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Mass fall-back and accretion in the central engine of gamma-ray bursts

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ABSTRACT
We calculate the rate of in-fall of stellar matter on an accretion disc during the collapse of a rapidly rotating massive star and estimate the luminosity of the relativistic jet that results from accretion on to the central black hole. We find that the jet luminosity remains high for about \(10^2\) s, at a level comparable to the typical luminosity observed in gamma-ray bursts (GRBs). The luminosity then decreases rapidly with time for about \(\sim 10^3\) s, roughly as \(\sim t^{-3}\); the duration depends on the size and rotation speed of the stellar core. The rapid decrease of the jet power explains the steeply declining X-ray flux observed at the end of most long-duration GRBs.

Observations with the Swift satellite show that, following the steep decline, many GRBs exhibit a plateau in the X-ray light curve (XLC) that lasts for about \(10^4\) s. We suggest that this puzzling feature is due to continued accretion in the central engine. A plateau in the jet luminosity can arise when the viscosity parameter \(\alpha\) is small, \(\sim 10^{-2}\) or less. A plateau is also produced by continued fall-back of matter – either from an extended stellar envelope or from material that failed to escape with the supernova ejecta. In a few GRBs, the XLC is observed to drop suddenly at the end of the plateau phase, while in others the XLC declines more slowly as \(\sim t^{-1} - t^{-2}\). These features arise naturally in the accretion model depending on the radius and mean specific angular momentum of the stellar envelope.

The total energy in the disc-wind accompanying accretion is found to be about \(10^{52}\) erg. This is comparable to the energy observed in supernovae associated with GRBs, suggesting that the wind might be the primary agent responsible for the explosion.

The accretion model thus provides a coherent explanation for the diverse and puzzling features observed in the early XLC of GRBs. It might be possible to use this model to invert gamma-ray and X-ray observations of GRBs and thereby infer basic properties of the core and envelope of the GRB progenitor star.

Key words: accretion, accretion discs – gamma-rays: bursts.

1 INTRODUCTION
Observations of gamma-ray bursts (GRBs) carried out by the NASA Swift satellite in the last two years have shown that the \(\gamma\)-ray prompt emission turns off abruptly after about a minute. The abrupt shut-off is evidenced by the rapidly declining X-ray flux (\(t^{-3}\) or faster) from about 80 to 300 s, which joins smoothly the prompt GRB light curve when extrapolated back in time and spectral band (Tagliaferri et al. 2005; Nousek et al. 2006; O’Brien et al. 2006). The abrupt decline suggests that the activity at the centre of the explosion declines very rapidly with time after about a minute of more or less steady activity.

At the same time, Swift observations have provided overwhelming evidence that the GRB central engine continues operating for hours and perhaps even days. There are two independent indications for this phenomenon. First, we often see a phase in the early X-ray light curve (XLC) of long GRBs, from \(\sim 10^7\) to \(10^8\) s, during which the flux declines slowly with time. We will refer to this as a ‘plateau’ in the light curve. There are many proposals to explain the plateau. The models are, however, highly constrained by a lack of correlation between X-ray and optical features for many GRBs. Therefore, a successful proposal must invoke continuing activity of the central engine to produce the X-ray plateau (e.g. Zhang et al. 2006; Panaitescu 2007, and references therein), whereas the simultaneous optical emission may be produced in the afterglow.

Secondly, in roughly a third of the observed GRBs, the X-ray flux is seen to increase suddenly and then to drop precipitously in what are referred to as ‘flares’. In some cases, the flux during these
flares increases by a factor of $\sim 10^2$ on a time-scale $\delta t \ll t$ (Burrows et al. 2005; Chincarini et al. 2007), which cannot be explained in terms of a density inhomogeneity in the external medium (Nakar & Granot 2006). Variable activity in the central engine is a more natural explanation. A sudden drop in the flux — e.g. in the case of GRB 070110 the XLC was nearly flat for 20 ks and then fell off as $r^{-5}$ (Troja et al. 2007) — is also not possible to understand other than as the result of highly variable central engine activity.

The above conflicting requirements, viz. (i) a sudden drop in activity at the end of the main GRB ($t \lesssim 10^2$ s), (ii) continued steady activity during an extended plateau ($t \sim 10^4$ s) and (iii) occasional dramatic flares, are challenging for models of the central engine. In this paper, we consider the currently most popular model of GRBs, which postulates ultra-rapid accretion of gas on to a newly formed black hole or neutron star (Narayan, Paczynski & Piran 1992; Popham, Woosley & Fryer 1999; Narayan, Piran & Kumar 2001). The accretion disc may be the result of (i) gas fall-back after a hypernova, as in the collapsar model (Woosley 1993; Paczynski 1998; MacFadyen & Woosley 1999), which is considered relevant for long-duration GRBs, or (ii) during the merger of a double neutron star or neutron-star-black hole binary (Eichler et al. 1989; Narayan et al. 1992), which is currently considered the most likely explanation for short-duration GRBs. We attempt to reconcile the accretion model for collapsars with Swift observations of long-duration GRBs.

In Section 2, we make use of a reasonably realistic model of the progenitor of a collapsar and estimate the rate at which gas is added to the accretion torus at the centre. The calculation is based on a crude free-fall model for the collapsing star and the results are to be taken in the spirit of an order of magnitude estimate. Then, in Section 3, we use this model to study the variation of accretion power with time. We show that the model naturally reproduces both the sudden shutdown of the prompt GRB emission and the extended plateau in the light curve. We speculate on possible scenarios for producing flares and suggest an explanation for the hypernova explosion associated with some GRBs. We conclude in Section 4 with a discussion and summary.

2 MASS FALL-BACK RATE

2.1 Particle trajectories

Consider an axisymmetric rotating star at the onset of core collapse. For simplicity, let us ignore pressure forces and take the trajectory of each particle to correspond to free-fall. At the beginning of the collapse, the velocity field is in the $\phi$ direction, and therefore particles start out at apo-centre. A particle that is initially located at $(r, \theta, \phi)$ with angular velocity $\Omega(r, \theta)$ will follow a trajectory with semimajor axis $a$ and eccentricity $e$ given by

$$a = \frac{r \Omega^2 \sin^2 \theta}{\Omega_k^2(1 - e^2)} = \frac{r}{1 + e}, \quad e = 1 - \frac{\Omega^2}{\Omega_k^2} \sin^2 \theta, \quad (1)$$

and its coordinates will vary with time $t$ as

$$x(t) = a(\cos \eta - \cos \phi) \sin \theta \cos \phi + a(1 - e^2)^{1/2} \sin \eta \sin \phi, \quad (2)$$
$$y(t) = a(\cos \eta - \cos \phi) \sin \theta \sin \phi - a(1 - e^2)^{1/2} \sin \eta \cos \phi, \quad (3)$$
$$z(t) = a(\cos \eta + \cos \phi) \cos \theta. \quad (4)$$

Here, $\eta$ is related to $r$ via

$$t = \Omega_k^{-1}(\eta + e \sin \eta)(1 + e)^{-3/2}, \quad \Omega_k = \left(\frac{G M_c}{r^3}\right)^{1/2}. \quad (5)$$

where $M_c$ is the mass enclosed within radius $r$, and $\eta = 0$ at $t = 0$.

The particle trajectory intersects the equatorial plane when $\cos \eta = -e$, or

$$t_0 = \Omega_k^{-1} \left[ \cos^{-1}(-e) + e(1 - e^2)^{1/2} \right] (1 + e)^{-3/2} + t_k(r), \quad (6)$$

and the distance of the particle from the centre at this time is

$$r_0(r, \theta, \phi) = r(1 - e). \quad (7)$$

The term $t_k(r)$ in equation (6) is the sound travel time from the centre to radius $r$, which is roughly the time it takes (from the start of collapse at the centre) for gas at $r$ to realize the loss of pressure support and to begin its fall towards the centre. Apart from ignoring pressure support during the collapse, the above analysis ignores a number of other effects, for example, a wind from the accretion disc which might inhibit fall-back, a shock generated by the bounce provided by a neutron star if one is created in the initial collapse, etc. Therefore, the results we present in this paper might have an error of a factor of a few. The purpose of this work is to identify a possible cause for the rapid decrease in central engine activity and the subsequent plateau and flare emission, and to help understand the dependence of accretion rate on stellar structure and rotation profile. It might perhaps also be useful for understanding certain aspects of 3D numerical simulations of collapsars.

2.2 Formation of an accretion disc

As particles intersect the equatorial plane during their free-fall, they will become part of a thick accretion disc that is centrifugally supported around the nascent black hole, provided they possess sufficient angular momentum. The formation of an accretion disc is likely required for launching a jet from an accreting black hole and producing a long-duration GRB via the collapsar model (e.g. MacFadyen & Woosley 1999); however, see Proga (2005) for an alternate point of view. Clearly, it is important to determine the conditions under which an accretion disc will form.

