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THE UNSOLVABILITY OF THE GÖDEL CLASS WITH IDENTITY

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The *Gödel class with identity* (GCI) is the class of closed, prenex quantificational formulas whose prefixes have the form $\forall\forall\exists\cdots\exists$ and whose matrices contain arbitrary predicate letters and the identity sign "=", but do not contain function signs or individual constants. The $\forall\forall\exists\cdots\exists$ class without identity was shown solvable over fifty years ago ([4], [12], [17]); slightly later, that class was shown to possess the stronger property of finite controllability ([5], [18]). (A class of formulas is solvable iff it is decidable for satisfiability; it is finitely controllable iff every satisfiable formula in it has a finite model.) At the end of [5], Gödel claims that the finite controllability of the GCI can be shown "by the same method" as he employed to show this for the class without identity. This claim has been questioned for nearly twenty years; in §1 below we give a brief history of investigations into it. The major result of this paper shows Gödel to have been mistaken: the GCI is unsolvable. §2 contains the basic construction, which yields a satisfiable formula in the GCI that lacks finite models. This formula may easily be exploited to encode undecidable problems into the GCI.

The *minimal Gödel class with identity* (MGCI) is the class of GCI formulas that contain one existential quantifier, i.e., the $\forall\forall\exists$ class with identity. In §3 the basic construction is elaborated to obtain the unsolvability of the MGCI. This settles the decision problem for all prefix classes of quantification theory with identity, given the following older results: the $\forall\exists\forall$ class is unsolvable, even without identity [11]; the $\exists\cdots\exists\forall\cdots\forall$ class and the $\exists\cdots\exists\forall\exists\cdots\exists$ class with identity are solvable ([16], [1]). Thus a prefix class is unsolvable iff it allows at least two universal quantifiers, at least one of which governs an existential quantifier. This dividing line differs from that in pure quantification theory, that is, quantification theory without identity. For here the $\exists\cdots\exists\forall\forall\exists\cdots\exists$ class is solvable (this is an easy consequence of the solvability of the $\forall\forall\exists\cdots\exists$ class), so that the minimal unsolvable classes are $\forall\forall\forall\exists$ [19] and $\forall\exists\forall$.

A class C of formulas is said to be *conservative* iff there is an effective mapping φ from the class of all quantificational formulas to C such that, for every F, F is satisfiable iff $\varphi(F)$ is satisfiable, and F is finitely satisfiable iff $\varphi(F)$ is finitely satisfiable. If C is conservative then the decision problem for C has maximum degree of unsolvability; moreover, C is also undecidable for finite satisfiability, and the class of formulas in C that have finite models is recursively inseparable from the class of

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formulas in C that are unsatisfiable [21]. In §4 the basic construction is further refined to show that the MGCI is conservative. Thus for quantification theory with identity, as for pure quantification theory, every unsolvable prefix class is conservative.

In §5 it is shown that the reduction of §4 can in fact be carried out with MGCI formulas whose nonlogical vocabulary contains, aside from monadic predicate letters, only one dyadic predicate letter. Thus the class of such formulas is conservative. This result, together with two easy consequences of it, settles the decision problem for all classes of formulas specified by prefix and similarity type. Details are given in §6.

§1. Background. Gödel's claim regarding the GCI seems to have been entirely ignored for over thirty years. Through the 1950s, there is no mention of the GCI or of the claim in the literature. In the early 1960s, Burton Dreben began to investigate the claim, and could not see how to prove it; Stål Aanderaa, at that time a student of Dreben's, devised several examples that exhibited prima facie difficulties in extending Gödel's method for the class without identity to the GCI. Dreben wrote to Gödel on May 24, 1966, asking for substantiation of the claim and presenting Aanderaa's examples. In a letter of July 19, 1966, Gödel replied that he could not recall the details, but he did remember the extension of his method as involving "no difficulty". Throughout the late 1960s, Dreben urged that the decision problem for the GCI be deemed open; by the early 1970s this view became widely accepted.

Also by the early 1970s, the nature of the difficulty with the GCI had been located. Let F be a formula in the GCI, and let $\mathfrak{A}$ be any model for F. A *distinguished element of level* 1 is an element of $\mathfrak{A}$ that is the sole exemplar of some property expressible using the predicate letters of F. For example, if F implies $\forall x \forall y(Zx \wedge Zy \rightarrow x = y) \wedge \exists w Zw$, where Z is a monadic predicate letter, then every model for F contains a unique element of which Z is true; this element is a distinguished element of level 1. For $k \geq 1$, a *distinguished element of level* $k + 1$ is an element of $\mathfrak{A}$ not among the distinguished elements of levels $\leq k$ that is the sole element bearing some particular relation (expressible using the predicate letters of F) to the distinguished elements of levels $\leq k$. Each of the known finite controllability proofs for the Gödel class without identity, including Gödel's own, can be adapted to yield the following:

> If F has a model $\mathfrak{A}$ that, for some k, contains no distinguished elements of level k, then F has a finite model.

This was shown, independently, by Dreben and Goldfarb, by Gurevich, and by Schütte in the early 1970s. A proof can be found in [3, p. 253]. Throughout the 1970s, many researchers sought to show the GCI finitely controllable by providing a bound on the levels of distinguished elements that a GCI formula could require. However, in 1979 the author showed that no primitive recursive function provides such a bound and, consequently, there is no primitive recursive decision procedure for the GCI, or even for the MGCI [6]. This made it clear that far more than Gödel's method would have to be used for the GCI, if it were to have a positive solution. Even so, many of those concerned with the class—including the author—were inclined to believe that it would turn out to be finitely controllable. In particular, the technique of [6] for obtaining GCI formulas that demand distinguished elements of large levels k cannot be extended to yield a GCI formula that demand distinguished elements of every level. Moreover, hopes for a positive solution were nourished when, in 1980, Gurevich, Shelah, and the author showed a subclass of the MGCI to be finitely controllable by a method that does not yield a primitive recursive decision procedure [7].

A brief look at the problems encountered in generating distinguished elements may help explain this misguided optimism, as well as aid in the understanding of the construction of §2. Imagine that we have shown that in any model $\mathfrak{A}$ for a GCI formula F there are distinguished elements of certain levels—let us call them $\mathbf{0}$, $\mathbf{1}, \ldots, \mathbf{k}$—and we wish to insure the existence of a unique element that bears a relation S to $\mathbf{k}$. This element will then be a distinguished element of the next higher level. It is trivial to obtain the existence of at least one element that bears S to $\mathbf{k}$. Uniqueness would follow if F could be made to imply $\forall x \forall y \forall z(Sxy \wedge Szy \rightarrow x = z)$; but since this requires three universal quantifiers, it outstrips the means allowed in GCI formulas. In a sense, the problem is to find a way of using existential quantifiers to capture a sufficient amount of the power of a third universal quantifier.

Now F can be made to imply

(i) $$\forall x \forall y[x \neq y \rightarrow \exists z(Sxz \wedge \neg Syz)].$$

If we also have

(ii) if an element bears S to $\mathbf{k}$ then it bears S to nothing else,

then uniqueness is forthcoming. For suppose a bears S to $\mathbf{k}$ and $b \neq a$. If in (i) x and y take the values a and b, then by (ii) the existential variable z must take the value $\mathbf{k}$, and we obtain $\neg Sb\mathbf{k}$. Thus only a bears S to $\mathbf{k}$. Now (ii) would follow if F could be made to imply $\forall x \forall y \forall z(Sxy \wedge Sxz \rightarrow y = z)$, but again this requires three universal quantifiers. In [6], (ii) is obtained by having F imply something of the form $\forall x \forall y(Sxy \rightarrow \exists z(\cdots)]$ such that, if x and y take values a and c, where a bears S to c and also bears S to $\mathbf{k}$, then the existential variable z must take a value among $\mathbf{0}, \ldots, \mathbf{k} - \mathbf{1}$; and this in turn can be used to force c to be identical with $\mathbf{k}$. However, such a strategy works only up to a point: for sufficiently large k, the existential variable cannot be required to take a value among the earlier distinguished elements. This limitation lent some plausibility to the belief that the GCI is finitely controllable.

The construction of §2 rests on a somewhat different strategy. To obtain (ii), F is made to imply a formula $\forall x \forall y(Sxy \rightarrow \exists z(\cdots)]$ in which the existential variable does not take as value a distinguished element of lower level. In fact, in some models its value need not be distinguished at all. However, its value can be required to bear certain relations to distinguished elements of lower levels, and this turns out to be enough. Further explanation at this point would be uninformative; let us now turn to the construction itself.

