Quantum modified mooses

The Harvard community has made this article openly available. Please share how this access benefits you. Your story matters

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Published Version</td>
<td>doi:10.1016/j.nuclphysb.2003.09.006</td>
</tr>
<tr>
<td>Citable link</td>
<td><a href="http://nrs.harvard.edu/urn-3:HUL.InstRepos:28410169">http://nrs.harvard.edu/urn-3:HUL.InstRepos:28410169</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>This article was downloaded from Harvard University’s DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <a href="http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA">http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA</a></td>
</tr>
</tbody>
</table>

Quantum Modified Mooses

Spencer Chang† and Howard Georgi‡

Lyman Laboratory of Physics

Harvard University

Cambridge, MA 02138

Abstract

We summarize our findings on the quantum moduli constraints and superpotentials of an infinite family of moose extensions of $n_f = n_c$ SUSY QCD. For $n_c = 2$, we perform concrete calculations using traditional integrating out techniques as well as Intriligator’s “integrating in” technique. Checking the constraints and superpotentials in the limits of setting Λ’s to zero or integrating out mass terms, we find that the quantum moduli constraints are local in theory space and are equivalent to a consistent structure of “splitting relations” amongst the different theories. Extending the results to arbitrary $n_c$, we show that the splitting relations, along with a set of rules for flowing from a high energy theory to a low energy theory, incorporate much of the physics of the moose chain. The relations can be used both to simplify perturbative calculations of symmetry breaking and to incorporate nonperturbative effects.

‡Research supported in part by the National Science Foundation under grant number NSF-PHY/98-02709.

†chang@physics.harvard.edu

‡georgi@physics.harvard.edu
1 Introduction

In the past year, the method of “deconstruction” \[1, 2\] has taken hold as a new way of model building and addressing unresolved problems in particle physics. It was first presented as a way of dynamically generating extra dimensions in certain energy regimes. Even far from the continuum limit, extra-dimensional properties (such as locality) persist in the “theory” space of these models. This has provided a new tool to address unresolved issues in novel ways and also by “deconstruction” of previously known extra-dimensional mechanisms \[3\].

Crucial to the “deconstruction” mechanism is the structure of the allowed vacua of the theory. It is therefore interesting to investigate models where the vacuum structure can be analyzed nonperturbatively. A particularly simple family of models is SUSY SU\(^{(n_c)}\)\(^{N}\) gauge theories, which in a moose diagram appears to be a discretization of an extra 5th dimension (see fig. 1).

![Figure 1: Chain Extension of SUSY QCD with \(n_f = n_c\)](image)

In this moose diagram, each solid circle represents an SU\((n_c)\) gauge group, each dashed circle a global SU\((n_c)\) group, and each Q a chiral superfield. Now, according to moose conventions \[4\], the field \(Q_i\) transforms as a \((n_c, \bar{n}_c)\) under \((SU(n_c)_i, SU(n_c)_{i+1})\). As one can see this theory is free of gauge anomalies and is a natural moose extension of \(n_f = n_c\) SUSY QCD. Now if \(< Q_i > = aI\) for \(i = 1, \cdots N - 1\), a perturbative analysis suggests that this theory deconstructs 5-d \(n_f = n_c\) SUSY QCD compactified on a \(S_1/Z_2\) orbifold. We wish to see if this feature is maintained nonperturbatively, and discovering the nonperturbative vacua for these models will be the main topic of this paper. Thus, in essence, we will attempt to extend Seiberg’s nonperturbative SUSY QCD results \[5\] to this interesting group of models. We will find that the results are simple, beautiful and useful. We will describe a simple set of “splitting functions” that incorporate the structure of the moduli space, and thus also much of the physics. These splitting functions can be used both to simplify perturbative calculations and to incorporate nonperturbative effects.

The layout of this paper is as follows: Sections 2-7 focus on \(n_c = 2\) — in Section 2 we give a quick review of the standard lore for SUSY QCD required for our analysis; in Section 3 we discuss some preliminaries; in Section 4 we look at simple example calculations and describe how an iterative process determines the results for all N; in Section 5 we discuss the computational results and describe the general splitting functions (and explain why we call them “splitting functions”); in Section 6 we demonstrate all the possible consistency checks we can perform; in Section 7, we discuss spontaneous symmetry breaking, emphasizing that our splitting relations are local in theory space; in Section 8, we generalize our results to SU\((n_c)^{N}\); in Section 9 we try to analyze power-law running; and in Section 10 we conclude and look at questions that should be addressed by further research. In addition, in the appendix we discuss an argument that completes the analysis of the constraints for SU\((2)\).
2 Standard Lore

For the purposes of this paper, the results from Seiberg, et al. [5] that we are most interested in are those for SUSY QCD with \( n_f \leq n_c \). Using arguments based on holomorphy and symmetry, they deduced that at the nonperturbative level, any generated superpotential had the form

\[
W = c_{n_f,n_c} \left( \frac{\Lambda^{3n_c-n_f}}{\text{det}(Q\bar{Q})} \right)^{\frac{1}{n_c-n_f}}.
\]

(1)

To determine if this is actually generated, they argued that for \( n_f = n_c - 1 \) this can be reliably calculated via an instanton calculation (since the gauge symmetry is entirely broken by the vev). Finnell and Pouliot’s calculation in \( SU(2) \) determines that \( c_{1,2} = 1 \) in \( \text{DR} \) [6]. Integrating out quarks gives recurrence relations for the \( c \)'s that fixes \( c_{n_f,n_c} = n_c - n_f \). Two things to note about this superpotential are that it doesn’t make sense for \( n_f \geq n_c \) (since either \( \text{det}(Q\bar{Q}) = 0 \) or the power diverges) and that for \( n_f < n_c \) the vacuum is pushed off to infinity.

In the limit \( n_f \to 0 \), (1) gives the gaugino condensation superpotential

\[
W_{\text{gaugino}} = n_c \Lambda^3.
\]

(2)

Notice that there are \( n_c \) inequivalent vacua (corresponding to a \( \theta \to \theta + 2\pi \) transformation where \( \Lambda^3 \to e^{2\pi i/n_c} \Lambda^3 \)) as expected by the Witten index analysis.

Finally, they discovered that at \( n_f = n_c = 2 \), the classical moduli constraint \( \text{Pf}(M) = 0 \) is modified. Incorporating the nonperturbative effects, they found that the full Quantum Modified Moduli Space (QMMS) constraint was

\[
\text{Pf}(M) = \Lambda^4.
\]

(3)

For one site with \( n_c \neq 2 \) the QMMS condition in our notation should be written as

\[
\text{det}(Q_0Q_1) = \text{det} Q_0 \text{det} Q_1 - \Lambda_{1}^{2n_c}.
\]

(4)

3 Preliminaries and a teaser

Consider the chain model we introduced in fig. 1. Just as in \( n_f = n_c \) SUSY QCD, there is an anomaly free \( U(1)_R \) symmetry under which each of the \( Q \)'s are uncharged. Assuming that this \( U(1)_R \) holds nonperturbatively, there can be no superpotential generated. Thus the moduli space is not lifted, and a quantum moduli space should exist. According to Luty and Taylor [7], the moduli space can be parameterized by a set of gauge invariant monomials. In
the chain model, this set comprises

\[ \det(Q_0), \det(Q_1), \cdots, \det(Q_N) \]
\[ \det(Q_0Q_1), \det(Q_1Q_2), \cdots, \det(Q_{N-1}Q_N) \]
\[ \cdots \]
\[ \det(Q_0Q_1 \cdots Q_{N-1}), \det(Q_1Q_2 \cdots Q_N) \]
\[ Q_0Q_1 \cdots Q_N. \]

