Optimal Constants in the Rosenthal Inequality for Random Variables with Zero Odd Moments.

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OPTIMAL CONSTANTS IN THE ROSENTHAL INEQUALITY FOR

RANDOM VARIABLES WITH ZERO ODD MOMENTS[1]

Marat Ibragimov

Department of Probability Theory, Tashkent State Economics University

Rustam Ibragimov[2]

Department of Economics, Harvard University

Abstract. We obtain estimates for the best constant in the Rosenthal inequality

$$E\left|\sum_{i=1}^{n}\xi_i\right|^{2m} \le C(2m)max\left(\sum_{i=1}^{n}E\xi_i^{2m}, \left(\sum_{i=1}^{n}E\xi_i^2\right)^m\right)$$ for independent random variables $\xi_1,...,\xi_n$

with l zero first odd moments, $l \ge 1$. The estimates are sharp in the extremal cases $l=1$ and $l=m$, that is, in the cases of random variables with zero mean and random variables with m zero first odd moments.

Rosenthal (1970) proved the following inequality:

$$E\left|\sum_{i=1}^{n}\xi_i\right|^{t} \le C(t)max\left(\sum_{i=1}^{n}E|\xi_i|^t, \left(\sum_{i=1}^{n}E\xi_i^2\right)^{t/2}\right) \tag{1}$$

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for all positive integers n and all independent random variables (r.v.'s) $\xi_1,...,\xi_n$ with $E\xi_i = 0$,

$E|\xi_i|^t < \infty$, $i=1,..., n$, $t > 2$, where $C(t)$ is a constant depending only on t. A number of papers

have focused on refinements and extensions of inequality (1) and related problems (see

Prokhorov, 1962; Sazonov, 1974; Nagaev and Pinelis, 1977; Pinelis, 1980, 1994; Pinelis and Utev,

1984; Johnson, Schechtman and Zinn, 1985; Utev, 1985; Hitczenko, 1990, 1994; Bestsennaya and

Utev, 1991, Ibragimov and Sharakhmetov, 1995, 1997, 2001a, b; Figiel, Hitczenko, Johnson,

Schechtman and Zinn, 1997; Ibragimov, 1997; de la Peña and Giné, 1999; and de la Peña,

Ibragimov and Sharakhmetov, 2003). Figiel et al. (1997) and Ibragimov and Sharakhmetov

(1995, 1997) derived the following expressions for the best constant $C^*_{sym}(t)$ in inequality (1)

for symmetric r.v.'s: $C^*_{sym}(t) = 1 + \dfrac{2^{t/2}\Gamma\left(\dfrac{t+1}{2}\right)}{\sqrt{\pi}}$, $2 < t < 4, C^*_{sym}(t) = E|\theta_1 - \theta_2|^t$, $t \geq 4$, where

$\Gamma(a) = \displaystyle\int_0^\infty x^{a-1}e^{-x}dx$ and θ_1, θ_2 are independent Poisson r.v.'s with parameter 0.5. The proof of

the expressions for $C^*_{sym}(t)$ in Ibragimov and Sharakhmetov (1995, 1997) relies on the work by

Utev (1985), who obtained, among other results, sharp upper and lower bounds on $E\left|\displaystyle\sum_{i=1}^n \xi_i\right|^t$,

$t \geq 4$, where $\xi_1,...,\xi_n$ are independent symmetric r.v.'s with finite tth moment, in terms of

$\displaystyle\sum_{i=1}^n E|\xi_i|^t$ and $\left(\displaystyle\sum_{i=1}^n E\xi_i^2\right)^{t/2}$. Bestsennaya and Utev (1991) derived a similar upper bound on even

moments of sums independent mean-zero r.v.'s $\xi_1,...,\xi_n$, from which the best constant in general

Rosenthal's inequality (1) in the case $t=2m$ can be deduced. Using a different proof technique, the

expression for the best constant in general inequality (1) for even moments $t=2m$ of sums of

mean-zero r.v.'s was independently obtained in Ibragimov and Sharakhmetov (2001a). Ibragimov and Sharakhmetov (2001b) obtained the best constant in the analogue of inequality (1) for nonnegative r.v.'s. The results in Ibragimov and Sharakhmetov (1995, 1997, 2001a, b) were also presented in Ibragimov (1997). de la Peña *et al.* derived sharp analogues of the Burkholder–Rosenthal inequalities and related estimates for the expectations of functions of sums of dependent nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments as well as for sums of multilinear forms.

The present paper deals with estimating the best constants in the Rosenthal's inequality for r.v.'s with l zero first odd moments. Namely, let $C_l^*(t)$ denote the best constants in inequality (1) for all positive integers n and all independent r.v.'s $\xi_1,...,\xi_n$ with $E\xi_i^{2s-1}=0$, $s=1,2,...,l$. Then the following theorem holds.

Theorem 1. If $t=2m$, $m \in N$, then

$$C_l^*(2m) \leq (2m)! \sum_{j=1}^{2m} \sum_{r=1}^{j} \sum \prod_{k=1}^{r} \frac{(m_k!)^{-j_k}}{j_k!}, \tag{2}$$

where the inner sum is taken over all natural $m_1>m_2>...>m_r$ and $j_1,...,j_r$ satisfying the conditions $m_1 j_1+...+m_r j_r=2m$, $j_1+...+j_r=j$, $m_i \neq 2s-1$, $i=1,2,...,r$, $s=1,2,...,l$.

