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On some points-and-lines problems and configurations

Noam D. Elkies

Abstract. We apply an old method for constructing points-and-lines configurations in the plane to study some recent questions in incidence geometry.

> What are known as "Points and Lines" puzzles are found very interesting by many people. The most familiar example, here given, to plant nine trees so that they shall form ten straight rows with three trees in every row, is attributed to Sir Isaac Newton, but the earliest collection of such puzzles is, I believe, in a rare little book that I possess — published in 1821 — *Rational Amusement for Winter Evenings*, by John Jackson. The author gives ten examples of "Trees planted in Rows."
>
> These tree-planting puzzles have always been a matter of great perplexity. They are real "puzzles," in the truest sense of the word, because nobody has yet succeeded in finding a direct and certain way of solving them. They demand the exercise of sagacity, ingenuity, and patience, and what we call "luck" is also sometimes of service.
>
> — H.E. Dudeney, *Amusements in Mathematics (1917) [8], page 56*

Introduction. Almost a century after Dudeney wrote these paragraphs, problems in incidence geometry continue to perplex both recreational and professional mathematicians, and the prospect of a uniform "direct and certain way of solving them" remains remote. Even for natural asymptotic questions, a wide gap often separates the best upper and lower bounds known. In this paper we construct some explicit point-and-line configurations that yield new lower bounds for two specific questions of this kind. Question 1, suggested by the recreational literature, asks: How many lines can meet n^2 points in the plane in at least n points each? Question 2 arises in the research literature [3]: If on each of N horizontal lines we choose (at most) N points, how many additional lines can contain N of these N^2 points? It turns out that an arrangement of 16 points in 15 lines of 4 (Figure 1 below), which has been known at least since 1908, naturally generalizes to configurations that not only give lower bounds for Question 1 but also improve on the previous records for Question 2. We also find a variation of this construction that yields a partial answer to Question 1 and a further improvement for the cases $N = 12m = 12, 24, 36, \ldots$ and $N = 12m - 1 = 11, 23, 35, \ldots$ of Question 2. By the construction in [3], the new results for Question 2 yield, for each $N \geq 5$, improved lower bounds on the exponent in the asymptotic "orchard-planting" problem with N-point lines. Each of these arrangements exploits dihedral symmetry: the lines include all axes of symmetry, and every point lies on one of the axes and at least one pair of lines symmetrical with respect to this axis. This approach is at least a century old (we give specific citations later), but might still produce further new examples and results for modern incidence geometry.

The rest of this paper is organized as follows. We first give some general background on this kind of points-and-lines problem. We then introduce Question 1, on plane arrangements of n^2 points with many n-point lines, and show the best configurations previously known. Next we present Brass's problem as Question 2, and observe that some of the configurations already known for Question 1 also answer Question 2. We proceed to modify the known constructions to obtain further improvements for both Questions. Finally we reconsider the symmetry of our configurations, which can be even greater than it appears. Most notably, the obvious fivefold

dihedral symmetry of Figure 1 extends to an action of the icosahedral group A_5 by projective linear transformations. This action, and an analogous action of the octahedral group S_4 on the real projective plane, leads us to further points-and-lines configurations related with the finite projective planes of orders ≤ 5. We expand the customary concluding Acknowledgements, to explain how we became aware of Question 2 and its connection with Question 1 even though such problems are quite far from our usual research work.

Definitions of T_k and t_k, and of $T_k^{(r)}$ and $t_k^{(r)}$; the exponents τ_k. For a finite set S of points in the plane, let $t_k(S)$ ($k = 2, 3, 4, \ldots$) be the number of lines meeting S in exactly k points, and $T_k(S) = \sum_{k' \geq k} t_{k'}(S)$ the number of lines meeting S in at least k points. For a positive integer n let

$$t_k(n) := \max_{|S|=n} t_k(S), \qquad T_k(n) := \max_{|S|=n} T_k(S), \tag{1}$$

so $t_k(n)$ (or $T_k(n)$) is the largest number of lines that can contain exactly (or at least) k points out of some configuration of n points in the plane. Clearly $t_k(n) \leq T_k(n)$. For $r > k$ we also let

$$t_k^{(r)}(n) := \max_{\substack{|S|=n \\ T_r(S)=0}} t_k(S), \qquad T_k^{(r)}(n) := \max_{\substack{|S|=n \\ T_r(S)=0}} T_k(S), \tag{2}$$

restricting S to point sets for which no line contains r or more points. For instance, the condition that no line contain more than k points (common in "orchard-planting" problems) corresponds to $r = k + 1$, and clearly in that case $t_k^{(r)}(n) = T_k^{(r)}(n)$. See for instance [4, p.315 ff.], where $t_k^{(k+1)}(n)$ is called $t_k^{\text{orchard}}(n)$.

