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# STABLE DIRECTIONS FOR EXCITED STATES OF NONLINEAR SCHRÖDINGER EQUATIONS 

Tai-Peng Tsai and Horng-Tzer Yau


#### Abstract

We consider nonlinear Schrödinger equations in $\mathbb{R}^{3}$. Assume that the linear Hamiltonians have two bound states. For certain finite codimension subset in the space of initial data, we construct solutions converging to the excited states in both non-resonant and resonant cases. In the resonant case, the linearized operators around the excited states are non-self adjoint perturbations to some linear Hamiltonians with embedded eigenvalues. Although self-adjoint perturbation turns embedded eigenvalues into resonances, this class of non-self adjoint perturbations turn an embedded eigenvalue into two eigenvalues with the distance to the continuous spectrum given to the leading order by the Fermi golden rule.


## 1 Introduction

Consider the nonlinear Schrödinger equation

$$
\begin{equation*}
i \partial_{t} \psi=(-\Delta+V) \psi+\lambda|\psi|^{2} \psi, \quad \psi(t=0)=\psi_{0} \tag{1.1}
\end{equation*}
$$

where $V$ is a smooth localized potential, $\lambda$ is an order one parameter and $\psi=\psi(t, x): \mathbb{R} \times \mathbb{R}^{3} \longrightarrow \mathbb{C}$ is a wave function. The goal of this paper is to study the asymptotic dynamics of the solution for initial data $\psi_{0}$ near some nonlinear excited state.

For any solution $\psi(t) \in H^{1}\left(\mathbb{R}^{3}\right)$ the $L^{2}$-norm and the Hamiltonian

$$
\begin{equation*}
\mathcal{H}[\psi]=\int \frac{1}{2}|\nabla \psi|^{2}+\frac{1}{2} V|\psi|^{2}+\frac{1}{4} \lambda|\psi|^{4} d x \tag{1.2}
\end{equation*}
$$

are constants for all $t$. The global well-posedness for small solutions in $H^{1}\left(\mathbb{R}^{3}\right)$ can be proved using these conserved quantities and a continuity argument.

We assume that the linear Hamiltonian $H_{0}:=-\Delta+V$ has two simple eigenvalues $e_{0}<e_{1}<0$ with normalized eigen-functions $\phi_{0}, \phi_{1}$. The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the equation

$$
\begin{equation*}
(-\Delta+V) Q+\lambda|Q|^{2} Q=E Q \tag{1.3}
\end{equation*}
$$

They are critical points to the Hamiltonian $\mathcal{H}[\psi]$ defined in (1.2) subject to the constraint that the $L^{2}$-norm of $\psi$ is fixed.

We may obtain two families of such bound states by standard bifurcation theory, corresponding to the two eigenvalues of the linear Hamiltonian. For any $E$ sufficiently close to $e_{0}$ so that $E-e_{0}$ and $\lambda$ have the same sign, there is a unique positive solution $Q=Q_{E}$ to (1.3) which decays exponentially as $x \rightarrow \infty$. See Lemma 2.1 of [21]. We call this family the nonlinear ground states and we refer to it as $\left\{Q_{E}\right\}_{E}$. Similarly, there is a nonlinear excited state family $\left\{Q_{1, E_{1}}\right\}_{E_{1}}$. We will abbreviate them as $Q$ and $Q_{1}$. From the same Lemma 2.1 of [21], these solutions are small and we have $\left\|Q_{E}\right\| \sim\left|E-e_{0}\right|^{1 / 2}$ and $\left\|Q_{1, E_{1}}\right\| \sim\left|E_{1}-e_{1}\right|^{1 / 2}$.

It is well-known that the family of nonlinear ground states is stable in the sense that if

$$
\inf _{\Theta, E}\left\|\psi(t)-Q_{E} e^{i \Theta}\right\|_{L^{2}}
$$

is small for $t=0$, it remains so for all $t$, see [13]. Let $\|\cdot\|_{L_{\text {loc }}^{2}}$ denote a local $L^{2}$ norm, for example the $L^{2}$-norm in a ball with large radius. One expects that this difference actually approaches zero in local $L^{2}$ norm, i.e.,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \inf _{\Theta, E}\left\|\psi(t)-Q_{E} e^{i \Theta}\right\|_{L_{\mathrm{loc}}^{2}}=0 \tag{1.4}
\end{equation*}
$$

If $-\Delta+V$ has only one bound state, it is proved in [17] [10] that the evolution will eventually settle down to some ground state $Q_{E_{\infty}}$ with $E_{\infty}$ close to $E$. Suppose now that $-\Delta+V$ has two bound states: a ground state $\phi_{0}$ with eigenvalue $e_{0}$ and an excited state $\phi_{1}$ with eigenvalue $e_{1}$. It is proved in 20] that the evolution with initial data $\psi_{0}$ near some $Q_{E}$ will eventually settle down to some ground state $Q_{E_{\infty}}$ with $E_{\infty}$ close to $E$. See also [3] for the one dimensional case, 18 for nonlinear Klein-Gorden equations with one unstable bound state.

If the initial data is not restricted to near the ground states, the problem becomes much more delicate due to the presence of the excited states. On physical ground, quantum mechanics tells us that excited states are unstable and all perturbations should result in a release of radiation and the relaxation of the excited states to the ground states. Since bound states are periodic orbits, this picture differs from the classical one where periodic orbits are in general stable.

There were extensive linear analysis for bound states of nonlinear Schrödinger and wave equations, see, e.g., [15, 16, 14, 5, 6, 22, 23]. A special case of Theorem 3.5 of [6], page 330, states that

Theorem A Let $H_{1}=-\Delta+V-E_{1}$. The matrix operator

$$
J H_{1}=\left[\begin{array}{cc}
0 & H_{1} \\
-H_{1} & 0
\end{array}\right], \quad J=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

is structurally stable if and only $e_{0}>2 e_{1}$.
The precise meaning of structural stability was given in [6]. Roughly speaking, it means that the operator remains stable under small perturbations. Theorem A will not be directly used in this paper.

As we will see later, the linearized operator around an excited state is a perturbation of $J H_{1}$. Thus, two different situations occurs:

1. Non-resonant case: $e_{0}>2 e_{1}$. $\quad\left(e_{01}<\left|e_{1}\right|\right)$.
2. Resonant case: $e_{0}<2 e_{1} . \quad\left(e_{01}>\left|e_{1}\right|\right)$.

Here $e_{01}=e_{1}-e_{0}>0$. In the resonant case, Theorem A says the linearized operator is in general unstable, which agrees with the physical picture. In the non-resonant case, however, the linearized operator becomes stable. The difference here is closely related to the fact that $2 e_{1}-e_{0}$ lies in the continuum spectrum of $H_{0}$ only in the resonant case.

In the resonant case, the unstable picture is confirmed for most data near excited states in our work [21]. We prove that, as long as the ground state component in $\psi_{0}-Q_{1}$ is larger than $\left\|\psi_{0}\right\|^{2}$ times the size of the dispersive part corresponding the continuous spectrum, the solution will move away from the excited states and relax and stabilize to ground states locally. Since $\left\|\psi_{0}\right\|^{2}$ is small, this assumption allows the dispersive part to be much larger than the ground state component.

There is a small set of data where [21] does not apply, namely, those data with ground state component in $\psi_{0}-Q_{1}$ smaller than $\left\|\psi_{0}\right\|^{2}$ times the size of the dispersive part. The aim of this paper is to show that this restriction is almost optimal: we will construct within this small set of initial data a "hypersurface" whose corresponding solutions converge to excited states.

This does not contradict with the physical intuition since this hypersurface in certain sense has zero measure and can not be observed in experiments. These solutions, however, show that linear instability does not imply all solutions to be unstable. In the language of dynamical systems, the excited states are one parameter family of hyperbolic fixed points and this hypersurface is contained in the stable manifold of the fixed points. We believe that this surface is the whole stable manifold.

We will also construct solutions converging to excited states in the nonresonant case, where it is expected since the linearized operator is stable. We now state our assumptions on the potential $V$ :
Assumption A0: $H_{0}:=-\Delta+V$ acting on $L^{2}\left(\mathbb{R}^{3}\right)$ has two simple eigenvalues $e_{0}<e_{1}<0$, with normalized eigenvectors $\phi_{0}$ and $\phi_{1}$.

Assumption A1: The bottom of the continuous spectrum to $-\Delta+V, 0$, is not a generalized eigenvalue, i.e., not an eigenvalue nor a resonance. There is a small $\sigma>0$ such that

$$
\left|\nabla^{\alpha} V(x)\right| \leq C\langle x\rangle^{-5-\sigma}, \quad \text { for }|\alpha| \leq 2
$$

Also, the functions $(x \cdot \nabla)^{k} V$, for $k=0,1,2,3$, are $-\Delta$ bounded with a $-\Delta$-bound $<1$ :

$$
\left\|(x \cdot \nabla)^{k} V \phi\right\|_{2} \leq \sigma_{0}\|-\Delta \phi\|_{2}+C\|\phi\|_{2}, \quad \sigma_{0}<1, \quad k=0,1,2,3
$$

Assumption A1 contains some standard conditions to assure that most tools in linear Schrödinger operators apply. In particular, it satisfies the assumptions of [24] so that the wave operator $W_{H}=\lim _{t \rightarrow \infty} e^{i t H_{0}} e^{i t \Delta}$ satisfies the $W^{k, p}$ estimates for $k \leq 2$. These conditions are certainly not optimal.

Let $e_{01}=e_{1}-e_{0}$ be the spectral gap of the ground state. In the resonant case $2 e_{01}>\left|e_{0}\right|$ so that $2 e_{1}-e_{0}$ lies in the continuum spectrum of $H_{0}$, we further assume

Assumption A2: For some $s_{0}>0$,

$$
\begin{equation*}
\gamma_{0} \equiv \inf _{|s|<s_{0}} \lim _{\sigma \rightarrow 0+} \operatorname{Im}\left(\phi_{0} \phi_{1}^{2}, \frac{1}{H_{0}+e_{0}-2 e_{1}+s-\sigma i} \mathbf{P}_{\mathrm{c}}^{H_{0}} \phi_{0} \phi_{1}^{2}\right)>0 . \tag{1.5}
\end{equation*}
$$

Note that $\gamma_{0} \geq 0$ since the expression above is quadratic. This assumption is generically true.

Let $Q_{1}=Q_{1, E_{1}}$ be a nonlinear excited state with $\left\|Q_{1, E_{1}}\right\|_{2}$ small. Since $\left(Q_{1}, E_{1}\right)$ satisfies (1.3), the function $\psi(t, x)=Q_{1}(x) e^{-i E_{1} t}$ is an exact solution of (1.1). If we consider solutions $\psi(t, x)$ of (1.1) of the form

$$
\psi(t, x)=\left[Q_{1}(x)+h(t, x)\right] e^{-i E_{1} t}
$$

with $h(t, x)$ small in a suitable sense, then $h(t, x)$ satisfies

$$
\partial_{t} h=\mathcal{L}_{1} h+\text { nonlinear terms }
$$

where $\mathcal{L}_{1}$, the linearized operator around the nonlinear excited state solution $Q_{1}(x) e^{-i E_{1} t}$, is defined by

$$
\begin{equation*}
\mathcal{L}_{1} h=-i\left\{\left(-\Delta+V-E_{1}+2 \lambda Q_{1}^{2}\right) h+\lambda Q_{1}^{2} \bar{h}\right\} . \tag{1.6}
\end{equation*}
$$

Theorem 1.1 Suppose $H_{0}=-\Delta+V$ satisfies assumptions A0-A1. Suppose either
(NR) $e_{0}>2 e_{1}$, or
(R) $e_{0}<2 e_{1}$, and the assumption A2 for $\gamma_{0}$ holds.

Then there are $n_{0}>0$ and $\varepsilon_{0}(n)>0$ defined for $n \in\left(0, n_{0}\right]$ such that the following holds. Let $Q_{1}:=Q_{1, E_{1}}$ be a nonlinear excited state with $\left\|Q_{1}\right\|_{L^{2}}=$ $n \leq n_{0}$, and let $\mathcal{L}_{1}$ be the corresponding linearized operator. For any $\xi_{\infty} \in$ $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right) \cap\left(W^{2,1} \cap H^{2}\right)\left(\mathbb{R}^{3}\right)$ with $\left\|\xi_{\infty}\right\|_{W^{2,1} \cap H^{2}}=\varepsilon, 0<\varepsilon \leq \varepsilon_{0}(n)$, there is a solution $\psi(t, x)$ of (1.1) and a real function $\theta(t)=O\left(t^{-1}\right)$ for $t>0$ so that

$$
\left\|\psi(t)-\psi_{\mathrm{as}}(t)\right\|_{H^{2}} \leq C \varepsilon^{2}(1+t)^{-7 / 4}
$$

where $C=C(n)$ and

$$
\psi_{\mathrm{as}}(t)=Q_{1} e^{-i E_{1} t+i \theta(t)}+e^{-i E_{1} t} e^{t \mathcal{L}_{1}} \xi_{\infty}
$$

To prove this theorem, a detailed spectral analysis of the linearized operator $\mathcal{L}_{1}$ is required. We shall classify the spectrum of $\mathcal{L}_{1}$ completely in
both non-resonant and resonant cases, see Theorems 2.1 and 2.2. It is wellknown that the continuous spectrum $\Sigma_{c}$ of $\mathcal{L}_{1}$ is the same as that of $J H_{1}$, i.e., $\Sigma_{c}=\left\{s i: s \in \mathbb{R},|s| \geq\left|E_{1}\right|\right\}$. The point spectrum of $\mathcal{L}_{1}$ is more subtle. By definition, $H_{1} \phi_{1}=-\left(E_{1}-e_{1}\right) \phi_{1}$ and $H_{1} \phi_{0}=-\left(E_{1}-e_{0}\right) \phi_{0}$, and thus the matrix operator $J H_{1}$ has 4 eigenvalues $\pm i\left(E_{1}-e_{1}\right)$ and $\pm i\left(E_{1}-e_{0}\right)$. In the non-resonant case, the eigenvalues of $\mathcal{L}_{1}$ are purely imaginary and are small perturbations of these eigenvalues. In the resonant case, the eigenvalues $\pm i\left(E_{1}-e_{0}\right)$ are embedded inside the continuum spectrum $\Sigma_{c}$. In general perturbation theory for embedded eigenvalues, they turn into resonances under self-adjoint perturbations. The operator $\mathcal{L}_{1}$ is however not a self-adjoint perturbation of $H_{1}$. In this case, we shall prove that the embedded eigenvalues $\pm i\left(E_{1}-e_{0}\right)$ split into four eigenvalues $\pm \omega_{*}$ and $\pm \bar{\omega}_{*}$ with the real part given approximately by the Fermi golden rule (see [12] Chap.XII.6):

$$
n^{4} \operatorname{Im}\left(\lambda \phi_{0} \phi_{1}^{2}, \frac{1}{-\Delta+V+e_{0}-2 e_{1}-0 i} \mathbf{P}_{\mathrm{c}} \lambda \phi_{1}^{2} \phi_{0}\right)
$$

Here $n$ is the size of $Q_{1}$, see (2.41). In particular, $e^{t \mathcal{L}_{1}}$ is exponentially unstable with the decay rate (or the blow-up rate) given approximately by the Fermi golden rule. In other words, although self-adjoint perturbation turns embedded eigenvalues into resonances, the non-self adjoint perturbations given by $\mathcal{L}_{1}$ turns an embedded eigenvalue into two eigenvalues with the shifts in the real axis given to the leading order by the Fermi golden rule. The dynamics of self-adjoint perturbation of embedded eigenvalues were studied in [19].

In the appendix we will prove the existence of solutions vanishing locally as $t \rightarrow \infty$, independent of the number of bound states in $H_{0}$. Although some weaker versions of this proposition are expected, it has never been proved in current form and we include it for completeness.

Proposition 1.2 Suppose $H_{0}=-\Delta+V$ satisfies assumption A1. There is an $\varepsilon_{0}>0$ such that the following holds. For any $\xi_{\infty} \in \mathbf{H}_{\mathrm{c}}\left(H_{0}\right) \cap\left(W^{2,1} \cap\right.$ $\left.H^{2}\right)\left(\mathbb{R}^{3}\right)$ with $0<\left\|\xi_{\infty}\right\|_{W^{2,1} \cap H^{2}}=\varepsilon \leq \varepsilon_{0}$, there is a solution $\psi(t, x)$ of (1.1) of the form

$$
\begin{aligned}
& \qquad \psi(t)=e^{-i t H_{0}} \xi_{\infty}+g(t), \quad(t \geq 0) \\
& \text { with }\|g(t)\|_{H^{2}} \leq C \varepsilon^{2}(1+t)^{-2}
\end{aligned}
$$

## 2 Linear analysis for excited states

As mentioned in $\S 1$, there is a family $\left\{Q_{1, E_{1}}\right\}_{E_{1}}$ of nonlinear excited states with the frequency $E_{1}$ as the parameter. They satisfy

$$
\begin{equation*}
(-\Delta+V) Q_{1}+\lambda\left|Q_{1}\right|^{2} Q_{1}=E_{1} Q_{1} \tag{2.1}
\end{equation*}
$$

Let $Q_{1}=Q_{1, E_{1}}$ be a fixed nonlinear excited state with $n=\left\|Q_{1, E_{1}}\right\|_{2} \leq$ $n_{0} \ll 1$. The linearized operator around the nonlinear bound state solution $Q_{1}(x) e^{-i E_{1} t}$ is defined in (1.6)

$$
\mathcal{L}_{1} h=-i\left\{\left(-\Delta+V-E_{1}+2 \lambda Q_{1}^{2}\right) h+\lambda Q_{1}^{2} \bar{h}\right\} .
$$

We will study the spectral properties of $\mathcal{L}_{1}$ in this section. It properties are best understood in the complexification of $L^{2}\left(\mathbb{R}^{3}, \mathbb{C}\right)$.

