Companion forms over totally real fields, II

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Abstract

We prove a companion forms theorem for mod $l$ Hilbert modular forms. This work generalises results of Gross and Coleman–Voloch for modular forms over $\mathbb{Q}$, and gives a new proof of their results in many cases.

1 Introduction

If $f \in S_k(\Gamma_1(N); \mathbb{F}_p)(\epsilon)$ is a mod $l$ cuspidal eigenform, where $l \nmid N$, there is a continuous, odd, semisimple Galois representation $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}_l)$ attached to $f$. A famous conjecture of Serre predicts that all continuous odd irreducible mod $l$ representations should arise in this fashion. Furthermore, the “strong Serre conjecture” predicts a minimal weight $k_\rho$ and level $N_\rho$, in the sense that $\rho \cong \rho_g$ for some eigenform $g$ of weight $k_\rho$ and level $N_\rho$ (prime to $l$), and if $\rho \cong \rho_f$ for some eigenform $f$ of weight $k$ and level $N$ prime to $l$ then $N_\rho | N$ and $k \geq k_\rho$. The question as to whether all continuous odd irreducible mod $l$ Galois representations are modular in this sense is still open, but the implication “weak Serre $\Rightarrow$ strong Serre” is essentially known (aside from a few cases where $l = 2$).

In solving the problem of weight optimisation it becomes necessary to consider the companion forms problem; that is, the question of when it can occur that we have $f = \sum a_n q^n$ of weight $2 \leq k \leq l$ with $a_l \neq 0$, and an eigenform $g = \sum b_n q^n$ of weight $k' = l + 1 - k$ such that $na_n = n^k b_n$ for all $n$. Serre conjectured that this can occur if and only if the representation $\rho_f$ is tamely ramified above $l$. This conjecture has been settled in most cases in the papers of Gross ([Gro90]) and Coleman-Voloch ([CV92]).

Our earlier paper [Gee04] generalised these results to the case of parallel weight Hilbert modular forms over totally real fields $F$ in which $l$ splits completely, by generalising the methods of [CV92]. In this paper we take a completely different and rather more conceptual approach: we construct our companion form by using a method of Ramakrishna to find an appropriate characteristic zero Galois representation, and then use recent work of Kisin ([Kis04]) to prove that the representation is modular. Note that our companion form is not necessarily of minimal prime-to-$l$ level, but that this is irrelevant for applications to Artin’s conjecture, and that in many cases a form of minimal level may be obtained from ours by the methods of [Jar99], [SW01], [Raj01] and [Fuj99]. In the case of weight $l$ forms, we avoid potential difficulties with weight 1 forms by constructing a companion form in weight $l$. 

1
2 Statement of the main results

Let \( l > 2 \) be a prime, and let \( F \) be a totally real field. We assume that if \( l > 3 \), \([F(\zeta_l) : F] > 3\) (note that this is automatic if \( l \) is unramified in \( F \)). Let \( \epsilon \) denote both the \( l \)-adic and mod \( l \) cyclotomic characters; this should cause no confusion. Let \( \rho : G_K \to \text{GL}_2(\mathbb{O}) \) be a continuous representation, where is \( K \) a finite extension of \( \mathbb{Q}_l \), and \( \mathbb{O} \) is the ring of integers in a finite extension of \( \mathbb{Q}_l \). We say that \( \rho \) is ordinary if it is Barsotti-Tate, coming from an \( l \)-divisible group which is an extension of an étale group by a multiplicative group, each of rank one as \( \mathbb{O} \)-modules. We say that it is potentially ordinary if it becomes ordinary upon restriction to an open subgroup of \( G_K \). We say that a Hilbert modular form of parallel weight 2 is (potentially) ordinary at a place \( v | l \) if its associated Galois representation is (potentially) ordinary at \( v \). These definitions agree with those in [Kis04]; they are slightly non-standard, but note that if the level is prime to \( l \) then this is equivalent to the \( U_v \)-eigenvalue being an \( l \)-adic unit. We say that a Hilbert modular form of parallel weight \( k \), \( 3 \leq k \leq l \) is ordinary at a place \( v | l \) if its \( U_v \)-eigenvalue is an \( l \)-adic unit. Finally, we say that a modular form is (potentially) ordinary if it is (potentially) ordinary at all places \( v | l \).

