Four-Partocle Anyon Exciton: Boson Approximation

The Harvard community has made this article openly available. Please share how this access benefits you. Your story matters

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Published Version</td>
<td>doi:10.1142/S0217984995000139</td>
</tr>
<tr>
<td>Citable link</td>
<td><a href="http://nrs.harvard.edu/urn-3:HUL.InstRepos:29407534">http://nrs.harvard.edu/urn-3:HUL.InstRepos:29407534</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>This article was downloaded from Harvard University’s DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <a href="http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA">http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA</a></td>
</tr>
</tbody>
</table>
Four-Particle Anyon Exciton: Boson Approximation

M. E. Portnoi and E. I. Rashba

Department of Physics, University of Utah, Salt Lake City, UT 84112

(October 30, 2013)

Abstract

A theory of anyon excitons consisting of a valence hole and three quasielectrons with electric charges ($-e/3$) is presented. A full symmetry classification of the $k = 0$ states is given, where $k$ is the exciton momentum. The energy levels of these states are expressed by quadratures of confluent hypergeometric functions. It is shown that the angular momentum $L$ of the exciton ground state depends on the distance between electron and hole confinement planes and takes the values $L = 3n$, where $n$ is an integer. With increasing $k$ the electron density shows a spectacular splitting on bundles. At first a single anyon splits off of the two-anyon core, and finally all anyons become separated.
I. INTRODUCTION

Recent progress in the intrinsic interband spectroscopy [1–4] of the Fractional Quantum Hall Effect (FQHE) [5] made a demand for developing a theory of optical phenomena accounting in an appropriate way the basic properties of FQHE phases. Such a theory has to reflect two essential facts: i) an exciton nature of the low-energy states in a strong magnetic field, and ii) the presence of an Incompressible Quantum Liquid (IQL) [6] underlying the FQHE. Therefore, the theory should describe an exciton against a background of an IQL. There are two approaches to this problem. The first one is based on exact numerical treatment of few-electron systems in a spherical geometry [7–12]. The second one is based on the anyon exciton (AE) model [13] which considers an exciton as a neutral entity consisting of a valence hole and several quasielectrons (QE’s) carrying fractional negative charges. These both approaches have a restricted applicability depending on the value of the parameter \( h/l \), here \( h \) is the separation between electron and hole confinement planes and \( l = (\hbar eH)^{1/2} / c \) is the magnetic length. Reliable calculations in a spherical geometry are accessible only for \( h/l \lesssim 1.5 \). On the contrary, the AE approach fails when \( h/l \) is small, but is exact in the opposite limit \( h \gg l \). Indeed, by electrostatic arguments the size of an AE ground state is about \( h \), and the model is applicable when the mean distance between anyons is large as compared to the anyon size which is about \( l \); in the same limit the perturbation exerted by a hole is small. There is our believe that both approaches will result in a similar picture in the intermediate region \( h/l \sim 1 \) and, therefore, taken together they will provide a description of excitons in the whole region of the \( h/l \) values. Experimental data on excitons are not available for the region \( h > l \) yet, and therefore a comparison with the experiment can not be performed at present. For this reason we abstain from discussing the involved problem of the transition probabilities [14] and concentrate on the energy spectra and electron density distribution. The relation between the results of the AE approach and of computations for
few-particle systems is discussed in the last Section.

The oversimplified model of an AE consisting of a hole and two semions (anyons having charges \((-e/2)\)) has been developed by us previously [13]. In that paper a multiple-branch spectrum of an AE has been established as a basic signature of the charge fractionalization in IQL’s. Recent data [14,15] have revealed the appearance of this structure also in few-particle computations. It is remarkable that several new exciton branches appear at very moderate values of \(h/l \lesssim 1\). In what follows we present for the first time a theory for a realistic case of three QE’s which corresponds to a \(\nu = 1/3\) IQL.

