Companion forms over totally real fields, II

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Abstract

We prove a companion forms theorem for mod $l$ Hilbert modular forms. This work generalises results of Gross and Coleman–Voloch for modular forms over $\mathbb{Q}$, and gives a new proof of their results in many cases.

1 Introduction

If $f \in S_k(\Gamma_1(N) ; \mathbb{F}_p)(\epsilon)$ is a mod $l$ cuspidal eigenform, where $l \nmid N$, there is a continuous, odd, semisimple Galois representation

$$\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\mathbb{F}_l)$$

attached to $f$. A famous conjecture of Serre predicts that all continuous odd irreducible mod $l$ representations should arise in this fashion. Furthermore, the “strong Serre conjecture” predicts a minimal weight $k_\rho$ and level $N_\rho$, in the sense that $\rho \cong \rho_g$ for some eigenform $g$ of weight $k_\rho$ and level $N_\rho$ (prime to $l$), and if $\rho \cong \rho_f$ for some eigenform $f$ of weight $k$ and level $N$ prime to $l$ then $N_\rho | N$ and $k \geq k_\rho$. The question as to whether all continuous odd irreducible mod $l$ Galois representations are modular in this sense is still open, but the implication “weak Serre $\Rightarrow$ strong Serre” is essentially known (aside from a few cases where $l = 2$).

In solving the problem of weight optimisation it becomes necessary to consider the companion forms problem; that is, the question of when it can occur that we have $f = \sum a_n q^n$ of weight $2 \leq k \leq l$ with $a_l \neq 0$, and an eigenform $g = \sum b_n q^n$ of weight $k' = l + 1 - k$ such that $na_n = n^k b_n$ for all $n$. Serre conjectured that this can occur if and only if the representation $\rho_f$ is tamely ramified above $l$. This conjecture has been settled in most cases in the papers of Gross ([Gro90]) and Coleman-Voloch ([CV92]).

Our earlier paper [Gee04] generalised these results to the case of parallel weight Hilbert modular forms over totally real fields $F$ in which $l$ splits completely, by generalising the methods of [CV92]. In this paper we take a completely different and rather more conceptual approach; we construct our companion form by using a method of Ramakrishna to find an appropriate characteristic zero Galois representation, and then use recent work of Kisin ([Kis04]) to prove that the representation is modular. Note that our companion form is not necessarily of minimal prime-to-$l$ level, but that this is irrelevant for applications to Artin’s conjecture, and that in many cases a form of minimal level may be obtained from ours by the methods of [Jar99], [SW01], [Raj01] and [Fuj99]. In the case of weight $l$ forms, we avoid potential difficulties with weight 1 forms by constructing a companion form in weight $l$. 

1
2 Statement of the main results

Let \( l > 2 \) be a prime, and let \( F \) be a totally real field. We assume that if \( l > 3 \), \([F(\zeta_l) : F] > 3\) (note that this is automatic if \( l \) is unramified in \( F \)). Let \( \epsilon \) denote both the \( l \)-adic and mod \( l \) cyclotomic characters; this should cause no confusion. Let \( \rho : G_K \to \GL_2(O) \) be a continuous representation, where is \( K \) a finite extension of \( \Q_l \), and \( O \) is the ring of integers in a finite extension of \( \Q_l \). We say that \( \rho \) is ordinary if it is Barsotti-Tate, coming from an \( l \)-divisible group which is an extension of an étale group by a multiplicative group, each of rank one as \( O \)-modules. We say that it is potentially ordinary if it becomes ordinary upon restriction to an open subgroup of \( G_K \). We say that a Hilbert modular form of parallel weight 2 is (potentially) ordinary at a place \( v \mid l \) if its associated Galois representation is (potentially) ordinary at \( v \). These definitions agree with those in [Kis04]; they are slightly non-standard, but note that if the level is prime to \( l \) then this is equivalent to the \( U_v \)-eigenvalue being an \( l \)-adic unit. We say that a Hilbert modular form of parallel weight \( k \), \( 3 \leq k \leq l \) is ordinary at a place \( v \mid l \) if its \( U_v \)-eigenvalue is an \( l \)-adic unit. Finally, we say that a modular form is (potentially) ordinary if it is (potentially) ordinary at all places \( v \mid l \).