Our main theorem is the following:

**Theorem 2.1.** Let \( g \) be an ordinary Hilbert modular eigenform of parallel weight \( k \), \( 2 \leq k \leq l \), and level coprime to \( l \). Let its associated Galois representation be \( \rho_g : G_F \to \text{GL}_2(\overline{\mathbb{Q}}) \), so that (by Theorem 2 of [Wil88]) we have, for all places \( v | l \),

\[
\rho_g|_{G_v} \simeq \left( \begin{array}{cc} \epsilon^{k-1} \psi_{v,1} & * \\ 0 & \psi_{v,2} \end{array} \right)
\]

for unramified characters \( \psi_{v,1}, \psi_{v,2} \). Suppose that the residual representation \( \overline{\rho}_g : G_F \to \text{GL}_2(\overline{\mathbb{F}}_l) \) is absolutely irreducible. Assume further that for all \( v | l \) we have that \( \epsilon^{k-1} \psi_{v,1} \neq \psi_{v,2} \), and that the representation \( \overline{\rho}_g|_{G_v} \) is tamely ramified, so that

\[
\overline{\rho}_g|_{G_v} \simeq \left( \begin{array}{cc} \epsilon^{k-1} \overline{\psi}_{v,1} & 0 \\ 0 & \overline{\psi}_{v,2} \end{array} \right).
\]

Assume in addition that if \( \epsilon^{k-2} \overline{\psi}_{v,1} = \overline{\psi}_{v,2} \), then the absolute ramification index of \( F_v \) is less than \( l - 1 \). If \( k = l \) then let \( k' = l \), and otherwise let \( k' = l + 1 - k \). Then there is a Hilbert modular form \( g' \) of parallel weight \( k' \) and level coprime to \( l \) satisfying

\[
\overline{\rho}_{g'} \simeq \overline{\rho}_g \otimes \epsilon^{k'-1}
\]

and the \( U_v \)-eigenvalue of \( g' \) is a lift of \( \overline{\psi}_{v,1}(\text{Frob}_v) \).

In fact, we work throughout with forms of parallel weight 2, and we use Hida theory to treat forms of more general (parallel) weight. In the case where \( \overline{\rho}_g(G_F) \) is soluble the Langlands-Tunnell theorem makes the proof straightforward, so we concentrate on the insoluble case, where we prove:

**Theorem 2.2.** Let \( \overline{\rho}_f : G_F \to \text{GL}_2(\overline{\mathbb{F}}_l) \) be an absolutely irreducible modular representation, coming from a Hilbert eigenform \( f \) of parallel weight 2, with associated Galois representation \( \rho_f : G_F \to \text{GL}_2(\overline{\mathbb{Q}}) \). Suppose that \( \overline{\rho}_f(G_F) \) is
insoluble. Suppose also that for every place $v$ of $F$ dividing $l$, $\rho_f|_{G_v}$ is potentially ordinary, and we have

$$\overline{\rho}_f|_{G_v} \simeq \begin{pmatrix} \epsilon^{k-1} \overline{\psi}_{v,1} & 0 \\ 0 & \overline{\psi}_{v,2} \end{pmatrix}$$

where $\overline{\psi}_{v,1}$, $\overline{\psi}_{v,2}$ are unramified characters, with $\epsilon^{k-2} \overline{\psi}_{v,1} \neq \overline{\psi}_{v,2}$. Assume in addition that if $\epsilon^{k-2} \overline{\psi}_{v,1} = \overline{\psi}_{v,2}$, then the absolute ramification index of $F_v$ is less than $l-1$.

If $k=l$ then let $k' = l$, and otherwise let $k' = l+1-k$. Then there is an eigenform $f'$ of parallel weight 2 which is potentially ordinary at all places $v|l$ such that the mod $l$ Galois representation $\overline{\rho}_{f'}$ associated to $f'$ satisfies

$$\overline{\rho}_{f'} \simeq \overline{\rho}_f \otimes \epsilon^{k'-1},$$

and such that at all places $v|l$ we have

$$\rho_{f'}|_{G_v} \simeq \begin{pmatrix} \epsilon \omega^{k'-2} \psi_{v,2} & * \\ 0 & \psi_{v,1} \end{pmatrix}$$

with $\psi_{v,i}$ an unramified lift of $\overline{\psi}_{v,i}$ for $i=1, 2$, and $\omega$ the Teichmüller lift of $\epsilon$.