A key question concerns the asymptotic behavior of $t_k^{(r)}(n)$ and $T_k^{(r)}(n)$ as $n \to \infty$ for fixed k, r. The question is trivial for $k = 2$: clearly $t_2^{(3)}(n) = \binom{n}{2} = T_2^{(r)}(n)$ for all $r > 2$. In general, for all k, r, n we have an elementary upper bound $T_k^{(r)}(n) \leq \binom{n}{2} / \binom{k}{2}$. For $k = 3$ this gives $T_3^{(r)}(n) \leq n^2/6 - O(n)$, which is known to be asymptotically sharp: certain configurations of torsion points on cubic curves even give $t_3^{(4)}(n) = n^2/6 - O(n)$ (see for instance [5, 6]). For $k \geq 4$, Erdős proposed long ago the conjecture that $t_k^{(k+1)}(n) = o(n^2)$ (this is "Conjecture 12" of [4, p.317]); more generally one might guess that $T_k^{(r)}(n) = o(n^2)$ for any fixed k, r with $4 \leq k < r$. [Note that the corresponding conjecture for $T_k(n)$ or even $t_k(n)$ is false, for instance because a $k \times m$ lattice array has at least $m^2/(k-1)$ lines of exactly k points (and even this is not optimal, see [15]); this is why we fix some finite upper bound r on the number of points in any line.] But it is not known that $t_k^{(k+1)} = o(n^2)$ for any $k \geq 4$, even though the best lower bounds on $T_k^{(r)}(n)$ are $C_{k,r} n^{\tau_k}$ with

$$2 = \tau_3 > \tau_4 \geq \tau_5 \geq \tau_6 \geq \cdots \to 1. \tag{3}$$

Our results include improvements on these τ_k for each $k \geq 5$ (though to be sure we are still nowhere near settling Erdős's conjecture). See Theorem 1, stated near the end of this paper.

Sets of n^2 points in the plane with many n-point lines. Anyone who has seen a magic square knows that $t_n^{(n+1)}(n^2) \geq 2n + 2$ for all $n > 1$: a square array of n^2 points in the plane forms $2n + 2$ lines of n, namely the n horizontal lines, n vertical lines, and 2 diagonals. For $n = 2$ this is clearly optimal because each of the six pairs of points has a two-point line through it. But for each $n > 2$ one can get more than $2n + 2$ lines. A famous configuration, known at least since the beginning of the twentieth century [7, p.175], shows that for $n = 4$ one may get

as many as 15 lines of 4 by using a double pentagram instead of a square. See Figure 1. (The closed and open circles indicate points on 3 and 5 lines respectively; more about this later.)

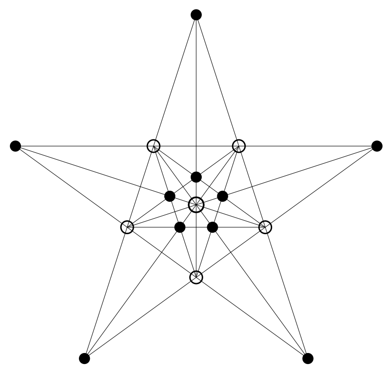

Figure 1: 16 points, 15 lines of 4

This construction readily generalizes to all even $n > 2$: replace the two nested pentagrams by two nested $(n+1)$-point stars, each formed from the longest diagonals of a regular $(n+1)$-gon, to obtain a configuration with $(n+1)$-fold dihedral symmetry consisting of n^2 points lying on $3n+3$ lines of n points each. Figure 2 shows the case $n = 6$ of this construction.

This suggests several questions, which we first raised in the interview [17, p.228]:

Question 1a: *Is this configuration optimal?*

That is, is $3n+3$ the maximal number of lines that can meet n^2 points in the plane in (at least) n points each? Using the notation of (1), we are asking: is $t_n(n^2) = T_n(n^2) = 3n+3$ for $n = 4, 6, 8, \ldots$? This might be known for $n = 4$, but is almost certainly open for every even $n \geq 6$.

Question 1b: *What happens for odd n?*

For $n = 3$ it has long been known that the maximum is 10, though over the complex numbers the famous configuration of nine flex points of a smooth cubic has 12 lines of three (it is probably mere coincidence that this is also the value of $3n+3$ for $n = 3$), which attains the upper bound $\binom{9}{2}/\binom{3}{2}$ exactly: the line through *every* pair of points goes through a third point of the configuration.[1] The 10-line configuration, mentioned by Dudeney in the passage quoted earlier from [8, p.56], is obtained from the 3×3 square array by moving an opposite pair of edge points halfway towards the center (Figure 3); we later return to this configuration as well.[2]

[1] Note too that the flexes of a smooth cubic in the plane are also its 3-torsion points. Over the real numbers, we already noted the use of torsion points on such curves in estimating $t_3^{(4)}(n)$. For more on points-and-lines arrangements in the complex plane and beyond, see [10, 14].

[2] Burr begins his article [5] by quoting the puzzle asking for this configuration from the same source (*Rational Amusement for Winter Evenings* (1821) by John Jackson), where it is given as a verse:

Your aid I want, nine trees to plant
  In rows just half a score;
And let there be in each row three.
  Solve this: I ask no more.

