Lattices and codes with long shadows

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Accessibility
Lattices and codes with long shadows

Noam D. Elkies

Introduction.

By a characteristic vector of an integral unimodular lattice $L \subset \mathbb{R}^n$ we mean a vector $w \in L$ such that $(v, w) \equiv (v, v) \mod 2$ for all $v \in L$. Such vectors are known to constitute a coset of $2L$ in $L$ whose norms are congruent to $n \mod 8$ (see e.g. [Se, Ch.V]); dividing this coset by 2 yields a translate of $L$ called the shadow of $L$ in [CS2]. If $L = \mathbb{Z}^n$ then $w \in L$ is characteristic if and only if all its coordinates are odd, so every characteristic vector of $\mathbb{Z}^n$ has norm at least $n$. In [El] we proved that if $L \not\cong \mathbb{Z}^n$ then $L$ has characteristic vectors of norm $\leq n - 8$, and described without proof all lattices for which $n - 8$ is the minimum. Here we prove this result, and along the way also obtain congruences and a lower bound on the kissing number of unimodular lattices with minimal norm 2. We then state and prove analogues of these results for self-dual codes, and relate them directly to the lattice problems via Construction A.

Estimates for unimodular lattices

Any integral lattice $L$ decomposes as the direct sum $\mathbb{Z}^r \oplus L_0$ where the $\mathbb{Z}^r$ is generated by the vectors of norm 1 and $L_0$ is a lattice of minimal norm $\geq 2$. [This $L_0$ is called the “reduced form” or “initial lattice” of $L$ in [CS1, p.414], the latter terminology suggesting the infinite family of lattices $L_0, L_0 \oplus \mathbb{Z}, L_0 \oplus \mathbb{Z}^2$, etc., of which $L_0$ is the initial member.] If $L$ (and thus also $L_0$) is unimodular then the shadow of $L$ is the orthogonal sum of the shadows of $\mathbb{Z}^r$ and $L_0$. Replacing $L$ by $L_0$ thus reduces both the rank of the lattice and the norm of its shortest characteristic vector by $r$, and does not change the difference between these two integers. We may thus restrict attention to lattices with no vectors of norm 1 for which that difference is 8, and at the end recover all such lattices by adding arbitrarily many $\mathbb{Z}$’s.

Theorem 1. Let $L$ be an integral unimodular lattice in $\mathbb{R}^n$ with no vectors of norm 1. Then:

i) $L$ has at least $2^n(23 - n)$ vectors of norm 2.

ii) Equality holds if and only if $L$ has no characteristic vectors of norm $< n - 8$.

iii) In that case the number of characteristic vectors of norm exactly $n - 8$ is $2^{n-1}n$.

