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Wess-Zumino-Witten terms in graphene Landau levels

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We consider the interplay between the antiferromagnetic and Kekulé valence bond solid orderings in the zero energy Landau levels of neutral monolayer and bilayer graphene. We establish the presence of Wess-Zumino-Witten terms between these orders: this implies that their quantum fluctuations are described by the deconfined critical theories of quantum spin systems. We present implications for experiments, including the possible presence of excitonic superfluidity in bilayer graphene.

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Introduction

A number of recent experimental [1–7] and theoretical [8–18] works have focused on the presence of antiferromagnetism (AF) in neutral monolayer and bilayer graphene in an applied magnetic field. It has also been argued that a non-magnetic state with lattice symmetry breaking in the Kekulé valence bond solid (VBS) pattern (see Fig. 1) is proximate to the AF state [12, 13, 16, 17]. Bilayer graphene offers a particularly attractive area for studying the interplay between the AF and VBS order because it may be possible to tune between them by applying a transverse electric field [1, 13, 17].

The presence of the competing AF and VBS orders sets up the possibility [17] of novel quantum criticality between these orders, similar to that found in insulating quantum spin models [19–29]. However these quantum spin models apply in the limit of very large on-site Coulomb repulsion between the electrons, and this is not the appropriate parameter regime for graphene. Here we examine a complementary limit of large magnetic field and moderate interactions, so that it is permissible to project onto an effective Hamiltonian acting only on the zero energy Landau levels. Such a limit has been widely used with considerable success in describing the properties of graphene. Our main new result is that the Landau level projected effective action for the AF and VBS orders has a topological Wess-Zumino-Witten (WZW) term [30–32] for both the monolayer and bilayer cases. The presence of this term implies [33–35] that the field theories of deconfined criticality [20, 21] apply to graphene, and we will discuss experimental implications.

Model and results

We begin by directly stating the Hamiltonian of the low energy graphene bands (see e.g. Refs. [15, 17] for details)

\[
H = v \begin{pmatrix} 0 & a^q \\ a^{\dagger q} & 0 \end{pmatrix}
\]

where \( v \) is a Fermi velocity, \( a = p_x - i p_y - (e/c)(A_x - A_y) \) with \((p_x, p_y)\) the electron momentum, \((A_x, A_y)\) is the vector potential of the applied magnetic field, the matrix acts on the graphene sublattice index, and \( q = 1 \) for monolayer graphene, while \( q = 2 \) for the bilayer case. For bilayers, the sublattice index co-incides with the layer index. For both monolayers and bilayers, there is an additional 2-fold valley degeneracy, along with the usual 2-fold spin degeneracy (in the absence of a Zeeman coupling).

The \( a, a^\dagger \) obey commutation relations proportional to those of the ladder operators of a harmonic oscillator, and so it is easy to diagonalize \( H \). In this manner we obtain \( q \) zero energy Landau levels, which are spanned by the orthonormal eigenfunctions \( \psi_\ell(r) \), where \( \ell = 1 \ldots qN_F \), with \( N_F \) the number of flux quanta. So we write the electron annihilation field operator projected to the zero
energy Landau levels as
\[ \Psi(r) = \sum_{\ell=1}^{qN_a} \psi_{\ell}(r)c_{\ell}. \tag{2} \]
where \(c_{\ell}\) is canonical fermion annihilator operator. In the zero energy Landau levels, the valley, sublattice, and layer indices all co-incide; henceforth we will refer to this as a valley index, and it can take 2 values. The fermion operators also carry a spin index with 2 possible values, and we do not explicitly display the spin or valley indices.

We now introduce Pauli matrices \(\sigma^{x,y,z}\) which act on the spin space, and a second set \(\rho^{x,y,z}\) which act on the valley space (here we follow the conventions of Ref. [17]). In terms of these matrices, the 3-component AF order is measured by \((\rho^x\sigma^x, \rho^y\sigma^y, \rho^z\sigma^z)\) while the 2-component VBS order parameter is \((\rho^x, \rho^y)\).

It is convenient to write the above matrices as
\[ \Gamma_1 = \rho^x\sigma^x, \quad \Gamma_2 = \rho^y\sigma^y, \quad \Gamma_3 = \rho^z\sigma^z, \quad \Gamma_4 = \rho^x, \quad \Gamma_5 = \rho^y. \]
and to notice that the 5 \(\Gamma_a\) matrices anti-commute and square to unity; indeed these are the 5 Dirac gamma matrices. Their 10 products \(i\Gamma_a\Gamma_b\) \((a \neq b)\) realize the Lie algebra of SO(5), and the 15 matrices \(\Gamma_a\) and \(i\Gamma_a\Gamma_b\) realize the Lie algebra of SU(4).