(5)

This gives us \( n_c^2 + \frac{N(N+3)}{2} \) moduli fields. A general vacuum breaks all of the gauge groups, which eats up \( N(n_c^2 - 1) \) of the scalars in the \( Q \)'s. This leaves \( N^2 - N(n_c^2 - 1) = N + n_c^2 \) independent moduli fields. Thus we expect there to be \( n_c^2 + \frac{N(N+3)}{2} - (N + n_c^2) = \frac{N(N+1)}{2} \) constraints amongst the moduli fields above. Classically, these constraints are just given by

\[ \det(Q_iQ_{i+1} \cdots Q_j) = \det(Q_i) \det(Q_{i+1}) \cdots \det(Q_j). \]

(6)

A standard procedure when analyzing these theories is to isolate one’s attention to an independent set of gauge invariant operators. However, in doing so, one throws out a significant amount of information, specifically the very constraints that we are trying to analyze! On the other hand, keeping all gauge invariant operators can make the problem intractable since manipulating the operators requires a knowledge of their interdependence.

With this in mind, in the next section, we choose a set of gauge invariant operators that is complete, but with just one expected constraint amongst them. The particular set of operators is

\[ \det(Q_0), \det(Q_1), \cdots, \det(Q_N), Q_0 \cdots Q_N. \]

(7)

In the later sections, when we discover the constraint that relates these operators, we will refer to it as the “highest” constraint.

However, before we descend to these technicalities, we will state one of our principal results for the general invariants and use it to do a nontrivial calculation. The basic splitting relation satisfied by these moduli spaces can be written in terms of determinants of strings of contiguous \( Q \)'s as (a determinant with no argument is defined to be 1)

\[ \det(Q_i \cdots Q_k) \det(Q_j \cdots Q_\ell) - \det(Q_i \cdots Q_\ell) \det(Q_j \cdots Q_k) \]
\[ = \det(Q_i \cdots Q_{j-2}) \det(Q_{k+2} \cdots Q_\ell) \prod_{m=j}^{k+1} \Lambda_m^{2n_c} \]
\[ \text{for } i \leq j - 1, j \leq k + 1, \text{ and } k \leq \ell - 1. \]

(8)

To attempt to convince the reader that there is something interesting going on in this relation, we will apply it in an example that is simple but nontrivial, the 2-site chain shown in fig. 2.

Applying the splitting relation, (8), for \( i = 0, j = k = 1 \) and \( \ell = 2 \) gives

\[ \det(Q_0Q_1) \det(Q_1Q_2) - \det(Q_0Q_1Q_2) \det(Q_1) = \Lambda_1^{2n_c} \Lambda_2^{2n_c}. \]

(9)
We will go very far out in moduli space in the classical $D$-flat direction

$$Q_0 = Q_2 = \begin{pmatrix} A_{mc} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_1 = \begin{pmatrix} \sqrt{v^2 I_{mc} + |A_{mc}|^2} & 0 \\ 0 & v I_{lc} \end{pmatrix}.$$  \hspace{1cm} (10)

where $I_{mc}$ and $I_{lc}$ are $m_c \times m_c$ and $\ell_c \times \ell_c$ identity matrices with $m_c + \ell_c = n_c$, and $A_{mc}$ is a diagonal $m_c \times m_c$ matrix. In perturbation theory, this breaks the symmetry down to a single $SU(\ell_c)$ and produces a QCD-like effective low energy theory with a QMMS. With the splitting relations, we will be able to write down the $\Lambda$ parameter in the low energy theory in a single step. In this case, the calculation is simple enough that one can check it perturbatively, but it is quite non-trivial.

In perturbation theory, the fields in the low energy theory are effectively frozen at their vacuum values except for the “quark fields” of the unbroken $SU(\ell_c)$ and we can go over to the low energy theory by replacing the high energy fields by

$$Q_0 = \begin{pmatrix} A_{mc} & 0 \\ 0 & \tilde{Q}_0 \end{pmatrix} \quad Q_1 = \begin{pmatrix} \sqrt{v^2 I_{mc} + |A_{mc}|^2} & 0 \\ 0 & v I_{lc} \end{pmatrix} \quad Q_2 = \begin{pmatrix} A_{mc} & 0 \\ 0 & \tilde{Q}_2 \end{pmatrix}. \hspace{1cm} (11)$$

Inserting (11) into (9), we find the splitting condition for the low energy theory,

$$\det(\tilde{Q}_0 \tilde{Q}_2) = \det(\tilde{Q}_0) \det(\tilde{Q}_2) - \frac{\Lambda_1^{2n_c} \Lambda_2^{2n_c}}{\det(A_{mc}^2) \det(v^2 I_{mc} + |A_{mc}|^2) v^{2\ell_c}}. \hspace{1cm} (12)$$

But this is the QMMS condition, (4), for the 1-site $SU(\ell_c)$ theory. Evidently, the $\tilde{\Lambda}$ parameter in the low energy theory is given by

$$\tilde{\Lambda}^{2\ell_c} = \frac{\Lambda_1^{2n_c} \Lambda_2^{2n_c}}{\det(A_{mc}^2) \det(v^2 I_{mc} + |A_{mc}|^2) v^{2\ell_c}}. \hspace{1cm} (13)$$

While it is possible to obtain this result directly using perturbation theory\(^1\), the splitting relations are a much simpler way of doing the analysis. We hope that this will encourage the reader to wade through the detailed analysis of $SU(2)$ models in sections 4-6. However, those impatient for more applications to moose chain models can skip directly to section 7 on page 11.

\(^1\)One has to keep in mind that the $\Lambda$ parameters depend upon the holomorphic coupling. In order to compare with perturbation theory, one has to relate the holomorphic coupling to the canonical coupling. See [8] for a nice discussion on this matter.
4 SU(2) Analysis

Before we begin, we should explain why we are first analyzing the case SU(2)^N instead of the more general case of SU(n_c)^N. We do this not only for simplicity, but because it is actually possible to add gauge invariant mass terms in the superpotential for the Q’s which are not allowed for larger n_c. This is just due to the fact that det(Q) is a mass term only for n_c = 2. On the other hand for n_c > 2, no such gauge invariant mass terms exist. In fact, though we have much more control in the SU(2) case, we expect the results we obtain for the SU(2) chains to be valid for SU(n_c) chains as well, and we will discuss the evidence for this later.

