Lattices and codes with long shadows

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Lattices and codes with long shadows

Noam D. Elkies

Introduction.

By a characteristic vector of an integral unimodular lattice \( L \subset \mathbb{R}^n \) we mean a vector \( w \in L \) such that \( (v, w) \equiv (v, v) \mod 2 \) for all \( v \in L \). Such vectors are known to constitute a coset of \( 2L \) in \( L \) whose norms are congruent to \( n \mod 8 \) (see e.g. [Se, Ch.V]); dividing this coset by 2 yields a translate of \( L \) called the shadow of \( L \) in [CS2]. If \( L = \mathbb{Z}^n \) then \( w \in L \) is characteristic if and only if all its coordinates are odd, so every characteristic vector of \( \mathbb{Z}^n \) has norm at least \( n \). In [El] we proved that if \( L \neq \mathbb{Z}^n \) then \( L \) has characteristic vectors of norm \( \leq n - 8 \), and described without proof all lattices for which \( n - 8 \) is the minimum. Here we prove this result, and along the way also obtain congruences and a lower bound on the kissing number of unimodular lattices with minimal norm 2. We then state and prove analogues of these results for self-dual codes, and relate them directly to the lattice problems via Construction A.

Estimates for unimodular lattices

Any integral lattice \( L \) decomposes as the direct sum \( \mathbb{Z}^r \oplus L_0 \) where the \( \mathbb{Z}^r \) is generated by the vectors of norm 1 and \( L_0 \) is a lattice of minimal norm \( \geq 2 \). [This \( L_0 \) is called the “reduced form” or “initial lattice” of \( L \) in [CS1, p.414], the latter terminology suggesting the infinite family of lattices \( L_0, L_0 \oplus \mathbb{Z}, L_0 \oplus \mathbb{Z}^2, \) etc., of which \( L_0 \) is the initial member.] If \( L \) (and thus also \( L_0 \)) is unimodular then the shadow of \( L \) is the orthogonal sum of the shadows of \( \mathbb{Z}^r \) and \( L_0 \). Replacing \( L \) by \( L_0 \) thus reduces both the rank of the lattice and the norm of its shortest characteristic vector by \( r \), and does not change the difference between these two integers. We may thus restrict attention to lattices with no vectors of norm 1 for which that difference is 8, and at the end recover all such lattices by adding arbitrarily many \( \mathbb{Z} \)'s.

**Theorem 1.** Let \( L \) be an integral unimodular lattice in \( \mathbb{R}^n \) with no vectors of norm 1. Then:

i) \( L \) has at least \( 2n(23 - n) \) vectors of norm 2.

ii) Equality holds if and only if \( L \) has no characteristic vectors of norm \( < n - 8 \).

iii) In that case the number of characteristic vectors of norm exactly \( n - 8 \) is \( 2^{n-1}n \).

**Proof:** We use theta series as in [El], though here we freely invoke modular
forms. For \( t \) in the upper half-plane \( H \) define

\[
\theta_L(t) := \sum_{v \in L} e^{\pi i |v|^2 t} = \sum_{k=0}^{\infty} N_k e^{\pi i k t},
\]

where \( N_k \) is the number of lattice vectors of norm \( k \), and

\[
\theta'_L(t) := \sum_{v \in L + w} e^{\pi i |v|^2 t} = \sum_{k=0}^{\infty} N'_k e^{\pi i k t / 4},
\]

where \( w \in L \) is any characteristic vector and \( N'_k \) is the number of characteristic vectors of norm \( k \), or equivalently the number of shadow vectors of norm \( k/4 \).

In [El] we noted the identity

\[
\theta_L(-1/t + 1) = (t/i)^{n/2} \theta'_L(t).
\]

By a theorem of Hecke (see e.g. [CS1, Ch.7, Thm.7]), \( \theta_L \) is a modular form of weight \( n/2 \) and can be written as a weighted-homogeneous polynomial \( P_L(\theta_Z, \theta_E^8) \) in the modular forms

\[
\theta_Z(t) := 1 + 2 \left(e^{\pi i t} + e^{4\pi i t} + e^{9\pi i t} + \cdots\right)
\]

of weight 1/2 and

\[
\theta_E^8(t) = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 e^{2\pi i m t}}{1 - e^{2\pi i m t}} = 1 + 240 e^{2\pi i t} + 2160 e^{4\pi i t} + \cdots
\]

of weight 4. From (3) it follows that \( \theta'_L \) is given by

\[
\theta'_L = P_L(\theta'_Z, \theta_E^8),
\]