Next, we introduce a 5-component unit vector \(n_a(r, \tau)\), where \(r = (x, y)\) are the spatial co-ordinates and \(\tau\) is imaginary time, representing the combined spacetime fluctuations of the AF and VBS orders. Then the imaginary time Lagrangian of the electrons projected to the zero energy Landau levels is
\[ \mathcal{L} = \sum_{\ell=1}^{qN_a} c_{\ell}^\dagger \frac{\partial c_{\ell}}{\partial \tau} - \lambda \int d^2r n_a(r, \tau) \Psi^\dagger(r, \tau) \Gamma_a \Psi(r, \tau), \tag{3} \]
where \(\lambda\) is the coupling of the electrons to the AF and VBS orders, and there is an implicit sum of \(a\) over 5 values, and also over the spin and valley indices.

Now we can state our primary result. We integrate over the the \(c_{\ell}\) electrons in \(\mathcal{L}\) and obtain an effective action for unit vector \(n_a(r, \tau)\). Apart from the usual terms of the O(5) non-linear sigma model considered in Ref. [16] (and anisotropies due to the Zeeman coupling and electron-electron interactions [16]), the effective action has a topological WZW term at level \(q\)
\[ \mathcal{S}_{WZW} = \frac{2\pi i q W[n_a]}{8 \pi^2} \int_0^1 du \int d^2r d\tau e_{abced} n_a \partial_x n_b \partial_y n_c \partial_z n_d \partial_u n_e \tag{4} \]
\[ W[n_a] = \frac{3}{8 \pi^2} \int_0^1 du \int d^2r d\tau e_{abced} n_a \partial_x n_b \partial_y n_c \partial_z n_d \partial_u n_e \]
Here we have introduced the extra co-ordinate \(u\), and \(n_a(r, \tau, u)\) is any function which smoothly extrapolates from the physical \(n_a(r, \tau)\) at \(u = 1\) to a fixed value (say) \(n_a = (1, 0, 0, 0, 0)\) at \(u = 0\). The choice of the extrapolation can only change \(W[n_a]\) by integers, and so \(e^{2\pi i q W}\) is well defined.

In the case of graphene in zero magnetic field and weak interactions, the same WZW term between the Néel and VBS orders is also present [36]. However, for the experimentally important case of bilayer graphene, there is no such WZW term for the AF and VBS orders at zero field and weak interactions [17, 37] (although, E.-G. Moon has noted such a term for the quantum spin Hall order [37]); so, in this case the zero energy Landau level projection is crucial for obtaining the topological coupling.

Such a WZW term has a strong impact in the interplay between the order parameters. As we will review below, it topologically links AF order to defects of the VBS order, and vice versa.

**Derivation**

We provide two derivations of Eq. (4).

First, pick any 3 of the 5 \(n_a\) components, say \(a = u, v, w\), and set the other 2 to zero. Then we have unit 3-vector field \(\vec{N} = (n_u, n_v, n_w)\). Now consider a static Skyrme texture in \(\vec{N}(r)\). Then by a computation parallel to that in Section III.B of K. Moon et al. [38] (and its generalization to \(q = 2\) [39]), the Skyrmion acquires a "charge". In the present situation the charge is measured by \(i\Gamma_a \Gamma_b \Gamma_w\) and its spatial density is
\[ \langle \Psi^\dagger(r) i\Gamma_a \Gamma_b \Gamma_w \Psi(r) \rangle = \frac{q}{2\pi} \vec{N} \cdot (\partial_x \vec{N} \times \partial_y \vec{N}) \tag{5} \]
where the angular brackets represent the expectation value over the occupied states in the zero energy Landau level perturbed by the texture in \(\vec{N}\) as in \(\mathcal{L}\). Now consider a VBS vortex i.e. a 2\(\pi\) vortex in \((n_u, n_v, n_w)\) applied to \(\mathcal{L}\). For a two-component order, the core of the vortex has a singularity, but this can be relieved by orienting \(n_a\) in a third direction, say \((\pm 1, 0, 0, 0, 0)\). Now the VBS vortex is equivalent to a half-Skyrmion in \(\vec{N} = (n_1, n_4, n_5)\), and after integrating Eq. (5) over all space, this vortex has \(\langle \sigma^z \rangle = \pm q\). Similarly, vortex cores in the directions \((0, \pm 1, 0, 0, 0)\) and \((0, 0, \pm 1, 0, 0)\) yield \(\langle \sigma^y \rangle = \pm q\) and \(\langle \sigma^z \rangle = \pm q\). So we reach the important conclusion that the VBS vortex has total spin \(S = q/2\), and has an associated \((q+1)\)-fold degeneracy. For \(q = 1\), note that this is precisely the situation considered in Ref. [40] for quantum spin models (see also [41]). Alternatively, we can examine the fate of \(S_{WZW}\) in the presence of such VBS vortices: following a computation by Grover and Senthil [35], we find that the WZW term reduces to the quantum Berry phase of a single spin with \(S = q/2\). From this we conclude that Eq. (5) implies Eq. (4).