Proof: We use theta series as in [El], though here we freely invoke modular
forms. For $t$ in the upper half-plane $H$ define
\[
\theta_L(t) := \sum_{v \in L} e^{\pi i |v|^2 t} = \sum_{k=0}^{\infty} N_k e^{\pi i k t}, \tag{1}
\]
where $N_k$ is the number of lattice vectors of norm $k$, and
\[
\theta'_L(t) := \sum_{v \in L + w} e^{\pi i |v|^2 t} = \sum_{k=0}^{\infty} N'_k e^{\pi i k t / 4}, \tag{2}
\]
where $w \in L$ is any characteristic vector and $N'_k$ is the number of characteristic vectors of norm $k$, or equivalently the number of shadow vectors of norm $k/4$. In [El] we noted the identity
\[
\theta_L \left( \frac{-1}{t} + 1 \right) = \left( \frac{t}{i} \right)^{n/2} \theta'_L(t). \tag{3}
\]
By a theorem of Hecke (see e.g. [CS1, Ch.7, Thm.7]), $\theta_L$ is a modular form of weight $n/2$ and can be written as a weighted-homogeneous polynomial $P_L(\theta_Z, \theta_{E_8})$ in the modular forms
\[
\theta_Z(t) := 1 + 2 \left(e^{\pi i t} + e^{4\pi i t} + e^{9\pi i t} + \cdots\right) \tag{4}
\]
of weight $1/2$ and
\[
\theta_{E_8}(t) = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 e^{2\pi i mt}}{1 - e^{2\pi i mt}} = 1 + 240 e^{2\pi i t} + 2160 e^{4\pi i t} + \cdots \tag{5}
\]
of weight $4$. From (3) it follows that $\theta'_L$ is given by
\[
\theta'_L = P_L(\theta'_Z, \theta_{E_8}), \tag{6}
\]
where
\[
\theta'_Z(t) = 2 \sum_{m=0}^{\infty} e^{\pi i (m+1/2)^2 t} = 2 e^{\pi i t / 4} \left(1 + e^{2\pi i t} + e^{6\pi i t} + e^{12\pi i t} + \cdots\right), \tag{7}
\]
and we used the fact that $\theta'_{E_8} = \theta_{E_8}$ because $E_8$ is an even lattice. Since $\theta'_Z(t) \sim 2 e^{\pi i t / 4}$ as $t \to \infty$, while $E_8(i \infty) = 1$ is nonzero, we see from (3) that the norm of the shortest characteristic vectors is simply the exponent of $X$ in the factorization of $P_L(X, Y)$.

\[\text{1} \text{We could now recover our theorem from [El] by observing that this exponent is at most } n, \text{ with equality if and only if } \theta_L \text{ is proportional to } \theta^n, \text{ etc.; but this is really the same proof because the crucial fact that } \theta_Z \text{ vanishes at one cusp and nowhere else is also an essential ingredient of Hecke’s theorem.}\]
In our setting \( N_0 = 1 \) and \( N_1 = 0 \). We first prove part (ii) of our theorem. If \( L \) has no characteristic vectors of norm \( < n - 8 \) then \( P_L(X,Y) \) is a linear combination of \( X^n \) and \( X^{n-8}Y \). The known values of \( N_0, N_1 \) determine this combination uniquely: we find that

\[
\theta_L = \theta^n_Z - \frac{n}{8} \theta^{n-8}_Z (\theta^8_Z - \theta_{E_8}) = 1 + 0 e^{\pi it} + 2n(23 - n) e^{2\pi it} + \cdots . \tag{8}
\]

Thus \( L \) indeed has \( 2n(23 - n) \) vectors of norm 2. Conversely if \( L \) is an integral unimodular lattice with \( N_1 = 0 \) and \( N_2 \leq 2n(23 - n) \) then \( n < 24 \) and \( P_L \) has at most 3 terms, whose coefficients are determined uniquely by \( N_0, N_1, N_2 \)

\[
\theta_L = \theta^n_Z - \frac{n}{8} \theta^{n-8}_Z (\theta^8_Z - \theta_{E_8}) + \frac{N_2 - (2n(23 - n))}{16^2} \theta^{n-16}_Z (\theta^8_Z - \theta_{E_8})^2 . \tag{9}
\]

But then by (9) we have

\[
N_{n-16}' = 2^{n-24}[N_2 - (2n(23 - n))]. \tag{10}
\]

Since \( N_{n-16}' \geq 0 \) we conclude that \( N_2 \geq 2n(23 - n) \) even for \( n < 24 \), as claimed in part (i) of the theorem; and equality occurs if and only if \( N_{n-16}' \) vanishes, whence the reverse implication in part (ii) follows. Finally to prove part (iii) we use (8) to compute

\[
\theta'_L = \frac{n}{8} \theta^{n-8}_Z (\theta^8_Z - \theta_{E_8}) = 2^{n-11} n e^{(n-8)\pi it} + \cdots \tag{11}
\]

so \( N_{n-8} = 2^{n-11} n \) as claimed. \( \square \)

