin I$. From Proposition 2 we have, $v_{ij} - p_j > v_{ik} - p_k - \epsilon$. Also, from the definition of CE, $v_{ik} - p_k^{\min} \geq v_{ij} - p_j^{\min}$. Adding these two equations and using the fact that, $p_j \geq p_j^{\min} + \delta$, we get $p_k > p_k^{\min} + \delta - \epsilon$.

Case 2: A buyer $\beta_i \in J$ is assigned an item $\alpha_k \notin I$ in CE. Let the assignment of $\beta_i$ in the auction be $\alpha_j \in I$. Again, from Proposition 2 we have $v_{ij} - p_j > v_{ik} - p_k - \epsilon$. Also, we have, $v_{ik} - p_k^{\min} \geq v_{ij} - p_j^{\min}$. Using the argument of case 1 again, we get $p_k > p_k^{\min} + \delta - \epsilon$.

The proof of the proposition is by repeatedly applying Lemma 1. Assume for contradiction that there is an item $\alpha_j$ for which $p_j \geq p_j^{\min} + w\epsilon$. Set $\delta = w\epsilon$ and $I = \{\alpha_j\}$ and apply Lemma 1 to discover another item $\alpha_k$, with $p_k > p_k^{\min} + (w - 1)\epsilon$ and which is assigned to a buyer in auction as $p_k > \epsilon$. Now, set $I = I \cup \{\alpha_k\}$ and $\delta = (w - 1)\epsilon$ and apply Lemma 1 again. This process can continue and we will end up discovering more than $n$ items (if $n \leq m$) or more than $m$ buyers (if $m \leq n$), which gives us a contradiction.

Proof (Proof of Proposition 4): Let the maximum utility of $\beta_i$ at the start of AVDA be $\pi_i$. Denote the maximum utility of $\beta_i$ at any iteration $t$ of AVDA as $\pi_i^t$. There are two cases.

Case 1: $\pi_i \leq 0$. Consider the first iteration when $t$ in AVDA, when $\pi_i^t \geq 0$. Since bid decrement is $\epsilon$, $\pi_i^t < \epsilon$. At this iteration, all items giving non-negative utility are in the improvement set of $\beta_i$. So, either $\beta_i$ is provisionally allocated an item or, he is unallocated and $\epsilon$-demands all items that give him non-negative utility. In the first case, by the rules of the auction, $\beta_i$ remains allocated throughout AVDA. In the latter case, all items giving him non-negative utility become universally allocated and thus their prices dont fall in AVDA and $\pi_i^t < \epsilon$ is always satisfied.

Case 2: $\pi_i > 0$. Since starting price of AVDA is $\geq p^{\min} + w\epsilon$, $\beta_i$ will have positive utility on some item in minimum CE. This means $\beta_i$ is assigned some item $\alpha_j$ in minimum CE. Let the utility of $\beta_i$ on $\alpha_j$ be $\pi_i^* \in \text{minimum CE}$.

Assume for contradiction that $\beta_i$ is unallocated at the end of AVDA. By the straightforward bidding strategy, if $\beta_i$ remains unallocated throughout AVDA, his maximum utility on any item will remain $\pi_i$ throughout AVDA (this is because as soon as utility on any item reaches $\geq \max(0, \pi_i - \epsilon)$, $\beta_i$ will $\epsilon$-demand it and that item becomes universally allo-
cated). Since starting price of AVDA is $p^{\min} + w\epsilon$, we have $\pi_i \leq \pi_i^\ast - w\epsilon$. This implies, $p_j \geq p_j^{\min} + w\epsilon$, where $p$ is the final price in AVDA. By proposition 3, this is a contradiction. This means $\beta_i$ is allocated in AVDA.

**Proof (Proof of Proposition 5):** Consider the following lemma.

**Lemma 2** Let $I$ be a set of items allocated with $p_j \leq p_j^{\min} - \delta \forall \alpha_j \in I$ and $\delta \geq \epsilon$. Let $J$ be the set of buyers allocated items from $I$ in AVDA ($J$ can be empty also). There exists a buyer $\notin J$ who is assigned an item $\alpha_k \notin I$ in AVDA such that $p_k < p_k^{\min} - \delta + \epsilon$.

**Proof:** Since $p_j^{\min} > p_j + \delta \geq \epsilon \forall \alpha_j \in I$, all the items in $I$ should be assigned in minimum CE. Let $K$ be the set of buyers assigned to items in $I$ in minimum CE. There are two cases. 