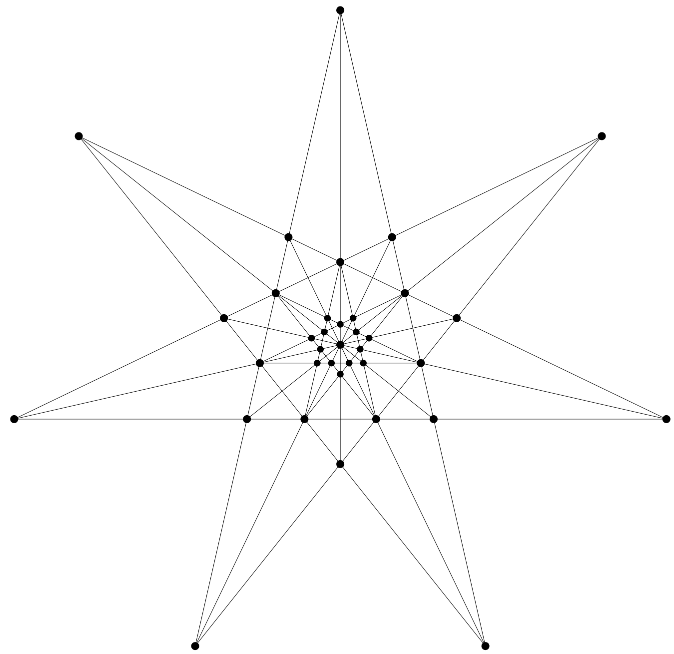

Figure 2: 36 points, 21 lines of 6

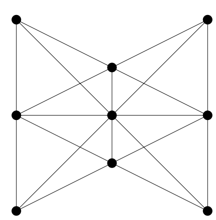

Figure 3: 9 points, 10 lines of 3

Some twenty years ago we constructed — with some "luck", as Dudeney might say — a sporadic arrangement of 25 points with 18 lines of five (Figure 4, also shown in [17, p.228]). The points on each edge of the triangle bisect and trisect the edge. Thus $t_n(n^2) \geq 3n+3$ also for $n = 5$. We construct a different such configuration later, from Figure 8. We do not know whether 18 lines is maximal, nor whether either 18-line configuration was known earlier.

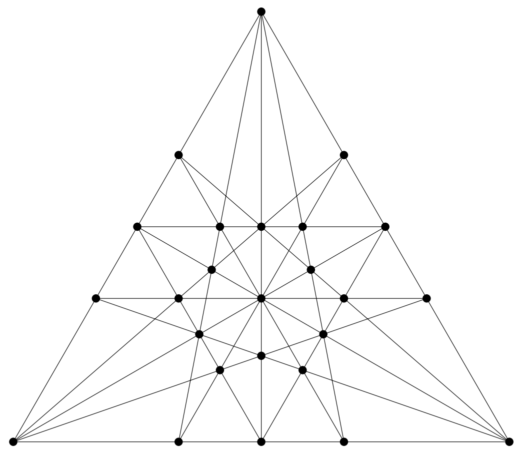

Figure 4: 25 points, 18 lines of 5

For odd $n \geq 7$, one can at least see quickly that the $2n+2$ lines of the square configuration are *not* optimal. We can already get $2n+2$ lines using only $(n-1)^2+3$ points: $(n-1)^2$ in a square array, one point in the center of the square (which has not been used yet because n is odd), and two points at infinity where the line at infinity meets the coordinate axes. Then we can use the remaining $2n-4$ points to form another $4\sqrt{n} - O(1)$ lines by putting them on diagonals that contain $n-2$, $n-3$, $n-4$, $\ldots$ points in the array. At the end, if points at infinity are deemed undesirable one may apply a projective transformation to put all n^2 points in the finite plane.[3] We shall show that $3n+1$ lines can always be attained, even under the "orchard" constraint that no line contain more than n points; that is, $t_n^{(n+1)}(n^2) \geq 3n+1$. We shall also show that for $n = 12m-1 = 11$, 23, 35, 47, etc., there are configurations of n^2 points in the plane with $3n+4$ lines each of which passes through at least n of the n^2 points. But these configurations necessarily contain some lines of $n+1$ points, so we obtain $T_n(n^2) \geq 3n+4$ for these values of n but not $t_n(n^2) \geq 3n+4$.

Parallel lines with many Brass transversals. P. Brass asks [3]:

Question 2: *Can there be N parallel lines l_i in the plane, and $M > N+4$ lines λ_j not parallel*

[3] Dudeney used much the same trick in his second solution [8, p.190] to the puzzle of placing 21 points in 12 lines of 5: the configuration is projectively equivalent to the 16 points (i,j) $(1 \leq i,j \leq 4)$ in the (x,y) plane, together with the points $(3/2, 5/2)$ and $(5/2, 3/2)$ and the three points at infinity contained in the four lines $x = i$, the four lines $y = j$, and the three lines $x = y$, $x - y = \pm 1$. The twelfth line is then $x + y = 4$. The use of projections in this context to bring points at infinity to the finite plane is noted explicitly in [1, p.105].

to the l_i, such that for each i we have

$$\#\Big(\bigcup_{j=1}^{M} l_i \cap \lambda_j\Big) \le N \tag{4}$$

(that is, there are at most N points on l_i through which some λ_j passes)?

We shall call such λ_j a collection of "Brass transversals" to the l_i. More generally, one may of course ask, for any N and N', for the maximal number of lines λ_j whose union intersects each of N parallel lines l_i in at most N' points. But the case $N = N'$ is of particular interest because Brass [3] gives an explicit recursive construction showing that a collection of M Brass transversals yields $t_N^{(N+1)}(n) = \Omega(n^{\log_N M})$ as $n \to \infty$.