Overall, our methodology, assumptions, and general notation follow that of Intriligator’s “integrating in” method [9]. The method is to introduce mass terms that reduce our theory to one whose low energy superpotential is known. Then to integrate back in the field and deduce the original superpotential, one just performs the inverse Legendre transformation. We will also abbreviate the notation for determinants, defining

\[ dQ \equiv \det Q. \]  

(14)

4.1 N=2 Chain

Let’s start with the smallest nontrivial model, the N=2 chain. If we add a mass term for Q_1 the low energy theory will contain two disconnected SU(2) gauge theories with n_f = 1. The superpotentials for these theories are known due to the Finnell-Pouliot calculation [6]. The terms we add are

\[ W_{\text{tree}} = m dQ_1 + \text{tr}(\lambda Q_0 Q_1 Q_2) \]  

(15)

where \( \lambda \) is a 2 by 2 matrix.\(^3\)

To integrate out Q_1, we solve the equation of motion

\[ 0 = (dW_{\text{tree}}/dQ_1)^T = m(Q_1)^{-1} dQ_1 + Q_2 \lambda Q_0 \]  

(16)

and get

\[ W_{\text{tree},d} = -m dQ_1 = -\frac{d(Q_2 \lambda Q_0)}{m} = -\frac{d\lambda d Q_0 d Q_2}{m}. \]  

(17)

Note that in the last equality, we have assumed that the large determinant breaks up into the three separate determinants. This is justified since Q_0 and Q_2 are not charged under a common gauge group, and thus we are only breaking up indices with no gauge dynamics between them. This point is especially important when this is generalized to larger N.

Now, we know the low energy superpotential of the 2 disconnected 1-site theories is

\[ W_d = \frac{A_{1,d}^5}{dQ_0} + \frac{A_{2,d}^5}{dQ_2} = m\frac{A_1^4}{dQ_0} + \frac{m A_2^4}{dQ_2}. \]  

(18)

where we have assumed the matching conditions

\[ A_{i,d}^5 = m A_i^4 \text{ for } i = 1, 2. \]  

(19)

\(^2\)Note: This derivation mirrors that of the original reference [9], recast in our own language.

\(^3\)The ‘integrating in’ method requires couplings in W_{\text{tree}} to all gauge invariants involving the massive field Q_1, which is why we’ve added the trace term.
The matching conditions are true if the gauge couplings of the high and low energy theories match at the scale $m$ and the $\theta$ parameters are equal. The matching occurs without threshold corrections in $\overline{DR}$.

So to integrate $Q_1$ back in, we perform the inverse Legendre transform. We do this by integrating $m$ and $\lambda$ out of

$$W_n = W_d + W_{tree,d} - W_{tree}$$

$$= \frac{m \Lambda_1^4}{dQ_0} + \frac{m \Lambda_2^4}{dQ_2} - \frac{d\lambda}{m} \frac{dQ_0 dQ_2}{dQ_0} - m dQ_1 - \text{tr}(\lambda Q_0 Q_1 Q_2).$$

(20)

$\lambda$’s equation of motion is

$$0 = \left(\frac{dW_n}{d\lambda}\right)^T = -\frac{dQ_0 dQ_2}{m} d\lambda \lambda^{-1} - Q_0 Q_1 Q_2$$

(21)

and substituting this back into $W_n$ gives

$$W_u = m \left( \frac{\Lambda_1^4}{dQ_0} + \frac{\Lambda_2^4}{dQ_2} - dQ_1 + \frac{d(Q_0 Q_1 Q_2)}{dQ_0 dQ_2} \right).$$

(22)

Integrating out $m$ gives a zero superpotential for the high energy theory and also enforces the constraint

$$d(Q_0 Q_1 Q_2) = dQ_0 dQ_1 dQ_2 - \Lambda_1^4 dQ_2 - \Lambda_2^4 dQ_0.$$  

(23)

Thus, there is a quantum modified moduli space and we have derived the highest constraint for it when $N = 2$. Notice that it has the form of the classical constraint plus nonperturbative terms. Now, as discussed before, we only expected to get information on one constraint. To deduce the two remaining constraints for $d(Q_0 Q_1)$ and $d(Q_1 Q_2)$ we will use indirect arguments that will be discussed later in the paper.

![Figure 3: N-site chain with a missing end link](image)

Now, we can start an iterative process to determine the highest constraint for the larger $N$ models. Given the highest constraint for an $N$-site chain, give a mass term to $Q_N$ and integrate out $Q_N$ to discover the superpotential for an $N$-site chain with one missing end link (see fig. 3). Once we have this superpotential, we can take a $(N+1)$-site chain and give a mass term to it’s $Q_N$ (along with the analogous trace term we put in above for $N=2$), and then the lower theory consists of the $N$-site chain with a missing end link plus a 1-site chain with a missing link. Both of those superpotentials are now known, so we can just integrate back in $Q_N$ to discover the highest constraint for the $(N+1)$-site chain. Inductively, this process can be repeated for all $N$. 

6
4.2 2-site Chain With A Missing End Link

Let’s see how we can integrate out an end link to determine the superpotential of a 2-site chain with a missing end link. From the added mass term $W = m dQ_2$, we can integrate out $Q_2$ by using the constraint for $d(Q_0Q_1Q_2)$ in the following way. In the low energy theory, we have the fields $Q_0$ and $Q_1$. To integrate out $Q_2$ we should think carefully on what set of independent fields we choose for the high energy theory. The obvious choice is the set $dQ_0, dQ_1, Q_0Q_1Q_2$ since it contains a complete set for the low energy theory as well.

To proceed, we must rewrite the mass term in terms of this independent set, and that’s where we use the constraint to get

$$ m dQ_2 = m \frac{d(Q_0Q_1Q_2) + \Lambda_4^4 dQ_0}{dQ_0dQ_1 - \Lambda_4^4}. $$

(24)

Now, we must integrate out $Q_0Q_1Q_2$ since this is the only independent in our set that does not appear in the low energy theory. This just sets $Q_0Q_1Q_2 = 0$, which leaves

$$ W = \frac{m \Lambda_4^4 dQ_0}{dQ_0dQ_1 - \Lambda_4^4} = \frac{\Lambda_5^5 dQ_0}{dQ_0dQ_1 - \Lambda_4^4}. $$

(25)

This is precisely the result obtained by Seiberg, et al. [10], where we used the matching condition in the last step of (25). This method can be applied for any $N$.

5 Results

Now that we can determine the superpotentials for a chain with a missing end link, it is straightforward to determine the highest QMMS constraints for all $N$-site chains. In section 7 and in the appendix, we will discuss evidence that this result for the highest constraint generalizes to the lower constraints in an entirely trivial way. The lower constraint on any contiguous set of link fields is obtained by simply ignoring the other links and sites. In other words, the constraints are local in theory space. The constraints have the form

$$ d(Q_i \cdots Q_j) = $$

$$ dQ_i dQ_{i+1} \cdots dQ_j + \sum_{\text{any number of neighbor contractions}} \left(-\Lambda_4^4\right)^{dQ_{k-1}dQ_k \cdots dQ_j} \equiv D(Q_i, \cdots, Q_j) $$

(26)

where the rule is that once a set of neighbors is contracted the corresponding $Q$‘s are gone, so these $Q$’s cannot be contracted with their other neighbors. So for instance, we would have

$$ d(Q_0Q_1Q_2Q_3) = D(Q_0, Q_1, Q_2, Q_3) $$

$$ = dQ_0 dQ_1 dQ_2 dQ_3 - \Lambda_4^4 dQ_2 dQ_3 - \Lambda_4^4 dQ_0 dQ_3 - \Lambda_4^4 dQ_0 dQ_1 + \Lambda_4^4 \Lambda_4^4. $$

(27)

