Asymptotic dynamics of nonlinear Schrödinger equations: Resonance-dominated and dispersion-dominated solutions

February, 2001

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Abstract

We consider a linear Schrödinger equation with a small nonlinear perturbation in $\mathbb{R}^3$. Assume that the linear Hamiltonian has exactly two bound states and its eigenvalues satisfy some resonance condition. We prove that if the initial data is near a nonlinear ground state, then the solution approaches to certain nonlinear ground state as the time tends to infinity. Furthermore, the difference between the wave function solving the nonlinear Schrödinger equation and its asymptotic profile can have two different types of decay: 1. The resonance dominated solutions decay as $t^{-1/2}$. 2. The radiation dominated solutions decay at least like $t^{-3/2}$.

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1 Introduction

Let H_0 be the Hamiltonian $H_0 = -\Delta + V - e_0$ with V a smooth localized potential and $e_0 < 0$ the ground state energy to $-\Delta + V$. Consider the nonlinear Schrödinger equation

$$i\partial_t \psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi, \qquad \psi(t=0) = \psi_0 \tag{1.1}$$

where λ is a small positive or negative parameter. Our goal is to understand its asymptotic evolution as $t \to \infty$. The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the nonlinear equation

$$(-\Delta + V)Q + \lambda|Q|^2 Q = EQ, \tag{1.2}$$

For any nonlinear bound state, $\psi_t = Qe^{-iEt}$ is a solution to the nonlinear Schrödinger equation. We may obtain a family of such bound states from minimizing the corresponding nonlinear energy functional:

$$\mathcal{H}[\phi] = \int \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}V|\phi|^2 + \frac{1}{4}\lambda|\phi|^4\, dx\ .$$

For each $N > 0$ and λ sufficiently small, subject to the constraint $\|\phi\|_{L^2}^2 = N$, there is a unique positive minimizer Q of $\mathcal{H}$, which solves (1.2) for some $E = E(N)$ and has exponential decay as $x \to \infty$. We call this family nonlinear ground states.

Instead of N, we can use E as the parameter. From now on we will refer to this continuous family as $\{Q_E\}_E$. Let

$$H_E = -\Delta + V - E + \lambda Q_E^2\ . \tag{1.3}$$

We have $H_E Q_E = 0$. Since λ is small, the spectral properties of H_E are similar to those of H_0.

Suppose the initial data of the nonlinear Schrödinger equation ψ_0 is near some Q_E. If $-\Delta + V$ has only one bound state, it was proved in [11] that the evolution will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. Suppose now that $-\Delta + V$ has multiple bound states, say, two bound states: a ground state ϕ_0 with eigenvalue e_0 and an excited state ϕ_1 with eigenvalue e_1, i.e., $H_0\phi_1 = e_{01}\phi_1$ where $e_{01} = e_1 - e_0$. The question is whether the evolution with initial data ψ_0 near some Q_E will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E? Furthermore, can we characterize the asymptotic behavior of the evolution?

The family of ground states is stable in the sense that if

$$\inf_{\Theta, E} \left\|\psi_0 - Q_E\, e^{i\Theta}\right\|_{L^2}$$

is small, it remains so for all t. Let $\|\cdot\|_{L^2,loc}$ denote local L^2 norm. One expects that this difference is actually approaching to zero, i.e.,

$$\lim_{t\to\infty}\inf_{\Theta,E}\left\|\psi_t - Q_E\, e^{i\Theta}\right\|_{L^2,loc} = 0 \tag{1.4}$$

In this paper, we shall answer this question positively in the case of two bound states. Furthermore, we estimate precisely the rate of relaxation for certain class of initial data.

We start with a simple question concerning the asymptotic dynamics around a fixed ground state profile at $t = \infty$. More precisely, we first fix an E and the corresponding nonlinear ground state Q_E. Our goal is to analyze the detailed asymptotic behavior of those solutions converging to this nonlinear ground state as $t \to \infty$. Although this problem seems to be simple, we found that there are two different types of behaviors for $\psi_t - Q_E e^{i\Theta}$: one is called resonance dominated solutions ; the other radiation dominated solutions. We first explain the Resonance condition on H_0 and the meanings of these solutions. Recall that the ground state and the excited state to H_0 are denoted by ϕ_0 and ϕ_1 respectively. The following two conditions are assumed for this paper.

Assumption A1: Resonance condition. Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. We assume that $2e_{01} > |e_0|$ so that $2e_{01}$ is in the continuum spectrum of H_0. Furthermore, for some constant $\gamma_0 > 0$ and all real s sufficiently small,

$$\left(\phi_0\phi_1^2\,,\, \mathrm{Im}\,\frac{1}{H_0 - 0i - 2\,e_{01} - s}\,\mathbf{P}_{\mathrm{c}}^{H_0}\,\phi_0\phi_1^2\right) > \gamma_0 > 0\ . \tag{1.5}$$

Assumption A2: For λ sufficiently small and E in a small neighborhood of e_0, the bottom of continuum spectrum to $-\Delta + V + \lambda Q_E^2$, 0, is not a generalized eigenvalue, i.e., not a resonance. Also, we assume that V satisfies the assumption in Yajima [15] so that the $W^{k,p}$ estimates for $k \le 2$ for the wave operator W_H holds: for a small $\sigma > 0$,

$$|\nabla^\alpha V(x)| \le C\,\langle x\rangle^{-5-\sigma}\,, \qquad \text{for } |\alpha| \le 2\ .$$

Also, the functions $(x\cdot\nabla)^k V$, for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with an $-\Delta$-bound < 1:

$$\left\|(x\cdot\nabla)^k V\phi\right\|_2 \le \sigma_0\,\|-\Delta\phi\|_2 + C\,\|\phi\|_2\,, \qquad \sigma_0 < 1, \quad k = 0, 1, 2, 3\ .$$

Fixed an E and its ground state Q_E. Let $\mathcal{L}^{(\mathrm{or})}$ be the operator obtained from linearizing the Schrödinger equation (1.1) around the trivial evolution Qe^{-iEt}, i.e.,

$$\mathcal{L}^{(\mathrm{or})}k = -i\left\{H_E k + \lambda Q^2(k + \bar{k})\right\}\ ,$$

where H_E is defined in (1.3). It is more convenient to work with operators orthogonal to the ground state Q. Let Π be the projection which eliminates the Q-direction:

$$\Pi h = h - (c_0 Q, h)Q \ , \quad c_0 = (Q,Q)^{-1}$$

and let X denote its image,

$$X = \Pi(L^2) \ .$$

Define the operator $\mathcal{L}$ acting on the space $X = \Pi L^2$ by

$$\mathcal{L}h = -i\left\{Hh + \lambda \Pi Q^2 \Pi(h + \bar{h})\right\} \ .$$

The operator can be written in matrix form

$$\mathcal{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \qquad L_- = H, \qquad L_+ = H + 2\lambda \Pi Q^2 \Pi.$$

With respect to $\mathcal{L}$, we define generalized "eigenspaces":

$$\mathtt{E}_\nu(\mathcal{L}) := \left\{\psi : \mathcal{L}^2\psi = -\nu^2\psi\right\} = \left\{u + iv : u, v \text{ real}, \ L_-L_+u = \nu^2 u, L_+L_-v = \nu^2 v\right\} \ .$$

Since the original Hamiltonian H_0 has only two bound states with the ground state projected out, there is exactly one value for ν, called κ, and $\dim_{\mathbb{R}} \mathtt{E}_\kappa = 2$. From simple perturbation theory, we know that $\kappa = e_{01} + O(\lambda)$. We can normalized u and v such that $(u, L_+u) = \kappa$ and $v = \kappa^{-1}L_+u$. Then $(u,v) = 1$ and the space $\mathtt{E}_\kappa(\mathcal{L})$ is just $\operatorname{span}_{\mathbb{R}}\{u, iv\}$.

Define the inner product

$$((\psi, \phi)) = (\operatorname{Re}\psi, \operatorname{Re}\phi) + (\operatorname{Im}\psi, \operatorname{Im}\phi) \ .$$

The space of continuum spectrum of $\mathcal{L}$ can be characterized by

$$\mathtt{H}_{\mathrm{c}}(\mathcal{L}) := \{\psi : \psi \perp \mathtt{E}_\nu(\mathcal{L}^*) \text{ for all } \nu\} \ ,$$

where $\mathcal{L}^*$ is the adjoint w.r.t. the inner product just defined. It is also clear from the definition that

$$X = \mathtt{E}_\kappa(\mathcal{L}) \oplus \mathtt{H}_{\mathrm{c}}(\mathcal{L}) \ .$$

Notice that this is not an orthogonal decomposition. Explicitly, the eigenspace $\mathtt{E}_\kappa(\mathcal{L}^*)$ is simply $\operatorname{span}_{\mathbb{R}}\{v, iu\}$. Hence the continuum space is characterized by

$$\mathtt{H}_{\mathrm{c}}(\mathcal{L}) = \{w_1 + iw_2 : w_1 \perp v, w_2 \perp u\} \ .$$

For any initial data ψ_0 near Q, we can solve γ_0 and ℓ_0 uniquely such that

$$\psi_0 = [Q + \gamma_0 Q + \ell_0]\, e^{i\theta_0}, \quad \ell_0 \perp Q \ .$$

It is more convenient to express ψ_0 in term of Q and $R = R_E$ defined by

$$R_E = \partial_E Q_E$$

We shall see in next section that R_E is of order λ^{-1}. We now rewrite the initial data as

$$\psi_0 = [Q + a_0 R + h_0]\, e^{i\theta_0}, \quad h_0 \perp Q \ .$$

(cf. Lemma 9.1) From the decomposition of X, we can write

$$h_0 = \alpha_0 u + \beta_0 iv + \eta_0, \quad \eta_0 \in \mathtt{H}_{\mathrm{c}}(\mathcal{L}) \ .$$

Denote the L^2 norm of ψ_0 by n. Since the evolution is nonlinear, we can always re-scale λ so that n, the L^2 norm of ψ_0, takes any fixed value. It is more convenient to allow n to be a parameter between, say, 1 and 10. Define $\mathcal{I}_\lambda$ to be the interval so that the L^2 norm of the ground state Q_E is between 1 and 10. Let $z_0 = \alpha_0 + i\beta_0$. Define the notations

$$\langle x \rangle = \sqrt{1 + x^2}, \qquad \{t\}_\varepsilon = \varepsilon^{-2} + 2\Gamma t, \quad \{t\}_\varepsilon \sim \max\left\{\varepsilon^{-2}, t\right\}$$

where Γ is a constant to be specified later on. For the moment, we remark that Γ is of order $2\lambda^2$ times the quantity in (1.5). When the subscript ε is understood, we shall drop it.

Theorem 1.1 (1) [**Resonance dominated solutions**] *There exists small parameters λ_0 and ε_0 such that for any $E \in \mathcal{I}_\lambda$ and $|\lambda| \le \lambda_0$ the following hold. Suppose that*

$$0 < |z_0| = \varepsilon \le \varepsilon_0,$$

$$\|\eta_0\|_{H^2 \cap W^{2,1}(\mathbb{R}^3)} \le C|z_0|^{3/2} \ .$$

Then we can find a small real number $a_0 = a_0(E, h_0)$ such that the solution $\psi(t)$ to the Schrödinger equation (1.1) *with the initial data $\psi_0 = Q + a_0 R + h_0$ can be decomposed as*

$$\psi(t) = [Q + a(t)R + h(t)]\, e^{i\Theta(t)} \tag{1.6}$$

with

$$a(0) = a_0, \quad h(0) = h_0, \quad \Theta(0) = 0,$$

and $a(t), h(t) \to 0$ *as* $t \to \infty$ *in the following sense: Let* $h(t) = \zeta(t) + \eta(t)$ *with* $\zeta(t)$ *the component in* $\mathtt{E}_\kappa(\mathcal{L})$ *and* $\eta(t)$ *the component in* $\mathtt{H}_{\mathrm{c}}(\mathcal{L})$*. Then we have*

$$
\begin{aligned}
&|a(t)| \leq C\,\{t\}^{-1} \\
&\zeta(t) \ldots \textit{local, smooth, } \|\zeta(t)\|_{L^2} \sim \{t\}^{-1/2} \ , \\
&\eta(t) \ldots \textit{dispersive wave, } \|\eta(t)\|_{L^4} \leq C\,\{t\}^{-3/4+\sigma} \ , \ \ \|\eta(t)\|_{L^2_{loc}} \leq Ct^{-1} \ ,
\end{aligned}
$$

where $\{t\} = \varepsilon^{-2} + t$*. Also,* $\Theta(t)/t \to -E$*.*

(2) **[Radiation dominated solutions]** *For any* $\chi_\infty \in H^2 \cap W^{2,1}(\mathbb{R}^3)$ *small, there exist solutions of the form*

$$\psi(x,t) = [Q(x) + a(t)R(x)]\ e^{i\Theta(t)} + \chi(x,t)$$

such that

$$|a(t)| \leq Ct^{-2}, \qquad \|\chi(\cdot,t)\|_{L^2_{loc}} \leq Ct^{-3/2}$$

and $\chi = \chi_1 + \chi_2$*, where* $\chi_1 = e^{i\Delta t}\chi_\infty$ *and* $\|\chi_2(\cdot,t)\|_{L^2} \leq o(t^{-3/2})$*.*

We have used the notation $f \sim g$ for $Cg \leq f \leq C^{-1}g$ for some constant C. Thus the decay of the excited state for solutions constructed in part (1) is given precisely. Consequently, the two types of solutions are completely different. Return to the general question concerning the asymptotic mass of the ground state profile at $t = \infty$ (1.4). Instead of minimizing the L^2 norm, we prefer to determine the ground state profile by the condition

$$\left(\psi_t - Q_{E(t)}\, e^{i\Theta(t)}\ ,\ Q_{E(t)}\right) = 0 \tag{1.7}$$

We shall prove that there is a unique solution to this equation provided that the initial data is near some $Q_{\mathrm{in}} = Q_{E_{\mathrm{in}}}$ (Lemma 9.2). Furthermore, we can determine the change between $E(t)$ and E_{in}.

Theorem 1.2 *Let* λ_0 *and* ε_0 *be given as in the first theorem. For any* $E_{\mathrm{in}} \in \mathcal{I}_\lambda$ *with* $\lambda \leq \lambda_0$ *the following properties hold:*

(1) *Suppose the initial data*

$$\psi_0 = Q_{\mathrm{in}} + a_{\mathrm{in}}R_{\mathrm{in}} + h_{\mathrm{in}}$$

satisfies that

$$\|\psi_0 - Q_{\mathrm{in}}\|_{H^2\cap W^{2,1}(\mathbb{R}^3)} \sim \|h_{\mathrm{in}}\|_{H^2\cap W^{2,1}(\mathbb{R}^3)} + |a_{\mathrm{in}}|\lambda^{-1} \leq \varepsilon_0^{3/2}\ .$$

Then there are solutions $E(t)$ and $\Theta(t)$ to (1.7) such that

$$\lim_{t\to\infty} E(t) = E_\infty, \qquad |E(t) - E_\infty| \le C/\langle t\rangle \ .$$

Furthermore, if we decompose ψ_t as in (1.6) with $Q = Q_\infty$ being the profile at time $t = \infty$ then we have

$$\begin{aligned} &|a(t)| \le C\,\{t\}^{-1}, \qquad h(t) = \zeta(t) + \eta(t)\ , \\ &\zeta(t)\ldots \textit{local, smooth, } \|\zeta(t)\|_{L^2} \le \{t\}^{-1/2}\ , \\ &\eta(t)\ldots \textit{dispersive wave, } \|\eta(t)\|_{L^4} \le C\,\{t\}^{-3/4},\ \ \|\eta(t)\|_{L^2_{loc}} \le Ct^{-1}\ . \end{aligned}$$

where $\{t\} = \{t\}_{\varepsilon_0}$.

(2) *Suppose that*

$$0 < \varepsilon = |z_{\rm in}| \le \varepsilon_0,$$

$$\|\eta_{\rm in}\|_{H^2\cap W^{2,1}(\mathbb{R}^3)} \le C\varepsilon^{3/2}, \qquad \lambda^{-1}|a_{\rm in}| \le C\varepsilon^2\ .$$

Then ψ_0 belongs to the class of resonance dominated solutions with respect to the final profile Q_{E_∞}.

We shall see in next section that R_E is of order λ^{-1}. This explains that $a_{\rm in}$ has to be order λ in order Theorem 1.2 to be correct. Notice that once we have proved the assertion (1) of Theorem 1.2, the second assertion follows from Theorem 1.1. The proof of Theorem 1.1 is complicated due to the construction of solutions to the Schrödinger equation with the boundary condition of h_0 at time $t = 0$ and that of a at the time $t = \infty$. If we are interested only in Theorem 1.2, we can omit this construction and the proof will be much simpler. We feel that this construction may be needed in other contents and so we keep it.

The resonance solutions for nonlinear Klein-Gordon equations were first proved in an important paper [12] by Soffer and Weinstein (see also [2]). They consider real solutions to the nonlinear Klein-Gordon equation

$$\partial_t^2 u + B^2 u = \lambda u^3, \qquad B^2 := (-\Delta + V + m^2), \tag{1.8}$$

with λ a small nonzero number. Assume that B^2 has only *one* eigenvector (the ground state) ϕ, $B^2\phi = \Omega^2\phi$, with the resonance condition $\Omega < m < 3\Omega$ (and some positivity assumption similar to that appears in assumption A1). Rewrite real solutions to equation (1.8) as $u = a\phi + \eta$ with

$$a(t) = \mathrm{Re}\, A(t)e^{i\Omega t}, \quad \mathrm{Re}\, A'(t)e^{i\Omega t} = 0.$$

Then A and η satisfy the equations

$$\dot{A} = \frac{1}{2i\Omega} e^{-i\Omega t} \left(\phi, \lambda(a\phi + \eta)^3\right) \tag{1.9}$$

$$(\partial_t^2 + B^2)\eta = P_c \lambda (a\phi + \eta)^3 \tag{1.10}$$

Theorem 1.1 in [12] states that *all* solutions decay as

$$A(t) \sim \langle t \rangle^{-1/4}, \quad \|\eta(t)\|_{L^\infty} \sim \langle t \rangle^{-3/4}.$$

In particular, the ground state is unstable and will decay as a resonance with rate $t^{-1/4}$.

We first remark that the proof in [12] has only established the upper bound $t^{-1/4}$. Furthermore, an universal lower bound of the form $t^{-1/4}$ is in fact incorrect. From the previous work of [1, 4], it is clear that radiation dominated solutions decaying much faster than $t^{-1/4}$ exist. Similar to Theorem 1.1, we have two cases:

1. $\eta(0) \ll A(0)$: the dominant term on the right side of (1.9) is $\lambda a^3 \phi^3$.
2. $\eta(0) \gg A(0)$: the dominant term is $\lambda \eta^3$.

In case 2, another type of solutions arises, namely, those with decay rate

$$A(t) \sim \langle t \rangle^{-2}, \quad \|\eta\|_{L^\infty} \sim \langle t \rangle^{-3/2}.$$

We shall sketch a construction of such solutions at the end of section 10.

Notice that all solutions in [12] decay as a function of t. Therefore, we can view [12] as a study of asymptotic dynamics around vacuum. Although most works concerning asymptotic dynamics of nonlinear evolution equations have been concentrated on cases with vacuum as the unique profile at $t = \infty$, more interesting and relevant cases are asymptotic dynamics around solitons (such as the Hartree equations [4], see also [2]). The soliton dynamics have extra complications involving translational invariance. The current setting of nonlinear Schrödinger equations with local potentials eliminates the translational invariance and constitutes an useful intermediate step. This simplifies greatly the analysis but preserves a key difficulty which we now explain.

Recall that we need to approximate the wave function ψ_t by nonlinear ground states for all t. Since we aim to show that the error between them decay like $t^{-1/2}$, we have to track the nonlinear ground states with accuracy at least like $t^{-1/2}$. While the nonlinear ground states approximating the wave function ψ_t can in principle be defined, say, via equation (1.4) or (1.6), neither characterizations are useful unless we know the wave functions ψ_t precisely. Furthermore, even assuming we can track the approximate ground states reasonably well, the

linearized evolution around these approximate ground states will be based on *time-dependent, non-self adjoint* operators $\mathcal{L}_t$. At this point we would like to mention the approach of [11] based on perturbation around the unitary evolution e^{itH_0}, where H_0 is the original self-adjoint Hamiltonian. While we do not know whether this approach can be extended to the current setting by combining with ideas of [12] (It was announced in [12] that its method can be extended to (1.1) as well.), such an approach can be difficult to extend to the Hartree or other equations with non-vanishing solitons. The main reason is that these dynamics are not perturbation of linear dynamics. We believe that perturbation around the profile at $t = \infty$ is a more nature setup. In this approach, at least we do not have to worry about the time dependence of approximate ground states in the beginning. But the linearized operator $\mathcal{L} = \mathcal{L}_\infty$ is still non-self adjoint and it does not commute with the multiplication by i. So calculations and estimates based on $\mathcal{L}$ are rather complicated. Our first idea is to map this operator to a self-adjoint operator by a bounded transformation in Sobolev spaces. This map simplifies many calculations and in a sense brings the problem back to the self-adjoint case at least as far as estimates are concerned. The price to pay is that the new operator involves square root of operators like H. From the standard formula for the square root of an operator, we can represent it as an integral of resolvents of H. Thus the linear analysis of the non-self-adjoint $\mathcal{L}$ is reduced to the analysis of resolvents of H where standard methods applied, see section 3.

The next step is to identify and calculate the leading oscillatory terms of the nonlinear systems involving the bound states components and the continuum spectrum components. The leading order terms however depend on the relative sizes of these components and thus we have two different asymptotic behaviors: the resonance dominated solutions and the radiation dominated solutions. Finally we represent the continuum spectrum component in terms of the bound states components and this leads to a system of ordinary differential-integral equations for the bound states components. This system can be put into a normal form and the size of the excited state component can be seen to decay as $t^{-1/2}$. Notice that the phase and the size of the excited state component decay differently. It is thus important to isolate the contribution of the phase in the system. Finally we estimate the error terms using estimates of the linearized operators.

The estimates obtained from integrating the equation from $t = \infty$ can be viewed as uniform bounds for all T, provided that the approximate nonlinear ground states for all T are known. On the other hand, to pin down the approximate nonlinear ground states, we need to have precise estimates on the wave functions. It is thus nature to consider continuity

method by assuming that the approximate nonlinear ground states and various estimates on the wave function are known up to time T. We then show that these estimates continue to hold up to time $T + \delta T$, with δT small but fixed, provided that all estimates are re-adjusted w.r.t. to the new nonlinear ground state at $T + \delta T$.

From this outline, it seems that the resonance dominated solutions and the radiation dominated solutions occur on equal footing. On the other hand, we believe that the radiation dominated solutions are in fact of lower dimension in the space of solutions. A proof is still lacking. Although we restrict to the small coupling constant case, it can be replaced by spectral assumptions on the operators $\mathcal{L}$ with some modifications. The details will appear in a future publication. This paper is divided into 10 sections:

Section 2. The set-up of the problem

Section 3. Properties of the linearized operator

Section 4. Main oscillation terms

Section 5. Estimates on dispersive wave

Section 6. Excited state equation and normal form

Section 7. Change of the mass of the ground state

Section 8. Contraction mapping

Section 9. Dynamical renormalization of mass

Section 10. Radiation dominated solution

It is a great pleasure to thanks M. Weinstein for explaining to us the beautiful idea in the work [12] and, in particular, to call our attention to the toy model in [12] which contains the basic idea of the resonance decaying in a very illuminating way.

2 The set-up of the problem

2.1 Ground state family

We first review the construction of the ground state family $\{Q_E\}_E$ mentioned in Section 1. The results we reviewed in this subsection follow from simple perturbation theory and we shall not give details. Denote the standard L_2 inner product by

$$(f, g) \equiv \int \bar{f} g \, d^3x \ .$$

Let $Q = Q_E$ satisfy

$$(-\Delta + V - e_0)Q + \lambda Q^3 = E'Q \ .$$

where $E' = E - e_0$. Let $Q_E = w\phi_0 + h$ with h real and orthogonal to ϕ_0. Then h satisfies

$$(H_0 - E')h + \lambda(w\phi_0 + h)^3 = E'w\phi_0,$$

We can solve w and h so that

$$w^2 = O(\lambda^{-1}E'), \qquad \|h\|_2 = O(\lambda w^3) \tag{2.1}$$

where we have used that the spectral gap of the operator $-\Delta + V - e_0$ is of order one.

Let $R = \partial_E Q_E$. Recall

$$L_+^{(\mathrm{or})} = -\Delta + V - E + 3\lambda Q^2.$$

Then by differentiating the equation of Q_E w.r.t E, we have

$$L_+^{(\mathrm{or})} R_E = Q_E$$

Denote by $|0\rangle$ the ground state to $L_+^{(\mathrm{or})}$ of norm 1. We also have

$$Q_E = w|0\rangle + O(\lambda w)$$

Hence

$$R = (L_+^{(\mathrm{or})})^{-1}\left[w|0\rangle + O(\lambda w)\right] = O(\lambda^{-1}w^{-1})\,|0\rangle + O(\lambda w) = O(\lambda^{-1}w^{-2})\,Q + O(w) \ . \tag{2.2}$$

2.2 The set-up of the problem

We consider solutions $\psi(t)$ to the equation (1.1) The picture is that the solution $\psi(t)$ can be decomposed to two parts: one represents a soliton, the other one the radiation. The radiation part will disperse to infinity. The soliton part will converge to a soliton Q_{N_∞}, while its L^2-norm is changing in time. Hence it is natural to consider solutions of the form

$$\psi = (Q_{E(t)} + h(t))e^{i\Theta(t)} \ , \tag{2.3}$$

and study its evolution. If we consider a minimization problem

$$\inf_{\Theta,E} \left\|\psi - Q_E\, e^{i\Theta}\right\|_{L^2} \ .$$

Then $\psi = (Q_E + h)\, e^{i\Theta}$ with $\operatorname{Im} h \perp Q_E$, $\operatorname{Re} h \perp R_E$. Hence we almost have $h(t) \perp Q_{E(t)}$ in our problem, with some small correction.

Since this setup would introduce a time-dependent linear operator, we try to find a good approximation with a fixed $Q = Q_E$. For this fixed Q, let

$$H = -\Delta + V - E + \lambda Q^2 \ ,$$

(we have $HQ = 0$), and let Π be the projection which eliminates the Q-direction:

$$\Pi h = h - (c_0 Q, h) Q \ , \quad c_0 = (Q, Q)^{-1}$$

Let X denote its image,

$$X = \Pi(L^2) \ .$$

If we assume

$$\psi = (Q + k(t)) e^{i[-Et + \theta(t)]} \ .$$

Then the equation for $k(t)$ is

$$\partial_t k = \mathcal{L}^{(\mathrm{or})} k - iF(k) - i\dot{\theta}(Q + k) \ , \tag{2.4}$$

where

$$\mathcal{L}^{(\mathrm{or})} k = -i \left\{ Hk + \lambda Q^2 (k + \bar{k}) \right\} \ ,$$

and

$$F(k) = \lambda Q (2|k|^2 + k^2) + \lambda |k|^2 k \ .$$

In view of (2.3), it is natural to assume

$$k = aR + h, \quad R = \partial_E Q_E \ .$$

Here $a(t)$ is small and compensates the change of L^2 norm of the soliton. Since we would like $h(t) \perp Q$ for all t. Hence we look for solutions of the form

$$\psi = (Q + aR + h)\, e^{i\Theta}, \quad (\Theta = -Et + \theta(t)) \ .$$

We note that ψ can be written in this form if ψ is sufficiently close to Q. Specifically, we let $e^{i\Theta}$ be the phase of $P_Q \psi$ and choose a so that $\|P_Q \psi\|_{L^2} = \|Q + aP_Q R\|_{L^2}$. Then h is obtained by $h = \psi\, e^{-i\Theta} - Q - aR \in X$.

We also note, by differentiating the equation of Q w.r.t. E,

$$\mathcal{L}^{(\mathrm{or})} R = -iQ \ .$$

We substitute $k = aR + h$ into (2.4) and obtain

$$\partial_t h = -\dot{a}R - aiQ + \mathcal{L}^{(\mathrm{or})}h - iF(k) - i\dot{\theta}(Q + aR + h) \ .$$

Since we would like $h \perp Q$, we choose $\dot{a}$ and $\dot{\theta}$ so that the right side of the above equation is perpendicular to Q. Using $h \perp Q$ and $Hh \perp Q$ we obtain the equations for a and θ:

$$\begin{aligned}
&\dot{a} = (c_1 Q, \operatorname{Im} F(k)) \ , \qquad c_1 = (Q, R)^{-1} \ , \\
&\dot{\theta} = -(1 + c_0 c_1^{-1} a)^{-1} \left[a + (c_0 Q, \lambda Q^2 (h + \bar{h})) + (c_0 Q, \operatorname{Re} F(k)) \right] \ ,
\end{aligned}$$

where $c_0 = (Q, Q)^{-1}$. Then the equation of h is

$$\begin{aligned}
\partial_t h &= \mathcal{L}h + \Pi F_{all} \ , \\
\mathcal{L}h &= -i \left\{ Hh + \Pi \lambda Q^2 \Pi (h + \bar{h}) \right\} \ , \\
F_{all} &= -i\dot{\theta}h - iF(k) - \left[(c_1 Q, \operatorname{Im} F) + ia\dot{\theta} \right] R_\Pi \ , \quad R_\Pi := \Pi R \ .
\end{aligned}$$

Here we have used the equation for $\dot{a}$.