II. THEORY

Let us consider an exciton consisting of a valence hole with a charge \((+e)\) and three QE’s with electrical charges \((-e/3)\) and statistical charges \(\alpha = -1/3\) (\(\alpha = 0\) for bosons and \(\alpha = 1\) for fermions). In the strong \(H\) limit, when the Coulomb energy \(\varepsilon_C = e^2/\epsilon l \ll \hbar \omega_c\), where \(\omega_c\) is the cyclotron frequency, it is convenient to use dimensionless variables scaled in units \(\varepsilon_C, l\) and \(e\). Instead of hole, \(r_h\), and anyon, \(r_i\), coordinates it is convenient to introduce the following two-dimensional coordinates:

\[
\mathbf{R} = \frac{1}{2}(r_h + \frac{1}{3}\sum_{i=1}^{3} r_i), \quad \rho = \frac{1}{3}\sum_{i=1}^{3} r_i - r_h, \quad r_{ij} = r_i - r_j, \quad ij = 12, 23, \text{ and } 31 .
\]  

(1)

Complex coordinates \(z_{ji} = x_{ji} + iy_{ji}\), as well as \(r_{ji}\), are not independent. Indeed:

\[
\mathbf{r}_{12} + \mathbf{r}_{23} + \mathbf{r}_{31} = 0 .
\]  

(2)

Since AE is a neutral entity it possesses an in-plane momentum \(\mathbf{k}\) [16]. As a basis for the exciton wave functions in the Halperin’s quasi wave function representation [17,18] the functions

\[
\Psi_{L,k} = \exp\{i\mathbf{k}\mathbf{R} + i(\rho_x Y - \rho_y X) - (\rho - \mathbf{d})^2/4\} P_{L}(\ldots \tilde{z}_{ji} \ldots) \prod_{ji}^{\mathbf{z}_{ji}} \exp\{-|\mathbf{z}_{ji}|^2/36\}
\]  

(3)

may be chosen. Here \(\mathbf{d} = \hat{z} \times \mathbf{k}\), and \((-\mathbf{d})\) is the exciton dipole moment. Exponential factors in Eq. 3 ensure the translational magnetic symmetry and belonging the function to the
ground Landau level both for anyons and a hole. A function $P_L(\ldots \bar{z}_{jl} \ldots)$ is a homogeneous polynomial of the degree $L$ which is symmetric in all coordinates $z_i$. The function $\Psi_{L,k}$ describes an exciton with the momentum $k$ and the internal angular momentum $(-L)$.

For a three anyon problem polynomials $P_L$ depend on three variables $\bar{z}_{12}, \bar{z}_{23},$ and $\bar{z}_{31}$. If to take into account the constraint $\bar{z}_{12} + \bar{z}_{23} + \bar{z}_{31} = 0$, one can show that the total number of linearly independent symmetric polynomials equals $[L/6] + 1$ for even $L$ and $[(L-3)/6]+1$ for odd $L$. Square brackets designate the integer part. Odd $L$ polynomials start with $L = 3$ because of Eq. 2. Even-$L$ polynomials $P_L$ may be classified as:

$$P_{L,M} = \bar{z}_{12}^{L-4M} \bar{z}_{23}^{2M} \bar{z}_{31}^{2M} + \bar{z}_{23}^{L-4M} \bar{z}_{31}^{2M} \bar{z}_{12}^{2M} + \bar{z}_{31}^{L-4M} \bar{z}_{12}^{2M} \bar{z}_{23}^{2M}, \quad (4)$$

where $M = 0, 1, \ldots [L/6]$. The only symmetric polynomial with $L = 3$ is a Vandermonde determinant in the variables $\bar{z}_{jl}$:

$$V = (\bar{z}_{12} - \bar{z}_{23})(\bar{z}_{23} - \bar{z}_{31})(\bar{z}_{31} - \bar{z}_{12}). \quad (5)$$

As distinct from the polynomials $P_{L,M}$ with even $L$, which are symmetric both in the bosonic permutations $\bar{z}_1 \leftrightarrow \bar{z}_2$ and permutations $\bar{z}_{12} \leftrightarrow \bar{z}_{23}$, etc., the polynomial $V$ is symmetric in the permutations of the first and antisymmetric in the permutations of the second type. All independent odd-$L$ polynomials can be represented in the form:

$$P_{L,M} = VP_{L-3,M}, \quad P_{3,0} = V. \quad (6)$$

To our best knowledge, in the previous studies only the even-$L$ polynomials have been taken into account [19]. When choosing polynomials $P_{L,M}$ we have not imposed the hard core constraint and defer the discussion of the related properties to what follows.