Given axisymmetry of the progenitor star about the rotation axis and mirror symmetry across the equator, when a particle hits the equatorial plane from above it will collide with another particle with the opposite sign of $v_{\phi}$ that started at the mirror location below the plane (Fig. 1). The velocities of the two particles when they collide at the equatorial ($x$-$y$) plane are given by

$$v_x = -\frac{r \Omega_k [\sin \theta \cos \phi + e \sin \phi]}{(1 - e)^{1/2}},$$
$$v_y = -\frac{r \Omega_k [\sin \theta \sin \phi - e \cos \phi]}{(1 - e)^{1/2}},$$
$$v_z = \frac{r \Omega_k \cos \theta}{(1 - e)^{1/2}}. \quad (8)$$

Collisions of gas blobs are highly inelastic, and so the two particles will merge and result in zero $z$ velocity. The dissipation of the energy associated with this component of the velocity will heat up the gas to nearly the virial temperature, and therefore the geometry of the infalling gas when it reaches the central region starts out as a thick disc (the disc thickness can become smaller at smaller radius if neutrino cooling is efficient). The surviving fluid velocity in the equatorial plane can be decomposed into azimuthal (circulation) and radial (in-fall) components:

$$v_{\phi} = -\frac{r \Omega_k \sin \theta}{(1 - e)^{1/2}}, \quad v_r = -\frac{r \Omega_k e}{(1 - e)^{1/2}}. \quad (9)$$

The velocity field immediately following the in-fall of stellar matter on the equatorial plane is convergent toward the centre of the star – the sign of $v_r$ is negative, independent of the initial particle position, and $|v_r| > v_{\phi}$. This leads to a rapid shrinking of the initial torus formed in the stellar collapse, as shown in Fig. 1. The rapid inward flow is terminated at a radius where the specific angular momentum of the in-falling gas is equal to the angular momentum in the black hole (black) until their angular momentum allows them to become centrifugally supported in an accretion disc (dark grey). The set of axes is shown for orientation; in our calculations, the origin of this coordinate system is at the centre of the star.

The descent from $r_{eq}$ to $r_h$, a 30 per cent drop in orbital radius, takes place on a local free-fall time-scale, which is much shorter than $t_{eq}$, and the ratio of the initial and final radius within the equatorial plane is almost independent of the initial particle position. Therefore, we take the rate at which gas settles into a centrifugally supported accretion disc as being the same as the rate at which stellar matter lands on the equatorial plane. The latter is obtained from equations (6) and (7), which map the initial particle position to the position on the disc at which the particle intersects the equatorial plane (note that $\phi \rightarrow \phi - \pi/2$). The Jacobian of the transformation $(r, \theta, \phi) \rightarrow (r_{eq}, \theta \rightarrow \pi/2, \phi - \pi/2)$ gives the mass fall-back rate per unit area on the disc.

2.3 Numerical results

The rate at which material falls back on the accretion disc depends on both the density profile of the star and its angular velocity profile. In this paper, we use a pre-GRB-collapse stellar model developed by Woosley & Heger (2006) for their collapsar simulations (model 16T1); the mass and radius of this progenitor star are $14 M_{\odot}$ and $5.18 \times 10^{15}$ cm (the core can be modelled as a polytrope of index 4.5). Fig. 2 shows the density profile for this star. Our choice of the rotation profile for the model (Fig. 2, lower right panel) is guided by the evolution calculation of Rockefeller, Fryer & Hui (2006).

In Fig. 3, we show for the above stellar model the rate at which mass is added to the disc, and the mean radius at which the gas goes into a circular orbit. In these calculations, we assumed that mass fall-back occurs only within a wedge extending $\pm 60^\circ$ from the equatorial plane. This allows crudely for the effect of a wind from the central disc which might prevent fall-back along the polar regions. We see that gas inflow on the disc starts out at a high rate and occurs initially at a small radius. With increasing time, the rate decreases rapidly ($\sim t^{-3}$), and much of the gas falls at a larger and larger distance from the centre. Material that lands on the equatorial plane at times earlier than about 20 s after core collapse falls directly into the black hole, whereas gas falling later than 20 s becomes centrifugally supported outside the Schwarzschild radius of the growing black hole. Therefore, we expect our model progenitor star to form an accretion disc at roughly this time after the initiation of core collapse. This is presumably also the time when a jet first forms.

2.4 Analytic scalings

It is instructive to try to understand the results shown in Fig. 3 using rough analytical estimates. The insights provided would be useful for determining the mass fall-back rate for other rotation and density profiles. We begin by expanding $t_{eq}$ (equation 6) to second order in $\Omega/\Omega_k$:

$$t_{eq} \approx t_a(r) + \frac{\pi}{2^{3/2}} \frac{1}{\Omega_k} \left[ 1 + \frac{3}{4} \left( \frac{\Omega \sin \theta}{\Omega_k} \right)^2 \right].$$  (11)

The difference, $\delta r$, between the polar and equatorial radii of an equal-collapse-time surface, i.e. the difference in $r$, for a fixed $t_{eq}$, between particles that start at $\theta = 0$ and those that start at $\theta = \pi/2$, is

$$\delta r \approx \frac{3\pi}{2^{7/2}} \frac{\Omega_k(r)}{\Omega_k(r)} d(t_e + 2^{-3/2} \Omega^{-1})/dr \sim H_t \Omega_k(r)^2,$$  (12)

where

$$H_t^{-1} \equiv \int \frac{d}{dr} \ln \left[ t_e + 2^{-3/2} \Omega^{-1} \right].$$  (13)

Here, we have assumed $t_e \sim \Omega_k^{-1}$, which is valid outside of a small core region. Since we see that $\delta r \ll r$, we may treat the equal-collapse-time surface as spherical, and thus the mass fall-back rate on to the disc is

$$m_{fb}(t) \equiv \pi \int dr \, r \, \Sigma(r, t) \sim \frac{dM(r)}{dr} \frac{dr}{dt_{eq}} \sim \frac{4\pi r^2 \rho(r) H_t}{t_{eq}} \sim \frac{4\pi r^2 \rho(r)}{t_{eq} \ln \Omega_k/\Omega_k}.$$  (14)

This simple analytical formula for the fall-back rate agrees to within a factor of 2 with the numerical result shown in Fig. 3. The formula shows that $m_{fb}(t)$ is insensitive to the rotation profile.
Figure 2. The top left panel shows the density $\rho$ of a 14 $M_\odot$ GRB progenitor star (model 16T1 of Woosley & Heger 2006), as a function of enclosed mass $M(r)$, and the top right panel shows $\rho$ as a function of $r$. The lower left panel shows $\tau \equiv d \ln \rho / d \ln r$, as a function of $M(r)$ (sharp glitches in $\tau$ are associated with composition changes), and the lower-right panel shows the angular velocity $\Omega$ (not from the evolutionary calculation of Woosley & Heger 2006). The vertical dotted line shows the part of the stellar core that falls directly to form a black hole; the region outside of this has sufficiently large specific angular momentum to form a disc.

Figure 3. The top left panel shows the rate at which gas rains down on the accretion disc – integrated over radius – as a function of time; model 16T1 of Woosley & Heger (2006) with the rotation profile shown in Fig. 2 was used for this calculation. The top right panel shows the variation with time of the mean radius of the circular orbit on which most of the fall-back gas lands; the distance is in units of the Schwarzschild radius of the central black hole (the black-hole mass at 23 s – from direct collapse of the core – is 5.7 $M_\odot$). The steep drop-off in the fall-back rate at $\sim 300$ s is the result of a steep decrease of density with $r$ for the stellar model of Woosley & Heger (2006) at $r \sim 2.9 \times 10^{10}$ cm (see Fig. 2). The lower left and right-hand panels show the evolution of black hole mass and spin.
in the star. However, the fall-back radius where the stellar matter circularizes has a strong dependence on \( \Omega \), and is given by

\[
r_{fb} \approx r \left( \frac{\Omega}{\Omega_a} \right)^2.
\]

This is the mass-weighted average fall-back radius for a spherical shell of gas; it is obtained from equations (1) and (7), and the time dependence of \( r_{fb} \) is determined by the relation between \( r \) and \( t_{eq} \) given in equation (11).