Next we compute $F(k) = F(h + aR)$:

$$F(k) = \lambda Q(2|h|^2 + h^2) + 2\lambda a Q R(2h + \bar{h}) + 3\lambda a^2 Q R^2 + \lambda (aR + h)^2 (aR + \bar{h}) \ .$$

With respect to $\mathcal{L}$, we can decompose X as

$$X = \mathtt{E}_\kappa \oplus \mathbf{P}_{\mathrm{c}}^{\mathcal{L}}(X) \ ,$$

where the direct sum is not orthogonal, and

$$\mathtt{E}_\kappa = \{ \alpha u + \beta i v : \alpha, \beta \in \mathbb{R} \} \ , \quad u, v \text{ real}, L_+ u = \kappa v, L_- v = \kappa u, (u, v) = 1.$$

We write

$$h(t) = \alpha(t) u + \beta(t) i v + \eta(t) \ .$$

Their equations are

$$\begin{aligned}
\dot{\alpha} &= \kappa\beta + (v, \operatorname{Re} F_{all}) \ , \\
\dot{\beta} &= -\kappa\alpha + (u, \operatorname{Im} F_{all}) \ , \\
\partial_t \eta &= \mathcal{L}\eta + \mathbf{P}_{\mathrm{c}}^{\mathcal{L}} \Pi (F_{all}) \ .
\end{aligned}$$

(Note $(v, \Pi f) = (v, f)$, and similarly for u.) The linear parts of the equations for $\alpha(t)$ and $\beta(t)$ form a rotation. To single out the rotation, we study

$$z(t) = \alpha(t) + i\beta(t) = p\mu^{-1}, \quad \mu(t) = e^{i\kappa t}.$$

Here μ^{-1} captures the rotation part and we expect a slower oscillation in p. Note that $z(t)$ can be obtained from $h(t)$ by

$$z(t) = (v, \operatorname{Re} h(t)) + i\,(u, \operatorname{Im} h(t))\ .$$

The function $p(t)$ satisfies

$$\begin{aligned}
\mu^{-1}\dot{p} &= (v, \operatorname{Re} F_{all}) + i(u, \operatorname{Im} F_{all}) \\
&= (v, \operatorname{Im}(F + \dot{\theta}h) - (c_1 Q, \operatorname{Im} F) R_\Pi) + i(u, -\operatorname{Re}(F + \dot{\theta}h) - a\dot{\theta}R_\Pi) \\
&= (v, \operatorname{Im} F) - (v, R_\Pi)\,(c_1 Q, \operatorname{Im} F) + i(u, -\operatorname{Re} F) + [(v, \operatorname{Im} h) + i(u, -\operatorname{Re} h)]\dot{\theta} - ic_3 a\dot{\theta} \\
&= (\tilde{v}, \operatorname{Im} F) + i(u, -\operatorname{Re} F) + [(v, \operatorname{Im} h) + i(u, -\operatorname{Re} h) - ic_3 a]\dot{\theta} \\
&= (\tilde{v}, (F - \bar{F})/2i) + i(u, -(F + \bar{F})/2) + [(v, (h - \bar{h})/2i) + i(u, -(h + \bar{h})/2) - ic_3 a]\dot{\theta} \\
&= -i\left[(\tilde{u}_+, F) + (\tilde{u}_-, \bar{F}) + \left\{(u_+, h) + (u_-, \bar{h}) + c_3 a\right\}\dot{\theta}\right]\ ,
\end{aligned}$$

where $c_3 = (u, R_\Pi)$, $\tilde{v} = v - (v, R_\Pi)c_1 Q \notin X$, and

$$\begin{aligned}
u_+ &= \tfrac{1}{2}(u + v) = O(1), \qquad & u_- &= \tfrac{1}{2}(u - v) = O(\lambda), \\
\tilde{u}_+ &= \tfrac{1}{2}(u + \tilde{v}) = O(1), \qquad & \tilde{u}_- &= \tfrac{1}{2}(u - \tilde{v}) = O(\lambda).
\end{aligned}$$

Their orders are derive from the simple facts: $(v, R_\Pi) = O(1)$, $c_2 = O(\lambda)$ and $c_3 = (u, R_\Pi) = O(1)$.

Summarizing, we have

$$\begin{aligned}
\psi &= (Q + aR + h)\, e^{i\Theta}, \\
h &= \zeta + \eta = (zu_+ + \bar{z}u_-) + \eta\ , \\
\dot{a} &= (c_1 Q, \operatorname{Im} F(k))\ , \qquad c_1 = (Q, R)^{-1}\ , \\
\dot{\theta} &= -(1 + c_0 c_1^{-1} a)^{-1}\ \left[a + (c_0 Q, \lambda Q^2 (h + \bar{h})) + (c_0 Q, \operatorname{Re} F(k))\right]\ , \qquad (2.5) \\
\partial_t \eta &= \mathcal{L}\eta + \mathbf{P}_{\mathrm{c}}{}^{\mathcal{L}}\Pi(F_{all}) \\
z(t) &= \alpha(t) + i\beta(t) = p\mu^{-1}, \quad \mu(t) = e^{i\kappa t}, \\
\mu^{-1}\dot{p} &= -i\left[(\tilde{u}_+, F) + (\tilde{u}_-, \bar{F}) + \left\{(u_+, h) + (u_-, \bar{h}) + c_3 a\right\}\dot{\theta}\right]\ ,
\end{aligned}$$

where

$$F_{all} = -i\dot{\theta}h - iF(h + aR) - \left[(c_1 Q, \operatorname{Im} F) + ia\dot{\theta}\right] R_\Pi\ , \quad R_\Pi := \Pi R\ , \qquad (2.6)$$

$$F(h + aR) = \lambda Q(2|h|^2 + h^2) + 2\lambda aQR(2h + \bar{h}) + 3\lambda a^2 QR^2 + \lambda(aR + h)^2(aR + \bar{h})\ . \qquad (2.7)$$

This is the equation we shall solve for the rest of this paper.

Convention: for a double-index $\alpha = (\alpha_0, \alpha_1)$, we denote

$$z^\alpha = z^{\alpha_0}\, \bar{z}^{\alpha_1}, \quad |\alpha| = \alpha_0 + \alpha_1, \quad [\alpha] = -\alpha_0 + \alpha_1 \ .$$

For example, $z^{(32)} = z^3 \bar{z}^2$. Hence $z^\alpha = \mu^{[\alpha]}\, p^\alpha$. In what follows $|\alpha| = 2$, $|\beta| = 3$, $|\gamma| = 4$.

3 Properties of the linearized operator

3.1 Spectral decomposition

Recall that, for given E,

$$H_E = -\Delta + V - E + \lambda Q_E^2$$

and $H_E Q_E = 0$. Recall Π be the projection which eliminates the Q-direction:

$$\Pi h = h - (h, Q) Q \ .$$

and the operator $\mathcal{L}$ acting on the space $X = \Pi L^2$ by

$$\mathcal{L} h = -i \left\{ H h + \Pi \lambda Q^2 \Pi (h + \bar{h}) \right\} \ .$$

We also have $L_- = H$, $L_+ = H + 2\Pi\lambda Q^2 \Pi$. With respect to $\mathcal{L}$, we can define generalized "eigenspaces":

$$\mathtt{E}_\nu(\mathcal{L}) := \left\{ \psi : \mathcal{L}^2 \psi = -\nu^2 \psi \right\} = \left\{ u + iv : u, v \text{ real}, \ L_- L_+ u = \nu^2 u, L_+ L_- v = \nu^2 v \right\} \ .$$

In our case, the only value for ν is κ. It is natural to define

$$\mathtt{H}_{\mathrm{c}}(\mathcal{L}) := \{ \psi : \psi \perp \mathtt{E}_\nu(\mathcal{L}^*) \text{ for all } \nu \} \ ,$$

since it is invariant under $\mathcal{L}$. The orthogonality is defined with respect to the inner product

$$((\psi, \phi)) = (\mathrm{Re}\, \psi, \mathrm{Re}\, \phi) + (\mathrm{Im}\, \psi, \mathrm{Im}\, \phi) \ ,$$

and we know

$$\mathtt{E}_\nu(\mathcal{L}^*) = \{ v + iu : u, v \text{ real}, \ u + iv \in \mathtt{E}_\nu(\mathcal{L}) \} \ .$$

It is also clear from the definition that

$$X = \bigoplus_\nu \mathtt{E}_\nu(\mathcal{L}) \oplus \mathtt{H}_{\mathrm{c}}(\mathcal{L}) \ .$$

Hence,

$$\mathtt{H}_{\mathrm{c}}(\mathcal{L}) = \left\{\phi_1 + i\phi_2 : \phi_1 \perp v, \phi_2 \perp u, \text{ for all } v,\, u \text{ such that } L_+L_-v = \nu^2 v, L_-L_+u = \nu^2 u\right\} \ .$$

In our situation, since we only have two simple eigenvalues for H_0 and the ground state is factored out, we can prove by perturbation argument that there are real u and v such that

$$\mathtt{E}_\kappa = \{au + biv : a, b \in \mathbb{R}\}\,, \quad L_+u = \kappa v, \quad L_-v = \kappa u, \quad (u, v) = 1 \ .$$

Since L_+L_- and L_-L_+ are not self-adjoint, it is not convenient to use them to characterize $\mathtt{H}_{\mathrm{c}}(\mathcal{L})$. We define two operators

$$B = \Pi(L_-)^{1/2}\Pi, \quad A = \sqrt{BL_+B}\,.$$

Note that both A and B are self-adjoint and **positive** on X. With these operators, if we define $w = \kappa^{-1/2}Bv$, we have w is the normalized eigenvector for the operator A and

$$u = \nu^{-1/2}Bw, \quad v = \nu^{1/2}B^{-1}w, \quad Aw = \kappa w, \quad (w, w) = 1.$$

Hence the continuum spectrum of $\mathtt{H}_{\mathrm{c}}(\mathcal{L})$ can be characterized by

$$\begin{aligned}
\mathtt{H}_{\mathrm{c}}(\mathcal{L}) &= \left\{\psi = \psi_1 + i\psi_2 : \psi_1, \psi_2 \text{ real }, \psi_1 \perp B^{-1}w_\nu, \psi_2 \perp Bw_\nu \text{ for all } \nu\right\} \\
&= \left\{\psi = \psi_1 + i\psi_2 : \psi_1, \psi_2 \text{ real }, B^{-1}\psi_1, B\psi_2 \in \mathtt{H}_{\mathrm{c}}(A)\right\} \\
&= \left\{\psi = B\phi_1 + iB^{-1}\phi_2 : \phi_1, \phi_2 \text{ real }, \phi_1, \phi_2 \in \mathtt{H}_{\mathrm{c}}(A)\right\} \ .
\end{aligned}$$

The maps B and B^{-1} change the differentiability of the functions. A better way is to use $A^{-1/2}B$ and $A^{1/2}B^{-1}$ instead. Let U be the operator

$$U(f + ig) = A^{1/2}B^{-1}f + iA^{-1/2}Bg, \quad \text{for real } f, g \in X. \tag{3.1}$$

We will prove in next subsection that these operators are bounded in Sobolev spaces and weighted L^2-spaces. We have established a one-to-one correspondence between the spectral decomposition of X with respect to $\mathcal{L}$ and that with respect to A:

$$\begin{aligned}
\psi \in \mathtt{E}_\kappa(\mathcal{L}) &\longleftrightarrow U\psi \in \mathtt{E}_\kappa(A), \\
\psi \in \mathtt{H}_{\mathrm{c}}(\mathcal{L}) &\longleftrightarrow U\psi \in \mathtt{H}_{\mathrm{c}}(A).
\end{aligned}$$

We have several ways to decompose $\mathcal{L}$ as products:

$$\begin{aligned}
\mathcal{L} &= V_1^{-1}\begin{bmatrix} 0 & 1 \\ -A^2 & 0 \end{bmatrix} V_1 & V_1 &= \begin{bmatrix} B^{-1} & 0 \\ 0 & B \end{bmatrix} \\
&= V_2^{-1}\begin{bmatrix} jA & 0 \\ 0 & -jA \end{bmatrix} V_2 & j &= \sqrt{-1}, \quad V_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \frac{1}{jA} \\ -jA & 1 \end{bmatrix} V_1 \\
&= U^{-1}\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} U & U &= \begin{bmatrix} A^{1/2}B^{-1} & 0 \\ 0 & A^{-1/2}B \end{bmatrix}
\end{aligned}$$

We prefer using the last one since it has a simpler form:

$$\mathcal{L} = U^{-1}(-iA)\,U \ ,$$

with U given in (3.1).

This decomposition will be especially useful when we integrate integrals of the form

$$\int_0^t e^{(t-s)\mathcal{L}}\,\mathbf{P}_{\mathrm{c}}^{\mathcal{L}}\,e^{2i\kappa s}\phi\,ds \ .$$

For this purpose, it is convenient to decompose

$$U = U_+ + U_-\mathtt{C}, \qquad U_\pm = A^{1/2}B^{-1} \pm A^{-1/2}B \ . \tag{3.2}$$

Here $\mathtt{C}$ is the conjugation operator, and both U_+ and U_- are self adjoint. Hence $[U, i] \neq 0$ but $[U_\pm, i] = 0$. More detailed properties of A and U are collected in next subsection.

3.2 Lemmas

Here we collect four lemmas. Recall

$$H = -\Delta - E + V + \lambda Q^2, \qquad A = (H^2 + \Pi H^{1/2}\lambda Q^2 H^{1/2}\Pi)^{1/2}.$$

Also, X is the subspace of all functions in $L^2(\mathbb{R}^3)$ that are orthogonal to Q, and Π is the orthogonal projection from $L^2(\mathbb{R}^3)$ onto X.

Lemma 3.1 (decay estimates for e^{iAt}) *For $q = 4, 8$,*

$$\left\|e^{-itA}\ \mathbf{P}_{\mathrm{c}}^{A}\,\Pi\phi\right\|_{L^q} \leq C\,|t|^{-3\left(\frac{1}{2}-\frac{1}{q}\right)}\,\|\phi\|_{L^{q'}} \ . \tag{3.3}$$

For smooth local functions ϕ and sufficiently large β, we have

$$\left\|\langle x\rangle^{-\beta}\ e^{-itA}\,\frac{1}{A-0i-2\kappa}\,\mathbf{P}_{\mathrm{c}}^{A}\,\Pi\,\langle x\rangle^{-\beta}\,\phi\right\|_{L^2} \leq C\,\langle t\rangle^{-6/5} \tag{3.4}$$

The estimate (3.4) for $A = \sqrt{H}$ was proved by Soffer-Weinstein [12].

Lemma 3.2

$$\left(Qu_+^2\,,\,\mathrm{Im}\,\frac{1}{A-0i-2\kappa}\,\mathbf{P}_{\mathrm{c}}^{A}\,\Pi Q u_+^2\right) = \left(\phi_0\phi_1^2\,,\,\mathrm{Im}\,\frac{1}{H_0-0i-2\kappa}\,\mathbf{P}_{\mathrm{c}}^{H_0}\,\phi_0\phi_1^2\right) + O(\lambda) > 0 \ . \tag{3.5}$$

This Lemma is a perturbation result. Notice that if $\lambda = 0$ then the statement of this lemma follows the assumption A.1. Since λ is small, by continuity it holds for small λ. We define

$$\Gamma \equiv 2\lambda^2\left(Qu_+^2\,,\,\mathrm{Im}\,\frac{1}{A-0i-2\kappa}\,\mathbf{P}_{\mathrm{c}}^{A}\,\Pi Q u_+^2\right) > 0 \ . \tag{3.6}$$

Lemma 3.3 (operator U) *(a) The operators U and U^{-1} are bounded operators in $W^{k,p} \cap X$ for $k \le 2$, $1 \le p < \infty$, and in $H^{0,r} \cap X$ for $r \le 3$. ($H^{0,r}$ is the weighted L^2 space with $\|f\|_{H^{0,r}} = \|\langle x \rangle^r f\|_{L^2}$.)*
(b) The commutator $[U, i]$ is a local operator in the sense

$$\|[U, i]\, \Pi\phi\|_{L^{8/7}} \le O(\lambda)\, \|\phi\|_{L^4} \ . \tag{3.7}$$

We denote the wave operators for $\mathcal{L}$ (resp. A and H) by $W_{\mathcal{L}}$, (resp. W_A and W_H).

Lemma 3.4 (wave operators) *The wave operators $W_{\mathcal{L}}$ and W_A exist and satisfy $W^{k,p}$ estimates for $k \le 2$, $1 \le p < \infty$: (Similar estimates hold for their adjoints.)*

$$\left\|W_{\mathcal{L}}\, \mathbf{P}_{\mathrm{c}}^{\mathcal{L}}\right\|_{(W^{k,p}, W^{k,p})} \le C, \qquad \left\|W_A\, \mathbf{P}_{\mathrm{c}}^{A}\, \Pi\right\|_{(W^{k,p}, W^{k,p})} \le C.$$

The statement on $W_{\mathcal{L}}$ was proved in [3], following the proof of [15]. Hence we only need to prove the statement on W_A.

Proof of these lemmas

We now proceed to prove these lemmas. To simplify the presentation, we will assume $\lambda > 0$. The proof for the case $\lambda < 0$ is exactly the same. Recall

$$H = -\Delta - E + V + \lambda Q^2, \qquad A = (H^2 + \Pi H^{1/2} \lambda Q^2 H^{1/2} \Pi)^{1/2}.$$

We also denote

$$H_* = -\Delta - E \ .$$

Note that, if W_A exists, we have the intertwining property that $f(A)P_c(A) = W_A f(H_*) W_A^*$ for suitable functions f. We also have similar property for $\mathcal{L}$.

Recall X is the space of all functions in $L^2(\mathbb{R}^3)$ that are orthogonal to Q, and Π is the orthogonal projection from $L^2(\mathbb{R}^3)$ onto X. In what follows we will only consider the restrictions of H and A on X. Hence we often omit the projection Π in the definition of A. (It should be noted, however, H_* acts on $L^2(\mathbb{R}^3)$.) We denote by $H^{0,r}$ the weighted L^2 space with norm $\|f\|_{H^{0,r}} = \|\langle x \rangle^r f\|_{L^2}$. In the remaining of the subsection, when we write L^2, $W^{k,p}$, or $H^{0,r}$, we often mean their intersection with X: $L^2 \cap X$, $W^{k,p} \cap X$, or $H^{0,r} \cap X$.

Assumption on V: We recall our Assumption A2. We assume that 0 is not an eigenvalue nor a resonance for $-\Delta + V$. We also assume that V satisfies the assumption in Yajima [15] so that the $W^{k,p}$ estimates for $k \le 2$ for the wave operator W_H holds: for a small $\sigma > 0$,

$$|\nabla^\alpha V(x)| \le C \langle x \rangle^{-5-\sigma}, \qquad \text{for } |\alpha| \le 2 \ .$$

Also, the functions $(x\cdot\nabla)^k V$, for $k=0,1,2,3$, are $-\Delta$ bounded with an $-\Delta$-bound <1:

$$\left\|(x\cdot\nabla)^k V\phi\right\|_2 \le \sigma_0 \left\|-\Delta\phi\right\|_2 + C\left\|\phi\right\|_2 , \qquad \sigma_0<1, \quad k=0,1,2,3 \ . \tag{3.8}$$

By the assumption, the following operators

$$H_*^{-1/2}(x\cdot\nabla)^k V H_*^{-1/2}, \quad (x\cdot\nabla)^k V H_*^{-1}, \quad H_*^{-1}(x\cdot\nabla)^k V \tag{3.9}$$

for $k=0,1,2,3$, are bounded in L^2.

Since Q is the ground state of H with V satisfying the previous assumptions, Q is a smooth function with exponential decay at infinity. Hence the above statements on V hold also for Q and Q^2. Since $V+\lambda Q^2$ and V have same properties, in what follows we will replace $V+\lambda Q^2$ in H by V and write $H=H_*+V$ to make the presentation simpler. So it should be kept in mind that the potential V in this subsection is in fact $V+\lambda Q^2$.

For two operators S and T, S is said to be *T-bounded* if ST^{-1} is a bounded operator. If both S and T are self-adjoint, this implies $T^{-1}S$ is also bounded. A deeper result says $S^{1/2}$ is $T^{1/2}$-bounded, see [RS2]. We say S and T are *mutually bounded* if both ST^{-1} and TS^{-1} are bounded operators. This is the case if $\|(S-T)T^{-1}\|_{(L^2,L^2)}=\theta<1$ for some θ. (It implies immediately $\|ST^{-1}\|<2$. Since $\|T\phi\|\le\|S\phi\|+\|(T-S)\phi\|\le\|S\phi\|+\theta\|T\phi\|$, we have $\|T\phi\|\le C\|S\phi\|$, which implies T is S-bounded.)

Lemma 3.5 *For each $k=\frac{1}{2},1,\frac{3}{2},2,3$, the operators H_*^k, H^k and A^k are mutually bounded.*

Proof. That H_*^k and H^k are mutually bounded follows from our assumption on V by standard argument. To show H^k and A^k are mutually bounded, it suffices to prove the cases $k=2$ and $k=3$ by the previous remark. We first show $\|(A^2-H^2)H^{-2}\|<1$, which implies the case $k=2$.

$$\left\|(A^2-H^2)H^{-2}\right\| = \left\|H^{1/2}\lambda Q^2 H^{1/2}H^{-2}\right\| \le \left\|H^{1/2}\lambda Q^2 H^{-1}\right\| \le \left\|H_*^{1/2}\lambda Q^2 H_*^{-1}\right\| \le 1/2.$$

The last inequality can be obtained by writing $H_*^{1/2}Q^2H_*^{-1}=H_*^{-1/2}Q^2+H_*^{1/2}[Q^2,H_*^{-1}]=H_*^{-1/2}Q^2+H_*^{-1/2}[Q^2,H_*]H_*^{-1}$, and noting $[Q^2,H_*]=\Delta Q^2+2\nabla Q^2\cdot\nabla$.

To prove the case $k=3$, it suffices to prove $A^6\le CH^6$ and $H^6\le CA^6$. Note

$$(fA^6f)=(fA^2A^2A^2f)\le(fA^2H^2A^2f) \ .$$

Since $A^2=H^2+H^{1/2}\lambda Q^2H^{1/2}$, we have

$$(fA^2H^2A^2f)\le C(fH^2H^2H^2f)+C(f(H^{1/2}\lambda Q^2H^{1/2})H^2(H^{1/2}\lambda Q^2H^{1/2})f)$$

where the cross terms are estimated by Schwarz inequality. To show that the last term is bounded by $C(fH^6f)$, we shall show that $H^{3/2}Q^2H^{-5/2}$ is bounded in X. Rewrite

$$\begin{aligned} H^{3/2}Q^2H^{-5/2} &= H^{3/2}H^{-2}Q^2H^{-1/2} + H^{3/2}[Q^2, H^{-2}]H^{-1/2} \\ &= H^{-1/2}Q^2H^{-1/2} + H^{-1/2}[Q^2, H^2]H^{-5/2} \end{aligned}$$

Since $[Q^2, H^2]$ is of the form $\sum_{|\alpha|\le 3} G_\alpha(x)\nabla^\alpha$, the operators on the right side of the equation are bounded in X. This shows $A^6 \le CH^6$. That $H^6 \le CA^6$ is proved similarly. **Q.E.D.**

Recall the standard formula:

$$T^{-\sigma} = \int_0^\infty \frac{1}{s+T}\,\frac{ds}{s^\sigma}, \qquad 0 < \sigma < 1\ . \tag{3.10}$$

The operator T in the above formula will be A^2 or H. Hence we also need to estimate operators of the form $\frac{H^m}{s+H^2}$. Clearly, for $s \ge 0$,

$$\left\| \frac{H^2}{s+H^2} \right\|_{\left(W^{k,p},W^{k,p}\right)} \le 1, \qquad \left\|H^{-1/2}\right\|_{\left(W^{k,p},W^{k,p}\right)} \le C\ . \tag{3.11}$$

Lemma 3.6 *Let $s \ge 0$. The operator $\dfrac{H}{s+H^2}$ is bounded in $W^{k,p} \cap X$ with*

$$\left\| \frac{H}{s+H^2} \right\|_{\left(W^{k,p},W^{k,p}\right)} \le C\,\langle s\rangle^{-1/2} \tag{3.12}$$

Also, for $k = \pm1, \pm2, \pm3$,

$$\left\| \langle x\rangle^k \frac{1}{s+H}\,\langle x\rangle^{-k} \right\|_{(L^2,L^2)} \le C\,\langle s\rangle^{-1}, \quad \left\| \langle x\rangle^k \frac{H}{s+H^2}\,\langle x\rangle^{-k} \right\|_{(L^2,L^2)} \le C\,\langle s\rangle^{-1/2}\ . \tag{3.13}$$

Proof. We can rewrite

$$\frac{H}{s+H^2} = \frac{1}{H+\sqrt{s}i} + \frac{1}{H-\sqrt{s}i}\ .$$

Therefore, to prove statements for $\frac{H}{s+H^2}$, it suffices to prove the corresponding statements for $\frac{1}{H\pm\sqrt{s}i}$. We first prove (3.12) for $k=0$. Let κ_1 denote the eigenvalue of the excited state of H and P_1 denote the projection onto the corresponding eigenspace. We can write

$$\left.\frac{1}{H+\sqrt{s}i}\right|_X = \frac{1}{\kappa_1+\sqrt{s}i}P_1 + W_H\,\frac{1}{p^2-E+\sqrt{s}i}\,W_H^*\,\mathbf{P_c}^H$$

where $p = -i\nabla$ and W_H is the wave operator of H. Note $E < 0$. Since W_H and W_H^* are bounded in $W^{k,p}$ for sufficiently nice V, ([15]), it is sufficient to prove that $\frac{1}{p^2-E\pm\sqrt{s}i}$ are bounded in $W^{k,p}$. However, $\frac{1}{p^2-E\pm\sqrt{s}i}$ are convolution operators with explicit Green functions:

$$\frac{C}{|x|}\,e^{-|x|(-E\pm\sqrt{s}i)^{1/2}}\ .$$

Since $|e^{-|x|(-E\pm\sqrt{s}i)^{1/2}}| \le e^{-c|x|\langle s\rangle^{1/4}}$, the L^1-norms of the Green functions are bounded by $\langle s\rangle^{-1/2}$. By Young's inequality we have

$$\left\|\frac{1}{p^2-E\pm\sqrt{s}i}\right\|_{(L^p,L^p)} \le C\,\langle s\rangle^{-1/2}\ ,$$

which proves (3.12) for $k=0$. For $k\ge 1$ and for $\phi\in W^{k,p}$, we have

$$\begin{aligned}\left\|H^{k/2}\frac{H}{s+H^2}\phi\right\|_{W^{k,p}} &\sim \left\|H^{k/2}\frac{H}{s+H^2}\phi\right\|_{L^p} = \left\|\frac{H}{s+H^2}H^{k/2}\phi\right\|_{L^p}\\ &\le C\,\langle s\rangle^{-1/2}\left\|H^{k/2}\phi\right\|_{L^p} \sim \langle s\rangle^{-1/2}\,\|\phi\|_{W^{k,p}}\ .\end{aligned}$$

This proves (3.12) for $k\ge 1$.

For (3.13), we prove the second part. The proof for the first part is similar. For $k>0$, since $\left[\langle x\rangle^k, \frac{1}{H+\sqrt{s}i}\right] = \frac{1}{H+\sqrt{s}i}\left[\langle x\rangle^k, H+\sqrt{s}i\right]\frac{1}{H+\sqrt{s}i}$ and

$$[\langle x\rangle^k, H+\sqrt{s}i] = 2\nabla^*(\nabla\langle x\rangle^k) - (\Delta\langle x\rangle^k)$$

we have

$$\left\|\left[\langle x\rangle^k, \frac{1}{H+\sqrt{s}i}\right]\langle x\rangle^{-k}\right\| \le C\left\|\frac{1}{H+\sqrt{s}i}(\nabla^*+1)\right\|\cdot\left\|\langle x\rangle^{k-1}\frac{1}{H+\sqrt{s}i}\ \langle x\rangle^{-k+1}\right\| \le C\,\langle s\rangle^{-1/2}$$

by induction in k. We have the same estimate for $\left[\langle x\rangle^k, \frac{1}{H-\sqrt{s}i}\right]$ and hence (3.13) holds for positive k. The proof for the case $k<0$ is similar. **Q.E.D.**

Recall $H^{0,r}$ is the weighted L^2 space with norm $\|f\|_{H^{0,r}} = \|\langle x\rangle^r f\|_{L^2}$.