§2. The basic construction. The bulk of this section is devoted to the construction of a GCI formula F that is satisfiable but has no finite models. As we shall see, once F is at hand it will be a simple matter to encode an undecidable problem into the GCI. The formula F contains the monadic predicate letter Z and the dyadic letters S, P_1, P_2, Q, N, R_1, and R_2. F is designed so that, in every model $\mathfrak{A}$ for F, there will be a unique element $\mathbf{0}$ of which Z is true, a unique element $\mathbf{1}$ that bears S to $\mathbf{0}$, a unique element $\mathbf{2}$ that bears S to $\mathbf{1}$, and so on ad infinitum. Thus Z acts as the predicate "is zero", and S as the successor relation. The other letters are used to insure the existence of such $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$, and are meant to act as follows. Elements of $\mathfrak{A}$ can be taken to represent pairs of integers. Suppose b represents $\langle p, q\rangle$; then P_1 holds between b and the element $\mathbf{p}$, P_2 between b and $\mathbf{q}$, Q between b and $\mathbf{q} + \mathbf{1}$, N between b and an element that represents $\langle p + 1, q\rangle$, R_1 between b and any element that represents $\langle q + 1, r\rangle$ for some r, and R_2 between b and any element that represents $\langle r, q + 1\rangle$ for some r.

Let F be a prenex form of $\forall x \forall y \exists z_0 H$, where H is the conjunction of the following eleven clauses:

(1) $Zx \wedge Zy \rightarrow x = y$,

(2) $Zz_0 \wedge \neg Sz_0x \wedge \bigwedge_{\delta=1,2} (P_\delta xz_0 \wedge P_\delta xy \rightarrow y = z_0)$,

(3) $\exists z Szx$,

(4) $\neg Zx \wedge x \neq y \rightarrow \exists z(Sxz \wedge \neg Syz)$,

(5) $\exists z[Nxz \wedge (Qxy \rightarrow Qzy) \wedge (R_1xy \rightarrow R_1zy) \wedge (R_2xy \rightarrow R_2zy)]$,

(6) $Nxy \rightarrow \exists z(P_2xz \wedge P_2yz)$,

(7) $Nxy \rightarrow \exists w \exists u(P_1xw \wedge Suw \wedge P_1yu)$,

(8) $Sxy \rightarrow \exists z(Qzx \wedge P_2zy \wedge P_1zz_0)$,

(9) $Qxy \rightarrow \exists z(P_1xz \wedge (Syz \rightarrow P_2xz))$,

(10) $\bigwedge_{\delta=1,2} [P_\delta xy \wedge \neg Zy \rightarrow \exists z \exists w(R_\delta zx \wedge P_2zw \wedge P_1zz_0 \wedge Syw)]$,

(11) $\bigwedge_{\delta=1,2} [R_\delta xy \rightarrow \exists z \exists w(P_1xz \wedge Swz \wedge (P_\delta yw \rightarrow P_2xz))]$.

LEMMA 1. *F is satisfiable.*

PROOF. Let $\pi: \mathbf{N}^2 \rightarrow \mathbf{N}$ be a bijective pairing function. Interpret the predicate letters of F over $\mathbf{N}$ as indicated two paragraphs back, where $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ are identified with $0, 1, 2, \ldots$ and an integer k is taken to represent $\langle p, q\rangle$ iff $k = \pi(p, q)$. These interpretations yield a model for F with universe $\mathbf{N}$. □

LEMMA 2. *F has no finite models.*

PROOF. Let $\mathfrak{A}$ be any model for F. We shall find distinct elements $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ of $\mathfrak{A}$ such that, for each integer p,

(A) for all c in $\mathfrak{A}$, Zc iff $c = \mathbf{0}$;

(B) for all c in $\mathfrak{A}$, $S\mathbf{p}c$ iff $p > 0$ and $c = \mathbf{p} - \mathbf{1}$;

(C) for all c in $\mathfrak{A}$, if $p > 0$ and $Sc\mathbf{p} - \mathbf{1}$ then $c = \mathbf{p}$; and

(D) for $\delta = 1, 2$ and all c, b in $\mathfrak{A}$, if $P_\delta c\mathbf{p}$ and $P_\delta cb$ then $b = \mathbf{p}$.

(An expression like "$P_\delta cb$" is short for "$\mathfrak{A} \models P_\delta cb$".)

By clauses (1) and (2) of F, there is a unique $\mathbf{0}$ in $\mathfrak{A}$ such that $Z\mathbf{0}$. Since the variable z_0 of F must always take $\mathbf{0}$ as its value, clause (2) of F yields (B)–(D) for $p = 0$.

As induction hypothesis, suppose $\mathbf{0}, \ldots, \mathbf{k}$ are distinct elements of $\mathfrak{A}$ obeying (A)–(D) for each $p \leq k$.

SUBLEMMA 1. *Let $c, d \in \mathfrak{A}$ and suppose Ncd. For each $p \leq k$, if $P_1c\mathbf{p} - \mathbf{1}$ then $P_1d\mathbf{p}$, and if $P_2d\mathbf{p}$ then $P_2c\mathbf{p}$.*

PROOF. Since Ncd, by clause (7) there exist a and b in $\mathfrak{A}$ with $P_1ca \wedge Sba \wedge P_1db$. If $P_1c\mathbf{p} - \mathbf{1}$, where $p \leq k$, then $a = \mathbf{p} - \mathbf{1}$ by (D), whence $b = \mathbf{p}$ by (C). Hence $P_1d\mathbf{p}$. Also, by clause (6), there exists e in $\mathfrak{A}$ such that $P_2ce \wedge P_2de$. If $P_2d\mathbf{p}$, where $p \leq k$, then by (D) $e = \mathbf{p}$, so that $P_2c\mathbf{p}$. □

SUBLEMMA 2. *Let $a, b \in \mathfrak{A}$ and suppose $Sa\mathbf{k}$ and Sab. Then $b = \mathbf{k}$.*

PROOF. Since Sab, by clause (8) there exists c_0 in $\mathfrak{A}$ with $Qc_0a \wedge P_2c_0b \wedge P_1c_0\mathbf{0}$. Iterated use of clause (5) yields the existence of $c_1,\ldots,c_k$ in $\mathfrak{A}$ such that Nc_ic_{i+1} for each i, $0 \leq i < k$, and Qc_ia for each i, $0 \leq i \leq k$. Since $P_1c_0\mathbf{0}$, iterated application of Sublemma 1 yields $P_1c_k\mathbf{k}$. By clause (9), there exists d in $\mathfrak{A}$ with $P_1c_kd \wedge (Sad \rightarrow P_2c_kd)$. By (D), $d = \mathbf{k}$. Since $Sa\mathbf{k}$, $P_2c_k\mathbf{k}$. Iterated application of Sublemma 1 yields $P_2c_0\mathbf{k}$. But P_2c_0b; by (D), $b = \mathbf{k}$. □

SUBLEMMA 3. *There is a unique a in $\mathfrak{A}$ such that $Sa\mathbf{k}$.*

PROOF. By clause (3) there is at least one a in $\mathfrak{A}$ with $Sa\mathbf{k}$. By (A) and (B), $\neg Za$. Let $b \in \mathfrak{A}$, $b \neq a$. By clause (4) there exists c in $\mathfrak{A}$ with $Sac \wedge \neg Sbc$. By Sublemma 2, $c = \mathbf{k}$. Thus $\neg Sb\mathbf{k}$. □

Now let $\mathbf{k+1}$ be the unique a such that $Sa\mathbf{k}$. By (B), $\mathbf{k+1}$ is distinct from $\mathbf{0}$, $\mathbf{1},\ldots,\mathbf{k}$.

SUBLEMMA 4. *Let $\delta = 1$ or 2, and let $c, b \in \mathfrak{A}$. Suppose $P_\delta c\mathbf{k+1}$ and $P_\delta cb$. Then $b = \mathbf{k+1}$.*

PROOF. By (A) and (D), $\neg Zb$. Hence by clause (10) there exist c_0, d in $\mathfrak{A}$ such that $R_\delta c_0c \wedge P_2c_0d \wedge P_1c_0\mathbf{0} \wedge Sbd$. Iterated use of clause (5) yields the existence of $c_1,\ldots,c_k$ in $\mathfrak{A}$ such that Nc_ic_{i+1} for each i, $0 \leq i < k$, and $R_\delta c_ic$ for each i, $0 \leq i \leq k$. Since $P_1c_0\mathbf{0}$, by Sublemma 1 we may infer $P_1c_k\mathbf{k}$. By clause (11) there exist e, e' in $\mathfrak{A}$ such that $P_1c_ke \wedge Se'e \wedge (P_\delta ce' \rightarrow P_2c_ke)$. By (D), $e = \mathbf{k}$. Thus $e' = \mathbf{k+1}$, so that $P_\delta ce'$. Hence $P_2c_k\mathbf{k}$. By Sublemma 1, $P_2c_0\mathbf{k}$. But P_2c_0d; hence, by (D), $d = \mathbf{k}$. Thus $Sb\mathbf{k}$, so $b = \mathbf{k+1}$. □

Sublemmas 2–4 show that (A)–(D) hold for all $p \leq k + 1$. Thus, by induction, there is an infinite sequence of distinct elements of $\mathfrak{A}$. □

To obtain unsolvability, we exploit the fact that every model for F contains an ω-sequence of elements on which S acts as the successor relation.