A good fraction of the gas in the core of the collapsing star will have insufficient centrifugal support and will directly form a central black hole. Specifically, any material that circularizes inside the innermost stable circular orbit (ISCO) of the black hole will fall into the hole on a dynamical time. The mass and dimensionless spin \( a_* \) of the initial black hole thus formed are given by the following conditions:

\[
R_{isco}(M_*, a_*) = r \frac{\Omega^2}{\Omega_a^2}, \quad a_* = \frac{c J_s}{GM_1},
\]

where \( M_* \) is the mass inside radius \( r \) in the pre-collapse star, \( J_s \) is the angular momentum of this mass and \( R_{isco}(M_*, a_*) \) is the radius of the ISCO for a black hole of mass \( M \) and spin parameter \( a_* \) (Bardeen, Press & Teukolsky 1972):

\[
R_{isco}(M_*, a_*) = \frac{GM_1}{c^2} \left\{ 3 + z_1 - [(3 - z_1)(3 + z_1 + 2z_2)]^{1/2} \right\},
\]

\[
z_1 = 1 + (1 - a_*^2)^{1/3} \left[ (1 + a_*^2)^{1/3} + (1 - a_*^2)^{1/3} \right]
\]

\[
z_2 = (3a_*^2 + z_1^2)^{1/2}.
\]

We solve these equations numerically to calculate the critical radius \( r \) inside which all the mass falls directly into the black hole, and thereby obtain the initial mass of the black hole \( M_{BH} \) and its initial spin \( a_* \). For the model shown in Fig. 2, we obtain \( M_{BH} = 5.7 M_* \) and \( a_0 = 0.45 \).

The rest of the material will have sufficient angular momentum to go into orbit around the black hole. Let us consider a model in which the pre-collapse density profile is of the form \( \rho(r) \propto r^{-\tau} \). The index \( \tau \equiv \frac{d \ln \rho}{d \ln r} \) is shown in Fig. 2 for the fiducial 14 \( M_* \) stellar model. The mass \( M_* \) enclosed within \( r \) increases with radius as \( r^{\tau'} \), where \( \tau' = 3 + \tau \) for \( \tau > -3 \) and \( \tau' = 0 \) when \( \tau < -3 \). Moreover, \( H_* \approx \frac{d \ln \Omega_1}{d \ln r} = (3 - \tau')/2 \approx 1/r \), and \( t_{eq} \approx \Omega_1^{-1} \propto r^{(3-\tau')/2} \). Using these scalings, we can calculate the time dependence of the stellar mass fall-back rate on to the disc (using equation 14) and the effective fall-back radius (equation 15)

\[
\dot{m}_{fb} \propto \frac{L^{3+\tau}}{t_{eq}^2} \propto t_{eq}^{2(3+\tau'+3)/(1-\tau')},
\]

\[
r_{fb} \propto \Omega^2 r^{2-\tau'} \propto t_{eq}^{2(4-\tau)/(1-\tau')} \Omega^2 \sim t_{eq}^{8/3} \Omega^2,
\]

where the final expression corresponds to \( \tau' = 0 \).

We see from Fig. 2 that \( \tau \sim -2.5 \) throughout most of the star except near the surface, in the outermost 0.5 \( M_* \) layer, where \( \tau < -5 \). Therefore, we expect \( m_{fb} \) to decline approximately as \( r^{0.5} \) while the interior of the star is collapsing, and \( m_{fb} \propto r^{-3} \) (or steeper) when the last \( 0.5 M_* \) of the star near the surface is accreted; the steep fall-off of \( m_{fb} \) starts at about 100 s after the beginning of the stellar collapse. MacFadyen, Woosley & Heger (2001) find \( m_{fb} \propto r^{-2.4} \) on a time-scale of 400–400 s (see their fig. 5) when a shell at a distance of \( \sim 3 \times 10^{10} \) cm falls to the centre; the density profile in this shell corresponds to \( \tau \sim -6 \) (fig. 2 in MacFadyen et al.) and therefore their numerical scaling of \( m_{fb} \) is roughly in agreement with our crude analytical scaling \( m_{fb} \propto r^{-3} \).

The 1D simulation of MacFadyen et al. (2001) has a strong forward shock that controls the fall-back of stellar material on to the central object and gives rise to \( \dot{m}_{fb} \propto t^{-1.5} \) for \( t \gtrsim 400 \) s; this situation is unlikely to apply to the 3D collapse to a black hole. Unfortunately, there are no 3D hydrodynamical or magnetohydrodynamical (MHD) simulation results published that have time coverage \( \gtrsim 10^2 \) s for us to check our analytical scaling. The 1D SNe simulation of Zhang, Woosley & Heger (2008) for a 25 \( M_* \) star of radius \( \sim 10^{12} \) cm finds that the accretion rate is nearly constant for about \( 10^3 \) s. This most likely results from the accretion of the extended envelope of the progenitor star as our analytical calculation suggests. We note that Zhang et al. (2008) do not see a phase at early times (\( t \lesssim 300 \) s) when the accretion undergoes a sharp decline as suggested by GRB observations.

### 3 MASS ACCRETION RATE AND JET LUMINOSITY

Fig. 2 shows that there are two distinct zones in the pre-collapse star: (i) the inner part of the stellar core (to the left of the vertical dotted line) which collapses directly to form a black hole and (ii) the outer part of the stellar core which goes into orbit around the black hole and then accretes viscously. In the following, we explore a scenario in which accretion of material in the inner part of zone ii produces the prompt GRB, while accretion of material near the surface of the star, where the density decreases rapidly with \( r \), gives rise to the steeply declining flux at the end of the \( \gamma \)-ray prompt emission.

#### 3.1 Approximate prescription for the accretion rate

To go from the mass fall-back rate \( \dot{m}_{fb} \) which we estimated in the previous section to the mass accretion rate on the black hole \( \dot{m}_{BH} \), we need to estimate (i) what fraction of the gas that falls on the accretion disc actually makes it to the black hole, and (ii) the time it takes the gas to accrete. Both issues have been discussed previously in the literature.

Narayan et al. (2001) showed that accretion in a fall-back disc can occur via two distinct modes. For high accretion rates and at small radii, neutrino emission and cooling are efficient and accretion occurs via a neutrino-dominated accretion flow (NDAF; Popham et al. 1999). In this mode of accretion, essentially all the mass accretes on to the black hole. However, for lower fall-back rates and/or at larger radii, the accretion is radiatively inefficient. We then have an advection-dominated accretion flow (ADAF; Narayan & Yi 1994), and only a fraction of the gas reaches the hole. A number of later investigators have studied the physics of these two regimes of accretion in the context of GRB models (Di Matteo, Ferna & Narayan 2002; Kohri & Mineshige 2002; Janiuk et al. 2004, 2007; Lee, Ramirez-Ruiz & Page 2004, 2005; Kohri, Narayan & Piran 2005, and references therein).

Kohri et al. (2005) present detailed estimates of the advection parameter \( f_{adv} \), viz., the fraction of the energy dissipated in the disc that is advected with the gas, as a function of the local accretion rate and the radius. We will take \( f_{adv} = 0.5 \), which corresponds to half the energy being radiated and half being advected, as the approximate boundary between the NDAF and ADAF regimes. From fig. 3 of Kohri et al. (2005), we find for the parameters of interest to us that the boundary is located roughly at

\[
\log m \approx \log \left( \frac{r}{R_s} \right) - 2.5, \quad m \equiv \frac{m}{M_5^{1.5}}, \quad R_s \equiv \frac{2GM_5}{c^2}.
\]
where \( r \) is the local radius and \( R_s \) is the Schwarzschild radius.

According to the prescription (20), when the logarithm of the mass accretion rate \( \dot{m} \) (in solar masses per second) at the outer perimeter of the disc at radius \( r_j \) is greater than \( \log (r_j/R_s) - 2.5 \), the accretion occurs via a NDAF and all the fall-back material accretes on to the black hole. However, when \( \log \dot{m} \) is smaller than this limit, accretion occurs at least partially via an ADAF.

An important feature of an ADAF is that it generally has a strong mass outflow (Stone, Pringle & Begelman 1999; Igumenshchev & Abramowicz 2000), which is believed to be driven by a positive Bernoulli constant (Narayan & Yi 1994, 1995). Because of this, at each radius a fraction of the accreting mass is lost in a wind, and so the net accretion rate decreases as we go to smaller radii. The exact functional form of this decrease is uncertain. In the most extreme case, we expect to have a convection-dominated accretion flow (CDAF; cf. Narayan, Igumenshchev & Abramowicz 2000; Quataert & Gruzinov 2000; Narayan, Piran & Kumar 2001), in which the accretion rate decreases linearly with \( r \). More generally, we expect a scaling of the form

\[
m(r) \approx \dot{m}_{\text{acc}} \left( \frac{r}{r_d} \right)^s, \quad 0 \leq s \leq 1,
\]

where \( \dot{m}_{\text{acc}} \) is the accretion rate in units of solar mass per second at the outer boundary of the disc and \( m(r) \) is accretion rate (M\(_\odot\) s\(^{-1}\)) at a radius \( r \). Pen, Matzner & Wong (2003) estimated \( s \sim 0.8 \) from 3D MHD simulations, while Yuan, Quataert & Narayan (2003) deduced \( s \sim 0.3 \) from modelling the radiatively inefficient accretion flow in the Galactic Centre source Sgr A*.