Definition 2.1 Identify $\mathbb{C}$ with $\mathbb{R}^{2}$ and $L^{2}=L^{2}\left(\mathbb{R}^{3}, \mathbb{C}\right)$ with $L^{2}\left(\mathbb{R}^{3}, \mathbb{R}^{2}\right)$. Denote by $\mathbb{C} L^{2}=L^{2}\left(\mathbb{R}^{3}, \mathbb{C}^{2}\right)$ the complexification of $L^{2}\left(\mathbb{R}^{3}, \mathbb{R}^{2}\right) . \mathbb{C} L^{2}$ consists of 2-dimensional vectors whose components are in $L^{2}$. We have the natural embedding

$$
f \in L^{2} \longrightarrow\left[\begin{array}{l}
\operatorname{Re} f \\
\operatorname{Im} f
\end{array}\right] \in \mathbb{C} L^{2}
$$

We equip $\mathbb{C} L^{2}$ with the natural inner product: For $f, g \in \mathbb{C} L^{2}, f=\left[\begin{array}{l}f_{1} \\ f_{2}\end{array}\right]$, $g=\left[\begin{array}{l}g_{1} \\ g_{2}\end{array}\right]$, we define

$$
\begin{equation*}
(f, g)=\int \bar{f} \cdot g d x=\int\left(\bar{f}_{1} g_{1}+\bar{f}_{2} g_{2}\right) d x \tag{2.2}
\end{equation*}
$$

Denote by $\mathbf{R E}$ the operator first taking the real part of functions in $\mathbb{C} L^{2}$ and then pulling back to $L^{2}$ :

$$
\mathbf{R E}: \mathbb{C} L^{2} \rightarrow L^{2}, \quad \mathbf{R E}\left[\begin{array}{l}
f \\
g
\end{array}\right]=(\operatorname{Re} f)+i(\operatorname{Re} g)
$$

Recall the matrix operator $J H_{1}$ defined in Theorem A. Since $H_{1} \phi_{1}=$ $-\left(E_{1}-e_{1}\right) \phi_{1}$ and $H_{1} \phi_{0}=-\left(E_{1}-e_{0}\right) \phi_{0}$, the matrix operator $J H_{1}$ has 4 eigenvalues $\pm i\left(E_{1}-e_{1}\right)$ and $\pm i\left(E_{1}-e_{0}\right)$ with corresponding eigenvectors

$$
\left[\begin{array}{c}
\phi_{1}  \tag{2.3}\\
-i \phi_{1}
\end{array}\right], \quad\left[\begin{array}{c}
\phi_{1} \\
i \phi_{1}
\end{array}\right], \quad\left[\begin{array}{c}
\phi_{0} \\
-i \phi_{0}
\end{array}\right], \quad\left[\begin{array}{c}
\phi_{0} \\
i \phi_{0}
\end{array}\right]
$$

Notice that

$$
\begin{equation*}
E_{1}-e_{1}=O\left(n^{2}\right), \quad E_{1}-e_{0}=e_{01}+O\left(n^{2}\right) \tag{2.4}
\end{equation*}
$$

The continuous spectrum of $J H_{1}$ is

$$
\begin{equation*}
\Sigma_{c}=\left\{s i: s \in \mathbb{R},|s| \geq\left|E_{1}\right|\right\} \tag{2.5}
\end{equation*}
$$

which consists of two rays on the imaginary axis.
The operator $\mathcal{L}_{1}$ in its matrix form

$$
\left[\begin{array}{cc}
0 & L_{-}  \tag{2.6}\\
-L_{+} & 0
\end{array}\right], \quad \text { with }\left\{\begin{array}{l}
L_{-}=-\Delta+V-E_{1}+\lambda Q_{1}^{2} \\
L_{+}=-\Delta+V-E_{1}+3 \lambda Q_{1}^{2}
\end{array}\right.
$$

is a perturbation of $J H_{1}$. By Weyl's lemma, the continuous spectrum of $\mathcal{L}_{1}$ is also $\Sigma_{c}$. The eigenvalues are more complicated. In both cases $\left(e_{01}<\left|e_{1}\right|\right.$ and $\left.e_{01}>\left|e_{1}\right|\right)$ they are near 0 and $\pm i e_{01}$. As we shall see, in both cases 0 is an eigenvalue of $\mathcal{L}_{1}$. The main difference between the two cases are the eigenvalues near $i e_{01}$ and $-i e_{01}$. If $e_{01}<\left|e_{1}\right|$, then $i e_{01}$ lies outside the continuous spectrum and $\mathcal{L}_{1}$ has an eigenvalue near $i e_{01}$ which is purely imaginary. On the other hand, if $e_{01}>\left|e_{1}\right|$, then $i e_{01}$ lies inside the continuous spectrum. Generically it splits under perturbation and the eigenvalues of $\mathcal{L}_{1}$ near $\pm i e_{01}$ have non-zero real parts.

We shall show that $L^{2}\left(\mathbb{R}^{3}, \mathbb{C}\right)$, as a real vector space, can be decomposed as the direct sum of three invariant subspaces

$$
\begin{equation*}
L^{2}\left(\mathbb{R}^{3}, \mathbb{C}\right)=S\left(\mathcal{L}_{1}\right) \oplus \mathbf{E}_{1}\left(\mathcal{L}_{1}\right) \oplus \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right) \tag{2.7}
\end{equation*}
$$

Here $S\left(\mathcal{L}_{1}\right)$ is the generalized null space, $\mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$ is a generalized eigenspaces and $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ corresponds to the continuous spectrum. Both $S\left(\mathcal{L}_{1}\right)$ and $\mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$ are finite dimensional.

Recall the Pauli matrices

$$
\sigma_{1}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], \quad \sigma_{2}=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right], \quad \sigma_{3}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]
$$

They are self-adjoint and

$$
\begin{equation*}
\sigma_{1} \mathcal{L}_{1}=\mathcal{L}_{1}^{*} \sigma_{1}, \quad \sigma_{3} \mathcal{L}_{1}=-\mathcal{L}_{1} \sigma_{3} \tag{2.8}
\end{equation*}
$$

where $\mathcal{L}_{1}^{*}=\left[\begin{array}{cc}0 & -L_{+} \\ L_{-} & 0\end{array}\right]$.
Let $R_{1}=\partial_{E_{1}} Q_{1, E_{1}}$. Direct differentiation of (2.1) with respect to $E_{1}$ gives $L_{+} R_{1}=Q_{1}$. Since $L_{-} Q_{1}=0$ and $L_{+} R_{1}=Q_{1}$, we have $\mathcal{L}_{1}\left[\begin{array}{c}0 \\ Q_{1}\end{array}\right]=0$ and $\mathcal{L}_{1}\left[\begin{array}{c}R_{1} \\ 0\end{array}\right]=-\left[\begin{array}{c}0 \\ Q_{1}\end{array}\right]$. We will show $\operatorname{dim}_{\mathbb{R}} S\left(\mathcal{L}_{1}\right)=2$, hence

$$
S\left(\mathcal{L}_{1}\right)=\operatorname{span}_{\mathbb{R}}\left\{\left[\begin{array}{c}
0  \tag{2.9}\\
Q_{1}
\end{array}\right],\left[\begin{array}{c}
R_{1} \\
0
\end{array}\right]\right\} .
$$

$\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ can be characterized as

$$
\begin{equation*}
\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)=\left\{\psi \in \mathbb{C} L^{2}:\left(\sigma_{1} \psi, f\right)=0, \forall f \in S\left(\mathcal{L}_{1}\right) \oplus \mathbf{E}_{1}\left(\mathcal{L}_{1}\right)\right\} . \tag{2.10}
\end{equation*}
$$

We will use (2.10) as a working definition of $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$. After we have proved the spectrum of $\mathcal{L}_{1}$ and the resolvent estimates, we will use the wave operator of $\mathcal{L}_{1}$ (see [1, 24, 25]) to show that (2.10) agrees with the usual definition of the continuous spectrum subspace. See $\S 2.5$.

The space $\mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$, however, has very different properties in the two cases, due to whether $\pm i\left(E_{1}-e_{0}\right)$ are embedded eigenvalues of $J H_{1}$. We will consider $\mathbf{E}_{1}=\mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$ as a subspace of $L^{2}\left(\mathbb{R}^{3}, \mathbb{R}^{2}\right)$ and denote by $\mathbb{C} \mathbf{E}_{1} \subset$ $\mathbb{C} L^{2}$ the complexification of $\mathbf{E}_{1}$. We will show that $\mathbb{C} \mathbf{E}_{1}$ is a direct sum of eigenspaces of $\mathcal{L}_{1}$ in $\mathbb{C} L^{2}$. We also have

$$
\begin{equation*}
\left(\sigma_{1} f, g\right)=0, \quad \forall f \in S\left(\mathcal{L}_{1}\right), \quad \forall g \in \mathbf{E}_{1}\left(\mathcal{L}_{1}\right) \tag{2.11}
\end{equation*}
$$

We have the following two theorems for the two cases.
Remark The case $e_{0}=2 e_{1}$ : The spectral property of $\mathcal{L}_{1}$ is not clear.
Theorem 2.1 (Non-resonant case) Suppose $e_{0}>2 e_{1}$, and the assumptions A0-A1 hold. Let $Q_{1}=Q_{1, E_{1}}$ be a nonlinear excited state with sufficiently small $L^{2}$-norm, and let $\mathcal{L}_{1}$ be defined as in (1.6).
(1) The eigenvalues of $\mathcal{L}_{1}$ are 0 and $\pm \omega_{*}$. The multiplicity of 0 is two. The other eigenvalues are simple. Here $\omega_{*}=i \kappa$, $\kappa$ is real, $\kappa=e_{01}+O\left(n^{2}\right)$. There is no embedded eigenvalue. The bottoms of the continuous spectrum are not eigenvalue nor resonance.
(2) The space $L^{2}=L^{2}\left(\mathbb{R}^{3}, \mathbb{C}\right)$, as a real vector space, can be decomposed as in (2.7). Here $S\left(\mathcal{L}_{1}\right)$ and $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ are given in (2.9) and (2.10), respectively; $\mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$ is the space corresponding to the perturbation of the eigenvalues $\pm i\left(E_{1}-e_{0}\right)$ of $J H_{1}$. We have the orthogonality relation (2.11).
(3) Let $\mathbb{C} \mathbf{E}_{1}$ denotes the complexification of $\mathbf{E}_{1}=\mathbf{E}_{1}\left(\mathcal{L}_{1}\right) . \mathbb{C} \mathbf{E}_{1}$ is 2-complex-dimensional. $\mathbf{E}_{1}$ is 2-real-dimensional. We have

$$
\begin{align*}
\mathbb{C E}_{1} & =\underset{\mathbb{C}}{\operatorname{span}}\{\Phi, \bar{\Phi}\},  \tag{2.12}\\
\mathbf{E}_{1} & =\underset{\mathbb{R}}{ }\left\{\begin{array}{l}
\text { span }
\end{array}\left\{\begin{array}{l}
u \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
v
\end{array}\right]\right\} .
\end{align*}
$$

Here $\Phi=\left[\begin{array}{c}u \\ -i v\end{array}\right]$ is an eigenfunction of $\mathcal{L}_{1}$ with eigenvalue $\omega_{*} . u$ and $v$ are real-valued $L^{2}$-functions satisfying $L_{+} u=-\kappa v, L_{-} v=-\kappa u$ and $(u, v)=1$.
$u$ and $v$ are perturbations of $\phi_{0} . \bar{\Phi}=\left[\begin{array}{c}u \\ i v\end{array}\right]$ is another eigenfunction with eigenvalue $-\omega_{*}$. We have $\mathcal{L}_{1} \Phi=\omega_{*} \Phi, \mathcal{L}_{1} \bar{\Phi}=-\omega_{*} \bar{\Phi}$.
(4) For any function $\zeta \in \mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$, there is a unique $\alpha \in \mathbb{C}$ so that

$$
\zeta=\mathbf{R E} \alpha \Phi,
$$

and we have $\mathcal{L}_{1} \zeta=\mathbf{R E} \omega_{*} \alpha \Phi$, $e^{t \mathcal{L}_{1}} \zeta=\mathbf{R E} e^{t \omega_{*}} \alpha \Phi$.
(5) We have the orthogonality relations in (2.10) and (2.11). Hence any $\psi \in L^{2}$ can be decomposed as (see (2.7))

$$
\psi=a\left[\begin{array}{c}
R_{1}  \tag{2.13}\\
0
\end{array}\right]+b\left[\begin{array}{c}
0 \\
Q_{1}
\end{array}\right]+c\left[\begin{array}{l}
u \\
0
\end{array}\right]+d\left[\begin{array}{l}
0 \\
v
\end{array}\right]+\eta
$$

with $\eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$,

$$
\begin{align*}
a & =\left(Q_{1}, R_{1}\right)^{-1}\left(Q_{1}, \operatorname{Re} \psi\right), & c=(u, v)^{-1}(v, \operatorname{Re} \psi), \\
b & =\left(Q_{1}, R_{1}\right)^{-1}\left(R_{1}, \operatorname{Im} \psi\right), & d=(u, v)^{-1}(u, \operatorname{Im} \psi) . \tag{2.14}
\end{align*}
$$

(6) Let $M_{1} \equiv \mathbf{E}_{1}\left(\mathcal{L}_{1}\right) \oplus \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$. We have

$$
M_{1} \equiv \mathbf{E}_{1}\left(\mathcal{L}_{1}\right) \oplus \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)=\left[\begin{array}{c}
Q_{1}^{\perp}  \tag{2.15}\\
R_{1}^{\perp}
\end{array}\right] .
$$

There is a constant $C_{2}>1$ such that, for all $\phi \in M_{1}$ and all $t \in \mathbb{R}$, we have

$$
\begin{equation*}
C_{2}^{-1}\|\phi\|_{H^{k}} \leq\left\|e^{t \mathcal{L}_{1}} \phi\right\|_{H^{k}} \leq C_{2}\|\phi\|_{H^{k}}, \quad(k=1,2) \tag{2.16}
\end{equation*}
$$

(7) Decay estimates: For all $\eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$, for all $p \in[2, \infty]$, one has

$$
\left\|e^{t \mathcal{L}_{1}} \eta\right\|_{L^{p}} \leq C|t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\eta\|_{L^{p^{\prime}}}
$$

Theorem 2.2 (Resonant case) Suppose $e_{0}<2 e_{1}$, and the assumptions A0-A2 hold. Let $Q_{1}=Q_{1, E_{1}}$ be a nonlinear excited state with sufficiently small $L^{2}$-norm, and let $\mathcal{L}_{1}$ be defined as in (1.6).
(1) The eigenvalues of $\mathcal{L}_{1}$ are $0, \pm \omega_{*}$ and $\pm \bar{\omega}_{*}$. The multiplicity of 0 is two. The other eigenvalues are simple. Here $\omega_{*}=i \kappa+\gamma, \kappa, \gamma>0$, $\kappa=e_{01}+O\left(n^{2}\right)$, and $\frac{3}{4} \lambda^{2} \gamma_{0} n^{4} \leq \gamma \leq C n^{4}$. ( $\gamma_{0}$ is given in (1.5)). There is no embedded eigenvalue. The bottoms of the continuous spectrum are not eigenvalue nor resonance.

There is an $\omega_{*}$-eigenvector $\Phi, \mathcal{L}_{1} \Phi=\omega_{*} \Phi$, which is of order one in $L^{2}$ and $\Phi-\left[\begin{array}{c}\phi_{0} \\ -i \phi_{0}\end{array}\right]$ is locally small in the sense that

$$
\left|\left(\phi, \Phi-\left[\begin{array}{c}
\phi_{0}  \tag{2.17}\\
-i \phi_{0}
\end{array}\right]\right)\right| \leq C n^{2}\left\|\langle x\rangle^{r} \phi\right\|_{L^{2}},
$$

for any $r>3$. However, $\Phi$ is not a perturbation of $\left[\begin{array}{c}\phi_{0} \\ -i \phi_{0}\end{array}\right]$ in $\mathbb{C} L^{2}$. In fact, $\Phi=\left[\begin{array}{l}u \\ v\end{array}\right]$ with $u-\phi_{0}$ and $v+i \phi_{0}$ of order one in $L^{2}$,

$$
u=\phi_{0}-\frac{1}{-\Delta+V-E_{1}-\kappa+\gamma i} \mathbf{P}_{\mathrm{c}}\left(H_{0}\right) \lambda \phi_{0} Q_{1}^{2}+O\left(n^{2}\right) \quad \text { in } L^{2},
$$

and $v=-L_{+} u / \omega_{*}$. Note $-E_{1}-\kappa=e_{0}-2 e_{1}+O\left(n^{2}\right)$.
(2) The space $L^{2}=L^{2}\left(\mathbb{R}^{3}, \mathbb{C}\right)$, as a real vector space, can be decomposed as in (2.7). Here $S\left(\mathcal{L}_{1}\right)$ and $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ are given in (2.9) and (2.10), respectively; $\mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$ is the space corresponding to the perturbation of the eigenvalues $\pm i\left(E_{1}-e_{0}\right)$ of $J H_{1}$. We have the orthogonality relation (2.11).
(3) Let $\mathbb{C} \mathbf{E}_{1}$ denotes the complexification of $\mathbf{E}_{1}=\mathbf{E}_{1}\left(\mathcal{L}_{1}\right) . \mathbb{C} \mathbf{E}_{1}$ is 4-complex-dimensional. $\quad \mathbf{E}_{1}$ is 4-real-dimensional. If we write $\Phi=\left[\begin{array}{l}u \\ v\end{array}\right]=$ $\left[\begin{array}{l}u_{1}+u_{2} i \\ v_{1}+v_{2} i\end{array}\right]$ with $u_{1}, u_{2}, v_{1}, v_{2}$ real-valued $L^{2}$ functions, we have

$$
\begin{align*}
\mathbb{C E}_{1} & =\operatorname{span}_{\mathbb{C}} \\
\mathbf{E}_{1} & =\operatorname{span}_{\mathbb{R}}\left\{\left[\begin{array}{c}
u_{1} \\
0
\end{array}\right],\left[\begin{array}{c}
u_{2} \\
0
\end{array}\right],\left[\begin{array}{c}
0 \\
v_{1}
\end{array}\right],\left[\begin{array}{c}
0 \\
v_{2}
\end{array}\right]\right\} . \tag{2.18}
\end{align*}
$$