Question 2 specifies $M > N + 4$ because $M = N + 4$ can be attained for each $N \ge 3$. Let l_i be the line $x = i$ for $i < N$, and the line at infinity for $i = N$; let λ_j be the line $y = j$ for $j \le N$; and let the remaining four transversals be the lines $y = x$, $y = x + 1$, $x + y = N$, and $x + y = N + 1$. These $N + 4$ lines meet l_N in 3 points, and l_i in N points for each $i < N$. For $N = 3$ this configuration is easily seen to be unique up to projective transformations. Figure 3 shows it in another guise, with $7 = 3+4$ Brass transversals to the three vertical lines; projecting one of these lines to infinity yields the case $N = 3$ of the construction described earlier in this paragraph. The resulting bound $t_3^{(4)}(n) = \Omega(n^{\log_3 7})$ is not interesting, because we already know that $t_3^{(4)}(n)$ is asymptotic to $n^2/6$. But in [4, p.317] we find that for $N \in [5, 17]$ the lower bound with exponent $\log_N(N+4)$ is the best exponent known, and for $N \ge 18$ it can be used with a different recursive construction due to Grünbaum [11] to obtain the record exponent $1+(1/(N-\gamma))$ with $\gamma \doteq 3.59$.

We improve this to

$$\log_N 2N = 1 + \frac{\log 2}{\log N} \tag{5}$$

for each $N = 5, 7, 9, \ldots$, using our configurations from Question 1 with $n = N - 1$ and $M = 2N$. Project the center of our n^2-point configuration to infinity; let the l_i be the $N = n + 1$ lines through this point at infinity, and let the λ_j be the remaining $M = 2N = 2n + 2$ lines. Then $\#(\cup_j l_i \cap \lambda_j) = N$ for each i, and the bound $t_N^{(N+1)}(n) = \Omega(n^\tau)$ with $\tau = \log_N 2N$ follows by [3].

We cannot quite do this for $N = 6$ using our sporadic 25-point configuration in Figure 4, because the six lines through the center are not equivalent. When l_i is one of the three axes of symmetry of the triangle, the λ_j meet l_i in only four points; but for the other three l_i (those parallel to the triangle's sides), there are seven points of intersection. Still, this configuration may be of use for Brass's construction because the inequality (4) remains true on average, even with a strict inequality: one might have expected eight points of intersection for l_i in the second group, but the two new points coincide because two of the λ_j are parallel to l_i and thus meet l_i in the same point at infinity (which is not one of the 25 points of our configuration).

Further refinements. For $N = 3$ the configuration that attains $N + 4 = 7$ Brass transversals is unique, and can be displayed symmetrically as shown on the left side of Figure 5 by projecting one of the transversals to infinity. This again suggests a generalization to arbitrary odd N: let l_i be the line through the origin making angle $(i/N)\pi$ with the horizontal; and let λ_j be the N pairs of lines parallel to the l_i at unit distance, together with the line at infinity, for a total of $M = 2N + 1$ transversals. Taking the indices of the l_i modulo N, we see that for each i' mod N the transversals parallel to $l_{i\pm i'}$ meet l_i at the point(s) $1/\sin((i'/N)\pi)$ units from the origin.

This gives N points of intersection for each line, and all the points with $i' = 0$ are on the line at infinity, which accounts for the $(2N+1)$-st transversal. The right side of Figure 5 shows the $N = 5$ case of this construction. Again we conclude by projecting the origin to infinity to obtain parallel lines l_i. For each $N = 5, 7, 9, \ldots$, this gives us an even better value $\log_N(2N+1)$ for the exponent τ_N of (3). Moreover, the set of N^2 points $l_i \cap \lambda_j$ meets the $3N+1$ lines l_i, λ_j in N points each, and meets no line in more than N points because the set is contained in the N lines l_i. Therefore $t_N^{(N+1)}(N^2) \geq 3N+1$. This gives a new lower bound on $t_N^{(N+1)}(N^2)$ for each odd $N \geq 7$. (We exclude $N = 5$, because then $3N+1 = 16$, but Figure 4 already attains 18.)

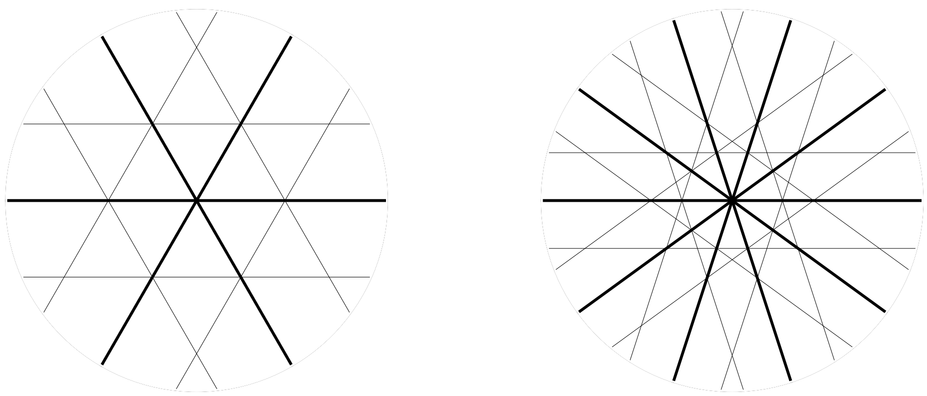

Figure 5: n lines, $2n+1$ Brass transversals including the line at infinity ($n = 3, 5$)