Alternatively, if we start with the final QMMS constraints, by using a superpotential with the mass term $m dQ_2$ and Lagrange multipliers enforcing the constraints, we can derive the same superpotential by integrating out $dQ_2, d(Q_1Q_2)$, and $Q_0Q_1Q_2$. Thus, this procedure is internally consistent.
The D functions that appear in these constraints (defined in (26)) satisfy the remarkable splitting property already discussed in section 3. Let us abbreviate

\[ D_{i,j} \equiv D(Q_i, \ldots, Q_j). \]  

(28)

For convenience, we will also define

\[ D_{i,i-1} = "D(" \equiv 1 \text{ and } D_{i,j} = 0 \text{ for } j < i - 1. \]  

(29)

The fundamental recursion relations (trivially derivable from the definitions, (26)) are

\[ D_{i,j} = D_{i,j-1} dQ_j - D_{i,j-2} \Lambda_j^4 \quad \text{and} \quad D_{i,j} = dQ_i D_{i+1,j} - \Lambda_{i+1}^4 D_{i+2,j}. \]  

(30)

With the definition (29), (30) is valid for all \( i \leq j \). From these one can derive the general splitting relation, for \( i \leq j - 1, j \leq k + 1, \) and \( k \leq \ell - 1 \)

\[ D_{i,k} D_{j,\ell} - D_{i,\ell} D_{j,k} = D_{i,j-2} D_{k+2,\ell} \prod_{m=j}^{k+1} \Lambda_m^4 \]  

(31)

We call these splitting relations because the product of \( \Lambda \)'s that appears on the right hand side is associated in a rather direct way with the splitting of the determinants of products of link fields. The point is that we can think of (31) in two different ways. Using (28) and the definition of the D functions, (31) is simply an identity. But using the constraints,

\[ D_{i,j} = d(Q_i, \ldots, Q_j), \]  

(32)

(31) becomes a dynamical statement about the determinants of the products. If any of the \( \Lambda \)'s in the product goes to zero, we can split each of the determinants into two pieces in such a way that the two terms on the left hand side of (31) cancel. For example, suppose \( \Lambda_m \to 0 \) for \( j < m < k + 1 \). This \( \Lambda \) characterizes the interaction between the links \( Q_m \) and \( Q_{m+1} \). If \( \Lambda_m \to 0 \), then the index shared by these two fields becomes a global index, and the determinant of a product of fields can be split at that index into the product of determinants:

\[
\begin{align*}
\det(Q_i \cdots Q_k) &\rightarrow \det(Q_i \cdots Q_m) \det(Q_{m+1} \cdots Q_k) \\
\det(Q_j \cdots Q_\ell) &\rightarrow \det(Q_j \cdots Q_m) \det(Q_{m+1} \cdots Q_\ell) \\
\det(Q_i \cdots Q_\ell) &\rightarrow \det(Q_i \cdots Q_m) \det(Q_{m+1} \cdots Q_\ell) \\
\det(Q_j \cdots Q_k) &\rightarrow \det(Q_j \cdots Q_m) \det(Q_{m+1} \cdots Q_k)
\end{align*}
\]  

(33)

If we put (33) into the left hand side of (31), the two terms cancel.

We will also see that the splitting relation (31) incorporates much of the perturbative physics we expect of the chain. This is interesting because if one instead started with (29) and (31), one could derive the forms of the D's (26). Therefore, the given constraints are the unique set that gives the physics contained within the splitting relation, which gives us confidence that we have found the correct constraints.
In the process of finding the constraints, (26), we’ve discovered the following superpotentials: for the N-site chain with a missing end as shown in fig. 3, there is a generated superpotential

$$W = \Lambda_5^N \frac{D(Q_0, Q_1, \cdots, Q_{N-2})}{D(Q_0, Q_1, \cdots, Q_{N-1})}$$ (34)

and for the N-site chain with its two end links missing (see fig. 4), there is a superpotential

$$W = \Lambda_5^1 D(Q_2, Q_3, \cdots, Q_{N-1}) + \Lambda_5^N D(Q_1, Q_2, \cdots, Q_{N-2}) \pm 2\sqrt{\Lambda_5^1 \Lambda_5^2 \cdots \Lambda_5^{N-1} \Lambda_5^N}$$ (35)

This last superpotential also agrees with the known result (for N=2) with similar interpretations of the terms as contributions both from instantons and gaugino condensation in the various SU(2) groups.

6 Consistency Checks

An easy consistency check we can perform is letting some $\Lambda_k$ go to zero. This is equivalent to setting $g_k = 0$, making SU(2)$_k$ a global symmetry. In this limit, we expect a chain to break up into two disjoint chains that don’t communicate. This behavior should be seen in the constraints and superpotentials we have derived. Looking at the N-site chain in this limit, we expect determinants to factorize in the following way

$$d(Q_i \cdots Q_j) \rightarrow d(Q_i \cdots Q_{k-1}) d(Q_k \cdots Q_j)$$ (36)

since the $Q_i \cdots Q_{k-1}$ and $Q_k \cdots Q_j$ do not interact any longer. The constraints should also factorize in this limit, and by using the splitting property for the D functions (31), we see that

$$D_{i,k-1} - D_{i,j} = D_{i,k-2} D_{k+1,j} \Lambda_k^4$$ (37)

which immediately implies

$$D(Q_i, \cdots, Q_j) \Lambda_k \rightarrow D(Q_i, \cdots, Q_{k-1}) D(Q_k, \cdots, Q_j)$$ (38)

as well. Thus the constraints for the N-site chain factor appropriately into the constraints of a (k-1)-site chain and the constraints of a (N-k)-site chain.

We can check the superpotentials as well. For an N-site chain with one missing end link, if we take the limit $\Lambda_k \rightarrow 0$, we get

$$W \rightarrow \frac{\Lambda_5^N D(Q_k, \cdots, Q_{N-2})}{D(Q_k, \cdots, Q_{N-1})}.$$ (39)
This is precisely the superpotential for a \((N-k)\)-site chain with a missing end link plus the superpotential for a \((k-1)\)-site chain (i.e. zero). For the \(N\)-site chain with both ends missing, the limit gives

\[
W \rightarrow \frac{\Lambda_1^5 \mathcal{D}(Q_2, \ldots, Q_{k-1})}{\mathcal{D}(Q_1, \ldots, Q_{k-1})} + \frac{\Lambda_N^5 \mathcal{D}(Q_k, \ldots, Q_{N-2})}{\mathcal{D}(Q_{k}, \ldots, Q_{N-1})}
\]

(40)

and this is precisely the superpotentials for a \((k-1)\)-site chain with a missing end link and a \((N-k)\)-site chain with a missing end link.