THEOREM 1. *The Gödel class with identity is unsolvable.*

PROOF. Let $J = \forall x\exists u\forall yK(x,u,y)$ be any $\forall\exists\forall$-formula of pure quantification theory; there is no loss of generality in supposing that the predicate letters of J are distinct from those of F. We construct a formula in the GCI that is satisfiable just in case J is satisfiable. Since the $\forall\exists\forall$ class of pure quantification theory is unsolvable, this yields the theorem.

Herbrand's theorem implies that J is satisfiable iff there is an interpretation of its predicate letters over $\mathbf{N}$ such that $K(p, p+1, q)$ is true for all integers p and q. Now let J' be a prenex equivalent of $F \wedge \forall x\forall y\exists u(Sux \wedge K(x,u,y))$ that is in the GCI. If J is satisfiable, then a model for J' can be obtained by adjoining, to the model for F given in the proof of Lemma 1 above, interpretations of the predicate letters of J over $\mathbf{N}$ such that $K(p, p+1, q)$ is true for all p and q. Conversely, if J' has a model $\mathfrak{A}$, then, since J' implies F, there are distinct elements $\mathbf{0},\mathbf{1},\mathbf{2},\ldots$ of $\mathfrak{A}$ that obey (A)–(D) for each integer p. And then, for all integers p and q, $K(\mathbf{p},\mathbf{p+1},\mathbf{q})$ is true in $\mathfrak{A}$. Thus the restriction of $\mathfrak{A}$ to $\{\mathbf{0},\mathbf{1},\mathbf{2},\ldots\}$ is a model for J. □

§3. Minimal Gödel class with identity. The construction of §1 may be refined so as to use only one existential quantifier. As before, every model for the formula we construct will contain elements $\mathbf{0},\mathbf{1},\mathbf{2},\ldots$ such that $\mathbf{0}$ is the unique element of which Z is true and, for each k, $\mathbf{k+1}$ is the unique element that bears S to $\mathbf{k}$. Additional monadic predicate letters B_1, B_2 and dyadic predicate letters L_1, L_2 will be used: $B_\delta c$ is to hold iff $P_\delta c\mathbf{0}$ holds, and $L_\delta c\mathbf{p}$ is to hold iff $P_\delta c\mathbf{p+1}$ holds, $\delta = 1, 2$. These new predicate letters enable us to eliminate the nested existential quantifiers used in §2.

Moreover, the elements **0, 1, 2**, ... are now going to be distinct from the elements that represent pairs. A new monadic predicate letter I will be true of the former elements and false of the latter. The last new predicate letter used is monadic D, true of an element only if it represents a pair $\langle p, p\rangle$.

Let $G = \forall x \forall y \exists z H$, where H is the conjunction of the following seventeen clauses:

(1) $Zx \wedge Zy \to x = y$,
(2) $Zx \to Ix \wedge \bigwedge_{\delta=1,2}(B_\delta y \equiv P_\delta yx)$,
(3) $\bigwedge_{\delta=1,2}(B_\delta x \wedge P_\delta xy \to Zy)$,
(4) $(Sxy \to \neg Zx \wedge Ix \wedge Iy) \wedge (L_1xy \to \neg Ix \wedge \neg B_1x)$,
(5) $Dx \to (P_1xy \equiv P_2xy)$,
(6) $x = y \wedge \neg Zx \to Zz$,
(7) $(Zx \wedge Iy \to Szy) \wedge (Zx \wedge \neg Iy \to P_1yz)$,
(8) $Ix \wedge \neg Zx \wedge Iy \wedge \neg Sxy \wedge x \neq y \to Sxz \wedge \neg Syz$,
(9) $\bigwedge_{\delta=1,2}(P_\delta yx \wedge \neg Zx \to Sxz \wedge L_\delta yz)$,
(10) $Nxy \to P_2xz \wedge P_2yz$,
(11) $Nyx \to P_1yz \wedge L_1xz$,
(12) $Sxy \to Qzx \wedge P_2zy \wedge B_1z$,
(13) $Qxy \wedge \neg Dx \to Nxz \wedge Qzy$,
(14) $Qyx \to P_1yz \wedge (Sxz \to P_2yz)$,
(15) $\bigwedge_{\delta=1,2}(L_\delta xy \to R_\delta zx \wedge P_2zy \wedge B_1z)$,
(16) $\bigwedge_{\delta=1,2}(R_\delta xy \wedge \neg Dx \to Nxz \wedge R_\delta zy)$,
(17) $\bigwedge_{\delta=1,2}(R_\delta yx \to P_1yz \wedge (L_\delta xz \to P_2yz))$.

LEMMA 1. *G is satisfiable.*

PROOF. Let the universe be $\mathbf{N} \cup \mathbf{N}^2$, and let π_1 and π_2 be the projection mappings on $\mathbf{N}^2$. Interpret the predicate letters of G over the universe so that, for $\delta = 1, 2$ and all a, b in the universe: Za iff $a = 0$, Ia iff $a \in \mathbf{N}$; $B_\delta a$ iff $a \in \mathbf{N}^2$ and $\pi_\delta a = 0$; Da iff $a \in \mathbf{N}^2$ and $\pi_1 a = \pi_2 a$; Sab iff $a, b \in \mathbf{N}$ and $a = b + 1$; $P_\delta ab$ iff $a \in \mathbf{N}^2$ and $\pi_\delta a = b$; $L_\delta ab$ iff $a \in \mathbf{N}^2$ and $\pi_\delta a = b + 1$; Qab iff $b \in \mathbf{N}$, $b > 0$, and $a = \langle p, b-1\rangle$ for some $p \leq b - 1$; Nab iff $a = \langle p, q\rangle$ and $b = \langle p+1, q\rangle$ for some integers p and q; R_1ab iff $a = \langle p, q\rangle$ and $b = \langle q+1, s\rangle$ for some integers p, q, s with $p \leq q$; and R_2ab iff $a = \langle p, q\rangle$ and $b = \langle s, q+1\rangle$ for some integers p, q, s with $p \leq q$. These interpretations yield a model for G. Indeed, define a two-place function φ on the universe thus:

$$\varphi(a,b) = \begin{cases} 0 & \text{if } a = b \neq 0, \\ b+1 & \text{if } a = 0 \text{ and } b \in \mathbf{N}, \\ a-1 & \text{if } a, b \in \mathbf{N},\ a \neq 0,\ a \neq b,\ a \neq b+1, \\ a-1 & \text{if } a \neq 0 \text{ and either } P_1ba \text{ or } P_2ba, \\ \pi_2 a & \text{if } Nab, \\ \pi_1 b & \text{if } Nba,\ Qba,\ R_1ba, \text{ or } R_2ba, \text{ or if } a = 0 \text{ and } b \in \mathbf{N}^2, \\ \langle 0, b\rangle & \text{if } Sab \text{ or } L_1ab \text{ or } L_2ab, \\ \langle \pi_1 a + 1, \pi_2 a\rangle & \text{if } \neg Da \text{ and either } Qab,\ R_1ab, \text{ or } R_2ab, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

It is a routine matter to check that this is a proper definition (that is, its clauses do not conflict) and that φ is a Skolem function for the existential variable of G (that is, $H[a, b, \varphi(a, b)]$ is true for all a and b in the universe under the interpretations of the predicate letters given above). □

LEMMA 2. *G has no finite models.*

PROOF. Let $\mathfrak{A}$ be any model for G. By clauses (1) and (6) of G, there is a unique element $\mathbf{0}$ of $\mathfrak{A}$ such that $Z\mathbf{0}$. By clauses (2) and (4), $I\mathbf{0}$ and $\neg S\mathbf{0}c$ for each c in $\mathfrak{A}$. For $\delta = 1, 2$ and any c, b in $\mathfrak{A}$, clauses (2) and (3) yield $(B_\delta c \equiv P_\delta c\mathbf{0}) \wedge (B_\delta c \wedge P_\delta cb \to Zb)$; hence if $P_\delta c\mathbf{0}$ and $P_\delta cb$ then $b = \mathbf{0}$.

Now suppose $\mathbf{0}, \ldots, \mathbf{k}$ are distinct elements of $\mathfrak{A}$ such that, for each $p \leq k$,

(A) for all c in $\mathfrak{A}$, Zc iff $c = \mathbf{0}$;

(B) for all c in $\mathfrak{A}$, $S\mathbf{p}c$ iff $p > 0$ and $c = \mathbf{p} - \mathbf{1}$;

(C) for all c in $\mathfrak{A}$, if $p > 0$ and $Sc\mathbf{p} - \mathbf{1}$ then $c = \mathbf{p}$;

(D) for $\delta = 1, 2$ and all c, b in $\mathfrak{A}$, if $P_\delta c\mathbf{p}$ and $\mathbf{P}_\delta cb$ then $b = \mathbf{p}$; and

(E) for all c in $\mathfrak{A}$, if $p > 0$ and $L_1 c\mathbf{p} - \mathbf{1}$ then $P_1 c\mathbf{p}$.