Recall $\sigma_{3}=\left[\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right]$. The other eigenvectors are $\bar{\Phi}, \sigma_{3} \Phi$ and $\sigma_{3} \bar{\Phi}$,

$$
\begin{equation*}
\mathcal{L}_{1} \Phi=\omega_{*} \Phi, \quad \mathcal{L}_{1} \bar{\Phi}=\bar{\omega}_{*} \bar{\Phi}, \quad \mathcal{L}_{1} \sigma_{3} \Phi=-\omega_{*}\left(\sigma_{3} \Phi\right), \quad \mathcal{L}_{1} \sigma_{3} \bar{\Phi}=-\bar{\omega}_{*}\left(\sigma_{3} \bar{\Phi}\right) \tag{2.19}
\end{equation*}
$$

(4) For any function $\zeta \in \mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$, there is a unique pair $(\alpha, \beta) \in \mathbb{C}^{2}$ so that

$$
\begin{equation*}
\zeta=\mathbf{R E}\left\{\alpha \Phi+\beta \sigma_{3} \Phi\right\}, \tag{2.20}
\end{equation*}
$$

and we have $\mathcal{L}_{1} \zeta=\mathbf{R E}\left\{\omega_{*} \alpha \Phi-\omega_{*} \beta \sigma_{3} \Phi\right\}$, $e^{t \mathcal{L}_{1}} \zeta=\mathbf{R E}\left\{e^{t \omega_{*}} \alpha \Phi+e^{-t \omega_{*}} \beta \sigma_{3} \Phi\right\}$.
(5) We have the orthogonality relations in (2.10) and (2.11). Moreover, $\sigma_{1} \bar{\Phi} \perp\left\{\bar{\Phi}, \sigma_{3} \Phi, \sigma_{3} \bar{\Phi}\right\}, \sigma_{1} \Phi \perp\left\{\Phi, \sigma_{3} \Phi, \sigma_{3} \bar{\Phi}\right\}$, and $\int \bar{u} v d x=0$, etc. For any function $\psi \in \mathbb{C} L^{2}$, if we decompose

$$
\psi=a\left[\begin{array}{c}
R_{1}  \tag{2.21}\\
0
\end{array}\right]+b\left[\begin{array}{c}
0 \\
Q_{1}
\end{array}\right]+\alpha_{1} \Phi+\alpha_{2} \bar{\Phi}+\beta_{1} \sigma_{3} \Phi+\beta_{2} \sigma_{3} \bar{\Phi}+\eta
$$

where $a, b, \alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2} \in \mathbb{C}$ and $\eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$, then we have

$$
\begin{array}{cl}
a=c_{1}\left(\sigma_{1}\left[\begin{array}{c}
0 \\
Q_{1}
\end{array}\right], \psi\right), & b=c_{1}\left(\sigma_{1}\left[\begin{array}{c}
R_{1} \\
0
\end{array}\right], \psi\right), \\
\alpha_{1}=c_{2}\left(\sigma_{1} \bar{\Phi}, \psi\right), & \alpha_{2}=\bar{c}_{2}\left(\sigma_{1} \Phi, \psi\right)  \tag{2.22}\\
\beta_{1}=-c_{2}\left(\sigma_{1} \sigma_{3} \bar{\Phi}, \psi\right), & \beta_{2}=-\bar{c}_{2}\left(\sigma_{1} \sigma_{3} \Phi, \psi\right)
\end{array}
$$

where $c_{1}^{-1}=\left(Q_{1}, R_{1}\right)$ and $c_{2}^{-1}=\left(\sigma_{1} \bar{\Phi}, \Phi\right)=\int 2 u v d x . \quad\left(\right.$ Note $\left.c_{1} \lambda>0.\right)$ The statement that $\psi \in \mathbf{E}_{1}$ is equivalent to that $a, b \in \mathbb{R}, \alpha_{1}=\alpha_{2}=\alpha / 2$, $\beta_{1}=\beta_{2}=\beta / 2$ and $\eta$ is real. In this case,

$$
\psi=a\left[\begin{array}{c}
R_{1}  \tag{2.23}\\
0
\end{array}\right]+b\left[\begin{array}{c}
0 \\
Q_{1}
\end{array}\right]+\mathbf{R E}\left\{\alpha \Phi+\beta \sigma_{3} \Phi\right\}+\eta
$$

with $a, b \in \mathbb{R}, \eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ real, $\alpha, \beta \in \mathbb{C}$, and

$$
\begin{equation*}
\alpha=P_{\alpha}(\psi) \equiv 2 c_{2}\left(\sigma_{1} \bar{\Phi}, \psi\right), \quad \beta=P_{\beta}(\psi) \equiv-2 c_{2}\left(\sigma_{1} \sigma_{3} \bar{\Phi}, \psi\right) \tag{2.24}
\end{equation*}
$$

$P_{\alpha}$ and $P_{\beta}$ are maps from $L^{2}$ to $\mathbb{C}$.
(6) There is a constant $C>1$ such that, for all $\eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ and all $t \in \mathbb{R}$, we have

$$
C^{-1}\|\eta\|_{H^{k}} \leq\left\|e^{t \mathcal{L}_{1}} \eta\right\|_{H^{k}} \leq C\|\eta\|_{H^{k}}, \quad(k=1,2)
$$

(7) Decay estimates: For all $\eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$, for all $p \in[2, \infty]$, one has

$$
\left\|e^{t \mathcal{L}_{1}} \eta\right\|_{L^{p}} \leq C|t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\eta\|_{L^{p^{\prime}}}
$$

where $C=C(n, p)$ depends on $n$.

Remark (i). In (6), we restrict ourselves to $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$, not $M_{1}$ as in Theorem 2.1. (ii). In (3), $\Phi$ is not a perturbaiton of $\left[\begin{array}{c}\phi_{0} \\ -i \phi_{0}\end{array}\right]$. Also, the $L^{2}$ functions $u_{1}$ and $u_{2}$ are independent of each other. So are $v_{1}$ and $v_{2}$. (iii) In (7) the constant depends on $n$ since there are eigenvalues which are very close to the continuous spectrum.

Since the proof of Theorem 2.1 is easier, we postpone it to the last subsection $\S 2.8$. We will focus on proving Theorem 2.2 in the following subsections.

### 2.1 Perturbation of embedded eigenvalues and their eigenvectors

In this subsection we study the eigenvalues of $\mathcal{L}_{1}$ near $i e_{01}$. By symmetry we get also the information near $-i e_{01}$.

For our fixed nonlinear excited state $Q_{1}=Q_{1, E_{1}}$, let $H=-\Delta+V-E_{1}+$ $\lambda Q_{1}^{2}$. (H is $L_{1}$ in (2.6).) Let $\widetilde{\phi}_{0}$ denote a positive normalized ground state of $H$, with ground state energy $-\rho$ which is very close to $-e_{01}$. Hence the
bottom of the continuous spectrum of $H$, which is close to $\left|e_{1}\right|$, is less than $\rho$. We have

$$
\begin{gather*}
H Q_{1}=0, \quad H \widetilde{\phi}_{0}=-\rho \widetilde{\phi}_{0} \\
Q_{1}=n \phi_{1}+O\left(n^{3}\right), \quad \widetilde{\phi}_{0}=\phi_{0}+O\left(n^{2}\right) \tag{2.25}
\end{gather*}
$$

We want to solve the eigenvalue problem $\mathcal{L}_{1} \Phi=\omega_{*} \Phi$ with $\omega_{*}$ near $i e_{01}$. Write $\Phi=\left[\begin{array}{l}u \\ v\end{array}\right]$. The problem has the form

$$
\left[\begin{array}{cc}
0 & H \\
-\left(H+2 \lambda Q_{1}^{2}\right) & 0
\end{array}\right]\left[\begin{array}{l}
u \\
v
\end{array}\right]=\omega_{*}\left[\begin{array}{l}
u \\
v
\end{array}\right]
$$

for some $\omega_{*}$ near $i e_{01}$ and for some complex $L^{2}$-functions $u, v$. We have

$$
H v=\omega_{*} u, \quad\left(H+2 \lambda Q_{1}^{2}\right) u=-\omega_{*} v
$$

Thus $H\left(H+2 \lambda Q_{1}^{2}\right) u=-\omega_{*}^{2} u$. Suppose $\omega_{*}=i \kappa+\gamma$ with $\gamma>0$. Since $\operatorname{Im}\left(-\omega_{*}^{2}\right)<0$ and $H$ is real, it is more convenient to solve $H\left(H+2 \lambda Q_{1}^{2}\right) \bar{u}=$ $-\bar{\omega}_{*}^{2} \bar{u}$ instead. If we decompose $\bar{u}=a \widetilde{\phi}_{0}+b Q_{1}+h$ with $h \in \mathbf{H}_{\mathrm{c}}(H)$, we find $b=0$ since $\bar{u} \in$ Image $H$. Since $a \neq 0$, we may assume $\bar{u}=\widetilde{\phi}_{0}+h$. Let $A=H 2 \lambda Q_{1}^{2}$ and $z=-\bar{\omega}_{*}^{2} \sim \rho^{2}$. (A small $\operatorname{Re} \omega_{*}>0$ corresponds to a small $\operatorname{Im} z>0$.) We have

$$
\left(H^{2}+A\right)\left(\widetilde{\phi}_{0}+h\right)=z\left(\widetilde{\phi}_{0}+h\right),
$$

i.e.,

$$
\begin{equation*}
z \widetilde{\phi}_{0}+z h=\rho^{2} \widetilde{\phi}_{0}+A \widetilde{\phi}_{0}+\left(H^{2}+A\right) h . \tag{2.26}
\end{equation*}
$$

Taking projection $\mathbf{P}_{\mathrm{c}}=\mathbf{P}_{\mathrm{c}}(H)$, we get

$$
z h=\mathbf{P}_{\mathrm{c}} A \widetilde{\phi}_{0}+\left(H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right) h
$$

If $\operatorname{Im} z \neq 0$,

$$
\begin{equation*}
h=-\left(H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}-z\right)^{-1} \mathbf{P}_{\mathrm{c}} A \widetilde{\phi}_{0} \tag{2.27}
\end{equation*}
$$

On the other hand, if $\operatorname{Im} z=0$, then $h$ is generically not in $L^{2}$. We will assume $\operatorname{Im} z \neq 0$ in this subsection. The non-existence of eigenvalues with $\operatorname{Im} z=0$ will be proved in next subsection.

Taking inner product of (2.26) with $\widetilde{\phi}_{0}$, we get

$$
z=\rho^{2}+\left(\widetilde{\phi}_{0}, A \widetilde{\phi}_{0}\right)+\left(\widetilde{\phi}_{0}, A h\right)
$$

Substituting (2.27), we get

$$
\begin{equation*}
z=\rho^{2}+\left(\widetilde{\phi}_{0}, A \widetilde{\phi}_{0}\right)-\left(\widetilde{\phi}_{0}, A\left(H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}-z\right)^{-1} \mathbf{P}_{\mathrm{c}} A \widetilde{\phi}_{0}\right) \tag{2.28}
\end{equation*}
$$

If $A$ is self-adjoint, then the signs of the imaginary parts of the two sides of the equation are different. Thus $z$ is real and generically $h$ is not in $L^{2}$. In our case, $A=H 2 \lambda Q_{1}^{2}$ is not self-adjoint. Recall $H \widetilde{\phi}_{0}=-\rho \widetilde{\phi}_{0}$. Equation (2.28) becomes the following fixed point problem

$$
\begin{equation*}
z=f(z) \tag{2.29}
\end{equation*}
$$

where

$$
\begin{align*}
f(z)=\rho^{2}- & \rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right) \\
& +\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2},\left(H^{2}+H \mathbf{P}_{\mathrm{c}} 2 \lambda Q_{1}^{2} \mathbf{P}_{\mathrm{c}}-z\right)^{-1} H \mathbf{P}_{\mathrm{c}} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right) \tag{2.30}
\end{align*}
$$

Let

$$
\begin{equation*}
R(z)=\left(H^{2}-z\right)^{-1} H=\frac{1}{2(H-\sqrt{z})}+\frac{1}{2(H+\sqrt{z})}, \tag{2.31}
\end{equation*}
$$

where $\sqrt{z}$ takes the branch $\operatorname{Im} \sqrt{z}>0$ if $\operatorname{Im} z>0$. We can expand $f(z)$ as

$$
\begin{equation*}
f(z)=\rho^{2}-\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)+\sum_{k=1}^{\infty} \rho 2 \lambda\left(\widetilde{\phi}_{0} Q_{1},\left[2 \lambda Q_{1} \mathbf{P}_{\mathrm{c}} R(z) \mathbf{P}_{\mathrm{c}} Q_{1}\right]^{k} Q_{1} \widetilde{\phi}_{0}\right) . \tag{2.32}
\end{equation*}
$$

Let

$$
\begin{aligned}
& z_{0}=\rho^{2}-\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right) \\
& z_{1}=z_{0}+4 \rho \lambda^{2}\left(\widetilde{\phi}_{0} Q_{1}^{2}, R\left(z_{0}+0 i\right) \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right)
\end{aligned}
$$

We have $\left|z_{1}-z_{0}\right| \leq C n^{4}$ from its explicit form, (cf. (2.35) of Lemma 2.3 below). We also have, by (2.25) and (1.5)

$$
\operatorname{Im} z_{1}=\operatorname{Im} 4 \rho \lambda^{2}\left(\widetilde{\phi}_{0} Q_{1}^{2}, \frac{1}{2(H-0 i)} \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right) \geq \frac{7}{4} e_{01} \lambda^{2} n^{4} \gamma_{0}+O\left(n^{6}\right)>0
$$

Let $r_{0}=\frac{1}{4}\left(\left(e_{01}\right)^{2}-\left|e_{1}\right|^{2}\right)$ be a length of order 1. Denote the regions

$$
\begin{equation*}
G=\left\{x+i y:\left|x-\rho^{2}\right|<r_{0}, 0<y<r_{0}\right\}, \tag{2.33}
\end{equation*}
$$

$$
\begin{equation*}
D=B\left(z_{1}, n^{5}\right)=\left\{z:\left|z-z_{1}\right| \leq n^{5}\right\} . \tag{2.34}
\end{equation*}
$$

Clearly $z_{0} \in G, z_{1} \in D \subset G$. Also observe that the real part of all points in $G$ are greater than $\left|E_{1}\right|^{2}$. We will solve the fixed point problem $(2.29)$ in $D$. We need the following two lemmas.

Lemma 2.3 Fix $r>3$. There is a constant $C_{1}>0$ such that, for all $z \in G$,

$$
\begin{gather*}
\left\|\langle x\rangle^{-r} \mathbf{P}_{\mathrm{c}} R(z) \mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r}\right\|_{\left(L^{2}, L^{2}\right)} \leq C_{1},  \tag{2.35}\\
\left\|\langle x\rangle^{-r} \mathbf{P}_{\mathrm{c}} \frac{d}{d z} R(z) \mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r}\right\|_{\left(L^{2}, L^{2}\right)} \leq C_{1}(\operatorname{Im} z)^{-1 / 2} . \tag{2.36}
\end{gather*}
$$

Here $\mathbf{P}_{\mathbf{c}}=\mathbf{P}_{\mathrm{c}}(H)$. Moreover, for $w_{1}, w_{2} \in G$,

$$
\begin{align*}
\|\langle x\rangle^{-r} \mathbf{P}_{\mathrm{c}}\left[R\left(w_{1}\right)-R\left(w_{2}\right)\right] & \mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r} \|_{\left(L^{2}, L^{2}\right)} \\
& \leq C_{1}\left(\max \left(\operatorname{Im} w_{1}, \operatorname{Im} w_{2}\right)\right)^{-1 / 2}\left|w_{1}-w_{2}\right| \tag{2.37}
\end{align*}
$$

Proof. We have

$$
\begin{equation*}
R(z)=\left(H^{2}-z\right)^{-1} H=\frac{1}{2(H-\sqrt{z})}+\frac{1}{2(H+\sqrt{z})} \tag{2.38}
\end{equation*}
$$

Since $\frac{1}{2(H+\sqrt{z})}$ is regular in a neighborhood of $\bar{G}$, it is sufficient to prove the lemma with $R(z)$ replaced by $R_{1}(z):=(H-\sqrt{z})^{-1}$.

That $\left\|\langle x\rangle^{-r} \mathbf{P}_{\mathrm{c}} R_{1}(z) \mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r}\right\|_{\left(L^{2}, L^{2}\right)} \leq C_{1}$ is well-known, see e.g. []], [7]. The estimate (2.36) will follow from (2.37) by taking limit. We now show (2.37) for $R_{1}(z)$. For any $w_{1}, w_{2} \in G$, we have $\left|\sqrt{w_{1}}-\sqrt{w_{2}}\right| \leq\left|w_{1}-w_{2}\right|$. Write $\sqrt{w_{1}}=a_{1}+i b_{1}$ and $\sqrt{w_{2}}=a_{2}+i b_{2}$. We may assume $0<b_{1}<b_{2}$. Let $w_{3} \in G$ be the unique number such that $\sqrt{w_{3}}=a_{1}+i b_{2}$.