**Case 1:** $J \subseteq K$. Clearly, there exists a buyer $\beta_i \notin K$, which demands an item from $I$ in minimum CE (else, we can decrease the prices of items in $I$ and still maintain CE). Since, $J \subseteq K$, $\beta_i \notin J$. By the definition of $I$, $\beta_i$ should have more than $\delta \geq \epsilon$ utility on that item at the end of AVDA. By Proposition 4, $\beta_i$ should be allocated at the end of AVDA. Clearly that assignment has to be an item $\alpha_k \notin I$. If the item demanded from $I$ by $\beta_i$ in minimum CE is $\alpha_j$, we can write, $v_{ik} - p_k > v_{ij} - p_j - \epsilon$ (from Proposition 2) and $v_{ij} - p_j^{\min} \geq v_{ik} - p_k^{\min}$ (from the definition of CE). Adding the two equations and using the fact that $p_j \leq p_j^{\min} - \delta$, we get, $p_k < p_k^{\min} - \delta + \epsilon$.

**Case 2:** There is a buyer $\beta_i \notin J$ who is assigned $\alpha_j \in I$ in minimum CE. Since $p_j \leq p_j^{\min} - \delta \leq \epsilon$, $\beta_i$ has $\geq \epsilon$ utility on $\alpha_j$ at the end of AVDA. By Proposition 4, $\beta_i$ should be assigned in AVDA. Let that item be $\alpha_k \notin I$. We can use arguments similar to case 1 and show $p_k < p_k^{\min} - \delta + \epsilon$.

The proof of the proposition is by repeatedly applying Lemma 2. First, assume for contradiction, that there exists an item $\alpha_j$ such that $p_j \leq p_j^{\min} - w\epsilon$. Set $I = \{\alpha_j\}$ and $\delta = w\epsilon$ and apply Lemma 2 to discover another item $\alpha_k$ which should be assigned in phase 1 to some buyer $\beta_i'$. Then set $I = I \cup \{\alpha_k\}$, $J = J \cup \{\beta_i'\}$ and $\delta = (w-1)\epsilon$. Continuing this process, we can discover more than $n$ items (if $n \leq m$) or more than $m$ buyers (if $m \leq n$). This gives us the contradiction.

**Proof (Proof of Proposition 6):** By Proposition 5 the final price of AVDA is greater than $p_j^{\min} - w\epsilon \forall \alpha_j \in A$. This means, buyers in $S$ will have utility less than zero at the end of AVDA. Since, AVDA is descending price, these buyers will never communicate.

**Proof (Proof of Proposition 7):** By Theorem 1, the final price of EVDA is the minimum CE price. By the condition in the auction, there does not exist a set of items which are weakly underdemanded at $p^{\min}$. So, $L(S, p^{\min}) > |S| \forall S \subseteq A^{p^{\min}}$.  

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Let there be a CE price \( q \neq p^{\min} \) and for which \( L(S, q) > |S| \forall S \subseteq A^q \). By the lattice nature of CE prices (Shapley and Shubik, 1972; Gul and Stacchetti, 1999), \( q \geq p^{\min} \). Define \( S' = \{ \alpha_i : q_i > p_i \} \). So, \( L(S', q) > |S'| \). By reducing the prices from \( q \) to \( p \), the buyers in \( L(S', q) \) will only demand items from \( S' \). Following (Demange et al., 1986), \( S' \) is overdemanded. This means \( q \) can not be a CE price. So, we reach a contradiction.

**Proof (Proof of Proposition 8):** First we prove \( \Rightarrow \). Consider an economy \( ASSIGN(B_{-j}) \). For this economy, we can write \( L(S, p) \geq |S| \forall S \subseteq A^p \). This means every item in \( A^p \) is demanded by some buyer in \( B \setminus \{ \beta_j \} \). Since \( p \) is a CE price, every subset \( S \subseteq A \) is not overdemanded. By removing \( \beta_j \), it will still not be overdemanded and we have shown that every item in \( A^p \) is demanded by some buyer. By Hall’s Theorem (1935), we can find a competitive equilibrium at this price with buyers in \( B \setminus \{ \beta_j \} \). So, \( p \) is a CE price for the economy \( ASSIGN(B_{-j}) \).

For \( \Leftarrow \), we know that if \( p \) is a UCE price then it is a CE price also. The necessary condition says that \( L(S, p) \geq |S| \forall S \subseteq A^p \). Assume for contradiction that for some \( S \subseteq A^p \), \( L(S, p) = |S| \). Consider any buyer \( \beta_j \in L(S, p) \). Clearly, in the economy \( ASSIGN(B_{-j}) \), \( L(S, p) < |S| \). This means \( p \) is not a CE price for the economy \( ASSIGN(B_{-j}) \). This is a contradiction by the definition of UCE price.