Fortunately the integral unimodular lattices of rank \( n < 24 \) are completely known, and those with \( N_1 = 0 \) are conveniently listed with their \( N_2 \) values in the table of \([CS1, pp.416–7]\). For \( n < 16 \) the shortest characteristic vector must have norm at least \( n - 8 \), so any unimodular lattice of minimal norm \( > 1 \) must have \( 2n(23 - n) \) vectors of norm 2; this is confirmed by the table. When \( 16 \leq n \leq 23 \) some lattices can have more than \( 2n(23 - n) \) such vectors, but it turns out there is always at least one lattice with \( N_2 = 2n(23 - n) \). (Can this be proved a priori?) Thus, as observed by J.H. Conway, the lattices of parts (ii), (iii) of our theorem are precisely the integral unimodular lattices of rank \( n < 24 \) with \( N_1 = 0 \) that minimize \( N_2 \) given \( n \). As noted in \([3]\), there are fourteen such lattices; in the following list, adapted from \([3]\), we label them as in the table of \([CS1]\) by the root system of norm-2 vectors:

<table>
<thead>
<tr>
<th>( n )</th>
<th>8</th>
<th>12</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_2 )</td>
<td>240</td>
<td>264</td>
<td>252</td>
<td>240</td>
<td>224</td>
<td>204</td>
<td>180</td>
<td>152</td>
<td>120</td>
<td>84</td>
<td>44</td>
<td>0</td>
</tr>
</tbody>
</table>

| \( L \) | \( D_8 \) | \( E_7^2 \) | \( A_{15} \) | \( D_8^2 \) | \( A_{11}E_6 \) | \( D_6^2 \) | \( A_2D_5 \) | \( D_4^2 \) | \( A_3^2 \) | \( A_2^2 \) | \( A_3^2 \) | \( O_23 \) |

We noted in \([3]\) that from our characterization of \( \mathbb{Z}^n \) we could also recover the fact that \( \mathbb{Z}^n \) is the only integral unimodular lattice of rank \( n \) for \( n < 8 \). Likewise from part (iii) of Theorem 1 we can recover the fact that every integral
unimodular lattice of rank $n < 12$ is either $\mathbb{Z}^n$ or $\mathbb{Z}^{n-8} \oplus E_8$. Indeed there would otherwise be such a lattice of rank 9, 10, or 11 with no vector of norm 1, but then by (iii) the lattice would have $N_{n-8}' = 2^{n-11}n$, which is impossible because $N_{n-8}'$ is an even integer for any lattice of rank $n \neq 8$.

Having obtained (11), we used $N_{n-16}' \geq 0$ to prove $N_2 \geq 2n(23 - n)$. Since $N_{n-16}'$ is always an even integer unless $n = 16$ and $0$ is a characteristic vector ($\iff L$ is its own shadow $\iff L$ is an even lattice), it follows that in fact

$$N_2 \equiv 2n(23 - n) \mod 2^{25-n} \quad (12)$$

for any even unimodular lattice with no vectors of norm 1, with the exception of the two even lattices $E_8^2$, $D_{16}^{+}$ of rank 16, which have $N_2 = 480$. (This is similar to the argument used in [CS1, Ch.19, p.440] to prove that there are no “extremal Type I lattices” with $16 \leq n \leq 22$, a fact which now also follows from part (i) of our Theorem.) The congruence (12) is confirmed by the table of [CS1], which also reveals that $N_{2} \equiv 2n(23 - n)$ is always a multiple of $2^{4}$ for $n = 22$ and $n = 23$; a “conceptual” (but far from easy) proof of this is found in [34, Thm. 4.4.2(3)]. Note that even though we have only proved (12) for $n < 24$, it in fact holds for all $n$, since $N_2$ is always an even integer.