For any $u, v \in L^{2}$ with $\|u\|_{2}=\|v\|_{2}=1$, let $u_{1}=\mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r} u$, $v_{1}=$ $\mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r} v$. We have $u_{1}, v_{1} \in L^{1} \cap L^{2}\left(\mathbb{R}^{3}\right)$ and

$$
\begin{aligned}
& \left|\left(u,\langle x\rangle^{-r} \mathbf{P}_{\mathrm{c}}\left[R_{1}\left(w_{1}\right)-R_{1}\left(w_{3}\right)\right] \mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r} v\right)\right| \\
& =\left|\int_{0}^{\infty}\left(u_{1}, e^{-i t\left(H-a_{1}\right)} v_{1}\right)\left(e^{-b_{1} t}-e^{-b_{2} t}\right) d t\right| \\
& \leq \int_{0}^{\infty}(1+t)^{-3 / 2}\left(e^{-b_{1} t}-e^{-b_{2} t}\right) d t \leq C\left(b_{2}^{1 / 2}-b_{1}^{1 / 2}\right) \leq b_{2}^{-1 / 2}\left(b_{2}-b_{1}\right) .
\end{aligned}
$$

Here we have used the decay estimate for $e^{-i t H}$ with $H=-\Delta+V-E_{1}-\lambda Q_{1}^{2}$, namely,

$$
\begin{equation*}
\left\|e^{-i t H} \mathbf{P}_{\mathrm{c}} \phi\right\|_{L^{\infty}} \leq C|t|^{-3 / 2}\|\phi\|_{L^{1}} \tag{2.39}
\end{equation*}
$$

under our Assumption A1. See [7, 8, 24].
We also have

$$
\begin{aligned}
& \left|\left(u,\langle x\rangle^{-r} \mathbf{P}_{\mathrm{c}}\left[R_{1}\left(w_{3}\right)-R_{1}\left(w_{2}\right)\right] \mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r} v\right)\right| \\
& =\left|\int_{0}^{\infty}\left(u_{1}, e^{-i t\left(H-s_{2}-i b_{2}\right)} v_{1}\right)\left(e^{i\left(a_{1}-a_{2}\right) t}-1\right) d t\right| \\
& \leq \int_{0}^{\infty}(1+t)^{-3 / 2} e^{-b_{2} t}\left|e^{i\left(a_{1}-a_{2}\right) t}-1\right| d t \leq C b_{2}^{-1 / 2}\left|a_{1}-a_{2}\right| .
\end{aligned}
$$

Since $\left|a_{1}-a_{2}\right|+\left|b_{1}-b_{2}\right| \sim\left|\sqrt{w_{1}}-\sqrt{w_{2}}\right| \leq\left|w_{1}-w_{2}\right|$, we conclude

$$
\left|\left(u,\langle x\rangle^{-r} \mathbf{P}_{\mathrm{c}}\left[R_{1}\left(w_{1}\right)-R_{1}\left(w_{2}\right)\right] \mathbf{P}_{\mathrm{c}}\langle x\rangle^{-r} v\right)\right| \leq C b_{2}^{-1 / 2}\left|w_{1}-w_{2}\right|
$$

Hence we have (2.37).
Q.E.D.

Lemma 2.4 Recall the regions $G$ and $D$ are defined in (2.33)-(2.34).
(1) $f(z)$ defined by (2.30) is well-defined and analytic in $G$.
(2) $\left|f^{\prime}(z)\right| \leq C n^{4}(\operatorname{Im} z)^{-1 / 2}$ in $G$ and $\left|f^{\prime}(z)\right| \leq 1 / 2$ in $D$.
(3) for $w_{1}, w_{2} \in G$,

$$
\left|f\left(w_{1}\right)-f\left(w_{2}\right)\right| \leq C n^{4}\left(\max \left(\operatorname{Im} w_{1}, \operatorname{Im} w_{2}\right)\right)^{-1 / 2}\left|w_{1}-w_{2}\right|
$$

(4) $f(z)$ maps $D$ into $D$.

Proof. By (2.35), the expansion (2.32) can be bounded by

$$
|f(z)| \leq C+C C_{1} n^{4}+C C_{1}^{2} n^{6}+\cdots
$$

and thus converges. Since every term in (2.32) is analytic, $f(z)$ is welldefined and analytic. We also get the estimates in (2). To prove (3), let $b=\max \left(\operatorname{Im} w_{1}, \operatorname{Im} w_{2}\right)$. Then from (2.35) $-(2.37)$,

$$
\left|f\left(w_{1}\right)-f\left(w_{2}\right)\right| \leq \sum_{k=1}^{\infty} C k C_{1}^{k} n^{2 k+2} b^{-1 / 2}\left|w_{1}-w_{2}\right| \leq C n^{4} b^{-1 / 2}\left|w_{1}-w_{2}\right|
$$

It remains to show (4). We first estimate $\left|f\left(z_{1}\right)-z_{1}\right|$. Write $z_{1}=z_{0}+a+b i$. Recall that $|a|<C n^{4}$ and $\frac{1}{4} \lambda^{2} \gamma_{0} n^{4}<|b|<C n^{4}$. Using (2.37) and (2.35) we have

$$
\begin{aligned}
\left|f\left(z_{1}\right)-z_{1}\right|= & \left(\widetilde{\phi}_{0} Q_{1}^{2},\left[R\left(z_{1}\right)-R\left(z_{0}+0 i\right)\right] \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right) \\
& +\sum_{k=2}^{\infty}\left(\widetilde{\phi}_{0} Q_{1},\left[Q_{1} \mathbf{P}_{\mathrm{c}} R\left(z_{1}\right) \mathbf{P}_{\mathrm{c}} Q_{1}\right]^{k} Q_{1} \widetilde{\phi}_{0}\right) \\
\leq & C n^{4} b^{-1 / 2}(|a|+|b|)+C C_{1}^{2} n^{6}+C C_{1}^{3} n^{8}+\cdots \leq C n^{6}
\end{aligned}
$$

Hence $\left|f\left(z_{1}\right)-z_{1}\right| \leq C n^{6}$. For any $z \in D$, we have

$$
\left|f(z)-z_{1}\right| \leq\left|f(z)-f\left(z_{1}\right)\right|+\left|f\left(z_{1}\right)-z_{1}\right| \leq \frac{1}{2}\left|z-z_{1}\right|+C n^{6} \leq\left|z-z_{1}\right|
$$

Hence $f(z) \in D$. This proves (4).
Q.E.D.

We are ready to solve (2.29) in $G$. By Lemma 2.4 (1), (2) and (4), the map $f \rightarrow f(z)$ is a contraction mapping in $D$ and hence has a unique fixed point $z_{*}$ in $D$. By (3), for any $z \in G$ we have $\left|f(z)-f\left(z_{*}\right)\right| \leq C n^{4}\left(\operatorname{Im} z_{*}\right)^{-1 / 2}\left|z-z_{*}\right| \leq$ $\frac{1}{2}\left|z-z_{*}\right|$. Hence there is no other fixed point of $f(z)$ in $G$.

By symmetry, there is another unique fixed point with negative imaginary part. Moreover, they have the size indicated in Theorem 2.2. We will prove in Subsections 2.2 and 2.3 that $\omega_{*}$ does not admit generalized eigenvectors and that there is no purely imaginary eigenvalue near $i e_{01}$, i.e., there is no embedded eigenvalue. Hence $\omega_{*}$, and $-\bar{\omega}_{*}$ are simple and are the only eigenvalues near $i e_{01}$.

We now look more carefully on $z_{*}$ and $u_{*}$, where $u_{*}$ denotes the unique solution of $H\left(H+2 \lambda Q_{1}^{2}\right) u_{*}=-\omega_{*}^{2} u_{*}$ with the form $u_{*}=\widetilde{\phi}_{0}+\bar{h}_{*}$.

Recall $\left|z_{1}-z_{*}\right| \leq n^{5}$ and

$$
z_{1}=\rho^{2}-\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)+4 \rho \lambda^{2}\left(\widetilde{\phi}_{0} Q_{1}^{2}, R\left(z_{0}+0 i\right) \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right)
$$

where $z_{0}=\rho^{2}-\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)$. Hence

$$
\begin{aligned}
\sqrt{z_{*}}= & \sqrt{z_{1}}+O\left(n^{5}\right) \\
= & \rho-\left(\widetilde{\phi}_{0} \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)+2 \lambda^{2}\left(\widetilde{\phi}_{0} Q_{1}^{2}, R\left(z_{0}+0 i\right) \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right) \\
& +\frac{1}{4 \rho}\left(\widetilde{\phi}_{0} \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)^{2}+O\left(n^{5}\right) .
\end{aligned}
$$

Since $z_{*}=-\bar{\omega}_{*}^{2}$, we have $\bar{\omega}_{*}=i \sqrt{z_{*}}$. Thus if we write $\omega_{*}=i \kappa+\gamma$, then

$$
\begin{align*}
& \kappa=\rho-\left(\widetilde{\phi}_{0} \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)+\frac{1}{4 \rho}\left(\widetilde{\phi}_{0} \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)^{2} \\
&+\operatorname{Re} 2 \lambda^{2}\left(\widetilde{\phi}_{0} Q_{1}^{2}, R\left(z_{0}+0 i\right) \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right)+O\left(n^{5}\right),  \tag{2.40}\\
& \gamma=- \operatorname{Im} 2 \lambda^{2}\left(\widetilde{\phi}_{0} Q_{1}^{2}, R\left(z_{0}+0 i\right) \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right)+O\left(n^{5}\right) .
\end{align*}
$$

By (2.31), (2.25), and expansion into series,

$$
\begin{align*}
\gamma & =\operatorname{Im} \lambda^{2}\left(\widetilde{\phi}_{0} Q_{1}^{2},\left(H-\sqrt{z_{0}}-0 i\right) \mathbf{P}_{\mathrm{c}} Q_{1}^{2} \widetilde{\phi}_{0}\right)+O\left(n^{5}\right) \\
& =\operatorname{Im} \lambda^{2} n^{4}\left(\phi_{0} \phi_{1}^{2}, \frac{1}{-\Delta+V-E_{1}-\sqrt{z_{0}}-0 i} \mathbf{P}_{\mathrm{c}} \phi_{1}^{2} \phi_{0}\right)+O\left(n^{5}\right) . \tag{2.41}
\end{align*}
$$

By (1.5), $\gamma \geq \lambda^{2} n^{4} \gamma_{0}+O\left(n^{5}\right)$.
By (2.27) and $A=H 2 \lambda Q_{1}^{2}$, we have

$$
\begin{equation*}
h_{*}=-\left(H^{2}+\mathbf{P}_{\mathrm{c}} H 2 \lambda Q_{1}^{2} \mathbf{P}_{\mathrm{c}}-z_{*}\right)^{-1} H \mathbf{P}_{\mathrm{c}} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0} \tag{2.42}
\end{equation*}
$$

where $\mathbf{P}_{\mathrm{c}}=\mathbf{P}_{\mathrm{c}}(H)$. We now expand the resolvent on the right side as in (2.32). Then by Lemma 2.3, we can derive $|(\phi, h)| \leq C n^{2}\left\|\langle x\rangle^{r} \phi\right\|_{2}$, for any $r>3$.

We now show that $h_{*}$ is bounded in $L^{2}$ with a bound uniform in $n$. Recall $\sqrt{z_{*}}=\kappa+i \gamma$ with $\kappa \sim e_{01}, \gamma>\frac{1}{2} \lambda^{2} \gamma_{0} n^{4}$. Since $Q_{1}=n \phi_{1}+O\left(n^{3}\right)$, by expansion and (2.25) we have

$$
\begin{align*}
h_{*} & =-\left(H^{2}-z_{*}\right)^{-1} H \mathbf{P}_{\mathrm{c}}(H) 2 \lambda \phi_{0} Q_{1}^{2}+O\left(n^{2}\right) \\
& =-\left(H-\sqrt{z_{*}}\right)^{-1} \mathbf{P}_{\mathrm{c}}(H) \lambda \phi_{0} Q_{1}^{2}+O\left(n^{2}\right) \\
& =-\frac{1}{-\Delta+V+s-\gamma i} \mathbf{P}_{\mathrm{c}}\left(H_{0}\right) \lambda \phi_{0} Q_{1}^{2}+O\left(n^{2}\right), \tag{2.43}
\end{align*}
$$

where $s=-E_{1}-\kappa=e_{0}-2 e_{1}+O\left(n^{2}\right)$. Here we have used the fact that

$$
\mathbf{P}_{\mathrm{c}}(H) \phi=\mathbf{P}_{\mathrm{c}}\left(H_{0}\right) \phi+n^{2} \sum_{k=1}^{N}\left(\psi_{k}^{*}, \phi\right) \psi_{k}
$$

for some local functions $\phi_{k}, \phi_{k}^{*}$ of order one. We will show that the leading term on the right of (2.43) is of order one in $L^{2}$. It follows from the same proof that $O\left(n^{2}\right)$ on the right is also in $L^{2}$ sense.

Observe that, for $f(p) \in L^{2}$ with $\|f\|_{2} \leq 1$,

$$
\begin{aligned}
& \int \frac{1}{p^{2}-s+\gamma i} f(p) \cdot \frac{1}{p^{2}-s-\gamma i} \bar{f}(p) d p \\
& =\int|f(p)|^{2} \frac{1}{\left(p^{2}-s\right)^{2}+\gamma^{2}} d p \\
& \leq C+C \int_{\sqrt{s} / 2}^{3 \sqrt{s} / 2} \frac{1}{(|r-\sqrt{s}|+\gamma)^{2}} d r \\
& =C+2 C \int_{0}^{\sqrt{s} / 2} \frac{1}{(r+\gamma)^{2}} d r \leq C+C / \gamma .
\end{aligned}
$$

Using wave operator for $-\Delta+V$, we have similar estimates if $p^{2}$ is replaced by $-\Delta+V$. Since $\lambda \phi_{0} Q_{1}^{2}=O\left(n^{2}\right)$,

$$
\left(h_{*}, h_{*}\right) \leq C n^{2} \gamma^{-1} n^{2} \leq C,
$$

where $C$ is independent of $n$. Since $u=\widetilde{\phi}_{0}+\bar{h}=\phi_{0}+\bar{h}+O\left(n^{2}\right)$, we have obtained the $u$ part of the estimates $\|\Phi\|_{L^{2}} \leq C$ and (2.17). The corresponding estimate for $v$ can be proved using $v=\left(-L_{+}\right) u / \omega_{*}$.

### 2.2 Resolvent estimates

In this subsection we study the resolvent $R(w)=\left(w-\mathcal{L}_{1}\right)^{-1}$. Note that $R(w)$ had a different meaning in the previous subsection. We will prove resolvent estimates along the continuous spectrum $\Sigma_{c}$ and determine all eigenvalues outside of $\Sigma_{c}$.

Let $L_{r}^{2}$ denote the weighted $L^{2}$ spaces for $r \in \mathbb{R}$ :

$$
L_{r}^{2}=\left\{f:\left(1+x^{2}\right)^{r / 2} f(x) \in L^{2}\left(\mathbb{R}^{3}\right)\right\}
$$

We will prove the following lemma.

Lemma 2.5 Let $R(w)=\left(w-\mathcal{L}_{1}\right)^{-1}$ be the resolvent of $\mathcal{L}_{1}$. Let $\mathbf{B}=$ $B\left(L_{r}^{2}, L_{-r}^{2}\right)$, the space of bounded operators from $L_{r}^{2}$ to $L_{-r}^{2}$ with $r>3$. Recall $\omega_{*}=i \kappa+\gamma$. For $\tau \geq\left|E_{1}\right|$ we have

$$
\begin{equation*}
\|R(i \tau \pm 0)\|_{\mathbf{B}}+\|R(-i \tau \pm 0)\|_{\mathbf{B}} \leq C(1+\tau)^{-1 / 2}+C\left(|\tau-\kappa|+n^{4}\right)^{-1} \tag{2.44}
\end{equation*}
$$

The constant $C$ is independent of $n$. We also have

$$
\begin{equation*}
\left\|R^{(k)}(i \tau \pm 0)\right\|_{\mathbf{B}}+\left\|R^{(k)}(-i \tau \pm 0)\right\|_{\mathbf{B}} \leq C(1+\tau)^{-(1+k) / 2}+C\left(|\tau-\kappa|+n^{4}\right)^{-1} . \tag{2.45}
\end{equation*}
$$

for derivatives, where $k=1,2$.

We first consider $R_{0}(w)=\left(w-J H_{1}\right)^{-1}$. Recall $H_{1}=-\Delta+V-E_{1}$. Since

$$
\begin{align*}
\left(w-J H_{1}\right)^{-1} & =\left[\begin{array}{cc}
w & -H_{1} \\
H_{1} & w
\end{array}\right]^{-1}=\frac{1}{H_{1}^{2}+w^{2}}\left[\begin{array}{cc}
w & H_{1} \\
-H_{1} & w
\end{array}\right] \\
& =\frac{1}{2}\left[\begin{array}{cc}
-i & 1 \\
-1 & -i
\end{array}\right]\left(H_{1}-i w\right)^{-1}+\frac{1}{2}\left[\begin{array}{cc}
i & 1 \\
-1 & i
\end{array}\right]\left(H_{1}+i w\right)^{-1}, \tag{2.46}
\end{align*}
$$

the estimates of $R_{0}(w)$ can be derived from that of $\left(H_{1}-i w\right)^{-1}$ and $\left(H_{1}+\right.$ $i w)^{-1}$. By assumption, the bottom of the continuous spectrum of $H_{1},-E_{1}$, is not an eigenvalue nor a resonance of $H_{1}$. Hence $\left(H_{1}-z\right)^{-1}$ is uniformly bounded in $\mathbf{B}$ for $z$ away from $e_{0}-E_{1}$ and $e_{1}-E_{1}$, see [7]. By (2.46) and (2.4), $R_{0}(w)$ is uniformly bounded in $\mathbf{B}$ for $w$ with $\operatorname{dist}\left(w, \Sigma_{p}\right) \geq n$, where $\Sigma_{p}=\left\{0, i e_{01},-i e_{01}\right\}$.

Write

$$
\mathcal{L}_{1}=J H_{1}+W, \quad W=\left[\begin{array}{cc}
0 & \lambda Q_{1}^{2} \\
-3 \lambda Q_{1}^{2} & 0
\end{array}\right] .
$$

For $R(w)=\left(w-\mathcal{L}_{1}\right)^{-1}$ we have

$$
\begin{equation*}
\left.R(w)=\left(1-R_{0}(w) W\right)^{-1} R_{0}(w)=\sum_{k=0}^{\infty}\left[R_{0}(w) W\right)\right]^{k} R_{0}(w) \tag{2.47}
\end{equation*}
$$

Since $R_{0}(w)$ is uniformly bounded in $\mathbf{B}$ for $w$ with $\operatorname{dist}\left(w, \Sigma_{p}\right)>n$, and $W$ is localized and small, (2.47) converges and $\left(w-\mathcal{L}_{1}\right)^{-1}$ is uniformly bounded in $\mathbf{B}$ for $w$ with $\operatorname{dist}\left(w, \Sigma_{p}\right)>n$ and we have

$$
\begin{equation*}
\|R(w)\|_{B} \leq C \operatorname{dist}\left(w, \Sigma_{p}\right)^{-1}, \quad\left(n \leq \operatorname{dist}\left(w, \Sigma_{p}\right) \leq 1\right) \tag{2.48}
\end{equation*}
$$

Recall $\Sigma_{c}=\left\{i s:|s| \geq\left|E_{1}\right|\right\}$ is the continuous spectrum of $J H_{1}$ and $\mathcal{L}_{1}$. For $w$ in the region

$$
\begin{equation*}
\left\{w: \operatorname{dist}\left(w, \Sigma_{p}\right) \geq n, w \notin \Sigma_{c}\right\} \tag{2.49}
\end{equation*}
$$

we have

$$
\left\|R_{0}(w)\right\|_{\left(L^{2}, L^{2}\right)} \leq C \operatorname{dist}\left(w, \Sigma_{c}\right)^{-1}
$$

By (2.47), and because $W$ is localized and small,

$$
\begin{aligned}
& \|R(w)\|_{\left(L^{2}, L^{2}\right)} \leq\left\|R_{0}(w)\right\|_{\left(L^{2}, L^{2}\right)}+ \\
& \quad+\sum_{k=1}^{\infty} C\left\|R_{0}(w)\right\|_{\left(L^{2}, L^{2}\right)}\left\{C n^{2}\left\|R_{0}(w)\right\|_{\mathbf{B}}\right\}^{k-1}\left\|R_{0}(w)\right\|_{\left(L^{2}, L^{2}\right)} \\
& \leq C \operatorname{dist}\left(w, \Sigma_{c}\right)^{-1}+C \operatorname{dist}\left(w, \Sigma_{c}\right)^{-2} .
\end{aligned}
$$

Hence $R(w)$ is uniformly bounded in $\left(L^{2}, L^{2}\right)$ in a neighborhood of $w$. In particular, there is no eigenvalue of $\mathcal{L}_{1}$ in the above region (2.49). Note that this region includes a neighborhood of the bottom of the continuous spectrum $\Sigma_{c}, \pm i E_{1}$, except those in $\Sigma_{c}$. Hence the eigenvalues can occur only in $\left\{w: \operatorname{dist}\left(w, \Sigma_{p}\right)<n\right\}$ or $\Sigma_{c}$.