3 Lifting theorems

Firstly, we prove a straightforward generalisation of the results of [Ram02] and [Tay03] to totally real fields. We begin by analysing the local representation theory at primes not dividing $l$. The next lemma is essentially contained in [Dia97]:

**Lemma 3.1.** Let $p \neq l$ be a prime, and let $K$ be a finite extension of $\mathbb{Q}_p$. Let $I_K$ denote the inertia subgroup of $G_K$. Let $\sigma : G_K \to \text{GL}_2(k)$ be a continuous representation, with $k$ a finite field of characteristic $l$, and assume that $l | \# \sigma(I_K)$.

Then either $p = 2$, $l = 3$, and $\text{proj}(\sigma(G_K)) \simeq A_4$ or $S_4$, or

$$\sigma \simeq \begin{pmatrix} \epsilon \chi & * \\ 0 & \chi \end{pmatrix}$$

with respect to some basis for some character $\chi$.

**Proof.** Note that $l | \# \sigma(I_K)$ if and only if $l | \# \text{proj}(\sigma(I_K))$. We must have $\sigma|_{I_K}$ indecomposable. If $\sigma$ is reducible, then $\sigma$ is a twist of a representation $(\psi^u)$ for some character $\psi$, with $u$ a cocycle representing a class in $H^1(G_K, k(\psi))$ whose image in $H^1(I_K, k(\psi))^{G_K}$ is non-zero; but the latter group is zero unless $\psi = \epsilon$.

If instead $\sigma$ is irreducible but $\sigma|_{I_K}$ is reducible, then $\sigma|_{I_K}$, being indecomposable, must fix precisely one element of $\mathbf{P}^1(k)$. But then $\sigma$ would also have to fix this element, a contradiction.

Assume now that $\sigma|_{I_K}$ is irreducible, and that $\sigma|_{P_K}$ is reducible, where $P_K$ is the wild inertia subgroup of $I_K$. Then $P_K$ must fix precisely two elements of $\mathbf{P}^1(k)$ (as $\sigma|_{I_K}$ is irreducible), so $\sigma$ is induced from a character on a ramified
that paper. is almost identical. We use the notation of [Tay03] for ease of comparison to those in section 1 of [Tay03], and the verification of the required properties \( \epsilon \) follows from the same argument as in the proof of Proposition 2.4 of [Dia97]. That \( l = 3 \) follows from \( l \mid \#\sigma(I_K) \). 

Let \( \overline{\rho} : G_F \to \text{GL}_2(k) \) be continuous, odd, and absolutely irreducible, with \( k \) a finite field of characteristic \( l \). Let \( S \) denote a finite set of finite places of \( F \) which contains all places dividing \( l \) and all places where \( \overline{\rho} \) is ramified, and let \( G_S \) denote the Galois group of the maximal extension of \( F \) unramified outside \( S \). A deformation of \( \overline{\rho} \) is a complete noetherian local ring \((R, \mathfrak{m})\) with residue field \( k \) and a continuous representation \( \rho : G_S \to \text{GL}_2(R) \) such that \((\rho \mod \mathfrak{m}) = \overline{\rho}\) and \( \epsilon^{-1} \det \rho \) has finite order prime to \( l \). We define deformations of \( \overline{\rho}|_{G_\nu} \) in a similar fashion.

Suppose that for each \( \nu \in S \) we have a pair \((C_\nu, L_\nu)\) satisfying the properties P1-P7 listed in section 1 of [Tay03]. Define \( H^1_{(L_\nu)}(G_S, \text{ad}^0 \overline{\rho}) \) and \( H^1_{(L_\nu)}(G_S, \text{ad}^0 \overline{\rho}) \) in the usual way.

**Lemma 3.2.** If \( H^1_{(L_\nu)}(G_S, \text{ad}^0 \overline{\rho}) = (0) \) then there is an \( S \)-deformation \((W(k), \rho)\) of \( \overline{\rho} \) such that for all \( \nu \in S \) we have \((W(k), \overline{\rho}|_{G_\nu}) \in C_\nu \).