where

\[
\theta'_Z(t) = 2 \sum_{m=0}^{\infty} e^{\pi i (m+\frac{1}{4})^2 t} = 2 e^{\pi i t / 4} \left(1 + e^{2\pi i t} + e^{6\pi i t} + e^{12\pi i t} + \cdots\right),
\]

and we used the fact that \( \theta'_E^8 = \theta_E^8 \) because \( E_8 \) is an even lattice. Since \( \theta'_Z(t) \sim 2 e^{\pi i t / 4} \) as \( t \to i\infty \), while \( E_4(i\infty) = 1 \) is nonzero, we see from (3) that the norm of the shortest characteristic vectors is simply the exponent of \( X \) in the factorization of \( P_L(X, Y) \) \(^1\).

\(^1\)We could now recover our theorem from [El] by observing that this exponent is at most \( n \), with equality if and only if \( \theta_L \) is proportional to \( \theta_Z^2 \), etc.; but this is really the same proof because the crucial fact that \( \theta_Z \) vanishes at one cusp and nowhere else is also an essential ingredient of Hecke’s theorem.
In our setting $N_0 = 1$ and $N_1 = 0$. We first prove part (ii) of our theorem. If $L$ has no characteristic vectors of norm $< n - 8$ then $P_L(X,Y)$ is a linear combination of $X^n$ and $X^{n-8}Y$. The known values of $N_0, N_1$ determine this combination uniquely: we find that

$$\theta_L = \theta^n_L - \frac{n}{8} \theta^{n-8}_Z (\theta^8_Z - \theta_{E_8}) = 1 + 0 e^{\pi i t} + 2n(23 - n) e^{2\pi i t} + \cdots . \quad (8)$$

Thus $L$ indeed has $2n(23 - n)$ vectors of norm 2. Conversely if $L$ is an integral unimodular lattice with $N_1 = 0$ and $N_2 \leq 2n(23 - n)$ then $n < 24$ and $P_L$ has at most 3 terms, whose coefficients are determined uniquely by $N_0, N_1, N_2$:

$$\theta_L = \theta^n_L - \frac{n}{8} \theta^{n-8}_Z (\theta^8_Z - \theta_{E_8}) + \frac{N_2 - (2n(23 - n))}{16^2} \theta^n_{-16} (\theta^8_Z - \theta_{E_8})^2 . \quad (9)$$

But then by (3) we have

$$N'_{n-16} = 2^{n-24}[N_2 - (2n(23 - n))]. \quad (10)$$

Since $N'_{n-16} \geq 0$ we conclude that $N_2 \geq 2n(23 - n)$ even for $n < 24$, as claimed in part (i) of the theorem; and equality occurs if and only if $N'_{n-16}$ vanishes, whence the reverse implication in part (ii) follows. Finally to prove part (iii) we use (3) to compute

$$\theta'_L = \theta^n'_L - \frac{n}{8} \theta^{n-8}_Z (\theta^8_Z - \theta_{E_8}) = 2^{n-11} n e^{(n-8)\pi i t} + \cdots , \quad (11)$$

so $N'_{n-8} = 2^{n-11} n$ as claimed. \(\square\)

Fortunately the integral unimodular lattices of rank $n < 24$ are completely known, and those with $N_1 = 0$ are conveniently listed with their $N_2$ values in the table of [CS1, pp.416–7]. For $n < 16$ the shortest characteristic vector must have norm at least $n - 8$, so any unimodular lattice of minimal norm $> 1$ must have $2n(23 - n)$ vectors of norm 2; this is confirmed by the table. When $16 \leq n \leq 23$ some lattices can have more than $2n(23 - n)$ such vectors, but it turns out there is always at least one lattice with $N_2 = 2n(23 - n)$. (Can this be proved a priori?) Thus, as observed by J.H. Conway, the lattices of parts (ii), (iii) of our theorem are precisely the integral unimodular lattices of rank $n < 24$ with $N_1 = 0$ that minimize $N_2$ given $n$. As noted in [CS1], there are fourteen such lattices; in the following list, adapted from [CS1], we label them as in the table of [CS1] by the root system of norm-2 vectors:

<table>
<thead>
<tr>
<th>$n$</th>
<th>8</th>
<th>12</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
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<td>$N_2$</td>
<td>240</td>
<td>264</td>
<td>252</td>
<td>240</td>
<td>224</td>
<td>204</td>
<td>180</td>
<td>152</td>
<td>120</td>
<td>84</td>
<td>44</td>
<td>0</td>
</tr>
<tr>
<td>$E_8$</td>
<td>24</td>
<td>$D_4$</td>
<td>$E_7$</td>
<td>$A_{15}$</td>
<td>$D_8^2$</td>
<td>$A_{11}E_6$</td>
<td>$D_6^2$</td>
<td>$A_9^2D_5$</td>
<td>$D_5^4$</td>
<td>$A_4^2$</td>
<td>$A_3^3$</td>
<td>$A_2^{12}$</td>
</tr>
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We noted in [CS1] that from our characterization of $\mathbb{Z}^n$ we could also recover the fact that $\mathbb{Z}^n$ is the only integral unimodular lattice of rank $n$ for $n < 8$. Likewise from part (iii) of Theorem 1 we can recover the fact that every integral
unimodular lattice of rank $n < 12$ is either $\mathbb{Z}^n$ or $\mathbb{Z}^{n-8} \oplus E_8$. Indeed there would otherwise be such a lattice of rank 9, 10, or 11 with no vector of norm 1, but then by (iii) the lattice would have $N'_{n-8} = 2^{n-11}n$, which is impossible because $N'_{n-8}$ is an even integer for any lattice of rank $n \neq 8$.

Having obtained (11), we used $N'_{n-16} \geq 0$ to prove $N_2 \geq 2n(23 - n)$. Since $N'_{n-16}$ is always an even integer unless $n = 16$ and 0 is a characteristic vector ($\iff L$ is its own shadow $\iff L$ is an even lattice), it follows that in fact

$$N_2 \equiv 2n(23 - n) \mod 2^{25-n}$$

(12)

for any even unimodular lattice with no vectors of norm 1, with the exception of the two even lattices $E_2^8$, $D_4^+$ of rank 16, which have $N_2 = 480$. (This is similar to the argument used in [CS1, Ch.19, p.440] to prove that there are no “extremal Type I lattices” with $16 \leq n \leq 22$, a fact which now also follows from part (i) of our Theorem.) The congruence (12) is confirmed by the table of [CS1], which also reveals that $N_2^{23-n}$ is always a multiple of $2^4$ even for $n = 22$ and $n = 23$; a “conceptual” (but far from easy) proof of this is found in [Bo, Thm. 4.4.2(3)]. Note that even though we have only proved (12) for $n < 24$, it in fact holds for all $n$, since $N_2$ is always an even integer.

### Estimates for self-dual binary codes

We recall some basic facts about binary linear codes; see for instance [CS1, Ch.3, §2.2]. Let $F = \mathbb{Z}/2\mathbb{Z}$ be the two-element field. We work in the vector space $F^n$, whose elements we regard as “words” of length $n$ whose “letters” are taken from the “alphabet” $F$. The (Hamming) “weight” $wt(w)$ of a word $w = (w_1, w_2, \ldots, w_n) \in F^n$ is $\#\{j : w_j = 1\}$ of nonzero coordinates of $w$. A “binary linear code” of length $n$ is a subspace $C \subset F^n$. A binary self-dual code is a linear code which is its own annihilator under the nondegenerate pairing $(\cdot, \cdot) : C \times C \to F$, defined by $(v, w) = \sum_{j=1}^{n} v_j w_j$. Such a code must have dimension $n/2$, and thus can only exist if $n$ is even, which we henceforth assume.

Note that under our pairing we have

$$\langle w, w \rangle = \langle w, 1^n \rangle \equiv wt(w) \mod 2$$

(13)

for all $w \in F^n$, where $1^n$ is the all-ones vector in $F^n$. Thus if $C$ is a self-dual code then $C \ni 1^n$ and all the words in $C$ have even weight.

The “weight enumerator” $W_C$ of $C$ is a generating function for the weight distribution of $C$:

$$W_C(x, y) := \sum_{c \in C} x^{n-wt(c)} y^{wt(c)}.$$  

(14)