Estimates for self-dual binary codes

We recall some basic facts about binary linear codes; see for instance [CS1, Ch.3, §2.2]. Let $F = \mathbb{Z}/2\mathbb{Z}$ be the two-element field. We work in the vector space $F^n$, whose elements we regard as “words” of length $n$ whose “letters” are taken from the “alphabet” $F$. The (Hamming) “weight” $w(w)$ of a word $w = (w_1, w_2, \ldots, w_n) \in F^n$ is $\# \{ j : w_j = 1 \}$ of nonzero coordinates of $w$. A “binary linear code” of length $n$ is a subspace $C \subset F^n$. A binary self-dual code is a linear code which is its own annihilator under the nondegenerate pairing $(\cdot, \cdot) : C \times C \to F$, defined by $(v, w) = \sum_{j=1}^{n} v_j w_j$. Such a code must have dimension $n/2$, and thus can only exist if $n$ is even, which we henceforth assume.

Note that under our pairing we have

$$(w, w) = (w, 1^n) \equiv wt(w) \mod 2 \quad (13)$$

for all $w \in F^n$, where $1^n$ is the all-ones vector in $F^n$. Thus if $C$ is a self-dual code then $C \supseteq 1^n$ and all the words in $C$ have even weight.

The “weight enumerator” $W_C$ of $C$ is a generating function for the weight distribution of $C$:

$$W_C(x, y) := \sum_{c \in C} x^{n-wt(c)} y^{wt(c)}. \quad (14)$$

For a binary self-dual code a theorem of Gleason (Thm.6 of [CS1, Ch.7]), analogous to Hecke’s theorem for theta series of lattices, states that $W_C$ is a weighted-homogeneous polynomial $P_W(x^2 + y^2, x^8 + 14x^4y^4 + y^8)$ in the weight enumerators of the double repetition code $z := \{(0,0), (1,1)\} \subset F^2$ and the extended Hamming code in $F^8$ respectively.
Analogous to the homomorphism \( v \mapsto |v|^2 \mod 2 \) from an integral lattice to \( \mathbb{Z}/2 \) we have for any self-dual code \( C \subset F^n \) a linear map from \( C \) to \( F \) taking any \( c \in C \) to \( \frac{1}{2} \text{wt}(c) \mod 2 \). We can use the pairing on \( F^n \) to represent any linear functional on a self-dual code by a unique coset of the code; thus we find a coset \( C' \) of \( C \) consisting of all \( c' \in F^n \) such that
\[
\frac{1}{2} \text{wt}(c) \equiv (c, c') \mod 2
\]
for all \( c \in C \). As in \([CS2]\) we call \( C' \) the shadow of \( C \), in analogy with the shadow of an integral unimodular lattice. Let
\[
W'_C(x, y) := \sum_{c \in C'} x^{n - \text{wt}(c)} y^{\text{wt}(c)}
\]
be the generating function for the weight distribution of \( C' \). Using discrete Poisson inversion as in the proof of the MacWilliams identity and the characterization \([15]\) of \( C' \) we find as in \([CS2]\)
\[
W'_C(x, y) = 2^{-n/2} \sum_{c \in C} (-1)^{\text{wt}(c)/2} (x + y)^{n - \text{wt}(c)} (x - y)^{\text{wt}(c)}
\]
\[
= 2^{-n/2} W_C(x + y, i(x - y)).
\]
Thus from \( W_C(x, y) = P_W(x^2 + y^2, x^8 + 14x^4y^4 + y^8) \) we obtain
\[
W'_C(x, y) = P_W(2xy, x^8 + 14x^4y^4 + y^8).
\]
Note that all the words in the shadow thus have weight congruent to \( n/2 \) mod 4. We could have also obtained this directly from the MacWilliams identity
\[
W_C(x, y) = 2^{-n/2} W_C(x + y, x - y),
\]
(which also underlies Gleason’s theorem); this would more closely parallel the analytic proof of \(|w|^2 \equiv n \mod 8 \) in \([3]\).