**Proof.** Identical to the proof of Lemma 1.1 of [Tay03].

**Lemma 3.3.** Suppose that \( \sum_{\nu \in S} \dim L_\nu \geq \sum_{v \in S \setminus \{ \infty \}} \dim H^0(G_v, \text{ad}^0 \overline{\rho}) \). Then we can find a finite set of places \( T \supseteq S \) and data \((C_\nu, L_\nu)\) for \( \nu \in T \setminus S \) satisfying conditions P1-P7 and such that \( H^1_{(L_\nu)}(G_T, \text{ad}^0 \overline{\rho}) = (0) \).

**Proof.** The proof of this lemma is almost identical to that of Lemma 1.2 of [Tay03]. We sketch a few of the less obvious details. In the case \( l = 5 \), \( \text{ad}^0 \overline{\rho}(G_F) \simeq A_5 \), we choose \( w \notin S \) such that \( NW \equiv 1 \mod 5 \) and \( \text{ad}^0 \overline{\rho}(\text{Frob}_w) \) has order 5 (such a \( w \) exists by Cebotarev’s theorem). Adding \( w \) to \( S \) with the pair \((C_w, L_w)\) of type E3 (see below), we may assume \( H^1_{(L_\nu)}(G_S, \text{ad}^0 \overline{\rho}) \cap H^1(\text{ad}^0 \overline{\rho}(G_F), \text{ad}^0 \overline{\rho}) = (0) \).

From here on, almost exactly the same argument as in [Tay03] applies, the only difference being that one must replace every occurence of “\(Q^\prime\)” with “\(F^\prime\).” Let \( K = F(\text{ad}^0 \overline{\rho}, \mu_l) \). The argument is essentially formal once one knows that there is an element \( \sigma \in \text{Gal}(K/F) \) such that \( \text{ad}^0 \overline{\rho}(\sigma) \) has an eigenvalue \( \epsilon(\sigma) \neq 1 \mod l \), that \( \text{ad}^0 \overline{\rho} \) is absolutely irreducible, and that \( \text{ad}^0 \overline{\rho} \) is not isomorphic to \( \text{ad}^0 \overline{\rho}(1) \). All of these assertions follow from our assumption that \(|F(\zeta_l) : F| > 3 \) if \( l > 3 \), with the proofs being similar to those in [Ram99] (note that one may replace the assumption that \( \overline{\rho}(G_Q) \subseteq \text{SL}_2(k) \) in [Ram99] with the assumption that \( \text{proj} \overline{\rho}(G_Q) \subseteq \text{PSL}_2(k) \) without affecting the proofs). For example, to check that \( \text{ad}^0 \overline{\rho} \) is not isomorphic to \( \text{ad}^0 \overline{\rho}(1) \) it is enough to prove that there is an element \( \sigma^\prime \in \text{Gal}(K/F) \) such that all of the eigenvalues of \( \text{ad}^0 \overline{\rho} \) are 1, and \( \epsilon(\sigma^\prime) \neq 1 \). The existence of \( \sigma \) and \( \sigma^\prime \) follows exactly as in the proof of Theorem 2 of [Ram99].

We now give examples of pairs \((C_\nu, L_\nu)\). Again, our pairs are very similar to those in section 1 of [Tay03], and the verification of the required properties is almost identical. We use the notation of [Tay03] for ease of comparison with that paper.
• E1. Suppose that \( v \nmid l \) and that \( l \nmid \#\mathfrak{p}(I_v) \). Take \( C_v \) to be the class of lifts of \( \mathfrak{p}|_{G_v} \) which factor through \( G_v/(I_v \cap \ker \mathfrak{p}) \) and let \( L_v \) be \( H^1(G_v/I_v, (\text{ad}^0 \mathfrak{p})^l_v) \). Then it is straightforward to see that properties P1-P7 are satisfied, and that

\[
\begin{align*}
H^2(G_v/(I_v \cap \ker \mathfrak{p}), \text{ad}^0 \mathfrak{p}) &\simeq H^2(G_v/I_v, (\text{ad}^0 \mathfrak{p})^l_v) = (0), \text{ as } G_v/I_v \simeq \mathbb{Z} \text{ has cohomological dimension 1}, \\
H^1(G_v/(I_v \cap \ker \mathfrak{p}), \text{ad}^0 \mathfrak{p}) &\subset H^1(G_v, \text{ad}^0 \mathfrak{p}), \\
\dim L_v &= \dim H^0(G_v, \text{ad}^0 \mathfrak{p}) \text{ (by the local Euler characteristic formula)}.
\end{align*}
\]