Our main theorem is the following:

**Theorem 2.1.** Let \( g \) be an ordinary Hilbert modular eigenform of parallel weight \( k \), \( 2 \leq k \leq l \), and level coprime to \( l \). Let its associated Galois representation be \( \rho_g : G_F \to \GL_2(\Q_l) \), so that (by Theorem 2 of [Wil88]) we have, for all places \( v \mid l \),

\[
\rho_g|_{G_v} \simeq \begin{pmatrix} \epsilon^{k-1} \psi_{v,1} & \ast \\ 0 & \psi_{v,2} \end{pmatrix}
\]

for unramified characters \( \psi_{v,1}, \psi_{v,2} \). Suppose that the residual representation \( \overline{\rho}_g : G_F \to \GL_2(\F_l) \) is absolutely irreducible. Assume further that for all \( v \mid l \) we have that \( \epsilon^{k-1} \overline{\psi}_{v,1} \neq \overline{\psi}_{v,2} \), and that the representation \( \overline{\rho}_g|_{G_v} \) is tamely ramified, so that

\[
\overline{\rho}_g|_{G_v} \simeq \begin{pmatrix} \epsilon^{k-1} \overline{\psi}_{v,1} & 0 \\ 0 & \overline{\psi}_{v,2} \end{pmatrix}.
\]

Assume in addition that if \( \epsilon^{k-2} \overline{\psi}_{v,1} = \overline{\psi}_{v,2} \), then the absolute ramification index of \( F_v \) is less than \( l-1 \). If \( k = l \) then let \( k' = l \), and otherwise let \( k' = l+1-k \). Then there is a Hilbert modular form \( g' \) of parallel weight \( k' \) and level coprime to \( l \) satisfying

\[
\overline{\rho}_{g'} \simeq \overline{\rho}_g \otimes \epsilon^{k'-1}
\]

and the \( U_v \)-eigenvalue of \( g' \) is a lift of \( \overline{\psi}_{v,1}(\Frob_v) \).

In fact, we work throughout with forms of parallel weight 2, and we use Hida theory to treat forms of more general (parallel) weight. In the case where \( \overline{\rho}_g(G_F) \) is soluble the Langlands-Tunnell theorem makes the proof straightforward, so we concentrate on the insoluble case, where we prove:

**Theorem 2.2.** Let \( \overline{\rho}_f : G_F \to \GL_2(\F_l) \) be an absolutely irreducible modular representation, coming from a Hilbert eigenform \( f \) of parallel weight 2, with associated Galois representation \( \rho_f : G_F \to \GL_2(\Q_l) \). Suppose that \( \overline{\rho}_f(G_F) \) is...
insoluble. Suppose also that for every place \( v \) of \( F \) dividing \( l \), \( \rho_f|_{G_v} \) is potentially ordinary, and we have

\[
\overline{\rho}_f|_{G_v} \cong \begin{pmatrix}
\epsilon^{k-1} \overline{\psi}_{v,1} & 0 \\
0 & \overline{\psi}_{v,2}
\end{pmatrix}
\]

where \( \overline{\psi}_{v,1}, \overline{\psi}_{v,2} \) are unramified characters, with \( \epsilon^{k-2} \overline{\psi}_{v,1} \neq \overline{\psi}_{v,2} \). Assume in addition that if \( \epsilon^{k-2} \overline{\psi}_{v,1} = \overline{\psi}_{v,2} \), then the absolute ramification index of \( F_v \) is less than \( l - 1 \).

If \( k = l \) then let \( k' = l \), and otherwise let \( k' = l + 1 - k \). Then there is an eigenform \( f' \) of parallel weight 2 which is potentially ordinary at all places \( v|l \) such that the mod \( l \) Galois representation \( \overline{\rho}_{f'} \) associated to \( f' \) satisfies

\[
\overline{\rho}_{f'} \cong \overline{\rho}_f \otimes \epsilon^{k'-1},
\]

and such that at all places \( v|l \) we have

\[
\rho_{f'}|_{G_v} \cong \begin{pmatrix}
\epsilon \omega^{k'-2} \psi_{v,2} & * \\
0 & \psi_{v,1}
\end{pmatrix}
\]

with \( \psi_{v,i} \) an unramified lift of \( \overline{\psi}_{v,i} \) for \( i = 1, 2 \), and \( \omega \) the Teichmuller lift of \( \epsilon \).