Lemma 3.7 *The operators $H^{1/2}A^{-1/2}$, $A^{-1/2}H^{1/2}$, $H^{-1/2}A^{1/2}$ and $A^{1/2}H^{-1/2}$ are bounded operators in $W^{k,p}\cap X$ and $H^{0,r}\cap X$.*

Proof. By (3.10) we can write

$$\begin{aligned}H^{1/2}A^{-1/2} &= H^{1/2}\int_0^\infty \frac{1}{s+H^2+H^{1/2}\lambda Q^2H^{1/2}}\,\frac{ds}{s^{1/4}}\\ &= H^{1/2}\int_0^\infty\left[\frac{1}{s+H^2}+\frac{1}{s+H^2}H^{1/2}\lambda Q\sum_{n=0}^\infty\left(\lambda Q\frac{H}{s+H^2}Q\right)^n QH^{1/2}\frac{1}{s+H^2}\right]\frac{ds}{s^{1/4}}\\ &= 1+\int_0^\infty\left[\frac{H}{s+H^2}\lambda Q\sum_{n=0}^\infty\left(\lambda Q\frac{H}{s+H^2}Q\right)^n Q\frac{H}{s+H^2}H^{-1/2}\right]\frac{ds}{s^{1/4}}\end{aligned}$$

Since $\left\|\frac{H}{s+H^2}\right\| \le \langle s\rangle^{-1/2}$ by Lemma 3.6, we have

$$\left\|H^{1/2}A^{-1/2}\right\|_{(W^{k,p},W^{k,p})} \le 1+C\int_0^\infty\langle s\rangle^{-1/2}\,\lambda\sum_{n=0}^\infty(\lambda\langle s\rangle^{-1/2})^n\langle s\rangle^{-1/2}\,\frac{ds}{s^{1/4}} \le 1+C\lambda\ .$$

Similarly

$$\left\|A^{-1/2}H^{1/2}\right\|_{(W^{k,p},W^{k,p})} \leq 1 + C\lambda \ .$$

Also, using (3.13), for $r \leq 3$ we have

$$\left\|H^{1/2}A^{-1/2}\right\|_{(H^{0,r},H^{0,r})} + \left\|A^{-1/2}H^{1/2}\right\|_{(H^{0,r},H^{0,r})} \leq 1 + C\lambda \ .$$

The above proves that $H^{1/2}A^{-1/2}$ and $A^{-1/2}H^{1/2}$ are bounded in $W^{k,p}$ and $H^{0,r}$. Indeed, we have proved

$$\left\|\langle x\rangle^3 \left(H^{1/2}A^{-1/2} - 1\right)\langle x\rangle^3\right\|_{(L^2,L^2)} + \left\|\langle x\rangle^3 \left(A^{-1/2}H^{1/2} - 1\right)\langle x\rangle^3\right\|_{(L^2,L^2)} \leq C\lambda \ . \tag{3.14}$$

We now consider $H^{-1/2}A^{1/2}$ and $A^{1/2}H^{-1/2}$. Since $A^{1/2} = A^2A^{-3/2} = A^2\int_0^\infty \frac{1}{s+A^2}\frac{ds}{s^{3/4}}$, we have

$$\begin{aligned} H^{-1/2}A^{1/2} &= H^{-1/2}(H^2 + H^{1/2}\lambda Q^2 H^{1/2})\int_0^\infty \frac{1}{s + H^2 + H^{1/2}\lambda Q^2 H^{1/2}}\frac{ds}{s^{3/4}} \\ &= (H^{3/2} + \lambda Q^2 H^{1/2})\int_0^\infty \frac{1}{s + H^2 + H^{1/2}\lambda Q^2 H^{1/2}}\frac{ds}{s^{3/4}} = I_1 + \lambda Q^2 I_2 \end{aligned}$$

The main term is I_1. The term I_2 is similar to $H^{1/2}A^{-1/2}$, and its integrand has a better decay in s for large s. Hence

$$\left\|\lambda Q^2 I_2\right\| \leq C\lambda \left\|I_2\right\| \leq C\lambda \ .$$

For the main term I_1,

$$I_1 = 1 + \int_0^\infty \frac{H^2}{s+H^2}Q\sum_{n=0}^\infty \left(Q\frac{H}{s+H^2}Q\right)^n Q\frac{H}{s+H^2}H^{-1/2}\frac{ds}{s^{3/4}} \ .$$

Hence

$$\|I_1\| \leq 1 + C\int_0^\infty \lambda\sum_{n=0}^\infty (\lambda\langle s\rangle^{-1/2})^n \langle s\rangle^{-1/2}\frac{ds}{s^{3/4}} \leq 1 + C\int_0^\infty \lambda\langle s\rangle^{-1/2}\frac{ds}{s^{3/4}} \leq 1 + C\lambda \ .$$

Here the norms are taken in $(W^{k,p}, W^{k,p})$ and $(H^{0,r}, H^{0,r})$. Hence we have proved Lemma 3.7. **Q.E.D.**

In fact, the last part of the above proof also shows

$$\left\|\langle x\rangle^3 \left(H^{-1/2}A^{1/2} - 1\right)\langle x\rangle^3\right\|_{(L^2,L^2)} + \left\|\langle x\rangle^3 \left(A^{1/2}H^{-1/2} - 1\right)\langle x\rangle^3\right\|_{(L^2,L^2)} \leq C\lambda \ . \tag{3.15}$$

The above lemma proves part (a) of Lemma 3.3 stating that U and U^{-1} are bounded in $W^{k,p}$ and $H^{0,r}$. Moreover, (3.14) and (3.15) mean that $U - 1$ and $U^{-1} - 1$ are "local" operators. In particular, they imply part (b) of Lemma 3.3 and that, for any $\phi \in L^2$,

$$(U^{\pm 1} - 1)e^{-itH_*}\phi \to 0 \qquad \text{in } L^2, \quad \text{as } t \to \infty \ . \tag{3.16}$$

We now prove Lemma 3.4. We only need to prove the statement on W_A. Notice

$$W_A = \lim_{t\to\infty} e^{itA}e^{-itH_*} = \lim_{t\to\infty} Ue^{t\mathcal{L}}U^{-1}e^{-itH_*} = \lim_{t\to\infty} Ue^{t\mathcal{L}}e^{-itH_*} + \lim_{t\to\infty} Ue^{t\mathcal{L}}(U^{-1}-1)e^{-itH_*} \ .$$

By (3.16) we have

$$W_A = \lim_{t\to\infty} Ue^{t\mathcal{L}}e^{-itH_*} = UW_{\mathcal{L}} \ .$$

The bounededness of W_A follows from that of U and $W_{\mathcal{L}}$. This proves Lemma 3.4.

We now prove Lemma 3.1. Since

$$e^{-itA}\,\mathbf{P}_{\mathrm{c}}^A\,\phi = W_A e^{-itH_*} W_A^*\,\mathbf{P}_{\mathrm{c}}^A\,\phi \ ,$$

the estimate (3.3) follows from the usual (L^p, L^q) estimate for e^{-itH_*} and the boundedness of W_A and $\mathbf{P}_{\mathrm{c}}^A$ in L^p-spaces. To prove (3.4), either we prove the bounededness of W_A in weighted spaces $H^{0,r}$, or we use the Mourre estimate. We will follow the second approach and the argument in [12].

Let $a = 2\kappa$. We consider intervals $\Delta = (a-r, a+r)$. Let $g_\Delta(t) = g_0((t-a)/r)$, where g_0 is a fixed smooth function with support in $(-2,2)$ and $g_0(t) = 1$ for $|t| < 1$. We will consider $g_\Delta(A)$ with r small enough. Let $D = xp + px$, $(p = -i\nabla)$, and the commutators

$$\mathrm{ad}_D^0(A) = A, \qquad \mathrm{ad}_D^{k+1}(A) = [\,\mathrm{ad}_D^k(A), D] \ .$$

We need to prove the following lemma.

Lemma 3.8 *For Δ small enough, the Mourre estimate*

$$g_\Delta(A)[iA, D]g_\Delta(A) \geq \theta g_\Delta(A)^2$$

holds for some $\theta > 0$. Also, $g_\Delta(A)\,\mathrm{ad}_D^k(A)g_\Delta(A)$ are bounded operators in L^2 for $k = 0,1,2,3$.

We will use the following lemma.

Lemma 3.9 *The operators*

$$H^{-3}D^kH^{m/2}\,\langle x\rangle^{-3} \qquad \textit{and} \qquad \langle x\rangle^{-3}\,H^{m/2}D^kH^{-3}$$

are bounded in L^2, for $k, m = 0,1,2,3$.

Proof. This is standard and we only sketch the proof. If m is even, we can compute the commutator $[D^k, H^{m/2}]$ explicitly and estimate

$$H^{-3}D^kH^{m/2}\langle x\rangle^{-3} = H^{-3}H^{m/2}D^k\langle x\rangle^{-3} + H^{-3}[D^k, H^{m/2}]\langle x\rangle^{-3} \ .$$

If m is odd, we write

$$H^{-3}D^kH^{m/2}\langle x\rangle^{-3} = \int_0^\infty H^{-3}D^kH^{(m+1)/2}\frac{1}{s+H}\langle x\rangle^{-3}\frac{ds}{\sqrt{s}}$$

and proceed as in the case m is even, by using (3.13). Here we have used the formula (3.10).

Q.E.D.

Proof of Lemma 3.8

Let $G = A - H$ and we write $A = H + G$. Since

$$[iA, D] = [H_* + V + G, iD] = -\Delta + [V + G, iD] = A - V - G + [V + G, iD]$$

and $g_\Delta(A)Ag_\Delta(A) \geq 2\theta g_\Delta(A)^2$ for some $2\theta > 0$, it suffices to show that, for $M = -V + [V, iD]$,$-G$ and $[G, D]$, the operators

$$g_\Delta(A)Mg_\Delta(A) = (g_\Delta(A)H_*^2)\,(H_*^{-2}MH_*^{-2})\,(H_*^2g_\Delta(A))$$

are bounded by $g_\Delta(A)^2$ and the bound goes to zero when the interval Δ shrinks to zero. Since both $g_\Delta(A)H_*^2 = (g_\Delta(A)A^2)(A^{-2}H_*^2)$ and $H_*^2g_\Delta(A) = (H_*^2A^{-2})(A^2g_\Delta(A))$ are bounded and converges to zero weakly when Δ shrinks to zero, this will be true if one can show that $H_*^{-2}MH_*^{-2}$ is compact. The case $M = -V + [V, iD]$ is standard and follows from our assumption, so we only consider $H_*^{-2}GH_*^{-2}$ and $H_*^{-2}[G, D]H_*^{-2}$.

We proceed to find an explicit form of G. By (3.10) with $T = A^2$, $\sigma = 1/2$, we write

$$\begin{aligned}
A^{-1} &= \int_0^\infty \frac{1}{s + H^2 + H^{1/2}\lambda Q^2H^{1/2}}\frac{ds}{\sqrt{s}} \\
&= \int_0^\infty \frac{1}{s+H^2} + \frac{1}{s+H^2}H^{1/2}\lambda Q\sum_{n=0}^\infty\left(\lambda Q\frac{H}{s+H^2}Q\right)^n QH^{1/2}\frac{1}{s+H^2}\frac{ds}{\sqrt{s}} \\
&= H^{-1} + H^{-1/2}\langle x\rangle^{-3}J_0\langle x\rangle^{-3}H^{-1/2}
\end{aligned}$$

where

$$J_0 = \int_0^\infty \langle x\rangle^3\frac{H}{s+H^2}\lambda Q\sum_{n=0}^\infty\left(\lambda Q\frac{H}{s+H^2}Q\right)^n Q\frac{H}{s+H^2}\langle x\rangle^3\frac{ds}{\sqrt{s}} \ .$$

By Lemma 3.6, $\|J_0\|_{(L^2,L^2)} \le \int_0^\infty \langle s\rangle^{-1/2} \cdot \lambda \cdot \langle s\rangle^{-1/2} s^{-1/2}\, ds \le C\lambda$. Hence

$$\begin{aligned} A &= A^2A^{-1} = (H^2 + H^{1/2}\lambda Q^2 H^{1/2}) \left(H^{-1} + H^{-1/2}\langle x\rangle^{-3} J_0 \langle x\rangle^{-3} H^{-1/2}\right) = H + G \\ G &= H^{1/2}\lambda Q^2 H^{-1/2} + H^{3/2}\langle x\rangle^{-3} J_0 \langle x\rangle^{-3} H^{-1/2} + H^{1/2}\lambda Q^2 \langle x\rangle^{-3} J_0 \langle x\rangle^{-3} H^{-1/2} \end{aligned} \tag{3.17}$$

Since $H^{-1/2}\langle x\rangle^{-1}$ and $\langle x\rangle^{-1} H^{-1/2}$ are compact, from (3.17) $H_*^{-2} G H_*^{-2}$ is compact. We can also write

$$H_*^{-2} G D H_*^{-2} = \left\{H_*^{-2} G H^{1/2}\langle x\rangle\right\} \cdot \left\{\langle x\rangle^{-1} H^{-1/2} D H_*^{-2}\right\}$$

The second operator is bounded by Lemma 3.9. The first is compact since its terms are of the form: $H^{-m} \cdot \langle x\rangle^{-k} \cdot$ (bounded operator). Similarly $H_*^{-2} D G H_*^{-2}$ is also compact. Hence we conclude the Mourre estimate.

To show that $g_\Delta(A)\,\mathrm{ad}_D^k(A) g_\Delta(A)$ are bounded for $k = 0, 1, 2, 3$, we rewrite

$$g_\Delta(A)\,\mathrm{ad}_D^k(A) g_\Delta(A) = (g_\Delta(A)A^3)\,(A^{-3}H^3)\,(H^{-3}\,\mathrm{ad}_D^k(A) H^{-3})\,(H^3 A^{-3})\,(A^3 g_\Delta(A))\ .$$

We only need to show that $H^{-3}\,\mathrm{ad}_D^k(A) H^{-3}$ are bounded since the other terms are bounded by Lemma 3.5. Recall $A = H + G$. It is standard to prove that $H^{-3}\,\mathrm{ad}_D^k(H) H^{-3}$ are bounded. For $H^{-3}\,\mathrm{ad}_D^k(G) H^{-3}$, since it is a sum of terms of the form

$$H^{-3} D^k G D^m H^{-3}, \qquad k + m \le 3\ ,$$

it suffices to show that these terms are bounded. By the explicit form (3.17) of G and Lemma 3.9, they are indeed bounded. For example,

$$\begin{aligned} &H^{-3} D^2 \left\{H^{3/2}\langle x\rangle^{-3} J_0 \langle x\rangle^{-3} H^{-1/2}\right\} D^1 H^{-3} \\ &\quad = \left\{H^{-3} D^2 H^{3/2}\langle x\rangle^{-3}\right\} J_0 \left\{\langle x\rangle^{-3} H^{-1/2} D^1 H^{-3}\right\}, \end{aligned}$$

a product of three bounded operators. We conclude that $g_\Delta(A)\,\mathrm{ad}_D^k(A) g_\Delta(A)$ are bounded for $k = 0, 1, 2, 3$. **Q.E.D.**

With Lemma 3.8, (cf. the remark in [12], p.27), the minimal velocity estimate in [5] and Theorem 2.4 of [10] implies

$$\left\| F(D \le \theta t/2)\, e^{-iAt} g_\Delta(A) \langle D\rangle^{-3/2} \right\|_{(L^2,L^2)} \le C \langle t\rangle^{-5/4}\,.$$

The same argument in [12] then gives the desired decay estimate (3.4) in Lemma 3.1.

Finally we prove Lemma 3.2. Let $\psi_\lambda = \mathbf{P}_{\mathrm{c}}^A \Pi Q u_+^2$ and $\psi_0 = \mathbf{P}_{\mathrm{c}}^{H_0} \phi_0 \phi_1^2$. Recall $H_0 = -\Delta + V - e_0$. We have $\psi_\lambda = \psi_0 + O(\lambda)$. We write $\psi_\lambda = \psi_0 + b\phi_1 + \eta$, where $\eta \in \mathtt{H}_{\mathrm{c}}(H_0)$ and

$b, \eta = O(\lambda)$. Rewrite

$$\left(\psi_\lambda\,,\, \mathrm{Im}\,\frac{1}{A-0i-2\kappa}\psi_\lambda\right) = \mathrm{Im}\, i\int_0^\infty \left(\psi_\lambda\,,\, e^{-it(A-0i-2\kappa)}\psi_\lambda\right)\, dt$$

$$= \mathrm{Im}\, i\int_0^\infty \left(\psi_\lambda\,,\, e^{-it(H_0-0i-2\kappa)}\psi_\lambda\right)\, dt \tag{3.18}$$

$$+\,\mathrm{Im}\, i\int_0^\infty\int_0^t \left(\psi_\lambda\,,\, e^{-i(t-s)(A-0i-2\kappa)}\left(\lambda Q^2+G\right)e^{-is(H_0-0i-2\kappa)}\psi_\lambda\right)\, ds\, dt\ . \tag{3.19}$$

The main term lies in (3.18). It is

$$\mathrm{Im}\, i\int_0^\infty \left(\psi_0\,,\, e^{-it(H_0-0i-2\kappa)}\psi_0\right)\, dt = \left(\psi_0\,,\, \mathrm{Im}\,\frac{1}{H_0-0i-2\kappa}\psi_0\right)$$

which is the desired main term in Lemma 3.2. We want to show that the rest of (3.18) and (3.19) are integrable and of order $O(\lambda)$. Recall we write $\psi_\lambda = \psi_0 + b\phi_1 + \eta$. For the term η in ψ_λ, by decay estimate we have

$$|\left(\psi_\lambda\,,\, e^{-it(H_0-0i-2\kappa)}\eta\right)| \le C\,\langle t\rangle^{-3/2}\,\|\psi_\lambda\|_{L^1\cap L^2}\,\|\eta\|_{L^1\cap L^2} \le C\,\langle t\rangle^{-3/2}\,\lambda\ , \tag{3.20}$$

hence this term is integrable. Also, since $H_0\phi_1 = e_{01}\phi_1$,

$$\left(\psi_\lambda\,,\, e^{-it(H_0-0i-2\kappa)}b\phi_1\right) = \left(\psi_\lambda\,,\, e^{-it(e_{01}-2\kappa-0i)}b\phi_1\right)\ ,$$

so we can integrate this oscillation term explicitly. (The boundary term at $t=\infty$ vanishes due to the decay of $e^{-it(-0i)}$.) We conclude that the rest of (3.18) are integrable and of order $O(\lambda)$.

For (3.19), it suffices to show its integrability since λQ^2+G gives the order $O(\lambda)$. Rewrite the last ψ_λ in (3.19) as $b\phi_1 + \mathbf{P}_{\mathrm{c}}^{H_0}\,\psi_\lambda$. For the part containing $b\phi_1$, we have

$$\left(\psi_\lambda\,,\, e^{-i(t-s)(A-0i-2\kappa)}\left(\lambda Q^2+G\right)e^{-is(H_0-0i-2\kappa)}b\phi_1\right)$$
$$= \left(\psi_\lambda\,,\, e^{-it(A-0i-2\kappa)}\,e^{is(A-e_{01})}\left(\lambda Q^2+G\right)b\phi_1\right)\ .$$

Integration in s gives

$$\left((A-e_{01})^{-1}\psi_\lambda\,,\, e^{-it(A-0i-2\kappa)}\left(\lambda Q^2+G\right)b\phi_1\right)\ .$$

Since e_{01} lies outside the continuous spectrum of A, the last expression is integrable in t following the same argument as (3.20). For the part containing $\mathbf{P}_{\mathrm{c}}^{H_0}\,\psi_\lambda$, since (λQ^2+G) is a "local" operator in the sense that it sends L^∞ functions to L^1, we have

$$|\left(\psi_\lambda\,,\, e^{-i(t-s)(A-0i-2\kappa)}\left(\lambda Q^2+G\right)e^{-is(H_0-0i-2\kappa)}\,\mathbf{P}_{\mathrm{c}}^{H_0}\,\psi_\lambda\right)| \le C\lambda\,\langle t-s\rangle^{-3/2}\,\langle s\rangle^{-3/2}\,\|\psi_\lambda\|^2_{L^1\cap L^2}$$

which can be integrated in s and t. Hence we have proved Lemma 3.2.

4 Main oscillation terms

We now identify the main oscillation terms in equation (2.5). We shall use the complex amplitude of the excited state, z, as the reference. Recall $z(t) = e^{-i\kappa t}p(t)$. Eventually we will have $p(t) \sim t^{-1/2}$ and its oscillation is much smaller than κ. The change of mass on the direction of the nonlinear ground state is given by a. We will also show that $a = O(z^2)$ and the order of the dispersive wave $\eta(t)$ is also of order $O(z^2)$. Assuming these orders, the second order term in F, $F^{(2)}$, is given explicitly by:

$$F^{(2)} = \lambda Q(2|\zeta|^2 + \zeta^2) = z^2\phi_{(20)} + z\bar{z}\phi_{(11)} + \bar{z}^2\phi_{(02)}$$

where $\zeta = zu_+ + \bar{z}u_-$ and

$$\begin{aligned}
\phi_{(20)} &= \lambda Q(u_+^2 + 2u_+u_-) = \lambda Q u_+^2 + O(\lambda^2), \\
\phi_{(11)} &= 2\lambda Q(u_+^2 + u_-^2 + u_+u_-) = O(\lambda), \\
\phi_{(02)} &= \lambda Q(u_-^2 + 2u_+u_-) = O(\lambda^2).
\end{aligned} \tag{4.1}$$

We shall write $F^{(2)} = z^\alpha\phi_\alpha$, where α is a double indices (ij) with $i + j = 2$ and $i, j \geq 0$. The repeated indices mean summation. We shall use β later on to denote double indices summing to 3 and γ for 4.

4.1 Main oscillation term in a and F

We start with identifying the main oscillation terms of $a(t)$. We have fixed the boundary condition of a at $t = \infty$ and set $a(\infty) = 0$. Thus we can rewrite the equation for $a(t)$ into the following equivalent integral equation:

$$a(t) = \int_\infty^t (c_1Q, \operatorname{Im} F(k))\, ds\ .$$

As the oscillation term of order z^2 in F comes from $F^{(2)}$, we have up to second order in z^2

$$(c_1Q, \operatorname{Im} F) \sim A^{(2)}$$

where

$$A^{(2)} = \big(c_1Q,\ \lambda Q \operatorname{Im}\zeta^2\big)$$

Since $\zeta = zu_+ + \bar{z}u_-$, we have

$$\operatorname{Im}\zeta = \operatorname{Im} z\,(u_+ - u_-), \quad \operatorname{Im}\zeta^2 = (\operatorname{Im} z^2)\,(u_+^2 - u_-^2)\ .$$

Therefore, we have

$$A^{(2)} = C_1 \ \mathrm{Im}\, z^2, \qquad C_1 = \big(c_1 Q,\, \lambda Q(u_+^2 - u_-^2)\big) = O(\lambda^2)\ ,$$

We can integrate $A^{(2)}$ by parts to have:

$$\begin{aligned}\int_\infty^t A^{(2)}\, ds &= C_1 \,\mathrm{Im} \int_\infty^t z^2\, ds = C_1 \,\mathrm{Im} \int_\infty^t \mu^{-2} p^2\, ds \\ &= C_1 \,\mathrm{Im}\, \frac{1}{-2i\kappa} \left\{ \mu^{-2} p^2 - \int_\infty^t \mu^{-2} 2p\dot{p}\, ds \right\}\end{aligned}$$

As we shall prove later on, the last integral is higher order term. The first term on the right hand side can be written explicitly as

$$C_1 \,\mathrm{Im}\, \frac{1}{-2i\kappa} \mu^{-2} p^2 = a_{20}(z^2 + \bar{z}^2)\ ,$$

where

$$a_{20} = \frac{C_1}{4\kappa} = \frac{\lambda}{4\kappa} \left\{ c_1 Q,\, Q(u_+^2 - u_-^2) \right\} = O(\lambda^2)\ , \tag{4.2}$$

We shall prove later on that the last term $a_{20}(z^2 + \bar{z}^2)$ is the main oscillatory term in a. We denote the rest by b, i.e.,

$$a = a_{20}(z^2 + \bar{z}^2) + b \tag{4.3}$$

As shall prove $a, b, \dot{a} = O(z^2)$, but $\dot{b} = O(z^3)$. In other words, b is the part of a that has slower oscillation. From the equation of a, we have the following equation for b:

$$\dot{b} = \big(c_1 Q, \mathrm{Im}(F - F^{(2)})\big) - 4\,\mathrm{Re}\, a_{20} z \mu^{-1} \dot{p}\ . \tag{4.4}$$

Assuming that b and η are of order z^2, we can decompose F into

$$F = F^{(2)} + F^{(3)} + \widetilde{F}^{(3)} + F^{(4)}$$

where $F^{(2)}$ and $F^{(3)}$ denote terms of order z^2 and z^3, respectively, and $F^{(4)}$ denotes higher order terms:

$$\begin{aligned}F^{(2)} &= \lambda Q(2|\zeta|^2 + \zeta^2) = z^2 \phi_{(20)} + z\bar{z}\phi_{(11)} + \bar{z}^2 \phi_{(02)} \\ F^{(3)} &= 2\lambda Q[(\zeta + \bar{\zeta})\eta^{(2)} + \zeta\bar{\eta}^{(2)}] + \lambda|\zeta|^2\zeta + 2\lambda a_{20}(z^2 + \bar{z}^2) QR(2\zeta + \bar{\zeta}) \\ \widetilde{F}^{(3)} &= 2\lambda b QR(2\zeta + \bar{\zeta}) \\ F^{(4)} &= 2\lambda Q[(\zeta + \bar{\zeta})\eta^{(3)} + \zeta\bar{\eta}^{(3)}] + \lambda Q\left[2|\eta|^2 + \eta^2\right] + 2\lambda a QR(2\eta + \bar{\eta}) \\ &\quad + 3\lambda a^2 QR^2 + \lambda\left[|k|^2 k - |\zeta|^2\zeta\right]\ ,\end{aligned} \tag{4.5}$$

We can also rewrite the equation of θ into

$$\dot{\theta} = c_2(z + \bar{z}) + F_\theta \ , \tag{4.6}$$

where $c_2 = -(c_0 Q, \lambda Q^2 u) = O(\lambda)$ and

$$F_\theta = \frac{-1}{1 + c_0 c_1^{-1} a} \left\{ c_0 c_1^{-1} c_2 a(z + \bar{z}) + \Big[a + (c_0 Q, \lambda Q^2(\eta + \bar{\eta})) + (c_0 Q, \operatorname{Re} F(k)) \Big] \right\}$$

Hence, since $z = \mu^{-1} p$,

$$\begin{aligned} \theta(t) &= \int_0^t 2c_2 \operatorname{Re}(z) + F_\theta \, ds = 2c_2 \operatorname{Re} z/(-i\kappa) + \int_0^t -2c_2 \operatorname{Re}(\mu^{-1}\dot{p})/(-i\kappa) + F_\theta \, ds \\ &= \frac{2c_2}{\kappa} \operatorname{Im} z + \int_0^t -\frac{2c_2}{\kappa} \operatorname{Im}(\mu^{-1}\dot{p}) + F_\theta \, ds \end{aligned} \tag{4.7}$$

4.2 Main oscillation term in η

We now identify the main oscillation term in η. We first rewrite the equation of η using the operator A as

$$\begin{aligned} \partial_t \eta &= \mathcal{L}\eta - \mathbf{P}_c^{\mathcal{L}} \, i\dot{\theta}\eta + \mathbf{P}_c^{\mathcal{L}} \, \Pi F^\sharp \ , \\ F^\sharp &= -i\dot{\theta}\zeta - iF(k) - [(c_1 Q, \operatorname{Im} F) + ia\dot{\theta}] R_\Pi \ . \end{aligned} \tag{4.8}$$

Notice that $F^\sharp$ and F^{all} differs only by the term $i\dot{\theta}\eta$. Observe also that $-i\dot{\theta}\zeta$ is not killed by $\mathbf{P}_c^{\mathcal{L}}$ since $[\mathcal{L}, i] \neq 0$. Let

$$\eta^\diamond = U\eta \ .$$

Since $\mathcal{L} = U^{-1}(-iA)U$ and $U \, \mathbf{P}_c^{\mathcal{L}} = \mathbf{P}_c^A \, U$, we have

$$\begin{aligned} \partial_t \eta^\diamond &= -iA\eta^\diamond - \mathbf{P}_c^A \, U i\dot{\theta} U^{-1} \eta^\diamond + \mathbf{P}_c^A \, U \Pi F^\sharp \\ &= -iA\eta^\diamond - i\dot{\theta}\eta^\diamond - \mathbf{P}_c^A \, [U, i]\dot{\theta} U^{-1}\eta^\diamond + \mathbf{P}_c^A \, U\Pi F^\sharp \ . \end{aligned}$$

Let $\tilde{\eta} = e^{i\theta}\eta^\diamond$ and use $U^{-1}\eta^\diamond = \eta$, we get

$$\partial_t \tilde{\eta} = -iA\tilde{\eta} + e^{i\theta} \, \mathbf{P}_c^A \, U\Pi F^\sharp - e^{i\theta} \, \mathbf{P}_c^A \, [U, i]\dot{\theta}\eta \ . \tag{4.9}$$

Hence

$$\tilde{\eta}(t) = e^{-iAt}\tilde{\eta}_0 + \int_0^t e^{-iA(t-s)} \, \mathbf{P}_c^A \, \left\{ e^{i\theta} U\Pi F^\sharp - e^{i\theta}[U, i]\dot{\theta}\eta \right\} \, ds \ . \tag{4.10}$$

Since $\eta = U^{-1} e^{-i\theta}\tilde{\eta}$ and U is bounded in Sobolev spaces, for the purpose of estimation we can treat η and $\tilde{\eta}$ the same.