• E2. (Note that our definitions here differ slightly from those in [Tay03]; we thank Richard Taylor for explaining this modification to us.) Suppose that \( l = 3 \), that \( v \mid 2 \), and that \( \text{ad}^0(\mathfrak{p})(G_v) \twoheadrightarrow S_4 \). Take \( C_v \) to be the class of lifts of \( \mathfrak{p}|_{G_v} \) which factor through \( G_v/(I_v \cap \ker \mathfrak{p}) \) and let \( L_v \) be \( H^1(G_v/I_v, (\text{ad}^0 \mathfrak{p})^l_v) \). The verification of properties P1-P7 is then as in [Tay03], except that to check that \( H^0(\mathfrak{p}(I_v), \text{ad}^0 \mathfrak{p}) = (0) \) for all \( i \geq 0 \) one uses the Hochschild-Serre spectral sequence and the fact that \( H^1(C_2 \times C_2, \text{ad}^0 \mathfrak{p}) = (0) \) for all \( i \geq 0 \).

• E3. Suppose that \( v \neq l \), that either \( Nv \neq 1 \) (mod \( l \)) or \( l \nmid \#\mathfrak{p}(G_v) \), and that with respect to some basis \( e_1, e_2 \) of \( k^2 \) the restriction \( \mathfrak{p}|_{G_v} \) has the form

\[
\begin{pmatrix}
\epsilon \chi & * \\
0 & \overline{\chi}
\end{pmatrix}.
\]

Take \( C_v \) to be the class of deformations of the form (with respect to some basis)

\[
\begin{pmatrix}
\epsilon \chi & * \\
0 & \chi
\end{pmatrix}
\]

with \( \chi \) lifting \( \overline{\chi} \), and take \( L_v \) to be the image of

\[
H^1(G_v, \text{Hom}(k e_2, k e_1)) \to H^1(G_v, (\text{ad}^0 \mathfrak{p})).
\]

That the pair \( (C_v, L_v) \) satisfies the properties P1-P7 follows from an identical argument to that in [Tay03]. An identical calculation to that in [Tay03] shows that \( \dim L_v = \dim H^0(G_v, \text{ad}^0 \mathfrak{p}) \).

• E4. Suppose that \( v \mid l \) and that with respect to some basis \( e_1, e_2 \) of \( k^2 \) \( \mathfrak{p}|_{G_v} \) has the form

\[
\begin{pmatrix}
\epsilon \chi_1 & 0 \\
0 & \chi_2
\end{pmatrix}.
\]

Suppose also that \( \overline{\chi}_1 \neq \overline{\chi}_2 \) and that \( \epsilon \overline{\chi}_1 \neq \overline{\chi}_2 \). Take \( C_v \) to consist of all deformations of the form

\[
\begin{pmatrix}
\epsilon \chi_1 & * \\
0 & \chi_2
\end{pmatrix}
\]

where \( \chi_1, \chi_2 \) are tamely ramified lifts of \( \overline{\chi}_1, \overline{\chi}_2 \) respectively. Let \( U^0 = \text{Hom}(k e_2, k e_1) \), and let \( L_v \) be the kernel of the map \( H^1(G_v, \text{ad}^0 \mathfrak{p}) \to H^1(I_v, \text{ad}^0 \mathfrak{p}/U^0)^G/I \). The verification of properties P1-P7 follows as
in [Tay03], and we may compute \( \dim L_v \) via a similar computation to that in the proof of Lemma 5 of [Ram02].