As an added complication, \( s \) probably depends on the degree of advection (i.e. the value of \( \dot{f}_{\text{acc}} \)). There is therefore considerable uncertainty regarding the value of \( s \), though a choice of \( s \sim 0.5 \) is probably reasonable. In the following, we leave \( s \) as a free parameter. Thus, we take \( \dot{m} \) to decrease as \( r^s \) so long as accretion occurs via an ADAF (\( \log m \) less than the limit in equation 20) and to be independent of \( r \) once the accretion flow becomes a NDAF (larger values of \( \log m \)). We also assume that all the mass that reaches the ISCO falls into the black hole. We then have three different regimes of accretion, each with its own prescription for the mass accretion rate:

\[ I: \log \dot{m}_{\text{BH}} = \log \dot{m}_{\text{acc}}, \quad \text{if } \log \left( \frac{r_d}{R_s} \right) - 2.5 \leq \log \dot{m}_{\text{acc}}, \]

\[ II: \log \dot{m}_{\text{BH}} = \frac{1}{1-s} \left[ \log \dot{m}_{\text{acc}} - s \log \left( \frac{r_d}{R_s} \right) + 2.5 \right], \quad \text{if } \log \left( \frac{R_{\text{isco}}}{R_s} \right) + s \log \left( \frac{r_d}{R_{\text{isco}}} \right) - 2.5 \leq \log \dot{m}_{\text{acc}} < \log \left( \frac{r_d}{R_s} \right) - 2.5, \]

\[ III: \log \dot{m}_{\text{BH}} = \log \dot{m}_{\text{acc}} - s \log \left( \frac{r_d}{R_{\text{isco}}} \right), \quad \text{if } \log \dot{m}_{\text{acc}} < \log \left( \frac{R_{\text{isco}}}{R_s} \right) + s \log \left( \frac{r_d}{R_{\text{isco}}} \right) - 2.5, \]

The three regimes correspond to (i) pure NDAF (equation 22), (ii) ADAF on the outside and NDAF on the inside (equation 23) and (iii) pure ADAF (equation 24). In these expressions, \( R_s \) is the Schwarzschild radius and \( R_{\text{isco}} \) is the radius of the ISCO corresponding to the current mass and spin of the black hole.

The disc radius \( r_d \) and the outer mass accretion rate \( \dot{m}_{\text{acc}} \) are determined by the current disc mass and angular momentum. Let \( M_d(t) \) be the mass of the disc at time \( t \) and \( J_d(t) \) the total angular momentum in the disc. The effective radius of the disc \( r_d \) is then defined by

\[ \frac{J_d}{M_d} = j(r_d) = (GM_{\text{BH}}r_d)^{1/2}, \]

where \( j(r) \) is the (Newtonian) specific angular momentum of an orbiting particle at radius \( r \). The mean rate at which mass empties from the disc as a result of accretion is

\[ \dot{m}_{\text{acc}} = \frac{M_d}{t_{\text{acc}}}, \]

where the accretion time-scale \( t_{\text{acc}} \) for a thick disc of vertical scale-height \( \sim r_d \) is approximately given in terms of the kinematic coefficient of viscosity \( v \) by

\[ t_{\text{acc}} \sim \frac{r_d^2}{v(r_d)} \left( \frac{\nu_k}{c_k} \right)^{2} \frac{1}{\alpha \sigma k} \sim \frac{2}{\alpha \sigma k}. \]

Here, \( \alpha \) is the standard dimensionless viscosity parameter (Shakura & Sunyaev 1973), which has a value of \( \sim 0.01-0.1 \), and the factor of 2 in the final expression is approximately correct for a fully radiatively inefficient accretion flow.

The mass and angular momentum of the disc change with time; they increase as a result of fall-back from the stellar envelope and decrease as a result of accretion. Thus, we may write

\[ \dot{m}_d = \dot{m}_b - \dot{m}_{\text{acc}}, \]

\[ J_d = J_b - J_{\text{acc}}. \]

The fall-back model described in Section 2 gives the fall-back terms \( \dot{m}_b \) and \( J_b \), while equation (26) gives \( \dot{m}_{\text{acc}} \). The angular momentum loss rate, \( J_{\text{acc}} \), due to accretion results from mass falling into the BH via \( j(R_{\text{isco}})\dot{m}_{\text{BH}} \) – plus angular momentum carried away by the wind. We assume that the specific angular momentum in the wind is equal to that of the gas in the disc at the radius from which the wind originates. It is then straightforward to integrate over radius and calculate the net angular momentum loss in the wind. Adding the two contributions, we have

\[ J_{\text{acc}} = j(R_{\text{isco}})\dot{m}_{\text{BH}} + \frac{2s}{(2s+1)} j(r_d)\dot{m}_{\text{acc}} \left[ 1 - \left( \frac{r_d}{r_s} \right)^{(2s+1)/2} \right], \]

where

\[ r_s = R_s \left[ 10^{-5} \dot{m}_{\text{acc}}(r_d/R_s)^{-s} \right]^{s/2}. \]

is the radius where a transition (if any) from NDAF to ADAF occurs; \( \dot{m}_{\text{acc}} = \dot{m}_{\text{acc}}/(1 \text{ M}_\odot \text{s}^{-1}) \).

The rate of increase of the black hole mass and angular momentum is

\[ \frac{dM_{\text{BH}}}{dt} = \dot{m}_{\text{BH}} \quad \text{and} \quad \frac{dJ_{\text{BH}}}{dt} = \dot{m}_{\text{BH}} j_{\text{isco}}. \]

where \( j_{\text{isco}} \) is the specific angular momentum of a particle on a circular orbit at the ISCO (see Bardeen et al. 1972):

\[ j_{\text{isco}} = (GM_{\text{BH}} R_{\text{isco}})^{1/2} \times \frac{R_{\text{isco}}^2 - a \cdot R_s (R_{\text{isco}} R_s/2)^{1/2} + a^2 R_s^2/4}{R_{\text{isco}}^2 - 3 R_{\text{isco}} R_s/2 + a \cdot R_s (R_{\text{isco}} R_s/2)^{1/2}}. \]

We solve equations (25)–(32) numerically and determine the accretion rate on to the BH and the evolution of the accretion disc.
However, when \( \dot{m}_{\text{BH}} = 0 \), or when \( t_{\text{acc}} \ll (r_0^3/GM_{\text{BH}})^{1/2} \), these coupled equations can be solved analytically as described in the next section.

### 3.2 Analytical solutions

A formal solution of equation (28) can be shown to be

\[
\dot{m}_d(t) = M_d(t_0) \exp \left( -\int_{t_0}^{t} dt_1 t_1^{-1/2} \right) + \int_{t_0}^{t} dt_1 \dot{m}_{\text{BH}}(t_1) \exp \left( -\int_{t_1}^{t} dt_2 t_2^{-1/2} \right). \tag{34}
\]

For \( t_{\text{acc}} \ll t \), i.e. rapid accretion, the first term on the right-hand side is very small and can be neglected. In this limit, the disc mass and the accretion rate are determined by the instantaneous value of the mass fall-back rate, i.e. \( \dot{m}_{\text{acc}} = M_d(t_{\text{acc}}) \), and the jet power (see Section 3.3) tracks the fall-back rate. This result comes in handy when we wish to understand various features in the early XLC of GRBs and their relationship to the structure of the progenitor star.

Another special case of considerable interest is where the stellar collapse leaves behind a reservoir of gas in a disc around the black hole, and no further mass is being added, i.e. \( \dot{m}_{\text{BH}} = 0 \). The accretion of gas from the reservoir on to the BH can keep the relativistic jet going for some period of time. We now estimate this accretion rate.

Equations (26)–(30) can be combined when \( \dot{m}_{\text{BH}} = 0 \) to obtain

\[
\dot{m}_d = -\frac{\alpha G^2 M_*^2 M_{\text{BH}}}{J_d^2}, \tag{35}
\]

and

\[
J_d = -\frac{2\alpha G M_*^2 M_{\text{BH}}}{2s + 1} \tag{36}
\]

In deriving equation (36) we assumed \( r_i \ll r_g \), and we neglected the angular momentum deposited on the BH in comparison to that carried away by the wind.

Equations (35) and (36) can be easily solved to yield

\[
\frac{J_d(t)}{J_d(t_0)} = \left[ \frac{M_d(t)}{M_d(t_0)} \right]^{2/3(s+1)} \quad \text{and} \quad \frac{r_d}{r_g} \propto (J_d/M_d)^2 \propto M_d^{2/3(s+1)}. \tag{37}
\]

We substitute this solution back into equation (35) to eliminate \( J_d \), and solve the resulting equation to find the accretion rate at \( r = r_d \):

\[
\frac{\dot{m}_d(t)}{\dot{m}_d(t_0)} = \left[ 1 + \frac{3}{2s + 1} \left( \frac{t - t_0}{t_{\text{acc}}^0} \right) \right]^{-1/2(s+4/3)} \tag{38}
\]

where

\[
t_{\text{acc}}^0 \equiv \frac{2}{\alpha G^2 M_{\text{BH}}} \left[ \frac{J_d(t_0)}{M_d(t_0)} \right]^{3/2} = t_{\text{acc}}(t_0). \tag{39}
\]

In the limit when \( s = 0 \), the accretion rate declines with time as \( t^{-4/3} \), consistent with the similarity solution described by Ogilvie (1999). The decline is faster for larger values of \( s \), with the fastest decline being \( t^{-2} \) for \( s = 1 \).