SUBLEMMA 1. *Let $c, d \in \mathfrak{A}$ and suppose Ncb. For each $p \leq k$, if $P_1 c\mathbf{p} - \mathbf{1}$ then $P_1 d\mathbf{p}$, and if $P_2 d\mathbf{p}$ then $P_2 c\mathbf{p}$.*

PROOF. Since Ncd, by clause (11) there exists b in $\mathfrak{A}$ such that $P_1 cb \wedge L_1 db$. If $P_1 c\mathbf{p} - \mathbf{1}$, where $p \leq k$, then $b = \mathbf{p} - \mathbf{1}$ by (D), so that $P_1 d\mathbf{p}$ by (E). By clause (10) there exists e in $\mathfrak{A}$ such that $P_2 ce \wedge P_2 de$. If $P_2 d\mathbf{p}$, where $p \leq k$, then $e = \mathbf{p}$ by (D), so that $P_2 c\mathbf{p}$. □

SUBLEMMA 2. *Let $a, b \in \mathfrak{A}$ and suppose $Sa\mathbf{k}$ and Sab. Then $b = \mathbf{k}$.*

PROOF. Since Sab, by clause (12) there exists c_0 in $\mathfrak{A}$ with $Qc_0 a \wedge P_2 c_0 b \wedge B_1 c_0$. By clause (2), $P_1 c_0 \mathbf{0}$. Iterated use of clause (13) yields the existence of $c_1, \ldots, c_j$ such that $Nc_i c_{i+1}$ for each i, $0 \leq i < j$, and $Qc_i a$ for each i, $0 \leq i \leq j$, and either $j = k$ or else $j < k$ and Dc_j. In the latter case we have $P_1 c_j \mathbf{j}$ by iterated use of Sublemma 1; by clause (5), then, $P_2 c_j \mathbf{j}$, so that $P_2 c_0 \mathbf{j}$ by Sublemma 1. But $P_2 c_0 b$; by (D), then, $b = \mathbf{j}$, whence $a = \mathbf{j} + \mathbf{1}$ by (C), and this is impossible. Hence $j = k$. Then, by Sublemma 1, $P_1 c_k \mathbf{k}$. Since $Qc_k a$, by clause (14) there exists d in $\mathfrak{A}$ such that $P_1 c_k d \wedge (Sad \to P_2 c_k d)$. By (D), $d = \mathbf{k}$. Since $Sa\mathbf{k}$ by hypothesis, $P_2 c_k \mathbf{k}$. By Sublemma 1, $P_2 c_0 \mathbf{k}$. Since $P_2 c_0 b$, by (D) $b = \mathbf{k}$. □

SUBLEMMA 3. *There is a unique a in $\mathfrak{A}$ such that $Sa\mathbf{k}$.*

PROOF. By clause (7) there is at least one a in $\mathfrak{A}$ with $Sa\mathbf{k}$. Now suppose $a \neq b$, $Sa\mathbf{k}$, and $Sb\mathbf{k}$. By Sublemma 2 and (B), $\neg Sab$. By clause (4), $\neg Za \wedge Ia \wedge Ib$. Hence, by clause (8), there exists c in $\mathfrak{A}$ such that $Sac \wedge \neg Sbc$. By Sublemma 2, $c = \mathbf{k}$. Thus $\neg Sb\mathbf{k}$, contrary to hypothesis. □

Now let $\mathbf{k} + \mathbf{1}$ be the unique a such that $Sa\mathbf{k}$. By (B), $\mathbf{k} + \mathbf{1}$ is distinct from $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{k}$.

SUBLEMMA 4. *Let $\delta = 1$ or 2, and let $c, b \in \mathfrak{A}$. Suppose $L_\delta c\mathbf{k}$ and $L_\delta cb$. Then $b = \mathbf{k}$.*

PROOF. Since $L_\delta cb$, by clause (15) there exists c_0 in $\mathfrak{A}$ with $R_\delta c_0 c \wedge P_2 c_0 b \wedge B_1 c_0$. By clause (2), $P_1 c_0 \mathbf{0}$. Iterated use of clause (16) yields the existence of $c_1, \ldots, c_j$ such that $Nc_i c_{i+1}$ for $0 \leq i < j$, $R_\delta c_i c$ for $0 \leq i \leq j$, and either $j = k$ or else $j < k$ and Dc_j. By reasoning as in the proof of Sublemma 2, we may infer that the latter case is impossible; hence $j = k$. By Sublemma 1, $P_1 c_k \mathbf{k}$. Since $R_\delta c_k c$, by clause (17) there exists d in $\mathfrak{A}$ such that $P_1 c_k d \wedge (L_\delta cd \to P_2 c_k d)$. By (D), $d = \mathbf{k}$. Since $L_\delta c\mathbf{k}$ by hypothesis, $P_2 c_k \mathbf{k}$. By Sublemma 1, $P_2 c_0 \mathbf{k}$. Since $P_2 c_0 b$, by (D) $b = \mathbf{k}$. □

SUBLEMMA 5. *Let* $\delta = 1$ or 2, *and let* $c, b \in \mathfrak{A}$. *Suppose* $P_\delta c\mathbf{k+1}$ *and* $P_\delta cb$. *Then* $b = \mathbf{k+1}$.

PROOF. Since $\neg Z\mathbf{k+1}$, by clause (2) $\neg B_\delta c$, so that $\neg Zb$. Two uses of clause (9) yield the existence of d and e in $\mathfrak{A}$ with $S\mathbf{k+1}d \wedge L_\delta cd$ and $Sbe \wedge L_\delta ce$. By Sublemma 2, $d = \mathbf{k}$; by Sublemma 4, then, $e = \mathbf{k}$. Since $Sb\mathbf{k}$, $b = \mathbf{k+1}$. □

SUBLEMMA 6. *Let* $c \in \mathfrak{A}$ *and suppose* $L_1 c\mathbf{k}$. *Then* $P_1 c\mathbf{k+1}$.

PROOF. By clause (4), $\neg Ic \wedge \neg B_1 c$. By clause (7) there exists b in $\mathfrak{A}$ such that $P_1 cb$; by clause (2), $\neg Zb$. Hence, by clause (9) there exists d in $\mathfrak{A}$ such that $Sbd \wedge Q_1 cd$. By Sublemma 4, $d = \mathbf{k}$. Hence $b = \mathbf{k+1}$, so that $P_1 c\mathbf{k+1}$. □

Sublemmas 2–6 show that the induction hypotheses (A)–(E) hold for each $p \leq k + 1$. By induction, then, there is an infinite sequence of distinct elements of $\mathfrak{A}$. □

THEOREM 2. *The minimal Gödel class with identity is unsolvable.*

PROOF. Let $J = \forall x \exists u \forall y K(x, u, y)$ be a formula in the $\forall\exists\forall$-class of pure quantification theory, whose predicate letters are distinct from those in G. Let J' be obtained from G by conjoining the following two additional clauses to the matrix:

(18) $Ix \wedge \neg Zx \wedge Iy \wedge \neg Sxy \wedge x \neq y \rightarrow K(z, x, y)$,

(19) $Syx \rightarrow K(x, y, y) \wedge K(x, y, x)$.

It suffices to show that J' is satisfiable iff J is satisfiable.

Suppose J is satisfiable. To the interpretations of the predicate letters of G over the universe $\mathbf{N} \cup \mathbf{N}^2$ given in the proof of Lemma 1, adjoin interpretations of the predicate letters of J over $\mathbf{N}$ that make $K(p, p + 1, q)$ true for all p and q. Since, for all a and b in the universe, Sab is true iff $a, b \in \mathbf{N}$ and $a = b + 1$, (19) is true for all values of x and y. If the antecedent of (18) is true for values a and b of x and y, then $a, b \in \mathbf{N}$, $a > 0$, and, by clause (8), z takes the value $a - 1$. Hence the consequent of (18) is true. Thus we have obtained a model for J'.