<u>Remark 1.</u> The value $(2m)!\sum\limits_{j=1}^{2m}\sum\limits_{r=1}^{j}\sum\limits\prod\limits_{k=1}^{r}\dfrac{(m_k!)^{-j_k}}{j_k!}$ in inequality (2) has a simple

combinatorial sense (e.g., Sachkov, 1996): it equals the number of partitions of a set consisting of

$2m$ elements into parts the number of elements in which is not equal to $2s-1$, $s=1,2,...,l$.

<u>Remark 2.</u> As follows from the results in Pinelis and Utev (1984), Bestsennaya and Utev

(1991) and Ibragimov and Sharakhmetov (1997, 2001a), bounds (2) are sharp for l=1 and l=m; in

addition, when l=m, the right-hand side of (2) equals to the best constant $C_{sym}^{*}(2m)$ in the

Rosenthal's inequality for symmetric r.v.'s. It is also interesting to note that, in the case l=0, the

expression on the right-hand side of (2), with the inner sum taken over all natural $m_1>m_2>...>m_r$

and $j_1,...,j_r$ satisfying the conditions $m_1j_1+...+m_rj_r=2m$, $j_1+...+j_r=j$, equals to the best constant

the analogue of inequality (1) for nonnegative r.v.'s (see Ibragimov and Sharakhmetov, 2001b).

Similar to Remark 1, the latter expression equals to the total number of partitions of a set

consisting of $2m$ elements (the $2m$-th Bell number).

Let us formulate some auxiliary results needed for the proof of Theorem 1. The following

lemma follows from Corollary 2 in Utev (1985) and the formula representing moments by semi

invariants.

Lemma 1. Let $\xi_1,...,\xi_n$ be independent r.v.'s with $E\xi_i^{2s-1}=0$, $s=1,2,...,l$. Set

$A_{k,n}=\sum\limits_{i=1}^{n}E\xi_i^{k}$, k=1, 2,..., $2m$, $B_n=A_{2,n}^{1/2}$. The following inequality holds:

$$E\left(\sum_{i=1}^{n}\xi_i\right)^{2m}\le(2m)!\sum_{r=0}^{2m}\sum\prod_{k=1}^{r}\frac{A_{m_k,n}^{j_k}(m_k!)^{-j_k}}{j_k!}, \tag{5}$$

where the inner sum is taken over all natural $m_1 > m_2 > ... > m_r$ and $j_1,...,j_r$, satisfying the conditions

$$m_1 j_1 + ... + m_r j_r = 2m, \quad j_1 + ... + j_r = j, \quad m_i \neq 2s - 1, \quad i = 1, 2, ..., r, \quad s = 1, 2, ..., l.$$

Let A_{2m}, B, $D > 0$. Denote $M_1^l(m, A_{2m}, B) = \sup_{n, \xi_k} ES_n^{2m}$, where $\sup$ is taken over

positive integers n and all independent r.v.'s $\xi_1, ..., \xi_n$ with $E\xi_i^{2s-1} = 0$, $s = 1, 2, ..., l$ and fixed

$A_{2m,n} = A_{2m}$, $B_n = B$; $M_2^l(m, A_{2m}, B) = \sup_{n, \xi_k} ES_n^{2m}$, where $\sup$ is taken over positive integers n

and all independent r.v.'s $\xi_1, ..., \xi_n$ with $E\xi_i^{2s-1} = 0$, $s = 1, 2, ..., l$, for which $A_{2m,n} \leq A_{2m}$, $B_n \leq B$;

$M^l(m, D) = \sup_{n, \xi_k} ES_n^{2m}$, where $\sup$ is taken over positive integers n and all independent r.v.'s

$\xi_1, ..., \xi_n$ with $E\xi_i^{2s-1} = 0$, $s = 1, 2, ..., l$, and fixed $max(A_{2m,n}, B_n^{2m}) = D$.

The following lemma is well-known (see, e.g., Pinelis and Utev, 1984).

Lemma 2. For $2 < s < 2m$

$$\left| A_{s,n} \right| \leq \left(A_{2m,n}^{s-2} B_n^{2(2m-s)} \right)^{1/(2m-2)}. \tag{6}$$

Relation (5) and Lemma 2 imply the following

Lemma 3. For A_{2m}, $B > 0$

$$M_i^l(m, A_{2m}, B) \leq (2m)! \sum_{j=1}^{2m} \left(\sum_{r=1}^{j} \sum \prod_{k=1}^{r} \frac{(m_k!)^{-j_k}}{j_k!} \right) \left(A_{2m}^{m-j} B^{2m(j-1)} \right)^{1/(m-1)},$$

$i=1,2,$ where the inner sum is taken over all natural $m_1>m_2>...>m_r$ and $j_1,...,j_r$, satisfying the conditions $m_1 j_1 + ... + m_r j_r = 2m$, $j_1 + ... + j_r = j$, $m_i \neq 2s-1$, $i = 1,2,...,r$, $s = 1,2,...,l$.

Proof of Theorem 1. From Lemma 3 and the evident inequality

$$M^l(m, D) \leq M_2^l(m, D, D^{1/2m})$$

it follows that

$$M^l(m, D) \leq (2m)! \sum_{j=1}^{2m} \sum_{r=1}^{j} \sum \prod_{k=1}^{r} \frac{(m_k!)^{-j_k} D}{j_k!},$$

where the inner sum is taken over all natural $m_1>m_2>...>m_r$ and $j_1,..., j_r$, satisfying the conditions $m_1 j_1 + ... + m_r j_r = 2m$, $j_1 + ... + j_r = j$, $m_i \neq 2s-1$, $i = 1,2,...,r$, $s = 1,2,...,l$. Since

$$C_l^*(2m) = \sup_{D>0} \frac{M^l(m,D)}{D}, \tag{7}$$

this implies (2).

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9

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