Combining equations (37) and (38), we obtain

\[
\frac{J_d(t)}{M_d(t)} = \frac{J_d(t_0)}{M_d(t_0)} \left[ 1 + \frac{3}{2s + 1} \left( \frac{t - t_0}{t_{\text{acc}}^0} \right) \right]^{1/3}, \tag{40}
\]

\[
r_{d}(t) = \frac{G M_{\text{BH}}}{r_{d}(t)} = r_{d}(t_0) \left[ 1 + \frac{3}{2s + 1} \left( \frac{t - t_0}{t_{\text{acc}}^0} \right) \right]^{2/3}, \tag{41}
\]

\[
\Omega_{\times}(t) = \sqrt{\frac{G M_{\text{BH}}}{r_d(t)}} = \Omega_{\times}(t_0) \left[ 1 + \frac{3}{2s + 1} \left( \frac{t - t_0}{t_{\text{acc}}^0} \right) \right]^{-1} t^{-1}. \tag{42}
\]

The accretion time, \( t_{\text{acc}} \approx 2/(\alpha G \Omega) \), asymptotically approaches \( 3/(2s + 1) \) for \( (t - t_0)/t_{\text{acc}}^0 \gg 1 \).

The accretion rate on to the BH is \( \dot{m}_{\text{BH}}(t) \approx \dot{m}_{\text{BH}}(t_0) \left[ 1 + \frac{3}{2s + 1} \left( \frac{t - t_0}{t_{\text{acc}}^0} \right) \right]^{-4(s+1)/3} \). \tag{43}

The rate declines as \( t^{-4/3} \) for \( s = 0 \), but much more steeply as \( t^{-8/3} \) when \( s = 1 \); it goes as \( t^{-2} \) for the intermediate value \( s = 0.5 \).

### 3.3 Prescription for the jet luminosity

To convert the mass accretion rate \( \dot{m}_{\text{BH}} \) to the power output \( L_{\text{jet}} \) in a relativistic jet, we need to estimate the jet efficiency factor \( \eta_j \):

\[
L_{\text{jet}} = \eta_j \dot{m}_{\text{BH}} c^2. \tag{44}
\]

Despite many years of study, the details of how relativistic jets are launched from accreting black holes are still poorly understood. The efficiency factor \( \eta_j \) is likely to depend on many details, but it is probably most sensitive to the spin of the black hole. In this paper, we make use of the following approximate prescription obtained by McKinney (2005) by fitting numerical results from GRMHD simulations:

\[
\eta_j \approx 0.07 \left( \frac{a_s}{1 + \sqrt{1 - a_s^2}} \right)^5. \tag{45}
\]

According to this prescription, the efficiency is a very steeply increasing function of the black hole spin: \( \eta_j \sim 10^{-4} \) for \( a_s \sim 0.5 \), \( \eta_j \sim 10^{-3} \) for \( a_s \sim 0.75 \) and \( \eta_j \sim 10^{-2} \) for \( a_s \gtrsim 0.9 \).

We note that McKinney’s simulations corresponded to a non-radiating accretion flow, i.e. an ADAF. Thus, for our problem, the above prescription is valid only for the ADAF phase of accretion; it is not clear what we should do when accretion occurs via a NADAF. For simplicity, we use the same prescription for the NADAF phase as well. Fortunately, most of the accretion in the fall-back disc occurs via an ADAF, so the error is probably not serious.

Given a model for the density and rotation profile of the precollapse star, the prescriptions given in Sections 3.1–3.3 allow us to estimate the jet luminosity \( L_{\text{jet}} \) as a function of time. We are thus ready to consider the implications of this model for GRBs.

### 3.4 Prompt GRB emission and rapid shutoff

We begin by discussing the first problem of interest to us, viz., the origin of the prompt GRB emission and the reason for the abrupt shutoff of this emission. As already mentioned, we associate the prompt GRB with accretion of the outer regions of the stellar core \( M > 8M_\odot \) in Fig. 2). Using the 14 M_\odot stellar model shown in Fig. 2, we calculated the mass accretion rate via equations (22)–(24) for different values of the parameter \( s \). We then computed the corresponding jet luminosity as a function of time using equation (44). The results are shown in Fig. 4. Also shown is the power-law index for the temporal decline of the jet power.

For an initial accretion rate of \( \sim 10^{-11} M_\odot \text{s}^{-1} \) (Fig. 3) and \( \eta_j \sim 10^{-2} \), we expect the jet power to be \( \sim 10^{51} \text{erg s}^{-1} \), as seen in the numerical results shown in Fig. 4. This is roughly consistent with the power observed in long GRBs. Thus, at least in terms of the overall energetics, the model gives fairly reasonable results.

The abrupt decline of the prompt emission after a period of activity is more challenging. The fastest possible decline is limited by the curvature of the \( \gamma \)-ray source surface and is given by \( t^{-2-\beta} \).
(Kumar & Panaitescu, 2000; $\beta$ is the spectral index of the radiation, i.e. $f_\nu \propto \nu^{-\beta}$) when the jet opening angle is larger than the inverse of the jet Lorentz factor. Declines of this order have been observed with the X-ray telescope aboard Swift (e.g. Tagliaferri et al. 2005; O’Brien et al. 2006). Such a rapid rate of decline is possible only if the intrinsic jet power itself declines faster than $t^{-2-\beta} \sim t^{-1}$. Thus, in order to explain the observed steep decline, we require the jet power in our model to satisfy $-\frac{d}{dt}\ln L_{\text{jet}} / (t - t_p) > 3$ for $t_p = 33$ s, where the time when gamma-ray emission is first observed.

Fig. 4 shows that at least some of our models do satisfy this requirement. Specifically, we find that the decline is faster than $t^{-1}$ after about 100 s whenever $s \gtrsim 0.5$. The rapid decline results from the steeply falling density profile in the outer part of the GRB progenitor star (see Fig. 2), coupled with the fact that a progressively decreasing fraction of the fallback mass reaches the black hole for larger values of $s$. The numerical results are consistent with the analytical scalings of Sections 2.4 and 3.2: $L_{\text{jet}} \propto m_{\text{fb}} r_{\text{fb}}^3 \propto t^{(\alpha - 2 + \beta)/3}$ for an ADAF. The results shown in Fig. 4 are for $\alpha = 0.1$. The jet power is not very sensitive to $\alpha$ because as long as $t_{\text{acc}} \propto \alpha^{-1}$ is less than $t$, $m_{\text{acc}} \approx m_{\text{fb}}$; for a larger $\alpha$, $t_{\text{acc}}$ is smaller, and that results in a slightly steeper decline of $L_{\text{jet}}(t)$ since $m_{\text{acc}}$ is now equal to $m_{\text{fb}}$ averaged over a smaller time period.

The sharp drop in $L_{\text{jet}}$ at $t \sim 10$ s results from the transition from a ADAF to a fully ADAF solution; the sharp drop is because the transition radius $(r_t)$ is a very steep function of $m_{\text{acc}}$ and $r_d$ for $s \gtrsim 0.5$ (see equation 31).

The duration of the steep decline is determined by the mass, radius and rotation rate of the progenitor star’s core; the mass and radius set the collapse time-scale $2(R^3/GM)^{3/2}$ and the rotation rate determines the fraction of the core that collapses directly to a black hole. The effect of core rotation on the jet luminosity is shown in Fig. 5. The peak jet power is much smaller when $\Omega$ in the core is either much larger, or much smaller, than the value shown in Fig. 2. For large $\Omega$, the accretion disc radius, and $r_{\text{acc}}$, is larger and, hence the jet power is smaller; much of the stellar mass in this case is ejected as a sub-relativistic, bi-polar, wind launched from the disc. Whereas for small $\Omega$, a larger fraction of the core collapses directly to a BH, and the total mass of gas available to form a disc and power the jet is smaller. When the core rotation rate is about eight times smaller than the value shown in Fig. 2, the angular momentum is insufficient to form a disc and the entire core collapses to a black hole. In this case, there would be no GRB. A variation in core rotation rate by a factor of 27 – the range considered in Fig. 5 – leads to a change in the peak jet power by a factor of $\sim 20$, and the duration over which the luminosity is high varies from $\sim 10$ to $10^2$ s; all time-scales are in the host-galaxy rest frame. The duration of the steep decline phase on the other hand is $\sim 400$ s for different $s$ values (Fig. 4) and for rotation speeds $\gtrsim \Omega$ the model considered in Fig. 2.