The circle $\{w:|w|=\sqrt{n}\}$ is in the resolvent set of $\mathcal{L}_{1}$. By 12] Theorem XII.6, the Cauchy integral

$$
P=\frac{1}{2 \pi i} \oint_{|w|=\sqrt{n}}\left(w-\mathcal{L}_{1}\right)^{-1} d w
$$

gives the $L^{2}$-projection onto the generalized eigenspaces with eigenvalues inside the disk $\{w:|w|<\sqrt{n}\}$. Moreover, the dimension of $P$ is an upper bound for the sum of the dimensions of those eigenspaces. However, since the
projection $P_{0}=(2 \pi i)^{-1} \oint_{|w|=\sqrt{n}} R_{0}(w) d w$ has dimension 2 (see (2.3)-(2.4)), and

$$
P-P_{0}=\frac{1}{2 \pi i} \oint_{|w|=\sqrt{n}} \sum_{k=1}^{\infty}\left[R_{0}(w) W\right]^{k} R_{0}(w) d w
$$

is convergent and bounded in $\left(L^{2}, L^{2}\right)$ by

$$
\begin{aligned}
& \leq C\left\|R_{0}(w)\right\|_{\left(L^{2}, L^{2}\right)} n^{2} \sum_{k=0}^{\infty}\left(C n^{2}\left\|R_{0}(w)\right\|_{\mathbf{B}}\right)^{k}\left\|R_{0}(w)\right\|_{\left(L^{2}, L^{2}\right)} \\
& \leq C n^{-1 / 2} C n^{2} n^{-1 / 2}=C n
\end{aligned}
$$

(here we have used (2.48)), the dimension of $P$ is also two. Since we already have two generalized eigenvectors $\left[\begin{array}{c}0 \\ Q_{1}\end{array}\right]$ and $\left[\begin{array}{c}R_{1} \\ 0\end{array}\right]$ with eigenvalue 0 , we have obtained all generalized eigenvectors with eigenvalues in the disk $|w|<\sqrt{n}$. Together with the results in $\S 2.1$, we have obtained all eigenvalues outside of $\Sigma_{c}: 0, \pm \omega_{*}$ and $\pm \overline{\omega_{*}}$.

We next study $R(w)=\left(w-\mathcal{L}_{1}\right)^{-1}$ for $w$ near $\pm i e_{01}:\left|w-i e_{01}\right|<n$ or $\left|w+i e_{01}\right|<n$. Let us assume $w=i \tau-\varepsilon$ with $\tau, \varepsilon>0$, thus $-w^{2}$ lies in $G$ (defined in (2.33)). The other cases are similar. Let $\left[\begin{array}{c}f \\ g\end{array}\right] \in \mathbb{C} L^{2}$. We want to solve the equation

$$
\left(w-\mathcal{L}_{1}\right)\left[\begin{array}{l}
u  \tag{2.50}\\
v
\end{array}\right]=\left[\begin{array}{l}
f \\
g
\end{array}\right]
$$

We have

$$
w u-H v=f, \quad w v+\left(H+2 \lambda Q_{1}^{2}\right) u=g .
$$

Cancelling $v$, we get (recall $A=H 2 \lambda Q_{1}^{2}$ )

$$
w^{2} u+\left(H^{2}+A\right) u=F, \quad F=w f+H g .
$$

Write $u=\alpha \widetilde{\phi}_{0}+\beta \widehat{Q}_{1}+\eta$ with $\eta \in \mathbf{H}_{\mathrm{c}}(H)$ and $\widehat{Q}_{1}=Q_{1} /\left\|Q_{1}\right\|_{2}$. Also denote $\zeta=\alpha \widetilde{\phi}_{0}+\beta \widehat{Q}_{1}$. We have

$$
\begin{aligned}
\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A\right) \eta & =\mathbf{P}_{\mathrm{c}} F-\mathbf{P}_{\mathrm{c}} A \zeta \\
\left(w^{2}+H^{2}+\mathbf{P}^{\perp} A\right) \zeta & =\mathbf{P}^{\perp} F-\mathbf{P}^{\perp} A \eta
\end{aligned}
$$

Here $\mathbf{P}_{\mathrm{c}}=\mathbf{P}_{\mathrm{c}}(H)$ and $\mathbf{P}^{\perp}=1-\mathbf{P}_{\mathrm{c}}$. Solving $\eta$ in terms of $\zeta$, we get

$$
\begin{equation*}
\eta=\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1}\left(\mathbf{P}_{\mathrm{c}} F-\mathbf{P}_{\mathrm{c}} A \zeta\right) \tag{2.51}
\end{equation*}
$$

Substituting the above into the $\zeta$ equation we get

$$
\begin{gather*}
\left(w^{2}+H^{2}+\mathbf{P}^{\perp} A-\mathbf{P}^{\perp} A\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1} \mathbf{P}_{\mathrm{c}} A\right) \zeta=\widetilde{F}  \tag{2.52}\\
\widetilde{F}=\mathbf{P}^{\perp} F-\mathbf{P}^{\perp} A\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1} \mathbf{P}_{\mathrm{c}} F
\end{gather*}
$$

Using $\widetilde{\phi}_{0}$ and $\widehat{Q}_{1}$ as basis, we can put (2.52) into matrix form

$$
\left[\begin{array}{cc}
a & b  \tag{2.53}\\
0 & w^{2}
\end{array}\right]\left[\begin{array}{l}
\alpha \\
\beta
\end{array}\right]=\left[\begin{array}{c}
\left(\widetilde{\phi}_{0}, \widetilde{F}\right) \\
\left(\widehat{Q}_{1}, \widetilde{F}\right)
\end{array}\right],
$$

where (recall $\left.H \widetilde{\phi}_{0}=-\rho \widetilde{\phi}_{0}, H \widehat{Q}_{1}=0\right)$

$$
a=w^{2}+\rho^{2}-\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)+\rho\left(\widetilde{\phi}_{0} 2 \lambda Q^{2},\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1} H \mathbf{P}_{\mathrm{c}} 2 \lambda Q_{1}^{2} \widetilde{\phi}_{0}\right)
$$

$$
b=-\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2} \widehat{Q}_{1}\right)+\rho\left(\widetilde{\phi}_{0} 2 \lambda Q^{2},\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1} H \mathbf{P}_{\mathrm{c}} 2 \lambda Q_{1}^{2} \widehat{Q}_{1}\right)
$$

Thus

$$
\left[\begin{array}{l}
\alpha  \tag{2.54}\\
\beta
\end{array}\right]=\left[\begin{array}{cc}
1 / a & -b /\left(a w^{2}\right) \\
0 & w^{-2}
\end{array}\right]\left[\begin{array}{c}
\left(\widetilde{\phi}_{0}, \widetilde{F}\right) \\
\left(\widetilde{Q}_{1}, \widetilde{F}\right)
\end{array}\right] .
$$

We now consider the case when $w$ is near the continuous spectrum $\Sigma_{c}$. We will assume $w=i \tau-\varepsilon$ with $\left|\tau-e_{01}\right|<n$ and $\varepsilon>0$ much smaller. The case $w=i \tau+\varepsilon$ follows similarly.

Let $z=-w^{2}$. It follows that $z \in G$ and $\operatorname{Re} z>0$ is small. The idea of what follows is to compare $z$ with $z_{*}$, the fixed point found in $\S 2.1$.

We have $a=-z+f(z)=\left(z_{*}-z\right)+\left(-f\left(z_{*}\right)+f(z)\right)$. Using Lemma 2.4 (3) with $w_{1}=z$ and $w_{2}=z_{*}$, we have

$$
|a| \geq\left|z-z_{*}\right|-\left|-f\left(z_{*}\right)+f(z)\right| \geq \frac{1}{2}\left|z-z_{*}\right|=\frac{1}{2}\left|w^{2}-\bar{\omega}_{*}^{2}\right| \geq C\left|w-\bar{\omega}_{*}\right| .
$$

Recall $\omega_{*}=i \kappa+\gamma$ with $\gamma \sim n^{4}$. Hence $|a| \geq C\left(|\tau-\kappa|+n^{4}\right)$. Thus

$$
|\alpha|+|\beta| \leq C\left(1+|a|^{-1}\right)\|\widetilde{F}\|_{L^{2}} \leq C\left(|\tau-\kappa|+n^{4}\right)^{-1}\left(\|f\|_{L^{2}}+\|g\|_{L^{2}}\right) .
$$

By (2.51) and $F=w f+H g$,

$$
\eta=\Omega w \mathbf{P}_{\mathbf{c}} f+\Omega H \mathbf{P}_{\mathbf{c}} g-\Omega w \mathbf{P}_{\mathbf{c}} A \zeta
$$

where $\Omega=\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1}$. Substituting the above into (2.51), we can solve $\eta$ and we have

$$
\|\eta\|_{L^{2}} \leq C\left(\|f\|_{L^{2}}+\|g\|_{L^{2}}\right)+C n^{2}\left(|\tau-\kappa|+n^{4}\right)^{-1}\left(\|f\|_{L^{2}}+\|g\|_{L^{2}}\right) .
$$

We conclude, for $u=\alpha \widetilde{\phi}_{0}+\beta \widehat{Q}_{1}+\eta$,

$$
\|u\|_{L^{2}} \leq\left(C+C\left(|\tau-\kappa|+n^{4}\right)^{-1}\right)\left(\|f\|_{L^{2}}+\|g\|_{L^{2}}\right)
$$

We can estimate $v$ similarly. Thus, for $\tau \in\left(e_{01}-n, e_{01}+n\right)$,

$$
\|R(i \tau \pm 0)\|_{\mathbf{B}} \leq C+C\left(|\tau-\kappa|+n^{4}\right)^{-1}
$$

For $\tau>e_{01}+n$ and $w=i \tau+0$, using $R(w)=\left(1+R_{0}(w) W\right)^{-1} R_{0}(w)$ and the fact that $\left\|R_{0}(w)\right\|_{\mathrm{B}} \leq C(1+\tau)^{-1 / 2}$, (see [7] Theorem 9.2), we have $\|R(i \tau+0)\|_{\mathbf{B}} \leq C \tau^{-1 / 2}$.

For $\tau \in\left[\left|E_{1}\right|, e_{01}-n\right]$, by the same argument we have $\|R(i \tau+0)\|_{\mathbf{B}} \leq C$.
The derivative estimates for the resolvent is obtained by induction argument and by differentiating the relation $R\left(1+W R_{0}\right)=R_{0}$ and using the relations $\left(1+W R_{0}\right)^{-1}=1-W R$ and $\left(1+R_{0} W\right)^{-1}=1-R W$. See the proof of (7] Theorem 9.2. We have proved Lemma 2.5.

### 2.3 Nonexistence of generalized $\omega_{*}$-eigenvector

We now show that $\omega_{*}$ is simple and $\Phi$ is the only generalized $\omega_{*}$-eigenvector, i.e., there is no vectors $\phi$ with $\left(\mathcal{L}_{1}-\omega_{*}\right) \phi \neq 0$ but $\left(\mathcal{L}_{1}-\omega_{*}\right)^{k} \phi=0$ for some $k \geq 2$. Suppose the contrary, then we may find a vector $\left[\begin{array}{l}u \\ v\end{array}\right]$ with $\left(\omega_{*}-\mathcal{L}_{1}\right)\left[\begin{array}{l}u \\ v\end{array}\right]=\left[\begin{array}{l}u_{*} \\ v_{*}\end{array}\right]$. That is, $w=\omega_{*}$ and $\left[\begin{array}{l}f \\ g\end{array}\right]=\left[\begin{array}{l}u_{*} \\ v_{*}\end{array}\right]$ in the system (2.50). We have $F=w u_{*}+H v_{*}=2 \omega_{*} u_{*}$. Since $u_{*}=\widetilde{\phi}_{0}+\bar{h}_{*}$ with $\bar{h}_{*} \in \mathbf{H}_{\mathrm{c}}(H)$, we have $\left(\widehat{Q}_{1}, \widetilde{F}\right)=\left(\widehat{Q}_{1}, F\right)=\left(\widehat{Q}_{1}, 2 \omega_{*} u_{*}\right)=0$. Hence $\beta=0$. Also

$$
\begin{aligned}
\left(\widetilde{\phi}_{0}, \widetilde{F}\right) & =\left(\widetilde{\phi}_{0}, F\right)-\left(\widetilde{\phi}_{0} H 2 \lambda Q_{1}^{2}\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1} \mathbf{P}_{\mathrm{c}} F\right) \\
& =2 \omega_{*}+\rho\left(\widetilde{\phi}_{0} 2 \lambda Q_{1}^{2}\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1} 2 \omega_{*} \bar{h}_{*}\right) \\
& =2 \omega_{*}[1+\rho(\Phi, \Omega \bar{\Omega} H \Phi)]
\end{aligned}
$$

where $\Omega=\left(w^{2}+H^{2}+\mathbf{P}_{\mathrm{c}} A \mathbf{P}_{\mathrm{c}}\right)^{-1}$ and $\Phi=\mathbf{P}_{\mathrm{c}} \widetilde{\phi}_{0} 2 \lambda Q_{1}^{2}$. Since the main term in $(\Phi, \Omega \bar{\Omega} H \Phi)$,

$$
\left(\Phi,\left(w^{2}+H^{2}\right)^{-1}\left(\bar{w}^{2}+H^{2}\right)^{-1} H \Phi\right)
$$

is positive, $\left(\widetilde{\phi}_{0}, \widetilde{F}\right)$ is not zero.
On the other hand, $a=\omega_{*}^{2}+f\left(-\omega_{*}^{2}\right)=-\bar{z}_{*}+f\left(\bar{z}_{*}\right)=0$. Hence there is no solution for $\alpha$. This shows $\omega_{*}$ is simple (and so are $-\omega_{*}, \pm \bar{\omega}_{*}$ ).

Once we have an eigenvector $\Phi$ with $\mathcal{L}_{1} \Phi=\omega_{*} \Phi$ and $\omega_{*}$ complex, then we have three other eigenvalues and eigenvectors as given in (2.19). Hence we have found all eigenvalues and eigenvectors of $\mathcal{L}_{1}, ~ \mathbb{C} E_{1}$ is the combined eigenspace of $\pm \omega_{*}$ and $\pm \bar{\omega}_{*}$. It is easy to check that $\mathbf{R E} \mathbb{C E}_{1}=\mathbf{E}_{1}$. We have proved parts (1)-(3) of Theorem 2.2 in $\S 2.1$ to $\S 2.3$.

### 2.4 Nonexistence of embedded eigenvalues

In this subsection we prove that there is no embedded eigenvalue $i \tau$ with $|\tau|>\left|E_{1}\right|$. Suppose the contrary, we may assume $\tau>-E_{1}>0$ and $\mathcal{L}_{1} \psi=$ $i \tau \psi$ for some $\psi \in \mathbb{C} L^{2}$. We will derive a contradiction.

Let $H_{*}=-\Delta-E_{1}$. We can decompose

$$
\mathcal{L}_{1}=J H_{*}+A, \quad A=\left[\begin{array}{cc}
0 & V+\lambda Q_{1}^{2}  \tag{2.55}\\
-V-3 \lambda Q_{1}^{2} & 0
\end{array}\right] .
$$

Hence $\left(i \tau-J H_{*}\right) \psi=A \psi$. By the same computation of (2.46) we have

$$
\left(w-J H_{*}\right)^{-1}=\left(H_{*}-i w\right)^{-1} M_{+}+\left(H_{*}+i w\right)^{-1} M_{-},
$$

where

$$
M_{+}=\frac{1}{2}\left[\begin{array}{cc}
-i & 1 \\
-1 & -i
\end{array}\right], \quad M_{-}=\frac{1}{2}\left[\begin{array}{cc}
i & 1 \\
-1 & i
\end{array}\right] .
$$

Thus, with $w=i \tau$, we have

$$
\begin{equation*}
\psi=\left(i \tau-J H_{*}\right)^{-1} A \psi=\left(H_{*}+\tau\right)^{-1} \phi_{+}+\left(H_{*}-\tau\right)^{-1} \phi_{-}, \tag{2.56}
\end{equation*}
$$

where $\phi_{+}=M_{+} A \psi$ and $\phi_{-}=M_{-} A \psi$. By Assumption A1 on the decay of $V$ and that $\psi \in L^{2}$, both $\phi_{+}, \phi_{-} \in L_{5+\sigma}^{2}$ with $\sigma>0$. Since $-\tau$ is outside the spectrum of $H_{*}$, we have $\left(H_{*}+\tau\right)^{-1} \phi_{+} \in L_{5+\sigma}^{2}$. Let $s=E_{1}+\tau>0$. We have $H_{*}-\tau=-\Delta-s$. By assumption $\psi \in \mathbb{C} L^{2}$, hence so is $\left(H_{*}-\tau\right)^{-1} \phi_{-}$. Therefore $\left(p^{2}-s\right)^{-1} \widehat{\phi_{-}}(p) \in L^{2}$. Since $\phi_{-} \in L_{5+\sigma}^{2}, \phi_{-}$is continuous and we can conclude

$$
\begin{equation*}
\left.\widehat{\phi_{-}}(p)\right|_{|p|=\sqrt{s}}=0 \tag{2.57}
\end{equation*}
$$

We now recall [11] page 82, Theorem IX.41: Suppose $f \in L_{r}^{2}$ with $r>1 / 2$ and let $B_{s} f=\left(\left(p^{2}-s\right)^{-1} \widehat{f}\right)^{\vee}$. Suppose $\left.\hat{f}(p)\right|_{|p|=\sqrt{s}}=0$. Then for any $\varepsilon>0$, one has $B_{s} f \in L_{r-1-2 \varepsilon}^{2}$ and $\left\|B_{s} f\right\|_{L_{r-1-2 \varepsilon}^{2}} \leq C_{r, \varepsilon, s}\|f\|_{L_{r}^{2}}$ for some constant $C_{r, \varepsilon, s}$.