For a second derivation of the WZW term from Eq. (3), we examine a diagrammatic expansion of \(\mathcal{L}\). Consider a situation where \(n_a\) is polarized near, say, \((0, 0, 0, 0, 1)\). Then, we can write \(n_a = (\pi_1, \pi_2, \pi_3, \pi_4, 1)\) where \(\pi_i \ll 1\) for \(i = 1 \ldots 4\). Then to zeroth order in the \(\pi_i\), the \(c_{\ell}\)
operators in $\mathcal{L}$ have the Green’s function

$$G = (i\omega + \lambda \Gamma_5)^{-1}$$  \hspace{1cm} (6)

where $\omega$ is the frequency of the electron propagator. We now proceed to integrate out the electrons, and derive an effective action for the $\pi_i$. At fourth order in the $\pi_i$, we consider the box diagram in Fig. 2; this can be evaluated by methods similar to those in [17], but with the $G$ above, and the vertices contributing the factors implied by Eq. (3). A computation described in the supplement yields the contribution

$$S_\pi = \frac{3q}{16\pi} \int d^2r d\tau \epsilon_{ijk} \partial_x \pi_i \partial_y \pi_j \partial_\tau \pi_k \partial_r \pi_m. \hspace{1cm} (7)$$

It can be checked that Eq. (4) reduces to $S_\pi$ for $n_{s} = (\pi_1, \pi_2, \pi_3, \pi_4, 1)$, and so $S_{\text{WZW}}$ is the explicitly $\text{SO}(5)$ invariant form of $S_\pi$.

**Theoretical consequences**

We now turn to a discussion of the theoretical consequences of the WZW term for the vicinity of the AF-VBS transition. For $q = 1$, it has been demonstrated in Refs. [34, 35, 40] that the O(5) non-linear sigma model with O(3)$\times$O(2) anisotropy and a level 1 WZW term is equivalent to the CP$^1$ model in 2+1 dimensions. This is the same model appearing in the AF-VBS transition of SU(2) quantum spin models [20, 21], and is a relativistic field theory with a U(1) gauge field and a 2-component complex scalar $z_a$. In terms of these fields, the AF order is $z_a \sigma_a^{x,y,z} z_{a'}$, with $s = x,y,z$; so the vector AF order has been ‘fractionalized’ into spinors $z_a$. Alternatively, we can also view the $z_a$ quanta as representing the vortices/anti-vortices in the VBS order [40] which, as we have just seen, carry spin $S = 1/2$.

Presently, the experimentally accessible case of the AF-VBS transition is in bilayer graphene, so we focus now on the $q = 2$ case. With a level 2 WZW term, the VBS vortices carry spin $S = 1$, and therefore we need complex scalar fields with 3 components: we write these as $Z_s$, with $s = x,y,z$. The field theory of the $Z_s$ quanta is now the CP$^2$ model with anisotropic quartic terms; such a field theory was considered in Ref. [42] in a different context:

$$\mathcal{L}_{cp} = |(\partial_\mu - iA_\mu)|Z_s|^2 + g|Z_s|^2 + u_1(|Z_s|^2)^2 + u_2(Z_s^2)(Z_s^*2).$$

Here $\mu$ is a spacetime index, $A_\mu$ is the emergent U(1) gauge field, $g$ is the coupling which tunes the AF to VBS transition, and $u_{1,2}$ are quartic couplings. In terms of the degrees of freedom in $\mathcal{L}_{cp}$, the 3-component AF order parameter is now $i\epsilon_{\mu\nu\rho}Z_s^\mu Z_s^\nu$, while the complex VBS order $(\rho^x + i\rho^y) \sim e^{i\theta}$ is the monopole operator in the U(1) gauge field [19, 21].