From the the definition of $\tilde{\eta}$ (4.10), the integrand is $e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U,i]\dot{\theta}\eta$. We first identify the main term in $F^\sharp$:

$$\begin{aligned} F^\sharp &= -i\dot{\theta}\zeta - iF(k) - [(c_1 Q, \operatorname{Im} F) + ia\dot{\theta}]R_\Pi \\ &= i\, z^\alpha \phi^\sharp_\alpha + F^{\sharp\sharp}. \end{aligned} \tag{4.11}$$

We have already decomposed F into orders in z. To identify the main term of $F^\sharp$, it remains to decompose $-i\dot{\theta}\zeta$ and $(c_1 Q, \operatorname{Im} F)R_\Pi$. From the equation of θ (2.5), we have

$$-i\dot{\theta}\zeta = -ic_2(z+\bar{z})(zu_+ + \bar{z}u_-) - iF_\theta\zeta$$

Also

$$(c_1 Q, \operatorname{Im} F)R_\Pi = (c_1 Q, \phi_{20} - \phi_{02})R_\Pi \operatorname{Im} z^2 + O(z^3).$$

Recall the decomposition of F in (4.5) and $F^{(2)} = z^\alpha \phi_\alpha$. Now we have (4.11) with

$$F^{\sharp\sharp} = -iF_\theta\zeta - i(F - F^{(2)}) - [(c_1 Q, \operatorname{Im}(F - F^{(2)})) + ia\dot{\theta}]R_\Pi$$

and $\phi^\sharp_\alpha$ are defined as follows:

$$\begin{aligned} \phi^\sharp_{20} &= -\phi_{20} - c_2 u_+ + \tfrac{1}{2}(c_1 Q, \phi_{20} - \phi_{02})R_\Pi = -\phi_{20} - c_2 u_+ + O(\lambda^2)\ , \\ \phi^\sharp_{11} &= -\phi_{11} - c_2(u_+ + u_-) \qquad\qquad\qquad\quad = O(\lambda)\ , \\ \phi^\sharp_{02} &= -\phi_{02} - c_2 u_- + \tfrac{1}{2}(c_1 Q, \phi_{20} - \phi_{02})R_\Pi = O(\lambda^2)\ . \end{aligned} \tag{4.12}$$

Here we have used $\phi_{20} = O(\lambda) = \phi_{11}$, $\phi_{02} = O(\lambda^2)$, $c_2 = O(\lambda)$ and $R_\Pi = O(1)$. Also note that they are all real.

Recall (3.2). Since $Ui = (U_+ + U_- C)i = i(U_+ - U_- C)$, We have

$$\mathbf{P}_{\mathrm{c}}^A\, e^{i\theta} U i z^\alpha \Pi \phi^\sharp_\alpha = \mathbf{P}_{\mathrm{c}}^A\, e^{i\theta} i(U_+ - U_- C) z^\alpha \Pi \phi^\sharp_\alpha = e^{i\theta} i z^\alpha \Phi_\alpha$$

where

$$\begin{aligned} \Phi_{20} &= \mathbf{P}_{\mathrm{c}}^A \left\{ U_+ \Pi \phi^\sharp_{20} - U_- \Pi \phi^\sharp_{02} \right\}, \quad \Phi_{11} = \mathbf{P}_{\mathrm{c}}^A \left\{ (U_+ - U_-)\Pi \phi^\sharp_{11} \right\}, \\ \Phi_{02} &= \mathbf{P}_{\mathrm{c}}^A \left\{ -U_- \Pi \phi^\sharp_{20} + U_+ \Pi \phi^\sharp_{02} \right\}\ . \end{aligned} \tag{4.13}$$

Hence we can rewrite the integrand in (4.10) as

$$e^{i\theta} i z^\alpha \Phi_\alpha + \mathbf{P}_{\mathrm{c}}^A \left\{ e^{i\theta} U \Pi F^{\sharp\sharp} - e^{i\theta}[U,i]\dot{\theta}\eta \right\}\ .$$

The leading term in $\tilde{\eta}$ is

$$
\begin{aligned}
(I) &\equiv \int_0^t e^{-iA(t-s)} e^{i\theta} i z^\alpha \Phi_\alpha \, ds \\
&= \int_0^t e^{-itA} e^{is(A-0i)} \left(\mu^{[\alpha]} (e^{i\theta} p^2)(s) i \Phi_\alpha \right) \, ds \qquad (\mu = e^{i\kappa s}) \\
&= e^{-itA} \left[\frac{e^{is(A-0i)} [e^{i\theta} z^\alpha](s)}{i(A - 0i + [\alpha]\kappa)} i \Phi_\alpha \right]_{s=0}^t + (II) \\
&= \tilde{\eta}^{(2)} - e^{-itA} (e^{i\theta} z^\alpha)(0) \widetilde{\eta}_\alpha + (II) \ , \qquad (4.14)
\end{aligned}
$$

where

$$
\tilde{\eta}^{(2)} = e^{i\theta} z^\alpha \widetilde{\eta}_\alpha, \qquad \widetilde{\eta}_\alpha = \frac{1}{A - 0i + [\alpha]\kappa} \Phi_\alpha \qquad (4.15)
$$

and (II) is the error from integration by parts,

$$
(II) = - \int_0^t e^{-i(t-s)A} \left\{ \mu^\alpha \frac{d}{ds} (e^{i\theta} p^\alpha) \widetilde{\eta}_\alpha \right\} \, ds \ .
$$

Also note the sign of $0i$ is so that the last two terms in (4.14) decay as $t \to \infty$.

We have identify the main oscillation term in $\tilde{\eta}$ and we denote the remaining term in by $\eta^{(3)}$:

$$
\tilde{\eta} = \tilde{\eta}^{(2)} + \tilde{\eta}^{(3)}. \qquad (4.16)
$$

Notice that $\tilde{\eta}^{(2)}$ is not in L^2 and this is not an L^2 decomposition. It is, however, very useful for local behavior as it identifies the local oscillation.

From (4.10) we obtain the equation for $\tilde{\eta}^{(3)}$:

$$
\begin{aligned}
\tilde{\eta}^{(3)}(t) &= e^{-iAt} \tilde{\eta}_0 - e^{-iAt} (e^{i\theta} z^\alpha)(0) \widetilde{\eta}_\alpha - \int_0^t e^{-iA(t-s)} \left\{ \mu^{[\alpha]} \frac{d}{ds} (e^{i\theta} p^\alpha) \widetilde{\eta}_\alpha \right\} \, ds \\
&\quad + \int_0^t e^{-iA(t-s)} \, \mathbf{P}_{\mathrm{c}}^A \left\{ e^{i\theta} U \Pi \left[F^{\sharp\sharp} + i\eta^2 \bar{\eta} \right] - e^{i\theta} [U, i] \dot{\theta} \eta \right\} \, ds \qquad (4.17) \\
&\quad + \int_0^t e^{-iA(t-s)} \, \mathbf{P}_{\mathrm{c}}^A \, e^{i\theta} U \Pi (-i\eta^2 \bar{\eta}) \, ds \\
&= \tilde{\eta}_1^{(3)} + \tilde{\eta}_2^{(3)} + \tilde{\eta}_3^{(3)} + \tilde{\eta}_4^{(3)} + \tilde{\eta}_5^{(3)} \ .
\end{aligned}
$$

We treat $\tilde{\eta}_5^{(3)}$ separately because $\eta^2 \bar{\eta}$ is a non-local term. Notice that $\tilde{\eta}^{(2)} \notin L^2$, but is still "orthogonal" to the eigenvector of A.

From the decomposition of $\tilde{\eta}$ and the relation $\eta = U^{-1} e^{-i\theta} \tilde{\eta}$, we have the corresponding decomposition for η:

$$
\eta(t) = \eta^{(2)}(t) + \eta^{(3)}(t) \ .
$$

where

$$\eta^{(2)} = U^{-1}e^{-i\theta}\tilde{\eta}^{(2)} = U^{-1}z^{\alpha}\widetilde{\eta}_{\alpha}, \quad \eta^{(3)} = U^{-1}e^{-i\theta}\tilde{\eta}^{(3)} \ .$$

Summarizing, we have decompose a, F and η into terms in order of z. The main oscillatory terms of order z^2 in a is $a_{20}(z^2+\bar{z}^2)$ and

$$a = a_{20}(z^2+\bar{z}^2) + b \tag{4.18}$$

with

$$\dot{b} = \big(c_1 Q, \mathrm{Im}(F - F^{(2)})\big) - 4\,\mathrm{Re}\, a_{20} z \mu^{-1}\dot{p} \ . \tag{4.19}$$

The nonlinear term F is decomposed into orders in (4.5) with the second order term $F^{(2)}$ explicitly given. We also rewrite the equation of θ into

$$\dot{\theta} = c_2(z+\bar{z}) + F_{\theta} \ , \tag{4.20}$$

where $c_2 = (c_0 Q, \lambda Q^2 u) = O(\lambda)$ and

$$F_{\theta} = \frac{-1}{1+c_0 c_1^{-1} a}\left\{c_0 c_1^{-1} c_2 a(z+\bar{z}) + \Big[a + (c_0 Q, \lambda Q^2(\eta+\bar{\eta})) + (c_0 Q, \mathrm{Re}\, F(k))\Big]\right\} \tag{4.21}$$

The dispersive wave η is related to $\tilde{\eta}$ by the relation $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$ with $\tilde{\eta}$ satisfying

$$\tilde{\eta}(t) = e^{-iAt}\tilde{\eta}_0 + \int_0^t e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^{A}\,\left\{e^{i\theta}U\Pi F^{\sharp} - e^{i\theta}[U,i]\dot{\theta}\eta\right\}\,ds \ . \tag{4.22}$$

Furthermore, it can be decomposed into $\tilde{\eta}^{(2)} + \tilde{\eta}^{(3)}$ with

$$\tilde{\eta}^{(2)} = e^{i\theta}z^{\alpha}\widetilde{\eta}_{\alpha}, \qquad \widetilde{\eta}_{\alpha} = \frac{1}{A - 0i + [\alpha]\kappa}\Phi_{\alpha}$$

and $\tilde{\eta}^{(3)}$ satisfies the equation (4.17). For Theorem 1.1 the boundary conditions are

$$\begin{aligned} a(\infty) = 0 = b(\infty) \\ 0 < |z_0| \le \varepsilon_0 \\ \|\eta_0\|_{H^2\cap W^{2,1}(\mathbb{R}^3)} \le |z_0|^2 \end{aligned} \tag{4.23}$$

We now collect a few properties of the operator U. We can expand $U_{\pm}$ in order of λ as

$$U_+ = 1 + O(\lambda), \quad U_- = O(\lambda).$$

Notice that $\mathbf{P}_{\mathrm{c}}^{A}\,Uc_2u_+ = O(\lambda^2)$ since Uu_+ is almost orthogonal to $\mathtt{H}_{\mathrm{c}}(A)$. Hence we have

$$\Phi_{20} = -\,\mathbf{P}_{\mathrm{c}}^{A}\,\Pi\phi_{20} + O(\lambda^2), \quad \Phi_{11} = O(\lambda), \quad \Phi_{02} = O(\lambda^2) \ .$$

We may decompose

$$U^{-1} = U_+^\diamond + U_-^\diamond \mathtt{C}, \qquad U_\pm^\diamond = \tfrac{1}{2}(BA^{-1/2} \pm B^{-1}A^{1/2}) \ . \tag{4.24}$$

then

$$\eta^{(2)} = \left(U_+^\diamond + U_-^\diamond \mathtt{C}\right) z^\alpha \widetilde{\eta}_\alpha = z^2\eta_{20} + z\bar{z}\eta_{11} + \bar{z}^2\eta_{02} \ ,$$

where

$$\eta_{20} = U_+^\diamond \widetilde{\eta}_{20} + U_-^\diamond \widetilde{\eta}_{02}, \quad \eta_{11} = U^{-1}\widetilde{\eta}_{11}, \quad \eta_{02} = U_+^\diamond \widetilde{\eta}_{02} + U_-^\diamond \widetilde{\eta}_{20} \ .$$

If we expand

$$U_+ = 1 + O(\lambda), \quad U_- = O(\lambda), \quad U_+^\diamond = 1 + O(\lambda), \quad U_-^\diamond = O(\lambda),$$

then we get, by (4.15) and (4.13),

$$\eta_{20} = \widetilde{\eta}_{20} + O(\lambda^2) = \frac{-1}{A - 0i - 2\kappa}\, \mathbf{P}_{\mathrm{c}}^A\, \Pi\phi_{20} + O(\lambda^2) \ , \quad \eta_{11} = O(\lambda) \ , \quad \eta_{02} = O(\lambda^2) \ .$$

5 Estimates of dispersive wave

We now estimate solutions to the equations (2.5) with the decompositions into main oscillatory and higher order terms in section 4. We first need to choose a suitable norm. Define

$$\{t\} = \varepsilon^{-2} + 2\Gamma t, \quad \{t\} \sim \max\left\{\varepsilon^{-2}, t\right\}.$$

and, for $\eta_{2-5}^{(3)} \equiv \sum_{j=2}^5 \eta_j^{(3)} = \eta^{(3)} - \eta_1^{(3)}$,

$$M(T) := \sup_{0\le t\le T} \left\{ \{t\}^{1/2} |z(t)| + \{t\}^{3/4-\sigma} \|\eta(t)\|_{L^4} + \{t\}^{1+\sigma/4} \left\|\eta_{2-5}^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \right\} \tag{5.1}$$

Recall that b, the slow oscillation part of a, is defined via (4.18) and satisfies the equation (4.19). We assume that the function b is in the following space $\mathcal{B}_T$:

$$\mathcal{B}_T = \left\{ b(t) : |b(t)| \le D\,\{t\}^{-1}, 0 \le t \le T \right\} \ , \tag{5.2}$$

where $D = 2B_{22}/\Gamma$ is some constant. The precise form of the constant is not important, it merely has to be an order one constant bigger than the true behavior of b which we shall derive. The class $\mathcal{B}$ is nonempty since it contains the constant function 0.

Our setting is thus given at the end of section 4 except we now assuming the estimate (5.2) on b (and thus on a too) up to time T. From now on, we fix T.

Theorem 5.1 *Suppose η, h, a, θ, z are solutions to the equations* (2.5)*. Assuming the estimate* (5.2) *on b, we have*

$$M(t) \leq 2 \qquad \textit{for all} \quad t \leq T \ .$$

Moreover, if we further assume $|z_0| = \varepsilon > 0$ and $\|\eta_0\| \leq \varepsilon^{3/2}$, then $|z(t)| \geq c\,\{t\}^{-1/2}$.

Our strategy is to show that $M(0) \leq 3/2$ and that $M(t) \leq 3/2$ if $M(t) \leq 2$. By continuity of $M(t)$, this would imply that $M(t) \leq 3/2$ for all $t \leq T$. So for the rest of this and next sections, we shall use freely that $M(t) \leq 2$ to prove that $M(T) \leq 3/2$. The proof of this theorem will be completed after Lemma 6.1. We now start the proof.

Due to the presence of the non-local term h^3 in F, we first need a global norm estimate on η, which we choose to be $\|\eta(t)\|_{L^4}$. Our goal is to prove that

$$\|\eta(t)\|_{L^4} \leq C\,\{t\}^{-3/4}\,\log\{t\}$$

which agrees with that of free evolution.

We first recall some basic facts concerning the Schrödinger equation which provide some useful feeling on the size of various quantities. Since we shall proceed with iteration scheme, our a, z and η do not solve the Schrödinger equation and we will not use these facts.

The H^1 norm of ψ_t is uniformly bounded if λ is sufficiently small. (It can be proved by using the conservation of the Hamiltonian and the Gagliado-Nirenberg inequality.) Hence $\|h(t)\|_{H^1}$ is uniformly bounded since $h = \psi e^{i\theta} - Q$. Assuming $z(t)$ is bounded, then $\zeta(t)$ and $\eta(t)$ are both bounded in H^1, uniformly in t, since $\zeta = zu_+ + \bar{z}u_-$ and $\eta = h - \zeta$.

Return to a global estimate for η. Since $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$ and U is bounded in Sobolev spaces, for the purpose of estimation we can treat η and $\tilde{\eta}$ the same. Recall $[U, i]$ is a local operator satisfying the estimate at Lemma 3.3.

We will need the following calculus lemma:

Lemma 5.2 *Let $0 < d < 1 < m$. $\{t\} \equiv \varepsilon^{-2} + 2\Gamma t$.*

$$\int_0^t |t-s|^{-d}\,\{s\}^{-m}\,ds \leq C\varepsilon^{2m-2}\,\{t\}^{-d} \ .$$

Also,

$$\int_0^t |t-s|^{-d}\,\{s\}^{-1}\,ds \leq C\,\{t\}^{-d}\log(\varepsilon^2\,\{t\}) \ .$$

If, instead, $d \geq m > 1$,

$$\int_0^t \langle t-s\rangle^{-d}\,\{s\}^{-m}\,ds \leq C\,\{t\}^{-m} \ .$$

Proof. Denote the first integral by (I). If $t \le \varepsilon^{-2}$, then $\{t\} \sim \varepsilon^{-2}$ and

$$(I) \sim \int_0^t |t-s|^{-d} \varepsilon^{2m}\, ds \lesssim \varepsilon^{2m}\varepsilon^{-2(1-d)} = \varepsilon^{2m-2}\varepsilon^{2d} \sim \varepsilon^{2m-2}\{t\}^{-d} \ .$$

If $t \ge \varepsilon^{-2}$, then $\{t\} \sim 2\Gamma t$ and

$$\begin{aligned}(I) &\le \int_0^{t/2} Ct^{-d}\{s\}^{-m}\, ds + \int_{t/2}^t C|t-s|^{-d} C\{t\}^{-m}\, ds \\ &\le Ct^{-d}\varepsilon^{2(m-1)} + Ct^{1-d}\{t\}^{-m} \sim C\{t\}^{-d}\varepsilon^{2(m-1)} + C\{t\}^{1-d-m} \\ &\le C\varepsilon^{2m-2}\{t\}^{-d} \ .\end{aligned}$$

For the second case, denote the second integral by (II). If $t \le \varepsilon^{-2}$, then $\{t\} \sim \varepsilon^{-2}$ and

$$(II) \sim \int_0^t \langle t-s\rangle^{-d}\varepsilon^{2m}\, ds \le \varepsilon^{2m}\cdot C \sim C\{t\}^{-m} \ .$$

If $t \ge \varepsilon^{-2}$, then $\{t\} \sim \Gamma\langle t\rangle$ and

$$(II) \sim \int_0^t \langle t-s\rangle^{-d}\Gamma^{-m}\langle s\rangle^{-m}\, ds \le C\Gamma^{-m}\langle t\rangle^{-m} \sim C\{t\}^{-m} \ .$$

We conclude the lemma. **Q.E.D.**

5.1 Estimates in L^4 and L^2

Lemma 5.3 *Suppose that $\tilde\eta$ is given by equation* (4.10) *and recall $\eta = U^{-1}e^{-i\theta}\tilde\eta$. Assuming the estimate* (5.2) *on b and $M(T) \le 2$, we have*

$$\|\eta(t)\|_{L^4} \le C\{t\}^{-3/4}\log\{t\} \ .$$

Moreover, we have

$$\|\eta(t)\|_{L^2} \le \varepsilon^{1/2} \ .$$

Proof. From the defining equation of $\tilde\eta$, we have

$$\|\widetilde\eta(t)\|_{L^4} \le C\|\widetilde\eta_0\|_{H^1\cap L^{4/3}}\langle t\rangle^{-3/4} + \int_0^t C|t-s|^{-3/4}\left\{\left\|F^\sharp(s)\right\|_{L^{4/3}} + |\dot\theta|\,\|\eta(t)\|_4\right\} ds \ .$$

From the definitions of $F^\sharp$ and ζ, we have

$$\left\|F^\sharp(s)\right\|_{L^{4/3}} + |\dot\theta|\,\|\eta(t)\|_4 \le C|\dot\theta|\,\|\zeta(t)\|_{4/3} + \|F(s)\|_{L^{4/3}} + C|a\dot\theta| \ ,$$

From the Holder inequality and the definition of $\|F(s)\|_{L^{4/3}}$ we can bound $\|F(s)\|_{L^{4/3}}$ by

$$\|F(s)\|_{L^{4/3}} \le C\left(\|h\|_{L^4}^2 + |a|\,\|h\|_{L^4} + |a|^2 + \left\|h^3\right\|_{L^{4/3}} + |a|^3\right) \ .$$

Since $\|h^3\|_{L^{4/3}} = \|h\|_{L^4}^3$ and

$$\|h(s)\|_{L^4} \le \|\zeta(s)\|_{L^4} + \|\eta(s)\|_{L^4} ,$$

we have from the assumption $M \le 2$ that

$$\|h(s)\|_{L^4} \le C \{s\}^{-1/2} + C \{s\}^{-3/4} \log \{s\} \le C \{s\}^{-1/2} \ ,$$

Therefore we have $\|F(s)\|_{L^{4/3}} \le C \{s\}^{-1}$ and thus

$$\left\|F^\sharp(s)\right\|_{L^{4/3}} \le C \{s\}^{-1} \ .$$

Since $\langle t\rangle^{-1} \le \varepsilon^{-2} \{t\}^{-1}$ and $\|\eta(t)\|_{L^4} \sim \|\widetilde{\eta}(t)\|_{L^4}$, we conclude

$$\|\eta(t)\|_{L^4} \le C\varepsilon^{-3/2} \|\eta_0\|_{H^1\cap L^{4/3}} \{t\}^{-3/4} + \int_0^t |t-s|^{-3/4} C \{s\}^{-1} \, ds \le C \{t\}^{-3/4} \log \{t\} \ .$$

Here we have used Lemma 5.2 in the last integration.

We now bound $\|\eta(t)\|_{L^2}$. Since

$$\frac{d}{2dt}(\tilde{\eta},\tilde{\eta}) = \mathrm{Re}(\tilde{\eta}, \partial_t\tilde{\eta}) = \mathrm{Re}(\tilde{\eta}, e^{i\theta}\, \mathbf{P}_{\mathrm{c}}^A\, U\Pi F^\sharp - e^{i\theta}\, \mathbf{P}_{\mathrm{c}}^A\, [U,i]\dot{\theta}\eta) \ ,$$

we have

$$\begin{aligned}\frac{d}{dt}\|\tilde{\eta}\|_{L^2}^2 &\le C\|\tilde{\eta}\|_{L^4} \cdot \left\{\left\|F^\sharp(s)\right\|_{L^{4/3}} + |\dot{\theta}|\, \|\eta(t)\|_4\right\} \\ &\le C \{t\}^{-3/4} \log \{t\}\, C \{t\}^{-1} = C \{t\}^{-7/4} \log \{t\} \ .\end{aligned}$$

In the second inequality we use our previous estimates. Hence

$$\|\tilde{\eta}\|_{L^2}^2 \le C\varepsilon^3 + \int_0^\infty C \{t\}^{-7/4} \log \{t\} \le \varepsilon \ ,$$

and we conclude $\|\tilde{\eta}(t)\|_{L^2} \le \varepsilon^{1/2}$, and so is $\|\eta(t)\|_{L^2}$. **Q.E.D.**

5.2 Local decay of $\eta^{(3)}$

Recall $\eta^{(3)} = U^{-1}e^{-i\theta}\tilde{\eta}^{(3)}$ and $\tilde{\eta}^{(3)}$ satisfies the equation (4.17). We want to show that $\tilde{\eta}^{(3)}$ is smaller than $\tilde{\eta}^{(2)}$ locally. Define the local L^2 norm by

$$\|f\|_{L^2_{\mathrm{loc}}} = \left\|\langle x\rangle^{-\beta_0} f\right\|_{L^2}$$

for a fixed sufficiently large $\beta_0 > 0$, which will become clear later on.

Lemma 5.4 *Assuming the estimate* (5.2) *on* b *and* $M(T) \leq 2$ *for all* T*, we have*

$$\begin{aligned} \left\| \tilde{\eta}^{(3)}_{2-5} \right\|_{L^2_{loc}} &\leq C \left\{ t \right\}^{-1-\sigma/2} \leq C\varepsilon^{\sigma/2} \left\{ t \right\}^{-1-\sigma/4} \ , \\ \left\| \eta^{(3)}_{2-5} \right\|_{L^2_{loc}} &\leq C\varepsilon^{\sigma} \left\{ t \right\}^{-1-\sigma/4} \end{aligned} \tag{5.3}$$

In particular, for a local function ϕ *we have*

$$\left| (\phi, \eta^{(3)}) \right| \leq C \left\{ t \right\}^{-1-\sigma/4} \ , \tag{5.4}$$

$$\left| (\phi, |\eta|^2 + |\eta|^3) \right| \leq C \left\| \eta \right\|^2_{L^2_{loc}} + C \left\| \eta \right\|_{L^2_{loc}} \left\| \eta \right\|^2_{L^4} \leq C \left\{ t \right\}^{-2} \ . \tag{5.5}$$

Proof. We first estimate $\tilde{\eta}^{(3)}_j$ appearing on the right side of the equation for $\tilde{\eta}^{(3)}$ (4.17). Note we did not include $\tilde{\eta}^{(3)}_1$ in the above Lemma:

$$\left\| \tilde{\eta}^{(3)}_1 \right\|_{L^2_{\text{loc}}} \leq C \left\| \eta_0 \right\|_{H^1 \cap L^{8/7}} \langle t \rangle^{-9/8} \ .$$

This term would be bounded by $\varepsilon^{\sigma/4} \left\{ t \right\}^{-9/8}$ if we assumed $\|\eta_0\| \leq \varepsilon^{2+\sigma}$. However, since we only assume $\|\eta_0\| \leq \varepsilon^{3/2}$, this term needs to be treated separately.