3 Lifting theorems

Firstly, we prove a straightforward generalisation of the results of [Ram02] and [Tay03] to totally real fields. We begin by analysing the local representation theory at primes not dividing \( l \). The next lemma is essentially contained in [Dia97]:

**Lemma 3.1.** Let \( p \neq l \) be a prime, and let \( K \) be a finite extension of \( \mathbb{Q}_p \). Let \( I_K \) denote the inertia subgroup of \( G_K \). Let \( \sigma : G_K \to \text{GL}_2(k) \) be a continuous representation, with \( k \) a finite field of characteristic \( l \), and assume that \( l \nmid \#(\sigma(I_K)) \).

Then either \( p = 2, l = 3 \), and \( \text{proj} \sigma(G_K) \cong A_4 \) or \( S_4 \), or

\[
\sigma \cong \begin{pmatrix}
\epsilon \chi & *\\
0 & \chi
\end{pmatrix}
\]

with respect to some basis for some character \( \chi \).

**Proof.** Note that \( l \nmid \#(I_K) \) if and only if \( l \nmid \#(\text{proj} \sigma(I_K)) \). We must have \( \sigma|_{I_K} \) indecomposable. If \( \sigma \) is reducible, then \( \sigma \) is a twist of a representation \( \begin{pmatrix} \psi & 0 \\ 0 & 1 \end{pmatrix} \) for some character \( \psi \), with \( u \) a cocycle representing a class in \( H^1(G_K, k(\psi)) \) whose image in \( H^1(I_K, k(\psi)) \) is non-zero; but the latter group is zero unless \( \psi = \epsilon \).

If instead \( \sigma \) is irreducible but \( \sigma|_{I_K} \) is reducible, then \( \sigma|_{I_K} \), being indecomposable, must fix precisely one element of \( \text{P}^1(k) \). But then \( \sigma \) would also have to fix this element, a contradiction.

Assume now that \( \sigma|_{I_K} \) is irreducible, and that \( \sigma|_{P_K} \) is reducible, where \( P_K \) is the wild inertia subgroup of \( I_K \). Then \( P_K \) must fix precisely two elements of \( \text{P}^1(k) \) (as \( \sigma|_{I_K} \) is irreducible), so \( \sigma \) is induced from a character on a ramified
quadratic extension of $K$, and thus $\sigma(I_K)$ has order $2p^r$ for some $r \geq 1$, a contradiction.

Finally, if $\sigma|_{P_K}$ is irreducible we must have $p = 2$. That $\text{proj } \sigma(G_K) \simeq A_4$ or $S_4$ follows from the same argument as in the proof of Proposition 2.4 of [Dia97]. That $l = 3$ follows from $l \not\mid \# \sigma(I_K)$. \hfill \Box

Let $\overline{\sigma} : G_F \to \text{GL}_2(k)$ be continuous, odd, and absolutely irreducible, with $k$ a finite field of characteristic $l$. Let $S$ denote a finite set of finite places of $F$ which contains all places dividing $l$ and all places where $\overline{\sigma}$ is ramified, and let $G_S$ denote the Galois group of the maximal extension of $F$ unramified outside $S$. A deformation of $\overline{\sigma}$ is a complete noetherian local ring $(R, m)$ with residue field $k$ and a continuous representation $\rho : G_S \to \text{GL}_2(R)$ such that $(\rho \bmod m) = \overline{\sigma}$ and $\epsilon^{-1} \det \rho$ has finite order prime to $l$. We define deformations of $\overline{\rho}|_{G_v}$ in a similar fashion.

Suppose that for each $v \in S$ we have a pair $(C_v, L_v)$ satisfying the properties P1-P7 listed in section 1 of [Tay03]. Define $H^1_{(L_v)}(G_S, \text{ad}^0 \overline{\rho})$ and $H^1_{(L_v)}(G_S, \text{ad}^0 \overline{\rho})$ in the usual way.