The jet luminosity declines as $\sim t^{-2}$ for larger $\Omega$ (see Fig. 5) due to the fact that $t_{\text{acc}}$ becomes greater than $t$, and this leads to the accretion rate on to the BH declining as $t^{-4\alpha+1/3}$ (equation 43) when $M_{\text{BH}}$ decreases rapidly (during the period the outer part of the star is collapsing). As we have already noted, a decline of jet-power as $\sim t^{-2}$ or slower is not consistent with early X-ray observations.

1. The fall-back radius $r_{\text{fb}} \propto \Omega^2$ (equation 15), so $t_{\text{acc}} \propto \Omega^{-1} r_{\text{fb}}^{-1} \propto \Omega^3$. 

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**Figure 4.** Left-hand panel shows the jet power $L_{\text{jet}}$ (this is the total jet power, not the isotropic equivalent luminosity) as a function of time for the $14 M_{\odot}$ pre-collapse stellar model and the rotation profile $\Omega(r)$ shown in Fig. 2. The time-axis has been shifted by $33$ s, so that $t = 0$ here is approximately when we see the start of the GRB; a relativistic jet produced during the initial $\sim 10$ s (in the rest frame of an observer at the centre of the star) of high accretion rate clears a polar cavity in the star and a short time thereafter, in the observer frame, $\gamma$-rays are generated. The five different curves correspond to five different values of the parameter $s$ (see equation 21 for definition); from top to bottom, the curves correspond to $s = 0$ (solid line), 0.25, 0.5, 0.75 and 1.0; $\alpha = 0.1$ for all of these curves. A smaller fraction of gas reaches the black hole for larger $s$ (the remainder leaves the system via a wind from the disc). This causes the overall luminosity to be lower and also the power to decline more rapidly. The steep drop-off of jet power at these curves. A smaller fraction of gas reaches the black hole for larger $s$ (the remainder leaves the system via a wind from the disc). This causes the overall luminosity to be lower and also the power to decline more rapidly. The steep drop-off of jet power at $t \sim 10$ s occurs when the accretion flow makes a transition from a ADAF in which all the gas reaches the black hole to an ADAF in which only a small fraction of the in-falling gas reaches the black hole. Right-hand panel shows $-\frac{d}{dt}\ln L_{\text{jet}} / (t - t_p)$, where $t_p = 33$ s. The different curves here correspond to different values of $s$, exactly as in the left-hand panel. For $-\frac{d}{dt}\ln L_{\text{jet}} / (t - t_p) > 3$, the observed light curve will be dominated by off-axis emission and will decline as $t^{-2-\beta} \sim t^{-3}$.
of GRBs (e.g. O’Brien et al. 2006). This constraint provides a limit on the rotation rate in the outer part of the progenitor star in our model.

3.5 Explanation for the plateau in the XLC

We now consider the second major puzzle in Swift observations of GRBs, viz., the presence of a plateau in the light curve for about $10^4$ s in $\sim$50 per cent of long-duration GRBs. First, we would like to understand what keeps the GRB engine operating for such a long time after the shutoff of the prompt burst. Secondly, we would like to explain how the engine is able to maintain a very shallow decline of luminosity $\sim t^{-0.5}$ during this entire time. Third, we would like to reproduce the sudden and sharp drop of luminosity at the end of the plateau seen in several GRBs, e.g. 060413, 060607A, 070110 (Fig. 2). The reason is that the fall-back radius of the system is larger by a factor of about 5 than the model considered in Fig. 2. The second solution (Sections 3.5.2 and 3.5.3) postulates continued fall-back of gas on the reservoir/accretion disc for about $10^5$ s, which is much longer than the free-fall time $\sim$500 s from the surface of our model star (Fig. 2).

3.5.1 Plateau as a result of small $\alpha$

Fig. 6 shows the long-term evolution of the jet luminosity $L_{\text{jet}}$ for different values of the viscosity parameter $\alpha$. For $\alpha = 0.1$ (our standard value), we find that $L_{\text{jet}}$ declines rapidly with time for $t \gtrsim 10^2$ s. However, for $\alpha \lesssim 10^{-2}$, $L_{\text{jet}}(t)$ has a plateau starting at about 200 s, and the duration of the plateau increases with decreasing $\alpha$. These results are for the pre-collapse stellar model of Woosley & Heger (2006) shown in Fig. 2, which is compact ($R_* = 5 \times 10^{10}$ cm) and has its density decreasing faster than $r^{-4}$ near the surface; the free-fall time at the surface of this model is about 500 s.

For $\alpha \gtrsim 0.1$, the viscous accretion time is shorter than the dynamical fall-back time (Fig. 6), and therefore the accretion rate on to the BH tracks the rate at which mass is added to the disc (see the analytical calculation in Section 3.2 and the discussion following equation 34). This leads to a rapid decrease of $L_{\text{jet}}$ for $t \lesssim 300$ s, which nicely explains the sudden shutoff of the prompt emission as described in Section 3.4. It is, however, a problem for the plateau. For $t \gtrsim 300$ s, when $\dot{m}_{\text{fb}} = 0$ (since the entire star has collapsed), the power continues to fall quite rapidly in this model. From our previous analysis (see equation 43), we expect the jet power to decline as $t^{-4 - 1/3}$, which is consistent with the numerical result shown by the solid curve in Fig. 6, and is too steep to produce a viable plateau.

For $\alpha \lesssim 10^{-2}$, the accretion time $t_{\text{acc}}$ is longer than the dynamical time (Fig. 6), and this allows the disc mass to build up to a substantial value ($M_\text{dis} \sim 1.0 M_\odot$). In this case, when $\dot{m}_{\text{fb}}$ drops sharply during the collapse of the outer, low density, envelope – the accretion rate $M_{\text{acc}}$ and the disc radius remain nearly constant for a time of the order of $t_{\text{acc}}$ (see equations 38 and 42). As a result, the jet-power remains nearly constant until $t \sim t_{\text{acc}}$. For $t \gtrsim t_{\text{acc}}$, we switch back to the canonical $L_{\text{jet}} \propto t^{-4 - 1/3}$ behaviour as described above.

We should note that, even with the small values of $\alpha$ considered here, a long-lasting plateau can be obtained only if $\Omega$ is larger by a factor of about 5 than the model considered in Fig. 2. The reason is that the fall-back radius $r_{\text{fb}} \propto \Omega^2$ (equation 19), and $t_{\text{acc}} \propto r_{\text{fb}}^{-5/2} \propto \Omega^{5/2}$ (equation 27). Since we require a long viscous time $t_{\text{acc}}$, we must have a large $\Omega$. Of course, a faster rotating star would need to be more compact in order to be bound, and this partially offsets the larger $t_{\text{acc}}$ we would get for larger $\Omega$. Nevertheless, a star rotating faster by a factor of 7 has $t_{\text{acc}} \gg t$ even for $\alpha = 0.01$ and will give a plateau in $L_{\text{jet}}$ lasting for $10^5$ s.

On the whole, we believe this is a reasonable scenario to explain the plateau in GRB light curves. However, it has a few problems. First, in order to have a plateau extending up to $10^5$ s as seen in many GRBs, we need to decrease $\alpha$ almost to $10^{-3}$. This value
will decline steeply as $L_{\text{jet}} \sim t^{-4(\alpha+1)/3}$, which is much too steep to explain the plateau. The only way we can avoid the luminosity drop is by having continued mass fall-back for the duration of the plateau $\sim 10^4 \, \text{s}$. We discuss this solution here and in the next section.

We describe the extended fall-back with three parameters: the amount of mass involved in the fall-back which is about $1 \, M_\odot$ for the model calculations presented here, the time dependence of the mass fall-back rate which we take to be a power law with a specified index and the angular momentum of the fall-back gas $\dot{m}_{\text{fb}}/m_{\text{fb}} \equiv j$ which we assume to be independent of time without restricting the solution space much. Once the fall-back is turned on (say at time $t_0$), the solution evolves quickly so that $t_{\text{acc}} \rightarrow 2j/(\alpha G M_{\text{BH}}^2)$ (see equations 38 and 39). Initially, it is possible for $\dot{m}_{\text{fb}}$ to fall rapidly, provided $\dot{m}_{\text{fb}}(t_0) < \dot{m}_{\text{acc}}(t_0)$ and $t_{\text{acc}}(t_0) < t_0$. However, on a time-scale of $\max(\dot{m}_{\text{fb}}, \dot{m}_{\text{acc}})$, a quasi-steady state is established with $\dot{m}_{\text{acc}}(t) \sim \dot{m}_{\text{fb}}(t)$ (see the discussion following equation 34), and the disc radius becomes $r_d \sim j^2/(\alpha GM_{\text{BH}})$. Thereafter, the accretion rate on to the BH is given by $\dot{m}_{\text{fb}}(R_*/r_d)$ and tracks the mass fall-back rate fairly well.