For $\tilde{\eta}^{(3)}_2$ and $\tilde{\eta}^{(3)}_3$, the two terms involving η_α, ($\eta_\alpha \notin L^2$), from the definition and the estimates Lemma 3.1 on A, we have

$$\left\| \tilde{\eta}^{(3)}_2 \right\|_{L^2_{\text{loc}}} \leq C \left\{ t \right\}^{-9/8} \ .$$

$$\left\| \tilde{\eta}^{(3)}_3 \right\|_{L^2_{\text{loc}}} \leq C \int_0^t \langle t-s \rangle^{-9/8} \left\{ s \right\}^{-3/2} \, ds \leq C \left\{ t \right\}^{-9/8} \ .$$

To estimate $\tilde{\eta}^{(3)}_4$, we recall the definition of $F^{\sharp\sharp}$ and rewrite

$$\begin{aligned} F^{\sharp\sharp} + i\eta^2\bar{\eta} &= \left(F^{\sharp} - iz^\alpha \phi^{\sharp}_\alpha \right) + i\eta^2\bar{\eta} \\ &= -i[\dot{\theta} - c_2(z+\bar{z})]\zeta - i\left\{ F - F^{(2)} - \eta^2\bar{\eta} \right\} \\ &\quad - \left[(c_1 Q, \mathrm{Im}(F - F^{(2)})) + ia\dot{\theta} \right] R_\Pi \ . \end{aligned}$$

Since $M \leq 2$, we have $|\dot{\theta} - c_2(z+\bar{z})| \leq C \left\{ s \right\}^{-1}$, $|a| \leq C \left\{ s \right\}^{-1}$, $\left\| \langle x \rangle^{\beta_0} \left(F - F^{(2)} - \eta^2\bar{\eta} \right) \right\|_{L^2} \leq C \left\{ s \right\}^{-3/2}$, and $|(c_1 Q, \mathrm{Im}(F - F^{(2)}))| \leq \left\{ s \right\}^{-3/2}$. Hence, by Lemma 3.1 we have

$$\begin{aligned} \left\| \tilde{\eta}^{(3)}_4 \right\|_{L^2_{\text{loc}}} &\leq \int_0^t \langle t-s \rangle^{-3/2} \left\| \langle x \rangle^{\beta_0} \left(F^{\sharp\sharp} + i\eta^2\bar{\eta} \right)(s) \right\|_{L^2} \, ds \\ &\leq \int_0^t \langle t-s \rangle^{-3/2} C \left\{ s \right\}^{-3/2} \, ds \\ &\leq C \left\{ t \right\}^{-3/2} \ . \end{aligned}$$

We now consider $\tilde{\eta}_5^{(3)}$. Denote the integrand $e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^A\,e^{i\theta}U\Pi(-i\eta^2\bar{\eta})$ by $X(t,s)$ and decompose the time interval into $(0,t-\sigma)$ and $(t-\sigma,t)$. From the triangle inequality, we have

$$\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} \le \int_0^{t-\sigma} \|X(t,s)\,ds\|_{L^2_{\mathrm{loc}}} + \int_{t-\sigma}^{t} \|X(t,s)\,ds\|_{L^2_{\mathrm{loc}}}$$

We can bound the L^2_{loc} norm by either L^8 or L^4 norm. Thus

$$\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} \le C\int_0^{t-\sigma} \|X(t,s)\|_{L^8}\,ds + C\int_{t-\sigma}^{t} \|X(t,s)\|_{L^4}\,ds$$

From Lemma 3.1, we have

$$\|X(t,s)\|_{L^8} = \left\|e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^A\,e^{i\theta}U\Pi(-i\eta^2\bar{\eta})\right\|_{L^8} \le \frac{1}{|t-s|^{9/8}}\left\|\eta^3\right\|_{L^{8/7}}$$

and

$$\|X(t,s)\|_{L^4} = \left\|e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^A\,e^{i\theta}U\Pi(-i\eta^2\bar{\eta})\right\|_{L^8} \le \frac{1}{|t-s|^{9/8}}\left\|\eta^3\right\|_{L^{4/3}}$$

From the Holder inequality and the global estimate of η in Lemma 5.3 we have

$$\begin{aligned}
&\left\|\eta^3(s)\right\|_{L^{8/7}} \le \|\eta\|_2^{1/2}\,\|\eta\|_4^{5/2} \le C\varepsilon^{1/4}\left(\{s\}^{-3/4}\ln\{s\}\right)^{5/2} \le C\varepsilon^{1/4}\,\{s\}^{-7/4}\ ,\\
&\left\|\eta^3(s)\right\|_{L^{4/3}} \le \|\eta\|_{L^4}^3 \le \left(\{s\}^{-3/4}\ln(2+s)\right)^3 \le C\,\{s\}^{-2}\ .
\end{aligned}$$

If $t \ge 2\varepsilon^{-2}$, then $\{t\}\sim t$. We choose $\sigma = t/2$ and thus

$$\begin{aligned}
\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} &\le C\int_0^{t/2}\frac{1}{|t-s|^{9/8}}C\,\{s\}^{-7/4}\,ds + C\int_{t/2}^{t}\frac{1}{|t-s|^{3/4}}C\,\{s\}^{-2}\,ds\\
&\le C\int_0^{t/2} C\,\{t\}^{-9/8}\,\{s\}^{-7/4}\,ds + C\int_{t/2}^{t}\frac{1}{|t-s|^{3/4}}C\,\{t\}^{-2}\,ds\\
&\le C\,\{t\}^{-9/8}\,(\varepsilon^{-2})^{-3/4} + C\,\{t\}^{-2}\,t^{1/4} \le C\varepsilon^{5/4}\,\{t\}^{-9/8}\ .
\end{aligned}$$

On the other hand, if $t < 2\varepsilon^{-2}$, then $\{t\}\sim\varepsilon^{-2}$ and we choose $\sigma = t$ to get

$$\begin{aligned}
\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} &\le C\int_0^{t}\frac{1}{|t-s|^{3/4}}C(\varepsilon^{-2})^{-2}\,ds\\
&\le C\varepsilon^4 t^{1/4} \le C\varepsilon^{7/2} \sim C\varepsilon^{5/4}\,\{t\}^{-9/8}\ .
\end{aligned}$$

Combining two cases, we conclude

$$\left\|\tilde{\eta}_5^{(3)}\right\|_{L^2_{\mathrm{loc}}} \le C\varepsilon^{5/4}\,\{t\}^{-9/8}\ .$$

The lemma follows from all the estimates on $\eta_j^{(3)}$, $j = 1,\cdots,5$. **Q.E.D.**

6 Excited state equation and normal form

6.1 Excited state equation

Return to the basic equation (2.5) and recall the equation for $\dot{p}$:

$$\dot{p} = -i\mu \left[(\tilde{u}_+, F) + (\tilde{u}_-, \bar{F}) + \left\{ (u_+, h) + (u_-, \bar{h}) + c_3 a \right\} \dot{\theta} \right] \ , \tag{6.1}$$

where $c_3 = (u, R_\Pi)$, $c_1 = (Q, R)^{-1}$. Recall that

$$z, p = O(z), \quad a, \eta = O(z^2), \quad \eta^{(3)} = O(z^3) \ .$$

We now expand the right hand side of the equation for $\dot{p}$ into terms in order of z:

$$\dot{p} = \mu \left\{ c_\alpha z^\alpha + d_\beta z^\beta + d_1 b z + d_2 b \bar{z} + P^{(4)} \right\} \tag{6.2}$$

The coefficients will be computed later on and their properties are summarized in the following lemma. The proof of this lemma is just straightforward computation and the reader can check it rather easily.

Lemma 6.1 *We can rewrite the equation of p into the form* (6.2) *such that the coefficients d_1, d_2 and all c_α are purely imaginary. Moreover,* $\operatorname{Re} d_{21} = -\Gamma + O(\lambda^3)$, *with*

$$\Gamma \equiv 2\lambda^2 \left(Q u_+^2, \operatorname{Im} \frac{1}{A - 0i - 2\kappa} \mathbf{P}_{\mathrm{c}}^A \, \Pi Q u_+^2 \right) \geq 0 \ . \tag{6.3}$$

Proof. There are two parts in p equation: $(\tilde{u}_+, F) + (\tilde{u}_-, \bar{F})$ and $\left\{ (u_+, h) + (u_-, \bar{h}) + c_3 a \right\} \dot{\theta}$. We first consider the second part:

$$\begin{aligned}
&\left\{ (u_+, h) + (u_-, \bar{h}) + c_3 a \right\} \dot{\theta} \\
&= \left\{ (u_+, \zeta + \eta^{(2)}) + (u_-, \bar{\zeta} + \bar{\eta}^{(2)}) + c_3 a \right\} \cdot \\
&\qquad \cdot \left\{ c_2 (z + \bar{z}) + (c_0 Q, \lambda Q^2 (\eta^{(2)} + \bar{\eta}^{(2)}) + a + (c_0 Q, \operatorname{Re} F^{(2)}) \right\} + \widetilde{P}_2^{(4)} \\
&= \left((u_+, \zeta) + (u_-, \bar{\zeta}) \right) \cdot c_2 (z + \bar{z}) \\
&\quad + \left[(u_+, \eta^{(2)}) + (u_-, \bar{\eta}^{(2)}) + c_3 a \right] \cdot c_2 (z + \bar{z}) \\
&\qquad + \left((u_+, \zeta) + (u_-, \bar{\zeta}) \right) \cdot \left[a + (c_0 Q, \lambda Q^2 (\eta^{(2)} + \bar{\eta}^{(2)})) + (c_0 Q, \operatorname{Re} F^{(2)}) \right] \\
&\qquad + P_2^{(4)} \ .
\end{aligned}$$

Here$\widetilde{P}_2^{(4)}$ and $P_2^{(4)}$ denote the remaining error terms. We first observe that line 4, lines 5–6, and $P_2^{(4)}$ are of orders $O(z^2)$, $O(z^3)$ and $O(z^4)$, respectively. Hence line 4 contributes to $c_\alpha z^\alpha$, line 5–6 to $d_\beta z^\beta$ and $d_1 b z + d_2 b \bar{z}$, and $P_2^{(4)}$ to $P^{(4)}$. We also observe that the coefficients in

line 4 are all real. Since there is a $-i\mu$ factor in front, their contribution to c_α are purely imaginary. Next we observe that the coefficients in lines 5–6 are real except those involving $\eta^{(2)}$. The terms with lowest λ-order are:

$$(u_+, \eta^{(2)}) \cdot c_2(z+\bar{z}) \ + \ (u_+, \zeta) \cdot (c_0 Q, \lambda Q^2(\eta^{(2)} + \bar{\eta}^{(2)})) \ .$$

The first part is of order λ^3 since $(u_+, \eta^{(2)}) = (P_\kappa^A u_+ + O(\lambda), \mathbf{P}_c^A \eta^{(2)} + O(\lambda^2))$, with $\mathbf{P}_c^A \eta^{(2)} = O(\lambda)$. The second part has order λ^2 terms, but not on $z^2\bar{z}$. We conclude that the contribution to $\mathrm{Re}\, d_{21}$ from the second part of p-equation is of order λ^3. Finally we observe that line 5–6 give a term $d_{1,2}bz + d_{2,2}b\bar{z}$, with real $d_{1,2} = O(1)$ and $d_{2,2} = O(\lambda)$. Since there is a $-i\mu$ factor in front, their contribution to d_1 and d_2 are purely imaginary.

Now we look at the first part of the p equation. The contribution to $c_\alpha z^\alpha$ is from $F^{(2)}$:

$$-i\left\{(\tilde{u}_+, z^\alpha \phi_\alpha) + (\tilde{u}_-, \bar{z}^\alpha \phi_\alpha)\right\} \ .$$

Clearly all coefficients of z^α are purely imaginary. Together with the analysis of second part of p-equation, we know c_α are purely imaginary.

The contribution to $d_1 bz + d_2 b\bar{z}$ is from $\widetilde{F}^{(3)}$:

$$-i\left\{(\tilde{u}_+, 2\lambda bQR(2\zeta + \bar{\zeta}) + (\tilde{u}_-, 2\lambda bQR(\zeta + 2\bar{\zeta})\right\}$$

with $\zeta = zu_+ + \bar{z}u_-$. Hence all coefficients of bz and $b\bar{z}$ are purely imaginary. Together with the analysis of second part of p equation, we know d_1 and d_2 are purely imaginary.

The contribution to $P^{(4)}$ is from $F^{(4)}$:

$$P_1^{(4)} = -i\left[(\tilde{u}_+, F^{(4)}) + (\tilde{u}_-, \bar{F}^{(4)})\right] \ ,$$

and we have $P^{(4)} = P_1^{(4)} - iP_2^{(4)}$.

The contribution to $d_\beta z^\beta$ is from $F^{(3)}$:

$$-i[(\tilde{u}_+, F^{(3)}) + (\tilde{u}_-, \bar{F}^{(3)}] \ .$$

We only consider d_{21}. A coefficient in $F^{(3)}$ has to have imaginary part in order to have a real part contribution to d_{21}. Hence only the first group of terms in $F^{(3)}$, $2\lambda Q[(\zeta + \bar{\zeta})\eta^{(2)}$, has contribution to $\mathrm{Re}\, d_{21}$. Also, when we decompose $\eta^{(2)} = z^\alpha \eta_\alpha$, we can disregard $\eta_{(11)}$ since it is real. Now, $\eta_{(20)} = O(\lambda)$, $\eta_{(02)} = O(\lambda^2)$, $u_+ = O(1)$ and $u_- = O(\lambda)$, hence the main part of $(\mathrm{Re}\, d_{21}) z^2\bar{z}$ is in

$$-i(\tilde{u}_+, \ 2\lambda Q(\bar{z}u_+) z^2 \eta_{(20)})$$

Summarizing our efforts, we have

$$\begin{aligned}\operatorname{Re} d_{21} &= \operatorname{Im}(u_+, 2\lambda Q u_+ \eta_{(20)}) + O(\lambda^3) = \operatorname{Im}(2\lambda Q u_+^2, \eta_{(20)}) + O(\lambda^3) \\ &= (2\lambda Q u_+^2, \operatorname{Im} \frac{-1}{A - 0i - 2\kappa} \mathbf{P}_{\mathrm{c}}^A \Pi \phi_{20}) + O(\lambda^3) \\ &= -(2\lambda Q u_+^2, \operatorname{Im} \frac{1}{A - 0i - 2\kappa} \mathbf{P}_{\mathrm{c}}^A \Pi \lambda Q u_+^2) + O(\lambda^3) \ .\end{aligned}$$

Q.E.D.

6.2 Normal form

From Lemma 3.2, we have $\Gamma > 0$.

Lemma 6.2 *We can rewrite the equation* (6.2) *of* p *into a* **normal form***:*

$$\dot{q} = \delta_{21} |q|^2 q + d_1 b q + g \ . \tag{6.4}$$

where q *is a perturbation of* p *given in the proof. The coefficient* δ_{21} *satisfying the relation*

$$\operatorname{Re} \delta_{21} = \operatorname{Re} d_{21} \tag{6.5}$$

If we assume the estimate (5.2) *on* b *and* $M(T) \leq 2$*, then the error term* $g(t)$ *given by* (6.13) *satisfies the bound*

$$|g(t)| \leq C_1 \{t\}^{-3/2-\sigma} \tag{6.6}$$

for some constant C_1*. Furthermore, there is a positive constant* σ *such that* $|q(t)|$ *is bounded by*

$$(1-\sigma) \{t\}^{-1/2} \leq |q(t)| \leq (1+\sigma) \{t\}^{-1/2} \ . \tag{6.7}$$

and hence

$$(1-2\sigma) \{t\}^{-1/2} \leq |z(t)| \leq (1+2\sigma) \{t\}^{-1/2} \ .$$

Proof of Theorem 5.1 From Lemma 6.1, we have

$$\{t\}^{1/2} |z(t)| \leq (1+2\sigma)$$

From Lemma 5.3 and Lemma 5.4 we can bound

$$\{t\}^{3/4-\sigma} \|\eta(t)\|_{L^4} + \{t\}^{1+\sigma/4} \left\| \eta_{2-5}^{(3)}(t) \right\|_{L^2_{\mathrm{loc}}}$$

by $C\varepsilon^{\sigma/4}$. Since ε is small, we have proved that $M(T) \leq 3/2$ and this concludes Theorem 5.1. **Q.E.D.**

We now prove Lemma 6.1.

Proof. We have

$$\dot{p} = \mu \left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)} \right]$$

and we want to obtain the normal form (6.4). We will repeatedly use the following formula:

$$\mu^m p^\alpha = \frac{d}{dt}\left(\frac{\mu^m p^\alpha}{i\kappa m}\right) - \frac{\mu^m p^\alpha}{i\kappa m} f_\alpha(z) \tag{6.8}$$

where, if $\alpha = (\alpha_0 \alpha_1)$, $(|\alpha| = \alpha_0 + \alpha_1 = 2, 3, 4 \cdots)$

$$\begin{aligned} f_\alpha(z) &= (\alpha_0 + \alpha_1 \mathtt{C})(p^{-1}\dot{p}) \\ &= (\alpha_0 + \alpha_1 \mathtt{C}) z^{-1} \left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)} \right] \end{aligned} \tag{6.9}$$

and $\mathtt{C}$ denotes the conjugation operator. The formula is equivalent to integration by parts.

We first remove $c_\alpha z^\alpha$. Let

$$p_1 = p - \frac{c_\alpha}{i\kappa(1 + [\alpha])} \mu z^\alpha.$$

Since $[\alpha]$ is even, $1 + [\alpha] \neq 0$. By (6.8)

$$\dot{p}_1 = \dot{p} - c_\alpha \mu z^\alpha - \frac{c_\alpha}{i\kappa(1 + [\alpha])} \mu z^\alpha f_\alpha(z)$$

Decomposing $f_\alpha(z)$ into two parts, we can write

$$\begin{aligned} d_\beta^+ z^\beta &= -\frac{c_\alpha}{i\kappa(1 + [\alpha])} \mu z^\alpha (\alpha_0 + \alpha_1 \mathtt{C}) z^{-1} c_{\tilde{\alpha}} z^{\tilde{\alpha}} \\ g_1 &= -\frac{c_\alpha}{i\kappa(1 + [\alpha])} \mu z^\alpha (\alpha_0 + \alpha_1 \mathtt{C}) z^{-1} \left[d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)} \right] \end{aligned}$$

and we get

$$\dot{p}_1 = \delta_\beta \mu z^\beta + d_1 \mu bz + d_2 \mu b\bar{z} + \mu P^{(4)} + g_1$$

with $\delta_\beta = d_\beta + d_\beta^+$. Since d_β^+ are purely imaginary, due to that c_α are purely imaginary, we have the key relation

$$\operatorname{Re} \delta_\beta = \operatorname{Re} d_\beta \tag{6.10}$$

Next we remove $d_2 \mu b\bar{z}$. Let

$$p_2 = p_1 - \frac{\mu d_2 b\bar{z}}{2i\kappa} \ .$$

We have

$$\begin{aligned}\dot{p}_2 &= \dot{p}_1 - \mu d_2 b\bar{z} - \frac{\mu^2 d_2}{2i\kappa}(\dot{b}\bar{p} + b\dot{\bar{p}}) \\ &= \mu\delta_\beta z^\beta + d_1\mu bz + (\mu P^{(4)} + g_1 + g_2)\ ,\end{aligned}$$

where

$$g_2 = -\frac{\mu^2 d_2}{2i\kappa}(\dot{b}\bar{p} + b\dot{\bar{p}})\ .$$

Now we deal with $\delta_\beta z^\beta$ terms. Let

$$p_3 = p_2 - \sum_{\beta\neq(21)} \frac{\delta_\beta \mu z^\beta}{i\kappa(1+[\beta])}.$$

Note $1+[\beta]\neq 0$ for $\beta\neq(21)$. We have

$$\dot{p}_3 = \dot{p}_2 - \mu\delta_\beta z^\beta + g_3 = \delta_{21}\mu z^2\bar{z} + \mu d_1 bz + (\mu P^{(4)} + g_1 + g_2 + g_3)\ ,$$

with

$$g_3 = -\sum_{\beta\neq(21)} \frac{\delta_\beta \mu^{1+[\beta]}}{i\kappa(1+[\beta])}\frac{d}{dt}(p^\beta).$$

Finally, since $\eta_1^{(3)} = U^{-1}e^{-i\theta}\, e^{-iAt}U\eta_0$ is larger than $\eta_{2-5}^{(3)}$ when time t is of order 1, we need to extract terms of order $O(z\eta_1^{(3)})$ from $\mu P^{(4)}$. Recall

$$\widetilde{\eta}_1^{(3)} = e^{-iAt}\widetilde{\eta}_0 = e^{-iAt}U\eta_0, \qquad \eta_1^{(3)} = U^{-1}e^{-i\theta}\,\widetilde{\eta}_1^{(3)} = U^{-1}e^{-i\theta}\, e^{-iAt}U\eta_0\ . \tag{6.11}$$

Also recall (6.1) and (6.2). In F we have a term $2\lambda Q((\zeta+\bar{\zeta})\eta_1^{(3)} + \zeta\overline{\eta_1^{(3)}})$, in h a term $\eta_1^{(3)}$ itself. Hence in $\mu P^{(4)}$, terms of order $O(z\eta_0)$ are exactly

$$\begin{aligned}P_{z\eta_0} \equiv &- i\mu(\tilde{u}_+, 2\lambda Q((\zeta+\bar{\zeta})\eta_1^{(3)} + \zeta\bar{\eta}_1^{(3)})) - i\mu\mathtt{C}(\tilde{u}_-, 2\lambda Q((\zeta+\bar{\zeta})\eta_1^{(3)} + \zeta\bar{\eta}_1^{(3)})) \\ &- i\mu\left\{(u_+, \eta_1^{(3)}) + \mathtt{C}(u_-, \eta_1^{(3)})\right\} c_2(z+\bar{z})\ .\end{aligned}$$

Clearly these terms can be summed in the form

$$\mu(z\phi + \bar{z}\phi, \eta_1^{(3)}) + \mathtt{C}\mu^{-1}(z\phi + \bar{z}\phi, \eta_1^{(3)}) \tag{6.12}$$

Here ϕ stand for different local smooth functions. Moreover, if we write $U^{-1} = U_3 + U_4\mathtt{C}$ with $U_3, U_4 = \frac{1}{2}(BA^{-1/2} \pm B^{-1}A^{1/2})$, both self-adjoint, by (6.11) and

$$U^{-1}(zf + \bar{z}g) = z(U_3 f + U_4\bar{g}) + \bar{z}(U_3 g + U_4\bar{f})$$

(cf. (7.2)), the above is equal to

$$\mu(z\phi + \bar{z}\phi, e^{-i\theta}e^{-iAt}U\eta_0) + \mathtt{C}\mu^{-1}(z\phi + \bar{z}\phi, e^{-i\theta}e^{-iAt}U\eta_0).$$

The first term can be written as

$$(\phi, e^{-i(A-0i-2\kappa)t}e^{-i\theta}\bar{p}U\eta_0) + (\phi, e^{-iAt}e^{-i\theta}pU\eta_0) = \frac{d}{dt}(p_{4,1}) + g_{4,1}$$

with

$$p_{4,1} = i\left(\frac{\phi}{A-0i-2\kappa}\,,\, e^{-i(A-0i-2\kappa)t}e^{-i\theta}\bar{p}U\eta_0)\right) + i\left(\frac{\phi}{A}\,,\, e^{-iAt}e^{-i\theta}pU\eta_0)\right)$$
$$g_{4,1} = -i\left(\frac{\phi}{A-0i-2\kappa}\,,\, e^{-i(A-0i-2\kappa)t}\frac{d}{dt}(e^{-i\theta}\bar{p})U\eta_0)\right) - i\left(\frac{\phi}{A}\,,\, e^{-iAt}\frac{d}{dt}(e^{-i\theta}p)U\eta_0)\right)$$

We can write the second term similarly as

$$\mathtt{C}(z\phi + \bar{z}\phi, e^{-i\theta}e^{-iAt}U\eta_0) = \frac{d}{dt}(\mathtt{C}p_{4,2}) + \mathtt{C}g_{4,2}$$

Thus we have

$$P_{z\eta_0} = \dot{p}_4 + g_4, \qquad p_4 := p_{4,1} + \mathtt{C}p_{4,2}, \qquad g_4 := (g_{4,1} + \mathtt{C}g_{4,2})\ .$$

Observe that $\frac{\phi}{A-0i-2\kappa}$ is not in L^2, hence we expect slower decay for p_4 and g_4. Specifically,

$$|p_4| \le \|\eta_0\| \{t\}^{-1/2} \langle t\rangle^{-1-\sigma}\,, \qquad |q_4| \le \|\eta_0\| \{t\}^{-1} \langle t\rangle^{-1-\sigma}\ .$$

Now we let

$$q = p_3 - p_4\ .$$

We have

$$\begin{aligned}\dot{q} &= \delta_{21}\mu z^2\bar{z} + \mu d_1 bz + (\mu P^{(4)} + g_1 + g_2 + g_3) - P_{z\eta_0} + g_4\\ &= \delta_{21}|p|^2p + d_1bp + ((\mu P^{(4)} - P_{z\eta_0}) + g_1 + g_2 + g_3 + g_4)\\ &= \delta_{21}|q|^2q + d_1bq + g\ ,\end{aligned}$$

where

$$g = ((\mu P^{(4)} - P_{z\eta_0}) + g_1 + g_2 + g_3 + g_4) + \delta_{21}(|p|^2p - |q|^2q) + d_1b(p-q)\ . \tag{6.13}$$

Hence we have arrived at the normal form. (Note that, although we kept $\dot{p}$ and $\dot{b}$ in the definition of g_2, it should be replaced by the corresponding equations (6.2) and (4.19).) Also note we have

$$q = p - \frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha - \frac{\mu d_2 b\bar{z}}{2i\kappa} - \sum_{\beta\neq(21)}\frac{\delta_\beta\mu z^\beta}{i\kappa(1+[\beta])} - p_4\ . \tag{6.14}$$

Finally, we can check the size of g satisfying the bound (6.6). To conclude this lemma, it remains to prove the estimate on q. But this follows from the next lemma. **Q.E.D.**

6.3 Decay and continuity estimates

In this subsection we present some calculus lemmas which deal with the decay of $q(t)$ and its continuity on the error term $g(t)$. We will write

$$q(t) = \rho(t)\, e^{i\omega(t)} \ ,$$

where $\rho = |q|$ and ω is the phase of q. Recall

$$\{t\} = \varepsilon^{-2} + 2\Gamma t, \quad \{t\} \sim \max\left\{\varepsilon^{-2}, t\right\}.$$

Before we proceed with proof, we first explain some simple facts of an ordinary differential equation.

Example. We consider real functions $r(t) > 0$ which solve

$$\dot{r}(t) = -\Gamma r(t)^3 - \varepsilon\,(1+t)^{-3} \ .$$

We have the following facts.

a. All solutions $r(t)$ satisfy

$$r(t) \le (C + 2\Gamma t)^{-1/2} \quad \text{with } r(0) = C^{-1/2} \ .$$

b. There is a number r_1 such that if $r(0) > r_1$, then $r(t) \sim (C + 2\Gamma t)^{-1/2}$.

c. There is a unique global solution $r_0(t)$ such that

$$r_0(t) \sim t^{-2} \quad \text{as } t \to \infty.$$

d. If $r(0) < r_0(0)$, then $r(t) = 0$ in finite time.

e. If $r(0) > r_0(0)$, then

$$\int_0^\infty r(s)^2 ds = \infty.$$

Lemma 6.3 *Let $\varepsilon_0 > 0$ be small. Suppose $\rho(t)$ satisfy*

$$\dot{\rho} = -\rho^3 + \widetilde{g}(t) \ ,$$

where $|\widetilde{g}(t)| < \varepsilon_0 \langle t\rangle^{-3/2-\sigma}$, $\sigma > 0$. We can find $0 < \rho_1 \le \rho_2$, (depending on ε_0), such that
(a) If $\rho_1 < \rho(0) < \rho_2$, then $|\rho(t)| \sim C\,\langle t\rangle^{-1/2}$.
(b) If $0 < \rho(0) < \rho_2$, then $|\rho(t)| \le C\,\langle t\rangle^{-1/2}$.

The above example shows that the conclusion of (a) cannot be expected to hold if $\rho(0)$ is too small. This is the main reason that there are two type of asymptotic behavior: the resonance dominated solutions given by case (a) and the radiation dominated ones by (b). The proof of this lemma is elementary and similar to the next lemma and thus we omit it.

Lemma 6.4 *Let $\Gamma > 0$, $\sigma > 0$, and $C_1 > 0$ be given constants, independent of ε_0. Suppose a positive function $\rho(t)$ satisfies*

$$\dot{\rho} = -\Gamma\rho^3 + \widetilde{g}(t), \ , \tag{6.15}$$

(a) Suppose $\rho(0) = \varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$ and

$$|\widetilde{g}(t)| \leq C_1 \{t\}^{-3/2-\sigma}, \quad \{t\} \equiv \varepsilon^{-2} + 2\Gamma t \ .$$

Then there is a constant $m = m(\varepsilon_0) > 1$ such that

$$m^{-1} \{t\}^{-1/2} \leq \rho(t) \leq m \{t\}^{-1/2} \ .$$

Moreover, $m(\varepsilon_0) \to 1^+$ as $\varepsilon_0 \to 0^+$.

(b) Suppose that $\rho(0) \leq \varepsilon_0$ and

$$|\widetilde{g}(t)| \leq C_1 \{t\}_0^{-3/2-\sigma}, \quad \{t\}_0 \equiv \varepsilon_0^{-2} + 2\Gamma t \ .$$

Then we have for some constant C

$$\rho(t) \leq C \{t\}_0^{-1/2} \ .$$

Proof. We first prove part (a). Let $\rho_+ = m \{t\}^{-1/2}$ and $\rho_- = m^{-1} \{t\}^{-1/2}$, with $m > 1$ to be determined. We have $\rho_+(0) > \rho(0) > \rho_-(0)$. Moreover,

$$\dot{\rho}_+ = -\Gamma m^{-2}\rho_+^3 = -\Gamma\rho_+^3 + \Gamma(1 - m^{-2})\rho_+^3 \geq -\Gamma\rho_+^3 + \widetilde{g} \ ,$$

if

$$\Gamma(1 - m^{-2})m^3 \{t\}^{-3/2} \geq C_1 \{t\}^{-3/2-\sigma} \ .$$

Also

$$\dot{\rho}_- = -\Gamma m^2\rho_-^3 = -\Gamma\rho_-^3 - \Gamma(m^2 - 1)\rho_-^3 \leq -\Gamma\rho_-^3 + \widetilde{g} \ ,$$

if

$$\Gamma(m^2 - 1)m^{-3} \{t\}^{-3/2} \geq C_1 \{t\}^{-3/2-\sigma} \ .$$

Since $\{t\}^{-\sigma} \le \varepsilon^{2\sigma}$, both inequalities hold if

$$\Gamma(1-m^{-2})m^3,\ \Gamma(m^2-1)m^{-3} \ge C_1\,\varepsilon_0^{2\sigma}\ .$$

This is true for $m>1$ arbitrarily close to 1 by choosing ε_0 sufficiently small. By comparison, we have $\rho_-(t) < \rho(t) < \rho_+(t)$ for all t.