Suppose J' is satisfiable; let $\mathfrak{A}$ be a model for it. Since J' implies G, there are elements $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ of $\mathfrak{A}$ that obey (A)–(E) for every integer p. By (B) and clause (4), $I\mathbf{p}$ for each p. Now for all integers p and q such that $q \neq p$ and $q \neq p + 1$, the antecedent of (18) holds when x has value $\mathbf{p+1}$ and y has value $\mathbf{q}$; by clause (8), in this case z has to take the value $\mathbf{p}$. Hence $K(\mathbf{p}, \mathbf{p+1}, \mathbf{q})$ is true in $\mathfrak{A}$. Moreover, when x has value $\mathbf{p}$ and y has value $\mathbf{p+1}$, then the antecedent of (19) holds, so that $K(\mathbf{p}, \mathbf{p+1}, \mathbf{p+1})$ and $K(\mathbf{p}, \mathbf{p+1}, \mathbf{p})$ are true in $\mathfrak{A}$. Thus $K(\mathbf{p}, \mathbf{p+1}, \mathbf{q})$ is true in $\mathfrak{A}$ for all integers p and q. We may conclude that the restriction of $\mathfrak{A}$ to $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$ is a model for J. □

§4. Conservativeness. Although the reduction just given of the $\forall\exists\forall$-class to the MGCI does not preserve finite satisfiability, it can be amended so as to do so. In fact, given an $\forall\exists\forall$-formula J, we may alter the construction of §3 thus: we introduce a monadic predicate letter W, along with new clauses that allow W to be true of an element n iff J has a model with universe $\{0, \ldots, n\}$; and we replace the clause $Zx \wedge Iy \rightarrow Szy$ of the formulas of §3 by $Zx \wedge Iy \wedge \neg Wy \rightarrow Szy$. Thus, if W holds of an element then that element need not have a successor. This will permit the MGCI formula to have a finite model.

In this section, however, we give a more intricate proof of conservativeness, so as to facilitate a further reduction—carried out in §5—to the class of MGCI formulas that contain only one dyadic predicate letter. The MGCI formulas we use in this proof all contain the same ten dyadic letters, whose intended interpretations are fixed. Nine of these letters were used in §3, namely, $S, P_1, P_2, L_1, L_2, N, Q, R_1$, and R_2. A new dyadic letter M is meant to hold between two elements of a model only if the first represents a pair $\langle p, q\rangle$ and the second a pair $\langle r, p\rangle$ for some p, q, and r. Another difference between the formulas below and those of §3 is this: the intended models for the formulas below contain three different elements that represent each pair $\langle p, q\rangle$; these elements will be identified with triples $\langle p, q, i\rangle$ for $i = 0, 1, 2$. A monadic letter E will be true of such a triple iff $i = 0$. We also use monadic letters Z, I, D, B_1, B_2 as in §3, and a monadic letter W with the role indicated above. Moreover, for every dyadic predicate letter Φ of the $\forall\exists\forall$-formula being reduced, we use two monadic letters A_Φ and A^*_Φ; given a model $\mathfrak{B}$ for that formula, if c represents a pair $\langle p, q\rangle$, then A_Φ is to be true of c iff $\mathfrak{B} \models \Phi pq$ and A^*_Φ is to be true of c iff $\mathfrak{B} \models \Phi qp$.

THEOREM 3. *The minimal Gödel class with identity is conservative.*

PROOF. Let $J = \forall x \exists u \forall y K(x, u, y)$ be an $\forall\exists\forall$-formula of pure quantification theory all of whose atomic subformulas have one of the forms Φxy, Φyx, Φuy, Φyu, where Φ is a dyadic predicate letter. The class of such formulas is conservative [22]. Hence it suffices to find an MGCI formula G_J that is satisfiable iff J is satisfiable, and that has a finite model iff J has a finite model.

Let $\mathbf{L}$ be the set of predicate letters of J, and let $K^*(v, w)$ be obtained from K by replacing atomic subformulas Φxy, Φyx, Φuy and Φyu by $A_\Phi v$, $A^*_\Phi v$, $A_\Phi w$ and $A^*_\Phi w$, respectively. Let H_J be

$$(Nxy \to K^*(x, y)) \wedge (Mxy \wedge Myx \to \bigwedge_{\Phi \in \mathbf{L}} (A_\Phi x \equiv A^*_\Phi y)).$$

Let H' be like the matrix of the formula G of §3, but for the following changes: clause (7) is replaced by the conjunction of

(7a) $Zx \wedge Iy \wedge \neg Wy \to Szy$,

(7b) $Zx \wedge Wy \to P_1 zx \wedge P_2 zy \wedge Ez$,

(7c) $Zx \wedge \neg Iy \to Nyz$;

clause (11) is replaced by

(11) $Nyx \to P_1 yz \wedge (\neg Wz \to L_1 yz) \wedge (Wz \to B_1 x)$;

and two new clauses are conjoined:

(18) $Ex \wedge Ey \wedge \neg Nxy \wedge \neg Nyx \to P_1 xz \wedge (P_2 yz \to Mxy)$;

(19) $(Nxy \to (Ex \equiv Ey)) \wedge (Wx \to \neg Zx) \wedge (Qxy \to Ex) \wedge (Ex \to \neg Ix)$.

Finally, let G_J be $\forall x \forall y \exists z (H' \wedge H_J)$.

LEMMA 1. *If J has a model then so does G_J.*

PROOF. Let $V = \mathbf{N} \times \mathbf{N} \times \{0, 1, 2\}$, and let the universe be $\mathbf{N} \cup V$. Let π_1, π_2, and π_3 be the projection functions on V. Interpret the predicate letters of H' so that, for $\delta = 1, 2$ and all a and b in the universe, Za iff $a = 0$; Ia iff $a \in \mathbf{N}$; $B_\delta a$ iff $a \in V$ and $\pi_\delta a = 0$; Da iff $a \in V$ and $\pi_1 a = \pi_2 a$; Ea iff $a \in V$ and $\pi_3 a = 0$; Sab iff $a, b \in \mathbf{N}$ and $a = b + 1$; $P_\delta ab$ iff $a \in V$ and $\pi_\delta a = b$; $L_\delta ab$ iff $a \in V$ and $\pi_\delta a = b + 1$; Qab iff $b \in \mathbf{N}$, $b > 0$, and $a = \langle p, b - 1, 0\rangle$ for some $p \leq b - 1$; Nab iff $a = \langle p, q, i\rangle$ and $b = \langle p + 1, q, i\rangle$ for some integers p, q, and i; $R_1 ab$ iff $a = \langle p, q, i\rangle$ and $b = \langle q + 1, s, j\rangle$ for some integers p, q, s with $p \leq q$ and some i and j with $i \equiv j + 1 \pmod 3$; $R_2 ab$ iff $a = \langle p, q, i\rangle$ and $b = \langle s, q + 1, j\rangle$ for some integers p, q, s with $p \leq q$ and some i and j with $i \equiv j + 1 \pmod 3$; and Mab iff $a = \langle p, q, 0\rangle$ and

$b = \langle s, p, 0\rangle$ for some integers p, q, and s. Moreover, interpret W to be true of no element.

These interpretations provide a model for $\forall x \forall y \exists z H'$. Indeed, a Skolem function φ for the existential variable z may be defined thus:

$$\varphi(a,b) = \begin{cases} 0 & \text{if } a = b \neq 0, \\ b + 1 & \text{if } a = 0 \text{ and } b \in \mathbf{N}, \\ \langle \pi_1 b + 1, \pi_2 b, \pi_3 b\rangle & \text{if } a = 0 \text{ and } b \in V, \\ a - 1 & \text{if } a, b \in \mathbf{N},\ a \neq 0,\ a \neq b,\ a \neq b + 1, \\ a - 1 & \text{if } a \neq 0 \text{ and either } P_1 ba \text{ or } P_2 ba, \\ \pi_2 a & \text{if } Nab, \\ \pi_1 a & \text{if } Nba,\ Qba,\ R_1 ba,\ R_2 ba, \text{ or } Ea \wedge Eb \wedge \\ & \neg Nab \wedge \neg Nba, \\ \langle 0, b, 0\rangle & \text{if } Sab, \\ \langle 0, b, i\rangle & \text{if } L_1 ab \text{ or } L_2 ab, \text{ where } i \equiv \pi_3 a + 1 \pmod 3, \\ \langle \pi_1 a + 1, \pi_2 a, \pi_3 a\rangle & \text{if } \neg Da \text{ and either } Qab,\ R_1 ab, \text{ or } R_2 ab, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Now suppose J is satisfiable. Then it has a model $\mathfrak{B}$ with universe $\mathbf{N}$ such that $\mathfrak{B} \models K(p, p + 1, q)$ for all integers p and q. Say that an element $a \in V$ represents $\langle p, q\rangle$ iff $\pi_1 a = p$ and $\pi_2 a = q$. For each $\Phi \in \mathbf{L}$, interpret the predicate letters A_Φ and A^*_Φ so that if a represents $\langle p, q\rangle$, then $A_\Phi a$ iff $\mathfrak{B} \models \Phi pq$ and $A^*_\Phi a$ iff $\mathfrak{B} \models \Phi qp$. We show that these interpretations provide a model for $\forall x \forall y H_J$. Suppose Nab. Then, for some p and q, a represents $\langle p, q\rangle$ and b represents $\langle p + 1, q\rangle$. Hence $A_\Phi a$ iff $\mathfrak{B} \models \Phi pq$, $A^*_\Phi a$ iff $\mathfrak{B} \models \Phi qp$, $A_\Phi b$ iff $\mathfrak{B} \models \Phi p + 1\ q$, and $A^*_\Phi b$ iff $\mathfrak{B} \models \Phi q\ p + 1$. Thus $K^*(a, b)$ is true. Now suppose $Mab \wedge Mba$. Then, for some p and q, a represents $\langle p, q\rangle$ and b represents $\langle q, p\rangle$. Hence $A_\Phi a$ iff $A^*_\Phi b$ for each $\Phi \in \mathbf{L}$.