For both the CP$^1$ and CP$^2$ models mentioned above, both first and second order transitions are possible between the AF and VBS states. A recent numerical study [16] on a single layer model indicates a first order transition for the parameters studied.

**Experimental implications**

Finally, we turn to experimental consequences for bilayer graphene. The defining characteristic of deconfined criticality is the presence of a gapless ‘photon’ excitation of an emergent U(1) gauge field [20]. This is associated with the $A_\mu$ above, and can also be interpreted as a ‘spin-wave’ excitation involving fluctuations of the angle $\theta$. Our definition of $\theta$ shows that it is the angular phase associated with off-diagonal-long-range order (ODLRO) in valley space. The valley anisotropy terms in graphene is very small [12, 16], because it is suppressed by powers of the lattice spacing to the magnetic length; so we expect a nearly gapless $\theta$ spin-wave mode to be present (and most of the remarks below to also apply) even in the case of a first-order transition.

Now recall the fact, noted earlier, that in the zero energy Landau levels the valley index coincides with the layer index of bilayer graphene (and also the sublattice index). So ODLRO in valley space is accompanied by ODLRO in the layer space i.e. $\theta$ is also the angular phase of interlayer excitonic superfluidity. Signatures of excitonic superfluidity have been observed in quantum Hall states in GaAs bilayers [43]. However, in the GaAs bilayers there is negligible tunneling of electron between the layers, and this crucial to the emergence of a U(1) symmetry which is broken by the excitonic condensate. So it might seem surprising that a similar superfluidity can be present in graphene bilayers, in the presence of very strong tunneling between the layers. The resolution is the identification of the layer and valley indices.
in the zero energy Landau levels of bilayer graphene: in the absence of intervalley scattering, and the irrelevancy of valley anisotropy terms to be presented below, there is also an emergent interlayer U(1) symmetry in bilayer graphene.

The counterflow electrical current can be written in terms of the gauge field

$$J_{\mu} - J_{b\mu} = \frac{4e}{2\pi} \epsilon_{\mu\nu\lambda} \partial_{\nu} A_{\lambda}$$  \hspace{1cm} (8)

where $J_{\mu}$ and $J_{b\mu}$ are the currents in the top and bottom layers, and $e$ is the charge of the electron. The factor of 4 is deduced from Eq. (5), which shows that a AF Skyrmion in $(n_1, n_2, n_3)$ has excitonic charge density $<\rho^+> = 4$ for $q = 2$. The counterflow conductivity can be computed from Eq. (8) using $L_{\phi}$: at the $g = g_c$ deconfined quantum critical point this implies a universal value of order the quantum unit of conductance $e^2/h$.

For $g > g_c$, in the VBS state, the conductivity should be computed using an effective action for $\theta$ which includes the influence of monopoles. Now the current is

$$J_{\mu} - J_{b\mu} = 4e \rho_s \partial_{\mu} \theta$$  \hspace{1cm} (9)

where $\rho_s$ is the stiffness of the excitonic superfluidity appearing the effective Lagrangian density

$$L_\theta = \frac{\rho_s}{2} (\partial_{\mu} \theta)^2 - y_3 \cos(3\theta),$$  \hspace{1cm} (10)

with $y_3$ the fugacity of tripled monopoles which are allowed by the 3-fold rotational symmetry of the underlying honeycomb lattice [19]. The stiffness vanishes as $\rho_s \sim (g - g_c)^{n}$ by the Josephson relation, where $n$ is the correlation length exponent. Away from the critical point, the bare value $y_3^0$ is proportional to the very small 3-fold valley anisotropy term [12, 16]; $y_3$ has a further suppression [21, 44] from the critical fluctuations of $L_{\phi}$ leading to $y_3 = y_3^0 (g - g_c)^{\Delta}$, where $\Delta$ is the scaling dimension of the tripled monopole operator. So the effective “interlayer tunneling” term, $y_3$, is highly suppressed near the deconfined quantum critical point. We also note that numerical studies on square lattice antiferromagnets have provided striking evidence for the emergent U(1) symmetry due to the suppression of monopoles [45], and there is direct evidence for the suppression of monopoles on the honeycomb lattice in the work of Block et al. [26].