**Lemma 3.2.** If $H^1_{(L_v)}(G_S, \text{ad}^0 \overline{\rho}) = (0)$ then there is an $S$-deformation $(W(k), \rho)$ of $\overline{\rho}$ such that for all $v \in S$ we have $(W(k), \overline{\rho}|_{G_v}) \in C_v$.

**Proof.** Identical to the proof of Lemma 1.1 of [Tay03]. \hfill \Box

**Lemma 3.3.** Suppose that $\sum_{v \in S} \dim L_v \geq \sum_{v \in S \cup \{\infty\}} \dim H^0(G_v, \text{ad}^0 \overline{\rho})$. Then we can find a finite set of places $T \supset S$ and data $(C_v, L_v)$ for $v \in T - S$ satisfying conditions P1-P7 and such that $H^1_{(L_v)}(G_T, \text{ad}^0 \overline{\rho}) = (0)$.

**Proof.** The proof of this lemma is almost identical to that of Lemma 1.2 of [Tay03]. We sketch a few of the less obvious details. In the case $l = 5$, $\text{ad}^0 \overline{\rho}(G_F) \simeq A_5$, we choose $w \notin S$ such that $\text{N}w \equiv 1 \bmod 5$ and $\text{ad}^0 \overline{\rho}(\text{Frob}_w)$ has order 5 (such a $w$ exists by Cebotarev’s theorem). Adding $w$ to $S$ with the pair $(C_w, L_w)$ of type E3 (see below), we may assume $H^1_{(L_v)}(G_S, \text{ad}^0 \overline{\rho}) \cap H^1(\text{ad}^0 \overline{\rho}(G_F), \text{ad}^0 \overline{\rho}) = (0)$.

From here on, almost exactly the same argument as in [Tay03] applies, the only difference being that one must replace every occurrence of “$Q$” with “$F$.”

Let $K = F(\text{ad}^0 \overline{\rho}, \mu_l)$. The argument is essentially formal once one knows that there is an element $\sigma \in \text{Gal}(K/F)$ such that $\text{ad}^0 \overline{\rho}(\sigma)$ has an eigenvalue $\epsilon(\sigma) \neq 1 \bmod l$, that $\text{ad}^0 \overline{\rho}$ is absolutely irreducible, and that $\text{ad}^0 \overline{\rho}$ is not isomorphic to $(\text{ad}^0 \overline{\rho})(1)$. All of these assertions follow from our assumption that $[F(\zeta_l) : F] > 3$ if $l > 3$, with the proofs being similar to those in [Ram99] (note that one may replace the assumption that $\overline{\rho}(G_Q) \supset S\text{L}_2(k)$ in [Ram99] with the assumption that $\text{proj } \overline{\rho}(G_Q) \supset P\text{S}\text{L}_2(k)$ without affecting the proofs). For example, to check that $\text{ad}^0 \overline{\rho}$ is not isomorphic to $(\text{ad}^0 \overline{\rho})(1)$ it is enough to prove that there is an element $\sigma' \in \text{Gal}(K/F)$ such that all of the eigenvalues of $\text{ad}^0 \overline{\rho}$ are 1, and $\epsilon(\sigma') \neq 1$. The existence of $\sigma$ and $\sigma'$ follows exactly as in the proof of Theorem 2 of [Ram99]. \hfill \Box

We now give examples of pairs $(C_v, L_v)$. Again, our pairs are very similar to those in section 1 of [Tay03], and the verification of the required properties is almost identical. We use the notation of [Tay03] for ease of comparison with that paper.
• E1. Suppose that \( v \nmid l \) and that \( l \nmid \# \overline{p}(I_v) \). Take \( C_v \) to be the class of lifts of \( \overline{p} \), which factor through \( G_v/(I_v \cap \ker \overline{p}) \) and let \( L_v \) be \( \text{H}^1(G_v/I_v, (\text{ad}^0 \overline{p})^l) \). Then it is straightforward to see that properties P1-P7 are satisfied, and that

\[
\begin{align*}
\text{H}^2(G_v/(I_v \cap \ker \overline{p}), \text{ad}^0 \overline{p}) & \cong \text{H}^2(G_v/I_v, (\text{ad}^0 \overline{p})^l) = (0), \quad \text{as } G_v/I_v \cong \mathbb{Z} \text{ has cohomological dimension 1}, \\
\text{H}^1(G_v/(I_v \cap \ker \overline{p}), \text{ad}^0 \overline{p}) & = L_v \subset \text{H}^1(G_v, \text{ad}^0 \overline{p}), \\
\dim L_v & = \dim \text{H}^0(G_v, \text{ad}^0 \overline{p}) \text{ (by the local Euler characteristic formula)}.
\end{align*}
\]