In the basis of polynomials $P_{L,M}$ the set of quantum numbers consists of $k$, $L$ and $M$. The only disadvantage of the functions $\Psi_{L,M,k}$ is that the scalar products $<LMk|LM'k> \neq 0$ for $M \neq M'$. As a result, the matrix $\hat{B}$ of these scalar products which is diagonal in $k$ and $L$ has a block-diagonal form; the block sizes increase with $L$. The Schrödinger equation is $\hat{H}\chi = \varepsilon\hat{B}\chi$. All integrations in scalar products and matrix elements of $\hat{H}$ should be
performed over four variables \( r_h, r_1, r_2, \) and \( r_3 \). It is convenient to choose the new variables \( R, \rho \) and three \( r_{ij} \) and take into account the constraint of Eq. 2 by the usual \( \delta \)-function transformation. It adds a new variable \( f \), but all calculation become symmetric in the anyon variables. As an example we show here the transformation of a matrix element with polynomials \( P_{L,M} \) substituted by monomials \( \bar{z}_{12}^{n_1} \bar{z}_{23}^{n_2} \bar{z}_{31}^{n_3} \). After the \( \delta \)-function transformation the matrix element takes the form:

\[
\begin{aligned}
<n_1 n_2 n_3 k | n_1' n_2' n_3' k> &= \int \frac{df}{(2\pi)^2} \mathcal{M}_{n_1 n_1'}^\alpha(f) \mathcal{M}_{n_2 n_2'}^\alpha(f) \mathcal{M}_{n_3 n_3'}^\alpha(f),
\end{aligned}
\]

where

\[
\mathcal{M}_{nn'}^\alpha(f) = 2\pi^{\left|n-n'\right|} \frac{\Gamma(\max\{n, n'\} + \alpha + 1)}{\left|n-n'\right|!} 2^{(n+n')/2+\alpha} 3^{(n+n')+2(\alpha+1)} \left(\frac{i}{2}\right)^{\left|n-n'\right|/2} \times \exp\{i\varphi_f(n'-n)\} \Phi(\max\{n, n'\} + \alpha + 1, \left|n-n'\right| + 1 ; -t). \]

(8)

Here \( t = 9f^2/2 \), \( \varphi_f \) is the azimuth of \( f \), and \( \Phi \) is a confluent hypergeometric function.

Quasi wave functions of Eq. 3 differ from bosonic functions only by a statistical factor \( \prod_{jl}(\bar{z}_{jl})^{-\alpha} \). It is of most importance for small values of \( |z_{ij}| \approx l \). For the same distances the deviation of the anyon-anyon interaction law from a Coulomb law [20] also should be taken into account and results in comparable contributions. In what follows we restrict ourselves with a Coulomb interaction in the Hamiltonian \( \hat{H} = V_{aa} + V_{ah} \), where \( V_{aa} = (1/9) \sum_{ij} r_{ij}^{-1} \) and \( V_{ah} = -(1/3) \sum_i \left| r_{hi} + h \hat{z} \right|^{-1} \), and omit the statistical factor in \( \Psi_{L,M,k} \). This last approximation strongly simplifies all calculations. Indeed, with \( \alpha = 0 \) all functions \( \Phi, \) Eq. 3, turn into polynomials if one performs the Kummer transformation, \( \Phi(\beta, \gamma; t) = e^t\Phi(\gamma - \beta, \gamma; -t) \) and takes into account the fact that \( \gamma - \beta \) are non-positive integers for all functions (3). As a result, integrals (3) can be calculated exactly. The matrix elements of \( V_{aa} \) differ from those of Eq. 7 by changing in one of the \( \mathcal{M}_{nn'}^\alpha \) functions \( \alpha \) for \( (\alpha - 1/2) \). This latter function can not be reduced to a polynomial, hence, one-fold integration over \( f \) should be performed numerically. The most complicated are the matrix elements of \( V_{ah} \). To simplify them it is convenient to perform a Fourier transformation \( V_{ah}(r) = \int dq V_{ah}(q) \exp(iqr)/(2\pi)^2 \). In this representation the integrand contains again
three factors $\mathcal{M}_{n_i n'_i}^{\alpha}$. One of them has $f$ as an argument, while two others have $(f \pm q/3)$. For $\alpha = 0$ the integrations over $f$ and over the angle between $f$ and $q$ may be performed. We choose $\Psi_{L,M,k}$ in such a way that the matrix elements of $\hat{H}$ are real, and the diagonal elements of $\hat{B}$ equal unity. The final expression for the matrix element of $V_{ah}$ is:

$$< LMk|V_{ah}|L'M'k> = -\int_0^{\infty} \exp(-3q^2/2 - qh) J_{|L-L'|}(kq) Q_{LM,L'M'}(q) dq ,$$  \hspace{1cm} (9)

where $J_{|L-L'|}(kq)$ are Bessel functions, and the functions $Q_{LM,L'M'}(q)$, real and symmetric in the $LM$ indices, are polynomials in $q$. If $k = 0$, the integral equals zero for $L \neq L'$ as it follows from the angular momentum conservation law. Therefore, for $k = 0$ the total Hamiltonian has a block-diagonal form. The lower polynomials are of a simple form: $Q_{00,00} = 1$, $Q_{20,00} = q^2/2$, $Q_{20,20} = 1 - q^2 + q^4/4$. Since the blocks are of the $1 \times 1$ size for the even-$L$ polynomials with $L < 6$ and the odd-$L$ polynomials with $L < 9$, the corresponding wave functions do not depend on $h$, and the energies may be expressed in terms of quadratures from transcendental and elementary functions. If $L$ is even and $6 \leq L < 12$, the coefficients of $2 \times 2$ secular equations may be expressed in the same terms, etc. With increasing $h$ the anyon-hole attraction becomes smaller, the effect being stronger the less is $L$. The block-diagonal structure of $\hat{H}$ allows the energy level intersections for $k = 0$, for $k \neq 0$ these level crossings turn into anticrossings.

In Fig. \[\] the energy spectrum $\varepsilon(k)$ is shown for two values of $h$. The following regularities are distinctly seen. With increasing $h$, the levels with higher $L$ values draw closer to the spectrum bottom. The branches with small $L$ values have a larger dispersion. The anticrossings are wider the larger the $k$ values and the lesser the differences $|L - L'|$ are. It is remarkable that the branches with a negative dispersion near $k = 0$ exist; it may appear even in the ground state, Fig. \[\]a. For a two-semion problem the dispersion near $k = 0$ is always positive \[\]3.

The most interesting property of AE’s which follows from our calculations is the distribution of the anyon density $D_t(r,k)$ around the hole
\begin{equation}
D_l(r, k) = \frac{1}{2\pi} \sum_{LM, L'M'} \cos((L - L')\theta) \\
\times \int_0^\infty dq \ q \ \exp(-3q^2/2) \ J_{|L-L'|}(q|r-d|) \ Q_{LM,L'M'}(q) \ \chi_{L'M'}^l(k) \chi_{LM}^l(k),
\end{equation}

where \(l\) numerates spectrum branches, and \(\theta\) is the angle between the vectors \(d - r\) and \(d\), \(d\) is the center-of-mass of the anyon density.

**III. ZERO MOMENTUM ANYON EXCITONS**

For \(k = 0\) the density distribution \(D(r, 0)\) is shown in Fig. 2 for the states with \(L \leq 6\). It has been mentioned above that \(D_L\) do not depend on \(h\) for \(L \leq 5\). The state \(L = 3\) is remarkable in some sense. It is the first to show a density minimum at \(r = 0\), and it generates all even-\(L\) polynomials, Eq. 3. For \(L = 6\) there are two eigenfunctions; they depend on \(h\). In Fig. 2 they are shown for \(h = 0\); the lower energy component is drawn by a solid line. With increasing \(L\) the functions \(D_{L,M}\) become broader, and as a result some of the states with large \(L\) values draw closer to the spectrum bottom when \(h\) is growing, Fig. 4. However, it is seen from Fig. 3 that only the states with \(L = 3n, n \geq 0\) are integers, reach the spectrum bottom. The bottom states described by even- and odd-\(L\) polynomials alternate. The periodicity with the period \(\Delta L = 3\) resembles the Laughlin’s result for a three-electron system. We attribute this periodicity in \(L\) to the quantization rule for identical particles, since the rotation angle between two exchange points (which enters into the semiclassical quantization rule) equals \(2\pi/3\).