This construction fails when N is even, because then the points at unit distance from the origin on l_i each lie on just one transversal (with $i' = N/2$). But we still achieve $M = 2N$ by discarding the line at infinity and rotating the other lines λ_j by an angle $\pi/2N$ about the origin. This improves on $N+4$ for all even $N \geq 6$. (Figure 6 shows the case $N = 6$.) We therefore attain $\tau_N = \log_N 2N$ for all even N, and have thus improved the exponent τ_N for all integers $N \geq 5$.

Our configuration with a double N-point star also required that N be odd, for a different reason: for even N, the longest diagonals of a regular N-gon that do not go through its center intersect each other in only $N-2$ points. But for large N the double-star construction has some flexibility that we can sometimes exploit to improve the configuration and allow some even N as well. Namely, we may match any of one star's rings of N intersection points with any ring at a different position on the other star. This can be done when the ratio between stars' circumradii is

$$\rho_N(i,j) := \sin\frac{i\pi}{N}\Big/\sin\frac{j\pi}{N}$$

for some distinct positive integers $i, j < N/2$, regardless of the parity of N. [So far, as in Figure 2 (with $N = 7$), we have always used $(i,j) = (1, (N-1)/2)$.] If $\rho_N(i,j) = \rho_N(i',j')$ for another pair (i',j') of integers in $(0, N/2)$, then the resulting double-star configuration has the same number of incidences with N fewer points. We may find such i, i', j, j' when $6|N$ and $N \geq 12$, using the identity

$$\sin\theta\,\sin(\frac{\pi}{2}-\theta) = \sin\theta\,\cos\theta = \frac{1}{2}\sin 2\theta = \sin\frac{\pi}{6}\,\sin 2\theta.$$

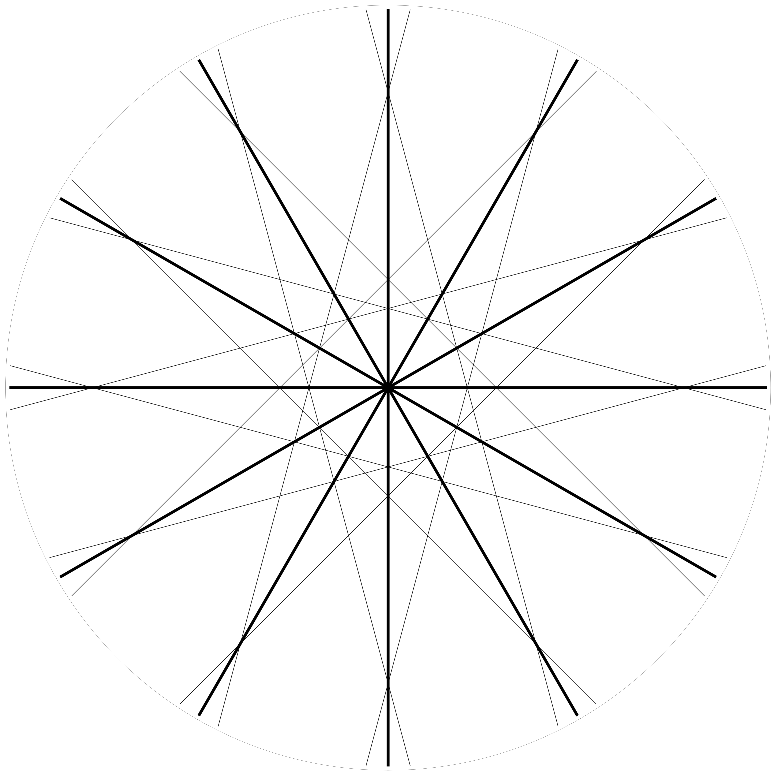

Figure 6: 6 lines, 12 Brass transversals

[That these are in fact the only solutions is a special case (and much easier than the full result) of [16, Thm. 4]; the authors of [16] report that the same theorem had already been obtained by Bol [2]. Unfortunately it is not possible to have a third pair (i'', j'').] This gives $(i', j, j') = (2i, N/6, (N/2) - 1)$. Moreover, when $N = 12m$, we may choose i, i', j, j' so that i and j' are odd while i' and j are even, for instance $(i, i', j, j') = (1, 2, 2m, 6m - 1)$. Figure 7 shows this when $N = 12$. Projecting the center to infinity then yields N parallel lines and $2N + 1$ Brass transversals (including the projection of the line at infinity, as before), with only $N - 1$ intersection points on each parallel line. We have thus obtained yet another improvement for the cases $N = 12m$ and $N = 12m - 1$ of Question 2.