• E2. (Note that our definitions here differ slightly from those in [Tay03]; we thank Richard Taylor for explaining this modification to us.) Suppose that \( l = 3 \), that \( v|2 \), and that \( (\text{ad}^0 \overline{p})(G_v) \to \mathbb{Z}_4 \). Take \( C_v \) to be the class of lifts of \( \overline{p} \) which factor through \( G_v/(I_v \cap \ker \overline{p}) \) and let \( L_v \) be \( \text{H}^1(G_v/I_v, (\text{ad}^0 \overline{p})^l) \). The verification of properties P1-P7 is then as in [Tay03], except that to check that \( \text{H}^0(\overline{p}(I_v), \text{ad}^0 \overline{p}) = (0) \) for all \( i \geq 0 \) one uses the Hochschild-Serre spectral sequence and the fact that \( \text{H}^i(C_2 \times C_2, \text{ad}^0 \overline{p}) = (0) \) for all \( i \geq 0 \).

• E3. Suppose that \( v \neq l \), that either \( \mathbb{N}v \neq 1 \) (mod l) or \( l|\# \overline{p}(G_v) \), and that with respect to some basis \( e_1, e_2 \) of \( k^2 \) the restriction \( \overline{p} \) has the form

\[
\begin{pmatrix}
\epsilon \chi & * \\
0 & \overline{\chi}
\end{pmatrix}.
\]

Take \( C_v \) to be the class of deformations of the form (with respect to some basis)

\[
\begin{pmatrix}
\epsilon \chi & * \\
0 & \chi
\end{pmatrix}
\]

with \( \chi \) lifting \( \overline{\chi} \), and take \( L_v \) to be the image of

\[
\text{H}^1(G_v, \text{Hom}(ke_2, ke_1)) \to \text{H}^1(G_v, (\text{ad}^0 \overline{p})).
\]

That the pair \((C_v, L_v)\) satisfies the properties P1-P7 follows from an identical argument to that in [Tay03]. An identical calculation to that in [Tay03] shows that \( \dim L_v = \dim \text{H}^0(G_v, \text{ad}^0 \overline{p}) \).

• E4. Suppose that \( v|l \) and that with respect to some basis \( e_1, e_2 \) of \( k^2 \) \( \overline{p} \) has the form

\[
\begin{pmatrix}
\epsilon \chi_1 & 0 \\
0 & \chi_2
\end{pmatrix}.
\]

Suppose also that \( \overline{\chi}_1 \neq \overline{\chi}_2 \) and that \( \epsilon \overline{\chi}_1 \neq \overline{\chi}_2 \). Take \( C_v \) to consist of all deformations of the form

\[
\begin{pmatrix}
\epsilon \chi_1 & * \\
0 & \chi_2
\end{pmatrix}
\]

where \( \chi_1, \chi_2 \) are tamely ramified lifts of \( \overline{\chi}_1, \overline{\chi}_2 \) respectively. Let \( U^0 = \text{Hom}(ke_2, ke_1) \), and let \( L_v \) be the kernel of the map \( \text{H}^1(G_v, \text{ad}^0 \overline{p}) \to \text{H}^1(I_v, \text{ad}^0 \overline{p}/U^0G_v/I_v) \). The verification of properties P1-P7 follows as
in [Tay03], and we may compute $\dim L_v$ via a similar computation to that in the proof of Lemma 5 of [Ram02].