The functions with \(L = 6M, M \geq 1\) are integers, depend on \(h\) strongly. For \(L = 6\) the electron density distribution \(D(r, 0)\) changes rapidly in the vicinity of \(h = 2\), Fig. 4. For \(h < 2\) the low energy component of the \(L = 6\) state is close to \(\Psi_{6,0}\) (for \(h = 0\) the overlap is 0.96), while for \(h > 2\) it approaches the hard core state \(\Psi_{6,1}\) (for \(h = 3\) the overlap is 0.97). The function \(\Psi_{6,1}\) is the first in the series of the even-\(L\) polynomials \(P_{6M,M} = (\bar{z}_{12}\bar{z}_{23}\bar{z}_{32})^{2M}, M \geq 1\), forming ground states for the large \(h\) values. The density \(D_{L,M}(r, 0), L = 6M,\) has a single maximum for each of these states. The maximum of
the density $\tilde{D}_{L,M}(r,0)$ found after averaging over $r_h$ is determined by a simple equation $r_L = \sqrt{2L}$; the maximum of $D_{L,M}(r,0)$ is pretty close to this point. In the vicinity of the maximum of $D_{L,M}(r,0)$ the three particle correlation function reaches the maximum for an equilateral triangle. It is just the result which is expected in the $L \gg 1$ limit from electrostatic arguments. The value of $r_{12}^2$, found both as a mean value in a $\Psi_{6M,M}$ state and by the semiclassical approach in terms of the distances between anyon guiding centers \cite{22}, satisfies the equation $L = 6M = r_{12}^2/6 - 2$.

The density distribution for $\Psi_{3,0}(r)$ shows a single maximum, Fig. 2. In the vicinity of it the three-particle correlation function reaches the maximum for an equilateral triangle configuration of anyons with $r_{12}^2 \approx 18$. In this respect the functions $P_{6,1}$ and $P_{3,0}$ show a similar behavior. A strong difference between them is reflected in the anyon two-particle correlation functions $W(r)$:

$$W_{6,1} = \frac{r^2}{192\pi}(1 + \frac{1}{2}(r/6)^4)\exp(-r^2/12), \ W_{3,0} = \frac{1}{48\pi}(1 + \frac{1}{6}(r/2)^4)\exp(-r^2/12).$$  \hspace{1cm} (11)

They are shown in Fig. \ref{fig:5}. The function $W_{6,1}$ has a hard core behavior, while $W_{3,0}$ reaches its absolute maximum at $r = 0$ and has the second maximum which is only by a factor 0.97 lower than the main one. Of course, all bottom state odd-$L$ polynomials with $L \geq 9$ show a hard core behavior. It follows from the above data that with increasing $h$ the charge fractionalization results in the broadening of the anyon density distribution in the low-energy $k = 0$ states. The dependence of the exciton binding energy on $h$ is close to the Coulomb law, but the numerator is considerably less than unity. In Fig. \ref{fig:3} the energy of an ordinary $k = 0$ magnetoexciton, $\varepsilon_{ME}(h)$, is shown for comparison. Because of the existence of a hidden symmetry, $\varepsilon_{ME}(h)$ gives an exact result in the $h \to 0$ limit. The AE model becomes exact in the opposite limit, $h \to \infty$. The both curves should be matched somewhere in the intermediate region, $h \sim 1$. 


IV. MOMENTUM DEPENDENCE OF THE EXCITON CHARGE DENSITY

Since the exciton dipole moment \( \mathbf{d} \) differs from \( \mathbf{k} \) only by the factor \( e l^2 \) and the rotation by \( \pi/2 \), Sect. [1], one can expect that with increasing \( k \) the electron density splits into bundles, their charges being multiples of \( e/3 \). The splitting of the electron shell into two well separated quasiparticles has been observed previously for a two-semion exciton [13]. For a three-anyon exciton the pattern are much more impressive. They are shown in Fig. 6 for \( h = 3 \) when the criterion of the large electron-hole separation is fulfilled. The distribution which is cylindrically symmetric for \( k = 0 \) transforms with increasing \( k \) into a distribution with a single split off anyon, \( k = 2 \) and 3. Two anyons constituting the exciton core show a slight but distinct splitting in a perpendicular direction. This core changes its shape with \( k \), but remains stable for a long. Finally, for rather large \( k \) values, it splits in the \( \mathbf{d} \) direction as it is seen in the last figure, \( k = 6 \). In all figures the center-of-mass of the electron density is at the point \( x = k \), and the asymmetric distribution of the density arises due to the odd-\( L \) polynomials. In the absence of them the distribution had to be symmetric with respect to the point \( \mathbf{d} \).