If \( c \in C \) has weight \( 2 \) then every codeword either contains or is disjoint from \( c \). Thus \( C \) decomposes as a direct sum of a double repetition code \( z \) generated by \( c \) and the self-dual code of length \( n - 2 \) consisting of codewords disjoint from \( c \). Iterating this we decompose \( C \) as \( C_0 \oplus z^r \), where \( r \) is the number of weight-2 words in \( C \), and \( C_0 \) is a self-dual code of length \( n - 2r \) with no words of weight 2. Now for any self-dual codes \( C_1, C_2 \), their direct sum \( C_1 \oplus C_2 \) has shadow
\[
(C_1 \oplus C_2)' = C_1' \oplus C_2'.
\]
Since the shadow of \( z \) is \( \{(0, 1), (1, 0)\} \) it follows that the shadow of \( z^r \) consists entirely of words of weight \( r \), and if \( C = C_0 \oplus z^r \) then the minimal weight of \( C_0 \) is \( r \) less than that of \( C' \).
Since $C'$ contains $w + 1^n$ whenever it contains $w$ it is clear that the minimal weight of $C'$ cannot exceed the value $n/2$ attained by $z^n/2$. This is much easier than proving the corresponding fact for characteristic vectors of unimodular lattices, but it does not show that $z^n/2$ is the only self-dual code whose shadow has minimal weight $n/2$. We prove this, as we did for lattices, by noting that such a code $C$ must have $W'_C(x, y) = (2xy)^{n/2}$, whence $W_C(x, y) = (x^2 + y^2)^{n/2}$. Since $C$ contains $n/2$ words of weight 2, then, it can only be $z^n/2$.

We have shown that the shadow of a binary self-dual code $C$ other than $z^{n/2}$ contains some words of weight $< n/2$. Thus $C'$ has minimal weight at most $(n-8)/2$. We next characterize all $C$ attaining this bound. If $C = C_0 \oplus z^r$ then $C$ attains the bound if and only if $C_0$ does, so we need only consider codes without weight-2 words.

**Theorem 1A.** Let $C$ be a binary self-dual code of length $n$ with no codewords of weight 2. Then:

i) $C$ has at least $n(22 - n)/8$ codewords of weight 4.

ii) Equality holds if and only if the shadow of $C$ contains no codewords of weight $< (n-8)/2$.

iii) In that case the number of codewords of weight exactly $(n-8)/2$ in the shadow is $2^{(n-14)/2} n$.

**Proof:** We can mimic the proof of Theorem 1. If the minimal weight of $C'$ is at least $(n-8)/2$ then $W'_C$ is a linear combination of $(xy)^{n/2}$ and $(xy)^{(n-8)/2}(x^8 + 14x^4y^4 + y^8)$, and thus $W_C(x, y)$ is a linear combination of $(x^2 + y^2)^{n/2}$ and $(x^2 + y^2)^{(n-8)/2}(x^8 + 14x^4y^4 + y^8)$. The condition that $C$ have no weight-2 codewords then forces

$$W_C(x, y) = (x^2 + y^2)^{n/2} - \frac{n}{8}(x^2 + y^2)^{(n-8)/2}((x^2 + y^2)^4 - (x^8 + 14x^4y^4 + y^8))$$

$$= x^n + 0 x^{n-2}y^2 + \frac{n(22-n)}{8} x^{n-4}y^4 + \ldots$$  \hspace{1cm} (21)

and

$$W_C(x, y) = (2xy)^{n/2} - \frac{n}{8}(2xy)^{(n-8)/2}[(2xy)^4 - (x^8 + 14x^4y^4 + y^8)]$$

$$= 2^{(n-14)/2}(x^2 + y^2)^{(n-8)/2}\left[nx^8 + (128 - 2n)x^4y^4 + ny^8\right].$$  \hspace{1cm} (22)