Note firstly that by local duality and the assumption that $\chi_1 \neq \chi_2$ we have $H^2(G_v, U^0) = 0$. Thus the short exact sequence

$$0 \to U^0 \to \text{ad}^0 \varpi \to \text{ad}^0 \varpi/U^0 \to 0$$

yields an exact sequence

$$H^1(G_v, \text{ad}^0 \varpi) \to H^1(G_v, \text{ad}^0 \varpi/U^0) \to 0.$$

Inflation-restriction gives us an exact sequence

$$0 \to H^1(G_v/I_v, (\text{ad}^0 \varpi/U^0)I_v) \to H^1(G_v, \text{ad}^0 \varpi/U^0) \to H^1(I_v, \text{ad}^0 \varpi/U^0) G_v/I_v \to 0,$$

and combining these two sequences shows that the map $H^1(G_v, \text{ad}^0 \varpi) \to H^1(I_v, \text{ad}^0 \varpi/U^0) G_v/I_v$ is surjective. Thus

$$\dim L_v = \dim H^1(G_v, \text{ad}^0 \varpi) - \dim H^1(I_v, \text{ad}^0 \varpi/U^0) G_v/I_v$$

$$= \dim H^1(G_v, \text{ad}^0 \varpi) - \dim H^1(G_v, \text{ad}^0 \varpi/U^0) + \dim H^1(G_v/I_v, (\text{ad}^0 \varpi/U^0) I_v)$$

$$= \dim H^1(G_v, \text{ad}^0 \varpi) - \dim H^1(G_v, \text{ad}^0 \varpi/U^0)$$

$$+ \dim H^0(G_v, \text{ad}^0 \varpi/U^0) \ (\text{by Lemma 3 of [Ram02]})$$

$$= \dim H^0(G_v, \text{ad}^0 \varpi) + \dim H^2(G_v, \text{ad}^0 \varpi) - \dim H^2(G_v, \text{ad}^0 \varpi/U^0)$$

$$+ [F_v : \mathbb{Q}_l] \ (\text{local Euler characteristic})$$

$$= [F_v : \mathbb{Q}_l] + \dim H^0(G_v, \text{ad}^0 \varpi).$$

• BT. Suppose that $v|l$ and that with respect to some basis $e_1, e_2$ of $k^2 \varpi|G_v$ has the form

$$\left( \begin{array}{cc} \epsilon \chi & 0 \\ 0 & \overline{\chi} \end{array} \right)$$

for some unramified character $\overline{\chi}$. Assume also that $\epsilon$ is not trivial (that is, that $F_v$ does not contain $\mathbb{Q}_l(\zeta_l)$). Take $C_v$ to consist of all flat deformations of the form

$$\left( \begin{array}{cc} \epsilon \chi_1 & * \\ 0 & \chi_2 \end{array} \right)$$

where $\chi_1, \chi_2$ are unramified lifts of $\overline{\chi}$. Then it follows from Corollary 2.5.16 of [Kis04] that there is an $L_v$ of dimension $[F_v : \mathbb{Q}_l] + \dim H^0(G_v, \text{ad}^0 \varpi)$ so that properties P1-P7 are all satisfied.

Set $\overline{\varpi} = \overline{\varpi}_f \otimes \chi^{l-1}$. We are now in a position to prove:

**Theorem 3.4.** There is a deformation $\rho$ of $\overline{\varpi}$ to $W(k)$ such that at all places $v|l$ we have $\rho|G_v$ potentially ordinary, and

$$\rho|G_v \simeq \left( \begin{array}{cc} \epsilon \omega^{k-2} \psi_{v,2} & * \\ 0 & \psi_{v,1} \end{array} \right)$$

with $\psi_{v,i}$ an unramified lift of $\overline{\psi}_{v,i}$ for $i=1, 2$, and $\omega$ the Teichmüller lift of $\epsilon$.  

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**Proof.** This follows almost at once from Lemma 3.3. By Lemma 3.1 we can choose $(C_v, L_v)$ for all $v | l$, with $\dim L_v = \dim H^0(G_v, \text{ad}^0 \mathfrak{p})$ (simply choose as in examples E1 or E3). At places $v \nmid l$, we choose $(C_v, L_v)$ as in examples E4 or BT, so that $\dim L_v = [F_v : Q_l] + \dim H^0(G_v, \text{ad}^0 \mathfrak{p})$. Then as $\sum_{v | l} [F_v : Q_l] = [F : Q_l]$, we have $\sum_{v \in S} \dim L_v = \sum_{v \in S \cup \{\infty\}} \dim H^0(G_v, \text{ad}^0 \mathfrak{p})$, so a deformation as in Lemma 3.3 exists. That the $\psi_{v, i}$ are unramified follows from the fact that they are tamely ramified lifts of unramified characters.