For a binary self-dual code a theorem of Gleason (Thm.6 of [CS1, Ch.7]), analogous to Hecke’s theorem for theta series of lattices, states that $W_C$ is a weighted-homogeneous polynomial $P_W(x^2 + y^2, x^8 + 14x^4y^4 + y^8)$ in the weight enumerators of the double repetition code $z := \{(0,0), (1,1)\} \subset F^2$ and the extended Hamming code in $F^8$ respectively.
Analogous to the homomorphism $v \mapsto |v|^2 \mod 2$ from an integral lattice to $\mathbb{Z}/2$ we have for any self-dual code $C \subset F^n$ a linear map from $C$ to $F$ taking any $c \in C$ to $\frac{1}{2} \text{wt}(c) \mod 2$. We can use the pairing on $F^n$ to represent any linear functional on a self-dual code by a unique coset of the code; thus we find a coset $C'$ of $C$ consisting of all $c' \in F^n$ such that

$$\frac{1}{2} \text{wt}(c) \equiv (c, c') \mod 2$$

for all $c \in C$. As in [CS2] we call $C'$ the shadow of $C$, in analogy with the shadow of an integral unimodular lattice. Let

$$W_C'(x, y) := \sum_{c \in C'} x^{n-\text{wt}(c)} y^{\text{wt}(c)}$$

be the generating function for the weight distribution of $C'$. Using discrete Poisson inversion as in the proof of the MacWilliams identity and the characterization (15) of $C'$ we find as in [CS2]

$$W_C'(x, y) = 2^{-n/2} \sum_{c \in C} (-1)^{\text{wt}(c)/2} (x+y)^{n-\text{wt}(c)} (x-y)^{\text{wt}(c)}$$

$$= 2^{-n/2} W_C(x+y, i(x-y)).$$

Thus from $W_C(x, y) = P_W(x^2 + y^2, x^8 + 14x^4y^4 + y^8)$ we obtain

$$W_C'(x, y) = P_W(2xy, x^8 + 14x^4y^4 + y^8).$$

Note that all the words in the shadow thus have weight congruent to $n/2 \mod 4$. We could have also obtained this directly from the MacWilliams identity

$$W_C(x, y) = 2^{-n/2} W_C(x+y, x-y),$$

(which also underlies Gleason’s theorem); this would more closely parallel the analytic proof of $|w|^2 \equiv n \mod 8$ in [El].

If $c \in C$ has weight 2 then every codeword either contains or is disjoint from $c$. Thus $C$ decomposes as a direct sum of a double repetition code $z$ generated by $c$ and the self-dual code of length $n-2$ consisting of codewords disjoint from $c$. Iterating this we decompose $C$ as $C_0 \oplus z^r$, where $r$ is the number of weight-2 words in $C$, and $C_0$ is a self-dual code of length $n-2r$ with no words of weight 2. Now for any self-dual codes $C_1, C_2$, their direct sum $C_1 \oplus C_2$ has shadow

$$(C_1 \oplus C_2)' = C_1' \oplus C_2'.$$
Since $C'$ contains $w + 1^n$ whenever it contains $w$ it is clear that the minimal weight of $C'$ cannot exceed the value $n/2$ attained by $z^n/2$. This is much easier than proving the corresponding fact for characteristic vectors of unimodular lattices, but it does not show that $z^n/2$ is the only self-dual code whose shadow has minimal weight $n/2$. We prove this, as we did for lattices, by noting that such a code $C$ must have $W_C(x, y) = (2xy)^{n/2}$, whence $W_C(x, y) = (x^2 + y^2)^{n/2}$. Since $C$ contains $n/2$ words of weight 2, then, it can only be $z^n/2$.

We have shown that the shadow of a binary self-dual code $C$ other than $z^{n/2}$ contains some words of weight $< n/2$. Thus $C'$ has minimal weight at most $(n-8)/2$. We next characterize all $C$ attaining this bound. If $C = C_0 \oplus z^r$ then $C$ attains the bound if and only if $C_0$ does, so we need only consider codes without weight-2 words.

**Theorem 1A.** Let $C$ be a binary self-dual code of length $n$ with no codewords of weight 2. Then:

i) $C$ has at least $n(22 - n)/8$ codewords of weight 4.

ii) Equality holds if and only if the shadow of $C$ contains no codewords of weight $< (n-8)/2$.

iii) In that case the number of codewords of weight exactly $(n-8)/2$ in the shadow is $2^{(n-14)/2}n$.

**Proof:** We can mimic the proof of Theorem 1. If the minimal weight of $C'$ is at least $(n-8)/2$ then $W_{C'}^z$ is a linear combination of $(xy)^{n/2}$ and $(xy)^{(n-8)/2}(x^8 + 14x^4y^4 + y^8)$, and thus $W_C(x, y)$ is a linear combination of $(x^2 + y^2)^{n/2}$ and $(x^2 + y^2)^{(n-8)/2}(x^8 + 14x^4y^4 + y^8)$. The condition that $C$ have no weight-2 codewords then forces

$$W_C(x, y) = (x^2 + y^2)^{n/2} - \frac{n}{8}(x^2 + y^2)^{(n-8)/2} ((x^2 + y^2)^4 - (x^8 + 14x^4y^4 + y^8)) = x^n + 0x^{n-2}y^2 + n(22 - n)\frac{8}{8}x^{n-4}y^4 + \ldots \text{ (21)}$$

and

$$W_C(x, y) = (2xy)^{n/2} - \frac{n}{8}(2xy)^{(n-8)/2} [(2xy)^4 - (x^8 + 14x^4y^4 + y^8)] = 2^{(n-14)/2}(xy)^{(n-8)/2} [nix^8 + (128 - 2n)x^4y^4 + ny^8]. \text{ (22)}$$