\[ \begin{array}{c}
\text{Figure 5: The consistency checks for } \Lambda_k \rightarrow 0. \text{ The filled gauge group is the one for which } \\
\text{ } \Lambda \rightarrow 0. \\
\text{Another consistency check is that adding a mass term for a } Q_k \text{ and integrating out that } \text{ } Q_k \text{ does properly flow down, upon using the matching condition } \text{[11]. We’ve already shown} \\
\end{array} \]
how this works for the N-site chain, so let’s see how the superpotentials reduce when we integrate out the quark superfield. For the N-site chain with a missing end link, integrating out the quark gives the superpotential

\[ W_{\text{low energy}} = \frac{m \Lambda^4_k D(Q_0, \cdots, Q_{k-2})}{D(Q_0, \cdots, Q_{k-1})} + \frac{m \Lambda^4_{k+1} D(Q_{k+2}, \cdots, Q_{N-1}) + \Lambda^5_N D(Q_{k+1}, \cdots, Q_{N-2}) \pm 2 \sqrt{m \Lambda^4_{k+1} \cdots \Lambda^4_{N-1} \Lambda^5_N}}{D(Q_{k+1}, \cdots, Q_{N-1})} \]

which is just the superpotentials for a k-site chain with a missing end link and a (N-k)-site chain missing both end links. Finally, for the N-site chain missing both end links, integrating out \( Q_k \) gives

\[ W_{\text{low energy}} = \frac{\Lambda^5_1 D(Q_2, \cdots, Q_{k-1}) + m \Lambda^4_k D(Q_1, \cdots, Q_{k-2}) \pm 2 \sqrt{m \Lambda^5_1 \Lambda^4_2 \cdots \Lambda^4_k}}{D(Q_1, \cdots, Q_{k-1})} + \frac{m \Lambda^4_{k+1} D(Q_{k+2}, \cdots, Q_{N-1}) + \Lambda^5_N D(Q_{k+1}, \cdots, Q_{N-2}) \pm 2 \sqrt{m \Lambda^4_{k+1} \cdots \Lambda^4_{N-1} \Lambda^5_N}}{D(Q_{k+1}, \cdots, Q_{N-1})} \]

and this is just the superpotentials for a k-site chain with both end links missing and a (N-k)-site chain with both ends links missing. Thus, the results come out as expected, thanks to some nontrivial algebra, the matching conditions (19), and use of the splitting relation (31).

For clarity, a graphical depiction of these two types of checks is given in the figures 5 and 6.

### 7 Spontaneous gauge symmetry breaking

In this section, we will see that by appropriately choosing parameters in a Moose-chain model and by going to the relevant region of moduli space, we can unravel the effect of spontaneous symmetry breaking in these models, and derive exact results for the matching of gauge couplings in spontaneous symmetry breaking.

For those who skimmed lightly over the previous three sections, let us recapitulate the results for the QMMS conditions for the moose chain. The basic constraint looks like

\[ d(Q_i \cdots Q_j) = dQ_i dQ_{i+1} \cdots dQ_j + \sum_{\text{any number of neighbor contractions}} dQ_i \cdots dQ_{k-1} dQ_k \cdots dQ_j \equiv D(Q_i, \cdots, Q_j) \]

We often abbreviate

\[ D_{i,j} \equiv D(Q_i, \cdots, Q_j), \quad D_{i,i-1} = "D()" \equiv 1, \quad D_{i,j} = 0 \text{ for } j < i - 1. \]
Then the general splitting relation, for $i \leq j - 1$, $j \leq k + 1$, and $k \leq \ell - 1$ is

$$D_{i,k} D_{j,\ell} - D_{i,\ell} D_{j,k} = D_{i,j-2} D_{k+2,\ell} \prod_{m=j}^{k+1} \Lambda_m^4.$$  \hspace{1cm} (45)

We now begin by considering spontaneous symmetry breaking in an $SU(2)$ chain. The basic idea is that when the $SU(2)_j \times \cdots \times SU(2)_{k+1}$ subgroup is spontaneously broken by vacuum values of the fields, $Q_j, \cdots, Q_k$ down to the diagonal $SU(2)_{j,k+1}$, the result should be a theory described by another moose of the same general form, but with the $j, \cdots, k+1$ sites

Figure 6: The consistency checks for integrating out a massive link. The bold link is the massive link.
collapsed into a single site. There should be some way of choosing the fields in the collapsed
theory to satisfy all the conditions that we have found for the moduli space in theories of
this general form. We will see that this is in fact possible, and that it provides interesting
information on the matching relations between the two theories.

More precisely, we consider a set of parameters such that \( \Lambda_j, \ldots, \Lambda_{k+1} \) are large compared
to the scale of our effective theory, but we go far out in moduli space to make

\[
D_{jk} = D(Q_j, \ldots, Q_k) = d(Q_j \cdots Q_k)
\]  

very large, so that

\[
\frac{\Lambda_j^4 \cdots \Lambda_{k+1}^4}{D(Q_j, \ldots, Q_k)^2}
\]  

is small compared to \( \Lambda_j^4, \ldots, \Lambda_{k+1}^4 \). The effective low energy theory should then describe a
theory with the \( SU(2)_j \times \cdots \times SU(2)_{k+1} \) gauge subgroup replaced by the diagonal, unbroken
\( SU(2)_{j,k+1} \) as shown in fig. 7.

There is no problem analyzing this in perturbation theory if the non-zero \( dQ_j \)'s are large
and the \( \Lambda \)'s are small. In this limit, we can explicitly write down the vacuum value of the
fields and work out the details of the super-Higgs mechanism. But we want more than that.
We want to understand this in the gauge invariant language of the QMMS conditions on the
high energy and low energy theory. This will give us detailed nonperturbative information.

\[
\begin{array}{ccccccc}
0 & Q_0 & 1 & \cdots & Q_{j-1} & j & \cdots & N
\end{array}
\]

\[
\downarrow
\]

\[
\begin{array}{ccccccc}
0 & \tilde{Q}_0 & 1 & \cdots & \tilde{Q}_{j-1} & \tilde{j} & \cdots & N
\end{array}
\]

Figure 7: The low energy theory below a symmetry breaking scale

Intuitively, what do we expect when the gauge groups break down to the diagonal? First
of all, fields that do not transform under the breaking groups should be unchanged upon
going to the low energy theory. This seems reasonable due to the locality in theory space
and the fact that these fields do not experience the dynamics of the broken gauge groups.
On the other hand, we don’t expect fields that transform only under the broken groups (i.e.
\( Q_j, \cdots, Q_k \)) to appear in the low energy fields except within composites. This is due to the
complementarity of the Higgs and the confining phases, which holds since the squarks are in
the fundamental representations.

In fig. 7 we have displayed these assumptions by leaving \( Q_0, \cdots, Q_{j-2} \) and \( Q_{k+2}, \cdots, Q_N \)
unchanged and labeling the links that are charged under the \( SU(2)_{j,k+1} \) group as \( \tilde{Q}_{j-1} \)
and \( \tilde{Q}_{k+1} \). These newly introduced tilde fields should satisfy some obvious constraints. In
particular, they should have all the right properties under the unbroken symmetries in the
low energy theory, and not transform under any symmetries that have disappeared in going to the low energy theory. However, explicitly writing $\tilde{Q}_{j-1}$ and $\tilde{Q}_{k+1}$ in terms of the high energy $Q$’s is not expected to be holomorphic and requires detailed knowledge about the Kähler potential. On the other hand, if we think in terms of gauge invariants, we will not encounter these subtleties. In fact, we’ll see that everything will naturally be holomorphic and cleanly defined, which will aid in interpreting results.