Since G_J is equivalent to $\forall x \forall y \exists z H' \wedge \forall x \forall y H_J$, the interpretations we have given provide a model for G_J with universe $\mathbf{N} \cup V$. □

LEMMA 2. *If J has a finite model then so does G_J.*

PROOF. If J has a finite model then for some $n > 0$ it has a model $\mathfrak{B}$ with universe $\{0, \ldots, n\}$ such that $\mathfrak{B} \models K(p, p + 1, q)$ whenever $p + 1, q \leq n$ and $\mathfrak{B} \models K(n, 0, q)$ whenever $q \leq n$. (This elementary fact about $\forall\exists\forall$ formulas is proved, for instance, in [3, p. 130].) We construct a model for G_J with universe $\{0, \ldots, n\} \cup \{0, \ldots, n\} \times \{0, \ldots, n\} \times \{0, 1, 2\}$. Let the interpretations of all the predicate letters of H' except N and W be the restrictions to this universe of the interpretations given in the proof of Lemma 1; let Wa iff $a = n$; and let Nab iff either Nab is true under the interpretation of Lemma 1 or else $a = \langle n, q, i\rangle$ and $b = \langle 0, q, i\rangle$ for some q and i. Then $\forall x \forall y \exists z H'$ is true; indeed, the Skolem function given in the proof of Lemma 1 needs only to be restricted to arguments from the finite universe and altered at two points, namely, $\varphi(a, b) = \langle 0, b, 0\rangle$ if $a = 0$ and $b = n$, and $\varphi(a, b) = \langle 0, \pi_2 b, \pi_3 b\rangle$ if $a = 0$, b is a triple, and $\pi_1 b = n$.

Now define interpretations of the letters A_Φ and A^*_Φ from the model $\mathfrak{B}$ as in the proof of Lemma 1. The verification that these interpretations provide a model for $\forall x \forall y H_J$ proceeds as before, except that now we may have Nab when a represents

$\langle n, q \rangle$ and b represents $\langle 0, q \rangle$. But since $\mathfrak{B} \models K(n, 0, q)$, it follows as before that $K^*(a, b)$ is true. Thus we have defined a model for G_J whose universe is finite. □

Let (A)–(E) be the five conditions given in the proof of Lemma 3, §3.

LEMMA 3. *Let $\mathfrak{A}$ be any model for G_J. Then either* (I) *$\mathfrak{A}$ contains distinct elements $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ such that (A)–(E) hold for each integer p, and also $\neg\, W\mathbf{p}$ for each p; or*

(II) *for some $n > 0$, $\mathfrak{A}$ contains distinct elements $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}$ such that* (A)–(E) *hold for each $p \leq n$, $\neg\, W\mathbf{p}$ for each $p < n$, and $W\mathbf{n}$.*

PROOF. By clauses (1) and (6) there is a unique element $\mathbf{0}$ of $\mathfrak{A}$ such that $Z\mathbf{0}$. Conditions (A)–(E) for $p = 0$ follow as in §3. Note that $\neg\, W\mathbf{0}$, by clause (19). Now suppose $\mathbf{0}, \ldots, \mathbf{k}$ are distinct elements of $\mathfrak{A}$ such that (A)–(E) hold for each $p \leq k$ and $\neg\, W\mathbf{p}$ for each $p < k$. It suffices to show that if $\neg\, W\mathbf{k}$ then there exists an element $\mathbf{k+1}$ of $\mathfrak{A}$, distinct from $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{k}$, such that (A)–(E) hold for $p = k + 1$.

Sublemmas 1 and 2 as in §3 can be proved as they were there. (The alteration in clause (11) made in this section does not affect the proof, since $\neg\, W\mathbf{p}$ for each $p < k$.) Moreover, on the assumption that $\neg\, W\mathbf{k}$, by clause (7a) there exists a in $\mathfrak{A}$ with $Sa\mathbf{k}$. An argument as in §3 then establishes that there is a unique a in $\mathfrak{A}$ with $Sa\mathbf{k}$; let $\mathbf{k+1}$ be that a. Sublemmas 4–6, which complete the proof that (A)–(E) hold for $p = k + 1$, are also shown as before, with one extra step in the proof of Sublemma 6 necessitated by the alteration in clause (7). We must show that if $\neg\, Ic$ then there exists b in $\mathfrak{A}$ with P_1cb. But by clause (7c), if $\neg\, Ic$ then there exists d in $\mathfrak{A}$ with Ncd. By clause (11), then, there exists b in $\mathfrak{A}$ with P_1cb.

LEMMA 4. *If G_J has a model then so does G; if G_J has a finite model then so does G.*

PROOF. Let $\mathfrak{A}$ be a model for G_J, with $\mathfrak{A}$ finite if G_J has a finite model. If (I) of Lemma 3 holds, let $\alpha = \omega$; if (II) holds let $\alpha = n + 1$. Since (II) must hold if $\mathfrak{A}$ is finite, it suffices to construct a model $\mathfrak{B}$ for J with universe $\{p \mid p < \alpha\}$. Indeed, for $p, q < \alpha$ let $\Gamma(p, q) = \{b \in \mathfrak{A} \mid P_1b\mathbf{p} \wedge P_2b\mathbf{q} \wedge Eb\}$, and for each $\Phi \in \mathbf{L}$ let $\mathfrak{B} \models \Phi pq$ iff there exists $c \in \Gamma(p, q)$ such that $A_\Phi c$. We shall show that $\mathfrak{B} \models J$.

SUBLEMMA 7. *Let $p, q < \alpha$ and $b, c \in \mathfrak{A}$.*

(a) *Suppose Nbc and $b \in \Gamma(p, q)$. If $p + 1 < \alpha$ then $c \in \Gamma(p + 1, q)$; if $p + 1 = \alpha$ then $c \in \Gamma(0, q)$.*

(b) *$\Gamma(p, q)$ is nonempty.*

(c) *If $b \in \Gamma(p, q)$ then, for each $\Phi \in \mathbf{L}$, $\mathfrak{B} \models \Phi qp$ iff $A^*_\Phi b$.*

(d) *If $b \in \Gamma(p, q)$ then, for each $\Phi \in \mathbf{L}$, $\mathfrak{B} \models \Phi pq$ iff $A_\Phi b$.*

PROOF. (a) By (19), $Eb \equiv Ec$; hence Ec. By clause (10) there exists d in $\mathfrak{A}$ with $P_2bd \wedge P_2cd$. By (D), $d = \mathbf{q}$; hence $P_2c\mathbf{q}$. If $p + 1 < \alpha$, then Sublemma 1 yields $P_1c\,\mathbf{p+1}$. If $p + 1 = \alpha$, then $p = n$ so that $W\mathbf{p}$. By clause (11) there exists d in $\mathfrak{A}$ with $P_1bd \wedge (We \rightarrow B_1c)$. By (D), $d = \mathbf{p}$; hence B_1c. By clause (2), $P_1c\mathbf{0}$.

(b) We show first that $\Gamma(0, q)$ is nonempty. If $q + 1 < \alpha$, then $Se\mathbf{q}$ for $e = \mathbf{q+1}$. By clause (12) there exists d in $\mathfrak{A}$ with $Qde \wedge P_2d\mathbf{q} \wedge B_1d$. By clauses (2) and (19), $P_1d\mathbf{0} \wedge Ed$; hence $d \in \Gamma(0, q)$. If $q + 1 = \alpha$, then $W\mathbf{q}$; by clause (7b) there exists d in $\mathfrak{A}$ with $P_1d\mathbf{0} \wedge P_2d\mathbf{q} \wedge Ed$; hence $d \in \Gamma(0, q)$. Now suppose $\Gamma(r, q)$ is nonempty and $r + 1 < \alpha$. Let $b \in \Gamma(r, q)$. Since Eb, by clause (19) $\neg\, Ib$; by clause (7c) there exists c in $\mathfrak{A}$ with Nbc; by part (a), $c \in \Gamma(r + 1, q)$. Thus $\Gamma(r + 1, q)$ is nonempty.