Note firstly that by local duality and the assumption that \( \overline{\chi}_1 \neq \overline{\chi}_2 \) we have \( H^2(G_v, U^0) = 0 \). Thus the short exact sequence
\[
0 \to U^0 \to \ad^0 \overline{\rho} \to \ad^0 \overline{\rho}/U^0 \to 0
\]
yields an exact sequence
\[
H^1(G_v, \ad^0 \overline{\rho}) \to H^1(G_v, \ad^0 \overline{\rho}/U^0) \to 0.
\]
Inflation-restriction gives us an exact sequence
\[
0 \to H^1(G_v/I_v, (\ad^0 \overline{\rho}/U^0)_{I_v}) \to H^1(G_v, \ad^0 \overline{\rho}/U^0) \to H^1(I_v, \ad^0 \overline{\rho}/U^0)_{G_v/I_v} \to 0,
\]
and combining these two sequences shows that the map \( H^1(G_v, \ad^0 \overline{\rho}) \to H^1(G_v/I_v, \ad^0 \overline{\rho}/U^0) \) is surjective. Thus
\[
\dim L_v = \dim H^1(G_v, \ad^0 \overline{\rho}) - \dim H^1(I_v, \ad^0 \overline{\rho}/U^0)_{G_v/I_v}
\]
\[
= \dim H^1(G_v, \ad^0 \overline{\rho}) - \dim H^1(I_v, \ad^0 \overline{\rho}/U^0)_{G_v/I_v}
\]
\[
+ \dim H^0(G_v, \ad^0 \overline{\rho}) \quad \text{(by Lemma 3 of [Ram02])}
\]
\[
= \dim H^0(G_v, \ad^0 \overline{\rho}) + \dim H^2(G_v, \ad^0 \overline{\rho}) - \dim H^2(G_v, \ad^0 \overline{\rho}/U^0)
\]
\[
+ [F_v : \mathbb{Q}_l] \quad \text{(local Euler characteristic)}
\]
\[
= [F_v : \mathbb{Q}_l] + \dim H^0(G_v, \ad^0 \overline{\rho}).
\]

- BT. Suppose that \( v|l \) and that with respect to some basis \( e_1, e_2 \) of \( k^2 \overline{\rho}|_{G_v} \) has the form
\[
\left( \begin{array}{cc}
\epsilon \overline{\chi} & 0 \\
0 & \overline{\chi}
\end{array} \right)
\]
for some unramified character \( \overline{\chi} \). Assume also that \( \epsilon \) is not trivial (that is, that \( F_v \) does not contain \( \mathbb{Q}_l(\zeta) \)). Take \( C_v \) to consist of all flat deformations of the form
\[
\left( \begin{array}{cc}
\epsilon \chi_1 & * \\
0 & \chi_2
\end{array} \right)
\]
where \( \chi_1, \chi_2 \) are unramified lifts of \( \overline{\chi} \). Then it follows from Corollary 2.5.16 of [Kis04] that there is an \( L_v \) of dimension \([F_v : \mathbb{Q}_l] + \dim H^0(G_v, \ad^0 \overline{\rho})\) so that properties P1-P7 are all satisfied.

Set \( \overline{\rho} = \overline{\rho}_f \otimes \epsilon^{k-1} \). We are now in a position to prove:

**Theorem 3.4.** There is a deformation \( \rho \) of \( \overline{\rho} \) to \( W(k) \) such that at all places \( v|l \) we have \( \rho|_{G_v} \) potentially ordinary, and
\[
\rho|_{G_v} \simeq \left( \begin{array}{cc}
\epsilon \omega^{k-2} \psi_{v,2} & * \\
0 & \psi_{v,1}
\end{array} \right)
\]
with \( \psi_{v,i} \) an unramified lift of \( \overline{\psi}_{v,i} \) for \( i = 1, 2 \), and \( \omega \) the Teichmüller lift of \( \epsilon \).
Proof. This follows almost at once from Lemma 3.3. By Lemma 3.1 we can choose \((C_v, L_v)\) for all \(v \mid l\), with \(\dim L_v = \dim H^0(G_v, \text{ad}^0 \mathfrak{p})\) (simply choose as in examples E1 or E3). At places \(v \nmid l\), we choose \((C_v, L_v)\) as in examples E4 or BT, so that \(\dim L_v = [F_v : \mathbb{Q}_l] + \dim H^0(G_v, \text{ad}^0 \mathfrak{p})\). Then as \(\sum_{v \mid l} [F_v : \mathbb{Q}_l] = [F : \mathbb{Q}]\), we have \(\sum_{v \in S} \dim L_v = \sum_{v \in S \cup \{\infty\}} \dim H^0(G_v, \text{ad}^0 \mathfrak{p})\), so a deformation as in Lemma 3.3 exists. That the \(\psi_{v,i}\) are unramified follows from the fact that they are tamely ramified lifts of unramified characters.