Now that we’ve refocused our attention to gauge invariants, it is simple to relate the low energy gauge invariants in terms of the high energy ones. It shouldn’t come as a surprise that the symmetries alone do not uniquely determine the relationship. However, we also have the restriction that the low energy gauge invariants satisfy their QMMS constraints. This along with the assumption that the untilded fields are unchanged uniquely determines the map between gauge invariants. Let us emphasize that it was not at all trivial that there was any consistent map between gauge invariants. That one exists and appears to be unique gives further evidence for the validity of the proposed constraints.

The gauge invariant quantities in the low energy theory that involve $\tilde{Q}_{j-1}$ and $\tilde{Q}_{k+1}$ must be chosen to satisfy

\begin{align}
\text{d}(Q_i \cdots \tilde{Q}_{j-1}) &= \left(D(Q_j, \cdots, Q_k)\right)^{-1} \text{d}(Q_i \cdots Q_k) \quad (48) \\
\text{d}(\tilde{Q}_{k+1} \cdots Q_\ell) &= \left(D(Q_j, \cdots, Q_k)\right)^{-1} \text{d}(Q_j \cdots Q_\ell) \quad (49) \\
\text{d}(Q_i \cdots \tilde{Q}_{j-1} \tilde{Q}_{k+1} \cdots Q_\ell) &= \left(D(Q_j, \cdots, Q_k)\right)^{-1} \text{d}(Q_i \cdots Q_\ell) \quad (50) \\
Q_0 \cdots \tilde{Q}_{j-1} \tilde{Q}_{k+1} \cdots Q_N &= \left(D(Q_j, \cdots, Q_k)\right)^{-1/2} Q_0 \cdots Q_N \quad (51)
\end{align}

Evidently, this incorporates all the right symmetry properties and is the unique choice that satisfies the properties stated above. Notice that the factors of $D(Q_j, \cdots, Q_k)$ in (48)-(51) are associated entirely with the fields $\tilde{Q}_{j-1}$ and $\tilde{Q}_{k+1}$.

We can rewrite (31) as

\begin{equation}
\frac{D_{i,\ell}}{D_{j,k}} = \frac{D_{i,k} D_{j,\ell}}{D_{j,k}^2} - D_{i,j-2} D_{k+2,\ell} \frac{\prod_{m=j}^{k+1} \Lambda_m^4}{D_{j,k}^2}.
\end{equation}

Comparing (48-51) with (52), we see that the general splitting relations become the splitting conditions in the effective low energy theory with the matching condition

\begin{equation}
\Lambda_{j,k+1} = \frac{\Lambda_j \cdots \Lambda_{k+1}}{D(Q_j, \cdots, Q_k)^{1/2}}.
\end{equation}

This is the appropriate nonperturbative form of the matching condition in this case. Note that this diagonal scale also agrees in form with the diagonal scale given in the gaugino condensation term of (35).
8 Higher $n_c$

It is noteworthy that the discussion in sections 6 and 7 about constraints, the splitting relation, and spontaneous symmetry breaking did not rely in any essential way on the fact that we were specialized to the case $n_c = 2$. As a matter of fact, the discussion can be easily generalized to arbitrary $n_c$ by simply changing $\Lambda^4 \rightarrow \Lambda^{2n_c}$. Thus it is reasonable to assume that these arguments are indeed valid for all $n_c$. However, for higher $n_c$, we must solve a more general problem, because now it is possible for the $SU(n_c)$ gauge groups to break down to a non-abelian subgroup, $SU(\ell_c)$ for $\ell_c < n_c$. This is a generalization of the simple 2-site example we worked out in section 3. Perturbatively, the general classical $D$-flat situation looks like

$$Q_i = \begin{pmatrix} A_{mc} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } 0 \leq i < j \text{ and } k < i \leq N$$

and

$$Q_i = \begin{pmatrix} \sqrt{v_i^2 I_{mc} + |A_{mc}|^2} & 0 \\ 0 & v_i I_{\ell_c} \end{pmatrix} \quad \text{for } j \leq i \leq k$$

where again $I_{mc}$ and $I_{\ell_c}$ are $m_c \times m_c$ and $\ell_c \times \ell_c$ identity matrices with $m_c + \ell_c = n_c$, and $A_{mc}$ is a diagonal $m_c \times m_c$ matrix. In perturbation theory, this produces an effective low energy theory like that in fig. 4 but in which the sites describe unbroken $SU(\ell_c)$ gauge subgroups, and now all the fields are suitably modified.

The gauge invariant way of describing this effect makes use of the product,

$$Q_0 \cdots Q_N.$$  \hfill (55)

It is the fact that some components of this matrix are large that signals spontaneous breaking of $SU(n_c) \rightarrow SU(\ell_c)$. To analyze this problem fully, we would find the form of the fields in terms of the values of the gauge invariant quantities (up to gauge transformations, of course), and then find appropriate forms for the low energy fields consistent with the splitting conditions. We will discuss this further in the next section. Here we will simply introduce the issues by showing how the splitting conditions work for one site and $A = aI_{mc}$. This is trivial and well understood from the early days of the subject, but it is worth saying in our current language.

The perturbative analysis of symmetry breaking in the one site case goes like this. To leading order the coupling does not change at the symmetry breaking threshold, so

$$(\Lambda/a)^{2n_c} = \left(\frac{\mu}{a}\right)^{2n_c} e^{-8\pi^2/g(\mu)^2} = (\tilde{\Lambda}/a)^{2\ell_c}$$

or

$$\tilde{\Lambda}^{2\ell_c} = \frac{\Lambda^{2n_c}}{a^{2m_c}}.$$  \hfill (57)

More generally, the result is

$$\tilde{\Lambda}^{2\ell_c} = \frac{\Lambda^{2n_c}}{\det A_{mc}^2}.$$  \hfill (58)

---

6This has already been proven for the 2 site case in [11].

7We are being sloppy here and below with phases. One could (and perhaps should) keep track of the phases and the $\theta$ parameters - but we will assume that everything is real and not worry about these niceties. They are easy to put in correctly because of holomorphy.
Thus in this case, it is clear how to make the connection between the high energy theory, with QMMS
\[ d(Q_0 Q_1) = dQ_0 dQ_1 - \Lambda^{2n_c} \] (59)
and the low energy theory. The gauge invariant product in the low energy theory is the projection onto the sector orthogonal to \( A_{mc} \)-
\[ Q_0 Q_1 \to \begin{pmatrix} B_{mc} & 0 \\ 0 & \tilde{Q}_0 \tilde{Q}_1 \end{pmatrix} \quad \text{and} \quad d(\tilde{Q}_0 \tilde{Q}_1) = \frac{d(Q_0 Q_1)}{\det B_{mc}}. \] (60)
The matrix \( B_{mc} \) is defined nonperturbatively in terms of the components of the gauge invariant product, \( Q_0 Q_1 \). These components might be fixed, for example, by Lagrange multiplier terms in the superpotential. The other gauge invariants would then be related by
\[ d\tilde{Q}_j = \frac{dQ_j}{\sqrt{\det B_{mc}}} \quad \text{for } j = 0 \text{ or } 1 \] (61)
Then the QMMS condition in the high energy theory can be written as
\[ d(\tilde{Q}_0 \tilde{Q}_1) = d\tilde{Q}_0 d\tilde{Q}_1 - \frac{\Lambda^{2n_c}}{\det B_{mc}} \] (62)
which is the QMMS condition for the low energy theory with
\[ \tilde{\Lambda}^{2\ell_2} = \frac{\Lambda^{2n_c}}{\det B_{mc}} \] (63)
which is the nonperturbative version of (58).