To prove part (b), note we can still define ρ_+ as a comparison function. Since $\rho_+(0) \ge \rho(0)$, the above argument still holds. **Q.E.D.**

The following lemma estimates the continuity of ρ in the error term $\widetilde{g}(t)$. Here we consider the case $\rho(0)=\varepsilon>0$ only.

Lemma 6.5 *Let $\Gamma>0$, $\sigma>0$, and $C_1>0$ be given constants, independent of ε_0. Suppose two positive function $\rho_1(t)$ and $\rho_2(t)$ satisfy*

$$\dot\rho_i = -\Gamma\rho_i^3 + \widetilde{g}_i(t) \quad (i=1,2)\ , \qquad \rho_1(0)=\rho_2(0)=\varepsilon\ ,$$

with $0<\varepsilon\le\varepsilon_0$ and

$$|\widetilde{g}_i(t)| \le C_1\,\{t\}^{-3/2-\sigma}\ , \quad |\boldsymbol{\delta}\widetilde{g}(t)| \le \boldsymbol{\delta}_0\,\{t\}^{-3/2-\sigma}\ ,$$

where $\boldsymbol{\delta}\widetilde{g} = \widetilde{g}_2 - \widetilde{g}_1$ and $\boldsymbol{\delta}_0 \ge 0$ is a small number. Then we have

$$|\rho_2(t)-\rho_1(t)| \le \boldsymbol{\delta}_0\varepsilon^\sigma(\Gamma\sigma)^{-1}\,\{t\}^{-(1+\sigma)/2}\ .$$

Note that the decay rate is improved.

Proof. Let $r=\boldsymbol{\delta}\rho=\rho_2-\rho_1$. Then r satisfies

$$\dot r = -Gr + \boldsymbol{\delta}\widetilde{g}, \quad r(0)=0, \quad G=\Gamma(3\rho^2+3\rho r+r^2)\ .$$

We have

$$r(t) = \int_0^t e^{-\int_s^t G(\tau)d\tau}\boldsymbol{\delta}\widetilde{g}(s)\,ds\ . \tag{6.16}$$

Note $|r(t)| < \rho_+(t)-\rho_-(t)$, hence

$$G(t) \ge \Gamma[3\rho_-^2 - 3\rho_+(\rho_+-\rho_-)] = 3\Gamma[m^{-2}-m(m-m^{-1})]\,\{t\}^{-1} \ge 2\Gamma\,\{t\}^{-1}$$

if m is sufficiently close to 1. Hence

$$\int_s^t G(\tau)d\tau \ge \int_s^t 2\Gamma\,\{\tau\}^{-1}\,d\tau = 2\Gamma\int_s^t (2\Gamma)^{-1}\,[\ln\{\tau\}]_{\tau=s}^t \ge \frac{1+\sigma}{2}\ (\ln\{t\}-\ln\{s\})$$

and

$$e^{-\int_s^t G(\tau)d\tau} \le \{t\}^{-(1+\sigma)/2} \cdot \{s\}^{((1+\sigma)/2} \ .$$

By (6.16) we have

$$\begin{aligned} |r(t)| &\le \int_0^t \{t\}^{-(1+\sigma)/2} \{s\}^{(1+\sigma)/2} \boldsymbol{\delta}_0 \{s\}^{-3/2-\sigma} \, ds \\ &= \boldsymbol{\delta}_0 \{t\}^{-(1+\sigma)/2} \int_0^t \{s\}^{-1-\sigma/2} \, ds \le \boldsymbol{\delta}_0 \varepsilon^\sigma (\Gamma\sigma)^{-1} \{t\}^{-(1+\sigma)/2} \ . \end{aligned}$$

Q.E.D.

7 Change of mass

For given Q_E and h_0, we want to find an $a(t) = a(E, h_0; t)$ which satisfies

$$a(\infty) = 0, \quad \dot{a} = (c_1 Q, \operatorname{Im} F(k)), \quad |a(t)| \le C\lambda \{t\}^{-1} \ .$$

7.1 The main oscillatory part of $a(t)$

We use the following equivalent integral equation for $a(t)$:

$$a(t) = \int_\infty^t (c_1 Q, \operatorname{Im} F(k)) \, ds \ .$$

Note $k = aR + h = aR + \zeta + \eta$. We want to perform several integrations by parts so that we get the form

$$a(t) = O(t^{-1}) + \int_\infty^t O(t^{-2}) \, ds = O(t^{-1}) \ .$$

As in Section 4, we decompose η and a as

$$\eta = \eta^{(2)} + \eta^{(3)}, \quad a = a_{20}(z^2 + \bar{z}^2) + b.$$

Recall $k = aR + \zeta + \eta$ and

$$\operatorname{Im} F(k) = \operatorname{Im} \left\{ \lambda Q h^2 + 2\lambda Q R a h + \lambda (aR + h)(aR + \bar{h}) h \right\} \ .$$

We also denote

$$\mu^{-1} \dot{p} = c_\alpha z^\alpha + d_\beta z^\beta + d_1 b z + d_2 b \bar{z} + P^{(4)} = c_\alpha z^\alpha + P^{(3,4)}$$

Using the decomposition of $F(k)$, we have

$$(c_1 Q, \operatorname{Im} F) = A^{(2)} + A^{(za)} + A^{(z\eta)} + A^{(z^3)} + A^{(4)} + A^{(5)} \ ,$$

where

$$
\begin{aligned}
A^{(2)} &= \big(c_1 Q,\ \lambda Q \operatorname{Im} \zeta^2\big) \\
A^{(za)} &= (c_1 Q,\ \operatorname{Im} 2\lambda Q R a \zeta) \\
A^{(z\eta)} &= (c_1 Q,\ \operatorname{Im} 2\lambda Q \zeta \eta) \\
A^{(z^3)} &= \big(c_1 Q,\ \operatorname{Im} \lambda |\zeta|^2 \zeta\big) \\
A^{(4)} + A^{(5)} &= \big(c_1 Q,\ \operatorname{Im} \big\{\lambda Q \eta^2 + 2\lambda Q R a \eta + \lambda \left[|k|^2 k - |\zeta|^2 \zeta\right]\big\}\big) \ , \\
A^{(4)} &= \Big(c_1 Q,\ \operatorname{Im} \Big\{\lambda Q (\eta^{(2)})^2 + 2\lambda Q R a \eta^{(2)} + \lambda \Big[2|\zeta|^2 (aR + \eta^{(2)}) + \zeta^2 (aR + \overline{\eta^{(2)}})\Big]\Big\}\Big) \\
A^{(5)} &= \big(c_1 Q,\ \operatorname{Im} \big\{\lambda Q [2\eta^{(2)} \eta^{(3)} + (\eta^{(3)})^2] + 2\lambda Q R a \eta^{(3)}\big\}\big) \\
&\quad + \Big(c_1 Q, \operatorname{Im} \Big\{\lambda \Big[2|\zeta|^2 \eta^{(3)} + \zeta^2 \overline{\eta^{(3)}} + \ell^2 \bar{\zeta} + 2|\ell|^2 \zeta + \ell^2 \bar{\ell}\Big]\Big\}\Big) \ , \quad \ell = aR + \eta \ .
\end{aligned}
$$

Since $\zeta = z u_+ + \bar{z} u_-$, hence

$$
\operatorname{Im} \zeta = \operatorname{Im} z \, (u_+ - u_-), \quad \operatorname{Im} \zeta^2 = (\operatorname{Im} z^2) \, (u_+^2 - u_-^2) \ .
$$

Therefore

$$
\begin{aligned}
A^{(2)} &= C_1 \ \operatorname{Im} z^2, \qquad C_1 = \big(c_1 Q,\ \lambda Q (u_+^2 - u_-^2)\big) = O(\lambda^2) \ , \\
A^{(za)} &= C_2 \, a \operatorname{Im} z, \qquad C_2 = (c_1 Q, 2\lambda Q R (u_+ - u_-)) = O(\lambda) \ .
\end{aligned}
$$

First we integrate $A^{(2)}$:

$$
\begin{aligned}
\int_\infty^t A^{(2)} \, ds &= C_1 \operatorname{Im} \int_\infty^t z^2 \, ds = C_1 \operatorname{Im} \int_\infty^t \mu^{-2} p^2 \, ds \\
&= C_1 \operatorname{Im} \frac{1}{-2i\kappa} \left\{\mu^{-2} p^2 - \int_\infty^t \mu^{-2} 2 p \dot{p} \, ds\right\} \\
&= \frac{C_1}{4\kappa} \, 2 \operatorname{Re} \left\{z^2 - 2 \int_\infty^t z \mu^{-1} \dot{p} \, ds\right\} \\
&= a_{20} (z^2 + \bar{z}^2) - 4 a_{20} \operatorname{Re} \int_\infty^t z \left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 b z + d_2 b \bar{z} + P^{(4)}\right] \, ds \\
&= a_{20} (z^2 + \bar{z}^2) + \int_\infty^t A_{2,3} + A_{2,4} + A_{2,5} \, ds \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
a_{20} &= \frac{C_1}{4\kappa} = \frac{\lambda}{4\kappa} \big(c_1 Q,\ Q (u_+^2 - u_-^2)\big) = O(\lambda^2) \ , \\
A_{2,3} &= -4 a_{20} \operatorname{Re} z c_\alpha z^\alpha \ , \\
A_{2,4} &= -4 a_{20} \operatorname{Re} \left[d_\beta z z^\beta + d_1 b z^2\right] \\
A_{2,5} &= -4 a_{20} \operatorname{Re} z P^{(4)} \ ,
\end{aligned}
$$

Note here we have used $\operatorname{Re} z d_2 b\bar{z} = 0$.

As mentioned in Section 4, the term $a_{20}(z^2+\bar{z}^2)$ is the main oscillatory part of a. We denote the remaining part as b and hence

$$a = a_{20}(z^2+\bar{z}^2) + b \ .$$

Although b is of the same order t^{-1}, its oscillation is slower since the right side of its equation

$$\dot{b} = (c_1 Q, \operatorname{Im}(F - F^{(2)})) - 4a_{20} \operatorname{Re} z\mu^{-1}\dot{p}$$

which we obtained from the above computation, is of order $O(z^3)$, (cf. (4.19)).

Another reason for using $b(t)$ is that it is better suited than $a(t)$ for the iteration scheme in Section 8. We have

$$\boldsymbol{\delta} a(t) \sim \boldsymbol{\delta}_0 \{t\}^{-1} \log\{t\}\,, \quad \boldsymbol{\delta} b(t) \sim \boldsymbol{\delta}_0 \{t\}^{-1} \ .$$

This difference arizes because it is harder to estimate the continuity of the phase.

7.2 Integration of $O(z^3)$ terms

we next integrate $A^{(za)}$:

$$\begin{aligned}
\int_\infty^t A^{(za)} &= \int_\infty^t C_2 a \operatorname{Im} z \\
&= C_2 \operatorname{Im} \frac{1}{-i\kappa}\left\{ az - \int_\infty^t \mu^{-1}\frac{d}{dt}(ap)\right\} \\
&= \frac{C_2}{2\kappa} 2\operatorname{Re}\left\{ az - \int_\infty^t z(c_1 Q, \operatorname{Im} F) + a(c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)})\right\} \\
&= c_{za} a(z+\bar{z}) + \int_\infty^t A_{za,3} + A_{za,4} + A_{za,5}\, ds \ ,
\end{aligned}$$

where

$$\begin{aligned}
c_{za} &= C_2/(2\kappa) = (c_1 Q, 2\lambda QR(u_+ - u_-))/(2\kappa) = O(\lambda) \\
A_{za,3} &= -c_{za}(z+\bar{z})\,(c_1 Q, \operatorname{Im} F^{(2)}) \\
A_{za,4} &= -c_{za}(z+\bar{z})\,(c_1 Q, \operatorname{Im}(2\lambda QRa\zeta + 2\lambda Q\zeta\eta^{(2)} + \lambda|\zeta|^2\zeta)) - 2c_{za} a \operatorname{Re} c_\alpha z^\alpha \\
A_{za,5} &= -c_{za}(z+\bar{z})\,(c_1 Q, \operatorname{Im} F^{(4)}) - 2c_{za} a \operatorname{Re}(d_\beta z^\beta + d_1 bz + d_2 b\bar{z} + P^{(4)})
\end{aligned}$$

We now integrate $A^{(z\eta)}$. Recall $U = U_+ + U_-\mathtt{C}$, see (3.2). We will also write $U^{-1} = U_3 + U_4\mathtt{C}$, with $U_3, U_4 = \frac{1}{2}(BA^{-1/2} \pm B^{-1}A^{1/2})$, both self-adjoint. We will also use the following formulas:

$$\operatorname{Re}\int dx f(\mathtt{C}g) = \operatorname{Re}\int dx(\mathtt{C}f)g, \quad \operatorname{Im}\int dx f(\mathtt{C}g) = -\operatorname{Im}\int dx(\mathtt{C}f)g, \tag{7.1}$$

and (using $U = U_+ + U_-\mathtt{C}$)

$$U(zf + \bar{z}g) = z(U_+ f + U_- \bar{g}) + \bar{z}(U_+ g + U_- \bar{f}) \tag{7.2}$$

Recall $\eta = U^{-1}\eta^\diamond = U^{-1}e^{-i\theta}\tilde{\eta}$. Thus, by (7.1),

$$\begin{aligned} A^{(z\eta)} &= (c_1 Q, \operatorname{Im} 2\lambda Q\zeta\eta) = \operatorname{Im}\int dx\, 2c_1\lambda Q^2\zeta\eta \\ &= \operatorname{Im}\int dx\, 2c_1\lambda Q^2 (zu_+ + \bar{z}u_-)\,(U_3 + U_4\mathtt{C})\eta^\diamond \\ &= \operatorname{Im}\int dx\, [(U_3 - U_4\mathtt{C})(z\phi_1 + \bar{z}\phi_2)]\,\eta^\diamond \end{aligned}$$

where

$$\phi_1 = 2c_1\lambda Q^2 u_+, \quad \phi_1 = 2c_1\lambda Q^2 u_-.$$

Hence, by (7.2), (think $U_+ = U_3$ and $U_- = -U_4$)

$$A^{(z\eta)} = \operatorname{Im}\int dx\, (z\phi_3 + \bar{z}\phi_4)\eta^\diamond\ ,$$

with

$$\phi_3 = U_3\phi_1 - U_4\phi_2, \quad \phi_4 = U_3\phi_2 - U_4\phi_1.$$

We rewrite

$$\eta^\diamond(\tau) = e^{-i\theta}\tilde{\eta} = e^{-i\theta}e^{-iA\tau}f(\tau)$$

where $f = e^{iA\tau}\tilde{\eta} = \tilde{\eta}_0 + \int_0^\tau e^{isA}\cdots ds$. The reason we work with f, instead of $\tilde{\eta}$, is that those terms in $\partial_\tau f$ of same order (z^2) are explicit and do not involve differentiation, compared to those in $\partial_\tau\eta$. Now we have

$$\begin{aligned} \int_\infty^t A^{(z\eta)} &= \operatorname{Im}\int_\infty^t (\phi_3, z\eta^\diamond) + (\phi_4, \bar{z}\eta^\diamond)\, d\tau \\ &= \operatorname{Im}\int_\infty^t \left(\phi_3, e^{-i\tau(A+\kappa)}\left(pe^{-i\theta}f\right)\right)\, d\tau + \operatorname{Im}\int_\infty^t \left(\phi_4, e^{-i\tau(A-\kappa)}\left(\bar{p}e^{-i\theta}f\right)\right)\, d\tau \end{aligned}$$

We deal with the first integral, which is equal to (note $f \in \mathtt{H}_{\mathrm{c}}(A)$)

$$\begin{aligned} &= \operatorname{Im}\left(\phi_3, \frac{1}{-i(A+\kappa)}e^{-i\tau(A+\kappa)}\left(pe^{-i\theta}f\right)\right) \\ &\quad - \operatorname{Im}\int_\infty^t \left(\phi_3, \frac{1}{-i(A+\kappa)}e^{-i\tau(A+\kappa)}\frac{d}{d\tau}\left(pe^{-i\theta}f\right)\right)\, d\tau \\ &= \operatorname{Re}\left(\frac{1}{A+\kappa}\,\mathbf{P}_{\mathrm{c}}^A\,\Pi\phi_3, ze^{-i\theta}\tilde{\eta}\right) \\ &\quad - \operatorname{Re}\int_\infty^t \left(\frac{1}{A+\kappa}\,\mathbf{P}_{\mathrm{c}}^A\,\Pi\phi_3, \left\{(\dot{p}/p - i\dot{\theta})ze^{-i\theta}\tilde{\eta} + ze^{-i\theta}\,\mathbf{P}_{\mathrm{c}}^A\,\left[e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U,i]\dot{\theta}\eta\right]\right\}\right)\, d\tau \\ &= \operatorname{Re}(\phi_5, z\eta^\diamond) - \operatorname{Re}\int_\infty^t \left(\phi_5, \left\{(\dot{p}/p - i\dot{\theta})z\eta^\diamond + z\,\mathbf{P}_{\mathrm{c}}^A\,\left[U\Pi F^\sharp - [U,i]\dot{\theta}\eta\right]\right\}\right)\, d\tau \end{aligned}$$

where

$$\phi_5 = \frac{1}{A+\kappa}\,\mathbf{P}_{\mathrm{c}}^A\,\Pi\phi_3\ .$$

We are careful in adding $\mathbf{P}_{\mathrm{c}}^A\,\Pi$ so that ϕ_5 makes sense. We can do so since $f \in \mathtt{H}_{\mathrm{c}}(A)$.

Similarly, the second integral is equal to

$$= \mathrm{Re}\,(\phi_6, \bar{z}\eta^\diamond) - \mathrm{Re}\int_\infty^t \Big(\phi_6, \Big\{(\overline{\dot{p}/p} - i\dot{\theta})\bar{z}\eta^\diamond + \bar{z}\,\mathbf{P}_{\mathrm{c}}^A\,\Big[U\Pi F^\sharp - [U,i]\dot{\theta}\eta\Big]\Big\}\Big)\,d\tau$$

with

$$\phi_6 = \frac{1}{A-\kappa}\,\mathbf{P}_{\mathrm{c}}^A\,\Pi\phi_4\ .$$

We can rewrite their leading terms in the form

$$\begin{aligned}
\mathrm{Re}(\phi_5, z\eta^\diamond) + \mathrm{Re}(\phi_6, \bar{z}\eta^\diamond) &= \mathrm{Re}\int dx\,(z\phi_5 + \bar{z}\phi_6)\,(U_+ + U_-\mathtt{C})\eta \\
&= \mathrm{Re}\int dx\,[(U_+ + U_-\mathtt{C})\,(z\phi_5 + \bar{z}\phi_6)]\,\eta \\
&= \mathrm{Re}\int dx\,(z\phi_8 + \bar{z}\phi_7)\eta. \\
&= \mathrm{Re}\,(z\phi_7 + \bar{z}\phi_8, \eta).
\end{aligned}$$

where we have used (7.1) and (7.2), with

$$\phi_8 = U_+\phi_5 + U_-\phi_6, \quad \phi_7 = U_+\phi_6 + U_-\phi_5\ .$$

The remaining integral has the integrand

$$\begin{aligned}
&= -\,\mathrm{Re}\Big(\phi_5, \Big\{(\dot{p}/p - i\dot{\theta})zU\eta + z\,\mathbf{P}_{\mathrm{c}}^A\,\Big[U\Pi F^\sharp - [U,i]\dot{\theta}\eta\Big]\Big\}\Big) \\
&\quad -\,\mathrm{Re}\Big(\phi_6, \Big\{(\overline{\dot{p}/p} - i\dot{\theta})\bar{z}U\eta + \bar{z}\,\mathbf{P}_{\mathrm{c}}^A\,\Big[U\Pi F^\sharp - [U,i]\dot{\theta}\eta\Big]\Big\}\Big) \\
&= -\,\mathrm{Re}\int dx\,\Big(\mu^{-1}\dot{p}\phi_5 + \mu\bar{\dot{p}}\phi_6 - i\dot{\theta}(z\phi_5 + \bar{z}\phi_6)\Big)\,U\eta \\
&\qquad + (z\phi_5 + \bar{z}\phi_6)\,\Big[U\Pi F^\sharp - [U,i]\dot{\theta}\eta\Big] \\
&= -\,\mathrm{Re}\int dx\,\big(\mu^{-1}\dot{p}\phi_5 + \mu\bar{\dot{p}}\phi_6\big)\,U\eta + (z\phi_5 + \bar{z}\phi_6)\,\Big[U\Pi F^\sharp - \dot{\theta}Ui\eta\Big] \\
&= A_{z\eta,3} + A_{z\eta,4} + A_{z\eta,5}
\end{aligned}$$

where $A_{z\eta,3}$ is from $F^\sharp = iz^\alpha\phi_\alpha^\sharp + F^{\sharp\sharp}$, defined in (4.11),

$$F^\sharp = -i\dot{\theta}\zeta - iF(k) - [(c_1Q, \mathrm{Im}\,F) + ia\dot{\theta}]R_\Pi := i\,z^\alpha\phi_\alpha^\sharp + F^{\sharp\sharp(3)} + F^{\sharp\sharp(4)}.$$

$$
\begin{aligned}
A_{z\eta,3} &= -\operatorname{Re}\int dx\,(z\phi_5+\bar z\phi_6)\,U\Pi i z^\alpha\phi^\sharp_\alpha\\
A_{z\eta,4} &= -\operatorname{Re}\int dx\,\left(\mu^{-1}c_\alpha z^\alpha\phi_5-\mu c_\alpha\bar z^\alpha\phi_6\right)U\eta^{(2)}\\
&\qquad+(z\phi_5+\bar z\phi_6)\,\left[U\Pi F^{\sharp\sharp(3)}-c_2(z+\bar z)Ui\eta^{(2)}\right]\\
A_{z\eta,5} &= -\operatorname{Re}\int dx\,\left(\mu^{-1}c_\alpha z^\alpha\phi_5-\mu c_\alpha\bar z^\alpha\phi_6\right)U\eta^{(3)}+\left(\mu^{-1}P^{(3,4)}\phi_5+\mu\overline{P^{(3,4)}}\phi_6\right)U\eta\\
&\qquad+(z\phi_5+\bar z\phi_6)\,\left[U\Pi F^{\sharp\sharp(4)}-\left(c_2(z+\bar z)Ui\eta^{(3)}+F_\theta Ui\eta\right)\right]
\end{aligned}
$$

We first observe that, although $e^{i\theta}$ appears in the computation, it does not show up in the final results. Also, $A_{z\eta,3}$ is a sum of monomials cz^β with $|\beta|=3$. There is no a or η in $A_{z\eta,3}$.

Summarizing, we have

$$
\begin{aligned}
a &= a_{20}\left(z^2+\bar z^2\right)+c_{za}a(z+\bar z)+\operatorname{Re}(z\phi_7+\bar z\phi_8,\eta)\\
&\quad+\int_\infty^t\left[A^{(z^3)}+A_{2,3}+A_{(za,3)}+A_{(z\eta,3)}\right]+\left[A^{(4)}+A_{2,4}+A_{(za,4)}+A_{(z\eta,4)}\right]\\
&\qquad+\left[A^{(5)}+A_{2,5}+A_{(za,5)}+A_{(z\eta,5)}\right]\,ds
\end{aligned}
$$

We now write

$$
\begin{aligned}
A_\beta z^\beta &= A^{(z^3)}+A_{2,3}+A_{(za,3)}+A_{(z\eta,3)}\\
&= \left(c_1Q,\,\operatorname{Im}\lambda|\zeta|^2\zeta\right)-4a_{20}\operatorname{Re}zc_\alpha z^\alpha-c_{za}(z+\bar z)\left(c_1Q,\operatorname{Im}F^{(2)}\right)\\
&\quad-\operatorname{Re}\int dx\,(z\phi_5+\bar z\phi_6)\,U\Pi i z^\alpha\phi^\sharp_\alpha
\end{aligned}
$$

From integration by parts, we have

$$
\begin{aligned}
\int_\infty^t A_\beta z^\beta\,ds &= a_\beta z^\beta-\int_\infty^t a_\beta z^\beta f_\beta(z)\,ds, \qquad a_\beta=\frac{A_\beta}{i[\beta]\kappa}\ ,\\
&= a_\beta z^\beta+\int_\infty^t A_{3,4}+A_{3,5}\,ds
\end{aligned}
$$

where f_β is defined in (6.9), and, $(\beta=(\beta_0,\beta_1))$

$$
\begin{aligned}
A_{3,4} &= -a_\beta z^\beta(\beta_0+\beta_1\mathtt{C})z^{-1}c_\alpha z^\alpha\\
A_{3,5} &= -a_\beta z^\beta(\beta_0+\beta_1\mathtt{C})z^{-1}P^{(3,4)}
\end{aligned}
$$

Note $[\beta]\neq 0$. Also note $\overline{a_\beta}=a_{\bar\beta}$.

Let $a^{(4)}+a^{(5)}$ denote the total of the remaining integrals. We have

$$
a(t)=a_{20}(z^2+\bar z^2)+a_\beta z^\beta+c_{za}a(z+\bar z)+\operatorname{Re}(z\phi_7+\bar z\phi_8,\eta)+a^{(4)}+a^{(5)} \tag{7.3}
$$

$$
a^{(4)}=\int_\infty^t A^{(4)}+A_{2,4}+A_{(za,4)}+A_{(z\eta,4)}+A_{3,4}\,ds\ , \tag{7.4}
$$

$$
a^{(5)}=\int_\infty^t A^{(5)}+A_{2,5}+A_{(za,5)}+A_{(z\eta,5)}+A_{3,5}\,ds\ .
$$

Now we consider the integrand of $a^{(4)}$. They are terms of order z^4.

$$
\begin{aligned}
A^{(4)} &= \Big(c_1 Q,\, \mathrm{Im}\left\{\lambda Q(\eta^{(2)})^2 + 2\lambda Q R a \eta^{(2)} + \lambda\left[2|\zeta|^2(aR+\eta^{(2)}) + \zeta^2(aR+\overline{\eta^{(2)}})\right]\right\}\Big) \\
A_{2,4} &= -4a_{20}\,\mathrm{Re}\left[d_\beta z z^\beta + d_1 b z^2\right] \\
A_{za,4} &= -c_{za}(z+\bar z)\left(c_1 Q, \mathrm{Im}(2\lambda Q R a\zeta + 2\lambda Q\zeta\eta^{(2)} + \lambda|\zeta|^2\zeta)\right) - 2c_{za} a\,\mathrm{Re}\, c_\alpha z^\alpha \\
A_{z\eta,4} &= -\,\mathrm{Re}\int dx\,\left(\mu^{-1}c_\alpha z^\alpha\phi_5 - \mu c_\alpha \bar z^\alpha \phi_6\right) U\eta^{(2)} \\
&\qquad + (z\phi_5 + \bar z\phi_6)\,\left[U\Pi F^{\sharp\sharp(3)} - c_2(z+\bar z)Ui\eta^{(2)}\right] \\
A_{3,4} &= -a_\beta z^\beta(\beta_0 + \beta_1 \mathtt{C}) z^{-1} c_\alpha z^\alpha
\end{aligned}
$$

We substitute $a = a_{20}(z^2+\bar z^2) + b$ and $\eta^{(2)} = z^\alpha \eta_\alpha$ in the above integrands. Note that although terms of order z^4 have the following forms

$$
z^4,\ a^2,\ \eta^2,\ z^2 a,\ z^2\eta,\ a\eta,\ z\eta^{(3)}\ .
$$

some of them do not occur in the above integrands. We have the following Lemma.