It remains to check that \(\rho_{G_v}\) is potentially ordinary. By the remarks in section 2.4.15 of [Kis04] it suffices to check that it is potentially Barsotti-Tate. This is immediate if we are in the case BT, so suppose we are considering deformations as in E4. By the proposition in section 3.1 of [PR94], \(\rho_{G_v}\) is potentially semistable, and it clearly has Hodge-Tate weights in \(\{0, 1\}\), so by Theorem 5.3.2 of [Bre00] it suffices to check that it is potentially crystalline. In order to check this, we consider the Weil-Deligne representation \(WD(\rho_{G_v})\) (see Appendix B of [CDT99] for the definition of \(WD(\sigma)\) for any potentially semistable \(p\)-adic representation \(\sigma\) of \(G_v\)). We need to check that the associated nilpotent endomorphism \(N\) is zero. As is well-known, \(N = 0\) unless \(WD(\rho_{G_v})\) is a twist of the Steinberg representation, which cannot happen because of our assumption that we are not in the BT case.

\(\square\)

Theorem 2.2 now follows immediately from:

**Theorem 3.5.** The representation \(\rho\) is modular.

Proof. This is an easy application of Theorem 3.5.5 of [Kis04]. We need to check that \(\rho\) is strongly residually modular. The representation \(\rho_{f} \otimes \omega^{k'-1}\) (where \(\omega\) is the Teichmuller lift of \(\epsilon\)) is certainly modular, with residual representation \(\overline{\rho}_{f}\). Furthermore, it is automatically potentially ordinary at all places \(v\mid l\) with \(\ell^{k'-2} \psi_{v,1} \neq \psi_{v,2}\). By Theorem 6.2 of [Jar04] and our assumption that if \(\ell^{k'-2} \psi_{v,1} = \psi_{v,2}\), then the absolute ramification index of \(F_v\) is less than \(l-1\), we may replace \(\rho_{f} \otimes \omega^{k'-1}\) with a modular lift of \(\pi\) which is potentially ordinary at all places \(v\mid l\). By construction, \(\rho\) is potentially ordinary at all places \(v\mid l\), so we are done.

\(\square\)

We now prove Theorem 2.1. Firstly, suppose that \(\overline{\rho}_{g}(G_F)\) is insoluble. Then Hida theory (see [Wil88] or [Hid88]) provides us with a weight 2 form \(f\) which satisfies the hypotheses of Theorem 2.2, and which has \(\overline{\rho}_{f} \simeq \overline{\rho}_{g}\) (that \(f\) is potentially ordinary follows as in the proof of Theorem 3.4). Then Theorem 2.2 provides us with a Hilbert modular form \(f'\) of parallel weight 2 with \(\overline{\rho}_{f'} \simeq \overline{\rho}_{f} \otimes \epsilon^{k'-1}\) and

\[\rho_{f'}|_{G_v} \simeq \begin{pmatrix} \epsilon \omega^{k'-2} \psi_{v,2} & * \\ 0 & \psi_{v,1} \end{pmatrix}\]

for all places \(v\mid l\), with \(\psi_{v,1}\) an unramified lift of \(\overline{\psi}_{v,1}\). Then Lemma 3.4.2 of [Kis04] shows that \(f'\) has \(U_v\)-eigenvalue \(\psi_{v,1}(\text{Frob}_v)\), an \(l\)-adic unit. The existence of \(g'\) now follows from Hida theory.

Now suppose that \(\overline{\rho}_{f}(G_F)\) is insoluble. Then there is a lift of \(\overline{\rho}_{f} \otimes \epsilon^{k'-1}\) to a characteristic zero representation, which comes from a Hilbert modular form of parallel weight 1 by the Langlands-Tunnell theorem (see for example Lemma 5.2 of [Kha05]). Such a form is necessarily ordinary in the sense of Hida theory, and the theorem follows by Hida theory as in the insoluble case.
References