### 9 Power-law running

Armed with our new calculational tools, we can attempt to address a practical example, that of power-law running. In the standard analysis [12], power-law running is not conventional running of the coupling constant at all, but rather just one-loop quantum corrections to the gauge coupling that are dependent upon the cutoff of the extra-dimensional gauge theory. Thus, it is difficult to know if power-law running is truly UV completion independent.

However, if we now consider a theory with a larger \( n_c \), we can try to separate the issue of power-law running from the physics of the cut-off. Classically, the constraints of \( D \)-flatness in these theories are
\[ Q_j^i Q_j - Q_{j+1}^i Q_{j+1}^j \ll I. \] (64)
Now, consider an \( SU(n_c) \) chain with equal gauge couplings at the sites and the following structure of vacuum values for the \( Q \)'s consistent with (64) (ignoring for the moment, the quantum modifications):
\[ Q_0 = Q_N = \begin{pmatrix} aI_{mc} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_j = \begin{pmatrix} \sqrt{v^2 + a^2 I_{mc}} & 0 \\ 0 & v I_{\ell_c} \end{pmatrix} \quad \text{for } 1 \leq j < N - 1. \] (65)
First suppose that \( a = 0 \). At energy scales far above \( gv \), the theory looks like a four dimensional theory. But at energies between \( gv \) and \( gv/N \), we are surrounded by the KK
modes of a five dimensional gauge theory with the fifth dimension confined to a region of size \( \frac{N}{g v} \). Finally, at energies below \( g v / N \), below the mass of the lightest KK mode, our wavefunction spreads over the whole fifth dimension, and we see only the massless unbroken \( SU(n_c) \) gauge theory. Again, for \( a \neq 0 \), the \( SU(n_c) \) gauge group is broken down to the diagonal \( SU(\ell_c) \) and the other gauge multiplets (all the broken KK modes included) outside this unbroken subgroup get a contribution to their mass squared.

However, now the question to ask is at what scale does this 5-d \( SU(n_c) \) gauge theory break down to \( SU(\ell_c) \)? To answer this question, we begin by analyzing the lowest mass eigenstates of the broken gauge bosons. Out of the gauge bosons in \( SU(n_c - \ell_c) \), the lowest lying state of the \( SU(n_c - \ell_c) \) gauge bosons has a first order contribution \( \frac{a^2}{2 N} \) whereas the lowest state of the \( X, Y \) off diagonal gauge bosons gets a contribution \( \frac{a^4}{4 N^2} + \frac{a^2}{N} \). Out of these scales, the scales in the \( X, Y \) masses are more important from the 5-d point of view. This is because as we go up in energy, only when we encounter the \( X, Y \) KK thresholds does the beta function change. Since only they contribute to power law running, the \( X, Y \) mass indicates when the 5-d theory sees the broken gauge structure. Out of the two terms in the \( X, Y \) mass, it turns out that we have to take the first term to dominate over the second in order to have a continuum limit in which the breaking scale is above the scale of the extra-dimension \((1/R)\). Thus, we will define the breaking scale

\[
v_b \equiv \frac{a^2}{2v} \tag{66}\]

and assume

\[
v \gg v_b \gg 1/R \sim v/N. \tag{67}\]

However, this theory does not have the continuum limit that we desire. The issue is that the continuum limit requires the following \( N \) scaling:

\[
v \sim N \quad a \sim \sqrt{N} \tag{68}\]

From the form of the vevs \( \Phi_{brane} \), the 5-d interpretation is of a bulk and two brane scalar fields getting vevs. The \( Q_1, \cdots Q_{N-1} \) vevs correspond to a bulk \( SU(n_c) \) adjoint getting a vev

\[
< \Phi_{balk} > = \begin{pmatrix} v_b I_{m_c} & 0 \\ 0 & 0 \end{pmatrix} \tag{69}\]

whereas the \( Q_0, Q_N \) vevs correspond to two brane \( SU(n_c) \) fundamentals transforming under a global \( SU(n_c) \), one on each brane with vevs

\[
< \Phi_{brane} > = \begin{pmatrix} a I_{m_c} & 0 \\ 0 & 0 \end{pmatrix}. \tag{70}\]

Neglecting the brane vevs, the bulk vev gives a universal contribution to all \( X, Y \) KK gauge bosons, which leads to the standard power-law running. However, when the bulk and brane vev contributions become equally important, the KK masses are not universally shifted and thus the model does not correspond to the standard power-law running setup. In the continuum limit, the brane vevs blow up and give a large threshold correction to the power law running (an explicit calculation confirms this). Therefore, in this model, the naive attempt at UV completing the standard power-law running setup fails, and thus the techniques formulated in this paper cannot be used to properly analyze power-law running.
10 Conclusion

In this paper, we have analyzed a set of N-site moose extensions of SUSY QCD. In the particular case of $SU(2)$ gauge groups, we have determined the superpotentials and the quantum modified moduli constraints of all N-site chains with or without end links. By integrating out link fields and setting $\Lambda$’s to zero, we have discovered that these results satisfy a restrictive network of interdependent checks. During the process, we noted that the success of the checks was highly dependent upon the mathematical properties of the constraints.

We found that it was particularly useful to recast our constraints in the form of “splitting relations” that are associated with splitting of determinants of products of link fields when a gauge coupling goes to zero. The splitting relations are particularly useful when we analyze regions of moduli space that correspond to a spontaneous breaking of some of the symmetries of the moose chain. When we combine the splitting relations with rules for writing the invariants in the low energy theory, below the symmetry breaking scale, we get nontrivial information about the physics of the low energy theory. We have shown how to use it to simplify perturbative analysis and to incorporate nonperturbative results. The success and $n_c$ independence of these checks gave compelling evidence to assume the QMMS constraints are valid for all $n_c$.

We tried to UV complete a model of power-law running in our setup, but the naive attempt did not have the desired continuum limit. Thus we were not able to use our calculational tools on this interesting problem. As an aside, shortly after this paper was submitted, a paper appeared which analyzed some conditions where power-law running corrections can be trusted [13].

There are directions for further analysis. Many of our calculations were specifically for the case of $SU(2)$, so it would be useful to strengthen the arguments for the form of the constraints for all $n_c$. Moving on to other theories, circular moose models have been important to “deconstructing” compact extra dimensions. Some work in this class of theories has already been done [14] [15] [16], but there may still be some interesting physics to work out in regards to their moduli spaces. In particular, we believe that the structure of the splitting relations is far more complicated, and contains far more information than it does for the linear chains.

Acknowledgements

Nima Arkani-Hamed, Andrew Cohen, and Girma Hailu collaborated with us on some of the early stages of this work. We thank them for many crucial discussions and suggestions. After the first version was submitted to the arXiv, Girma Hailu posted two papers containing some overlapping work [17] [18]. We also thank Arthur Hebecker, Erich Poppitz, Yuri Shirman, and Alexander Westphal for pointing out issues in the power-law running section. Some of this work was done at the Aspen Center for Physics, and HG is very grateful to the staff of the Center and to the organizers of the “Advances in Field Theory and Applications to Particle Physics” workshop, Andrew Cohen, David Kaplan, Ann Nelson and Matt Strassler, for the opportunity to work in this stimulating environment. SC would also like to thank Csaba...
Csáki for looking over an early draft of the paper and for providing helpful comments. This paper is based upon research that is supported in part by the National Science Foundation under grant number NSF-PHY/98-02709 and by a National Science Foundation Graduate Fellowship.