If the mass fall-back is turned off (say at time $t_1$), the jet power will revert to the asymptotic scaling $L_{\text{jet}} \propto t^{-4(\alpha+1)/3}$ (equation 43). However, there are two distinct sub-cases possible. (i) If $t_{\text{acc}} \approx 2j/(\alpha G M_{\text{BH}}^2) \ll t_1$, the jet power will first undergo a sharp drop by a factor of $t_1/t_{\text{acc}}$ on a time-scale of $t_1$ (see equation 38) and will only then settle down to the asymptotic $t^{-4(\alpha+1)/3}$ decline. (ii) On the other hand, if $t_{\text{acc}} \gtrsim t_1$, the plateau will smoothly transition to the asymptotic decline without an intermediate sharp drop.

Fig. 7 shows $L_{\text{jet}}$ for three different models of the extended fall-back, as detailed in the caption. We see that the jet power closely follows the time dependence of $\dot{m}_{\text{fb}}$ as long as $\dot{m}_{\text{fb}} \neq 0$. For instance, when $\dot{m}_{\text{fb}} \propto t^{-0.4}$, $L_{\text{jet}} \propto t^{-0.4}$ as well, and so is the case when the fall-back declines as $t^{-1.2}$. This means that if the observed X-ray plateau were to arise due to central engine activity, we require a long-lasting, nearly constant, mass fall-back rate, perhaps something like $\dot{m}_{\text{fb}} \sim t^{-0.5}$.

The second interesting result is that the behaviour of the light curve at the end of the plateau depends sensitively on the angular momentum of the fall-back material. This is to be expected, of course, since the angular momentum determines the fall-back radius and thereby the accretion time. When the angular momentum is small (solid line in Fig. 7), we have $t_{\text{acc}} \ll t_1$. This causes a dramatic drop in the jet luminosity as described above for Case (i). Such a model can explain GRBs like 060413, 060607A, 070110 (Liang et al. 2007), which show a sudden drop at the end of the plateau. On the other hand, when the angular momentum is larger (dashed line), there is a smooth transition from the plateau to the asymptotic $t^{-4(\alpha+1)/3}$ tail (Case ii), as is seen in other bursts.

### 3.5.3 Two scenarios for continued fall-back

The discussion in the previous section was couched in terms of a parametrized model of the mass fall-back. Here, we consider two specific scenarios that might produce continued fall-back of gas.

Our first scenario invokes a progenitor star with an outer envelope which extends out to $R_* \gtrsim 2 \times 10^{11} \, \text{cm}$. We assume that this envelope survives the initial implosion/explosion of the star. The size of the envelope is dictated by the requirement that we must have continued fall-back until $t \sim 10^4 \, \text{s}$. Thus, the dynamical time has to be $\sim 10$ times longer than at the outer edge of the Woosley & Heger (2006) star, which means that the envelope must have a radius of about a factor of 4 larger than the outer radius of their
model. Moreover, the star should have a density structure similar to their model inside $r \sim 5 \times 10^{10}$ cm, including a very steep density gradient from $r \sim 3$ to $5 \times 10^{10}$ cm in order to produce the rapidly falling light curve at the end of the prompt $\gamma$-ray emission. The envelope must thus be a distinct entity sitting on top of the Woosley and Heger star, and must have a flatter density profile.

How flat must the density profile in the envelope be? In Section 2.4, we derived a scaling for the mass fall-back rate on to the central disc, which gives $m_{fb} \propto r^0 (t^{-1})$ for $\rho(r) \propto r^{-2} (r^{-3})$ (see equation 19). If we assume for simplicity that the jet power is proportional to $m_{fb}$, then a shallow plateau light curve $\sim t^{-0.5}$ requires an envelope density profile substantially shallower than $r^{-3}$. Actually, the accretion rate on to the black hole is smaller than $m_{fb}$ by a factor of $r_{fb} \propto r^{1/3} \Omega^{-2}$ (equations 19 and 22–24). If $r_{fb}$ increases with time, as is likely unless the envelope has a constant specific angular momentum, then we will need an envelope with density going as $r^{-2}$ or even shallower. In any case, the envelope must rotate rapidly enough to form a centrifugally supported disc when the gas falls back.

The amount of He in the 16T1 pre-collapse stellar model of Woosley & Heger (2006) is 0.37 M$_\odot$ and it is concentrated near the surface. A somewhat more massive and extended He envelope is what is needed if we want continued fall-back lasting for $\sim 10^3$ s to produce an X-ray plateau. We note that the wind mass-loss rates from massive stars are uncertain. Some observations suggest that the typically assumed mass-loss rates are too high by a factor of a few (Smith et al. 2007; Smith 2007). The implication being that massive stars may retain a small fraction of their envelope – perhaps more often than generally assumed. Woosley & Heger (2006) suggest that their 16T1 model is likely to produce a SN Ic upon collapse, and a slightly more extended envelope probably would not modify that conclusion.

One of the most attractive features of the envelope scenario is that it naturally produces a sudden and dramatic drop in the fall-back rate when the outermost layers of the envelope have fallen back. Then, depending on whether $t_{acc}$ is smaller or larger than $t$, which is determined by the angular momentum of the fall-back gas (Section 3.5.2), we can have either a large drop in the luminosity (solid line in Fig. 7) or a smooth roll-over to a power-law tail (dashed line). Thus, the model may be able to accommodate most observed plateau light curves.

Our second scenario involves a bona fide supernova explosion in which a substantial fraction of the outer layers of the star is accelerated outward on a relatively short time-scale. However, some of the envelope material fails to escape to infinity and is accreted on to the BH. This material contributes to an extended episode of fall-back and causes a plateau in the light curve. This model is again subject to a number of requirements. A total fall-back mass of $\sim 0.5 M_\odot$ is needed in order to explain the XLC during the plateau. The requirement on the time dependence of $m_{fb}$ is fairly stringent – it should be no steeper than $r^{-0.5}$, which is a serious problem. The canonical fall-back rate of marginally bound ejecta is $r^{-5/3}$ (Chevalier 1989); a very similar time dependence, $m_{fb} \propto t^{-1.7}$, was also seen in a 1D collapsar simulation by MacFadyen et al. (2001). In addition, it is very difficult to see how one could have fall-back stopping abruptly, as required to explain the sudden drop in luminosity at the end of the plateau in a few bursts (e.g. Troja et al. 2007). It would require that the layers that fall back to have a density

![Figure 7](http://mnras.oxfordjournals.org/)

**Figure 7.** Top left figure shows jet power versus time for three different models. The solid and dashed lines correspond to $m_{fb} \propto t^{-0.4}$ with $j = 2.8 \times 10^{18}$ and $8.4 \times 10^{18}$ cm$^2$ s$^{-1}$, respectively. The dotted line corresponds to $m_{fb} \propto r^{-1.2}$ with $j = 2.8 \times 10^{18}$ cm$^2$ s$^{-1}$; these large values of specific angular momentum correspond to surface layers of the progenitor star. All the models assume $a = 0.1$, $s = 0.75$, $t_1 = 10^3$ s (end of fall-back), and correspond to the stellar model shown in Fig. 2. Note the sharp drop in $L_{jet}$ at the end of the plateau in the solid curve. Top right figure shows the mass fall-back rate $m_{fb}$ corresponding to the three models. Bottom left figure shows the dependence of the accretion time $t_{acc}$ as a function of time. Bottom right figure shows the mass contained in the disc as a function of time.
profile varying as \(\sim r^{-2}\) (to reproduce the shallow light curve) out to a radius of \(\sim 2 \times 10^3\) cm (to fit the plateau time-scale), and then to cut-off abruptly, presumably because everything outside of this radius escapes to infinity.

### 3.6 Flares

A major feature of many GRB light curves is the presence of one or more X-ray flares. Assuming a flare corresponds to a sudden increase in the jet luminosity, we see from equation (44) that we require either \(\eta_j\) or \(m_{BH}\) to change suddenly. We have assumed that \(\eta_j\) is determined primarily by the BH spin, which is not likely to change abruptly. Therefore, we require a sudden burst in the mass accretion rate. One possibility is a viscous instability in the disc (Piran 1978). The other is a sudden enhancement in the mass fallback rate, e.g. if the material in the stellar envelope is held up for a while by the mass outflow from the disc and then suddenly finds a way to accrete.

Even if there is an abrupt jump in \(m_{fb}\), the accretion rate \(\dot{m}_{BH}\) will still be smoothed on the accretion time-scale \(t_{acc}\). However, as Figs 6 and 7 show, \(t_{acc}/t\) is often less than unity. Therefore, sudden changes in the jet power are possible in this model.