If $n < 24$ and $C$ is any binary self-dual code of length $n$ containing no words of weight 2 and $n(22 - n)/8 + d$ words of weight 4 then its weight enumerator exceeds (21) by

$$\frac{d}{16}(x^2 + y^2)^{(n-16)/2} [x^8 + 14x^4y^4 + y^8 - (x^2 + y^2)^4]^2, \text{ (23)}$$

so the weight enumerator of the shadow $C'$ exceeds (22) by

$$\frac{d}{16}(2xy)^{(n-16)/2}(x^8 - 2x^4y^4 + y^8)^2. \text{ (24)}$$

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Thus $C'$ contains $2^{(n-24)/2}d$ words of weight $(n-16)/2$, from which we find that $d$ is a nonnegative multiple of $2^{(24-n)/2}$.

Alternatively we could deduce Theorem 1A from Theorem 1 via Construction A [CS1, Ch.7, §2]. Recall that this construction associates to a self-dual code $C \subset F^n$ the unimodular integral lattice

$$L_C := \{2^{-1/2}v \mid v \in \mathbb{Z}^n, v \mod 2 \in C\}. \quad (25)$$

The theta series of this lattice is given by

$$\theta_{L_C}(t) = W_C(\theta_{2^1}(2t), \theta_{2^2}(2t)); \quad (26)$$

in particular $L_C$ has no vectors of norm 1 if and only if $C$ has no codewords of weight 2 (NB $L_z \cong \mathbb{Z}^2$), and the $N_2(L_C) - 2n$ is $2^4$ times the number of weight-4 codewords in $C$. Moreover the set of characteristic vectors of $L_C$ is

$$\{2^{1/2}v \mid v \in \mathbb{Z}^n, v \mod 2 \in C'\} \quad (27)$$

(in effect the shadow of $L_C$ is obtained by applying Construction A to the shadow of $C$), so the norm of the shortest characteristic vectors is half the minimal weight of $C'$. Applying Theorem 1 to $L_C$ thus yields Theorem 1A immediately. \(\square\)

Thus also the codes $C$ of parts (ii), (iii) of Theorem 1A are precisely those for which $L_C$ is one of the 14 lattices listed in connection with Theorem 1. Of course not every such lattice arises because $n$ must be even; moreover, the root system can only involve $A_1$, $D_{2m}$, $E_7$ and $E_8$ if the lattice arises from construction A. This leaves only the seven lattices with root systems $E_8$, $D_{12}$, $E_7^2$, $D_8^2$, $D_6^4$, $D_4^4$, $A_7^2$ of rank 8, 12, 14, 16, 18, 20, 22 respectively. It turns out that each of those lattices arises as $L_C$ for a unique code $C$ [PS]. For instance the first of these arises from the extended Hamming code, and the last from what might be called the shorter binary Golay code; these are the shortest self-dual binary codes having minimal weight 4 and 6 respectively. Again it so happens that whenever there is a self-dual code of length $n < 24$ with minimal weight at least 4, there is such a code (this time unique) with only $n(22-n)/8$ words of weight 4, so Conway’s description of our fourteen lattices also applies mutatis mutandis to our seven codes.

Can we go past $n - 8$?

Our results suggest the following questions:

For any $k > 0$ is there $N_k$ such that every integral unimodular lattice all of whose characteristic vectors have norm $\geq n - 8k$ is of the form $L_0 \oplus \mathbb{Z}^r$ for some lattice $L_0$ of rank at most $N_k$?
For any $k > 0$ is there $n_k$ such that every binary self-dual code of whose shadow has minimal norm $\geq (n - 8k)/2$ is of the form $C_0 \oplus z^r$ for some code $C_0$ of length at most $n_k$?

Of course a positive answer for lattices would imply one for codes, and vice versa for a negative answer, via Construction A, with $n_k \leq N_k$ in the former case. Even $k = 2$ seems difficult.

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References


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