(c) Suppose first that $b \in \Gamma(p, q)$ and $c \in \Gamma(q, p)$; we show that $Mbc \wedge Mcb$. Suppose that Nbc. By part (a) and condition (D), $q = p$ and either $p + 1 < \alpha$ and $q = p + 1$, or else $p + 1 = \alpha$ and $q = 0$. The former case is, obviously, impossible; in the latter case we have $p = n = 0$, which is also impossible. Thus $\neg Nbc$. Similarly, $\neg Ncb$. By clause (18), then, there exists d in $\mathfrak{A}$ with $P_1bd \wedge (P_2cd \rightarrow Mbc)$. By (D), $d = \mathbf{p}$; hence Mbc. Symmetric reasoning shows that Mcb.

Now, by definition of $\mathfrak{B}$ there exists $c \in \Gamma(q, p)$ with $\mathfrak{B} \models \Phi qp$ iff $A_\Phi c$. By what was just shown, $Mcb \wedge$ Mbc. Thus $\forall x \forall y H_J$ implies $A_\Phi c \equiv A_\Phi^* b$. Hence $\mathfrak{B} \models \Phi qp$ iff $A_\Phi^* b$.

(d) If $A_\Phi b$ then $\mathfrak{B} \models \Phi pq$ by definition. Suppose $\mathfrak{B} \models \Phi pq$, and let d be any member of $\Gamma(q, p)$. By part (c), $A_\Phi^* d$. Moreover, as in (c), $Mbd \wedge Mdb$. Thus $\forall x \forall y H_J$ implies $A_\Phi b \equiv A_\Phi^* d$. Hence $A_\Phi b$. □

SUBLEMMA 8. *Let* $p, q < \alpha$. *Then* $\mathfrak{B} \models K(p, p+1, q)$ *whenever* $p + 1 < \alpha$, *and* $\mathfrak{B} \models K(p, 0, q)$ *if* $p + 1 = \alpha$.

PROOF. Let $b \in \Gamma(p, q)$. Since Eb, by clause (19) $\neg Ib$. By clause (7c) there exists c in $\mathfrak{A}$ with Nbc. Thus $\forall x \forall y H_J$ implies $K^*(b, c)$. Let $r = p + 1$ if $p + 1 < \alpha$; let $r = 0$ if $p + 1 = \alpha$. By Sublemma 7(a), $c \in \Gamma(r, q)$. By Sublemma 7(c) and (d), $\mathfrak{B} \models \Phi pq$ iff $A_\Phi b$, $\mathfrak{B} \models \Phi qp$ iff $A_\Phi^* b$, $\mathfrak{B} \models \Phi rq$ iff $A_\Phi c$, and $\mathfrak{B} \models \Phi qr$ iff $A_\Phi^* c$. By the construction of $K^*(b, c)$, it follows that $\mathfrak{B} \models K(p, r, q)$. □

Sublemma 8 shows that $\mathfrak{B}$ is a model for J. □

§5. One dyadic letter. In this section we show how to reduce the formulas G_J constructed in §4 to MGCI formulas that contain only one dyadic predicate letter R. For this, we use new monadic letters C_i for $i = 0, 1, 2$, and C_i^j for $i = 0, 1, 2$ and $j = 1, 2, 3$. In the intended model, C_i holds of integers p such that $p \equiv i \pmod 3$, and C_i^j holds of triples b such that $\pi_j b \equiv i \pmod 3$. Let $\sigma 0 = 1$, $\sigma 1 = 2$, and $\sigma 2 = 0$. For each dyadic letter Ψ of G_J, we define a formula $\Psi^*(v, w)$ that contains just R and the new monadic letters. (In these definitions, the conjunctions $\bigwedge$ are for $i = 0, 1, 2$.) Let

$S^*(v, w)$ be $Rvw \wedge Iv \wedge Iw$,

$P_1^*(v, w)$ be $Rvw \wedge \bigwedge (C_i v \equiv C_i^1 w)$,

$P_2^*(v, w)$ be $Rwv \wedge \bigwedge (C_i v \equiv C_i^2 w)$,

$L_1^*(v, w)$ be $Rvw \wedge \bigwedge (C_i v \equiv C_{\sigma i}^1 w)$,

$L_2^*(v, w)$ be $Rwv \wedge \bigwedge (C_i v \equiv C_{\sigma i}^2 w)$,

$Q^*(v, w)$ be $Rwv \wedge \bigwedge (C_{\sigma i} v \equiv C_i^2 w)$,

$R_1^*(v, w)$ be $Rvw \wedge \bigwedge (C_{\sigma i}^3 v \equiv C_i^3 w)$,

$R_2^*(v, w)$ be $Rwv \wedge \bigwedge (C_{\sigma i}^3 v \equiv C_i^3 w)$,

$N^*(v, w)$ be $Rvw \wedge Rwv \wedge \bigwedge [(C_i^3 v \equiv C_i^3 w) \wedge (C_i^2 v \equiv C_i^2 w) \wedge (C_i^1 v \equiv C_{\sigma i}^1 w)]$,

$M^*(v, w)$ be $Rvw \wedge C_0^3 v \wedge C_0^3 w \wedge \neg N^*(v, w) \wedge \neg N^*(w, v)$.

Now let G_J^* be obtained from G_J by replacing, for every dyadic predicate letter Ψ and all variables v and w, each atomic subformula Ψvw by the formula $\Psi^*(v, w)$. Since G_J^* comes from G_J by replacement of predicate letters, if it has a model then so does G_J, so that J has a model; and if it has a finite model then so does G_J, so that J has a finite model. Thus we need only show that if J has a model then so does G_J^*, and if J has a finite model then so does G_J^*.

Suppose J has a model. Then it has a model $\mathfrak{B}$ with universe $\mathbf{N}$ such that $\mathfrak{B} \models K(p, p+1, q)$ for all integers p and q. Let the universe be $\mathbf{N} \cup V$, where $V = \mathbf{N} \times \mathbf{N} \times \{0, 1, 2\}$. Interpret the predicate letters of G_J (including the dyadic letters that do not occur in G_J^*) as in the proof of Lemma 1, §4; interpret the new monadic letters C_i and C_i^j as indicated at the start of this section, and interpret R so that, for all a and b in the universe, Rab iff

$$(\ddagger) \qquad Sab \vee P_1ab \vee P_2ba \vee L_1ab \vee L_2ba \vee Qba \vee Nab \vee Nba \vee R_1ab \vee R_2ba \vee Mab.$$

It is easily checked that, for every dyadic letter Ψ of G_J except M, and all a and b in the universe, $\Psi^*(a,b)$ is true iff Ψab is true. Moreover, $M^*(a,b)$ is true iff $Mab \wedge \neg Nab \wedge \neg Nba$ is true. Now if the interpretations of Lemma 1, §4, are altered so that Mab now holds iff $Mab \wedge \neg Nab \wedge \neg Nba$ held under the original interpretations, then the result is still a model for G_J, since $\neg Nxy \wedge \neg Nyx$ occurs in the antecedent of clause (18). It follows that the interpretation of R just given, along with the interpretations of the monadic letters, provides a model for G_J^* with universe $\mathbf{N} \cup V$.

Now suppose J has a finite model. Then it has a model $\mathfrak{B}$ with universe $\{0,\dots,n\}$ such that $\mathfrak{B} \models K(p,p+1,q)$ whenever $p+1, q \le n$ and $\mathfrak{B} \models K(n,0,q)$ whenever $q \le n$. Without loss of generality we may assume that $n+1 \equiv 0 \pmod 3$. For if $n+1 \not\equiv 0 \pmod 3$ then we can expand $\mathfrak{B}$ to a suitable model with universe $\{0,\dots,3n+2\}$ by making p indiscernible from q whenever $p \equiv q \pmod{n+1}$. We now show that G_J^* has a model with universe $\{0,\dots,n\} \cup (\{0,\dots,n\} \times \{0,\dots,n\} \times \{0,1,2\})$. Interpret the predicate letters of G_J over this universe as in Lemma 2, §4, interpret the monadic letters C_i and C_i^j as indicated above, and interpret R so that Rab holds iff ($\ddagger$) of the proof immediately above holds. It follows that, for every dyadic predicate letter Ψ of G_J save M and all elements a and b of the universe, $\Psi^*(a,b)$ is true iff Ψab is true; and, moreover, $M^*(a,b)$ is true iff $Mab \wedge \neg Nab \wedge \neg Nba$. From this and Lemma 2, §4, we may conclude that these interpretations provide a finite model for G_J^*.