In GaAs bilayers [43], the excitonic superfluidity is most directly observed in counterflow experiments, where electric currents flow in opposite direction in the two layers. This would be technically more difficult in bilayer graphene, given the close spacing of the layers, but experiments of this type would be ideal. In the bilayer graphene experiments of Weitz et al. [1], there is a Zeeman coupling to the magnetic field (whose consequences have been studied earlier [21]), and an electric field is applied transverse to the layers. The electric field provides a small breaking of the layer-exchange symmetry. In the presence of such a symmetry breaking, there is a coupling between the counterflow and parallel current modes, and a vestige of the counterflow superfluidity would also be present in a measurement of the total current in both layers. Weitz et al. observe a phase transition out of the (presumed) AF state, signaled by the enhancement of the conductivity. We propose that this enhancement is due to the coupling to counterflow superfluidity.

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[18] K. Dho, E. Shimshoni, and E. Berg,
Supplement: Wess-Zumino-Witten terms in graphene Landau levels
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DIAGRAMMATIC DERIVATION OF WZW TERM

We give a detailed derivation of Eq. (7). We work in Landau gauge and separate the quantum number \( l \) in Eq. (2) into \( n \) and \( X \), where \( n \) is the Landau level index and \( X \) is the center of the magnetic oscillator. For monolayer graphene \( (q = 1) \) \( n \) is 0, and for bilayer graphene \( (q = 2) \) \( n \) is over 0 and 1. There are \( N_q \) of \( X \) labels, so the total number of states in the zero energy Landau level is \( qN_q \).

The spatial wavefunction \( \psi_{n,X}(r) \) in Eq. (2) is:

\[
\psi_{n,X}(r) = e^{iX_{\alpha}x_\alpha} \varphi_{n,X}(x),
\]

where \( \varphi_{n,X}(x) \) are the \( n \)-th harmonic oscillator eigenfunction:

\[
\varphi_0,X(x) = \frac{1}{\pi^{1/4}} e^{-(x-x)^2/2},
\]

\[
\varphi_1,X(x) = \frac{\sqrt{2}(x-x)}{\pi^{1/4}} e^{-(x-x)^2/2}.
\]

Note that we work in the unit where the magnetic length \( l_B = 1 \). We also set the volume of the system as 1 to avoid volume factors in Fourier transform.

Now we can write the Hamiltonian in \( cn,X \) basis.

\[
H = -\lambda \int d^2r \ n_{\alpha}(r) \Psi^\dagger(r) \Gamma_\alpha \Psi(r)
\]

\[
= -\lambda \int d^2r \ d^2q \ e^{-iq \cdot r} n_{\alpha}(-q) \sum_{n,X,n',X'} \epsilon_{n,X}^\dagger \varphi_{n,X}(x) e^{-iX_{\alpha}x_\alpha} \Gamma_\alpha \ e^{iX'_{\alpha}x_\alpha} \varphi_{n',X'}(x) c_{n',X'}
\]

\[
= -\lambda \int d^2q \ n_{\alpha}(-q) \sum_{n,n',X} e^{-iq x} \ e^{iX_{\alpha}q_{\alpha}/2} \Gamma_\alpha \ c_{n',X+q_\alpha/2} F_{n,n'}(q)
\]

(S3)

The form factors $F_{n,n'}(q)$ are calculated as,

\[
F_{00}(q) = e^{-q^2/4} \\
F_{01}(q) = \frac{-iq_x - q_y}{\sqrt{2}} e^{-q^2/4} \\
F_{10}(q) = \frac{-iq_x + q_y}{\sqrt{2}} e^{-q^2/4} \\
F_{11}(q) = (1 - q^2/2)e^{-q^2/4}.
\]  

The difference in Eq. (S3) between $q = 1$ and $q = 2$ case comes from the summation limit of $n$ and $n'$.