In our case, we have $f=\phi_{-}, \varepsilon=\sigma / 2$ and $r=5+\sigma$. We conclude $\left(H_{*}-\tau\right)^{-1} \phi_{-}=B_{s} f \in L_{4}^{2}$. Thus $\psi \in L_{4}^{2}$.

However, since $\left(z-\mathcal{L}_{1}\right) \psi=(z-i \tau) \psi$, we have $R(z) \psi=(z-i \tau)^{-1} \psi$. Thus we have

$$
\left\|(z-i \tau)^{-1} \psi\right\|_{L_{-r}^{2}} \leq C\|\psi\|_{L_{4}^{2}}
$$

where the constant $C$ remains bounded as $z \rightarrow i \tau$ by Lemma 2.5. This is clearly a contradiction. Thus $\psi$ does not exist.

### 2.5 Absence of eigenvector and resonance at bottom of continuous spectrum

In this subsection we show that there is no eigenvector and resonance at $\pm i E_{1}$. We want to show that, for $n=\left\|Q_{1, E_{1}}\right\|_{L^{2}}$ sufficiently small, the null space of $\mathcal{L}_{1} \pm i E_{1}$ in $X=L_{-r}^{2}, r>1 / 2$, is zero. Let us consider the case at $i\left|E_{1}\right|$. Suppose otherwise, we have a sequence $Q_{1, E_{1}(k)} \rightarrow 0$ and $\psi_{k}$ so that

$$
\left(\mathcal{L}_{1, E_{1}(k)}+i E_{1}(k)\right) \psi_{k}=0, \quad\left\|\psi_{k}\right\|_{X}=1
$$

As in the previous subsection, we write $\mathcal{L}_{1, E_{1}(k)}=J H_{*}+A_{k}$, where $H_{*}=$ $-\Delta-E_{1}(k)$ and $A_{k}=J V+\left[\begin{array}{cc}0 & 1 \\ -3 & 0\end{array}\right] \lambda Q_{1, E_{1}(k)}^{2}$. By (2.56) with $\tau=\left|E_{1}(k)\right|$ we have

$$
\psi_{k}=\left(i \tau-J H_{*}\right)^{-1} A_{k} \psi_{k}=(-\Delta+2 \tau)^{-1} M_{+} A_{k} \psi_{k}+(-\Delta)^{-1} M_{-} A_{k} \psi_{k}
$$

in $X$. Note that $(-\Delta+2 \tau)^{-1} M_{+} A_{k}$ and $(-\Delta)^{-1} M_{-} A_{k}$ are compact operators in $X$, with a bound uniform in $k$. Since $X$ is a reflexive Banach space, we can find a subsequence, which we still denote by $\psi_{k}$, converging weakly to some $\psi_{*} \in X$. Thus $\tau \rightarrow\left|e_{1}\right|,(-\Delta+2 \tau)^{-1} M_{+} A_{k} \psi_{k} \rightarrow\left(-\Delta-2 e_{1}\right)^{-1} M_{+} J V \psi_{*}$ and $(-\Delta)^{-1} M_{-} A_{k} \psi_{k} \rightarrow(-\Delta)^{-1} M_{+} J V \psi_{*}$ strongly in X. Thus

$$
\psi_{*}=\left(-\Delta-2 e_{1}\right)^{-1} M_{+} J V \psi_{*}+(-\Delta)^{-1} M_{+} J V \psi_{*}
$$

and $\psi_{k} \rightarrow \psi_{*}$ strongly. Hence $\left\|\psi_{*}\right\|_{X}=\lim \left\|\psi_{k}\right\|_{X}=1$ and $\left(J H_{1}+i e_{1}\right) \psi_{*}=0$ by (2.56) again. This contradiction to our assumption shows our claim.

## Another proof:

We will use the resolvent estimates Lemma 2.5 to give a proof, without using that $\mathcal{L}_{1}$ is a perturbation of $J H_{1}$. Suppose the contrary that we have $\psi \in X$ which satisfies $\psi \neq 0, \mathcal{L}_{1} \psi=i \tau \psi$, with $\tau=\left|E_{1}\right|$. Write $\mathcal{L}_{1}=J H_{*}+A$ as before and let $R(z)=\left(z-\mathcal{L}_{1}\right)^{-1}$ and $R_{0}(z)=\left(z-J H_{*}\right)^{-1}$. We have $\left(i \tau-J H_{*}\right) \psi=A \psi$, hence

$$
\begin{equation*}
\psi=R_{0}(i \tau) A \psi \quad \text { in } X \tag{2.58}
\end{equation*}
$$

Let $w=\sigma_{1} A \psi=A^{*} \sigma_{1} \psi$. (Note $\left.A^{*}=\sigma_{1} A \sigma_{1}\right)$. We have that $V w \in L_{r}^{2}$. By Lemma 2.5 the $L_{-r}^{2}$-norm of $R(z) V w$ is uniformly bounded as $z \rightarrow i \tau$. We will derive a contradiction.

Recall the resolvent identity $R(z) A=-1-\left(1-R_{0}(z) A\right)^{-1}$. Hence $\rho(z) \equiv$ $\left(1-R_{0}(z) A\right)^{-1} w=R(z) A w+w$ is also uniformly bounded in $L_{-r}^{2}$ as $z \rightarrow i \tau$.

Recall [7] Lemma 2.3 that $(-\Delta-z)^{-1}=(-\Delta)^{-1}+O(\sqrt{|z|})$ in $\mathcal{B}\left(H_{s}^{-1}, H_{-s}^{1}\right)$ for $s>1$. Hence for $z$ near $i \tau$ we have by (2.56)

$$
R_{0}(z)=R_{0}(i \tau)+O(\sqrt{|z-i \tau|})
$$

in $\mathcal{B}\left(H_{s}^{-1}, H_{-s}^{1}\right)$. Therefore

$$
\begin{aligned}
(w, w) & =\left(A^{*} \sigma_{1} \psi,\left(1-R_{0}(z) A\right) \rho(z)\right) \\
& =\left(A^{*} \sigma_{1} \psi,\left[1-R_{0}(i \tau) A\right] \rho(z)\right)+\left(A^{*} \sigma_{1} \psi, O(\sqrt{|z-i \tau|}) A \rho(z)\right)
\end{aligned}
$$

Since $\sigma_{1} J=-J \sigma_{1}, \sigma_{1}\left(z-J H_{*}\right)=\left(z+J H_{*}\right) \sigma_{1}=\left(z-J H_{*}\right)^{*} \sigma_{1}$. Hence $\sigma_{1} R_{0}(i \tau)=R_{0}(i \tau)^{*} \sigma_{1}$ and we have by (2.58)

$$
\sigma_{1} \psi=\sigma_{1} R_{0}(i \tau) A \psi=R_{0}(i \tau)^{*} \sigma_{1} A \psi=R_{0}(i \tau)^{*} A^{*} \sigma_{1} \psi
$$

Hence

$$
\left(A^{*} \sigma_{1} \psi, R_{0}(i \tau) A \rho(z)\right)=\left(A^{*} R_{0}(i \tau)^{*} A^{*} \sigma_{1} \psi, \rho(z)\right)=\left(A^{*} \sigma_{1} \psi, \rho(z)\right) .
$$

Hence $\left(A^{*} \sigma_{1} \psi,\left[1-R_{0}(i \tau) A\right] \rho(z)\right)=0$. Since $\rho(z)$ is uniformly bounded in $L_{r}^{2}$, we have

$$
(w, w)=\left(A^{*} \sigma_{1} \psi, O(\sqrt{|z-i \tau|}) A \rho(z)\right)=O(\sqrt{|z-i \tau|})
$$

as $z \rightarrow i \tau$. Thus $w=0$. Hence $\left(i \tau-J H_{*}\right) \psi=0$. If we write $\psi=\left[\begin{array}{l}u \\ v\end{array}\right]$, then

$$
i \tau u-(-\Delta+\tau) v=0, \quad(-\Delta+\tau) u+i \tau v=0
$$

One gets $\Delta(u-i v)=0$ immediately. Since $u, v \in X$, we conclude $u=i v$ and $(-\Delta+2 \tau) u=0$. Hence $u, v=0$. This finishes the proof.

### 2.6 Proof of Theorem [2.2 (4)-(6)

We first show the orthogonality conditions. Recall $\sigma_{1}=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$. It is self-adjoint in $\mathbb{C} L^{2}$. Let $\mathcal{L}_{1}^{*}$ be the adjoint of $\mathcal{L}_{1}$ in $\mathbb{C} L^{2}$. We have $\mathcal{L}_{1}^{*}=$ $\left[\begin{array}{cc}0 & -L_{+} \\ L_{-} & 0\end{array}\right]$ and $\mathcal{L}_{1}^{*}=\sigma_{1} \mathcal{L}_{1} \sigma_{1}$.

Suppose $\mathcal{L}_{1} f=\omega_{1} f$ and $\mathcal{L}_{1} g=\omega_{2} g$ with $\bar{\omega}_{1} \neq \omega_{2}$. We have $\mathcal{L}_{1}^{*} \sigma_{1} f=$ $\sigma_{1} \mathcal{L}_{1} f=\omega_{1} \sigma_{1} f$ and hence
$\omega_{2}\left(\sigma_{1} f, g\right)=\left(\sigma_{1} f, \omega_{2} g\right)=\left(\sigma_{1} f, \mathcal{L}_{1} g\right)=\left(\mathcal{L}_{1}^{*} \sigma_{1} f, g\right)=\left(\omega_{1} \sigma_{1} f, g\right)=\bar{\omega}_{1}\left(\sigma_{1} f, g\right)$.
Hence we must have $\left(\sigma_{1} f, g\right)=0$. Therefore we have $\sigma_{1} \bar{\Phi} \perp \bar{\Phi}, \sigma_{3} \Phi, \sigma_{3} \bar{\Phi}$, $\sigma_{1} \Phi \perp \Phi, \sigma_{3} \Phi, \sigma_{3} \bar{\Phi}$, etc. If we write $u=u_{1}+i u_{2}, v=v_{1}+i v_{2}$ and $\Phi=\left[\begin{array}{l}u \\ v\end{array}\right]$, then we have

$$
\begin{equation*}
\int \bar{u} v d x=0 . \tag{2.59}
\end{equation*}
$$

In other words, $\left(u_{1}, v_{1}\right)+\left(u_{2}, v_{2}\right)=0$ and $\left(u_{1}, v_{2}\right)=\left(u_{2}, v_{1}\right)$.
If $f \in S\left(\mathcal{L}_{1}\right)$ and $\mathcal{L}_{1} g=\omega_{2} g$ with $\omega_{2} \neq 0$. We have $\left(\mathcal{L}_{1}^{*}\right)^{2} \sigma_{1} f=0$, hence

$$
\left(\sigma_{1} f, \omega_{2}^{2} g\right)=\left(\sigma_{1} f, \mathcal{L}_{1}^{2} g\right)=\left(\left(\mathcal{L}_{1}^{*}\right)^{2} \sigma_{1} f, g\right)=(0, g)
$$

Hence $\left(\sigma_{1} f, g\right)=0$. In terms of components, we get $\left(Q_{1}, u_{1}\right)=\left(Q_{1}, u_{2}\right)=0$, $\left(R_{1}, v_{1}\right)=\left(R_{1}, v_{2}\right)=0$. The above shows (2.22). The rest of (4) and (5) follows directly.

To prove (6), we first prove the following spectral gap

$$
\begin{equation*}
\left.L_{+}\right|_{\left\{Q_{1}, v_{1}, v_{2}\right\}^{\perp}}>\frac{1}{2}\left|e_{1}\right|,\left.\quad L_{-}\right|_{\left\{R_{1}, u_{1}, u_{2}\right\}^{\perp}}>\frac{1}{2}\left|e_{1}\right| . \tag{2.60}
\end{equation*}
$$

We will show the first assertion. Note that by (2.17) we have

$$
v_{1}=\mathbf{P}_{\mathrm{c}}\left(L_{-}\right) v_{1}+O\left(n^{2}\right), \quad v_{2}=-\phi_{0}+\mathbf{P}_{\mathrm{c}}\left(H_{1}\right) v_{2}+O\left(n^{2}\right)
$$

in $L^{2}$. In particular $\left\|v_{2}\right\|_{L^{2}} \geq 1 / 2$, and $\left(v_{1}, L_{-} v_{1}\right) \geq\left(v_{1}, L_{-} \mathbf{P}_{\mathrm{c}}\left(L_{-}\right) v_{1}\right)-$ $C n^{2} \geq-C n^{2}$. By (2.59)

$$
\left(v_{1}, L_{-} v_{1}\right)+\left(v_{2}, L_{-} v_{2}\right)=\left(v, L_{-} v\right)=(v, \omega u)=0
$$

Hence $\left(v_{2}, L_{+} v_{2}\right)=\left(v_{2}, L_{-} v_{2}\right)+O\left(n^{2}\right) \leq C n^{2}$. We also have $\left(Q_{1}, L_{+} Q_{1}\right)=$ $\left(Q_{1}, L_{-} Q_{1}\right)+O\left(n^{4}\right)=0+O\left(n^{4}\right)$. Let $Q_{1}^{\prime}=Q_{1}-\left(Q_{1}, v_{2}\right) v_{2} /\left\|v_{2}\right\|_{2}^{2}$. We have $Q_{1}^{\prime} \perp v_{j}$ and $Q_{1}^{\prime}=Q_{1}+O\left(n^{3}\right)$ by (2.17) again. Hence $\left(Q_{1}^{\prime}, L_{+} Q_{1}^{\prime}\right)=$
$\left(Q_{1}, L_{+} Q_{1}\right)+O\left(n^{4}\right)=O\left(n^{4}\right) \leq C n^{2}\left(Q_{1}^{\prime}, Q_{1}^{\prime}\right)$. We conclude that $\left.L_{+}\right|_{\operatorname{span}\left\{Q_{1}, v_{2}\right\}} \leq$ $C n^{2}$. Since $L_{+}$is a perturbation of $H_{1}$, it has exactly two eigenvalues below $\frac{1}{2}\left|e_{1}\right|$. By minimax principle we have $\left.L_{+}\right|_{\left\{Q_{1}, v_{2}\right\}^{\perp}}>\frac{1}{2}\left|e_{1}\right|$. This shows the first assertion of (2.60). The second assertion is proved similarly.

Let $\mathbf{Q}(\psi)$ denote the quadratic form: (see e.g. [22, 23])

$$
\begin{equation*}
\mathbf{Q}(\psi)=\left(f, L_{+} f\right)+\left(g, L_{-} g\right), \quad \text { if } \psi=f+i g \tag{2.61}
\end{equation*}
$$

One can show for any $\psi \in L^{2}$

$$
\begin{equation*}
\mathbf{Q}\left(e^{t \mathcal{L}_{1}} \psi\right)=\mathbf{Q}(\psi), \quad \text { for all } t \tag{2.62}
\end{equation*}
$$

by direct differentiation in $t$. By (2.60) one has

$$
\mathbf{Q}(\eta) \sim\|\eta\|_{H^{1}}^{2}, \quad \text { for any } \eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)
$$

Thus

$$
\left\|e^{t \mathcal{L}_{1}} \eta\right\|_{H^{1}}^{2} \sim \mathbf{Q}\left(e^{t \mathcal{L}_{1}} \eta\right)=\mathbf{Q}(\eta) \sim\|\eta\|_{H^{1}}^{2}
$$

Similarly, we have by (2.60) and the above relation

$$
\|\eta\|_{H^{3}}^{2} \sim\left\|\mathcal{L}_{1} \eta\right\|_{H^{1}}^{2} \sim \mathbf{Q}\left(\mathcal{L}_{1} \eta\right)
$$

Since $\mathbf{Q}\left(\mathcal{L}_{1} \eta\right)=\mathbf{Q}\left(e^{t \mathcal{L}_{1}} \mathcal{L}_{1} \eta\right)$, we have $\|\eta\|_{H^{3}} \sim\left\|e^{t \mathcal{L}_{1}} \eta\right\|_{H^{3}}$. By interpolation we have $\|\eta\|_{H^{2}} \sim\left\|e^{t \mathcal{L}_{1}} \eta\right\|_{H^{2}}$. We have proven (6).