Thus J has a model iff G_J^* has a model, and J has a finite model iff G_J^* has a finite model. This yields

THEOREM 4. *The class of formulas in the minimal Gödel class with identity whose nonlogical vocabulary contains, aside from monadic predicate letters, just one dyadic predicate letter is conservative.* □

§6. Prefix-similarity classes. A *prefix-similarity class* is a class of prenex quantificational formulas specified by form of quantifier prefix and number and degree of predicate letters.[1] If Π denotes a prefix form and p and q are integers, then $\Pi(p,q)$ is the class of formulas with identity whose prefixes have form Π and which contain at most p monadic predicate letters, q dyadic predicate letters, and no k-adic predicate letters for $k \ge 3$; and $\Pi(\infty,q)$ is the union of the classes $\Pi(p,q)$. Note that, for any Π, the class $\Pi(\infty,0)$ is subsumed by monadic quantification theory with identity, and hence is solvable. Moreover, if Π is bounded (that is, contains at most r quantifiers for some r), then for all integers p and q the class $\Pi(p,q)$ contains only

[1] For quantification theory extended by the inclusion of function symbols, the specification of prefix-similarity classes also includes the number and degree of such symbols. See [9] for an exhaustive list of solvable and unsolvable prefix-similarity classes that allow at least one function symbol.

finitely many different formulas, up to alphabetic variants and truth-functionally equivalent matrices; hence $\Pi(p,q)$ is solvable.

Theorem 4 of §5 states that the class $\forall\forall\exists(\infty,1)$ is conservative. Now the class $\forall\exists\forall(\infty,1)$ is also conservative [10]. From the positive results for prefix classes of quantification theory with identity given at the beginning of this paper, and from the remarks of the preceding paragraph, it follows that these two are minimal unsolvable prefix-classes with bounded prefix form.

Now in pure quantification theory, the minimal undecidable prefix-similarity classes with bounded prefix form are $\forall\forall\forall\exists(\infty,1)$ and $\forall\exists\forall(\infty,1)$, and both of these are conservative ([20] and [10]). Thus the dividing line between solvable and unsolvable prefix-similarity classes differs, tracking the difference in the dividing line between solvable and unsolvable prefix classes noted at the beginning of this paper.

For pure quantification theory, the minimal unsolvable prefix-similarity classes with unbounded prefix form are the following: $\forall\cdots\forall\exists(0,1)$ [14]; $\forall\exists\forall\cdots\forall(0,1)$ [2]; $\forall\exists\cdots\exists\forall(0,1)$ [15]; $\exists\cdots\exists\forall\exists\forall(0,1)$ [20]; $\exists\cdots\exists\forall\forall\forall\exists(0,1)$ [20]; $\forall\forall\forall\exists\cdots\exists(0,1)$ [13]; and $\forall\exists\forall\exists\cdots\exists(0,1)$ [8]. Moreover, each of these classes is conservative. For quantification theory with identity, the last three classes can be collapsed into two; for, as we now show, it follows from Theorem 4 that the classes $\exists\cdots\exists\forall\forall\exists(0,1)$ and $\forall\forall\exists\cdots\exists(0,1)$ are conservative. Thus our results settle the decision problem for all prefix-similarity classes of quantification theory with identity.

Theorem 5. *The class* $\exists\cdots\exists\forall\forall\exists(0,1)$ *is conservative.*

Proof. Let $F = \forall x\forall y\exists zH$ be any formula in the class $\forall\forall\exists(\infty,1)$; let R be the dyadic predicate letter of F, and let $P_1,\ldots,P_m$ be the monadic letters of F. For any variable v let $D(v)$ be the formula $\bigwedge_{1\le i\le m} v \ne w_i$, and let $K = [D(z) \wedge (D(x) \wedge D(y) \to H')]$, where H' is obtained from H by replacing each atomic subformula $P_i v$ with Rvw_i. Finally, let $G = \exists w_1\cdots\exists w_m\forall x\forall y\exists zK$. Thus $G \in \exists\cdots\exists\forall\forall\exists(0,1)$.

Suppose F has a model $\mathfrak{A}$ with universe U. Let $e_1,\ldots,e_m$ be distinct objects not in U. Let $\mathfrak{B}$ be the structure with universe $U \cup \{e_1,\ldots,e_m\}$ such that, for all a and b in this universe, $\mathfrak{B} \models Rab$ iff either $a,b \in U$ and $\mathfrak{A} \models Rab$ or else $a \in U$, $b = e_i$, and $\mathfrak{A} \models P_i a$. Clearly $\mathfrak{B} \models \forall x\forall y\exists zK[e_1,\ldots,e_m]$; hence $\mathfrak{B}$ is a model for G.

Now suppose G has a model $\mathfrak{B}$ with universe V. Let $e_1,\ldots,e_m$ be elements of V such that $\mathfrak{B} \models \forall x\forall y\exists zK[e_1,\ldots,e_m]$, and let $U = V - \{e_1,\ldots,e_m\}$. Since $\forall x\forall y\exists zK$ implies $\exists zD(z)$, U is nonempty. Let $\mathfrak{A}$ have universe U and, for a and b in U, let $\mathfrak{A} \models Rab$ iff $\mathfrak{B} \models Rab$, and let $\mathfrak{A} \models P_i a$ iff $\mathfrak{B} \models Rae_i$. Then $\mathfrak{A} \models F$.

Thus F has a model iff G has a model, and F has a finite model iff G has a finite model. □

Theorem 6. *The class* $\forall\forall\exists\cdots\exists(0,1)$ *is conservative.*

Proof. Let F be a formula in $\forall\forall\exists(\infty,1)$ whose sole dyadic letter is R. Let F' be obtained from F by replacing each atomic subformula Rvw by $Rvw \vee (v = w \wedge Pv)$, where P is a new monadic letter. Then F and $\forall x(\neg Rxx) \wedge F'$ are satisfiable over the same universes. For if $\forall x(\neg Rxx) \wedge F'$ is satisfiable over U then, since F' comes from F by replacement of a predicate letter, F is satisfiable over U. Conversely, any model for F can be transformed into one for $\forall x(\neg Rxx) \wedge F'$ by interpreting P as true of any element a such that $\mathfrak{A} \models Raa$ and reinterpreting R so that Rab is true iff $\mathfrak{A} \models Rab$ and $a \ne b$.

Suppose that $F' = \forall x \forall y \exists z H$, and let $P_1, \ldots, P_m$ be the monadic predicate letters of F'. Let $D(v)$ be $\bigwedge_{1 \le i \le m} v \ne w_i$, let H' be obtained from H by replacing each atomic subformula $P_i v$ with Rvw_i, and let K be the conjunction of the following clauses:

(1) $D(z)$,

(2) $Rw_1w_1 \wedge (Rxx \wedge Ryy \to x = y)$,

(3) $\bigwedge_{1 \le i < m} m[Rw_iw_{i+1} \wedge (Rw_ix \wedge Rw_iy \to x = y)]$,

(4) $D(x) \wedge D(y) \to H'$.

Finally, let G be $\forall x \forall y \exists z \exists w_1 \cdots \exists w_m K$. Thus $G \in \forall\forall\exists \cdots \exists(0, 1)$.

Suppose that $\forall x(\neg Rxx) \wedge F'$ has a model $\mathfrak{A}$ with universe U. Let $e_1, \ldots, e_m$ be distinct objects not in U. Let $\mathfrak{B}$ have universe $U \cup \{e_1, \ldots, e_m\}$, and, for all a and b in this universe, let $\mathfrak{B} \models Rab$ iff either $a, b \in U$ and $\mathfrak{A} \models Rab$, or $a = b = e_1$, or $a = e_i$ and $b = e_{i+1}$ for some i, $1 \le i < m$, or $a \in U$, $b = e_i$, and $\mathfrak{A} \models P_ia$. Note that since $\mathfrak{A} \models \forall x(\neg Rxx)$, $\mathfrak{B} \models Raa$ iff $a = e_1$; also, for $1 \le i < m$, $\mathfrak{B} \models Re_ia$ iff $a = e_{i+1}$. It follows quickly that $\mathfrak{B} \models K[e_1, \ldots, e_m]$, so that $\mathfrak{B}$ is a model for G.

Now suppose that $\mathfrak{B}$ is a model for G with universe V. By (2) there is a unique $e_1 \in V$ with $\mathfrak{B} \models Re_1e_1$; by (3) there are a unique $e_2 \in V$ with $\mathfrak{B} \models Re_1e_2$, a unique $e_3 \in V$ with $\mathfrak{B} \models Re_2e_3, \ldots$, and a unique $e_m \in V$ with $\mathfrak{B} \models Re_{m-1}e_m$. Moreover, the existential variables $w_1, \ldots, w_m$ must always take values $e_1, \ldots, e_m$; that is, $\mathfrak{B} \models \forall x \forall y \exists z K[e_1, \ldots, e_m]$. Let $U = V - \{e_1, \ldots, e_m\}$. By clause (1), V is nonempty. Let $\mathfrak{A}$ have universe U, and for all $a, b \in U$ let $\mathfrak{A} \models Rab$ iff $\mathfrak{B} \models Rab$ and $\mathfrak{A} \models P_ia$ iff $\mathfrak{B} \models Rae_i$. Then $\mathfrak{A} \models \forall x(\neg Rxx)$, since $e_1 \notin V$; and, by clause (4) of G, $\mathfrak{A} \models F'$.

Thus F has a model iff G has a model, and F has a finite model iff G has a finite model. □

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