Lemma 7.1 *After the substitution $a = a_{20}(z^2+\bar z^2) + b$ and $\eta^{(2)} = z^\alpha\eta_\alpha$, the integrand of $a^{(4)}$, (7.4), can be summed into the form*

$$
B_{22}|z|^4 + \mathrm{Re}\left\{A_{40}z^4 + A_{31}z^3\bar z + A_{b2} b z^2\right\}\ .
$$

There are no terms of the form b^2, $b|z|^2$, or $z\eta^{(3)}$. Moreover, we have

$$
B_{22} = \frac{c_1}{2}\Gamma + O(\lambda^4), \quad A_{40}, A_{31} = O(\lambda^3), \quad A_{b2} = O(\lambda)\ . \tag{7.5}
$$

Proof. The first part is obtained by direct inspection. Note we deal with $A_{z\eta,4}$ in the same way we deal with $A^{(z\eta)}$. The orders of the coefficients are also obtained by direct check, with the following table in mind:

$$
\begin{gathered}
c_\alpha = \lambda, \quad d_{30}, d_{12}, d_{03} = \lambda^2, \quad d_{21} = 1, \quad d_1 = 1 \\
c_1 = \lambda, \quad R = \lambda^{-1}, \\
a_{20} = \lambda^2, \quad a_\beta = \lambda^2, \quad c_{za} = \lambda, \quad \phi_7, \phi_8 = \lambda^2 \\
\zeta \sim z + \lambda\bar z, \quad aR \sim \rho^2, \quad \eta \sim \lambda\rho^2
\end{gathered}
$$

We note that most $C|z|^4$ terms with $C = O(\lambda^2)$ are killed by the Im operator. The only term that survives, due to resonance, and becomes the main term in B_{22}, is from the last term of $A^{(4)}$:

$$
\left(c_1 Q,\, \mathrm{Im}\,\lambda\zeta^2\overline{\eta^{(2)}}\right) = \left(c_1 Q,\, \mathrm{Im}\,\lambda z^2 u_+^2 \overline{z^2\eta_{20}}\right) + O(\lambda^4)\,|z|^4 + \sum_{|\gamma|=4,\,\gamma\neq(22)} O(\lambda^3)\, z^\gamma\ ,
$$

where the first term is equal to $\frac{c_1}{2}\Gamma|z|^4$. We also note that the most dangerous term, $|\zeta|^2\zeta = z^2\bar{z}u_+^3 + O(\lambda\rho^3)$, in all integrands, only contributes $O(\lambda^4)$ to B_{22}. **Q.E.D.**

7.3 Integration of $O(z^4)$ terms

Next we proceed to integrate out those oscillatory terms in $a^{(4)}$.

$$\int_\infty^t A_{40}z^4 + A_{31}z^3\bar{z} + A_{b2}bz^2\,ds = \frac{A_{40}z^4}{-4i\kappa} + \frac{A_{31}z^3\bar{z}}{-2i\kappa} + \frac{A_{b2}bz^2}{-2i\kappa}$$
$$- \int_\infty^t \frac{A_{40}z^4}{-4i\kappa}f_{40}(z) + \frac{A_{31}z^3\bar{z}}{-2i\kappa}f_{31}(z) + \frac{A_{b2}\mu^{-2}}{-2i\kappa}\frac{d}{ds}(bp^2)\,ds$$

where $\frac{d}{ds}(bp^2) = \dot{b}p^2 + 2bp\dot{p} = O(z^5)$. Let

$$a_{40} = \frac{A_{40}}{-4i\kappa} = O(\lambda^3), \quad a_{31} = \frac{A_{31}}{-2i\kappa} = O(\lambda^3), \quad a_{b2} = \frac{A_{b2}}{-2i\kappa} = O(\lambda)\ .$$

Then

$$a^{(4)} = \mathrm{Re}\left\{a_{40}z^4 + a_{31}z^3\bar{z} + a_{b2}bz^2\right\} + \int_\infty^t A_{4,5}\,ds$$
$$A_{4,5} = -\,\mathrm{Re}\left\{a_{40}z^4 f_{40}(z) + a_{31}z^3\bar{z}f_{31}(z) + a_{b2}\frac{d}{ds}(bp^2)\right\}$$

Since $\eta_1^{(3)} = U^{-1}e^{-i\theta}\,e^{-iAt}U\eta_0$ is larger than $\eta_{2-5}^{(3)}$ when time t is of order 1, we also need to integrate out terms of order $z^2\eta_1^{(3)}$, as we did for (6.12) in the equation for p. Specifically, we have terms of the form

$$A_{z^2\eta_0} \equiv \sum_{|\alpha|=2} \mathrm{Re}(\phi, z^\alpha\eta_1^{(3)}) = \frac{d}{dt}(a_{z^2\eta_0}) + A_{z^2\eta_0,5}$$

(for different ϕ), where $a_{z^2\eta_0}$ and $A_{z^2\eta_0,5}$ are similar to p_4 and g_4, respectively,

$$a_{z^2\eta_0} \sim \sum_{|\alpha|=2} \mathrm{Re}\,(\phi, z^\alpha e^{-i\theta}\,\frac{1}{A - 0i + [\alpha]\kappa}\,e^{-iAt}U\eta_0),$$
$$A_{z^2\eta_0,5} \sim \sum_{|\alpha|=2} \mathrm{Re}\,(\phi, \mu^{[\alpha]}\frac{d}{ds}\left(p^\alpha e^{-i\theta}\right)\,\frac{1}{A - 0i + [\alpha]\kappa}\,e^{-iAt}U\eta_0),$$

(for different ϕ). Here $\frac{1}{A-0i+[\alpha]\kappa}$ indicates that we have some resonance effect, (as in p_4 and g_4), and hence these terms has slower dacay in t:

$$|a_{z^2\eta_0}| \le \|\eta_0\|\,\{t\}^{-1}\,\langle t\rangle^{-1-\sigma}\,, \qquad |A_{z^2\eta_0,5}| \le \|\eta_0\|\,\{t\}^{-3/2}\,\langle t\rangle^{-1-\sigma}\ .$$

Now we define

$$B_5 = B_5(z, b, \eta^{(2)}, \eta^{(3)}) = A^{(5)} + A_{2,5} + A_{(za,5)} + A_{(z\eta,5)} + A_{4,5} - A_{z^2\eta_0} + A_{z^2\eta_0,5}\ .$$

B_5 is of order $O(z^5)$. It is a complicated polynomial in its arguments, which including U, U^{-1} and $((A-\kappa)^{-1}\phi,\cdot)$, but it does not contain $e^{i\theta}$.

We conclude

$$\begin{aligned} a &= a_{20}\left(z^2+\bar{z}^2\right)+b \\ b &= c_{za}a(z+\bar{z})+\mathrm{Re}(z\phi_7+\bar{z}\phi_8,\eta)+\mathrm{Re}\left\{a_{40}z^4+a_{31}z^3\bar{z}+a_{b2}bz^2\right\}+a_{z^2\eta_0} \\ &\quad +\int_\infty^t B_{22}|z|^4+B_5\,ds\ . \end{aligned}$$

We have obtained the main estimate Theorem 5.1 assuming the estimate (5.2) on b. We now need to prove the existence of the solution and check the assumption on b. Define the mapping $S(b)(t)=S_T(b)(t)$:

$$\begin{aligned} S(b)(t) &= \Big[c_{za}a(z+\bar{z})+\mathrm{Re}(z\phi_7+\bar{z}\phi_8,\eta)+\mathrm{Re}\left\{a_{40}z^4+a_{31}z^3\bar{z}+a_{b2}bz^2\right\}+a_{z^2\eta_0}\Big]_T^t \\ &\quad +\int_T^t B_{22}|z|^4+B_5\,ds\ . \end{aligned} \tag{7.6}$$

Recall the class $\mathcal{B}_T$ of b

$$\mathcal{B}_T=\left\{b(t):|b(t)|\le D\left\{t\right\}^{-1},\,0\le t\le T\right\}\ , \tag{7.7}$$

where $D=2B_{22}/\Gamma=O(\lambda)$. Our goal is to show that $S(b)$ maps $\mathcal{B}_T$ into itself. More precisely, we have the following Lemma.

Lemma 7.2 *Suppose that $M(t)\le 2$ and $|b(t)|\le D\left\{t\right\}^{-1}$, $0\le t\le T$. Recall $D=2B_{22}/\Gamma=O(\lambda)$. Then we have*

$$|S(b)(t)|\le C_1(D)\left\{t\right\}^{-3/2}+\frac{B_{22}}{2\Gamma}\left\{t\right\}^{-1}\le\frac{D}{2}\left\{t\right\}^{-1}\ .$$

Hence S map $\mathcal{B}_T$ to itself. There is a similar statement for $a(t)$.

Assuming the previous bound on b, we can also estimate θ. Since $\theta(t)$ is given by (cf. 4.7)

$$\theta(t)=\frac{2c_2}{\kappa}\,\mathrm{Im}\,z+\int_0^t-\frac{2c_2}{\kappa}\,\mathrm{Im}(\mu^{-1}\dot{p})+F_\theta\,ds$$

we have

$$|\theta(t)|\le C\log\left\{t\right\}\ .$$

This estimate will not be used in the rigorous proof, but it provides an idea about its size.

8 Contraction map

We review what we have so far. Up to now, we have not shown the existence of the solution to the equations setup in section 4 with the boundary condition (4.23). Notice we cannot conclude this from the existence to the Schrödinger equation as the boundary condition of a is set at the time $t = \infty$ (4.23). We have obtained the main estimate Theorem 5.1 assuming the estimate (7.7) on b. We also proved a bound on the map S in Lemma 7.2, again, assuming an estimate on b. We now need to prove the existence of the solution and prove this bound on b. We shall achieve both goals simultaneously by proving the map S is a contraction map on the space $\mathcal{B} = \mathcal{B}_{T=\infty}$. Once we have proved this, we have constructed rigorously the solution needed in part (1) of Theorem 1.1 and established all the upper bounds in part (1). To conclude part (1) of Theorem 1.1, it remains to prove the lower bound in ζ. The size of ζ is given by z and thus by ρ. From part (s) of Lemma 6.4, we conclude the lower bound of ζ provided that we can check the bound on g. Since g is given explicitly in (6.12) and we can estimate all terms in g from the upper bounds in part (1) of Theorem 1.1. This concludes part (1) of Theorem 1.1 assuming that S is a contraction on the space $\mathcal{B}$. The rest of this section is a proof that S is a contraction.

We first recall the setting. For each $b \in \mathcal{B}$, we can solve our system to get

$$z, p, q = S_1(b), \quad \eta = S_2(b), \quad \theta = S_3(b) \ .$$

More specifically, for fixed E and $h_0 \perp Q$, our system is

$$\begin{aligned}
\dot{q} &= \delta_{21}|q|^2 q + ic_4 bq + g(b, \eta, \theta; t) \ , \\
&\qquad q = p + O(z^2), \text{ given by (6.14)}, \ z = \mu^{-1} p \ , \\
&\qquad q(0) \text{ given by } p(0) = z(0) = (v, \operatorname{Re} h_0) + i(u, \operatorname{Im} h_0) \ , \\
\tilde{\eta}(t) &= e^{-iAt}\tilde{\eta}_0 + \int_0^t e^{-iA(t-s)}\, \mathbf{P}_{\mathrm{c}}^A \left\{ e^{i\theta} U \Pi F^\sharp - e^{i\theta}[U, i]\dot{\theta}\eta \right\} ds \ , \\
\eta &= U^{-1} e^{-i\theta}\tilde{\eta}, \quad \eta(0) = \mathbf{P}_{\mathrm{c}}^{\mathcal{L}}\, h_0 \ , \\
\theta(t) &= \frac{2c_2}{\kappa} \operatorname{Im} z + \int_0^t \widetilde{F}_\theta(b, \eta, \theta; s)\, ds, \qquad \widetilde{F}_\theta = -\frac{2c_2}{\kappa} \operatorname{Im}(\mu^{-1}\dot{p}) + F_\theta \ .
\end{aligned}$$

Recall $F^\sharp$ is given by (4.8), and that $q = p + c\mu z^\alpha + c\mu b\bar{z} + c\mu z^\beta - p_4$, for some constants c. In this system all a are already replaced by $a_{20}(z^2 + \bar{z}^2) + b$. Hence we do not have a. We will also write

$$\rho = |q|, \quad q = \rho\, e^{i\omega} \ .$$

From the equation of q we can write ω explicitly as

$$\omega(t) = \int_0^t \left[\operatorname{Im} \delta_{21}\rho^2 + c_4 b + \operatorname{Im}(g/q)\right] \, ds \tag{8.1}$$

Assuming the estimate on b in Lemma 7.2, we can bound ω by

$$|\omega(t)| \leq C \log \{t\} \ .$$

This provides an idea on the size of ω, but we shall not need it in the following rigorous proof for b.

Return to the proof that $S(b)$ is a contraction. Given $b, b' \in \mathcal{B}$, we can define z, η, θ and z', η', θ' correspondingly. We use $\boldsymbol{\delta}$ to denote the difference of quantities in these two sets. For example,

$$\boldsymbol{\delta} b = b - b', \quad \boldsymbol{\delta}\eta = \eta - \eta', \quad \boldsymbol{\delta}|z| = |z| - |z'| \ .$$

If

$$|\boldsymbol{\delta} b(t)| \leq \boldsymbol{\delta}_0 \{t\}^{-1} \ ,$$

we want to show

$$|\boldsymbol{\delta} S(b)(t)| \leq \tfrac{1}{2}\boldsymbol{\delta}_0 \{t\}^{-1} \ . \tag{8.2}$$

Let us define

$$\begin{aligned} N(T) = \sup_{0\leq t\leq T} \Big\{ & \{t\}^{1/2+\sigma} |\boldsymbol{\delta}|z|(t)| + \{t\}^{1/2} (\log\{t\})^{-1} |\boldsymbol{\delta} z(t)| \\ & + \{t\}^{3/4-\sigma} \|\boldsymbol{\delta}\eta(t)\|_{L^4} + \{t\}^{1+\sigma/4} \left\|\boldsymbol{\delta}\eta^{(3)}_{2-5}(t)\right\|_{L^2_{\mathrm{loc}}} \\ & + (\log\{t\})^{-1} |\boldsymbol{\delta}\theta(t)| \Big\} \end{aligned}$$

The key here is that $\boldsymbol{\delta}|z|$ has a faster decay than $|z|$. We first want to show that $N(T) \leq C\boldsymbol{\delta}_0$ uniformly for all T. Our strategy is to show that

$$N(T) \leq C\varepsilon^\sigma N(T) + C\boldsymbol{\delta}_0 \tag{8.3}$$

for all T. After this is proved, by choosing ε_0 sufficiently small we have $N(T) \leq C\boldsymbol{\delta}_0$. We also have

$$\begin{aligned} |\boldsymbol{\delta} S(b)(t)| \leq & \{t\}^{-1} N \{t\}^{-1/2} \log\{t\} + \{t\}^{-1/2} \boldsymbol{\delta}_0 \{t\}^{-1} \\ & + \int_\infty^t \{s\}^{-3/2} \ N \{s\}^{-1/2-\sigma} + \{s\}^{-2-\sigma} (N + \boldsymbol{\delta}_0) \, ds \\ \leq & \, C\boldsymbol{\delta}_0 \{t\}^{-1-\sigma} \leq \tfrac{1}{2}\boldsymbol{\delta}_0 \{t\}^{-1} \end{aligned}$$

which shows (8.2). Now we focus on proving (8.3).

Let $t \leq T$. We will write $N = N(T)$. Also, when we say things like $|\boldsymbol{\delta}(\rho^2)| \leq C|\rho\boldsymbol{\delta}\rho|$, what we really means is $|\boldsymbol{\delta}(\rho^2)| \leq C|\rho\boldsymbol{\delta}\rho| + C|\rho'\boldsymbol{\delta}\rho|$.

1. We first estimate $\boldsymbol{\delta}g$. By (6.13) and the definitions of $P^{(4)}$, g_1, g_2, and g_3, we have

$$|\boldsymbol{\delta}g(t)| \leq C(N + \boldsymbol{\delta}_0)\left\{t\right\}^{-3/2-\sigma}$$

Note $C(N + \boldsymbol{\delta}_0) \leq C_1$ if $\boldsymbol{\delta}_0$ is sufficiently small. Hence

$$m^{-1}\left\{t\right\}^{-1/2} \leq \rho(t), \rho'(t) \leq m\left\{t\right\}^{-1/2} \ .$$

By Lemma 6.5,

$$|\boldsymbol{\delta}\rho(t)| \leq C(N + \boldsymbol{\delta}_0)\varepsilon^\sigma\left\{t\right\}^{-(1+\sigma)/2}$$

2. From the equation for ω (8.1), we can bound the variation of ω by

$$\begin{aligned}
|\boldsymbol{\delta}\omega(t)| &\leq \int_0^t C|\boldsymbol{\delta}(\rho^2)| + C|\boldsymbol{\delta}b| + C|\boldsymbol{\delta}g|\left\{s\right\}^{1/2} + C|g/q^2||\boldsymbol{\delta}q|\, ds \\
&\leq \int_0^t C\left\{s\right\}^{-1-\sigma}\varepsilon^\sigma N + C\left\{s\right\}^{-1}\boldsymbol{\delta}_0 + C(N + \boldsymbol{\delta}_0)\left\{s\right\}^{-3/2-\sigma+1/2} \\
&\qquad\qquad + C(N + \boldsymbol{\delta}_0)\left\{s\right\}^{-3/2-\sigma+1} N\left\{s\right\}^{-1/2}\log\left\{s\right\}\, ds \\
&\leq [C\varepsilon^\sigma N + C\boldsymbol{\delta}_0]\log\left\{t\right\}
\end{aligned}$$

Hence the variation of z is bounded by

$$|\boldsymbol{\delta}z(t)| \leq |\rho\boldsymbol{\delta}\omega| + |\boldsymbol{\delta}\rho| \leq (C\varepsilon^\sigma N + C\boldsymbol{\delta}_0)\left\{t\right\}^{-1/2}\log\left\{t\right\} \ .$$

3. Since $\theta(t) = c_2[(z + \bar{z})]_0^t + \int_0^t \widetilde{F}_\theta\, ds$, we have

$$|\boldsymbol{\delta}\theta(t)| \leq C|\boldsymbol{\delta}z| + \int_0^t \boldsymbol{\delta}\widetilde{F}_\theta\, ds \leq (C\varepsilon^\sigma N + C\boldsymbol{\delta}_0)\log\left\{t\right\} \ .$$

4. To estimate $\boldsymbol{\delta}\eta$,

$$\boldsymbol{\delta}\eta = \int_0^t e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^A\,\boldsymbol{\delta}\left\{e^{i\theta}U\Pi\left[F^\sharp + i\eta^2\bar{\eta}\right] - e^{i\theta}[U, i]\dot{\theta}\eta\right\}\, ds$$

we note

$$\left\|\boldsymbol{\delta}\left\{e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U, i]\dot{\theta}\eta\right\}(s)\right\|_{L^{4/3}} \leq C|\boldsymbol{\delta}\theta|\left\{s\right\}^{-1} + (C\varepsilon^\sigma N + C\boldsymbol{\delta}_0)\left\{s\right\}^{-1}$$

hence

$$\|\boldsymbol{\delta}\eta(t)\|_{L^4} \leq \int_0^t \frac{C}{|t-s|^{3/4}}(C\varepsilon^\sigma N + C\boldsymbol{\delta}_0)\left\{s\right\}^{-1}\log\left\{s\right\}\, ds \leq C\varepsilon^\sigma(N + \boldsymbol{\delta}_0)\left\{s\right\}^{-3/4+\sigma}$$

5. To estimate $\boldsymbol{\delta}\eta^{(3)}_{2-5}$,

$$\begin{aligned}\boldsymbol{\delta}\tilde{\eta}^{(3)}_{2-5}(t) &= -\int_0^t e^{-iA(t-s)}\left\{\mu^{[\alpha]}\boldsymbol{\delta}\left\{\frac{d}{ds}(e^{i\theta}p^{\alpha})\right\}\widetilde{\eta}_{\alpha}\right\}\,ds \\ &\quad + \int_0^t e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^{A}\,\boldsymbol{\delta}\left\{e^{i\theta}U\Pi\left[F^{\sharp\sharp}+i\eta^2\bar{\eta}\right]-e^{i\theta}[U,i]\dot{\theta}\eta\right\}\,ds \\ &\quad + \int_0^t e^{-iA(t-s)}\,\mathbf{P}_{\mathrm{c}}^{A}\,\boldsymbol{\delta}\left\{e^{i\theta}U\Pi(-i\eta^2\bar{\eta})\right\}\,ds \\ &= \boldsymbol{\delta}\tilde{\eta}^{(3)}_{3}+\boldsymbol{\delta}\tilde{\eta}^{(3)}_{4}+\boldsymbol{\delta}\tilde{\eta}^{(3)}_{5}\ .\end{aligned}$$

Hence

$$\left\|\boldsymbol{\delta}\tilde{\eta}^{(3)}_{2-5}(t)\right\|_{L^2_{\mathrm{loc}}} \le \left(C\varepsilon^{\sigma}N+C\boldsymbol{\delta}_0\right)\{s\}^{-1-3\sigma/4}$$

Note that we estimated $\|\eta\|_{L^2}$ in section 5 in order to estimate $\||\eta|^2\eta\|_{L^{8/7}}$, which is used in estimating $\left\|\tilde{\eta}^{(3)}_5\right\|_{L^2_{\mathrm{loc}}}$. However, we do not need $\|\boldsymbol{\delta}\eta\|_{L^2}$ here since

$$\left\|\boldsymbol{\delta}(|\eta|^2\eta)\right\|_{L^{8/7}} \le C\,\|\eta\|^{1/2}_{L^2}\,\|\eta\|^{3/2}_{L^4}\,\|\boldsymbol{\delta}\eta\|_{L^4}\ .$$

Compare with Subsection 5.3. Other estimates are similar to those in that subsection.

Since $\eta^{(3)}_{2-5}=U^{-1}e^{-i\theta}\tilde{\eta}^{(3)}_{2-5}(t)$,

$$\left\|\boldsymbol{\delta}\eta^{(3)}_{2-5}(t)\right\|_{L^2_{\mathrm{loc}}} \le |\boldsymbol{\delta}\theta|\left\|\tilde{\eta}^{(3)}_{2-5}(t)\right\|_{L^2_{\mathrm{loc}}}+\left\|\boldsymbol{\delta}\tilde{\eta}^{(3)}_{2-5}(t)\right\|_{L^2_{\mathrm{loc}}} \le C\varepsilon^{\sigma/4}(N+\boldsymbol{\delta}_0)\{s\}^{-1-\sigma/2}$$

5. Summarizing, we have shown (8.3). Choosing ε_0 sufficiently small we have $N\le C\boldsymbol{\delta}_0$.

6. The above shows that the map $S(b)$ is a contraction mapping. hence there is a fixed point $b=S(b)$, which gives us a solution.

9 Dynamical renormalization of mass

As explained in the introduction, we expect that the solution to be of the form

$$\psi(t)=[Q_{E(t)}+h(t)]e^{i\Theta(t)}$$

with a changing $E(t)$, and we use $[Q_E+aR_E+h]e^{i\Theta(t)}$ as an approximation. In Theorem 5.1 we conclude that $M(T)\le 2$ assuming $|b(t)|\le D\{t\}^{-1}$ for $t\in[0,T]$. We cannot extend this result beyond T. The reason is that, since $E(t)$ is changing and we expect that $E(t)$ will converge to some E_∞ which is likely to be different from $E(0)$, after certain time $[Q_E+aR_E+h]e^{i\Theta(t)}$ is no longer a good approximation and $|a(t)|$ will no longer decay. To overcome this difficulty, for each time step $T=k\Delta T$, we propose to re-choose $E=E_k$ so

that the new $a(t)$ satisfies $a(T) = 0$. Then we prove that the new $a(t)$ has the same estimate as the old $a(t)$ if our time increment ΔT is sufficiently small. Then we estimate the change of $E(t)$, especially taking into account of its oscillation, by studying $a(t)$, and prove that $E(t)$ does converge. The rest of the proof essentially follows that of Theorem 1.1.

9.1 Renormalization lemma

Lemma 9.1 *Suppose $\|\psi - Q_E\|_2 \ll 1$, then we can rewrite*

$$\psi = ((1+a)Q_E + k)e^{i\theta} = (Q_E + a'R_E + h)e^{i\theta}$$

uniquely with $h, k \perp Q$. Moreover, a, a', θ, k and h are small.

Proof. First we can write $\psi = (1+\alpha)Q_E + k_1$, with $k_1 \perp Q_E$, and α a complex number. Since $\psi - Q_E$ is small, α and k_1 are both small. Hence we can find small real a and θ such that $(1+\alpha) = (1+a)e^{i\theta}$. Let $k = k_1 e^{-i\theta}$. Then we have $\psi = ((1+a)Q_E + k)e^{i\theta}$. We have $R_E = c_1^{-1}c_0 Q_E + \Pi_E R_E$, and hence $aQ_E = c_1 c_0^{-1} a(R_E - \Pi_E R_E)$. Let $a' = c_1 c_0^{-1} a$ and $h = k - a'\,\Pi_E R_E$. We have $h \perp Q_E$ and $aQ_E + k = a'R_E + h$. **Q.E.D.**

Lemma 9.2 *Let τ_0 and δ_0 be sufficiently small and suppose $E \in \mathcal{I}_\lambda$ with dist$(E, \partial\mathcal{I}_\lambda) > C\tau_0\lambda$. Suppose that $\psi = [Q_E + aR_E + h]\, e^{i\Theta}$, with $|\lambda^{-1}a| = \tau < \tau_0$, $\|h\|_{L^2} = \delta < \delta_0$. Then we can find $\widetilde{E}$ with $|\widetilde{E} - (E+a)| \le \tau\lambda/2$ such that, if we write uniquely*

$$\psi = [Q_{\widetilde{E}} + \widetilde{a}R_{\widetilde{E}} + \widetilde{h}]\, e^{i\widetilde{\Theta}}, \quad \widetilde{h} \perp Q_{\widetilde{E}} \tag{9.1}$$

using Lemma 9.1, then $\widetilde{a} = 0$ and we have $\left\|h - \widetilde{h}\right\|_{L^2} < \tau/2$, and $|\Theta - \widetilde{\Theta}| < \tau/2$. In fact, $(\widetilde{E}, \widetilde{\Theta})$ is the unique solution of

$$\left(\psi - Q_{E'}\, e^{i\Theta'}\,,\, Q_{E'}\right) = 0\,, \qquad (|E - E'| \le 2\lambda\tau_0), \tag{9.2}$$

for $\|\psi - Q_E\|_{L^2}$ sufficiently small.

Proof. We will define a sequence $\{E_k\}_{k=1,2,3\cdots}$ which converges to $\widetilde{E}$. Let $E_1 = E + a$. We can write uniquely

$$\psi = [Q_{E_1} + a_1 R_{E_1} + h_1]\, e^{i\Theta_1}, \quad h_1 \perp Q_{E_1}$$

using Lemma 9.1. Denote $\delta Q = Q_E + aR_E - Q_{E_1}$ and $\delta\Theta = \Theta - \Theta_1$. We have $\|Q_E - Q_{E_1}\| \le O(aR) \le C|a|\lambda^{-1} \le C\tau$, and $\|\delta Q\| \le Ca^2(\partial_E^2 Q) \le Ca^2\lambda^{-2} \le C\tau^2$. Note we have

$$a_1 R_{E_1} + h_1 = Q_{E_1}(e^{i\delta\Theta} - 1) + [\delta Q + h]\, e^{i\delta\Theta}\ .$$

Hence $0 = \mathrm{Im}(Q_{E_1}, a_1 R_{E_1} + h_1) = O(\delta\Theta) + O((Q_{E_1}, \delta Q + h))$, and hence

$$|\delta\Theta| \le O(\delta Q) + C|(Q_E, h)| + C|(Q_{E_1} - Q_E, h)| \le C\tau^2 + 0 + C\tau\delta \ .$$

Also,

$$O(\lambda^{-1})\, a_1 = (Q_{E_1}, a_1 R_{E_1} + h_1) = c_0^{-1}(e^{i\delta\Theta} - 1) + O((Q_{E_1}, \delta Q + h)) \ ,$$

hence

$$\lambda^{-1}|a_1| \le C|e^{i\delta\Theta} - 1| + C\tau^2 + C\tau\delta \le C\tau^2 + C\tau\delta \ .$$

Finally,

$$\begin{aligned}
\|h_1 - h\|_{L^2} &\le \left\|h_1 - h\, e^{i\delta\Theta}\right\|_{L^2} + \left\|h\, e^{i\delta\Theta} - h\right\|_{L^2} \\
&= \left\|(Q_{E_1}(e^{i\delta\Theta} - 1) + (\delta Q)e^{i\delta\Theta} - a_1 R_{E_1}\right\|_{L^2} + \|h\|_{L^2}\ |e^{i\delta\Theta} - 1| \\
&\le C\tau^2 + C\tau\delta \ .
\end{aligned}$$

If we choose $C\tau + C\delta \le 1/3$, then we have

$$\lambda^{-1}|a_1|,\ |\delta\Theta|,\ \|h_1 - h\|_{L^2} \le \tau/3 \ .$$

Now for $k \ge 2$ we define $E_k = E_{k-1} + a_{k-1}$, and define a_k, h_k and Θ_k correspondingly by Lemma 9.1. We follow the previous estimates to get

$$\lambda^{-1}|a_k|,\ |\Theta_k - \Theta_{k-1}|,\ \|h_k - h_{k-1}\|_{L^2} \le 3^{-k}\tau$$

Note that the size of h_k may increase in the process, but since

$$C|a_k| + C\,\|h_k\|_{L^2} \le C\frac{|a_{k-1}|}{3} + C(\|h_{k-1}\|_{L^2} + \frac{|a_{k-1}|}{3}) \le C|a_{k-1}| + C\,\|h_{k-1}\|_{L^2} \le \cdots \le 1/3$$

our condition for estimates is always satisfied. Hence $E + \sum_k a_k$ converges to a limit $\widetilde{E}$ with $|\widetilde{E} - E| \le 3\lambda\tau/2$ and $|\widetilde{E} - E - a| \le \lambda\tau/2$. Writing ψ in the form (9.1) with respect to $\widetilde{E}$, we have $\widetilde{a} = \lim_k a_k = 0$ and $|\widetilde{\Theta} - \Theta| \le \sum_k |\Theta_{k+1} - \Theta_k| \le \tau/2$. We conclude $(\widetilde{E}, \widetilde{\Theta})$ is a solution of (9.2).