The well outlined profiles of the electron density seen in Fig. 6 may be somewhat smeared by the oscillatory screening inherent in IQL’s [23]. Nevertheless, the basic pattern of the charge separation in an exciton should strongly influence the \( k \) dependence of the magnetoroton-assisted recombination processes since charge density excitations are left in a crystal afterwards.

V. DISCUSSION

The above theory is based on a Coulomb interaction between all the particles and neglecting the statistical factor \( \prod_{j \ell} (\bar{z}_{j \ell})^\alpha \) in Eq. 3. For small \( L \) values the theory results in dense states which can not be described by the AE model, and for larger \( L \) values in more loose states to which the AE model should be applicable. The \( L_3 \) state or, more probable,
the $L_6$ state are the first candidates. If one takes into account the statistical factor, the effective repulsion of anyons should increase. Therefore, we expect that the critical value of $h$, $h_{cr}$, when the loose states reach the spectrum bottom must decrease. According to Fig. 3, $h_{cr} \approx 2$ in the AE model, while the computations in the spherical geometry result in $h_{cr} \approx 1$ for $\nu = 1/3$ [14][15]. These data are in a reasonable agreement. The more intriguing question is whether the realistic anyon form-factors and quasipotentials may result in some new types of the $k = 0$ ground states, e.g., the two-scale states like the low-energy component of the $L = 6$ state in Fig. 3, $h = 1$. It has been proposed by Apalkov and Rashba [9], that the two-scale exciton observed by them in computations for a $\nu = 1/3$ IQL consisted of a two-anyon core and a split off anyon (it resembles the exciton shown in our Fig. 3, $k = 2$, if described in the $(L,k)$ rather than $k$ representation). Such a configuration may be energetically favorable for a $k = 0$ exciton if the two-QE repulsion at small distances is suppressed well below its Coulomb value. This hypothesis is in agreement with the data of Béran and Morf [20]. Therefore, the configurations of anyon shells of AE’s with moderate values of $h$ are expected to be sensitive to the QE interaction law, and calculations based on realistic quasipotentials are desirable.

VI. ACKNOWLEDGMENTS

We are grateful to Prof. A. L. Efros for numerous suggestive discussions and to E. V. Tsiper for valuable comments related to odd-$L$ polynomials. The support by Subagreement No. KK3017 from QUEST of UCSB is acknowledged.
REFERENCES

* Also at A. F. Ioffe Institute, 194021 St. Petersburg, Russia.

# Also at L. D. Landau Institute for Theoretical Physics, 117940 Moscow, Russia.


FIGURES

FIG. 1. Anyon exciton dispersion law $\varepsilon(k)$ for two values of $h$. Mention the opposite signs of the ground state dispersion for both curves. Anticrossings become tiny with increasing $h$. Numbers show $L$ values. For more detail see text.

FIG. 2. Axisymmetric electron density distributions $D_L(r,0)$ for the states with $L \leq 6$. Two $L = 6$ states are shown for $h = 0$; the lowest state is shown by a solid line. Numbers show $L$ values.

FIG. 3. The energy $\varepsilon(0)$ and the electron density $D_L(0,0)$ at the point $r_h$ where the hole resides plotted vs $h$ for the ground state of an exciton with $k = 0$. The ground state energy of an anyon exciton is shown by a solid line; the dots on it show the positions of the intersections between the energy levels with different $L$-values. For comparison the energy of a conventional magnetoexciton $\varepsilon_{ME}$ with $k = 0$ is shown by a dashed line. Numbers near the $D_L(0,0)$ curve show the $L$ values. Only the states with $L = 3n$ reach the spectrum bottom (as an exclusion the state $L = 2$ appears as a bottom state in an extremely narrow region of the $h$ values).

FIG. 4. Electron density distribution $D(r,0)$ in the $L = 6$ states with $k = 0$ for three values of $h$. The density in the lower energy state is shown by solid a line. Consecutive numbers, $l$, of the energy levels are shown near the curves.

FIG. 5. Normalized anyon pair correlation function $W(r)$ for the states $\Psi_{6,1}$ and $\Psi_{3,0}$; $k = 0$.

FIG. 6. Electron density distribution in an anyon exciton for different values of $k$. A hole is in the origin, the $x$ axis is chosen in the $d$ direction. The center of the electron density distribution is at $x = k$. 
DENSITY, $D(r,0)$

COORDINATE, $r$