We can also use this configuration to partly answer Question 1b, as follows. Each of the lines through the center has a pair of points each of which lies on just one of the transversals. (These N pairs of points are marked by closed circles in Figure 7.) There are 2^N sets of N points containing one point from each of these pairs; choosing one of these sets and removing it leaves $(N - 1)^2$ points with $2N$ lines of $N - 1$ points and $N + 1$ lines of N. We have thus shown that $T_n(n^2) \geq 3n + 4$ for $n = N - 1 = 12m - 1$, as promised earlier.

Returning to Question 2, we collect all our results and use them in Brass's recursive construction [3], obtaining:

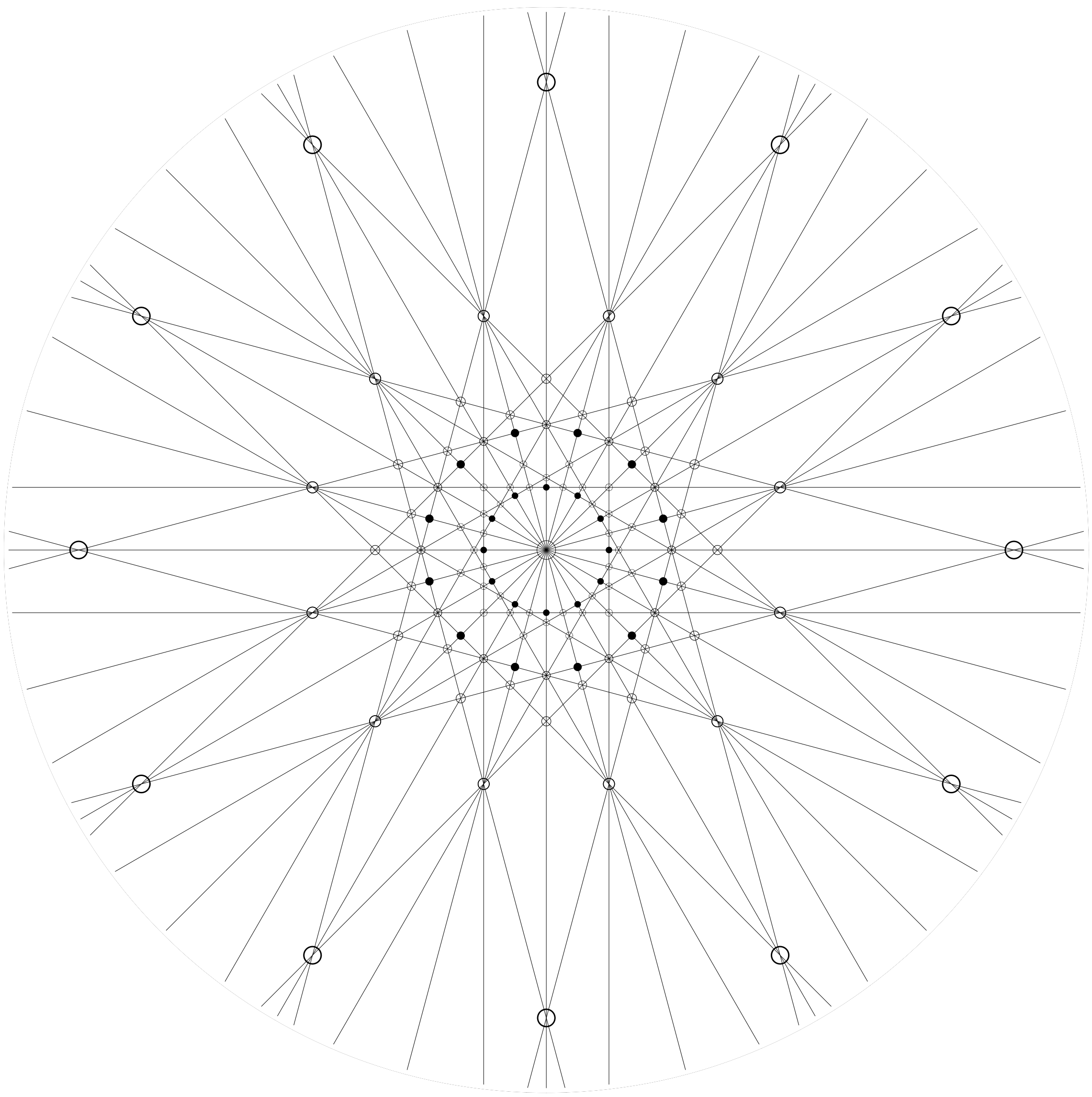

Figure 7: $(N, M) = (11, 25)$ or $(12, 25)$; also, 121 points, 37 lines of at least 11 (see text)

Theorem 1. *For $N \geq 5$, let*

$$M = M(N) = \begin{cases} 2N+3, & \text{if } N \equiv -1 \bmod 12; \\ 2N+1, & \text{if } N \equiv 0,1,3,5,7, \text{ or } 9 \bmod 12; \\ 2N, & \text{if } N \equiv 2,4,6,8, \text{ or } 10 \bmod 12. \end{cases} \tag{6}$$

Then for each $N \geq 5$ we have $t_N^{(N+1)}(n) = \Omega(n^{\tau_N})$ where $\tau_N = \log_N M$.