If $n < 24$ and $C$ is any binary self-dual code of length $n$ containing no words of weight 2 and $n(22 - n)/8 + d$ words of weight 4 then its weight enumerator exceeds (21) by

$$\frac{d}{16}(x^2 + y^2)^{(n-16)/2}\left[x^8 + 14x^4y^4 + y^8 - (x^2 + y^2)^4\right]^2,$$  \hspace{1cm} (23)

so the weight enumerator of the shadow $C'$ exceeds (22) by

$$\frac{d}{16}(2xy)^{(n-16)/2}(x^8 - 2x^4y^4 + y^8)^2.$$  \hspace{1cm} (24)

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Thus $C'$ contains $2^{(n-24)/2}d$ words of weight $(n - 16)/2$, from which we find that $d$ is a nonnegative multiple of $2^{(24-n)/2}$.

Alternatively we could deduce Theorem 1A from Theorem 1 via Construction A [CS1, Ch.7, §2]. Recall that this construction associates to a self-dual code $C \subset F^n$ the unimodular integral lattice

$$L_C := \{ 2^{-1/2}v \mid v \in \mathbb{Z}^n, v \mod 2 \in C \}. \quad (25)$$

The theta series of this lattice is given by

$$\theta_L(t) = W_C(\theta_{\mathbb{Z}}(2t), \theta'_{\mathbb{Z}}(2t)); \quad (26)$$

in particular $L_C$ has no vectors of norm 1 if and only if $C$ has no codewords of weight 2 (NB $L_2 \cong \mathbb{Z}^2$), and the $N_2(L_C) - 2n$ is $2^4$ times the number of weight-4 codewords in $C$. Moreover the set of characteristic vectors of $L_C$ is

$$\{ 2^{1/2}v \mid v \in \mathbb{Z}^n, v \mod 2 \in C' \} \quad (27)$$

(in effect the shadow of $L_C$ is obtained by applying Construction A to the shadow of $C$), so the norm of the shortest characteristic vectors is half the minimal weight of $C'$. Applying Theorem 1 to $L_C$ thus yields Theorem 1A immediately. □

Thus also the codes $C$ of parts (ii), (iii) of Theorem 1A are precisely those for which $L_C$ is one of the 14 lattices listed in connection with Theorem 1. Of course not every such lattice arises because $n$ must be even; moreover, the root system can only involve $A_1$, $D_{2m}$, $E_7$ and $E_8$ if the lattice arises from construction A. This leaves only the seven lattices with root systems $E_8$, $D_{12}$, $E_7^2$, $D_8^2$, $D_6^5$, $D_5^3$, $A_1^2$ of rank 8, 12, 14, 16, 18, 20, 22 respectively. It turns out that each of those lattices arises as $L_C$ for a unique code $C$ [ES]. For instance the first of these arises from the extended Hamming code, and the last from what might be called the shorter binary Golay code; these are the shortest self-dual binary codes having minimal weight 4 and 6 respectively. Again it so happens that whenever there is a self-dual code of length $n < 24$ with minimal weight at least 4, there is such a code (this time unique) with only $n(22 - n)/8$ words of weight 4, so Conway’s description of our fourteen lattices also applies *mutatis mutandis* to our seven codes.

**Can we go past $n - 8$?**

Our results suggest the following questions:

For any $k > 0$ is there $N_k$ such that every integral unimodular lattice all of whose characteristic vectors have norm $\geq n - 8k$ is of the form $L_0 \oplus \mathbb{Z}^r$ for some lattice $L_0$ of rank at most $N_k$?
For any $k > 0$ is there $n_k$ such that every binary self-dual code of whose shadow has minimal norm $\geq (n - 8k)/2$ is of the form $C_0 \oplus z^r$ for some code $C_0$ of length at most $n_k$?

Of course a positive answer for lattices would imply one for codes, and vice versa for a negative answer, via Construction A, with $n_k \leq N_k$ in the former case. Even $k = 2$ seems difficult.

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References