To show the uniqueness of (9.2), we first note that it is locally unique by inspecting the equations obtained by taking derivatives of (9.2) with respect to E' and Θ'. Now suppose we may write

$$\psi = [Q_1 + h_1]e^{i\Theta_1} = [Q_2 + h_2]e^{i\Theta_2}$$

with $h_1 \perp Q_1$ and $h_2 \perp Q_2$, . Then we have $[Q_1 + h_1] = [Q_2 + h_2]e^{i\delta\Theta}$. Since both h_1 and h_2 are small, taking projection on Q_1 we get $\delta\Theta$ is small. The local uniqueness implies the claim.

Q.E.D.

9.2 Proof of Theorem 1.2

First we proceed an induction argument to find the desired renormalization at each time step and the corresponding estimates. Denote the space of initial data

$$Y = H^2 \cap W^{2,1}(\mathbb{R}^3).$$

Our initial datum is $\psi_0 = Q_{E_{\rm in}} + a_{\rm in} R_{E_{\rm in}} + h_{\rm in}$, with $\|\psi_0 - Q_{E_{\rm in}}\|_Y \le C^{-1}\varepsilon$. Applying Lemma 9.1 to ψ_0, we can find E_0, h_0, and Θ_0 such that $\psi_0 = [Q_{E_0} + h_0]\, e^{i\Theta_0}$. Moreover, we have $|E_0 - E_{\rm in}| \le C^{-1}|a_{\rm in}| \le C^{-1}\lambda\varepsilon$, and hence $\left\|Q_{\rm in} - Q_0 e^{i\Theta_0(0)}\right\|_Y \le \frac{1}{2}\varepsilon$. This is the beginning of our induction argument. We will choose $\Delta T > 0$ sufficiently small. Then we define $T_k = k\Delta T$ and will define E_k, h_k, etc. at time T_k. Also, we abbreviate $Q_k = Q_{E_k}$, $R_k = R_{E_k}$, etc..

Note, Theorem 5.1 remains valid if we replace the assumption $|b(t)| \le D\,\{t\}^{-1}$ by $|a(t)| \le D\,\{t\}^{-1}$. Also recall that $D = O(\lambda)$ by Lemma 7.1.

Assume for $T_k = k\Delta T$ we have found E_k, $a_k(t)$, $h_k(t)$, $\Theta_k(t)$ so that

$$\begin{cases} \psi(t) = [Q_k + a_k(t)R_k + h_k]\, e^{i\Theta_k}, \quad E_k \in \mathcal{I}_\lambda\ , \\ h_k(t) \perp Q_k \quad \text{for } t \in [0, T_k]\ ; \quad \left\|\psi_0 - Q_k e^{i\Theta_k(0)}\right\|_Y \le \varepsilon\ , \\ |a_k(t)| \le D\,\{t\}^{-1}, \quad a_k(T_k) = 0\ . \end{cases} \tag{9.3}$$

This is our induction hypothesis. By Theorem 5.1 we have $M_k(T_k) \le 2$. Since $a_k(T_k) = 0$, we have

$$a_k(t) = \left[a_{20}(E_k)(z_k^2 + \bar{z}_k^2) + \cdots\right]_T^t + \int_T^t B_{22}(E_k)|z_k|^4 + \cdots \tag{9.4}$$

The direct estimate of Lemma 7.2 gives us

$$|a_k(t)| \le \frac{1}{2} D\,\{t\}^{-1} \qquad \text{for } t \in [0, T_k]\ .$$

Thus $|a_k(0)| \le \frac{1}{2}D\varepsilon^2$ and we have

$$\begin{aligned} \left\|\psi_0 - Q_k e^{i\Theta_k(0)}\right\|_Y &\le \left\|\psi_0 - Q_0 e^{i\Theta_0(0)}\right\|_Y + \left\|Q_0 e^{i\Theta_0(0)} - Q_k e^{i\Theta_k(0)}\right\|_Y \\ &\le \frac{1}{2}\varepsilon + \lambda^{-1}|a_k(0)| \le \frac{3}{4}\,\varepsilon\ . \end{aligned}$$

By continuity, there is a $\sigma > 0$ such that

$$|a_k(t)| \le D\,\{t\}^{-1} \qquad \text{for } t \in [0, T_k + \sigma]\ .$$

Theorem 5.1 ensures that

$$M_k(T_k + \sigma) \le 2\ .$$

We now return to (9.4). Since the derivatives of the terms outside the integral (say, $z_k^2 + \bar{z}_k^2$) are bounded by $\{t\}^{-1}$, for $t \in [T_k, T_k + \sigma]$ we have

$$\begin{aligned} |a_k(t)| &\le CT^{-1}(t-T) + CT^{-2}(t-T) \\ &\le C_3 T^{-1}(t-T) \le \frac{1}{2} D \{T\}^{-1} \le \frac{5}{8} D \{t\}^{-1} \end{aligned} \tag{9.5}$$

if $(t-T) \le D/(2C_3)$ and $(t-T) \le \{T\}/4 \le \varepsilon^{-2}/4$. Now we define

$$\Delta T = \min\left\{D/(2C_3)\,,\, \varepsilon^{-2}/4\right\} \ .$$

Recall $T_{k+1} = T_k + \Delta T$. We have thus proved that $\sigma \ge \Delta T$ and

$$|a_k(t)| \le \ \frac{5}{8} D \{t\}^{-1}\,, \qquad \text{for } t \in [0, T_{k+1}]$$

and $M_k(T_{k+1}) \le 2$. By Lemma 9.2, we can find E_{k+1} with

$$|E_{k+1} - E_k| \le \tfrac{3}{2} |a_k(T_{k+1})| \ , \tag{9.6}$$

such that we can rewrite

$$\psi(t) = [Q_{k+1} + a_{k+1}(t) R_{k+1} + h_{k+1}(t)]\, e^{i\Theta_{k+1}(t)}, \quad h_{k+1}(t) \perp Q_{k+1}$$

with $a_{k+1}(T_{k+1}) = 0$. Now we compare these two sets of data with respect to E_k and E_{k+1}, for $t \in [0, T_{k+1}]$. As in the proof of lemma 9.2, we have

$$\begin{aligned} |\Theta_k(t) - \Theta_{k+1}(t)| &\le C|(Q_{k+1}, h_k(t))| = C|(Q_k + O(\lambda^{-1} a_k(T_{k+1})), h_k(t))| \\ &\le 0 + C\lambda \{T\}^{-1} \{t\}^{-1/2} \le C\lambda \{t\}^{-3/2} \end{aligned} \tag{9.7}$$

and since

$$a_{k+1}(t) R_{k+1} + h_{k+1}(t) = [Q_k + a_k(t) R_k + h_k(t)]\, e^{i\Theta_k(t) - i\Theta_{k+1}(t)} - Q_{k+1}$$

Taking inner product with Q_{k+1} we get

$$\lambda^{-1} |a_{k+1}(t) - a_k(t)| \le C|\Theta_k(t) - \Theta_{k+1}| + C|a_k(t)|^2 + C\lambda \{T\}^{-1} \|h_k(t)\| \le C\lambda \{t\}^{-3/2}$$

Hence

$$|a_{k+1}(t)| \le |a_k(t)| + |a_{k+1}(t) - a_k(t)| \le \frac{5}{8} D \{t\}^{-1} + \frac{1}{8} D \{t\}^{-1} = \frac{3}{4} D \{t\}^{-1} \ ,$$

if ε is sufficiently small. By (9.6) and (9.7), we have

$$\left\|\psi_0 - Q_{k+1} e^{i\Theta_{k+1}(0)}\right\|_Y \le \left\|\psi_0 - Q_k e^{i\Theta_k(0)}\right\|_Y + C\lambda \{t\}^{-1} \le \varepsilon \ .$$

Also note

$$|E_0 - E_{k+1}| \le \frac{3}{2}\,|a_{k+1}(0)| \le 2D\varepsilon^2 \le C\lambda\varepsilon^2 \ .$$

Hence $E_{k+1} \in \mathcal{I}_\lambda$. We have thus proved the induction hypothesis (9.3) for $t = T_{k+1}$. The induction argument is completed.

Next we proceed to show that E_k has a limit. Let $n = 2^k$ and $T = T_n$. Since $a_n(T) = 0$, we have

$$a_n(t) = \left[a_{20}(E_n)(z_n^2 + \bar{z}_n^2) + \cdots\right]_T^t + \int_T^t B_{22}(E_n)|z_n|^4 + \cdots$$

for $t \in [0, T_n]$. In particular,

$$|a_n(T/2)| \le C\,\{T\}^{-1} + \int_{T/2}^T |B_{22}|\ \{s\}^{-2}\ ds \le 2D\,\{T\}^{-1}$$

(Note that the argument here differs from (9.5).) Hence, by considering $\psi = \psi(T/2)$, $E = E_n$ and $\widetilde{E} = E_{n/2}$ in Lemma 9.2, we have

$$|E_{n/2} - E_n| \le 2|a_n(T/2)| \le 4D\,\{T\}^{-1} \ .$$

This estimate shows that the sequence E_{2^k} converges to a limit E_∞. Moreover, choose k_1 so that $2\Gamma 2^{k_1}\Delta T > \varepsilon^{-2}$ and we have

$$|E_0 - E_\infty| \le |E_0 - E_{k_1}| + |E_{k_1} - E_\infty| \le C\lambda\varepsilon^2 + CD\sum_{k>k_1}\left\{2^k\Delta T\right\}^{-1} \le C\lambda\varepsilon^2 \ .$$

Write $Q_\infty = Q_{E_\infty}$, $R_\infty = R_{E_\infty}$, and

$$\psi(t) = \left[Q_\infty + a_\infty(t)R_\infty + h_\infty(t)\right]\ e^{i\Theta_\infty(t)}$$

and compare this set of data with data at T_k, $t \in [0, T_k]$. As before we get $|a_\infty(t) - a_k(t)| \le CD\,\{t\}^{-3/2}$ and hence

$$|a_\infty(t)| \le \frac{3}{4}D\,\{t\}^{-1} \ .$$

Since k is arbitrary, this estimate is true for all $t \in [0, \infty)$. Since $|E_\infty - E_0| \le O(\lambda\varepsilon^2)$, we have $\|\psi_0 - Q_\infty\|_Y \le \varepsilon$. Theorem 5.1 shows that

$$M_\infty(T) \le 2 \qquad \text{for all } T \ .$$

The first part of Theorem 1.2 is thus proved.

Suppose the assumption of part (2) of Theorem 1.2 holds, that is,

$$0 < |z_{\rm in}| = \varepsilon \le \varepsilon_0, \quad \|\eta_{\rm in}\|_Y \le C\varepsilon^{3/2}, \quad \lambda^{-1}|a_{\rm in}| \le C\varepsilon^2.$$

Let $a_\infty = a_\infty(0)$, $z_\infty = z_\infty(0)$, and $\eta_\infty = \eta_\infty(0)$. Apply Lemma 9.2 to $(\psi, E) = (\psi_0, E_{\rm in})$ and $(\psi, E) = (\psi_0, E_\infty)$ respectively. In both cases, $\widetilde{E} = E_0$. We have

$$|a_{\rm in}| \le C\lambda\varepsilon^2, \quad |a_\infty| \le C\lambda\varepsilon^2$$
$$|E_0 - E_{\rm in}| \le C|a_{\rm in}|, \quad |E_0 - E_\infty| \le C|a_\infty|$$

Therefore $Q_{\rm in} - Q_\infty$ is of order $\lambda^{-1}|a_{\rm in} - a_\infty| = O(\varepsilon^2)$, and hence so is $h_{\rm in} - h_\infty$. Denote $P_\infty = P_c(E_\infty)\Pi_\infty$ and $P_{\rm in} = P_c(E_{\rm in})\Pi_{\rm in}$, we have

$$\begin{aligned}\|P_\infty h_\infty\|_Y &\le \|P_{\rm in} h_{\rm in}\|_Y + \|P_{\rm in}(h_\infty - h_{\rm in})\|_Y + \|(P_\infty - P_{\rm in})h_\infty\|_Y \\ &\le \varepsilon^{3/2} + O(\varepsilon^2) + |E_\infty - E_{\rm in}|\varepsilon \le 2\varepsilon^{3/2}\ .\end{aligned}$$

By the last statement of Theorem 5.1, we have $|z_\infty(t)| \ge c\,\{t\}^{-1/2}$ for all time t. The last statement of Theorem 1.2 is proved. **Q.E.D.**

10 Radiation dominated solutions

In this section we prove part (2) of Theorem 1.1. For a given profile ξ_∞, we want to construction solutions of the form (1.6) such that

$$\chi(x,t) = h(x,t)e^{i\Theta(t)} \longrightarrow e^{it\Delta}\xi_\infty$$

as t goes to infinity. Let W denote the wave operator for $\mathcal{L}$,

$$W : L^2 \to \mathtt{H}_{\rm c}(\mathcal{L}), \qquad W\phi = \lim_{t\to\infty} e^{-t\mathcal{L}}\, e^{-i(-\Delta - E)t}\phi\ .$$

We have

$$e^{t\mathcal{L}}W\chi \longrightarrow e^{i(\Delta+E)t}\,\chi, \qquad \text{as } t \to \infty,$$

for general $\chi \in L^2$. See subsection 3.2. For the profile χ_∞ given in part (2) of Theorem 1.1, we let

$$\xi_\infty = UW\chi_\infty\ .$$

We recall that we set

$$\psi = (Q + aR + h)\, e^{i[-Et+\theta(t)]}\ ,$$

and h satisfies

$$\partial_t h = -\dot{a}R - aiQ + \mathcal{L}^{(\mathrm{or})}h - iF(aR+h) - i\dot{\theta}(Q + aR + h) \ .$$

To ensure $h(t) \perp Q$ all the time, we set

$$\begin{aligned}
\dot{a} &= (c_1 Q, \operatorname{Im} F(k)) \ , \qquad c_1 = (Q,R)^{-1} \ , \\
\dot{\theta} &= -(1 + c_0 c_1^{-1} a)^{-1} \left[a + (c_0 Q, \lambda Q^2 (h + \bar{h})) + (c_0 Q, \operatorname{Re} F(k)) \right] \ .
\end{aligned}$$

(Recall $c_0 = (Q,Q)^{-1}$.) Then the equation of h is

$$\begin{aligned}
\partial_t h &= \mathcal{L}h + \Pi F_{all} \ , \\
\mathcal{L}h &= -i \left\{ Hh + \Pi \lambda Q^2 \Pi (h + \bar{h}) \right\} \ , \\
F_{all} &= -i\dot{\theta}h - iF - \left[(c_1 Q, \operatorname{Im} F) + ia\dot{\theta} \right] R_\Pi \ , \quad R_\Pi := \Pi R \ , \\
F &= F(aR + h) \\
&= \lambda Q(2|h|^2 + h^2) + 2\lambda a Q R(2h + \bar{h}) + 3\lambda a^2 Q R^2 + \lambda (aR + h)^2 (aR + \bar{h}) \ .
\end{aligned}$$

We first rewrite the equation of h. The equation for h is

$$\begin{aligned}
\partial_t h &= \mathcal{L}h - i\dot{\theta}h + \Pi F^\sharp \ , \\
F^\sharp &= -iF(k) - [(c_1 Q, \operatorname{Im} F) + ia\dot{\theta}] R_\Pi \ .
\end{aligned}$$

Let

$$h^\diamond = Uh \ .$$

Since $\mathcal{L} = U^{-1}(-iA)U$, we have

$$\begin{aligned}
\partial_t h^\diamond &= -iAh^\diamond - Ui\dot{\theta}U^{-1}h^\diamond + U\Pi F^\sharp \\
&= -iAh^\diamond - i\dot{\theta}h^\diamond - [U,i]\dot{\theta}U^{-1}h^\diamond + U\Pi F^\sharp \ .
\end{aligned}$$

Let $\widetilde{h} = e^{i\theta}h^\diamond$ and use $U^{-1}h^\diamond = h$, we get

$$\partial_t \widetilde{h} = -iA\widetilde{h} + e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U,i]\dot{\theta}h \ . \tag{10.1}$$

For a specificed datum ξ_∞ at infinity, we define a solution by the equations

$$\begin{aligned}
\widetilde{h}(t) &= \xi(t) + g(t) \\
\xi(t) &= e^{-iAt}\xi_\infty \\
g(t) &= \int_\infty^t e^{-iA(t-s)} \left\{ e^{i\theta}U\Pi F^\sharp - e^{i\theta}[U,i]\dot{\theta}h \right\} ds \ .
\end{aligned}$$

We also let $h = U^{-1}e^{-i\theta}\widetilde{h}$.

Suppose we have a solution, then the main term in $F^\sharp$ is of order t^{-3}, hence $g \sim t^{-2}$, $a \sim t^{-2}$, $\dot\theta \sim t^{-3/2}$, $\theta \sim t^{-1/2}$.

Note that $\xi(t)$ is fixed. We consider ξ_∞ small in the following norm

$$\|\xi_\infty\|_{H^2\cap W^{2,1}} \le \varepsilon$$

with ε sufficiently small. Then

$$\|\xi(t)\|_{H^2} \le C_1\varepsilon, \qquad \|\xi(t)\|_{W^{2,\infty}} \le C_1\varepsilon|t|^{-3/2}. \tag{10.2}$$

Also, for the leading term $\lambda|\xi|^2\xi$ in F, we have, for $t > 1$,

$$\begin{aligned}
\left\||\xi|^2\xi\right\|_{L^2} &\le C\,\|\xi(t)\|_\infty^2\,\|\xi(t)\|_2 \le C\varepsilon^3\,\langle t\rangle^{-3}\,,\\
\left\|\nabla^2(|\xi|^2\xi)\right\|_{L^2} &\le C\,\|\xi(t)\|_\infty^2\,\left\|\nabla^2\xi(t)\right\|_2 + C\,\|\xi(t)\|_\infty\,\|\nabla\xi(t)\|_4^2\\
&\le C(\langle t\rangle^{-3/2})^2\cdot 1 + C\,\langle t\rangle^{-3/2}\cdot(\langle t\rangle^{-3/4})^2 \le C\varepsilon^3\,\langle t\rangle^{-3}\,.
\end{aligned}$$

It $t \le 1$, we simply have $\||\xi|^2\xi\|_{H^2} \le C\,\|\xi\|_{H^2}^3 \le C\varepsilon^3$. We conclude

$$\left\||\xi|^2\xi\right\|_{H^2} \le C\varepsilon^3\,\langle t\rangle^{-3}\,. \tag{10.3}$$

Similarly, if $\|g(t)\|_{H^2} \le C\varepsilon\,\langle t\rangle^{-2}$, one can prove, for example,

$$\left\||\xi+g|^2(\xi+g)\right\|_{H^2} \le C\varepsilon^3\,\langle t\rangle^{-3}\,, \quad \|Q\xi g\|_{H^2} \le C\varepsilon^2\,\langle t\rangle^{-7/2}\,, \quad \text{etc.}$$

Now we proceed to construct a solution. For convenience, we introduce a new variable $\omega = \dot\theta$. (It should not be confused with the ω used in Section 8.) We will define a Cauchy sequence on the space

$$\begin{aligned}
\mathcal{A} = \Big\{&(\omega,\theta,a,g) : [0,\infty) \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times H^2,\\
&|\omega(t)| \le \varepsilon^{1/2}\,\langle t\rangle^{-3/2}\,,\ |\theta(t)| \le \varepsilon^{1/2}\,\langle t\rangle^{-1/2}\,,\ |a(t)| \le \varepsilon\,\langle t\rangle^{-2}\,,\ \|g(t)\|_{H^2} \le \varepsilon\,\langle t\rangle^{-2}\Big\}
\end{aligned}$$

Here $H^2 = W^{2,2}(\mathbb{R}^3)$. We define our map by

$$\begin{aligned}
\omega^\triangle(t) &:= -(1+c_0c_1^{-1}a)^{-1}\,\left[a + (c_0Q, \lambda Q^2(h+\bar h)) + (c_0Q, \operatorname{Re}F)\right]\\
\theta^\triangle(t) &:= \int_\infty^t \omega\,ds\\
a^\triangle(t) &:= \int_\infty^t (c_1Q, \operatorname{Im}F)\,ds\\
g^\triangle(t) &:= \int_\infty^t e^{-iA(t-s)}\left\{e^{i\theta}U\Pi\left\{-iF - [(c_1Q,\operatorname{Im}F) + ia\omega]R_\Pi\right\} - e^{i\theta}[U,i]\omega h\right\}\,ds\,,
\end{aligned}$$

where

$$F = F(aR + h), \quad h(t) := U^{-1} e^{-i\theta} \left(e^{-iAt} \xi_\infty + g(t) \right) \ .$$

Our initial data are

$$\omega(t) \equiv 0, \quad \theta(t) \equiv 0, \quad a(t) \equiv 0, \quad g(t) \equiv 0 \ .$$

Given $(\omega, \theta, a, g) \in \mathcal{A}$, using this assumption and (10.2)–(10.3), we have

$$\begin{aligned}
\left\| |g|^2 g \right\|_{H^2} &\le \|g\|_{H^2}^3 \le C\varepsilon^3 \langle t \rangle^{-6} \\
\|F\|_{H^2} &\le C\varepsilon^2 \langle t \rangle^{-3} \\
|\omega^\triangle(t)| &\le C\varepsilon \langle t \rangle^{-3/2} \le \varepsilon^{1/2} \langle t \rangle^{-3/2} \\
|\theta^\triangle(t)| &\le \int_\infty^t \varepsilon^{1/2} \langle s \rangle^{-3/2} \, ds \le \varepsilon^{1/2} \langle t \rangle^{-1/2} \\
|a^\triangle(t)| &\le \int_\infty^t C\varepsilon^2 \langle s \rangle^{-3} \, ds \le \varepsilon \langle t \rangle^{-2} \\
\left\| g^\triangle(t) \right\|_{H^2} &\le \int_\infty^t C\varepsilon^2 \langle s \rangle^{-3} + C\varepsilon^{3/2} \langle s \rangle^{-3} \, ds \le \varepsilon \langle t \rangle^{-2}
\end{aligned}$$

We have shown that $(\omega^\triangle, \theta^\triangle, a^\triangle, g^\triangle) \in \mathcal{A}$, that is, our mapping maps $\mathcal{A}$ into itself.

Next we show that it is a contraction. Given $(\omega_1, \theta_1, a_1, g_1), (\omega_2, \theta_2, a_2, g_2) \in \mathcal{A}$, we denote

$$\boldsymbol{\delta}_0 = \sup_t \left\{ \langle t \rangle^3 |\boldsymbol{\delta}\omega(t)|^2 + \langle t \rangle |\boldsymbol{\delta}\theta(t)|^2 + \langle t \rangle^2 |\boldsymbol{\delta}a(t)| + \langle t \rangle^2 \|\boldsymbol{\delta}g(t)\|_{H^2} \right\}$$

we know $\boldsymbol{\delta}_0 \le 8\varepsilon$. Notice that $-i|\xi|^2\xi$ is cancelled in $\boldsymbol{\delta}F$.

$$\begin{aligned}
\left\| \boldsymbol{\delta}(|g|^2 g) \right\|_{H^2} &\le \|g\|_{H^2}^2 \|\boldsymbol{\delta}g\|_{H^2} \le C\varepsilon^2 \boldsymbol{\delta}_0 \langle t \rangle^{-6} \\
\|\boldsymbol{\delta}F\|_{H^2} &\le C\varepsilon \boldsymbol{\delta}_0 \langle t \rangle^{-7/2} \le C\varepsilon \boldsymbol{\delta}_0 \langle t \rangle^{-3} \\
|\boldsymbol{\delta}\omega^\triangle(t)| &\le C\boldsymbol{\delta}_0 \langle t \rangle^{-3/2} \le \frac{1}{8}\boldsymbol{\delta}_0^{1/2} \langle t \rangle^{-3/2} \\
|\boldsymbol{\delta}\theta^\triangle(t)| &\le \int_\infty^t \boldsymbol{\delta}_0^{1/2} \langle s \rangle^{-3/2} \, ds \le \frac{1}{8}\boldsymbol{\delta}_0^{1/2} \langle t \rangle^{-1/2} \\
|\boldsymbol{\delta}a^\triangle(t)| &\le \int_\infty^t C\varepsilon\boldsymbol{\delta}_0 \langle s \rangle^{-3} \, ds \le \frac{1}{8}\boldsymbol{\delta}_0 \langle t \rangle^{-2} \\
\left\| \boldsymbol{\delta}g^\triangle(t) \right\|_{H^2} &\le \int_\infty^t C\varepsilon \boldsymbol{\delta}_0 \langle s \rangle^{-3} + C\varepsilon^{1/2}\boldsymbol{\delta}_0 \langle s \rangle^{-3} \, ds \le \frac{1}{8}\boldsymbol{\delta}_0 \langle t \rangle^{-2}
\end{aligned}$$

Therefore we have

$$\sup_t \left\{ \langle t \rangle^3 |\boldsymbol{\delta}\omega^\triangle(t)|^2 + \langle t \rangle |\boldsymbol{\delta}\theta^\triangle(t)|^2 + \langle t \rangle^2 |\boldsymbol{\delta}a^\triangle(t)| + \langle t \rangle^2 \left\| \boldsymbol{\delta}g^\triangle(t) \right\|_{H^2} \right\} \le \frac{1}{2}\boldsymbol{\delta}_0 \ .$$

These show that our map is a contraction mapping. We conclude that we do have solutions $\widetilde{h}$ with the main profile $e^{iAt}\xi_\infty$.

Now

$$\chi(x,t) = e^{i\Theta}h = e^{i(-Et+\theta)}U^{-1}e^{-i\theta}\widetilde{h} = e^{-iEt}\left\{U^{-1} + [U, e^{i\theta}]\, e^{-i\theta}\right\}(e^{-itA}\xi_\infty + g)$$

Since $\|g(t)\|_{L^2} = O(t^{-2})$ and $[U, e^{i\theta}] = O(\theta(t)) = O(t^{-1/2})$, we have

$$\chi(x,t) \to e^{-iEt}U^{-1}e^{-itA}\xi_\infty, \qquad \text{as } t \to \infty.$$

However,

$$\begin{aligned} e^{-iEt}U^{-1}e^{-itA}\xi_\infty &= e^{-iEt}U^{-1}e^{-itA}UW\chi_\infty = e^{-iEt}e^{t\mathcal{L}}W\chi_\infty \\ &\longrightarrow e^{-iEt}e^{i(\Delta+E)t}\chi_\infty = e^{i\Delta t}\chi_\infty \end{aligned}$$

as $t \to \infty$. Hence part (2) of Theorem 1.1 is proved.

Remark on radiation dominated solutions to Klein-Gordon equations

We now sketch a construction for radiation dominated solutions to Klein-Gordon equations. We follow the notation in the introduction. For a specified profile $\eta_\pm$, let

$$u = \xi + g$$

where

$$\xi(t) = e^{iBt}\eta_+ + e^{-iBt}\eta_-$$

and g denotes the rest. Then we have

$$\begin{aligned} &(\partial_t^2 + B^2)\xi = 0, \\ &(\partial_t^2 + B^2)g = \lambda(\xi + g)^3 \ . \end{aligned}$$

Hence $g(t)$ satisfies

$$g(t) = \int_\infty^t \left\{e^{iB(t-s)} - e^{-iB(t-s)}\right\}\frac{1}{2iB}\lambda(\xi+g)^3 ds \ ,$$

Since the main source term is

$$\left\|\xi^3\right\|_2 \le \|\xi\|_\infty^2 \|\xi\|_2 \le Ct^{-3}$$

we get $\|g\|_2 \le \int_\infty^t Cs^{-3}\, ds = Ct^{-2}$. Then we proceed as in section 10 to construct one such solution by a contraction mapping argument.

Acknowledgments

It is a great pleasure to thank M.I. Weinstein for explaining to us the beautiful ideas of the work [12]. Part of this work was done when both authors were visiting the Center for Theoretical Sciences, Taiwan, in the summer 2000.

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