A numerical table of these new τ_N for $5 \leq N \leq 30$ follows:

N	5	6	7	8	9	10	11	12	13	14	15	16	17
τ_N	1.489	1.386	1.391	1.333	1.340	1.301	1.342	1.295	1.284	1.262	1.268	1.250	1.254

N	18	19	20	21	22	23	24	25	26	27	28	29	30
τ_N	1.239	1.244	1.231	1.235	1.224	1.241	1.224	1.221	1.212	1.215	1.208	1.210	1.203

Table 1

These values of τ_N, like the ones previously known, approach 1 as $N \to \infty$, but much more slowly: $\tau_N - 1 \approx \log 2/\log N$, while the previous results had $\tau_N - 1 \approx 1/N$. Unlike those previous τ_N, the values in Table 1 are quite far from the monotonic descent described in (3). For instance, our lower bound on $t_6^{(r)}(n)$ (any $r > 7$) uses configurations with many 7-point lines, and for $N = 8, 9, 10$ and $r > 11$ our lower bound on $t_N^{(r)}(n)$ uses configurations with many 11-point lines! Evidently the asymptotic behavior of $t_k^{(r)}(n)$ remains "a matter of great perplexity", as Dudeney described it almost 90 years ago. Can one improve on Theorem 1 by showing that $\tau_N \geq \tau_{N'}$ when $N \leq N'$? Can one exploit the extra line l_{N+1} in our configuration for the case $N = 12m - 1$ of Question 2 to obtain a further asymptotic improvement? Can the configurations arising from the identity $\rho_N(i, 2i) = \rho_N(N/6, (N/2) - 1)$ be exploited also in the cases $N = 18, 30, 42, \ldots$ when $N/6$ is odd?

One can attempt similar constructions with three or more nested stars, or only one. The only such variation we have found that bears on the questions that motivated us here is a triple pentagram. Adding to the old 16-point configuration of Figure 1 a third star, and also each of the five points where the line at infinity meets parallel sides of the three stars, we obtain 26 points spanning 21 lines of 15. See Figure 8. The open circles mark the 10 points each of which is contained in only three of the 21 lines; removing any one of these leaves 25 points in 18 lines of 5, in a configuration distinct from Figure 4.

More about Figure 1 and symmetries. We saw that the solution of the puzzle "to plant nine trees so that they shall form ten straight rows with three trees in every row" is more symmetrical than it appears from its usual presentation in Figure 3: this presentation has only 4 symmetries, but the projection shown on the left side of Figure 5 exhibits the 12-element group of symmetries of the regular hexagon. Likewise, our initial configuration of 16 points in 15 lines of 4 (Figure 1) turns out to be even more symmetrical than it looks: its group of projective symmetries is the alternating group A_5, acting transitively on the 15 lines and dividing the points into orbits of size 6 and 10. (The six-point orbit consists of the central point and the five points of the middle ring, each of which lies on 5 four-point lines; these are the points drawn as open circles in Figure 1.) To see this, let A_5 act on the $2 \cdot 6$ vertices of a regular icosahedron in $\mathbf{R}^3$, and map those vertices to 6 points in $\mathbf{P}^2$ while preserving a fivefold symmetry of the icosahedron. The other 10 points