A Determination of Lower Constraints

This appendix contains another bit of evidence in favor of our assertion that the lower constraints are local in theory space, satisfying (20). For the N-site chain, “integrating in” has determined the highest constraint

\[ d(Q_0 \cdots Q_N) = D(Q_0, Q_1, \cdots, Q_N). \]  

(71)

Through an argument based on anomaly matching, the constraints on the smaller determinants can be deduced as well. For the purposes of this section, we take general \( SU(n_c) \) groups instead of the specific case of \( SU(2) \) to show the generality of the anomaly matching.

First of all, the elementary degrees of freedom in this theory have the following charges under the non-anomalous global symmetries:

<table>
<thead>
<tr>
<th>Field</th>
<th>( SU(n_c)_0 )</th>
<th>( SU(n_c)_{N+1} )</th>
<th>( U(1) )</th>
<th>( U(1)_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_0 )</td>
<td>( n_c )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( Q_i )</td>
<td>1</td>
<td>1</td>
<td>((-1)^i)</td>
<td>-1</td>
</tr>
<tr>
<td>( Q_N )</td>
<td>1</td>
<td>( \bar{n}_c )</td>
<td>((-1)^N)</td>
<td>-1</td>
</tr>
<tr>
<td>( \lambda_j )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(72)

where \( i = 1, \cdots, N - 1 \), \( j = 1, \cdots, N \), and the \( U(1)_R \) charge is for the fermionic component.

This is to be compared with the global charges of the chosen set of gauge invariant composites:

<table>
<thead>
<tr>
<th>Field</th>
<th>( SU(n_c)_0 )</th>
<th>( SU(n_c)_{N+1} )</th>
<th>( U(1) )</th>
<th>( U(1)_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_0 \cdots Q_N )</td>
<td>( n_c )</td>
<td>( \bar{n}_c )</td>
<td>((1 + (-1)^N)/2)</td>
<td>-1</td>
</tr>
<tr>
<td>( dQ_j )</td>
<td>1</td>
<td>1</td>
<td>((-1)^j n_c)</td>
<td>-1</td>
</tr>
</tbody>
</table>

(73)

where \( j = 0, \cdots, N \) and again the \( U(1)_R \) charge is for the fermionic component.

Now, (71) suggests that there is always a vacuum where the \( U(1) \) is preserved. Specifically where the vevs of the determinants are all zero and \( \langle Q_0 \cdots Q_N \rangle \sim \Lambda_1^2 \Lambda_3^2 \cdots \Lambda_N^2 \) if \( N \) is odd and zero if \( N \) is even. If in addition the anomalies match between composites and elementary fields, this would convince us that the \( U(1) \) preserving vacuum does in fact exist.
In terms of the elementary fields we get the following global anomalies:

<table>
<thead>
<tr>
<th>Global Anomaly</th>
<th>N odd</th>
<th>N even</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1) = U(1)^3$</td>
<td>0</td>
<td>$n^2$</td>
</tr>
<tr>
<td>$U(1)SU(n_c)^2_0$</td>
<td>$n/2$</td>
<td>$n/2$</td>
</tr>
<tr>
<td>$U(1)SU(n_c)^2_{N+1}$</td>
<td>$-n/2$</td>
<td>$n/2$</td>
</tr>
<tr>
<td>$U(1)U(1)^2_R$</td>
<td>0</td>
<td>$n^2$</td>
</tr>
<tr>
<td>$U(1)^2U(1)_R$</td>
<td>$-(N + 1)n^2$</td>
<td>$-(N + 1)n^2$</td>
</tr>
<tr>
<td>$U(1)_R = U(1)^3_R$</td>
<td>$-n^2 - N$</td>
<td>$-n^2 - N$</td>
</tr>
<tr>
<td>$U(1)_R SU(n_c)^2_0 = U(1)<em>R SU(n_c)^2</em>{N+1}$</td>
<td>$-n/2$</td>
<td>$-n/2$</td>
</tr>
<tr>
<td>$SU(n_c)^3_D = -SU(n_c)^3_{N+1}$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Let’s now address the global anomalies on the composite side. For $N$ even, let’s assume the $U(1)$ preserving vacuum is allowed (i.e. where all vevs vanish). In this case, (74) sets a linear combination of the $dQ_j$ with $U(1)$ charge $n_c$ to zero. Thus, in matching anomalies we can leave out $dQ_N$. The remaining composites $dQ_0, dQ_1, \cdots, dQ_{N-1}$ and $Q_0 \cdots Q_N$ have the exact global anomalies as in the rightmost column above.

When $N$ is odd, the proposed vacuum that breaks $SU(n_c)_0 \times SU(n_c)_{N+1} \rightarrow SU(n_c)_D$ and preserves $U(1)$ sets $\text{tr}(Q_0Q_1 \cdots Q_N)$ to zero. This leaves an adjoint under the $SU(n_c)_D$. This adjoint plus all of the $dQ_j$ give the following global anomalies:

<table>
<thead>
<tr>
<th>Composite Global Anomaly</th>
<th>N odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1) = U(1)^3$</td>
<td>0</td>
</tr>
<tr>
<td>$U(1)SU(n_c)^2_D$</td>
<td>0</td>
</tr>
<tr>
<td>$U(1)U(1)^2_R$</td>
<td>0</td>
</tr>
<tr>
<td>$U(1)^2U(1)_R$</td>
<td>$-(N + 1)n^2$</td>
</tr>
<tr>
<td>$U(1)_R = U(1)^3_R$</td>
<td>$-n^2 - N$</td>
</tr>
<tr>
<td>$U(1)_R SU(n_c)^2_D$</td>
<td>$-n$</td>
</tr>
<tr>
<td>$SU(n_c)^3_D$</td>
<td>0</td>
</tr>
</tbody>
</table>

These anomalies are also precisely those of the elementary fields.

This analysis has suggested two things: 1) that the full set of QMMS constraints has an allowed vacuum where the $U(1)$ is preserved and 2) that the set of composites shown above are truly a complete set (i.e. the other determinants can be written in terms of them). That vacuum’s existence along with known limits of the constraints actually completely determines the QMMS constraints for the smaller determinants. For instance, consider the constraint for $d(Q_0Q_1)$. The limits $\Lambda_i \rightarrow 0$ determine that $d(Q_0Q_1) = dQ_0dQ_1 - \Lambda_1^4 + X$ where $X$ has the form

$$X \sim \sum (\prod_i a_i^\gamma_i)\Lambda_1^4$$

where $a_i \equiv \Lambda_1^4/(dQ_{i-1}dQ_i)$. However, if the $U(1)$ preserving vacuum truly exists, $X$ must be identically zero. Thus, the constraint is just that for the original one site case!
The analysis for the other lower determinants works in the same fashion. In each case, the constraint is exactly like the highest constraint for a lower site model as given in (26). Although it might seem surprising that the non-highest constraints would take on their naïve form, previous work and observations suggests that it is to be expected [19].

References