As in the main text, consider the situation $n_a(r) = (\pi_1, \pi_2, \pi_3, \pi_4, 1)$ where $n_a$ is polarized near $(0, 0, 0, 0, 1)$. Substituting this to Eq. (S3) we get the Hamiltonian,

\[
H = H_0 + H_{\pi} = -\lambda \sum_{n,X} \bar{c}_{n,X}^\dagger \Gamma_3 c_{n,X} - \lambda \sum_{i=1}^4 \int d^2q \, \pi_i(-q) \sum_{n,n',X} e^{-iq_x X} c_{n,X-q_y/2}^\dagger \Gamma_i c_{n',X+q_y/2} F_{n,n'}(q).
\]  

From the above equation, we integrate out the fermions to get an effective theory for the fluctuating order parameters. The coupling between the order parameters appear at fourth order of one-loop expansion. In momentum space, the four point coupling between the order parameter fields are,

\[
S_1 = \int \prod_{\alpha=1}^3 dp_\alpha K_{p_1 p_2 p_3}^{jkm;i} \pi_j(p_1)\pi_k(p_2)\pi_m(p_3)\pi_i(-p_1 - p_2 - p_3).
\]  

Among these terms, we are most interested in the topological term,

\[
S_\pi = iK \int d^2r d\tau \epsilon_{ijkm} \pi_i \partial_x \pi_j \partial_y \pi_k \partial_\tau \pi_m.
\]  

To extract the coefficient $K$ from $S_1$ we expand $K_{p_1 p_2 p_3}^{jkm;i}$ in powers of momenta and frequency, and consider the terms linear in $p_1 p_2 p_3$:

\[
K_{p_1 p_2 p_3}^{jkm;i} = \cdots + K_{\alpha \beta \gamma}^{jkm;i} p_1^\alpha p_2^\beta p_3^\gamma + \cdots.
\]  

Here, $\alpha, \beta, \gamma$ are spacetime indices $\tau, x, y$. In real space, these terms correspond to the derivative expansion.

\[
S_1 = \cdots + i \sum_{i,j,k,m=1}^4 K_{\alpha \beta \gamma}^{jkm;i} \int d^2r d\tau \partial_\alpha \pi_i \partial_j \pi_k \partial_\tau \pi_m + \cdots
\]  

Comparing Eq. (S7) with Eq. (S9), we obtain the expression for $K$ in terms of $K_{\alpha \beta \gamma}^{jkm;i}$,

\[
24K = \epsilon_{\alpha \beta \gamma} \epsilon_{ijkm} K_{\alpha \beta \gamma}^{jkm;i}.
\]  

Note the summation of repeated indices are implicit, and thus the right-hand side of the above equation consists of 144 terms.

Now we only need to calculate $K_{\alpha \beta \gamma}^{jkm;i}$ to obtain $K$. This can be done by calculating the box diagrams in Fig. 2. The kinetic energy of fermions are quenched, and the propagators are momentum independent. Therefore, the momentum dependence comes from the vertices and frequency dependence comes from the propagator.

Recalling Eq. (S5), the inverse Green’s function is written as $G^{-1} = -i\omega + H_0$, where $\omega$ is the Matsubara frequency of the fermions. The effective action of order $\pi^4$ in perturbation theory, which is of our interest according to Eq. (S9), is:

\[
S[\pi^4] = \frac{1}{4} \text{tr} (GH_\pi GH_\pi GH_\pi).
\]
Considering the momentum flow in Fig. 2, we obtain the expression for \( K^{k_{\alpha\beta\gamma}} \).

\[
K^{k_{\alpha\beta\gamma}} = \frac{1}{4} \left( \partial_{p_1} \partial_{p_2} \partial_{p_3} \text{tr} \left( G(k) H_\alpha^1(p_1) G(k + p_1) H_\alpha^k(p_2) G(k + p_1 + p_2) \right. \right.
\quad \times \left. H_\alpha^m(p_3) G(k + p_1 + p_2 + p_3) H_\alpha^{i\beta\gamma}(p_1 = p_2 = p_3 = 0) \right) \right)_{p_1 = p_2 = p_3 = 0} \tag{S12}
\]

We used a shorthand notation \( H_\alpha^a \) for \( H_\alpha^a \equiv H_\alpha(\pi_i = a, \pi_{i\neq a} = 0) \).

We get the value of \( K \) by calculating Eq. (S12). In zero temperature, the frequency integral can be done analytically,

\[
K = \frac{1}{2\pi} \int d\omega \sum_X \frac{2q^{5}}{(\omega^2 + \lambda^2)^{\frac{3}{2}}}
= \sum_X \frac{3q}{8}
= \frac{3q}{16\pi}. \tag{S13}
\]

The individual diagram is independent of \( X \), merely reflecting momentum conservation. Therefore the \( X \) summation is just multiplying \( N_\Phi \), which equals \( \text{(sample area)}/(2\pi) \). This gives the result in Eq. (7).