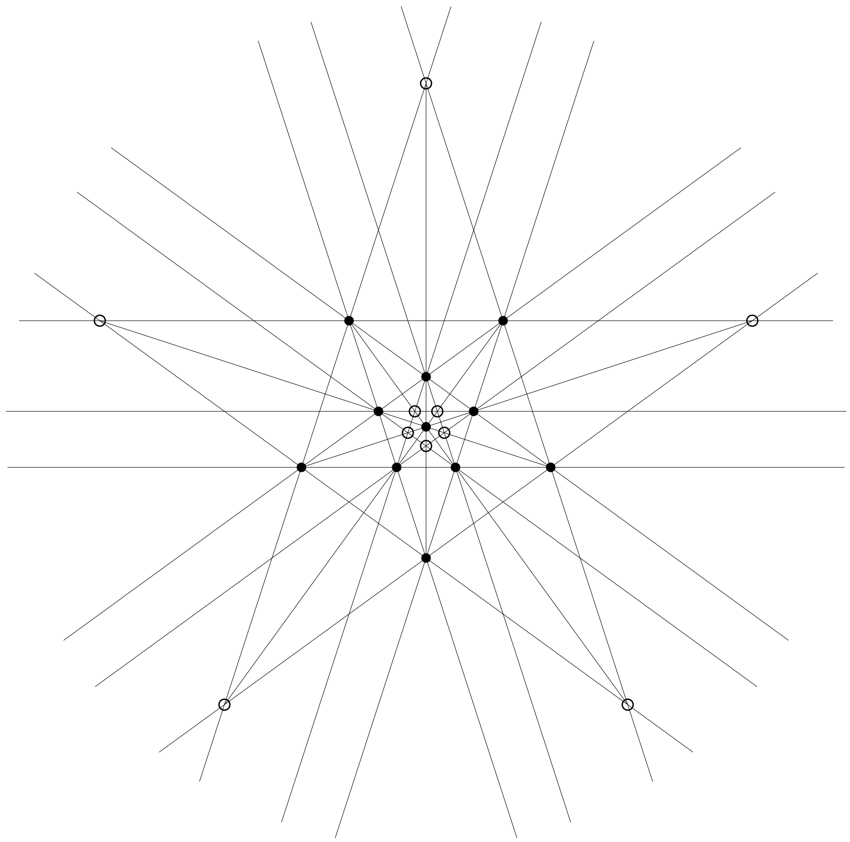

Figure 8: 26 points (5 at infinity), 21 lines of 5

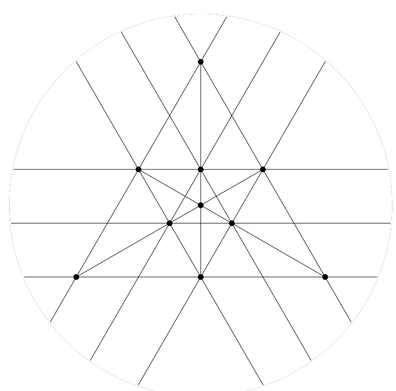

Figure 9: 13 points (3 at infinity), 13 lines

are the pairs of face centers, and the 15 lines are dual to the pairs of edge centers.[4] Let P_6, P_{10}, and P_{15} be the 6-, 10-, and 15-point orbits of points under this action of A_5, and L_6, L_{10}, L_{15} the corresponding orbits of lines. Then for $i, j \in \{6, 10, 15\}$ there exists a point in P_i contained in some line of L_j if and only if $i = 15$ or $j = 15$, in which case there are 30 such points. Figure 1 shows 60 of these incidences. If we instead consider the 21 points and 21 lines of $P_6 \cup P_{15}$ and $L_6 \cup L_{15}$, we find 90 incidences. These are contained among the 105 incidences in the finite projective plane of order 4; the 15 missing incidences are between each point of P_{15} and its dual line. Likewise the 31 points and 31 lines of $P_6 \cup P_{10} \cup P_{15}$ and $L_6 \cup L_{10} \cup L_{15}$ show 150 of the 186 incidences of the projective plane of order 5.

A similar configuration arises from the regular cube or octahedron, with symmetry group S_4, again larger than can be shown in any plane projection. The vertices, faces and edges of a regular octahedron yield $3 + 4 + 6$ points and as many lines, shown in Figure 9. The 48 incidences are among the 52 in the finite projective plane of order 3, lacking only the incidences between each face point and its dual line. To explain this, note that the points are the images of the $2 \cdot 13$ nonzero points $(x_1, x_2, x_3) \in \mathbf{Z}^3$ with each $|x_i| \leq 1$, and likewise the lines are $x_1y_1 + x_2y_2 + x_3y_3 = 0$ with each $y_i \in \{-1, 0, 1\}$. These remain distinct when reduced mod 3, and the only new incidences mod 3 are the four with

$$(x_1 : x_2 : x_3) = (y_1 : y_2 : y_3) = (\pm 1 : \pm 1 : \pm 1).$$

We can similarly relate the configurations of 21 or 31 points and lines of the previous paragraph with the corresponding finite projective planes, by recognizing them as points and lines with small coordinates in $\mathbf{Z}[\varphi]$ where $\varphi = (\sqrt{5} + 1)/2$, and then reducing these coordinates modulo the prime ideal $2\mathbf{Z}[\varphi]$ or $\sqrt{5}\mathbf{Z}[\varphi]$ respectively.

Account and acknowledgements. Last year I traveled to Calgary for the Workshop in Discrete Geometry in honor of the 50th birthday of Károly Bezdek, and found my way to the lecture room just in time for the problem session. I intended to present an open "tree-planting" problem in incidence geometry (Question 1) that I had wondered about for some time. The first few cases lead to appealing configurations; I had no better reason than pure curiosity for asking the question in general, but this meeting seemed a natural venue to raise the problem, and a reasonable one to hope for new information. That incidence geometry was an appropriate topic was confirmed when Peter Brass, who was among the first to present a problem at this session, asked a question of a similar flavor (Question 2), though his interest in it was more than recreational: a positive answer would yield an asymptotic improvement to a construction in his recent paper [3]. I thought that one of the "appealing configurations" I was about to show (Figure 1) might work, and after some hurried scribbling verified that projecting its center point to infinity answers the first odd instance ($N = 5$) of Brass's question. Later experimentation showed that the natural generalization of this configuration (as in Figure 2 for $n = 6$) yields such an answer for all odd $N > 3$, and afterwards led to the further refinements described in the Introduction and illustrated in Figures 5, 6, and 7.

I thank the organizers of the Calgary Workshop in Discrete Geometry, for inviting me to participate in the workshop; Peter Brass, for extended e-mail correspondence on these problems, including references to his paper [3] and the relevant sections of [4]; and the referee, for directing

[4] We noted this online at [9]. This page links to a picture of the images of the vertices, face centers, and edge centers, and of their dual lines; it also mentions Question 1 and the configurations for the cases $n = 4, 6, 8, \ldots$ and $n = 5$.

me to references [13, 15] and suggesting a rearrangement of the exposition. This paper is based on research supported in part by NSF grant DMS-0501029.

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