### 2.7 Wave operator and decay estimate

It remains to prove the decay estimate (7). We will use the wave operator. We will compare $\mathcal{L}_{1}$ with $J H_{*}$, where $H_{*}=-\Delta-E_{1}$. Recall we write $\mathcal{L}_{1}=J H_{*}+A$ in $\S 2.4$, (2.55). Keep in mind that $H_{*}$ has no bound states and $A$ is local. Define $W_{+}=\lim _{t \rightarrow+\infty} e^{-t \mathcal{L}_{1}} e^{t J H_{*}}$. Let $R(z)=\left(z-\mathcal{L}_{1}\right)^{-1}$ and $R_{*}(z)=\left(z-J H_{*}\right)^{-1}$. We have

$$
\begin{aligned}
& W_{+} f-f \\
& =\lim _{\varepsilon \rightarrow 0+} \int_{\left|E_{1}\right|}^{+\infty} R(i \tau+\varepsilon) A\left[R_{*}(i \tau-\varepsilon)-R_{*}(i \tau+\varepsilon)\right] f d \tau \\
& -\lim _{\varepsilon \rightarrow 0+} \int_{\left|E_{1}\right|}^{+\infty} R(-i \tau+\varepsilon) A\left[R_{*}(-i \tau-\varepsilon)-R_{*}(-i \tau+\varepsilon)\right] f d \tau
\end{aligned}
$$

Yajima [24, 25] was the first to give a general method for proving the $\left(W^{k, p}, W^{k, p}\right)$ estimates for the wave operators of self-adjoint operators. This method was extended by Cuccagna [4] to non-selfadjoint operators in the form we are considering. (He also used idea from Kato [9]). One key ingredient in this approach is the resolvent estimates near the continuous spectrum, which in many cases can be obtained by the Jensen-Kato [7] method. (See [24] Lemmas 3.1 and 3.2 and (4] Lemmas 3.9 and 3.10). In our current setting, this estimate is provided by the Lemma 2.5. We can thus follow the proof of [罒 to obtain that $W_{+}$is an operator from $\mathbb{C} L^{2}$ onto $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$. Furthermore, $W_{+}$and its inverse (restricted to $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ ) are bounded in $\left(L^{p}, L^{p}\right)$-norm for any $p \in[1, \infty]$. (Note this bound depends on $n$ since our bound on $R(w)$ depends on $n$.) By the intertwining property of the wave operator we have

$$
e^{t \mathcal{L}_{1}} \mathbf{P}_{\mathbf{c}}=W_{+} e^{t J H_{*}}\left(W_{+}\right)^{*} \mathbf{P}_{\mathbf{c}}
$$

The decay estimate in (7) follows from the decay estimate of $e^{t J H_{*}}$.
The proof of Theorem 2.2 is complete.

### 2.8 Proof of Theorem [2.1]

By the same Cauchy integral argument as in Subsection 2.2, the only eigenvalues of $\mathcal{L}_{1}$ are inside the disks $\{w:|w|<\sqrt{n}\},\left\{w:\left|w-i e_{01}\right|<\sqrt{n}\right\}$ and $\left\{w:\left|w+i e_{01}\right|<\sqrt{n}\right\}$. Moreover, their dimensions are 2,1 and 1 , respectively, the same as that of $J H_{1}$. It counts the dimension of (generalized) eigenspaces of $\mathcal{L}_{1}$ in $\mathbb{C} L^{2}$. It also counts the dimensions of the restriction of these spaces in $L^{2}=L^{2}\left(\mathbb{R}^{3}, \mathbb{R}^{2}\right)$ as a real-valued vector space.

By (2.9), we already have two generalized eigenvectors near 0. Hence we have everything near 0 .

Since the dimension is 1 near $i e_{01}$, there is only a simple eigenvalue $\omega_{*}$ near $i e_{01}$. We have $\omega_{*}=i e_{01}+O\left(n^{2}\right)$ since the difference between $\mathcal{L}_{1}$ and $J H_{1}$ is of order $O\left(n^{2}\right) . \omega_{*}$ has to be purely imaginary, otherwise $-\bar{\omega}_{*}$ is another eigenvalue near $i e_{01}$, cf. ( $\overline{2.19)}$, and the dimension can not be 1. (This also follows from the Theorem of Grillakis.)

By the same arguments in $\S 2.2-2.4$ we can prove resolvent estimates and the non-existence of embedded eigenvalues. Also, the bottoms of the continuous spectrum are not eigenvalue nor resonance.

Let $\Phi$ be an eigenvector corresponding to $\omega_{*}$. Since $\mathcal{L}_{1} \Phi=\omega_{*} \Phi$ and $\bar{\omega}_{*}=-\omega_{*}$, we have $\mathcal{L}_{1} \bar{\Phi}=-\omega_{*} \bar{\Phi}$. Hence the (unique) eigenvalue near $-i e_{01}$ is $-\omega_{*}$ with eigenvector $\bar{\Phi}$. Write $\Phi=\left[\begin{array}{c}u \\ -i v\end{array}\right]$. We may assume $u$ is real. Writing out $\mathcal{L}_{1} \Phi=i \kappa \Phi$ we get $L_{-} v=-\kappa u$ and $L_{+} u=-\kappa v$. Hence $v$ is also real. We can normalize $u$ so that $(u, v)=1$ or -1 . Since $\Phi$ is a perturbation of $\left[\begin{array}{c}\phi_{0} \\ -i \phi_{0}\end{array}\right]$, we have $(u, v)=1$.

With this choice of $u, v$, let $\mathbb{C} \mathbf{E}_{1}$ and $\mathbf{E}_{1}$ be defined as in (2.12). $\mathbb{C} \mathbf{E}_{1}$ is the combined eigenspace corresponding to $\pm \omega_{*}$. Clearly $\mathbf{R E} \mathbb{C} \mathbf{E}_{1} \subset \mathbf{E}_{1}$. Since

$$
a\left[\begin{array}{l}
u \\
0
\end{array}\right]+b\left[\begin{array}{l}
0 \\
v
\end{array}\right]=\mathbf{R E} \alpha \Phi, \quad \alpha=a+b i
$$

we have $\mathbf{R E} \mathbb{C} \mathbf{E}_{1}=\mathbf{E}_{1}$. That the choice of $\alpha$ is unique can be checked directly. The statement that if $\zeta=\mathbf{R E} \alpha \Phi$ then $\mathcal{L}_{1} \zeta=\mathbf{R E} \omega_{*} \alpha \Phi$ and $e^{t \mathcal{L}_{1}} \zeta=\mathbf{R E} e^{t \omega_{*}} \alpha \Phi$ is clear. We have proved (3) and (4).

Clearly, $S\left(\mathcal{L}_{1}\right), \mathbf{E}_{1}\left(\mathcal{L}_{1}\right)$ and $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ defined as in (2.9), (2.12) and (2.10) are invariant subspaces of $L^{2}$ under $\mathcal{L}_{1}$, and we have the decomposition (2.7). This is (2).

For (5), note that (2.10) is by definition. For (2.11), we have

$$
\begin{gathered}
\left(Q_{1}, u\right)=\left(Q_{1},(-\kappa)^{-1} L_{-} v\right)=\left(L_{-} Q_{1},(-\kappa)^{-1} v\right)=0 \\
\left(R_{1}, v\right)=\left(R_{1},(-\kappa)^{-1} L_{+} u\right)=(-\kappa)^{-1}\left(L_{+} R_{1}, u\right)=(-\kappa)^{-1}\left(Q_{1}, u\right)=0
\end{gathered}
$$

(2.14) comes from the orthogonal relations directly.

The first statement of (6) is because of (5). For the rest of (6), We first prove the following spectral gap

$$
\begin{equation*}
\left.L_{+}\right|_{\left\{Q_{1}, v\right\}^{\perp}}>\frac{1}{2}\left|e_{1}\right|,\left.\quad L_{-}\right|_{\left\{R_{1}, u\right\}^{\perp}}>\frac{1}{2}\left|e_{1}\right| . \tag{2.63}
\end{equation*}
$$

Since $L_{+}$is a perturbation of $H_{1}$, it has exactly two eigenvalues below $\frac{1}{2}\left|e_{1}\right|$. Notice that $\left(Q_{1}, L_{+} Q_{1}\right)=\left(Q_{1} L_{-} Q_{1}\right)+O\left(n^{4}\right)=O\left(n^{4}\right)$ and $\left(v, L_{+} v\right)=$ $(v,-\kappa u)=-\kappa$. Since $Q_{1}=n \phi_{1}+O\left(n^{3}\right)$ and $v=\phi_{0}+O\left(n^{2}\right)$, one has $\left(Q_{1}, v\right)=O\left(n^{3}\right)$. Thus one can show $\left.L_{+}\right|_{\operatorname{span}\left\{Q_{1}, v\right\}} \leq C n^{2}$. If there is a $\phi \perp Q_{1}, v$ with $\left(\phi, L_{+} \phi\right) \leq \frac{1}{2}\left|e_{1}\right|(\phi, \phi)$, then we have $\left.L_{+}\right|_{\operatorname{span}\left\{Q_{1}, v, \phi\right\}} \leq \frac{1}{2}\left|e_{1}\right|$, which contradicts with the fact that $L_{+}$has exactly two eigenvalues below $\frac{1}{2}\left|e_{1}\right|$ by minimax principle. This shows the first part of (2.63). The second part is proved similarly.

Recall the quadratic form $\mathbf{Q}(\psi)$ defined in (2.61) in §2.6. Also recall (2.62) that $\mathbf{Q}\left(e^{t \mathcal{L}_{1}} \psi\right)=\mathbf{Q}(\psi)$ for all $t$ and all $\psi \in L^{2}$. By the spectral gap (2.63) one has

$$
\begin{equation*}
\mathbf{Q}(\eta) \sim\|\eta\|_{H^{1}}^{2}, \quad \mathbf{Q}\left(\mathcal{L}_{1} \eta\right) \sim\|\eta\|_{H^{3}}^{2}, \quad \text { for any } \eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right) \tag{2.64}
\end{equation*}
$$

For $\psi \in M_{1}$, we can write $\psi=\zeta+\eta$, where $\zeta=\mathbf{R E} \alpha \Phi, \alpha \in \mathbb{C}$ and $\eta \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$. Notice that, by orthogonality in (2.10),

$$
\mathbf{Q}(\psi)=-|\alpha|^{2} \kappa(u, v)+\mathbf{Q}(\eta)
$$

which is not positive definite, (recall $(u, v)=1)$. However,

$$
\begin{equation*}
\|\psi\|_{H^{1}}^{2} \sim|\alpha|^{2}+\|\eta\|_{H^{1}}^{2} . \tag{2.65}
\end{equation*}
$$

To see it, one first notes that $\|\psi\|_{H^{1}}^{2}$ is clearly bounded by the right side. Because of (2.14), one has $|\alpha|^{2} \leq C\|\psi\|_{H^{1}}^{2}$. One also has $\|\eta\|_{H^{1}}^{2} \leq C\|\phi\|_{H^{1}}^{2}+$ $C|\alpha|^{2}$. Hence (2.65) is true.

Therefore for $\psi=(\boldsymbol{R E} \alpha \Phi)+\eta$ we have

$$
\begin{aligned}
\left\|e^{t \mathcal{L}_{1}} \psi\right\|_{H^{1}}^{2} & \sim\left\|e^{t \mathcal{L}_{1}} \mathbf{R E} \alpha \Phi\right\|_{H^{1}}^{2}+\left\|e^{t \mathcal{L}_{1}} \eta\right\|_{H^{1}}^{2} & & (\text { by }(2.65)) \\
& \sim\left|e^{-i t \omega_{*}} \alpha\right|^{2}+\mathbf{Q}\left(e^{t \mathcal{L}_{1}} \eta\right) & & (\text { by }(4),(2.64)) \\
& \sim|\alpha|^{2}+\mathbf{Q}(\eta) & & (\text { by }(2.62))
\end{aligned}
$$

Hence we have $\left\|e^{t \mathcal{L}_{1}} \psi\right\|_{H^{1}}^{2} \sim\|\psi\|_{H^{1}}^{2}$ for all $t$. By an argument similar to that in $\S 2.6$, we have $\left\|e^{t \mathcal{L}_{1}} \psi\right\|_{H^{k}} \sim\|\psi\|_{H^{k}}$ for $k=3,2$. We have shown (6).

The decay estimate in (7) is obtained as in Theorem 2.2 (7). The constant $C$, however, is independent of $n$ in the non-resonant case. The proof of Theorem 2.1 is complete.

## 3 Solutions converging to excited states

In this section we prove Theorem 1.1 using Theorems 2.1 and 2.2. Since the proof for the non-resonant case is easier, we will first prove the resonant case and then sketch the non-resonant case. Note that we could follow the approach of Theorem 1.5 of [20] if we had the transform $\mathcal{L}_{1} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}=$ $-U^{-1} i A U \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}$ as in 20]. However, it is not easy to define $A$ and $U$ for $\mathcal{L}_{1}$ and hence we choose another approach. This approach also gives another proof for Theorem 1.5 of [20].

Fix $E_{1}$ and $Q_{1}=Q_{1, E_{1}}$. Let $\mathcal{L}_{1}$ be the corresponding linearized operator, and $\mathbf{P}_{M_{1}}, \mathbf{P}_{\mathbf{E}_{1}}$ and $\mathbf{P}_{\mathrm{c}}{ }^{\mathcal{L}_{1}}$ the corresponding projections with respect to $\mathcal{L}_{1}$. For any $\xi_{\infty} \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ with small $H^{2} \cap W^{2,1}$ norm, we want to construct a solution $\psi(t)$ of the nonlinear Schrödinger equation (1.1) with the form

$$
\psi(t)=\left[Q_{1}+a(t) R_{1}+h(t)\right] e^{-i E_{1} t+i \theta(t)}
$$

where $a(t), \theta(t) \in \mathbb{R}$ and $h(t) \in M_{1}=\mathbf{E}_{1} \oplus \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$. Substituting the above ansatz into (1.1) and using $\mathcal{L}_{1} i Q_{1}=0$ and $\mathcal{L}_{1} R_{1}=-i Q_{1}$, we get

$$
\partial_{t} h=\mathcal{L}_{1} h+i^{-1} F\left(a R_{1}+h\right)-i \dot{\theta}\left(Q_{1}+a R_{1}+h\right)-a i Q_{1}-\dot{a} R_{1}
$$

where

$$
\begin{equation*}
F(k)=\lambda Q_{1}\left(2|k|^{2}+k^{2}\right)+\lambda|k|^{2} k, \quad k=a R_{1}+h . \tag{3.1}
\end{equation*}
$$

The condition $h(t) \in M_{1}$ can be satisfied by requiring that $h(0) \in M_{1}$ and

$$
\begin{align*}
& \dot{a}=\left(c_{1} Q_{1}, \operatorname{Im}(F+\dot{\theta} h)\right),  \tag{3.2}\\
& \dot{\theta}=-\left[a+\left(c_{1} R_{1}, \operatorname{Re} F\right)\right] \cdot\left[1+\left(c_{1} R_{1}, R_{1}\right) a+\left(c_{1} R_{1}, \operatorname{Re} h\right)\right]^{-1}, \tag{3.3}
\end{align*}
$$

where $c_{1}=\left(Q_{1}, R_{1}\right)^{-1}$ and $F=F\left(a R_{1}+h\right)$. The equation for $h$ becomes

$$
\partial_{t} h=\mathcal{L}_{1} h+P_{M} F_{\text {all }}, \quad F_{\text {all }}=i^{-1}\left(F+\dot{\theta}\left(a R_{1}+h\right)\right)
$$

The proofs of the two cases diverge here. For the resonant case we decompose, using the decomposition of $M_{1}$ and (2.20) of Theorem 2.2,

$$
h(t)=\zeta(t)+\eta(t), \quad \zeta(t)=\mathbf{R E}\left\{\alpha(t) \Phi+\beta(t) \sigma_{3} \Phi\right\},
$$

where $\alpha(t), \beta(t) \in \mathbb{C}$ and $\eta(t) \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$. Note

$$
\mathcal{L}_{1} \zeta=\mathbf{R E}\left\{\omega_{*} \alpha \Phi-\omega_{*} \beta \sigma_{3} \Phi\right\} .
$$

Recall $\omega_{*}=i \kappa+\gamma$ with $\kappa, \gamma>0$. Taking the projections $P_{\alpha}$ and $P_{\beta}$ defined in (2.24) of Theorem 2.2 of the $h$-equation, we have

$$
\begin{align*}
& \dot{\alpha}=\omega_{*} \alpha+P_{\alpha} F_{\text {all }},  \tag{3.4}\\
& \dot{\beta}=-\omega_{*} \beta+P_{\beta} F_{\text {all }} . \tag{3.5}
\end{align*}
$$

Taking projection $\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}}$ we get the equation for $\eta$,

$$
\partial_{t} \eta=\mathcal{L}_{1} \eta+\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}} i^{-1} \dot{\theta} \eta+\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}} \widetilde{F}, \quad \widetilde{F}=i^{-1}\left(F+\dot{\theta}\left(a R_{1}+\zeta\right)\right)
$$

We single out $\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}} i^{-1} \dot{\theta} \eta$ since it is a global linear term in $\eta$ and cannot be treated as error. Let

$$
\widetilde{\eta}=\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} e^{i \theta} \eta .
$$

Note $\eta=\widetilde{\eta}+\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}}\left(1-e^{i \theta}\right) \eta$ and $\mathbf{P}_{\mathbf{c}}\left(1-e^{i \theta}\right)$ is a bounded map from $\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right) \cap H^{2}$ into itself with its norm bounded by $C|\theta|$. Hence if $\theta$ is sufficiently small, we can solve $\eta$ in terms of $\widetilde{\eta}$ by expansion:

$$
\begin{equation*}
\eta=U_{\theta} \widetilde{\eta}, \quad U_{\theta} \equiv \sum_{j=0}^{\infty}\left[\mathbf{P}_{\mathrm{c}}\left(1-e^{i \theta}\right)\right]^{j} \tag{3.6}
\end{equation*}
$$

The equation for $\widetilde{\eta}$ is

$$
\begin{aligned}
\partial_{t} \widetilde{\eta}= & \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} e^{i \theta}\left(i \dot{\theta} \eta+\partial_{t} \eta\right) \\
= & \mathcal{L}_{1} \widetilde{\eta}+\left\{\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} e^{i \theta} \mathcal{L}_{1}-\mathcal{L}_{1} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} e^{i \theta}\right\} \eta \\
& +\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} e^{i \theta}\left\{i \dot{\theta} \eta-\dot{\theta} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} i \eta+\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} \widetilde{F}\right\}
\end{aligned}
$$

Note that $i \dot{\theta} \eta-\dot{\theta} \mathbf{P}_{\mathrm{c}}{ }^{\mathcal{L}_{1}} i \eta=\left(1-\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\right) i \dot{\theta} \eta$ and

$$
\begin{aligned}
\left\{\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} e^{i \theta} \mathcal{L}_{1}-\mathcal{L}_{1} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} e^{i \theta}\right\} \eta & =\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\left[e^{i \theta}, \mathcal{L}_{1}\right] \eta=\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} \sin \theta\left[i, \mathcal{L}_{1}\right] \eta \\
& =\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} \sin \theta 2 \lambda Q_{1}^{2} \bar{\eta} .
\end{aligned}
$$

Hence we have

$$
\partial_{t} \widetilde{\eta}=\mathcal{L}_{1} \widetilde{\eta}+\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\left\{\sin \theta 2 \lambda Q_{1}^{2} \bar{\eta}+e^{i \theta}\left(1-\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\right) i \dot{\theta} \eta+e^{i \theta} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} \widetilde{F}\right\}
$$

For a given profile $\xi_{\infty}$, let

$$
\begin{equation*}
\widetilde{\eta}(t)=e^{t \mathcal{L}_{1}} \xi_{\infty}+g(t) \tag{3.7}
\end{equation*}
$$

We have the equation

$$
\begin{equation*}
\partial_{t} g=\mathcal{L}_{1} g+\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\left\{\sin \theta 2 \lambda Q_{1}^{2} \bar{\eta}+e^{i \theta}\left(1-\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\right) i \dot{\theta} \eta+e^{i \theta} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} \widetilde{F}\right\} . \tag{3.8}
\end{equation*}
$$

We want $g(t) \rightarrow 0$ as $t \rightarrow \infty$ in some sense.
Summarizing, we write the solution $\psi(t)$ in the form

$$
\begin{align*}
\psi(t)=\{ & Q_{1}+a(t) R_{1}+\mathbf{R E}\left\{\alpha(t) \Phi+\beta(t) \sigma_{3} \Phi\right\}  \tag{3.9}\\
& \left.+U_{\theta(t)}\left(e^{t \mathcal{L}_{1}} \xi_{\infty}+g(t)\right)\right\} e^{-i E_{1} t+i \theta(t)}
\end{align*}
$$

with $a(t), \theta(t), \alpha(t), \beta(t)$ and $g(t)$ satisfying (3.2), (3.3), (3.4), (3.5), and (3.8), respectively.