It remains to check that $\rho_{G_v}$ is potentially ordinary. By the remarks in section 2.4.15 of [Kis04] it suffices to check that it is potentially Barsotti-Tate. This is immediate if we are in the case BT, so suppose we are considering deformations as in E4. By the proposition in section 3.1 of [PR94], $\rho_{G_v}$ is potentially semistable, and it clearly has Hodge-Tate weights in $\{0, 1\}$, so by Theorem 5.3.2 of [Bre00] it suffices to check that it is potentially crystalline. In order to check this, we consider the Weil-Deligne representation $WD(\rho_{G_v})$ (see Appendix B of [CDT99] for the definition of $WD(\sigma)$ for any potentially semistable $p$-adic representation $\sigma$ of $G_v$). We need to check that the associated nilpotent endomorphism $N$ is zero. As is well-known, $N = 0$ unless $WD(\rho_{G_v})$ is a twist of the Steinberg representation, which cannot happen because of our assumption that we are not in the BT case.

Theorem 2.2 now follows immediately from:

**Theorem 3.5.** The representation $\rho$ is modular.

**Proof.** This is an easy application of Theorem 3.5.5 of [Kis04]. We need to check that $\rho$ is strongly residually modular. The representation $\rho_f \otimes \omega^{k-1}$ (where $\omega$ is the Teichmuller lift of $\epsilon$) is certainly modular, with residual representation $\overline{\rho}_f$; furthermore, it is automatically potentially ordi\narily at all places $v | l$ with $\epsilon^{k-2} \overline{\psi}_{v, 1} \neq \overline{\psi}_{v, 2}$. By Theorem 6.2 of [Jar04] and our assumption that if $e^{k-2} \overline{\psi}_{v, 1} = \overline{\psi}_{v, 2}$ the absolute ramification index of $F_v$ is less than $l - 1$, we may replace $\rho_f \otimes \omega^{k-1}$ with a modular lift of $\overline{\sigma}$ which is potentially ordinary at all places $v | l$. By construction, $\rho$ is potentially ordinary at all places $v | l$, so we are done.

We now prove Theorem 2.1. Firstly, suppose that $\overline{\sigma}_g(G_F)$ is insoluble. Then Hida theory (see [Wil88] or [Hid88]) provides us with a weight 2 form $f$ which satisfies the hypotheses of Theorem 2.2, and which has $\overline{\rho}_f \simeq \psi_g$ (that $f$ is potentially ordinary follows as in the proof of Theorem 3.4). Then Theorem 2.2 provides us with a Hilbert modular form $f'$ of parallel weight 2 with $\overline{\rho}_{f'} \simeq \overline{\rho}_f \otimes \epsilon^{k-1}$ and

$$\rho_{f'}|_{G_v} \simeq \begin{pmatrix} \epsilon \omega^{k-2} \psi_{v, 2} & \ast \\
0 & \psi_{v, 1} \end{pmatrix}$$

for all places $v | l$, with $\psi_{v, 1}$ an unramified lift of $\overline{\psi}_{v, 1}$. Then Lemma 3.4.2 of [Kis04] shows that $f'$ has $U_v$-eigenvalue $\psi_{v, 1}(\text{Frob}_v)$, an $l$-adic unit. The existence of $g'$ now follows from Hida theory.

Now suppose that $\overline{\rho}_f(G_F)$ is insoluble. Then there is a lift of $\overline{\rho}_f \otimes \epsilon^{k-1}$ to a characteristic zero representation, which comes from a Hilbert modular form of parallel weight 1 by the Langlands-Tunnell theorem (see for example Lemma 5.2 of [Kha05]). Such a form is necessarily ordinary in the sense of Hida theory, and the theorem follows by Hida theory as in the insoluble case.
References