Yet another possibility is a gravitational instability in the disc. There could be substantial mass in the reservoir, and this gas could become self-gravitating and go unstable. A gravitational instability could, in principle, produce features in the light curve on time-scales faster than the viscous time \(t_{acc}\).

A final possibility is that a strong magnetic flux may accumulate around the black hole during the accretion and may then repeatedly stop and restart the accretion, causing flares in the X-ray emission (Proga & Zhang 2006). This model is close in spirit to the ‘Magnetically Arrested Model’ described by Narayan, Igumenshchev & Abramowicz (2003).

### 3.7 Supernova

A feature of our model is that there is no conventional supernova explosion. Except for some material along the polar axes which may be punched out by the jets, the rest of the stellar mass falls back either directly into the black hole or on to an accretion disc. So, how do we explain the supernova-like optical light curves that have been seen in a few GRBs (e.g. Hjorth et al. 2003; Stanek et al. 2003; Modjaz et al. 2006)?

Even though we do not have the usual bounce and outgoing shock that are present in neutron-star-forming supernovae, our fall-back model does have mass and energy flowing out of the system. As we discussed in Section 3.1, only a fraction of the fall-back mass accretes on to the BH. The rest is ejected in a disc wind. Kohri et al. (2005) discussed this possibility that this wind might boost the energy output of a normal supernova and perhaps convert a failed supernova to a supernova-like optical light curve in the optical; Woosley has suggested that the production of \(^{56}\)Ni in the disc wind could give rise to a bright supernova (MacFadyen & Woosley 1999; Pruet, Guiles & Fuller 2002; see Woosley & Bloom 2006, for a review).

### 4 DISCUSSION

This work was motivated by two basic questions posed by the extensive and excellent Swift observations of GRBs. (1) Why does the \(\gamma\)-ray/X-ray flux undergo a sharp decline (flux decreasing as \(t^{-3}\) or faster) about 1 min after the start of the burst, even though the central engine itself is apparently active for hours? (2) After the sharp decline, how does the power from the engine remain nearly constant for a period of \(\sim 10^3\) s and compare the model predictions with GRB observations. We compute the mass in-fall rate during the collapse of the massive star, and the viscous accretion rate on to the central BH using ideas from Narayan et al. (2001) and Kohri et al. (2005). We then estimate the relativistic jet luminosity from the mass accretion rate and the BH spin, using a prescription proposed by McKinney (2005).

We find that the jet has a luminosity of about \(10^{51}\) erg s\(^{-1}\), lasting for about 10–20 s (Fig. 4), which is of the order the power and duration of a typical long GRB. The luminosity arises from the fallback and accretion of the outer half of the core of the progenitor star; the material here has a sufficient angular momentum to go into orbit, whereas the material from the inner half of the core collapses directly to form the BH. When the outer layers of the stellar core ~ where the density falls off rapidly with \(\rho\) ~ are accreted via the ADAF process, the jet luminosity drops rapidly. This provides a straightforward and natural explanation for the steep decline of the early XLC observed by Swift.

Fluence of GRBs can be used to constrain the rotation rate in the core of their progenitor stars to within a factor of 10 or better; too
large or too small $\Omega$ in the core results in a small jet luminosity (Section 3.4), and causes the temporal decay of the light curve at the end of the prompt emission, for a few hundred seconds, to be either too shallow or too steep (Fig. 5).

In our model, the plateau in the XLC requires continued accretion to power the jet. One possibility is that the viscous time-scale in the disc is so long that it takes a time $\sim 10^4$ s for the material in the disc to accrete. A viscosity parameter $\alpha \lesssim 10^{-2}$ is required. In this scenario, although mass fall-back ceases in a few hundred seconds, accretion continues on for a few hours (Fig. 6). While the model succeeds in producing an extended plateau, it causes the fall-off of the early GRB light curve to be less steep – more like $r^{-2}$ than $r^{-3}$ (Fig. 6) – in conflict with observations. The decline at the end of the plateau is also fairly shallow, inconsistent with observations of some GRBs.

A second and, in our opinion, more likely scenario is that there is continued mass fall-back for the entire duration of the plateau (Fig. 7). In this case, the accretion time-scale itself is short (i.e. $\alpha$ is large), so the light curve reflects the mass fall-back rate. One possibility is that the fall-back is due to material that fails to be ejected by the supernova explosion. This idea has two problems. First, it is hard to see why there should be an extended period of a fairly constant fall-back rate as required by the observations, whereas we expect the fall-back rate to vary much more steeply as $t^{-5/3}$ (Chevalier 1989). Secondly, it is hard to see how we can have a sudden cut-off in the fairly shallow, inconsistent with observations of some GRBs.

Another possibility is that the progenitor star has a core-envelope structure, with the core producing the early GRB and the envelope producing the plateau. The core must have a radius of $\sim$ few $\times 10^{10}$ cm to explain the duration of the GRB, and the envelope must have a radius of $\sim$ few $\times 10^{11}$ cm to explain the duration of the plateau. There should be a large density ($\rho$) contrast between the core and the envelope in order to explain the sharp cut-off of the prompt emission, and the density profile in the envelope must be fairly shallow, $\rho \sim r^{-2}$, in order to obtain a shallow plateau. Depending on the rotation profile of the star, various kinds of light curves – including ones in which we have a very rapid cut-off of the X-ray luminosity at the end of the plateau – are possible. This scenario is therefore capable of explaining almost all observed cases. GRBs that do not have a plateau in their light curve are also easily explained; these presumably had progenitors with only a core and no envelope (like the model shown in Fig. 2).

A nice feature of this model is that we could, in principle, use GRB observations to deduce the density and rotation structure of the progenitor star. On the other hand, it is not clear that evolved massive stars do have the kind of core-envelope structure we need to explain a typical GRB X-ray plateau. (We are not aware of any pre-supernova models in the literature with the required properties.)

An important implication of the accretion model of GRB central engines is that the accretion flows are advection-dominated and thus have strong outflows/winds. We have estimated the total energy in the wind for the pre-collapse stellar model of Fig. 2 and find it to be about $2 \times 10^{52}$ erg. This is sufficient to explode the star, and might explain the observed energetics of supernovae associated with GRBs.

A generic prediction of the late fall-back model for the X-ray plateau is that brighter GRBs should have a weaker (lower luminosity) and shorter duration plateau. The reason is that a stronger GRB, with its stronger jet and wind, is likely to eject more of the stellar envelope during the main burst. Indeed, recent simulations of relativistic jet-induced supernovae support this prediction, with more luminous explosions expelling more of the stellar envelope and leaving less material available for accretion (e.g. Tominaga 2007).

In order to test this prediction, we have looked at a sample of bursts with known redshift and isotropic equivalent luminosities (Butler & Kocevski 2007). From this set of bursts, we have selected two subsets, those with distinct X-ray plateaus (GRBs 070110, 060614, 050315, 060607A, 060729, 070810A) and those clearly lacking X-ray plateaus (GRBs 071020, 070318, 061007, 050922C, 050826, 070411). Consistent with the prediction of the fall-back model, we find that the average peak isotropic equivalent luminosity (per frequency interval) of the subset of bursts with distinct X-ray plateaus is nearly four times lower than that of the subset of bursts lacking an X-ray plateau. Moreover, using the same sample of bursts, we find that the average peak isotropic equivalent luminosity (per frequency interval) of bursts with an X-ray plateau is lower than that of bursts without an X-ray plateau at the >10 per cent level of significance, assuming normal distributions for the luminosities of each of these populations of bursts.

A related prediction is that the plateau should be absent, or at least weak, in those cases where we see a bright supernova event associated with a GRB. The idea is that a bright supernova implies powerful ejection, and there should be less material available for accretion. We have only one well-observed case of a GRB-supernova association in the Swift sample of bursts (GRB 030329), and the XLC did not have a plateau. While this observation is consistent with the late fall-back model, we note that it is just a single object and therefore the result is not very significant.

It is interesting to note that X-ray plateaus are not seen for short-duration GRBs (see Nakar 2007 for an excellent review). This is consistent with our model. If short GRBs are the result of the merger of double neutron star binaries (the currently popular model), then there is no material in an extended envelope in the progenitor to produce late fall-back.

The model described in this paper is obviously incomplete. We have not provided any quantitative explanation for the X-ray flares seen during the plateau phase (and even later) in many GRBs. The only qualitative idea we have offered is that the flares reflect an instability in the accretion disc. We have also simplified the model considerably by postulating a direct proportionality between the jet power at the point where it is launched from the BH and the observed luminosity. Several factors could seriously modify this relation. First, the tunnelling of the jet through the stellar material may be inefficient, and so the power that escapes from the surface of the star may be a small (and variable) fraction of the jet power at the base. Second, the efficiency with which the escaping jet power is converted to radiation (the physics of which is poorly understood) may be variable. Finally, the beaming of the jet may be different for the prompt GRB and the X-ray plateau, and may even vary during the plateau. We have ignored these complications in the interests of simplicity.

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