The main term of $F$ is

$$
F_{0}=\lambda Q_{1}\left(2|\xi|^{2}+\xi^{2}\right)+\lambda|\xi|^{2} \xi, \quad \xi(t)=U_{\theta(t)} e^{t \mathcal{L}_{1}} \xi_{\infty}
$$

Notice that, if $\left\|\xi_{\infty}\right\|_{H^{2} \cap W^{2,1}} \leq \varepsilon \ll 1$, then $\xi(t)$ satisfies
$\|\xi(t)\|_{H^{2}} \leq C(n) \varepsilon, \quad\|\xi(t)\|_{W^{2, \infty}} \leq C(n) \varepsilon|t|^{-3 / 2}, \quad\left\||\xi|^{2} \xi(t)\right\|_{H^{2}} \leq C(n) \varepsilon^{3}\langle t\rangle^{-3}$.

Here we have used the boundedness and decay estimates for $e^{t \mathcal{L}_{1}} \mathbf{P}_{c}^{\mathcal{L}_{1}}$ in Theorem 2.2 (6)-(7). Since $Q_{1}$ is fixed, it does not matter that the constant depends on $n$. The main term of $F_{0}$ is quadratic in $\xi$. Hence

$$
\left\|F_{0}(t)\right\|_{H^{2}} \leq C \varepsilon^{2}\langle t\rangle^{-3}
$$

As it will become clear, we have the freedom to choose $\xi_{\infty}$ and $\beta_{0}=\beta(0)$. We require that $\xi_{\infty} \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)$ and

$$
\begin{equation*}
\left\|\xi_{\infty}\right\|_{H^{2} \cap W^{2,1}} \leq \varepsilon, \quad\left|\beta_{0}\right| \leq \varepsilon^{2} / 4 \tag{3.10}
\end{equation*}
$$

with $\varepsilon$ sufficiently small. With given $\xi_{\infty}$ and $\beta_{0}$, we will define a contraction mapping $\Omega$ in the following space

$$
\begin{aligned}
\mathcal{A}=\{ & (a, \theta, \alpha, \beta, g):[0, \infty) \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C} \times\left(\mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right) \cap H^{2}\right), \\
& |a(t)|,|\alpha(t)|,|\beta(t)|, \leq \varepsilon^{7 / 4}(1+t)^{-2}, \\
& \left.\|g(t)\|_{H^{2}} \leq \varepsilon^{7 / 4}(1+t)^{-7 / 4}, \quad|\theta(t)| \leq 2 \varepsilon^{7 / 4}(1+t)^{-1}\right\}
\end{aligned}
$$

For convenience, we introduce a variable $b=\dot{\theta}$. Our map $\Omega$ is defined by

$$
\Omega:(a, \theta, \alpha, \beta, \eta) \longrightarrow\left(a^{\Delta}, \theta^{\Delta}, \alpha^{\Delta}, \beta^{\triangle}, \eta^{\Delta}\right)
$$

where, with $c_{1}=\left(Q_{1}, R_{1}\right)^{-1}$ and $F=F(a R+h)$ defined in (3.1),

$$
\begin{aligned}
& h(t)=\zeta(t)+\eta(t) \\
& \zeta(t)=\operatorname{RE}\left\{\alpha(t) \Phi+\beta(t) \sigma_{3} \Phi\right\}, \quad \eta(t)=U_{\theta(t)}\left(e^{t \mathcal{L}_{1}} \xi_{\infty}+g(t)\right) \\
& b(t)=-\left[a+\left(c_{1} R_{1}, \operatorname{Re} F\right)\right] \cdot\left[1+\left(c_{1} R_{1}, R_{1}\right) a+\left(c_{1} R_{1}, \operatorname{Re} h\right)\right]^{-1} \\
& a^{\triangle}(t)=\int_{\infty}^{t}\left(c_{1} Q_{1}, \operatorname{Re}(F+b h)\right) d s \\
& \theta^{\triangle}(t)=\int_{\infty}^{t} b(s) d s \\
& \alpha^{\triangle}(t)=\int_{\infty}^{t} e^{\omega_{*}(t-s)} P_{\alpha} i^{-1}(F+b(a R+h)) d s \\
& \beta^{\triangle}(t)=e^{-\omega_{*} t} \beta_{0}+\int_{0}^{t} e^{-\omega_{*}(t-s)} P_{\beta} i^{-1}(F+b(a R+h)) d s \\
& g^{\triangle}(t)=\int_{\infty}^{t} e^{\mathcal{L}_{1}(t-s)} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\left\{\sin \theta 2 \lambda Q_{1}^{2} \bar{\eta}+e^{i \theta}\left(1-\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\right) i b \eta+\right. \\
&\left.\quad+e^{i \theta} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} i^{-1}(F+b(a R+\zeta))\right\} d s .
\end{aligned}
$$

We will use Strichartz estimate for the term $\sin \theta 2 \lambda Q_{1}^{2} \bar{\eta}$ in the $g$-integral:

$$
\begin{equation*}
\left\|\int_{\infty}^{t} e^{\mathcal{L}_{1}(t-s)} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} f(s, \cdot) d s\right\|_{L_{x}^{2}} \leq C(n)\left\{\int_{\infty}^{t}\|f(s, \cdot)\|_{L_{x}^{r^{\prime}}}^{q^{\prime}} d s\right\}^{1 / q^{\prime}} \tag{3.11}
\end{equation*}
$$

for $\frac{3}{r}+\frac{2}{q}=\frac{3}{2}, 2<q \leq \infty$. Here ' means the usual conjugate exponent. Eq. (3.11) can be proved by either using wave operator to map $e^{t \mathcal{L}_{1}}$ to $e^{-i t\left(-\Delta-E_{1}\right)}$, or by using the decay estimate Theorem 2.2 (7) and repeating the usual proof for Strichartz estimate. We will also use

$$
\|\phi\|_{H^{2}} \sim\left\|\mathcal{L}_{1} \phi\right\|_{L^{2}} \quad \text { for } \phi \in \mathbf{H}_{\mathrm{c}}\left(\mathcal{L}_{1}\right)
$$

which follows from the spectral gap (2.60). Since $\sin \theta 2 \lambda Q_{1}^{2} \bar{\eta}$ is local and bounded by $C(n) \varepsilon^{2}(1+t)^{-1} \cdot \varepsilon(1+t)^{-3 / 2}$, by choosing $q$ large we have

$$
\begin{aligned}
& \left\|\int_{\infty}^{t} e^{\mathcal{L}_{1}(t-s)} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} \sin \theta 2 \lambda Q_{1}^{2} \bar{\eta} d s\right\|_{H^{2}} \\
& \leq C(n)\left\|\int_{\infty}^{t} e^{\mathcal{L}_{1}(t-s)} \mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}} \mathcal{L}_{1} \sin \theta 2 \lambda Q_{1}^{2} \bar{\eta} d s\right\|_{L_{x}^{2}} \\
& \leq C(n)\left\{\int_{\infty}^{t} \varepsilon^{3}(1+s)^{-(5 / 2) q^{\prime}} d s\right\}^{1 / q^{\prime}}=C(n) \varepsilon^{3}(1+t)^{-5 / 2+1 / q^{\prime}} .
\end{aligned}
$$

In particular, we get $C \varepsilon^{3}(1+t)^{-7 / 4}$ by choosing $q=4$.
Note $|b(t)| \leq 2|a(t)|$. Since $t-s<0$ in the integrand of $\alpha, \operatorname{Re} \omega_{*}(t-s)<0$ and the $\alpha$-integral converges. Similarly $\operatorname{Re} \omega_{*}(t-s)>0$ in the integrand of
$\beta$ and hence the $\beta$-integration converges. Observe that we have the freedom of choosing $\beta_{0}$ and $\xi_{\infty}$. Since $e^{-\omega_{*} t} \beta_{0}$ decays exponentially, the main term of $\beta(t)$ when $t$ large is given by $F_{0}$, not $e^{-\omega_{*} t} \beta_{0}$. Direct estimates show that

$$
\begin{gathered}
|\alpha(t)| \leq C(n) \varepsilon^{2}(1+t)^{-3}, \quad|\beta(t)| \leq \varepsilon^{2} e^{-\gamma t} / 4+C(n) \varepsilon^{2}(1+t)^{-3} \\
|a(t)|,|b(t)| \leq C(n) \varepsilon^{2}(1+t)^{-2}, \quad|\theta(t)| \leq C(n) \varepsilon^{2}(1+t)^{-1} \\
\|g(t)\|_{H^{2}} \leq C(n) \varepsilon^{2}(1+t)^{-7 / 4}
\end{gathered}
$$

It is easy to check that the map $\Omega$ is a contraction if $\varepsilon$ is sufficiently small. Thus we have a fixed point in $\mathcal{A}$, which gives a solution to the system (3.2), (3.3), (3.4), (3.5), and (3.8). Since it lies in $\mathcal{A}$, we also have the desired estimates. We obtain $\alpha(0), a(0)$ and $\theta(0)$ as functions of $\xi_{\infty}$ and $\beta_{0}$.

Recall $\psi_{\text {as }}(t)=Q_{1} e^{-i E_{1} t+i \theta(t)}+e^{-i E_{1} t} e^{t \mathcal{L}_{1}} \xi_{\infty}$ and we have

$$
\psi(t)=\left[Q_{1}+U_{\theta(t)} e^{t \mathcal{L}_{1}} \xi_{\infty}\right] e^{-i E_{1} t+i \theta(t)}+O\left(t^{-7 / 4}\right) \quad \text { in } H^{2}
$$

Since $\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}}\left(1-e^{i \theta}\right)=O(\theta(t))=O\left(t^{-1}\right)$, by the definition (3.6) of $U_{\theta}$,

$$
\begin{aligned}
U_{\theta(t)} e^{t \mathcal{L}_{1}} \xi_{\infty} & =\left[1+\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\left(1-e^{i \theta}\right)\right] e^{t \mathcal{L}_{1}} \xi_{\infty}+O\left(t^{-2}\right) \\
& =\left(2-e^{i \theta}\right) e^{t \mathcal{L}_{1}} \xi_{\infty}+\left(1-\mathbf{P}_{\mathrm{c}}^{\mathcal{L}_{1}}\right)\left(1-e^{i \theta}\right) e^{t \mathcal{L}_{1}} \xi_{\infty}+O\left(t^{-2}\right)
\end{aligned}
$$

in $H^{2}$. Since $\left(1-\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}}\right)$ is a local operator, $\left(1-\mathbf{P}_{\mathbf{c}}^{\mathcal{L}_{1}}\right)\left(1-e^{i \theta}\right) e^{t \mathcal{L}_{1}} \xi_{\infty}=$ $O\left(t^{-1} \cdot t^{-3 / 2}\right)$. Also, $e^{i \theta}\left(2-e^{i \theta}\right)=1+O\left(\theta^{2}\right)=1+O\left(t^{-2}\right)$. Hence we have $\psi(t)-\psi_{\text {as }}(t)=O\left(t^{-7 / 4}\right)$ in $H^{2}$. We have proven Theorem 1.1 under assumption (R).

We now sketch the proof for the non-resonant case. The only difference is that we define $\zeta(t)$ as $\mathbf{R E} \alpha(t) \Phi$ and write $\psi(t)$ in the form

$$
\psi(t)=\left\{Q_{1}+a(t) R_{1}+\mathbf{R E}(\alpha(t) \Phi)+U_{\theta(t)}\left(e^{t \mathcal{L}_{1}} \xi_{\infty}+g(t)\right)\right\} e^{-i E_{1} t+i \theta(t)}
$$

The function $\alpha(t)$ still satisfies (3.4) but with a purely imaginary eigenvalue $\omega_{*}$. The previous proof will go through if we remove all terms related to $\beta$.

## 4 Appendix: Vanishing solutions

In this appendix we prove Proposition 1.2. Recall $H_{0}=-\Delta+V$. The propagator $e^{-i H_{0} t}$ is bounded in $H^{s}, s \geq 0$, and satisfies the decay estimate of the form (2.39):

$$
\begin{equation*}
\left\|e^{-i t H_{0}} \mathbf{P}_{\mathrm{c}}^{H_{0}} \phi\right\|_{L^{\infty}} \leq C|t|^{-3 / 2}\|\phi\|_{L^{1}} \tag{4.1}
\end{equation*}
$$

under our assumption A1. See [7, 8, 24].
For any $\xi_{\infty} \in \mathbf{H}_{\mathrm{c}}\left(H_{0}\right)$ with small $H^{2} \cap W^{2,1}$ norm, we want to construct a solution $\psi(t)$ of (1.1) with the form

$$
\begin{equation*}
\psi(t)=e^{-i H_{0} t} \xi_{\infty}+g(t), \quad g(t)=\text { error } \tag{4.2}
\end{equation*}
$$

Let $\xi(t)=e^{-i H_{0} t} \xi_{\infty}$. Suppose $\left\|\xi_{\infty}\right\|_{H^{2} \cap W^{2,1}}=\varepsilon, 0<\varepsilon \leq \varepsilon_{0}$, we have by (4.1),

$$
\|\xi(t)\|_{H^{2}} \leq C_{1} \varepsilon,\|\xi(t)\|_{W^{2, \infty}} \leq C_{1} \varepsilon|t|^{-3 / 2}, \quad\left\||\xi|^{2} \xi(t)\right\|_{H^{2}} \leq C_{1} \varepsilon^{3}(1+t)^{-3}
$$

for some constant $C_{1}$.
The error term $g(t)$ satisfies

$$
\partial_{t} g=-i H_{0} g+F
$$

with $g(t) \rightarrow 0$ as $t \rightarrow \infty$ in certain sense, and

$$
\begin{equation*}
F(t)=-i \lambda|\psi|^{2} \psi=-i|\xi(t)+g(t)|^{2}(\xi(t)+g(t)), \quad \xi(t)=e^{-i H_{0} t} \xi_{\infty} \tag{4.3}
\end{equation*}
$$

We define a solution by (4.3) and

$$
\begin{equation*}
g(t)=\int_{\infty}^{t} e^{-i H_{0}(t-s)} F(s) d s \tag{4.4}
\end{equation*}
$$

Note that our $g(t)$ belongs to $L^{2}$ and is not restricted to the continuous spectrum component of $H_{0}$. Also note that the main term in $F$ is $|\xi|^{2} \xi(t)$, which is of order $t^{-3}$ in $H^{2}$. Hence $g(t) \sim t^{-2}$.

We define a contraction mapping in the following class

$$
\mathcal{A}=\left\{g(t):[0, \infty) \rightarrow H^{2}\left(\mathbb{R}^{3}\right),\|h(t)\|_{H^{2}} \leq C_{1} \varepsilon^{3}(1+t)^{-2}\right\}
$$

This class is not empty since it contains the zero function. We also define the norm

$$
\|g\|_{\mathcal{A}}:=\sup _{t>0}(1+t)^{2}\|g(t)\|_{H^{2}}
$$

For $g(t) \in \mathcal{A}$ we define

$$
\Omega: g(t) \rightarrow g^{\triangle}(t)=-i \lambda \int_{\infty}^{t} e^{-i H_{0}(t-s)}\left(|\xi+g|^{2}(\xi+g)\right)(s) d s
$$

It is easy to check that

$$
\begin{aligned}
\left\|g^{\triangle}(t)\right\|_{H^{2}} & \leq \int_{t}^{\infty}\|F(t)\|_{H^{2}} d s \\
& \leq \int_{t}^{\infty} C_{1} \varepsilon^{3}\langle s\rangle^{-3}+C \varepsilon^{5}\langle s\rangle^{-7 / 2} d s \leq C_{1} \varepsilon^{3}\langle t\rangle^{-2}
\end{aligned}
$$

if $\varepsilon_{0}$ is sufficiently small. This shows that the map $\Omega$ maps $\mathcal{A}$ into itself. Similarly one can show $\left\|\Omega g_{1}-\Omega g_{2}\right\|_{\mathcal{A}} \leq\left\|g_{1}-g_{2}\right\|_{\mathcal{A}}$, if $g_{1}, g_{2} \in \mathcal{A}$. Therefore our map $\Omega$ is a contraction mapping and we have a fixed point. Hence we have a solution $\psi(t)$ of the form (4.2) with $e^{-i t H_{0}} \xi_{\infty}$ as the main profile.

Remark. The above existence result holds no matter how many bound states $H_{0}$ has. The situation is different if we linearize around a nonlinear excited state. In that case, the propagator $e^{t \mathcal{L}_{1}},\left(\mathcal{L}_{1}\right.$ is the linearized operator $)$, may not be bounded in whole $L^{2}$.

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